EXERCISES, JUNE 13, 2014

1. Linear systems, orthogonality etc

Ex. 1.1. *In* $V_4(\mathbb{R})$, *let* $W = L((1,2,1,-1),(1,1,-1,-1),(-1,1,5,1)$. *Find dimension and a basis of* W. *Find dimension and a basis of* $W^\perp$.

Ex. 1.2. *Find a basis of* $V_4(\mathbb{R})$ *containing the vectors* $(1,1,-1,1)$ *and* $(1,2,0,1)$.

Ex. 1.3. *In* $V_4(\mathbb{R})$, *with usual dot product. (a) Find a orthonormal basis* $\{e_1,e_2,e_3,e_4\}$ *of* $V_4(\mathbb{R})$ *such that* e_1 *is parallel to* $(0,3,4,0)$. *(b) Compute the distance between* $(1,0,0,0)$ *and* $L((0,3,4,0))^\perp$.

Ex. 1.4. *In* $V_4(\mathbb{R})$, *with the usual dot product. Let* $u_1 = (1,2,-2,3)$ *and* $u_2 = (2,3,-1,-1)$. *(a) Find a orthogonal basis* $\{v_1,v_2,v_3,v_4)$ *of* $V_4(\mathbb{R})$ *such that* $L(v_1,v_2) = L(u_1,u_2)$. *(b) Find an element* $u \in L(u_1,u_2)$ *and an element* $w \in L(u_1,u_2)^\perp$ *such that* $(1,0,0,0) = u + w$. *(c) Compute* $d((1,0,0,0),L(u_1,u_2))$ *and the point of* $L(u_1,u_2)$ *nearest to* $(1,0,0,0)$.

Ex. 1.5. *(a) In* V_3, *find a system of linear equations whose space of solutions is the line* $L((1,2,-1)) = \{t(1,2,-1) \mid t \in \mathbb{R}\}$. *(b) In* $V_4(\mathbb{R})$ *find a system of linear equations whise space of solutions is the two-dimensional linear subspace* $W = L((1,1,-1,0),(1,2,1,1)) = \{t(1,1,-1,0)+s(1,2,1,1) \mid t,s \in \mathbb{R}\}$.

Ex. 1.6. *In* $V_2(\mathbb{R})$ *define an inner product by*

$$< (x_1,x_2),(y_1,y_2) >= 3x_1y_1 - 2x_1y_2 - 2x_2y_1 + 3x_2y_2.$$

(a) Verify that this is in fact an inner product. (b) Find the orthogonal complement of $L((1,1))$ *with respect to this inner product. (c) Find the distance between* $(1,0)$ *and* $L((1,1))$ *with respect to this inner product. (d) Find an orthonormal basis of* $V_2(\mathbb{R})$ *with respect to this inner product.*

2. Determinants

Ex. 2.1. *Let* $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *(a) Compute* A^{-1}.
(b) Compute $\det(2A^2B)$, $\det(4A+B)$, $\det(2(A^3B^{-2}))$.

Ex. 2.2. *For t varying in $\mathbb{R}$ let us consider the matrix* $M_t = \begin{pmatrix} 1 & 0 & -1 & t \\ t & t+2 & 1 & t+2 \\ t+1 & 0 & 1 & 0 \\ 1 & 0 & 2 & t \end{pmatrix}$.
(a) Compute $\det M_t$ *as a function of t and find the values of t such that* $rk(M_t) = 4$.
(b) For t varying in $\mathbb{R}$, let L_t and R_t be the lines of cartesian equations

$$L_t : \begin{cases} x_1 - x_3 = t \\ tx_1 + (t+2)x_2 + x_3 = t+2 \end{cases} \qquad \text{and} \qquad R_t : \begin{cases} (t+1)x_1 + x_3 = 0 \\ x_1 + 2x_3 = t \end{cases} .$$

Find the values of t such that the lines L_t and R_t meet at a point (hint: use (a)).

Ex. 2.3. *For t varying in $\mathbb{R}$ let* $A_t = \begin{pmatrix} t & t & t & -t \\ 1 & 0 & t^2-t & -1 \\ 1 & 0 & 0 & 1 \\ t^3 & t & t & -t \end{pmatrix}$.
(a) Compute $\det(A_t)$ *and find the values of t such that* $rk(A_t) = 4$.
(b) Find the values of t such that the system of linear equations having A_t as augmented *matrix has solutions (hint: use (a)) For such values of t solve the system.*

3. Eigenvalues and eigenvectors

Ex. 3.1. *Let* $A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 3 & 0 & 5 \\ 0 & 5 & -2 & 5 \\ 4 & 0 & 0 & 2 \end{pmatrix}$.
(a) Find the eigenvalues of A.
(b) Let $T_A : V_4 \to V_4$ be the linear transformation defined by $T_A(X) = A\,X$. Is there a basis of V_4 composed by eigenvectors of T_A? If the answer is yes, produce explicitly such a basis.
(c) Is there a matrix C such that $C^{-1}AC$ is a diagonal matrix? If the answer is yes produce explicitlt such a matrix.

Ex. 3.2. *Let* $\mathcal{B} = \{(0,1,-1),\ (0,1,1), (2,3,0)\}$.
(a) Prove that $\mathcal{B}$ is a basis of V_3.
Now Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ *and let $T : V_3 \to V_3$ be the linear transformation such that* $m^{\mathcal{B}}_{\mathcal{B}}(T) = A$.
(b) Find the eigenvalues of T.
(c) Find a basis of each eigenspace of T.
(d) Is there a basis of V_3 composed of eigenvectors of T? If the answer is yes , produce explicitly such a basis.
(e) Is there a matrix C such that $C^{-1}AC$ is a diagonal matrix? If the answer is yes, produce explicitly such a matrix.

Ex. 3.3. *Let* $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -1 & -2 & 0 \end{pmatrix}$.
(a) Find the eigenvalues of A. *(b(Find a basis of each eigenspace of the transformation* $T_A : V_3 \to V_3$*, where, as susal,* $T_A(X) = AX$. *(c) Let* $v = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Ex. 3.4. *Let* $T : V_3 \to V_3$ *the linear transformation such that:* $T((1,1,0)) = (2,2,0)$, $T((0,-1,1)) = (0,-3,3)$ *e* $T((1,0,-1)) = (0,1,-1)$.
(a) Compute $rk(T)$ *and a basis of* $Im(V_3)$.
(b) A basis of $N(T)$. *(c) Eigenvalues and eigenvectors of* T. *(d)* $T((1,0,0))$.

Ex. 3.5. *(a) For* t *varying in* $\mathbb{R}$, *let* $A_t = \begin{pmatrix} 1 & t+2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$. *Let* $T_{A_t} : V_3 \to V_3$ *be the linear transformation defined, as usual,* $T)A_t(X) = A_t X$. *Find the values of* t *such that* T_{A_t} *is diagonalizable (that is, there is a basis of* $V_3(\mathbb{R})$ *composed by eigenvectors of* T_{A_t}.
(b) Same question for $B_t = \begin{pmatrix} 1 & t+2 \\ 2 & 3 \end{pmatrix}$.

Ex. 3.6. *Let* $V = L((1,0,1,1),(0,0,1,-1))$. *Let* $R_V : V^4 \to V^4$ *be the reflection with respect to* V *and let* $P_{V^\perp} : V^4 \to V^4$ *be the projection of* V^4 *on* $V^\perp$. *(a) Find the range and the eigenvalues of the linear transformation* $P_{V^\perp} \circ R_V$. *(b) Find a base of each eigenspace of* $P_{V^\perp} \circ R_V$.

Ex. 3.7. *Let* $T : V^3 \to V^3$ *be the linear transformation such that:* $T((1,1,0)) = (2,2,0)$, $T((0,-1,1)) = (0,-3,3)$ *e* $T((1,0,-1)) = (0,1,-1)$. *(a) Compute* $rk(T)$ *and find a basis of* $T(V_3)$. *(b) Find a basis of* $N(T)$. *(c) Find the eigenvalues and eigenvectors of* T. *(d) Compute* $T((1,0,0))$.