EXERCISES ON DETERMINANTS, EIGENVALUES AND EIGENVECTORS

1. Determinants

Ex. 1.1. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (a) Compute A^{-1} .

(b) Compute $\det(2A^2B)$, $\det(4A+B)$, $\det(2(A^3B^{-2}))$.

Ex. 1.2. For t varying in
$$\mathbb{R}$$
 let us consider the matrix $M_t = \begin{pmatrix} 1 & 0 & -1 & t \\ t & t+2 & 1 & t+2 \\ t+1 & 0 & 1 & 0 \\ 1 & 0 & 2 & t \end{pmatrix}$.

(a) Compute det M_t as a function of t and find the values of t such that $rk(M_t) = 4$.

(b) For t varying in \mathbb{R} , let L_t and R_t be the lines of cartesian equations

$$L_t: \begin{cases} x_1 - x_3 = t \\ tx_1 + (t+2)x_2 + x_3 = t+2 \end{cases} \quad and \quad R_t: \begin{cases} (t+1)x_1 + x_3 = 0 \\ x_1 + 2x_3 = t \end{cases}$$

Find the values of t such that the lines L_t and R_t meet at a point (hint: use (a)).

Ex. 1.3. For t varying in
$$\mathbb{R}$$
 let $A_t = \begin{pmatrix} t & t & t & -t \\ 1 & 0 & t^2 - t & -1 \\ 1 & 0 & 0 & 1 \\ t^3 & t & t & -t \end{pmatrix}$.

(a) Compute $det(A_t)$ and find the values of t such that $rk(A_t) = 4$.

(b) Find the values of t such that the system of linear equations having A_t as augmented matrix has solutions (hint: use (a)) For such values of t solve the system.

Ex. 1.4. Let
$$A \in \mathcal{M}_{4,4}$$
 be a matrix whose rows are $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$ and sup-

pose that det A = -3. Let $C \in \mathcal{M}_{4,4}$ l be the matrix whose rows are $\begin{pmatrix} 2A_1 \\ 3A_1 + 4A_2 \\ -2A_1 + 2A_2 + A_4 \\ A_1 + A_2 + 3A_3 \end{pmatrix}$. Compute det(C) and det $(\frac{1}{2}C)$.

2. EIGENVALUES AND EIGENVECTORS

Ex. 2.1. Let
$$A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 3 & 0 & 5 \\ 0 & 5 & -2 & 5 \\ 4 & 0 & 0 & 2 \end{pmatrix}$$
.

(a) Find the eigenvalues of A.

(b) Let $T_A : V_4 \to V_4$ be the linear transformation defined by $T_A(X) = A X$. Is there a basis of V_4 composed by eigenvectors of T_A ? If the answer is yes, produce explicitly such a basis.

(c) Is there a matrix C such that $C^{-1}AC$ is a diagonal matrix? If the answer is yes produce explicitly such a matrix.

Ex. 2.2. Let
$$\mathcal{B} = \{(0, 1, -1), (0, 1, 1), (2, 3, 0)\}.$$

(a) Prove that \mathcal{B} is a basis of V_3 . Now Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and let $T : V_3 \to V_3$ be the linear transformation

such that $m_{\mathcal{B}}^{\mathcal{B}}(T) = A$.

(b) Find the eigenvalues of T.

(c) Find a basis of each eigenspace of T.

(d) Is there a basis of V_3 composed of eigenvectors of T? If the answer is yes, produce explicitly such a basis.

(e) Is there a matrix C such that $C^{-1}AC$ is a diagonal matrix? If the answer is yes, produce explicitly such a matrix.

Ex. 2.3. Let
$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -1 & -2 & 0 \end{pmatrix}$$
.

(a) Find the eigenvalues of A. (b(Find a basis of each eigenspace of the transformation $T_A : V_3 \to V_3$, where, as susal, $T_A(X) = AX$. (c) Let $\begin{pmatrix} -1 \end{pmatrix}$

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
. Compute $A^{151} v$.

Ex. 2.4. Let $T: V_3 \to V_3$ the linear transformation such that: T((1,1,0)) = (2,2,0), T((0,-1,1)) = (0,-3,3) e T((1,0,-1)) = (0,1,-1).(a) Compute rk(T) and a basis of $Im(V_3)$.

(b) A basis of N(T). (c) Eigenvalues and eigenvectors of T. (d) T((1,0,0)).

Ex. 2.5. Let $T: V_4 \to V_4$ be a linear transformation such that: (a) $T((1,0,-1,2)) = (2,0,-2,4), \quad T((0,1,1,1) = (0,-5,-5,-5), \quad T((1,2,-1,0) = (2,4,-2,0); \quad (b) (2,2,-2,2) \text{ and } (1,0,0,0) \text{ are eigenvectors of } T;$ (c) det T = 100.

(i) Find the eigenvalues of T and a basis of each eigenspace of T.

(ii) Is there a basis of V_4 composed of eigenvectors of T? If the answer is yes, produce explicitly such a basis. (iv) Compute Tr(T).

3

Ex. 2.6. (a) For t varying in \mathbb{R} , let $A_t = \begin{pmatrix} 1 & t+2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$. Let $T_{A_t} : V_3 \to$

 V_3 be the linear transformation defined, as usual, $T A_t(X) = A_t X$. Find the values of t such that T_{A_t} is diagonalizable (that is, there is a basis of $V_3(\mathbb{R})$ composed by eigenvectors of T_{A_t} .

 $V_{3}(\mathbb{R}) \text{ composed by eigenvectors of } T_{A_{t}}.$ (b) Same question for $B_{t} = \begin{pmatrix} 1 & t+2\\ 2 & 3 \end{pmatrix}.$

Ex. 2.7. Let V = L((1,0,1,1), (0,0,1,-1)). Let $R_V : V^4 \to V^4$ be the reflection with respect to V and let $P_{V^{\perp}} : V^4 \to V^4$ be the projection of V^4 on V^{\perp} . (a) Find the range and the eigenvalues of the linear transformation $P_{V^{\perp}} \circ R_V$. (b) Find a base of each eigenspace of $P_{V^{\perp}} \circ R_V$. Is $P_{V^{\perp}} \circ R_V$ diagonalizable?. (c) Is $P_{V^{\perp}} \circ R_V$ a symmetric linear transformation?

Ex. 2.8. For all the matrices A of the previous exercises, take the diagonalizable ones and compute their 132th power and their (-132)th power.

Ex. 2.9. Let
$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 3 & -2 & 1 & 1 \\ -3 & -1 & -4 & -1 \\ -6 & -2 & -2 & -5 \end{pmatrix}$$
 and let $v = \begin{pmatrix} -2 \\ 1 \\ 5 \\ 0 \end{pmatrix}$. (a) Verify

that -3 is an eigenvalue of A.

(b) Find all eigenvalues of A (hint: use the previous point). Is A diagonalizable?

(c) Let $C \in \mathcal{M}_4$ be an invertible matrix. Find the eigenvalues of the matrix $C^{34} A((C^{34})^{-1})$.

Ex. 2.10. Let $T: V_3 \to V_3$ be a linear transformation with a double eigenvalue and a simple eigenvalue. Show that there is a basis \mathcal{B} of V_3 such that the representative matrix $m_{\mathcal{B}}^{\mathcal{B}}$ is upper triangular.

Ex. 2.11. Let $v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ Find (if any) all matrices $A \in$

 $\mathcal{M}_{3,3}(\mathbb{R})$ such that: Tr(M) = 3, Mv = 2v, Mw = -w and A has an eigenvector orthogonal both to v and w.

Ex. 2.12. Let $T : V^3 \to V^3$ be the linear transformation such that: T((1,1,0)) = (2,2,0), T((0,-1,1)) = (0,-3,3) e T((1,0,-1)) = (0,1,-1).(a) Compute rk(T) and find a basis of $T(V_3)$. (b) Find a basis of N(T). (c) Find the eigenvalues and eigenvectors of T. (d) Compute T((1,0,0)).

Ex. 2.13. Let a, b, c, d, e be real numbers, and let us consider the matrix $\begin{pmatrix} a+e & b & c & d \end{pmatrix}$

$$A_{a,b,c,d,e} = \begin{pmatrix} a & b+e & c & d \\ a & b & c+e & d \\ a & b & c & d+e \end{pmatrix}.$$
 Find all eigenvalues of $A_{a,b,c,d,e}$

(in function of a, b, c, d, e).

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4

3. (Skew-)Hermitian and (skew-)symmetric transformations

Ex. 3.1. Find all symmetric matrices $A \in \mathcal{M}_{3,3}(\mathbb{R})$ such that tr(A) = -1, det A = -45, 5 is an eigenvalue of A, and $\begin{pmatrix} 1\\ 2\\ -1 \end{pmatrix}$ is an eigenvector of 5.

Ex. 3.2. Find all symmetric matrices $A \in \mathcal{M}_{3,3}(\mathbb{R})$ whose eigenvalues are 2 and -3 and such that the eigenspace of 2 is E(2) = L(((1, 1, -3, 4), (1, 0, 0, 2))).

Ex. 3.3. (a) Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find (if any) all symmetric lmatrices $A \in \mathcal{M}_{2,2}(\mathbb{R})$ such that -3 is an eigenvalue of A and Av = 2v.

(b) Let $w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Is there a symmetric matrix $B \in \mathcal{M}_{2,2}(\mathbb{R})$ such that Av = 2v and Aw = 4w?

(c) Is there a hermitian matrix $E \in \mathcal{M}_{2,2}(\mathbb{C})$ such that Av = 2v and Aw = 4w?

(d) Is there a skew-symmetric matrix $C \in \mathcal{M}_{2,2}(\mathbb{R})$ such that Cv = 2v?

(e) is there a skew-hermitian matrix $D \in \mathcal{M}_{2,2}(\mathbb{C})$ such that Dv = 2v? (d) Is there a skew-hermitian matrix $F \in \mathcal{M}_{2,2}(\mathbb{C})$ such that Fv = iv?

Ex. 3.4. Let $\mathcal{B} = \{(1,2), (1,3)\}$. Sia $T : V_2 \to V_2$ be the linear transformation such that $m_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 9 & 13 \\ -6 & -9 \end{pmatrix}$. Moreover, let $S : \mathcal{V}_2 \to \mathcal{V}_2$ be the linear transformation such that $m_{\mathcal{B}}^{\mathcal{B}}(S) = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$. Is T a symmetric linear transformation?. Is S a symmetric linear transformation?.

Ex. 3.5. Let v = (1, 2, 1). Let us consider the function $T : V_3 \to V_3$ defined as follows: $T(X) = X \times v$.

(a) Prove that T is a linear transformation. (b) Is T symmetric?

(b) Compute the matrix representing T with respect to the canonical basis of V_3 and compare the result with the answer given to (a)

(c) Find an orthonormal basis of V_3 $\mathcal{B} = \{v_1, v_2, v_3\}$ with v_1 parallel to v and compute $m_{\mathcal{B}}^{\mathcal{B}}(T)$.