EXERCISES ON QUADRATIC FORMS, SYMMETRIC MATRICES AND CONICS

1. (Skew-)Hermitian and (skew-)symmetric transformations

Ex. 1.1. *For $c \in \mathbb{R}$, let us consider the quadratic form*

$$Q_c(x,y,z,t) = (2+c)x^2+xy+xz+xt+(2+c)y^2+yz+yt+(2+c)z^2+zt+(2+c)t^2$$

Reduce Q_e to diagonal form.

Ex. 1.2. *(a) Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find (if any) all symmetric lmatrices $A \in \mathcal{M}_{2,2}(\mathbb{R})$ such that -3 is an eigenvalue of A and $Av = 2v$.*
(b) Let $w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Is there a symmetric matrix $B \in \mathcal{M}_{2,2}(\mathbb{R})$ such that $Av = 2v$ and $Aw = 4w$?
(c) Is there a hermitian matrix $E \in \mathcal{M}_{2,2}(\mathbb{C})$ such that $Av = 2v$ and $Aw = 4w$?
(d) Is there a skew-symmetric matrix $C \in \mathcal{M}_{2,2}(\mathbb{R})$ such that $Cv = 2v$?
(e) is there a skew-hermitian matrix $D \in \mathcal{M}_{2,2}(\mathbb{C})$ such that $Dv = 2v$? (d) Is there a skew-hermitian matrix $F \in \mathcal{M}_{2,2}(\mathbb{C})$ such that $Fv = i\,v$?

Ex. 1.3. *Let $v = \begin{pmatrix} 1+i \\ i \end{pmatrix}$.*
(a) Find (if any) all hermitian matrices $A \in \mathcal{M}_{2,2}(\mathbb{C})$ such that $Av = 2v$.
(b) Find (if any) all skew-hermitian matrices $B \in \mathcal{M}_{2,2}(\mathbb{C})$ such that $Bv = 2v$.
(c) Find (if any) all hermitian matrices $C \in \mathcal{M}_{2,2}(\mathbb{C})$ such that $Cv = iv$.
(d) Find (if any) all skew-hermitian matrices $D \in \mathcal{M}_{2,2}(\mathbb{C})$ such that $Av = iv$.

Ex. 1.4. *(a) Is the set of all symmetric $n \times n$ real matrices a linear subspace of $\mathcal{M}_{n,n}(\mathbb{R})$? Let V be a real linear inner product space. Is the set of all symmetric linear transformations from V to V a linear subspace of $\mathcal{L}(V,V)$?*

Ex. 1.5. *Let $\mathcal{B} = \{A^1, A^2, A^3, A^4\}$ be an orthogonal basis of V_4 such that $\| A^1 \| = 3$, $\| A^2 \| = 5$, $\| A^3 \| = 2$, $\| A^4 \| = 2$. Let $A = (A^1\ A^2\ A^3\ A^4)$ be the matrix whose columns are A^1, A^2, A^3, A^4. Compute $|\det A|$.*

Ex. 1.6. *Let $\mathcal{B} = \{(1,2),(1,3)\}$. Sia $T : V_2 \to V_2$ be the linear transformation such that $m^{\mathcal{B}}_{\mathcal{B}}(T) = \begin{pmatrix} 9 & 13 \\ -6 & -9 \end{pmatrix}$. Moreover, let $S : \mathcal{V}_2 \to \mathcal{V}_2$ be the* *linear transformation such that* $m^{\mathcal{B}}_{\mathcal{B}}(S) = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$. *Is* T *a symmetric linear transformation?. Is* S *a symmetric linear transformation?.*

Ex. 1.7. *Let* $v = (1, 2, 1)$. *Let us consider the function* $T : V_3 \to V_3$ *defined as follows:* $T(X) = X \times v$.
(a) Prove that T *is a linear transformation . (b) Is* T *symmetric?*
(b) Compute the matrix representing T *with respect to the canonical basis of* V_3 *and compare the result with the answer given to (a)*
(c) Find an orthonormal basis of V_3 $\mathcal{B} = \{v_1, v_2, v_3\}$ *with* v_1 *parallel to* v *and compute* $m^{\mathcal{B}}_{\mathcal{B}}(T)$.

2. Real quadratic forms. Conics

Ex. 2.1. *Let* $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. *Let us consider the real quadratic form* $Q(X) = XAX^T$.
(a) Reduce Q *to diagonal form by means of an orthonormal basis (that is: find an orthonormal basis* $\mathcal{B}$ *of* V_4 *and real numbers* λ_i, $i = 1, 2, 3, 4$ *such that* $Q(X) = \sum_{i=1}^{4} \lambda_i x'^2$, *where* x'_i *are the components of* X *with respect to the basis* $\mathcal{B}$.
(b) Is Q *a positive real quadratic form (this mean that* $Q(X) \geq 0$ *for all* $X \in V_4$ *and equality holds if and only if* $X = O$*).*
(c) Compute the maximum and minimum value Q *on the unite sphere* S^3. *Compute the points of maximum and the points of minimum.*

Ex. 2.2. *For* a, b, c *varying in* $\mathbb{R}$, *let us consider the function* $F_{a,b,c} : [0, 2\pi] \to \mathbf{R}$, $F_{a,b,c}(\theta) = a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta)$. *Assume that max* $F_{a,b,c} = 5$, *min* $F_{a,b,c} = -2$ *and that* $\theta = \pi/3$ *is a point of maximum. Compute* a, b *and* c. *(Hint: think* F *as* $Q : S^1 \to \mathbb{R}$, *where* S^1 *is the unit circle and* Q *is a quadratic form).*

Ex. 2.3. *Find the* $a \in \mathbb{R}$ *such that the equation* $2xy = 4x + 7y + a = 0$ *defines a pair of lines.*

Ex. 2.4. *(a) For* k *varying in* $\mathbb{R}$ *let* $\mathcal{C}_k$ *be the conic defined by the equation*

$$5x^2 - 6xy + 5y^2 - 10x + 6y - k = 0$$

Find, in function of k, *what type of conic is* $\mathcal{C}_k$.

Ex. 2.5. *Reduce the following real quadratic forms* Q *to diagonal form. Establish if they are positive, negative or what. Find explicitly (if any) non-zero vectors* X *such that* $Q(X) > 0$ *or* $Q(X) < 0$ *or* $Q(X) = 0$. *Find the maximum and minimum value of* Q *on the unit sphere and find the points of maximum and the points of minimum.*
(a) $Q(x, y) = x^2 - 3xy + y^2$.

(b) $Q(x, y, z) = -2x^2 - 5y^2 + 12yz + 7z^2$.
(c) $Q(x, y, z) = 4xy + 3y^2 + z^2$.
(d) $Q(x, y, z) = 25x^2 - 7y^2 + 48yz + 7z^2$.
(e) $Q(x, y, z) = \sqrt{2}(x^2 + y^2 + z^2) + 2x(y - z)$
(f) $Q(x, y, z) = 5x^2 - y^2 + z^2 + 4xy + 6xz$
(g) $Q(x, y, z) = 2(xy + yz + zx)$

Ex. 2.6. *Let* $Q(x_1, x_2, x_3, x_4, x_5) = \sum_{1=1}^{5} x_i^2 + \sum_{i<j} x_i x_j$. *Is Q positive?*

Ex. 2.7. *Reduce the equations of teh following conics to canonical form, find the symmetry axes, the center, the foci.. (everything in coordinates with respect to the canonical basis).*
(a) $5x^2 + 5y^2 - 6xy + 16\sqrt{2}x + 38 = 0$.
(b) $5x^2 - 8xy + 5y^2 + 18x - 18y + 9 = 0$
(c) $2x^2 - 2\sqrt{3}xy + 2x + 2\sqrt{3}y - 5 = 0$
(d) $x^2 + 2xy + y^2 + 2x - 2y = 0$
(e) $5x^2 + 8xy + 5y^2 - 18x - 18y + 9 = 0$