EXERCISES ON DETERMINANTS, EIGENVALUES AND EIGENVECTORS

1. Determinants

Ex. 1.1. *Let* $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *(a) Compute* A^{-1}.
(b) Compute $\det(2A^2B)$, $\det(4A+B)$, $\det(2(A^3B^{-2}))$.

Ex. 1.2. *For t varying in* $\mathbb{R}$ *let us consider the matrix* $M_t = \begin{pmatrix} 1 & 0 & -1 & t \\ t & t+2 & 1 & t+2 \\ t+1 & 0 & 1 & 0 \\ 1 & 0 & 2 & t \end{pmatrix}$.
(a) Compute $\det M_t$ *as a function of* t *and find the values of* t *such that* $rk(M_t) = 4$.
(b) For t *varying in* $\mathbb{R}$, *let* L_t *and* R_t *be the lines of cartesian equations*

$$L_t : \begin{cases} x_1 - x_3 = t \\ tx_1 + (t+2)x_2 + x_3 = t+2 \end{cases} \quad \text{and} \quad R_t : \begin{cases} (t+1)x_1 + x_3 = 0 \\ x_1 + 2x_3 = t \end{cases} .$$

Find the values of t such that the lines L_t *and* R_t *meet at a point (hint: use (a)).*

Ex. 1.3. *For* t *varying in* $\mathbb{R}$ *let* $A_t = \begin{pmatrix} t & t & t & -t \\ 1 & 0 & t^2-t & -1 \\ 1 & 0 & 0 & 1 \\ t^3 & t & t & -t \end{pmatrix}$.
(a) Compute $\det(A_t)$ *and find the values of t such that* $rk(A_t) = 4$.
(b) Find the values of t such that the system of linear equations having A_t *as* augmented *matrix has solutions (hint: use (a)) For such values of t solve the system.*

Ex. 1.4. *Let* $A \in \mathcal{M}_{4,4}$ *be a matrix whose rows are* $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$ *and suppose that* $\det A = -3$. *Let* $C \in \mathcal{M}_{4,4}$ *l be the matrix whose rows are* $\begin{pmatrix} 2A_1 \\ 3A_1 + 4A_2 \\ -2A_1 + 2A_2 + A_4 \\ A_1 + A_2 + 3A_3 \end{pmatrix}$. *Compute* $\det(C)$ *and* $\det(\frac{1}{2}C)$.

2. Eigenvalues and eigenvectors

Ex. 2.1. *Let* $A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 3 & 0 & 5 \\ 0 & 5 & -2 & 5 \\ 4 & 0 & 0 & 2 \end{pmatrix}$.
(a) Find the eigenvalues of A.
(b) Let $T_A : V_4 \to V_4$ *be the linear transformation defined by* $T_A(X) = A\,X$. *Is there a basis of* V_4 *composed by eigenvectors of* T_A*? If the answer is yes, produce explicitly such a basis.*
(c) Is there a matrix C *such that* $C^{-1}AC$ *is a diagonal matrix? If the answer is yes produce explicitlt such a matrix.*

Ex. 2.2. *Let* $\mathcal{B} = \{(0,1,-1),\ (0,1,1),(2,3,0)\}$.
(a) Prove that $\mathcal{B}$ *is a basis of* V_3.
Now Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ *and let* $T : V_3 \to V_3$ *be the linear transformation such that* $m_{\mathcal{B}}^{\mathcal{B}}(T) = A$.
(b) Find the eigenvalues of T.
(c) Find a basis of each eigenspace of T.
(d) Is there a basis of V_3 *composed of eigenvectors of* T*? If the answer is yes , produce explicitly such a basis.*
(e) Is there a matrix C *such that* $C^{-1}AC$ *is a diagonal matrix? If the answer is yes, produce explicitly such a matrix.*

Ex. 2.3. *Let* $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -1 & -2 & 0 \end{pmatrix}$.
(a) Find the eigenvalues of A. *(b(Find a basis of each eigenspace of the transformation* $T_A : V_3 \to V_3$*, where, as susal,* $T_A(X) = AX$. *(c) Let* $v = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. *Compute* $A^{151}\,v$.

Ex. 2.4. *Let* $T : V_3 \to V_3$ *the linear transformation such that:* $T((1,1,0)) = (2,2,0)$, $T((0,-1,1)) = (0,-3,3)$ *e* $T((1,0,-1)) = (0,1,-1)$.
(a) Compute $rk(T)$ *and a basis of* $Im(V_3)$.
(b) A basis of $N(T)$. *(c) Eigenvalues and eigenvectors of* T. *(d)* $T((1,0,0))$.

Ex. 2.5. *Let* $T : V_4 \to V_4$ *be a linear transformation such that:*
(a) $T((1,0,-1,2)) = (2,0,-2,4)$, $T((0,1,1,1) = (0,-5,-5,-5)$, $T((1,2,-1,0) = (2,4,-2,0)$; *(b)* $(2,2,-2,2)$ *and* $(1,0,0,0)$ *are eigenvectors of* T; *(c)* $\det T = 100$.
(i) Find the eigenvalues of T *and a basis of each eigenspace of* T.
(ii) Is there a basis of V_4 *composed of eigenvectors of* T*? If the answer is yes, produce explicitly such a basis. (iv) Compute* $Tr(T)$.

Ex. 2.6. *(a) For* t *varying in* $\mathbb{R}$*, let* $A_t = \begin{pmatrix} 1 & t+2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$. *Let* $T_{A_t} : V_3 \to V_3$ *be the linear transformation defined, as usual,* $T)A_t(X) = A_t X$*. Find the values of* t *such that* T_{A_t} *is diagonalizable (that is, there is a basis of* $V_3(\mathbb{R})$ *composed by eigenvectors of* T_{A_t}*.*
(b) Same question for $B_t = \begin{pmatrix} 1 & t+2 \\ 2 & 3 \end{pmatrix}$.

Ex. 2.7. *Let* $V = L((1,0,1,1),(0,0,1,-1))$*. Let* $R_V : V^4 \to V^4$ *be the reflection with respect to* V *and let* $P_{V^\perp} : V^4 \to V^4$ *be the projection of* V^4 *on* $V^\perp$*. (a) Find the range and the eigenvalues of the linear transformation* $P_{V^\perp} \circ R_V$*. (b) Find a base of each eigenspace of* $P_{V^\perp} \circ R_V$*. Is* $P_{V^\perp} \circ R_V$ *diagonalizable?. (c) Is* $P_{V^\perp} \circ R_V$ a symmetric *linear transformation?*

Ex. 2.8. *For all the matrices* A *of the previous exercises, take the* diagonalizable *ones and compute their* $132th$ *power and their* $(-132)th$ *power.*

Ex. 2.9. *Let* $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 3 & -2 & 1 & 1 \\ -3 & -1 & -4 & -1 \\ -6 & -2 & -2 & -5 \end{pmatrix}$ *and let* $v = \begin{pmatrix} -2 \\ 1 \\ 5 \\ 0 \end{pmatrix}$. *(a) Verify that* -3 *is an eigenvalue of* A*.*
(b) Find all eigenvalues of A *(hint: use the previous point). Is* A *diagonalizable?*
(c) Let $C \in \mathcal{M}_4$ *be an invertible matrix. Find the eigenvalues of the matrix* $C^{34} A\,((C^{34})^{-1})$.

Ex. 2.10. *Let* $T : V_3 \to V_3$ *be a linear transformation with a double eigenvalue and a simple eigenvalue. Show that there is a basis* $\mathcal{B}$ *of* V_3 *such that the representative matrix* $m^{\mathcal{B}}_{\mathcal{B}}$ *is upper triangular.*

Ex. 2.11. *Let* $v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ *and* $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *Find (if any) all matrices* $A \in \mathcal{M}_{3,3}(\mathbb{R})$ *such that:* $Tr(M) = 3$, $M\,v = 2v$, $M\,w = -w$ *and* A *has an eigenvector orthogonal both to* v *and* w*.*

Ex. 2.12. *Let* $T : V^3 \to V^3$ *be the linear transformation such that:* $T((1,1,0)) = (2,2,0)$, $T((0,-1,1)) = (0,-3,3)$ *e* $T((1,0,-1)) = (0,1,-1)$. *(a) Compute* $rk(T)$ *and find a basis of* $T(V_3)$*. (b) Find a basis of* $N(T)$*. (c) Find the eigenvalues and eigenvectors of* T*. (d) Compute* $T((1,0,0))$*.*