ADDENDUM CONCERNING LINEAR TRANSFORMATIONS AND MATRICES, II

1. Linear transformations and matrices (continuation)

In the next Theorem, we consider the linear space $\mathcal{L}(V_n, V_m)$ of all linear transformations from V_n to V_m (see Section 4, Theorem 16.4 Vol. I, corresponding to Theorem 2.4 Vol. II) and the linear space $M_{m,n}$ of all $m \times n$ matrices. We define the following function

$$\mathcal{T} : M_{m,n} \to \mathcal{L}(V_n, V_m), \qquad A \mapsto T_A$$

Theorem 1.1 (Correspondence between matrices and linear transformations, provisional form)**.** *The above function $\mathcal{T}$ is a bijective linear transformation (terminology: a bijective linear transformation is called an* isomorphism of linear spaces*). Hence it is invertible and its inverse is linear.*

Proof. It is easy to see that $\mathcal{T}$ is a linear transformation (exercise!) and that it is injective (exercise!). To prove that it is surjective let $T \in \mathcal{L}(V_n, V_m)$: we have to prove that there exists a (unique, by the injectivity) $A \in \mathcal{M}_{m,n}$ such that $T = T_A$. To see this, we note that given

$$X^t = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

we have that $X^t = x_1E^1 + \cdots x_nE^n$ (see Remark (e)). Therefore $T(X) = T(x_1E^1 + \cdots x_nE^n) = x_1T(E^1) + \cdots + x_nT(E^n)$. Let A be the matrix whose colums are $A^1 := T(E^1)$, ... ,$A^n := T(E^n)$. Then $T(X) = x_1A^1 + \cdots + x_nA^n = AX = T_A(X)$, see Remark (a). Therefore $T = T_A = \mathcal{T}(A)$. Hence $\mathcal{T}$ is surjective. $\square$

From the previous theorem it follows

Corollary 1.2 (Matrix representation with respect to canonical bases)**.** *Any linear transformation $T : V_n \to V_m$ is of the form T_A for a (unique) matrix $A \in \mathcal{M}_{M,n}$. In other words: $T(X) = AX$ for all $X \in V_n$ (seen as a column vector). Following the book, we will denote*

$$A = m(T)$$

We have the following definition

Definition 1.3. *$m(T)$ is called the matrix representing the linear transformation T (with respect to the canonical bases of V_n and V_m).*

Example 1.4. Let us consider the identity map $I : V_n \to V_n$. We have that $m(I) = I_n$, the identity matrix of order n. This is obvious, since $I(X) = X = I_n X^t$. Analogously, let c be a scalar and $T_c : V_n \to V_n$ the "multiplication by c" (or "omothety") linear trasformation defined as $T_c(X) = cX$. Then

$$m(T_c) = cI_n = \begin{pmatrix} c & 0 & \dots & 0 \\ \dots & & & \\ \dots & & & \\ 0 & \dots & 0 & c \end{pmatrix}$$

Indeed $T_c(X) = cX = (cI_n)X^t$

Example 1.5. Let $R_\theta : V_2 \to V_2$ be the rotation (counterclockwise) of angle θ of V_2. Then

$$m(R_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(Exercise)

A very important feature of the correspondence between linear transformations and matrices is that matrix multiplication corresponds to composition of functions

Theorem 1.6. *Let $T : V_n \to V_m$ and $S : V_k \to V_m$ be linear transformations. Let us consider the composition $TS : V_k \to V_m$. Then*

$$m(TS) = m(T)m(S)$$

Proof. This follows immediately from the associativity of matrix multiplication (see Section 15 in the book). Indeed, let $A = m(T)$ and $B = m(S)$. From Theorem 1.1, the assertion of the present Theorem is equivalent to the assertion

$$T_{AB} = T_A T_B$$

that is

$$(AB)X^t = A(BX^t) \qquad \text{for any } \; X \in V_k$$

which is a particular case of the associativity property of matrix multiplication. □

2. Invertible matrices and their inverses

We have see that the identity matrices I_n are neutral elements with respect to matrix multiplication. It is therefore natural to ask which matrices have an inverse element with respect to multiplication.

Definition 2.1. *Let $A \in \mathcal{M}_{n,n}$ be a square matrix. A is said to be invertible if there is another matrix $B \in \mathcal{M}_{n,n}$ such that $AB = BA = I_n$.*

Remark 2.2. If A is invertible the matrix B is unique. Indeed, if B' is another such matrix, then $B' = B'I_n = B'(AB) = (B'A)B = I_nB = B$.

Definition 2.3. *If A is invertible then the matrix B is called the inverse of A, and denoted A^{-1}.*

It is not hard to imagine that invertible matrices correspond to invertible linear transformations:

Proposition 2.4. *Let $A \in \mathcal{M}_{n,n}$. The following are equivalent:*
(a) A is invertible,
(b) the linear transformation $T_A : V_n \to V_n$ is invertible, and $(T_A)^{-1} = T_{A^{-1}}$;
(c) $rk(A) = n$.

Proof. (a) $\Leftrightarrow$ (b) Assume that A is invertible. Then, by Theorem 3.3 and Example 1.4,

$$I = T_{I_n} = T_{AA^{-1}} = T_A T_{A^{-1}}.$$

Analogously,

$$I = T_{A^{-1}} T_A.$$

Therefore T_A is invertible.
Conversely, assume that T_A is invertible. Then we know that $(T_A)^{-1}$ is a linear transformation too (Prop. 1.4 of Addendum, I). Hence, by Theorem 1.1 there is matrix B such that $(T_A)^{-1} = T_B$. By Theorem 3.3 we have that $AB = m(T_A T_B) = m(I) = I_n$ and $BA = m(T_B T_A) = m(I) = I_n$. Therefore A is invertible and $B = A^{-1}$.
(b) $\Leftrightarrow$ (c) The linear transformation T_A is invertible if and only if it is bijective. by Corollary 1.5 of Addendum I, this happens if and only if it is surjective, that is $rk(T_A) = n$. But, by definition, $rk(A) = rk(T_A)$. □

The following proposition ensures that, in order to check invertibility and find the inverse of a matrix, it is sufficient to check only *one* of the conditions $AB = I_n$ and $BA = I_n$.

Proposition 2.5. *(a) Let $A \in \mathcal{M}_{n,n}$. If there is a matrix $B \in \mathcal{M}_{n,n}$ such that $AB = I_n$ then A is invertible and $B = A^{-1}$.*
(b) Let $A \in \mathcal{M}_{n,n}$. If there is a matrix $B \in \mathcal{M}_{n,n}$ such that $BA = I_n$ then A is invertible and $B = A^{-1}$.

Proof. (a) If $AB = I_n$ then $T_A T_B = T_{AB} = T_{I_n} = I$ (Theorem 3.3 and Example 1.4. This implies that T_A is surjective since, for all $X \in V_n$, $X = (T_A T_B)(X) = T_A(T_B(X)$, hence there is a Y such that $X = T_A(Y)$. By Cor. 1.4 of Addendum I, T_A is bijective, hence invertible. Therefore, by Prop. 2.4 A is invertible.

(b) If $BA = I_n$ then $T_B T_A = T_{BA} = T_{I_n} = I$. This implies that T_A is injective since, for all $X, X' \in V_n$, if $T_A(X) = T_A(X')$ then $X = T_B(T_A(X)) = T_B(T_A(X')) = X'$. Then by Corollary 1.5 of Addemdum I, T_A is bijective, hence invetrible. Therefore, by Prop. 2.4 A is invertible.

Remark 2.6 (Inverse matrix and linear systems)**.** Let us consider a *square* linear system, that is a system of linear equations such that the number of equations equals the number of unknowns. In other words, a linear system

$AX = B$ where $A \in \mathcal{M}_{n,n}$ is a square matrix. Then we know that for all $B \in \mathcal{V}_n$ there is a solution if and only if A has rank n and in this case the solution is actually unique. Now A has rank n if and only if it is invertible and in the case the unique solution is

$$X = A^{-1}B.$$

This is simply obtained multiplying both members of $AX = B$ by A^{-1} on the left. Note the analogy with a linear equation

$$ax = b$$

where $a, b \in \mathbb{R}$. Under the condition $a \neq 0$, which means that a is invertible with respect to the multiplication of real numbers, then there is always a solution, such solution is unique, and more precisely such solution is

$$x = a^{-1}b.$$

2.1. **Computation of the inverse matrix.** Given an invertible matrix $A \in \mathcal{M}_{n,n}$, Prop. 2.5 assures that, in order to find its inverse, it is enough to solve the matricial equation

$$AX = I_n \tag{1}$$

where the unknown X is a $n \times n$ matrix. In the next examples we show how to solve such equation using row elimination.

Example 2.7. *Let* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. *Equation (1) can be solved by finding the two columns of* X, *denoted, as usual,* X^1 *and* X^2. *Therefore equation (1) is equivalent to the two systems*

$$AX^1 = E^1 \qquad \textit{and} \qquad AX^2 = E^2$$

that is

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} x_{11} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} x_{21} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} 1 \\ 3 \end{pmatrix} x_{12} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} x_{22} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This can be summarized in the single equation

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} (x_{11}, x_{12}) + \begin{pmatrix} 2 \\ 4 \end{pmatrix} (x_{21}, x_{22}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This can be seen as a usual system of linear equations

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} X_1 + \begin{pmatrix} 2 \\ 4 \end{pmatrix} X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

that is

$$\begin{cases} X_1 & + & 2X_2 & = & (1,0) \\ 3X_1 & + & 4X_2 & = & = (0,1) \end{cases}$$

where the unknowns X_1 and X_2 are the rows of the inverse matrix. This can be solved in the usual way

$$\begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 3 & 4 & | & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -2 & | & -3 & 1 \end{pmatrix}$$

This corresponds to the system

$$\begin{cases} X_1 & + & 2X_2 & = & (1,0) \\ & & -2X_2 & = & (-3.1) \end{cases}$$

Solving as usual we get $X_2 = (3/2, -1/2)$*, and* $X_1 = (1,0) - 2X_2 = (1,0) + (-3,1) = (-2,1)$. *Therefore the inverse matrix is*

$$A^{-1} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Check that it is really the inverse matrix!

Example 2.8. $A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

Arguing as before, we are lead to solve the system of linear equations

$$\begin{cases} X_1 & + & 2X_2 & + & X_3 & = & (1,0,0) \\ -2X_1 & + & 2X_2 & + & 3X_3 & = & (0,1,0) \\ X_1 & + & X_2 & + & X_3 & = & (0,0,1) \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ -2 & 2 & 3 & | & 0 & 1 & 0 \\ 1 & 1 & 1 & | & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 6 & 5 & | & 2 & 1 & 0 \\ 0 & -1 & 0 & | & -1 & 0 & 1 \end{pmatrix}$$

Note that from this calculation if follows that $rk(A) = 3$ *(exercise!), that is that* A *is invertible. Solving we have* $X_2 = (1, 0, -1)$,
$X_3 = 1/5\big((2,1,0) - 6X_2\big) = 1/5((2,1,0) - (6,0,-6)\big) = 1/5(-4,1,6) = (-4/5, 1/5, 6/5)$,
$X_1 = (1,0,0) - 2X_2 - X_3 = (1,0,0) - 2(1,0,-1) - (-4/5, 1/5, 6/5) = (-1/5, -1/5, 4/5)$.
Therefore the inverse matrix is

$$A^{-1} = \begin{pmatrix} -1/5 & -1/5 & 4/5 \\ 1 & 0 & -1 \\ -4m/5 & 1/5 & 6/5 \end{pmatrix}$$

Check that this is really the inverse matrix!

3. Correspondence between matrices and linear transformations: general version

It turns out that Theorem 1.1 , Corollary 1.2 and Theorem 3.3 are particular cases of much more general statements. The point is that, rather than using the usual coordinates (that is the components with respect to the canonical basis, formed by the usual unit coordinate vectors) one can use the coordinates with respect to an arbitrary basis. The general formulation of Corollary 1.2 is Theorem 16.13 of Vol. I (2.13 of Vol. II) plus Theorem 16.16 of Vol. I (Theorem 2.16 of Vol. II). Before stating these results we introduce the following setup:

(1) *Let V and W be finite-dimensional linear spaces, of dimension respectively n and m.*
(2) *Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ and $\mathcal{C} = \{f_1, \ldots, f_n\}$ be bases of V and W respectively.*
(3) *Let $A \in \mathcal{M}_{m,n}$ be a $m \times n$ matrix.*
(4) *Given a vector $v \in V$, let*

$$X^t_{\mathcal{B},v} = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

be the column vector of components of v with respect to the basis $\mathcal{B}$. In other words, $x_1, \ldots, x_n$ are the unique scalars such that $v = x_1e_1 + \cdots + x_ne_n$.

(5) *Given a vector $w \in W$, let*

$$Y^t_{\mathcal{C},w} = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{pmatrix}$$

be the column vector of components of w with respect to the basis $\mathcal{C}$. In other words, $y_1, \ldots, y_m$ are the unique scalars such that $w = y_1f_1 + \cdots + y_nf_m$.

Definition 3.1. *(1) In the previous setting, we define a linear transformation*

$$T : V \to W$$

as follows. Let $v \in V$ then

$$Y^t_{\mathcal{C},T(v)} = A\, X^t_{\mathcal{B},v}$$

In words: we define T by defining, for all $v \in V$, the (column) vector of components of $T(v)$ with respect to the basis $\mathcal{C}$. This is, by definition, the product of the matrix A times the (column) vector of components of v with respect to the basis $\mathcal{B}$.

(Note that, by construction, the (column) vectors

$$Y^t_{\mathcal{C},T(e_1)}, \cdots, Y^t_{\mathcal{C},T(e_n)}$$

are the columns

$$A^1, \ldots, A^n$$

of the matrix A.)

(2) The linear transformation T is called the linear transformation represented by the matrix A with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ and we denote

$$A = m_{\mathcal{C}}^{\mathcal{B}}(T).$$

The analogue of Corollary 1.2 in this more general setting is the following

Theorem 3.2 (Matrix representation, general form)**.** *Let*

$$T : V \to W$$

be a linear trasformation. let $\mathcal{B}$ be a bses of V and let $\mathcal{C}$ be a basis of W. Then there is a unique matrix $A \in \mathcal{M}_{m,n}$ such that

$$A = m_{\mathcal{C}}^{\mathcal{B}}(T).$$

The matrix A is the one whose colums are the (column) vector of components of $T(e_1), \ldots, T(e_n)$ with respect to the basis $\mathcal{C}$.

Theorem 3.3 (Composition of transformations corresponds to matrix multiplication)**.** *Keeping the notation of the previous Theorem, let U be another finite-dimensional linear space, of dimension k, and let $\mathcal{D}$ be a basis of U. Furthermore let $S : U \to V$ be a linear transformation. Then*

$$m_{\mathcal{C}}^{\mathcal{D}}(TS) = m_{\mathcal{C}}^{\mathcal{B}}(T) m_{\mathcal{B}}^{\mathcal{D}}(S)$$

The proofs are similar to those of Theorem 1.1, Corollary 1.2 and Theorem 3.3, and they are omitted. As a useful exercise, you should try at least to outline them.

Example 3.4. Let $T = Pr_{L((1,2))} : V_2 \to V_2$ be the projection along $L(1,2)$. Let $S = Ref_{L((1,2))}$ be the reflection with respect to $L((1,2))$. Let $\mathcal{B} = \{(1,2),(2,-1)\}$. Then:

$$m_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \textit{and} \qquad m_{\mathcal{B}}^{\mathcal{B}}(S) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is as follows: let $v \in V_2$, and write v in components with respect to the basis $\mathcal{B}$:

$$v = x_1(1,2) + x_2(2,-1)$$

Then, by definition of orthogonal decomposition (note that $\{(2,-1)\}$ is a basis of $(L(1,2))^{\perp}$),

$$T(v) = x_1(1,2)$$

Therefore the (column) vector of components of $T(v)$ with respect to $\mathcal{B}$ is

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(Alternatively, one can note that $T((1,2)) = (1,2) = 1(1,2) + 0(2,-1)$ and $T((2,-1)) = (0,0) = 0(1,2) + 0(2,-1)$, so that the columns $m_{\mathcal{B}}^{\mathcal{B}}(T)$ are respectively $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$).

Concerning S, note that

$$S(v) = x_1(1,2) - x_2(2,-1)$$

At this point, the computation of the matrix representing S with respect to the basis $\mathcal{B}$ goes as above.

Example 3.5 (Projections and reflections)**.** More generally, let W be a k-dimensional subspace of V_n and let $T : V_n \to V_n$ be the projection onto W, namely $T(v) = Pr_W(v)$. Let $\{w_1, \ldots, w_k\}$ be a basis of W and let $\{u_{k+1}, \ldots, u_n\}$ be a basis of $W^\perp$. Let $\mathcal{B} = \{w_1, \ldots, w_k, u_{k+1}, \ldots, u_n\}$. It is a basis of V_n (why?). Then

$$m^{\mathcal{B}}_{\mathcal{B}}(T) = diag(1, \cdots, 1, 0, \cdots, 0)$$

where *diag* means *diagonal matrix* and the 1's are k.
Moreover let $S = Ref_W : V_n \to V_n$ be the reflection with respect to W. Then

$$m^{\mathcal{B}}_{\mathcal{B}}(S) = diag(1, \cdots, 1, -1, \cdots, -1)$$

where the 1's are k and the -1's are $n - k$.

Example 3.6. We consider the following linear transformation of V_3. Let $v \in V_3$ be the vector of cylindrical coodinates (see Apostol, Vol. I, Section 14.18)

$$(\rho, \theta, z)$$

namely

$$v = (x, y, z) = (\rho \cos\theta, \rho \sin\theta, z)$$

Then $T(v)$ is the vector of cylindrical coordinates

$$(\rho, \theta + \theta_0, z)$$

where $\theta_0 \in [0, 2\pi)$ is a fixed angle. Hence

$$T(v) = (\rho\cos(\theta + \theta_0), \rho\sin(\theta + \theta_0), z)$$

In practice, T is nothing else than the rotation of angle θ_0 around the z-axis. The rotation is counterclockwise, with respect to the orientation of the z axis given by $(0, 0, 1)$. Using the trigonometric addition formulas, the explicit formula for $T(v)$ is:

$$T(x, y, z) = (x\cos\theta_0 + y\sin\theta_0, -x\sin\theta_0 + y\cos\theta_0, z)$$

In matrix notation:

$$T(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & 0 \\ \sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Therefore the matrix $m^{\mathcal{E}}_{\mathcal{E}}(T)$, the matrix representing T with respect to the canonical basis (both in the domain and in the target space) is

$$m^{\mathcal{E}}_{\mathcal{E}}(T) = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & 0 \\ \sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3.7. Let $W = L(1, 2, -3)$. We consider the following transformation of V_3. Given $v \in V_3$, $T(v)$ is the vector contained in the same plane orthogonal to $(1, 2, 3)$ containing v, obtained rotating v of an angle equal to θ, with $\theta \in [0, 2\pi)$. The rotation will be counterclockwise with respect to the orientation of W given by the vector $(1, 2, -3)$. In practice, let us denote $v = (1, 2, -3)$ and let $\{u_1, u_2\}$ be an *orthonormal* basis of the plane $W^\perp$, such that the determinant of the matrix having as columns u_1, u_2, v (in this order) is positive[1]. Let us consider the following basis of V_3: $\mathcal{B} = \{u_1, u_2, v\}$. Then, exactly for the same reason of the previous example, the matrix representing T with respect to the basis $\mathcal{B}$ is very simple, namely

$$m^{\mathcal{B}}_{\mathcal{B}}(T) = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & 0 \\ \sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In fact: $T(u_1) = \cos\theta_0 u_1 + \sin\theta_0 u_2 + 0v$, $T(u_2) = \sin\theta_0 u_1 + \cos\theta_0 u_2 + 0v$, $T(v) = 0u_1 + 0u_2 + 1v$. Note that two vectors u_1 and u_2 as requested are obtained, for example, as follows: solving the equation

$$x + 2y - 3z = 0$$

we get that $W^\perp = L((2, 1, 0), (-3, 0, 1))$. Orthogonalizing we get the basis $W^\perp = L((2, 1, 0), (-3/5, 6/5, 1))$. Dividing by the norms, we get $\{1/\sqrt{5}(2, 1, 0), 5/\sqrt{46}(-3, 6, 1)\} = \{u_1, u_2\}$. Since, as one can easily check, the determinant of the matrix whose columns are u_1, u_2, v is negative, we exchange the order of u_1 and u_2 and find

$$\mathcal{B} = \{u_2, u_1, v\}$$

Example 3.8. Let V be a n-dimensional linear space and let $\mathcal{B}$ be any basis of V. Let, as above, $I : V \to V$ the identity transformation and, for a given scalar c, $T_c : V \to V$ the linear transformation $v \mapsto cv$. Then

$$m^{\mathcal{B}}_{\mathcal{B}}(I) = I_n \quad \textit{and} \quad m^{\mathcal{B}}_{\mathcal{B}}(T_c) = cI_n \tag{2}$$

This is because, if X are is the vector of components of a given vector $v \in V$ with respect to the basis $\mathcal{B}$, then cX is the vector of components of the vector cv with respect to the *same* basis $\mathcal{B}$. Note that the basis of the source and the target space has to be the same, otherwise (2) is false.

Proposition 3.9 (Inverse transformation and inverse matrix)**.** *Let $T : V \to W$ be an invertible trasformation, and let $\mathcal{B}$ and $\mathcal{C}$ be respectively bases of V and W. The matrix representing the inverse transformation $T^{-1} : W \to V$ with respect to the bases $\mathcal{C}$ and $\mathcal{B}$ is*

$$m^{\mathcal{C}}_{\mathcal{B}}(T^{-1}) = (m^{\mathcal{B}}_{\mathcal{C}}(T))^{-1}$$

Proof. This follows from Theorem 3.3, because

$$I_n = m^{\mathcal{B}}_{\mathcal{B}}(id_V) = m^{\mathcal{B}}_{\mathcal{B}}(T^{-1} \circ T) = m^{\mathcal{C}}_{\mathcal{B}}(T^{-1}) m^{\mathcal{B}}_{\mathcal{C}}(T)$$

[1] this means that the vectors u_1, u_2, v satisfy phisicists' "right-hand rule"

and
$$I_n = m^{\mathcal{C}}_{\mathcal{C}}(id_W) = m^{\mathcal{C}}_{\mathcal{C}}(T \circ T^{-1}) = m^{\mathcal{B}}_{\mathcal{C}}(T) m^{\mathcal{C}}_{\mathcal{B}}(T^{-1})$$
□

Example 3.10 (Change-of-basis matrices)**.** Let $C \in \mathcal{M}_{n,n}$ be square matrix of maximal rank, namely n. We know that the columns of C, $C^1, ..., C^n$ are n independent vectors of V_n, hence (why?) a basis of V_n, which we call $\mathcal{B}$. I claim that
$$C = m^{\mathcal{B}}_{\mathcal{E}}(id)$$
where $id : V_n \to V_n$ is the identity transformation. This follows from the definition: given a vector $v \in V_n$ such that its components with respect to $\mathcal{B}$ are $x'_1, \ldots, x_n\prime$, that is $v = x'_1 C^1 + \cdots x'_n C^n$, then the column vector of the usual coordinates of $id(v) = v$ is exactly
$$C(X')^t = x'_1 C^1 + \cdots x'_n C^n$$
Therefore it follows from Proposition 3.9 that
$$m^{\mathcal{E}}_{\mathcal{B}}(id) = C^{-1}$$
This is nothing new: if we want the components of $X = (x_1, \cdots, x_n)$ with respect to te basis $\mathcal{B}$ we have to solve the system
$$C(X')^t = X^t$$
whose solution is
$$(X')^t = C^{-1} X^t$$

Example 3.11. Let $v_1 = (1, 2, -1)$, $v_2 = (0, 1, 2)$ and $v_3 = (1, 0, 3)$. It is easy to check that the matrix
$$C = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}$$
has non-zero determinant, hence $rk(C) = 3$. Therefore $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis of V_3 and
$$C = m^{\mathcal{B}}_{\mathcal{E}}(Id)$$
Let $v = (x, y, z)$ be a vector of V_3. To find the components of (x, y, z) with respect to the basis $\mathcal{B}$ we need to solve the system
$$x' \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y' \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + z' \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
The unique (why?) solution is
$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{8}x & \frac{1}{4}y & -\frac{1}{8}z \\ \frac{3}{4}x & \frac{1}{2}y & \frac{1}{4}z \\ \frac{5}{8}x & -\frac{1}{4}y & \frac{1}{8}z \end{pmatrix}$$

4. Change of basis

In this section we address the following question: *what is the relation between matrices representing the same transformation with respect to different bases?* The answer is the following

Theorem 4.1 (Change of basis, general form)**.** *Let $T : V \to W$ be a linear transformation. Let also $\mathcal{B}$,$\mathcal{U}$ and $\mathcal{C}$,$\mathcal{V}$ be respectively two bases of V and two bases of W. Then*

$$m^{\mathcal{U}}_{\mathcal{V}}(T) = m^{\mathcal{C}}_{\mathcal{V}}(id_W)\, m^{\mathcal{B}}_{\mathcal{C}}(T)\, m^{\mathcal{U}}_{\mathcal{B}}(id_V)$$

Proof. This follows because, in the diagram

$$\begin{tikzcd} V_{\mathcal{B}} \arrow[r, "T"] & W_{\mathcal{C}} \arrow[d, "id_W"] \\ V_{\mathcal{U}} \arrow[u, "id_V"] \arrow[r, "T"] & W_{\mathcal{V}} \end{tikzcd}$$

the composition $id_W \circ T \circ id_V = T$. Then the assertion follows from Theorem 3.3. □

As a particular case we have

Theorem 4.2. *Let $T : V \to V$ be a linear transformation. Let also $\mathcal{B}$ and $\mathcal{U}$ be two bases of V. Then*

$$m^{\mathcal{U}}_{\mathcal{U}}(T) = (m^{\mathcal{U}}_{\mathcal{B}}(id))^{-1}\, m^{\mathcal{B}}_{\mathcal{B}}(T)\, m^{\mathcal{U}}_{\mathcal{B}}(id)$$

Proof. From the previous Theorem we have that

$$m^{\mathcal{U}}_{\mathcal{U}}(T) = m^{\mathcal{B}}_{\mathcal{U}}(id)\, m^{\mathcal{B}}_{\mathcal{B}}(T)\, m^{\mathcal{U}}_{\mathcal{B}}(id)$$

and, by Example 3.10, $m^{\mathcal{B}}_{\mathcal{U}}(id) = (m^{\mathcal{U}}_{\mathcal{B}}(id))^{-1}$. □

As a (fundamental) example, we have

Theorem 4.3. *Let $T : V_n \to V_n$ be a linear transformation. let $\mathcal{E}$ be the canonical basis of V_n and let $\mathcal{U}$ another basis of V_n. Let*

$$A = m^{\mathcal{E}}_{\mathcal{E}}(T) \qquad D = m^{\mathcal{U}}_{\mathcal{U}}(T) \qquad \textit{and} \qquad C = m^{\mathcal{U}}_{\mathcal{E}}(id)$$

(note that C is the matrix whose columns are the usual coordinates of the vectors of $\mathcal{U}$, see Example 3.11)*. Then*

$$D = C^{-1}\, A\, C$$

Conversely

$$A = C\, D\, C^{-1}$$

Example 4.4. Let us consider the two transformations of Example 3.4, namely $T = Pr_{L((1,2))} : V_2 \to V_2$ be the projection along $L(1,2)$ and $S = Ref_{L((1,2))}$ be the reflection with respect to $L((1,2))$. Let

$$= \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

We have that

$$m^{\mathcal{E}}_{\mathcal{E}}(T) = C\,diag(1,0)\,C^{-1} = \begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} \cdot \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix} = \begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & -\frac{1}{5}\end{pmatrix}$$

This says that

$$Pr_{L((1,2))}(\begin{pmatrix}x\\ y\end{pmatrix}) = \begin{pmatrix}\frac{1}{5}x + \frac{2}{5}y\\ \frac{2}{5}x - \frac{1}{5}y\end{pmatrix}$$

You can check this by computing $Pr_{L((1,2))}(\begin{pmatrix}x\\ y\end{pmatrix})$ in the usual way.

Do the same with S.

Example 4.5. Let us consider the rotation around a line in V_3 of Example 3.7 and let us assume that the angle θ_0 is $\pi/6$. Then

$$D = m^{\mathcal{B}}_{\mathcal{B}}(T) = \begin{pmatrix}\frac{1}{6} & -\frac{\sqrt{2}}{3} & 0\\ \frac{\sqrt{2}}{3} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{pmatrix}$$

where the basis $\mathcal{B}$ is

$$\mathcal{B} = \{\frac{5}{\sqrt{46}}(-3,6,1), \frac{1}{\sqrt{5}}(2,1,0), (1,2,-3)\}$$

Therefore the matrix of T with respect to the canonical basis of V_3 – that is: the law defining T in our usual coordinates – is

$$m^{\mathcal{E}}_{\mathcal{E}}(T) = C\,D\,C^{-1}$$

where $C = m^{\mathcal{B}}_{\mathcal{E}}(id)$ is the matrix whose columns are the vectors of the basis $\mathcal{B}$.

Example 4.6. Let $u_1 = (1,1,-2)$, $u_2 = (1,0,3)$, $u_3 = (2,0,-1)$. As above, it is easy to check that

$$\mathcal{B} = \{u_1, u_2, u_3\}$$

is a basis of V_3. Let $T : V_3 \to V_3$ be the unique linear transformation such that

$$T(u_1) = (2,2,1) \qquad T(u_2) = (3,-1,2) \qquad T(u_3) = (3,7,1)$$

How do we compute $N(T)$ and $T(V_3)$? Concerning $T(V_3)$ there is no problem, it is $L((2,2,1),(3,-1,2),(3,7,1))$. A simple calculation with gaussian elimination shows that this space has dimension 2 and that, for example, $\{(2,2,1),(3,-1,2)\}$ is a basis of $T(V_3)$.

Now we compute the null-space. By the nullity+ rank Theorem, we know that $\dim N(T) = 1$. The simplest way of finding a generator of$N(T)$ is the following: $N(T)$ is:

the space of vectors $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ *such that* $T(\lambda^1 u_1 + \lambda_2 u_2 + \lambda_3 u_3) = \lambda_1 T(u_1) + \lambda_2 T(u_2) + \lambda_3 T(u_3) = \lambda_1(2,2,1) + \lambda_2(3,-1,2) + \lambda_3(3,7,1) = (0,0,0)$

Therefore we are $(\lambda_1, \lambda_2, \lambda_3)$ are solutions of the linear system

$$\begin{pmatrix} 2 & 3 & 3 \\ 2 & -1 & 7 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One finds easily that the space of solutions is $L((-3, 1, 1)$. Therefore, by the above description of $N(T)$:

$$N(T) = L(-3u_1 + u_2 + u_3) = L((0, -3, 8))$$

NOTE: $N(T)$ **is not** $L((-3, 1, 1))$.

Example 4.7. By the way, given the T of the previous example, what is $M^{\mathcal{E}}_{\mathcal{E}}(T)$? The matrix

$$D = \begin{pmatrix} 2 & 3 & 3 \\ 2 & -1 & 7 \\ 1 & 2 & 1 \end{pmatrix}$$

is not $m^{\mathcal{E}}_{\mathcal{E}}(T)$. We have rather

$$D = \begin{pmatrix} 2 & 3 & 3 \\ 2 & -1 & 7 \\ 1 & 2 & 1 \end{pmatrix} = m^{\mathcal{B}}_{\mathcal{E}}(T)$$

(the columns of D are the usual coordinates of $T(u_1)$, $T(u_2)$, $T(u_3)$). The diagram

$$\begin{tikzcd} (V_3)_{\mathcal{E}} \arrow[r, "T"] \arrow[d, "id"] & (V_3)_{\mathcal{E}} \\ (V_3)_{\mathcal{B}} \arrow[ur, "T"] & \end{tikzcd}$$

shows that We have at our disposal

$$C = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ -2 & 3 & -1 \end{pmatrix} = m^{\mathcal{B}}_{\mathcal{E}}(id)$$

Therefore

$$m^{\mathcal{E}}_{\mathcal{E}}(T) = D\,C^{-1} = \begin{pmatrix} 2 & 3 & 3 \\ 2 & -1 & 7 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ -2 & 3 & -1 \end{pmatrix}^{-1}$$

Exercise: finish the calculation.
NOTE: one could have gotten to the same matrix by a simple-minded reasoning as follows: express $(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1(x, y, z), \lambda_2(x, y, z), \lambda_3(x, y, z))$ [2].

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[2]this amounts to finding the inverse matrix C^{-1}, because it amounts to solve the system

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ -2 & 3 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Then

$$T(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 2\lambda_1(x,y,z) + 3\lambda_2(x,y,z) + 3\lambda_3(x,y,z) \\ 2\lambda_1(x,y,z) - 1\lambda_2(x,y,z) + 7\lambda_3(x,y,z) \\ \lambda_1(x,y,z) + 2\lambda_2(x,y,z) + \lambda_3(x,y,z) \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 3 & 3 \\ 2 & -1 & 7 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1(x,y,z) \\ \lambda_2(x,y,z) \\ \lambda_3(x,y,z) \end{pmatrix} = D\, C^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

5. Exercises

Ex. 5.1. Let $\mathcal{B} = \{u, v, w\}$ be a basis of V_3.
(a) Is there a linear transformation $T : V_3 \to V_3$ such that $T(u) = w$, $T(v) = w$ and $T(w) = 3v - 2w$? If the anwer is yes:
(b) Find dimensions and bases of $N(T)$ and $T(V_3)$.
(c) Compute $m^{\mathcal{B}}_{\mathcal{C}}(T)$.
(d) Is there a linear transformation $S : V_3 \to V_3$ such that $S(u) = v$, $S(v) = w$, $S(3u - 2v) = u$? If the answer is yes answer to questions (b) and (c) as above.

Ex. 5.2. Let $u = (1, 1, -1)$, $v = (1, 1, 0)$, $w - (1, -1, 1)$. For t varying in $\mathbb{R}$, let $S_t : V_3 \to V_3$ be the linear transformation defined by $S_t(u) = (1, 0, -1)$, $S_t(v) = (1, t + 1, 1)$, $S_t(w) = (1, 4, t + 2)$.
(a) Find the value of t such that S_t is injective.
(b) For $t = -5$ find a basis of $N(S_t)$, a basis of $S_t(V_3)$ and a a system of cartesian equations whose space of solutions is $N(S_t)$.
(c) Compute $m^{\mathcal{B}}_{\mathcal{E}}(S_t)$, where $\mathcal{B} = \{u, v, w\}$ and $\mathcal{E}$ is the canonical basis.

Ex. 5.3. Let $T : V_3 \to V_2$ be the linear transformation such that $T((1, 0, -1) = (2, -1)$, $T((0, 1, 1)) = (0, 1)$ and $T((0, 1, 0)) = (1, 0)$.
(a) Find $N(T)$.
(b) Find a line L in V_3 passing trough $P = (3, 4, 5)$ such that $T(L)$ is a point.
(c) Find a plane Π in V_3 passing trough $P = (3, 4, 5)$ such that $T(\Pi)$ is a line.
(d) Is there a plane M of V_3 such that $T(M)$ is a point?
(e) Let $\mathcal{B} = \{\{(1, 0, -1), (0, 1, 1), (0, 1, 0)\}$. Let $\mathcal{C} = \{(2, -1), (0, 1)\}$. Let $\mathcal{E}_3$ and $\mathcal{E}_2$ be the canonical base sof V_3 and V_2. Compute $m^{\mathcal{B}}_{\mathcal{E}_2}(T)$ and $m^{\mathcal{B}}_{\mathcal{C}}(T)$.

Ex. 5.4. Let us consider the linear transformation

$$R_{L((1,2))} R_{L((1,3))} : V_2 \to V_2,$$

. This has the solution

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = C^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where R_U denotes the reflection with respect to the linear subspace U. Compute its null-space and range.

Ex. 5.5. Let us consider the linear transformations

$$P_{L((1,2-1),(1,1,1))}R_{L((1,0,-1),(11,3))} : V_3 \to V_3$$

and

$$P_{L((1,2,-1),(1,1,1))}P_{L((1,0,-1),(1,1,3))} : V_3 \to V_3,$$

where, as above, R_U denotes the reflexion with respect to the linear subspace U, and P_V denotes the projection on the linear subspace V. Compute null-space and range of such transformations.

Ex. 5.6. For t varying in $\mathbb{R}$ let $u_t = (1, t+1, 1)$, $v_t = (1, t+2, 2)$ and $w_t = (2, 1, t+1)$. Let $S_t : V_3 \to V_3$ be the linear transformation such that $S(E^1) = u_t$, $S(E_2) = v_t$, $S(E^3) = w_t$, where $\{E^1, E^2, E^3\}$ are the unit coordinate vectors.
(a) Find for which $t \in \mathbb{R}$ the transformation S_t is surjective.
(b) For all $t \in \mathbb{R}$ such that S_t is not surjective, find a basis of $S_t(V_3)$.
(c) Find, if possible, a vector $v \in V_3$ such that $S_{-1}(v) = (1, 0, 0)$.

Ex. 5.7. Let V be a linear space and let $\{v_1, v_2\} \subset V$ be a linearly independent set made of two elements. Let $T : V \to V$ be a linear transformation such that $T(v_1 + 2v_2) = 2v_1 - v_2$, and $T(v_1 - v_2) = v_1 + 3v_2$.
(a) Express $T(v_2)$ as linear combination of v_1 and v_2.
(b) Is there a $u \in V$ such that $T(u) = v_1$? If the answer is yes, find it.

Ex. 5.8. True/false? (Then explain all anwers)
(a) For a matrix $A \in \mathcal{M}_{5,6}$, $T(X) = AX^t$ defines a linear transformation $T : V_5 \to V_6$.
(b) Every linear transformation $T : V_6 \to V_4$ is surjective.
(c) Every linear transformation $T : V_4 \to V_6$ is injective.
(d) Every linear transformation $T : V_6 \to V_4$ such that $\dim N(T) = 2$ is surjective.
(e) Every linear transformation $T : V_4 \to V_6$ such that $\dim T(V_4) = 4$ is surjective.
(f) If $\dim V = \dim W$ a linear transformation $T :\to W$ is injective if and only if it is surjective.

Ex. 5.9. Let V and W be linear spaces and $T : V \to W$ a linear transformation. Let $v_1, ..., v_k \in V$.
(a) Prove that if $T(v_1), ..., T(v_k)$ are linearly independent then $v_1, ..., v_k$ are linearly independent.
(b) Prove that if $v_1, ..., v_k$ are linearly independent and T is injective then $T(v_1), ..., T(v_k)$ are linearly independent.

Ex. 5.10. Let $u_1 = (1, 0, 0, 1)$, $u_2 = (1, -1, 1, 0)$, $u_3 = (2, -3, 0, 1)$, and $u_4 = (2, 0, 0, -1)$. Let $T : V_4 \to V_3$ such that $T(u_1) = (1, 2, 1)$, $T(u_2) = (1, 0, 2)$, $T(u_3) = (1, 4, 0)$, $T(u_4) = (4, 2, 7)$.
Find dimension and bases of $N(T)$ and $T(V_4)$.

Ex. 5.11. Let $u_1 = (1,3)$, $u_2 = (1,1)$, $v_1 = (0,2)$, $v_2 = (1,2)$. Let $T : V_2 \to V_2$ be the linear transformation such that $T(u_1) = 2v_1 + v_2$ and $T(u_2) = -3v_1 - 2u_2$. Given $(x,y) \in V_2$, who is $T((x,y))$?

Ex. 5.12. Do linear transformations with the below properties exist? If the answer is yes exhibit an example.
(a) $T : V_2 \to V_4$ such that $T(V_2) = L((1,0,1,0),(0,1,0,1),(1,0,0,0))$;
(b) $S : V_4 \to V_3$ surjective and such that $N(S) = L((1,2,-1,1))$.

Ex. 5.13. For$t$ varying in $\mathbb{R}$, let $A_t = \begin{pmatrix} t & t+1 & t+3 \\ -1 & 0 & 2 \\ 2 & 0 & t+1 \end{pmatrix}$.
(a) For t varying in $\mathbb{R}$, compute $\dim(N(T_{A_t}))$ e $\dim(T_{A_t}(V_3))$.
(b) Exhibit a basis $\mathcal{B}$ of $T_{A_{-1}}(V_3)$ and find a basis of V^3 containing $\mathcal{B}$.

Ex. 5.14. (a) Let $T : V_n \to V_m$ be a linear transformation. Prove that, via T, the image of a parallelogram is either a parallelogram, or a segment or a point. For each case exhibit an example
(b) Describe the image of the unit square of V_2 (that is $[0,1] \times [0,1]$) via the linear transformations T_A, where:
(i) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, (ii) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, (iii) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, (iv) $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Ex. 5.15. Find three vectors $v_1, v_2, v_3 \in V_3$ such that $\begin{cases} v_2 + v_3 = (1,0,0) \\ v_1 + 2v_3 = (0,1,0) \\ v_1 + 2v_2 = (0,0,1) \end{cases}$

Is the solution unique?

Ex. 5.16. Let $A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix}$. (a) Compute A^{-1}.
(b) Let $C = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$. Find all matrices B such that $AB = C$.

Ex. 5.17. Let $C = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & -3 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.
(a) Compute C^{-1}.
(b) Compute $(C^2)^{-1}$ (without computing C^2).

Ex. 5.18. Let $A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$.
(a) Compute the inverse of A.

(b) For $t \in \mathbb{R}$, let $B_t = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 2 & t & 0 \end{pmatrix}$. Find the values of t such hat there is a matrix $X \in \mathcal{M}_{3,3}$ such that $B_t X = A$.

Ex. 5.19. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
(a) Compute the inverses of A and B.
(b) Without computing AB and BA, compute the inverses of AB and BA.

Ex. 5.20. Let us consider the vectors $A_1 = (0,0,1,1)$, $A_2 = (1,-2,0,1)$, $A_3 = (0,1,2,1)$, $A_4 = (0,0,1,0)$.
(a) Prove that $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$ is a basis of V_4.
(b) For all $(x,y,z,t) \in V_4$ find (in function of x,y,z,t) the components of (x,y,z,t) with respect to the basis $\mathcal{B}$.

Ex. 5.21. For $t \in \mathbb{R}$, let $A_t = \begin{pmatrix} t & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 2+t & 0 \\ t & 1 & 0 & t+3 \end{pmatrix}$.
(a) Find for which t the matrix A_t is invertible.
(b) Find for which t the system of linear equations of 4 equations in 3 unknowns having A_t as *augmented* matrix has solutions. For such t's, find explicitly the solutions.

Ex. 5.22. For the matrix A_t of the previous exercise find the values of t such that there exists a vector $B_t \in V_4$ such that the system $A_t X = B_t$ has no solutions.

Ex. 5.23. Let $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 2 & -1 & 3 \end{pmatrix}$.
(a) Find a matrix D such that $AD = B$.
(b) Find a matrix E such that $EA = B$ and a matrix F such that $FA = C$.

Ex. 5.24. Let $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}$.
(a) Find A^{-1}. (b) Is there a matrix X such that $AX = B$? Is it unique? (c) Is there a matrix Y such that $YA^{-1} = B$? Is it unique? (d) Is there a matrix Z such that $BZ = A$? Is it unique? (e) Is there a matrix T such that $BT = C$? Is it unique?