ADDENDUM CONCERNING LINEAR TRANSFORMATIONS AND MATRICES, I

Up to Section 5 you should study the material in the book with no changes. You should also study Section 9 (Linear transformation with prescribed values) with no changes (but keeping in mind the examples provided in the lectures).

1. Replacement of sections 6 and 7 : Inverses and one-to-one transformations

Concerning the topic of sections 6 and 7, we will will content ourselves of the following simpler version.

We recall the following well known concepts concerning functions.

Let X and Y be two sets and $T : X \to Y$ a function. We recall that:
• T is said to be *injective* (or *one-to-one*) if the following condition holds: *let* $x, x' \in X$*, with* $x \neq x'$*. Then* $T(x) \neq T(x')$. In words: T sends distinct elements of X to distinct elements of Y.
• T is said to be *surjective* if *for each* $y \in Y$ *there is a* $x \in X$ *such that* $T(X) = y$*.* In words: every element of Y is the value, via T, of an element of X.
• T is said to be *bijective* if it is injective and surjective. This means that for each $y \in Y$ there is a UNIQUE $x \in X$ such that $T(x) = y$. In words: every element of Y corresponds, via T, to a unique element of X.
• T is said to be *invertible* if *there is a function* $S : Y \to X$ *such that* $ST = I_X$ *and* $TS = I_Y$ (where I_X and I_Y denote the identity functions of X and Y). We have the following proposition:

Proposition 1.1. *(a)* T *is invertible if and only if it is bijective.*
(b) In this case S *is the unique function with the above properties. It is denoted* T^{-1}*, the inverse of* T*.*
(c) Assume that T *is bijective. Let* $S : Y \to X$ *such that* $TS = I_Y$*. Then* $S = T^{-1}$*. In particular,* $ST = I_X$*.*
(d) Assume that T *is bijective. Let* $S : Y \to X$ *such that* $ST = I_X$*. Then* $S = T^{-1}$*. In particular* $TS = I_Y$*.*

Proof. (a) If T is bijective one defines S as follows: given $y \in Y$, $S(y)$ is be the unique $x \in X$ such that $T(x) = y$. Conversely, if $ST = I_X$ then T is injective, since if $T(x) = T(x')$ then $S(T(x)) = S(T(x'))$. But $S(T(x)) = x$ and $S(T(x')) = x'$. If $TS = I_Y$, then T is surjective, since, for each $y \in Y$, $y = T(S(y))$.
(b) This follows from the proof of point (a).
(c) Assume that T is bijective. Since $T(S(y)) = y$ for all $y \in Y$, $S(y)$ must

be the unique $x \in X$ such that $T(x) = y$.
(d) is similar. □

Note that, without the hypothesis of bijectivity, (c) and (d) of the previous proposition are false (see for example Exercise 16.8-27 Vol. I, corresponding to 2.8.27 Vol. II).

In general, injectivity, surjectivity and bijectivity are quite unpredictable properties of functions. However, for *linear transformations*, especially in the finite-dimensional case, everything is much simpler. In the rest of the section we will exploit this point. We begin with injectivity.

Proposition 1.2. *Let V and W be linear spaces and $T : V \to W$ be a linear transformation. Then T is injective if and only if $N(T) = \{O_V\}$.*

Proof. We know that, since T is linear, $T(O_V) = O_W$. If T is injective, there is no other $v \in V$ such that $T(v) = O_W$. Therefore $N(T) = \{O_V\}$. For the other implication, let us assume that $N(T) = \{O_V\}$. Let $v, v' \in V$ such that $T(v) = T(v')$. This can be rewritten as $T(v) - T(v') = O_W$ or, since T is linear, $T(v - v') = O_W$, that is $v - v' \in N(T)$. But we assumed that $N(T) = \{O_V\}$. Hence $v - v' = O_V$, that is $v - v'$. Therefore T is injective. □

The next proposition deals with the finite-dimensional case

Proposition 1.3. *Let V and W be finite-dimensional linear spaces and $T : V \to W$ be a linear transformation. Then the following are equivalent:*
(a) T is injective;
(b) $\dim T(V) = \dim V$;
(c) If $\{e_1, \dots, e_n\}$ is a basis of V then $\{T(e_1), \dots, T(e_n)\}$ is a basis of W.

Proof. (a)⇔(b) follows from Prop. 1.2 and the Nullity + Rank theorem. In fact, $N(T) = \{O\}$ if and only if $\dim N(T) = 0$. Since, by Nullity + Rank, $dimT(V) = \dim V - \dim N(T)$, T is injective if and only if $\dim T(V) = \dim V$.
(b)⇒(c) is as follows. In the first place we note that, since $e_1, \dots, e_n$ span V, then in any case $T(e_1), \dots, T(e_n)$ span $T(V)$, because, given $v \in V$, $v = \sum c_i e_i$. by the linearity if T, $T(v) = \sum T(c_i e_i) = \sum ciT(e_i)$. If $\dim T(V) = \dim V = n$ then $\{T(e_1), \dots, T(e_n)\}$ is a basis, since it is a spanning set formed by n elements. (c)⇒(b) is obvious. □

Concerning bijectivity and invertibility, we start by recording the following easy fact, which does not need finite-dimensionality

Proposition 1.4. *Let V and W be linear spaces and $T : V \to W$ be a linear transformation. If T is invertible then also $T^{-1} : W \to V$ is a linear transformation.*

Proof. Let $w_1, w_2 \in W$ and let v_1, v_2 be the unique elements of V such that $T(v_1) = w_1$ and $T(v_2) = w_2$. hence $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. Since T is linear, $T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$. Therefore

$$T^{-1}(w_1 + w_2) = v_1 + v_2 = T^{-1}(w_1) + T^{-1}(w_2).$$

Moreover, let $c \in \mathbb{R}$. We have that $T(c\,v_1) = c\,T(v_1) = c\,w_1$. Therefore

$$T^{-1}(c\,w_1) = c\,v_1 = c\,T^{-1}(w_1) \qquad \square$$

In the finite-dimensional case, Proposition 1.3 has the following consequences

Corollary 1.5. *Let V and W be finite-dimensional linear spaces and $T : V \to W$ be a linear transformation.*
(a) If T is bijective (or, equivalently, invertible) then $\dim V = \dim W$. *(b) Conversely, assume that* $\dim V = \dim W$. *Then the following are equivalent*
(i) T is injective;
(ii) T is surjective,
(iii) T is bijective.

Proof. It is sufficient to prove the equivalence of (i) and (ii). Assume that T is injective. Then, by Prop. 1.3, $\dim T(V) = \dim V = \dim W$. Therefore $T(V) = W$, that is T is surjective.
Assume that T is surjective, that is $\dim T(V) = \dim W (= \dim V)$. By nullity+rank, this implies that $\dim N(T) = 0$. Thus Prop. 1.3 implies that T is injective. $\square$

For example, let $T : V_3 \to V_3$ defined by $T((x, y, z)) = (x - 2y + 3z, x + y + z, x - y - z)$. By the previous Corollary and Theorem 1.3 T is bijective (hence invertible) if and only if $N(T) = \{O\}$. $N(T)$ is the space of solutions of the system of linear equations

$$\begin{cases} x - 2y + 3z = 0 \\ x + y + z = 0 \\ x - y - z = 0 \end{cases}$$

By what we studied in the first semester, hence $N(T) = \{O\}$ means that this system has only the trivial solution $(0, 0, 0)$ (or, equivalently, that the columns of the system are linearly independent). This can be checked by computing the determinant

$$\det \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} = -13$$

Since the determinant is non-zero then (go back to the lectures of the first semester, or to Chapter 15 of Vol. I !) $(0, 0, 0)$ is the only solution. Therefore T is bijective. Later on we'll see an efficient way to compute the inverse transformation T^{-1}.

2. Matrices and linear transformations: supplementary notes

It is conceptuallly easier to study the remaining sections of the chapter on linear transformation and matrices as follows: read the beginning of Section 10 for generalities about matrices. Then skip, for the moment, the Theorem and the subsequent examples, and go directly Section 13 (Linear spaces of matrices). Then skip, for the moment, Section 14 and go directly to Section 15 (Multiplication of matrices). At this point go back to the Theorem and Examples of Section 10 and Section 14, which are about the correspondence between matrices and linear transformations. Here are some supplementary notes about this material, which hopefully may helpful to understand the meaning of these results. Note: the contents of Section 11 (Construction of a matrix representation in diagonal form) should be *skipped*.

We will use the following notation: $\mathcal{M}_{m,n}$ will denote the set of all $m \times n$ matrices. Equipped with the operations of matrix addition and scalar multiplication $\mathcal{M}_{m,n}$ is in fact a linear space (Section 13.)

Moreover, given a matrix $A \in \mathcal{M}_{m,n} = (a_{ij})$, the *transpose of* A, is the matrix $A^t \in \mathcal{M}_{n,m}$ defined as $A^t = (a_{ji})$. In practice, the columns of A are the rows of A^t and the rows of A are the columns of A^t.

Example 2.1. *Let* $A = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 2 & 1 \end{pmatrix}$. *Then* $A^t = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ -3 & 1 \end{pmatrix}$.

In the sequel it will more confortable to write n-yuples of V_n as *column vectors*, that is matrices with one column. Given $X = (x1, \dots, x_n) \in V_n$ the corresponding column vector is $X^t = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$.

Definition 2.2 (Standard linear transformation associated to a $m \times n$ matrix)**.** *Let* $A \in \mathcal{M}_{M,n}$. *The standard linear transformation associated to* A *is the linear trasformation*

$$T_A : V_n \to V_m$$

defined as follows. We see the elements of V_n *and* V_m *as column vectors*

$$X^t = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \in \mathcal{M}_{n,1} \qquad\qquad Y^t = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_m \end{pmatrix} \in \mathcal{M}_{m,1}$$

Then T_A *is defined as*

$$T_A(X^t) = AX^t$$

where AX^t *denoted the multiplication of the* $m \times n$ *matrix* A *with the* $n \times 1$ *matrix (= column vector of length* n*)* X^t. *The result is a* $m \times 1$ *matrix (=column vector of length* m*). In coordinates:*

$$T_A(X^t) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \\ \dots & & \\ \dots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \cdot \\ \cdot \\ \cdot \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$$

Here are some remarks:
(a) The column vector $T_A(X^t) = AX^t$ can be written also as

$$x_1 \begin{pmatrix} a_{11} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{pmatrix} = x_1 A^1 + \cdots + x_n A^n.$$

(b) A system of linear equations

$$\begin{cases} A_1 \cdot X = b_1 \\ \dots \\ \dots \\ \dots \\ A_m \cdot X = b_m \end{cases}$$

can be written in compact form as

$$AX^t = B$$

where B is the (column) vector of constant terms. This has the conceptual advantage of seeing a system composed by many equations as a single *vector* equation, that is an equation whose unknown is a vector. For example,

$$\begin{cases} 2x + 3y - z = 3 \\ 2x + y + 2z = 4 \end{cases} \qquad \Leftrightarrow \qquad \begin{pmatrix} 2 & 3 & -1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

(c) $T_A(V_n)$, the range of T_A, is, by definition, the subspace of V_m formed by the $B \in V_m$ such that the system of linear equations $AX = B$ (see the above remarks) has some solutions. This shows that $T_A(V_n) = L(A^1, \dots, A^n)$.
(d) The linear transformation T_A is nothing else but the "linear transformation defined by linear equations" of Example 4 of Section 1 of the textbook. As remarked in Example 4 of Section 2 of the book, $N(T_A)$, the null-space of T_A, is the subspace of V_n formed by the solutions of the homogenous system $AX = 0$.
(e) Denoting

$$E^1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \dots\dots, E^n = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

the "coordinate unit vectors" of V_n – that is the vectors of the so-called "canonical basis" of V_n written as column vectors – then

$$AE^i = A^i$$

(where A^i are the column vectors of A).

3. The rank of a matrix

Let us start with the following

Definition 3.1. *Let $A \in \mathcal{M}_{M,n}$ be a matrix. The rank of A, denoted $rk(A)$, is defined as the rank of the linear transformation $T_A : V_n \to V_m$, $X \mapsto AX^t$* (compare Def. 2.2).

By Remark (c) after Definition 2.2 we know that $rk(A) = \dim L(A^1, \dots, A^n)$, where $A^1, \dots, A^n$ are the columns of A. In other words, $rk(A)$ is the maximal number of independent columns of A (see Thms 15.5 and 15.7 of Vol. I, corresponding to Thms 1.5 and 1.7 of Vol. II). We have the remarkable

Proposition 3.2. *$rk(A) = \dim L(A_1, \dots, A_m)$, where $A_1, \dots, A_m$ are the rows of A. In other words, the maximal number of independent columns of A equals the maximal number of independent rows of A.*

Proof. By the nullity + rank Theorem, $rk(T_A) = n - \dim N(T_A)$. On the other hand, by definition,

$$N(T_A) = L(A_1, \dots, A_m)^{\perp}.$$

This is because $N(T_A)$ is the set of solution of the homogeneous system

$$\begin{cases} A_1 \cdot X = 0 \\ \dots \\ \dots \\ A_m \cdot X = 0 \end{cases}$$

Therefore $\dim N(T_A) = \dim L(A_1, \dots, A_m)^{\perp} = n - \dim L(A_1, \dots, A_m)$, see Lemma 3.4 below. (Note that the rows $A_1, \dots, A_m$ have $n(=$ number of columns od $A)$ components, hence they are vectors of V_n). Putting evething together $Rk(A) = rk(T_A) = n - \dim N(T_A) = n - (n - \dim L(A_1, \dots, A_m)) = \dim L(A_1, \dots A_m)$. □

Example 3.3. *Let $A_1 = (1, 2, 3)$, $A_2 = (3, 4, 5)$ and let $A_3 := A_1 + A_2 = (4, 6, 8)$. Let us consider the matrix*

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{pmatrix}$$

Since the maximal number of independent rows is 2 then the maximal number of independent columns is 2. In particular, the columns are dependent. Exercise: check this!

Lemma 3.4. *Let V be a finite-dimensional linear space and let $W \subset V$ be a linear subspace of V. Then $\dim W^{\perp} = \dim V - \dim W$.*

Proof. Let $n = \dim V$ and $k = \dim W$. Let $\{w_1, \dots, w_k\}$ be a basis of W. We complete it to a basis of V: $\mathcal{B} = \{w_1, \dots, , w_k, u_{k+1}, \dots, u_n\}$ (Apostol, Vol. I, Thm 15.7(b)). Applying Gram-Schmidt orthogonalization to $\mathcal{B}$ (Apostol, Vol. I, Thm 15.13) we find an orthogonal basis $\{e_1, \cdots, e_k, f_{k+1}, \ldots, f_n\}$ such that $L(e_1, \ldots, e_k) = L(w_1, \ldots, w_k) = W$. Therefore $\{f_{k+1}, \ldots, f_n\}$ is an orthogonal basis of $W^\perp$. In particular $\dim W^\perp = n - k$. □

4. Computation of the rank of a matrix, with application to systems of linear equations

We have the following easy result, summarizing the qualitive behaviour of systems of linear equations. We will need the followimg terminology: given a linear system $AX = B$, with $A \in \mathcal{M}_{m,n}$ and $B \in \mathcal{M}_{m,1}$ (see Remark (b) after Definition 2.2), we denote $A|B$ the $m \times (n+1)$-matrix whose first n columns are the columns of A and the last one is B. This is called the *augmented* or *complete* matrix of the linear sistem.

Theorem 4.1 (Rouché-Capelli)**.** *Let $AX^t = B$ be a linear system. Then*
(a) A has some solutions if and only if $rk(A) = rk(A|B)$ (if this happens the system is sometimes called *compatible*)*,*
(b) In this case, the set of all solutions of the system is of the form $v + W = \{v + w \mid w \in W\}$, where $v \in V_n$ is a solution of the system and W is the linear subspace of V_n formed by all solutions of the homogeneous system $AX^t = O$. In particular, there is a unique solution if and only if (a) holds and $\dim W = 0$.
(c) $dimW = n - rk(A)$.

Proof. (a) The system has some solutions if and only if the column vector B is a linear combinations of the columns $A^1, \ldots, A^n$. This means exactly that the rank(= number of independent columns) of $A|B$ is the same as the rank of A.
(b) Let v_1 and v two solutions of the system, that is $Av_1^t = B$ and $Av^t = B$. Then $A(v_1^t - v_2^t) = 0$. therefore $v_1 - v$ is a solution of the homogeneous system $AX^t = O$, that is $v_1 - v \in W$. Therefore $v_1 = v + w$ for a $w \in W$.
(c) This is just a restatement of the nullity + rank Theorem. □

In order to solve a linear system by computing the rank of the matrices A and $A|B$ one can use the row-elimination method of Gauss-Jordan. Here are some examples (see also the examples given in the lectures and those in the book at Section 18).

Example 4.2. $\begin{cases} x + 2y + z + t = 1 \\ x + 3y + z - t = 2 \\ x + 4y + z - 3t = 3 \\ 2x + y \quad + z = 2 \end{cases}$.

We will make use of the following modifications of the equations of the system:
(a) exchanging to equations;
(b) multiplying an equation by a non-zero scalar,
(c) adding to an equation a scalar multiple of another equation.

Clearly such modifications produce *equivalent*(= having the same solutions) systems. Since the equations correspond to the rows of the associated augmented matrix $A|B$, the above modifications correspond to modifications of the rows of $A|B$. Note that, even if after operating one such modifications the rows of the modified matrix do change, the linear span of the rows remains the same. Therefore such modifications leave unchanged the rank.

$$A|B = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 4 & 1 & -3 & 3 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & -4 & 2 \\ 0 & -3 & -2 & -1 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & -2 & -7 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we arrived to a matrix in *r*ow-echelon form, corresponding to the equivalent system

$$\begin{cases} x & + & 2y & + & z & + & t & = 1 \\ & & y & & & - & 2t & = 1 \\ & & & - & z & - & 7t & = 3 \end{cases}$$

Let us denote $A'X = B'$ this new system. We have that $rk(A|B) = rk(A'|B') = 3$, since the non-zero rows of a row-ladder matrix are clearly independent (exercise!). For the same reason, $rk(A) = rk(A') = 3$. Therefore the system has solutions (note that, in general, there are solutions if and only if, in the final ladder matrix there is no row of the form $\begin{pmatrix} 0 & \dots & 0 & a \end{pmatrix}$ with $a \neq 0$). Even before computing the explicit solutions, we know that the set of solutions will have the form

$$v + W, \qquad \text{with} \quad \dim W = 1$$

since $n - rk(A|B) = 4 - 3 = 1$. We compute the solutions starting from the last equation: $z = -3 - 7t$, $y = 1 + 2t$, $x = 1 - 2y - z - t = 1 - 2(1 + 2t) - (-3 - 7t) - t = 2 + 2t$. Therefore the solutions are the 4-tuples of the form

$$\begin{pmatrix} 2+2t \\ 1+2t \\ -3-7t \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ -7 \\ 1 \end{pmatrix} = v + w, \quad \textit{where} \quad w \in W = L(\begin{pmatrix} 2 \\ 2 \\ -7 \\ 1 \end{pmatrix}$$

Example 4.3. $$\begin{cases} x + 2y + z + t = 1 \\ x + 3y + z - t = 2 \\ x + 4y + z - 3t = 2 \\ 2x + y \quad + z = 2 \end{cases}.$$

$$A|B = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 4 & 1 & -3 & 2 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & -4 & 1 \\ 0 & -3 & -2 & -1 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & -2 & -7 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The system has no solution because the last equation is $0 = -1$. This corresponds to the fact that $rk(A) = 3$ while $rk(A|B) = 4$. *In general, the rank of a matrix in row-echelon form is the number of non-zero rows).*

Example 4.4. $\begin{cases} x + 2y + z + t = 1 \\ x + 3y + z - t = 2 \\ x + 4y + z - 2t = 3 \\ 2x + y \quad + z = 2 \end{cases}$.

$$A|B = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 4 & 1 & -2 & 3 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & -3 & 2 \\ 0 & -3 & -2 & -1 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & -2 & -7 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In this case $rk(A) = rk(A|B) = 4$. Therefore there is a *unique* solution, since $\dim W = 4 - 4 = 0$ (this simply means that the four columns of A are independent). *Note that, since $rk(A) = 4$ for all possible vectors of constant terms $B' \in V_4$ the system $AX = B'$ has a unique solution!)* Exercise: find the solution.

5. Exercises

Ex. 5.1. Let $T : V_4 \to V_4$ be the linear transformation $T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}) = \begin{pmatrix} x_1 + x_3 \\ 2x_1 + x_2 + 2x_3 + 2x_4 \\ x_1 + x_2 + x_3 + 4x_4 \\ x_1 + x_2 + x_3 + 2x_4 \end{pmatrix}$

(a) Find a basis of $T(V_4)$.

(b) Let $v = (-3, -3, 0, 0)$. Does v belong to $T(V_4)$? Is case of positive

answer, find the components of v with respect to the basis of $T(V_4)$ found in (a).
(c) Find a basis of $N(T)$.

Ex. 5.2. For t varying in $\mathbb{R}$, let us consider the linear system

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ x_1 + 2x_2 = 0 \\ x_1 + x_2 + (t-1)x_3 = 2 \\ x_1 + x_2 - x_3 = t \end{cases}$$

Find the values of t such that the system has solutions, and those such that the system has a unique solution. For such values of t, solve the system.

Ex. 5.3. Let us consider the lines of V_3 $L : \begin{cases} x+y=0 \\ x+2y+z=0 \end{cases}$ and $M : \begin{cases} x-y=1 \\ x+5y+z=0 \end{cases}$. What is the correct statement among the following: (a) they meet at a point; (b) they are parallel; (c) they don't meet but they are not parallel (in which case they are called *skew lines*).

Ex. 5.4. Solve the following systems (non necessarily with row elimination!):

(a) $\begin{cases} 2x_1 - x_2 + x_3 = 1 \\ 3x_1 + x_2 - x_3 = 3 \\ x_1 + 2x_2 - x_3 = -2 \end{cases}$

(b) $\begin{cases} 4x+y+z+2v+3w=0 \\ 14x+2y+2z+7v+11w=0 \\ 15x+3y+3z+6v+10w=0 \end{cases}$

(c) $\begin{cases} 5x+4y+7z=3 \\ x+2y+3z=1 \\ x-y-z=0 \\ 3x+3y+5z=2 \end{cases}$ (d) $\begin{cases} 19x-y+5z+t=3 \\ 18x+5z+t=1 \\ 6x+9y+t=1 \\ 12x+18y+3t=3 \end{cases}$

Ex. 5.5. Let us consider the homogeneous linear system $\begin{cases} x_1+x_2+x_4=0 \\ x_1+2x_3+x_4=0 \\ x_2-2x_3=0 \end{cases}$.
(a) Find the dimension and a basis of the space of solutions;
(b) Find the dimension and a basis of the linear span of the columns of the linear system.
(c) Find the dimension and a basis of the linear span of the rows of the linear system.

Ex. 5.6. For which values of $t \in \mathbb{R}$ the system

$$\begin{cases} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ x_1 & + & (t+4)x_2 & - & 3x_3 & = & 1/2 \\ -2x_1 & + & (t-2)x_2 & + & (2t-6)x_3 & = & 5/2 \end{cases}$$

has respectively no solutions, a unique solution, infinitely many solutions?

Ex. 5.7. Find for which values of $t, a \in R$, the system

$$\begin{cases} x_1 + x_2 + tx_3 & = 1 \\ 2x_1 + tx_2 + x_3 & = -1 \\ 6x_1 + 7x_2 + 3x_3 & = a \end{cases}$$

has respectively no solution, a unique solution, infinitely many solutions. For this last case, describe the set of solutions.