EXERCISES ON QUADRATIC FORMS

Ex. 0.1. For $c \in \mathbb{R}$, let us consider the quadratic form $Q_c(x, y, z, t) = (2+c)x^2 + xy + xz + xt + (2+c)y^2 + yz + yt + (2+c)z^2 + zt + (2+c)t^2$ Reduce Q_e to diagonal form.

Ex. 0.2. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Let us consider the real quadratic form

 $Q(X) = XAX^T.$

(a) Reduce Q to diagonal form by means of an orthonormal basis (that is: find an orthonormal basis \mathcal{B} of V_4 and real numbers λ_i , i = 1, 2, 3, 4 such that $Q(X) = \sum_{i=1}^{4} \lambda_i x^{\prime 2}$, where x'_i are the components of X with respect to the basis \mathcal{B} .

(b) Is Q a positive real quadratic form (this mean that $Q(X) \ge 0$ for all $X \in V_4$ and equality holds if and only if X = O).

(c) Compute the maximum and minimum value Q on the unite sphere S^3 . Compute the points of maximum and the points of minimum.

Ex. 0.3. For a, b, c varying in \mathbb{R} , let us consider the function $F_{a,b,c}$: $[0, 2\pi] \to \mathbb{R}$, $F_{a,b,c}(\theta) = a \cos^2(\theta) + b \cos(\theta) \sin(\theta) + c \sin^2(\theta)$. Assume that max $F_{a,b,c} = 5$, min $F_{a,b,c} = -2$ and that $\theta = \pi/3$ is a point of maximum. Compute a, b and c. (Hint: think F as $Q : S^1 \to \mathbb{R}$, where S^1 is the unit circle and Q is a quadratic form).

Ex. 0.4. (a) Let C be an orthogonal matrix. Prove that det C = +1 or det C = -1. (Hint: use the fact that, for any square matrix, det $A = \det A^t$. (b) Let $\mathcal{B} = \{A^1, A^2, A^3, A^4\}$ be an orthogonal basis of V_4 such that $|| A^1 || = 3$, $|| A^2 || = 5$, $|| A^3 || = 2$, $|| A^4 || = 2$. Let $A = (A^1 A^2 A^3 A^4)$ be the matrix whose columns are A^1 , A^2 , A^3 , A^4 . Compute $|\det A|$.

Ex. 0.5. Let v = (1, 2, 1). Let us consider the function $T : V_3 \to V_3$ defined as follows: $T(X) = X \times v$.

(a) Prove that T is a linear transformation. (b) Is T symmetric?

(b) Compute the matrix representing T with respect to the canonical basis of V_3 and compare the result with the answer given to (a)

(c) Find an orthonormal basis of V_3 $\mathcal{B} = \{v_1, v_2, v_3\}$ with v_1 parallel to v and compute $m_{\mathcal{B}}^{\mathcal{B}}(T)$.

Ex. 0.6. Reduce the following real quadratic forms Q to diagonal form. Establish if they are positive, negative or what. Find explicitly (if any) nonzero vectors X such that Q(X) > 0 or Q(X) < 0 or Q(X) = 0. Find the maximum and minimum value of Q on the unit sphere and find the points of maximum and the points of minimum. (a) $Q(x, y) = x^2 - 3xy + y^2$. (b) $Q(x, y) = -2x^2 - 5y^2 + 12yz + 7z^2$

b) maximum and the points of minimum. (a) $Q(x, y) = x^2 - 3xy + y^2$. (b) $Q(x, y, z) = -2x^2 - 5y^2 + 12yz + 7z^2$. (c) $Q(x, y, z) = 4xy + 3y^2 + z^2$. (d) $Q(x, y, z) = 25x^2 - 7y^2 + 48yz + 7z^2$. (e) $Q(x, y, z) = \sqrt{2}(x^2 + y^2 + z^2) + 2x(y - z)$ (f) $Q(x, y, z) = 5x^2 - y^2 + z^2 + 4xy + 6xz$ (g) Q(x, y, z) = 2(xy + yz + zx)

Ex. 0.7. Let
$$Q(x_1, x_2, x_3, x_4, x_5) = \sum_{1=1}^5 x_i^2 + \sum_{i < j} x_i x_j$$
. Is Q positive?

Ex. 0.8. Reduce the equations of teh following conics to canonical form, find the symmetry axes, the center, the foci. (everything in coordinates with respect to the canonical basis).

(a) $5x^2 + 5y^2 - 6xy + 16\sqrt{2}x + 38 = 0.$ (b) $5x^2 - 8xy + 5y^2 + 18x - 18y + 9 = 0$ (c) $2x^2 - 2\sqrt{3}xy + 2x + 2\sqrt{3}y - 5 = 0$ (d) $x^2 + 2xy + y^2 + 2x - 2y = 0$ (e) $5x^2 + 8xy + 5y^2 - 18x - 18y + 9 = 0$