UNIVERSITÀ DEGLI STUDI DI ROMA "TOR VERGATA"
Corso di Laurea in Engineering Sciences
Linear Algebra and Geometry 2011/2012
Solutions

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[Apo67] **Exercise 14.9.11**.

(a) In a plane given a nonzero vector c there are only two vectors that make with c a given fixed angle. Denote these two vectors by v, v' respectively. By hypothesis we then have either $r(t) = f(t)v$ or $r(t) = f(t)v'$. It follows that either $r'(t) = f'(t)v$ or $r'(t) = f'(t)v'$, differentiating again we have that either $r''(t) = f''(t)v$ or $r''(t) = f''(t)v'$ from which the conclusion follows immediately.

(b) For instance, one can take the curve $r(t) = (e^t \cos(t), e^t \sin(t), e^t)$. $r(t)$ makes a constant angle of $\pi/4$ with the vector $(0, 0, 1)$, and r'' satisfies the required properties [Do verify these statements]. In order to find such a curve, one may start taking $z(t) = e^t$, in a such a way that $z''(t) \neq 0$ for every t. After that, one may choose $x(t)$ and $y(t)$ such that $(x(t), y(t), z(t))$ is contained in the cone $x^2 + y^2 - z^2 = 0\ldots$

[Apo67] **Exercise 14.9.13**. Write the curve as $r(t) = (x(t), y(t))$. Since the curve must have negative slope we have that $x'y' < 0$, or in other words they have opposite sign, moreover $0 \leq \theta, \phi \leq \pi/2$. It follows that

$$\frac{x}{y} = \cot(\theta)$$

and

$$-\frac{y'}{x'} = \tan(\phi)$$

but since

$$3\tan(\phi) = 4\cot(\theta)$$

we have

$$3\frac{y'}{x'} = 4\frac{x}{y}$$

and then

$$-3yy' = 4xx' \ .$$

Integrating both terms of the above equality we get

$$\frac{x^2}{6} + \frac{y^2}{8} = c$$

for some real constant c to be determined. But since the curve must contain the point $(3/2, 1)$ we must have $c = 1/2$.

[Apo67] **Exercise 14.9.15**.

(a) Differentiating both sides of the equality $r'(t) = A \times r(t)$ we get $r''(t) = A \times r'(t)$. The conclusion follow now easily by the properties of the cross product.

(b) First of all we are going to prove that the $\|r(t)\|$ is constant. Indeed

$$(r(t)\cdot r(t))' = 2r'(t)\cdot r(t) = 0$$

by the properties of the cross product. It follows that $\|r(t)\| = \|r(0)\| = \|B\|$. We have then

$$v(t) = \|r'(t)\| = \|A\times r(t)\| = \|A\|\|B\|\sin(\theta)\ .$$

(c) $r(t)$ makes a constant angle with A and this angle is equal to θ. First of all observe that $r(t)\cdot A$ is constant since

$$(r(t)\cdot A)' = r'(t)\cdot A = 0\ .$$

We have then $r(t)\cdot A = r(0)\cdot A = B\cdot A$, and

$$\frac{r(t)\cdot A}{\|r(t)\|\|A\|} = \frac{r(t)\cdot A}{\|B\|\|A\|} = \frac{B\cdot A}{\|B\|\|A\|} = \cos(\theta)\ .$$

It follows that $r(t)$ lies on the plane $z = \|A\|\cos(\theta)$ on a circle of radius $\|B\|\sin(\theta)$, since it has constant length equal to $\|B\|\cos(\theta)$ and makes a constant angle with A equal to θ. The curve $r(t)$ satisfies then

$$r(t) = (\|B\|\sin(\theta)\cos\alpha(t), \|B\|\sin(\theta)\sin\alpha(t), \|A\|\cos(\theta))$$

and since $v(t) = \|A\|$ we have $\alpha(t) = \|A\|t$.

[Apo67] **Exercise 14.9.16**.

(a) We have

$$Y'(t) = X'(u(t))u'(t)$$

and then

$$T_Y(t) = \frac{Y'(t)}{\|Y'(t)\|} = \frac{u'(t)X'(u(t))}{|u'(t)|\|X'(u(t))\|} = \operatorname{sgn}(u')T_X(u(t))$$

$\operatorname{sgn}(u') = +1$ if and only if u is strictly increasing and $\operatorname{sgn}(u') = -1$ if and only if u is strictly decreasing.

(b) From the equality

$$T_Y(t) = \operatorname{sgn}(u')T_X(u(t))$$

we have

$$T_Y'(t) = \operatorname{sgn}(u')u'(t)T_X'(u(t))$$

and then

$$N_Y(t) = \frac{T_Y'(t)}{\|T_Y'(t)\|} = \frac{|u'|T_X'(u(t))}{|u'|\|T_X'(u(t))\|} = N_X(u(t))\ .$$

[Apo67] **Exercise 14.15.3**. By Exercise 14.9.15 we have that $\|r(t)\| = \|B\|$, moreover since $r'(t) = A\times r(t)$, $v(t) = \|B\|\|A\|\sin(\theta)$ where again θ is constant by Exercise 14.9.15. By [Apo67, (14.22)] we have

$$k(t) = \frac{\|r'(t)\times r''(t)\|}{v^3(t)} = \frac{\|r'(t)\times(A\times r'(t))\|}{\|B\|^3\|A\|^3\sin(\theta)^3} = \frac{\|B\|\|A\|\sin(\theta)\|B\|\|A\|^2\sin(\theta)}{\|B\|^3\|A\|^3\sin(\theta)^3} = \frac{1}{\sin(\theta)}$$

since $\|A\| = \|B\| = 1$.

[Apo67] **Exercise 14.15.11**. We may write $r'(t) = (5\cos\alpha(t), 5\sin\alpha(t))$, we have then

$$2t = k(t) = \frac{\|T'(t)\|}{v(t)} = \frac{|\alpha'(t)|}{5}$$

since the curve never crosses the y-axis $\alpha'(t)$ must be negative and then

$$\alpha'(t) = -10t\ .$$

Finally

$$\alpha(t) = \alpha(0) - 5t^2 = \pi/2 - 5t^2 \ .$$

[Apo67] **Exercise 14.15.12**. We may write $T(t) = (\cos\alpha(t), \sin\alpha(t))$. Since the curve never crosses the x-axis we have $|\alpha'(t)| = \alpha'(t)$ and then

$$4t = k(t) = \frac{\|T'(t)\|}{2} = \frac{\alpha'(t)}{2}$$

and $\alpha(t) = 4t^2$. Finally we have

$$r'(t) = v(t)T(t) = (2\cos(4t^2), 2\sin(4t^2))$$

and $r'(\sqrt(\pi)/4) = (\sqrt(2), \sqrt(2))$.

References

[Apo67] Tom M. Apostol. *Calculus. Vol. I: One-variable calculus, with an introduction to linear algebra.* Second edition. Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1967.