ADDENDUM CONCERNING LINEAR TRANSFORMATIONS AND MATRICES

Up to Section 5 you should study the material in the book with no changes. You should also study Section 9 (Linear transformation with prescribed values) with no changes (but keeping in mind the examples provided in the lectures).

1. Replacement of sections 6 and 7 : Inverses and one-to-one transformations

Concerning the topic of sections 6 and 7, we will will content ourselves of the following simpler version.

We recall the following well known concepts concerning functions.

Let X and Y be two sets and $T : X \to Y$ a function. We recall that:

• T is said to be *injective* (or *one-to-one*) if the following condition holds: *let* $x, x' \in X$, *with* $x \neq x'$. *Then* $T(x) \neq T(x')$. In words: T sends distinct elements of X to distinct elements of Y.

• T is said to be *surjective* if *for each* $y \in Y$ *there is a* $x \in X$ *such that* $T(X) = y$. In words: every element of Y is the value, via T, of an element of X.

• T is said to be *bijective* if it is injective and surjective. This means that for each $y \in Y$ there is a UNIQUE $x \in X$ such that $T(x) = y$. In words: every element of Y corresponds, via T, to a unique element of X.

• T is said to be *invertible* if *there is a function* $S : Y \to X$ *such that* $ST = I_X$ *and* $TS = I_Y$ (where I_X and I_Y denote the identity functions of X and Y). We have the following proposition:

Proposition 1.1. *(a) T is invertible if and only if it is bijective.*
(b) In this case S is the unique function with the above properties. It is denoted T^{-1}, the inverse of T.
(c) Assume that T is bijective. Let $S : Y \to X$ such that $TS = I_Y$. Then $S = T^{-1}$. In particular, $ST = I_X$.
(d) Assume that T is bijective. Let $S : Y \to X$ such that $ST = I_X$. Then $S = T^{-1}$. In particular $TS = I_Y$.

Proof. (a) If T is bijective one defines S as follows: given $y \in Y$, $S(y)$ is be the unique $x \in X$ such that $T(x) = y$. Conversely, if $ST = I_X$ then T is injective, since if $T(x) = T(x')$ then $S(T(x)) = S(T(x'))$. But $S(T(x)) = x$ and $S(T(x')) = x'$. If $TS = I_Y$, then T is surjective, since, for each $y \in Y$, $y = T(S(y))$.
(b) This follows from the proof of point (a).
(c) Assume that T is bijective. Since $T(S(y)) = y$ for all $y \in Y$, $S(y)$ must

be the unique $x \in X$ such that $T(x) = y$.
(d) is similar. $\square$

Note that, without the hypothesis of bijectivity, (c) and (d) of the previous proposition are false (see for example Exercise 16.8-27 Vol. I, corresponding to 2.8.27 Vol. II).

In general, injectivity, surjectivity and bijectivity are quite unpredictable properties of functions. However, for *linear transformations*, especially in the finite-dimensional case, everything is much simpler. In the rest of the section we will exploit this point. We begin with injectivity.

Proposition 1.2. *Let V and W be linear spaces and $T : V \to W$ be a linear transformation. Then T is injective if and only if $N(T) = \{O_V\}$.*

Proof. We know that, since T is linear, $T(O_V) = O_W$. If T is injective, there is no other $v \in V$ such that $T(v) = O_W$. Therefore $N(T) = \{O_V\}$. For the other implication, let us assume that $N(T) = \{O_V\}$. Let $v, v' \in V$ such that $T(v) = T(v')$. This can be rewritten as $T(v) - T(v') = O_W$ or, since T is linear, $T(v - v') = O_W$, that is $v - v' \in N(T)$. But we assumed that $N(T) = \{O_V\}$. Hence $v - v' = O_V$, that is $v - v'$. Therefore T is injective. $\square$

The next proposition deals with the finite-dimensional case

Proposition 1.3. *Let V and W be finite-dimensional linear spaces and $T : V \to W$ be a linear transformation. Then the following are equivalent:*
(a) T is injective;
(b) $\dim T(V) = \dim V$;
(c) If $\{e_1, \dots, e_n\}$ is a basis of V then $\{T(e_1), \dots, T(e_n)\}$ is a basis of W.

Proof. (a)$\Leftrightarrow$(b) follows from Prop. 1.2 and the Nullity + Rank theorem. In fact, $N(T) = \{O\}$ if and only if $\dim N(T) = 0$. Since, by Nullity + Rank, $dim T(V) = \dim V - \dim N(T)$, T is injective if and only if $\dim T(V) = \dim V$.
(b)$\Rightarrow$(c) is as follows. In the first place we note that, since $e_1, \dots, e_n$ span V, then in any case $T(e_1), \dots, T(e_n)$ span $T(V)$, because, given $v \in V$, $v = \sum c_i e_i$. by the linearity if T, $T(v) = \sum T(c_i e_i) = \sum ci T(e_i)$. If $\dim T(V) = \dim V = n$ then $\{T(e_1), \dots, T(e_n)\}$ is a basis, since it is a spanning set formed by n elements. (c)$\Rightarrow$(b) is obvious. $\square$

Concerning bijectivity and invertibility, we start by recording the following easy fact, which does not need finite-dimensionality

Proposition 1.4. *Let V and W be linear spaces and $T : V \to W$ be a linear transformation. If T is invertible then also $T^{-1} : W \to V$ is a linear transformation.*

Proof. Let $w_1, w_2 \in W$ and let v_1, v_2 be the unique elements of V such that $T(v_1) = w_1$ and $T(v_2) = w_2$. hence $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. Since T is linear, $T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$. Therefore

$$T^{-1}(w_1 + w_2) = v_1 + v_2 = T^{-1}(w_1) + T^{-1}(w_2).$$

Moreover, let $c \in \mathbb{R}$. We have that $T(c\,v_1) = c\,T(v_1) = c\,w_1$. Therefore

$$T^{-1}(c\,w_1) = c\,v_1 = c\,T^{-1}(w_1) \qquad \square$$

In the finite-dimensional case, Proposition 1.3 has the following consequences

Corollary 1.5. *Let V and W be finite-dimensional linear spaces and $T : V \to W$ be a linear transformation.*
(a) If T is bijective (or, equivalently, invertible) then $\dim V = \dim W$. (b) Conversely, assume that $\dim V = \dim W$. Then the following are equivalent
(i) T is injective;
(ii) T is surjective,
(iii) T is bijective.

Proof. It is sufficient to prove the equivalence of (i) and (ii). Assume that T is injective. Then, by Prop. 1.3, $\dim T(V) = \dim V = \dim W$. Therefore $T(V) = W$, that is T is surjective.
Assume that T is surjective, that is $\dim T(V) = \dim W (= \dim V)$. By nullity+rank, this implies that $\dim N(T) = 0$. Thus Prop. 1.3 implies that T is injective. $\square$

For example, let $T : V_3 \to V_3$ defined by $T((x, y, z)) = (x - 2y + 3z, x + y + z, x - y - z)$. By the previous Corollary and Theorem 1.3 T is bijective (hence invertible) if and only if $N(T) = \{O\}$. $N(T)$ is the space of solutions of the system of linear equations

$$\begin{cases} x - 2y + 3z = 0 \\ x + y + z = 0 \\ x - y - z = 0 \end{cases}$$

By what we studied in the first semester, hence $N(T) = \{O\}$ means that this system has only the trivial solution $(0, 0, 0)$ (or, equivalently, that the columns of the system are linearly independent). This can be checked by computing the determinant

$$\det \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} = -13$$

Since the determinant is non-zero then (go back to the lectures of the first semester, or to Chapter 15 of Vol. I !) $(0, 0, 0)$ is the only solution. Therefore T is bijective. Later on we'll see an efficient way to compute the inverse transformation T^{-1}.

2. Matrices and linear transformations: supplementary notes

It is conceptuallly easier to study the remaining sections of the chapter on linear transformation and matrices as follows: read the beginning of Section 10 for generalities about matrices. Then skip, for the moment, the Theorem and the subsequent examples, and go directly Section 13 (Linear spaces of matrices). Then skip, for the moment, Section 14 and go directly to Section 15 (Multiplication of matrices). At this point go back to the Theorem and Examples of Section 10 and Section 14, which are about the correspondence between matrices and linear transformations. Here are some supplementary notes about this material, which hopefully may helpful to understand the meaning of these results. Note: the contents of Section 11 (Construction of a matrix representation in diagonal form) should be *skipped*.

We will use the following notation: $\mathcal{M}_{m,n}$ will denote the set of all $m \times n$ matrices. Equipped with the operations of matrix addition and scalar multiplication $\mathcal{M}_{m,n}$ is in fact a linear space (Section 13.)

Definition 2.1 (Standard linear transformation associated to a $m \times n$ matrix)**.** *Let $A \in \mathcal{M}_{M,n}$. The standard linear transformation associated to A is the linear trasformation*

$$T_A : V_n \to V_m$$

defined as follows. We see the elements of V_n and V_m as column vectors

$$X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \in \mathcal{M}_{n,1} \qquad\qquad Y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_m \end{pmatrix} \in \mathcal{M}_{m,1}$$

Then T_A is defined as

$$T_A(X) = AX$$

where AX denoted the multiplication of the $m \times n$ matrix A with the $n \times 1$ matrix (= column vector of length n) X. The result is a $m \times 1$ matrix (=column vector of length m). In coordinates:

$$T_A(X) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \\ \dots & & \\ \dots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \cdot \\ \cdot \\ \cdot \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$$

Here are some remarks:

(a) The column vector $T_A(X) = AX$ can be written also as

$$x_1 \begin{pmatrix} a_{11} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{pmatrix} = x_1 A^1 + \cdots + x_n A^n.$$

(b) A system of linear equations

$$\begin{cases} A_1 \cdot X = b_1 \\ \dots \\ \dots \\ \dots \\ A_m \cdot X = b_m \end{cases}$$

can be written in compact form as

$$AX = B$$

where B is the (column) vector of constant terms. This has the conceptual advantage of seeing a system composed by many equations as a single *vector* equation, that is an equation whose unknown is a vector. For example,

$$\begin{cases} 2x + 3y - z = 3 \\ 2x + y + 2z = 4 \end{cases} \qquad \Leftrightarrow \qquad \begin{pmatrix} 2 & 3 & -1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

(c) $T_A(V_n)$, the range of T_A, is, by definition, the subspace of V_m formed by the $B \in V_m$ such that the system of linear equations $AX = B$ (see the above remarks) has some solutions. This shows that $T_A(V_n) = L(A^1, \dots, A^n)$.
(d) The linear transformation T_A is nothing else but the "linear transformation defined by linear equations" of Example 4 of Section 1 of the textbook. As remarked in Example 4 of Section 2 of the book, $N(T_A)$, the null-space of T_A, is the subspace of V_n formed by the solutions of the homogenous system $AX = 0$.
(e) Denoting

$$E^1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \dots\dots, E^n = \begin{pmatrix} 0 \\ . \\ . \\ . \\ . \\ 1 \end{pmatrix}$$

the "coordinate unit vectors" of V_n – that is the vectors of the so-called "canonical basis" of V_n written as column vectors – then

$$AE^i = A^i$$

(where A^i are the column vectors of A).

In the next Theorem, we consider the linear space $\mathcal{L}(V_n, V_m)$ of all linear transformations from V_n to V_m (see Section 4, Theorem 16.4 Vol. I, corresponding to Theorem 2.4 Vol. II) and the linear space $M_{m,n}$ of all $m \times n$

matrices. We define the following function

$$\mathcal{T} : M_{m,n} \to \mathcal{L}(V_n, V_m), \qquad A \mapsto T_A$$

Theorem 2.2 (Correspondence between matrices and linear transformations, provisional form)**.** *The above function $\mathcal{T}$ is a bijective linear transformation (terminology: a bijective linear transformation is called an* isomorphism of linear spaces*). Hence it is invertible and its inverse is linear.*

Proof. It is easy to see that $\mathcal{T}$ is a linear transformation (exercise!) and that it is injective (exercise!). To prove that it is surjective let $T \in \mathcal{L}(V_n, V_m)$: we have to prove that there exists a (unique, by the injectivity) $A \in \mathcal{M}_{m,n}$ such that TT_A. To see this, we note that given

$$X = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

we have that $X = x_1 E^1 + \cdots x_n E^n$ (see Remark (e)). Therefore $T(X) = T(x_1 E^1 + \cdots x_n E^n) = x_1 T(E^1) + \cdots + x_n T(E^n)$. Let A be the matrix whose colums are $A^1 := T(E^1)$, ... ,$A^n := T(E^n)$. Then $T(X) = x_1 A^1 + \cdots + x_n A^n = AX = T_A(X)$, see Remark (a). Therefore $T = T_A = \mathcal{T}(A)$. Hence $\mathcal{T}$ is surjective. □

From the previous theorem it follows

Corollary 2.3 (Matrix representation with respect to canonical bases)**.** *Any linear transformation $T : V_n \to V_m$ is of the form T_A for a (unique) matrix $A \in \mathcal{M}_{M,n}$. In other words: $T(X) = AX$ for all $X \in V_n$ (seen as a column vector). Following the book, we will denote*

$$A = m(T)$$

We have the following definition

Definition 2.4. *$m(T)$ is called the matrix representing the linear transformation T (with respect to the canonical bases of V_n and V_m).*

Example 2.5. *Let us consider the identity map $I : V_n \to V_n$. We have that $m(I) = I_n$, the identity matrix of order n. This is obvious, since $IX = X = I_n X$. Analogously, let c be a scalar and $T_c : V_n \to V_n$ the "multiplication by c" (or "omothety") linear trasformation defined as $T_c(X) = cX$. Then*

$$m(T_c) = cI_n = \begin{pmatrix} c & 0 & \dots & 0 \\ \dots & & & \\ \dots & & & \\ 0 & \dots & 0 & c \end{pmatrix}$$

Indeed $T_c(X) = cX = (cI_n)X$

Example 2.6. *Let $R_\theta : V_2 \to V_2$ be the rotation (counterclockwise) of angle θ of V_2. Then*

$$m(R_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(Exercise)

A very important feature of the correspondence between linear transformations and matrices is that matrix multiplication corresponds to composition of functions

Theorem 2.7. *Let $T : V_n \to V_m$ and $S : V_k \to V_m$ be linear transformations. Let us consider the composition $TS : V_k \to V_m$. Then*

$$m(TS) = m(T)m(S)$$

Proof. This follows immediately from the associativity of matrix multiplication (see Section 15 in the book). Indeed, let $A = m(T)$ and $B = m(S)$. From Theorem 2.2, the assertion of the present Theorem is equivalent to the assertion

$$T_{AB} = T_A T_B$$

that is

$$(AB)X = A(BX) \qquad \text{for any } \; X \in V_k$$

which is a particular case of the associativity property of matrix multiplication. □

2.1. **Correspondence between matrices and linear transformations: general version.** It turns out that Theorem 2.2 , Corollary 2.3 and Theorem 2.7 are particular cases of much more general statements. The point is that, rather than using the usual coordinates (that is the components with respect to the canonical basis, formed by the usual unit coordinate vectors) one can use the coordinates with respect to an arbitrary basis. The general formulation of Corollary 2.3 is Theorem 16.13 of Vol. I (2.13 of Vol. II) plus Theorem 16.16 of Vol. I (Theorem 2.16 of Vol. II). Before stating these results we introduce the following setup:

(1) *Let V and W be finite-dimensional linear spaces, of dimension respectively n and m.*
(2) *Let $\mathcal{B} = \{e_1, \dots, e_n\}$ and $\mathcal{C} = \{f_1, \dots, f_n\}$ be bases of V and W respectively.*
(3) *Let $A \in \mathcal{M}_{m,n}$ be a $m \times n$ matrix.*
(4) *Given a vector $v \in V$, let*

$$X_{\mathcal{B},v} = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

be the column vector of components of v with respect to the basis $\mathcal{B}$. In other words, $x_1, \ldots, x_n$ are the unique scalars such that $v = x_1 e_1 + \cdots + x_n e_n$.

(5) *Given a vector $w \in W$, let*

$$Y_{\mathcal{C},w} = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{pmatrix}$$

be the column vector of components of w with respect to the basis $\mathcal{C}$. In other words, $y_1, \ldots, y_m$ are the unique scalars such that $w = y_1 f_1 + \cdots + y_n f_m$.

Definition 2.8. *(1) In the previous setting, we define a linear transformation*

$$T : V \to W$$

as follows. Let $v \in V$ then

$$Y_{\mathcal{C},T(v)} = A\, X_{\mathcal{B},v}$$

In words: we define T by defining, for all $v \in V$, the (column) vector of components of $T(v)$ with respect to the basis $\mathcal{C}$. This is, by definition, the product of the matrix A times the (column) vector of components of v with respect to the basis $\mathcal{B}$.

(Note that, by construction, the (column) vectors

$$Y_{\mathcal{C},T(e_1)}, \cdots, Y_{\mathcal{C},T(e_n)}$$

are the columns

$$A^1, \ldots, A^n$$

of the matrix A.)

(2) The linear transformation T is called the linear transformation represented by the matrix A with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ and we denote

$$A = m^{\mathcal{B}}_{\mathcal{C}}(T).$$

The analogue of Corollary 2.3 in this more general setting is the following

Theorem 2.9 (Matrix representation, general form)**.** *Let*

$$T : V \to W$$

be a linear trasformation. let $\mathcal{B}$ be a bses of V and let $\mathcal{C}$ be a basis of W. Then there is a unique matrix $A \in \mathcal{M}_{m,n}$ such that

$$A = m^{\mathcal{B}}_{\mathcal{C}}(T).$$

The matrix A is the one whose colums are the (column) vector of components of $T(e_1), \ldots, T(e_n)$ with respect to the basis $\mathcal{C}$.

Theorem 2.10. *Keeping the notation of the previous Theorem, let U be another finite-dimensional linear space, of dimension k, and let $\mathcal{D}$ be a basis of U. Furthermore let $S : U \to V$ be a linear transformation. Then*

$$m^{\mathcal{D}}_{\mathcal{C}}(TS) = m^{\mathcal{B}}_{\mathcal{C}}(T)m^{\mathcal{D}}_{\mathcal{B}}(S)$$

The proofs are similar to those of Theorem 2.2, Corollary 2.3 and Theorem 2.7, and they are omitted. As a useful exercise, you should try at least to outline them.

Example 2.11. *Let V be a n-dimensional linear space and let $\mathcal{B}$ be any basis of V. Let, as above, $I : V \to V$ the identity transformation and, for a given scalar c, $T_c : V \to V$ the linear transformation $v \mapsto cv$. Then*

$$m^{\mathcal{B}}_{\mathcal{B}}(I) = I_n \quad \text{and} \qquad m^{\mathcal{B}}_{\mathcal{B}}(T_c) = cI_n \tag{1}$$

This is because, if X are is the vector of components of a given vector $v \in V$ with respect to the basis $\mathcal{B}$, then cX is the vector of components of the vector cv with respect to the same *basis $\mathcal{B}$. Note that the basis of the source and the target space has to be the same, otherwise (1) is false.*

Example 2.12. *Let $T = Pr_{L((1,2))} : V_2 \to V_2$ be the projection along $L(1, 2)$. Let $S = Ref_{L((1,2))}$ be the reflection with respect to $L((1, 2))$. Let $\mathcal{B} = \{(1, 2), (2, -1)\}$. Then:*

$$m^{\mathcal{B}}_{\mathcal{B}}(T) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad m^{\mathcal{B}}_{\mathcal{B}}(S) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Exercise!

3. The rank of a matrix

Let us start with the following

Definition 3.1. *Let $A \in \mathcal{M}_{M,n}$ be a matrix. The rank of A, denoted $rk(A)$, is defined as the rank of the linear transformation $T_A : V_n \to V_m$, $X \mapsto AX$* (compare Def. 2.1)*.*

By Remark (c) after Definition 2.1 we know that $rk(A) = \dim L(A^1, \dots, A^n)$, where $A^1, \dots, A^n$ are the columns of A. In other words, $rk(A)$ is the maximal number of independent columns of A (see Thms 15.5 and 15.7 of Vol. I, corresponding to Thms 1.5 and 1.7 of Vol. II). We have the remarkable

Proposition 3.2. $rk(A) = \dim L(A_1, \dots, A_m)$*, where $A_1, \dots, A_m$ are the rows of A. In other words, the maximal number of independent columns of A equals the maximal number of independent rows of A.*

Proof. By the nullity + rank Theorem, $rk(T_A) = n - \dim N(T_A)$. On the other hand, by definition,

$$N(T_A) = L(A_1, \dots, A_m)^{\perp}.$$

This is because $N(T_A)$ is the set of solution of the homogeneous system

$$\begin{cases} A_1 \cdot X = 0 \\ \dots \\ \dots \\ A_m \cdot X = 0 \end{cases}$$

Therefore $\dim N(T_A) = \dim L(A_1, \dots, A_m)^{\perp} = n - \dim L(A_1, \dots, A_m)$, see the addendum on orthogonal complements (note that the rows $A_1, \dots, A_m$ have $n(=$ number of columns od $A)$ components, hence they are vectors of V_n). Putting evething together $Rk(A) = rk(T_A) = n - \dim N(T_A) = n - (n - \dim L(A_1, \dots, A_m)) = \dim L(A_1, \dots A_m)$. □

Example 3.3. *Let $A_1 = (1,2,3)$, $A_2 = (3,4,5)$ and let $A_3 := A_1 + A_2 = (4,6,8)$. Let us consider the matrix*

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{pmatrix}$$

Since the maximal number of independent rows is 2 then the maximal number of independent columns is 2. In particular, the columns are dependent. Exercise: check this!

4. Computation of the rank of a matrix, with application to systems of linear equations

We have the following easy result, summarizing the qualitive behaviour of systems of linear equations. We will need the followimg terminology: given a linear system $AX = B$, with $A \in \mathcal{M}_{m,n}$ and $B \in \mathcal{M}_{m,1}$ (see Remark (b) after Definition 2.1), we denote $A|B$ the $m \times (n+1)$-matrix whose first n columns are the columns of A and the last one is B. This is called the *augmented* or *complete* matrix of the linear sistem.

Theorem 4.1 (Rouché-Capelli). *Let $AX = B$ be a linear system. Then*
(a) A has some solutions if and only if $rk(A) = rk(A|B)$ (if this happens the system is sometimes called *compatible*),
(b) In this case, the set of all solutions of the system is of the form $v + W = \{v + w \mid w \in W\}$, where $v \in V_n$ is a solution of the system and W is the linear subspace of V_n formed by all solutions of the homogeneous system $AX = O$. In particular, there is a unique solution if and only if (a) holds and $\dim W = 0$.
(c) $dimW = n - rk(A)$.

Proof. (a) The system has some solutions if and only if the column vector B is a linear combinations of the columns $A^1, \dots, A^n$. This means exactly that the rank(= number of independent columns) of $A|B$ is the same as the rank of A.
(b) Let v_1 and v two solutions of the system, that is $Av_1 = B$ and $Av = B$.

Then $A(v_1 - v_2) = 0$. therefore $v_1 - v$ is a solution of the homogeneous system $AX = O$, that is $v_1 - v \in W$. Therefore $v_1 = v + w$ for a $w \in W$.
(c) This is just a restatement of the nullity + rank Theorem. $\square$

In order to solve a linear system by computing the rank of the matrices A and $A|B$ one can use the row-elimination method of Gauss-Jordan. Here are some examples (see also the examples given in the lectures and those in the book at Section 18).

Example 4.2. $\begin{cases} x + 2y + z + t = 1 \\ x + 3y + z - t = 2 \\ x + 4y + z - 3t = 3 \\ 2x + y \quad + z = 2 \end{cases}$.

We will make use of the following modifications of the equations of the system:
(a) exchanging to equations;
(b) multiplying an equation by a non-zero scalar,
(c) adding to an equation a scalar multiple of another equation.
Clearly such modifications produce *equivalent*(= having the same solutions) systems. Since the equations correspond to the rows of the associated augmented matrix $A|B$, the above modifications correspond to modifications of the rows of $A|B$. Note that, even if after operating one such modifications the rows of the modified matrix do change, the linear span of the rows remains the same. Therefore such modifications leave unchanged the rank.

$$A|B = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 4 & 1 & -3 & 3 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & -4 & 2 \\ 0 & -3 & -2 & -1 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & -2 & -7 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we arrived to a matrix in $r$ow-echelon form, corresponding to the equivalent system

$$\begin{cases} x & + & 2y & + & z & + & t & = 1 \\ & & y & & & - & 2t & = 1 \\ & & & - & z & - & 7t & = 3 \end{cases}$$

Let us denote $A'X = B'$ this new system. We have that $rk(A|B) = rk(A'|B') = 3$, since the non-zero rows of a row-ladder matrix are clearly independent (exercise!). For the same reason, $rk(A) = rk(A') = 3$. Therefore the system has solutions (note that, in general, there are solutions if and only if, in the final ladder matrix there is no row of the form $\begin{pmatrix} 0 & \dots & 0 & a \end{pmatrix}$ with $a \neq 0$). Even before computing the explicit solutions, we know that the set of solutions will have the form

$$v + W, \qquad \text{with} \quad \dim W = 1$$

since $n - rk(A|B) = 4 - 3 = 1$. We compute the solutions starting from the last equation: $z = -3 - 7t$, $y = 1 + 2t$, $x = 1 - 2y - z - t = 1 - 2(1 + 2t) - (-3 - 7t) - t = 2 + 2t$. Therefore the solutions are the 4-tuples of the form

$$\begin{pmatrix} 2+2t \\ 1+2t \\ -3-7t \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ -7 \\ 1 \end{pmatrix} = v + w, \quad \textit{where} \quad w \in W = L(\begin{pmatrix} 2 \\ 2 \\ -7 \\ 1 \end{pmatrix}$$

Example 4.3. $\begin{cases} x + 2y + z + t = 1 \\ x + 3y + z - t = 2 \\ x + 4y + z - 3t = 2 \\ 2x + y \quad + z = 2 \end{cases}$.

$$A|B = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 4 & 1 & -3 & 2 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & -4 & 1 \\ 0 & -3 & -2 & -1 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & -2 & -7 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

The system has no solution because the last equation is $0 = -1$. This corresponds to the fact that $rk(A) = 3$ while $rk(A|B) = 4$. *In general, the rank of a matrix in row-echelon form is the number of non-zero rows).*

Example 4.4. $\begin{cases} x + 2y + z + t = 1 \\ x + 3y + z - t = 2 \\ x + 4y + z - 2t = 3 \\ 2x + y \quad + z = 2 \end{cases}$.

$$A|B = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 4 & 1 & -2 & 3 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & -3 & 2 \\ 0 & -3 & -2 & -1 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -7 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & -2 & -7 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In this case $rk(A) = rk(A|B) = 4$. Therefore there is a *unique* solution, since $\dim W = 4 - 4 = 0$ (this simply means that the four columns of A are independent). *Note that, since $rk(A) = 4$ for all possible vectors of constant terms $B' \in V_4$ the system $AX = B'$ has a unique solution!)* Exercise: find the solution.

5. Invertible matrices and their inverses

We have see that the identity matrices I_n are neutral elements with respect to matrix multiplication. It is therefore natural to ask which matrices have an inverse element with respect to multiplication.

Definition 5.1. *Let $A \in \mathcal{M}_{n,n}$ be a square matrix. A is said to be invertible if there is another matrix $B \in \mathcal{M}_{n,n}$ such that $AB = BA = I_n$.*

Remark 5.2. If A is invertible the matrix B is unique. Indeed, if B' is another such matrix, then $B' = B'I_n = B'(AB) = (B'A)B = I_nB = B$.

Definition 5.3. *If A is invertible then the matrix B is called the inverse of A, and denoted A^{-1}.*

It is not hard to imagine that invertible matrices correspond to invertible linear transformations:

Proposition 5.4. *Let $A \in \mathcal{M}_{n,n}$. The following are equivalent:*
(a) A is invertible,
(b) the linear transformation $T_A : V_n \to V_n$ is invertible, and $(T_A)^{-1} = T_{A^{-1}}$;
(c) $rk(A) = n$.

Proof. (a) $\Leftrightarrow$ (b) Assume that A is invertible. Then, by Theorem 2.7 and Example 2.5,

$$I = T_{I_n} = T_{AA^{-1}} = T_A T_{A^{-1}}.$$

Analogously,

$$I = T_{A^{-1}} T_A.$$

Therefore T_A is invertible.
Conversely, assume that T_A is invertible. Then we know that $(T_A)^{-1}$ is a linear transformation too (Prop. 1.4). Hence, by Theorem 2.2 there is matrix B such that $(T_A)^{-1} = T_B$. By Theorem 2.7 we have that $AB = m(T_A T_B) = m(I) = I_n$ and $BA = m(T_B T_A) = m(I) = I_n$. Therefore A is invertible and $B = A^{-1}$.
(b) $\Leftrightarrow$ (c) The linear transformation T_A is invertible if and only if it is bijective. by Corollary 1.5 his happens if and only if it is surjective, that is $rk(T_A) = n$. But, by definition, $rk(A) = rk(T_A)$. □

The following proposition ensures that, in order to check invertibility and find the inverse of a matrix, it is sufficient to check only *one* of the conditions $AB = I_n$ and $BA = I_n$.

Proposition 5.5. *(a) Let $A \in \mathcal{M}_{n,n}$. If there is a matrix $B \in \mathcal{M}_{n,n}$ such that $AB = I_n$ then A is invertible and $B = A^{-1}$.*
(b) Let $A \in \mathcal{M}_{n,n}$. If there is a matrix $B \in \mathcal{M}_{n,n}$ such that $BA = I_n$ then A is invertible and $B = A^{-a}$.

Proof. (a) If $AB = I_n$ then $T_A T_B = T_{AB} = T_{I_n} = I$ (Theorem 2.7 and Example 2.5. This implies that T_A is surjective since, for all $X \in V_n$, $X = (T_A T_B)(X) = T_A(T_B(X)$, hence there is a Y such that $X = T_A(Y)$. By Cor 1.5 T_A is bijective, hence invertible. Therefore, by Prop. 5.4 A is invertible.

(b) If $BA = I_n$ then $T_B T_A = T_{BA} = T_{I_n} = I$. This implies that T_A is injective since, for all $X, X' \in V_n$, if $T_A(X) = T_A(X')$ then $X = T_B(T_A(X)) = T_B(T_A(X')) = X'$. Then by Corollary 1.5 T_A is bijective, hence invetrible. Therefore, by Prop. 5.4 A is invertible.

Remark 5.6 (Inverse matrix and linear systems)**.** Let us consider a *square* linear system, that is a system of linear equations such that the number of equations equals the number of unknowns. In other words, a linear system $AX = B$ where $A \in \mathcal{M}_{n,n}$ is a square matrix. Then we know that for all $B \in \mathcal{V}_n$ there is a solution if and only if A has rank n and in this case the solution is actually unique. Now A has rank n if and only if it is invertible and in the case the unique solution is

$$X = A^{-1}B.$$

This is simply obtained multiplying both members of $AX = B$ by A^{-1} on the left. Note the analogy with a linear equation

$$ax = b$$

where $a, b \in \mathbb{R}$. Under the condition $a \neq 0$, which means that a is invertible with respect to the multiplication of real numbers, then there is always a solution, such solution is unique, and more precisely such solution is

$$x = a^{-1}b.$$

5.1. **Computation of the inverse matrix.** Given an invertible matrix $A \in \mathcal{M}_{n,n}$, Prop. 5.5 assures that, in order to find its inverse, it is enough to solve the matricial equation

$$AX = I_n \tag{2}$$

where the unknown X is a $n \times n$ matrix. In the next examples we show how to solve such equation using row elimination.

Example 5.7. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Equation (2) can be solved by finding the two columns of X, denoted, as usual, X^1 and X^2. Therefore equation (2) is equivalent to the two systems

$$AX^1 = E^1 \qquad \text{and} \qquad AX^2 = E^2$$

that is

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} x_{11} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} x_{21} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 3 \end{pmatrix} x_{12} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} x_{22} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This can be summarized in the single equation

$$\begin{pmatrix}1\\3\end{pmatrix}(x_{11},x_{12})+\begin{pmatrix}2\\4\end{pmatrix}(x_{21},x_{22})=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

This can be seen as a usual system of linear equations

$$\begin{pmatrix}1\\3\end{pmatrix}X_1+\begin{pmatrix}2\\4\end{pmatrix}X_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

that is

$$\begin{cases}X_1 & + & 2X_2 & = & (1,0)\\ 3X_1 & + & 4X_2 & = & =(0,1)\end{cases}$$

where the unknowns X_1 and X_2 are the rows of the inverse matrix. This can be solved in the usual way

$$\begin{pmatrix}1&2&|&1&0\\3&4&|&0&1\end{pmatrix}\to\begin{pmatrix}1&2&|&1&0\\0&-2&|&-3&1\end{pmatrix}$$

This corresponds to the system

$$\begin{cases}X_1 & + & 2X_2 & = & (1,0)\\ & & -2X_2 & = & (-3.1)\end{cases}$$

Solving as usual we get $X_2=(3/2,-1/2)$, and $X_1=(1,0)-2X_2=(1,0)+(-3,1)=(-2,1)$. Therefore the inverse matrix is

$$A^{-1}=\begin{pmatrix}X_1\\X_2\end{pmatrix}=\begin{pmatrix}-2&1\\3/2&-1/2\end{pmatrix}$$

Check that it is really the inverse matrix!

Example 5.8. $A=\begin{pmatrix}1&2&1\\-2&2&3\\1&1&1\end{pmatrix}.$

Arguing as before, we are lead to solve the system of linear equations

$$\begin{cases}X_1 & + & 2X_2 & + & X_3 & = & (1,0,0)\\ -2X_1 & + & 2X_2 & + & 3X_3 & = & (0,1,0)\\ X_1 & + & X_2 & + & X_3 & = & (0,0,1)\end{cases}$$

$$\begin{pmatrix}1&2&1&|&1&0&0\\-2&2&3&|&0&1&0\\1&1&1&|&0&0&1\end{pmatrix}\to\begin{pmatrix}1&2&1&|&1&0&0\\0&6&5&|&2&1&0\\0&-1&0&|&-1&0&1\end{pmatrix}$$

Note that from this calculation if follows that $rk(A)=3$ (exercise!), that is that A is invertible. Solving we have $X_2=(1,0,-1)$,
$X_3=1/5\big((2,1,0)-6X_2\big)=1/5((2,1,0)-(6,0,-6)\big)=1/5(-4,1,6)=(-4/5,1/5,6/5)$,
$X_1=(1,0,0)-2X_2-X_3=(1,0,0)-2(1,0,-1)-(-4/5,1/5,6/5)=$

$(-1/5, -1/5, 4/5)$.
Therefore the inverse matrix is

$$A^{-1} = \begin{pmatrix} -1/5 & -1/5 & 4/5 \\ 1 & 0 & -1 \\ -4m/5 & 1/5 & 6/5 \end{pmatrix}$$

Check that this is really the inverse matrix!

6. Exercises

Ex. 6.1. Let $T : V_4 \to V_4$ be the linear transformation $T(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}) = \begin{pmatrix} x_1 + x_3 \\ 2x_1 + x_2 + 2x_3 + 2x_4 \\ x_1 + x_2 + x_3 + 4x_4 \\ x_1 + x_2 + x_3 + 2x_4 \end{pmatrix}$
(a) Find a basis of $T(V_4)$.
(b) Let $v = (-3, -3, 0, 0)$. Does v belong to $T(V_4)$? Is case of positive answer, find the components of v with respect to the basis of $T(V_4)$ found in (a).
(c) Find a basis of $N(T)$.

Ex. 6.2. Let $\mathcal{B} = \{u, v, w\}$ be a basis of V_3.
(a) Is there a linear transformation $T : V_3 \to V_3$ such that $T(u) = w$, $T(v) = w$ and $T(w) = 3v - 2w$? If the anwer is yes:
(b) Find dimensions and bases of $N(T)$ and $T(V_3)$.
(c) Compute $m_{\mathcal{C}}^{\mathcal{B}}(T)$.
(d) Is there a linear transformation $S : V_3 \to V_3$ such that $S(u) = v$, $S(v) = w$, $S(3u - 2v) = u$? If the answer is yes answer to questions (b) and (c) as above.

Ex. 6.3. Let $u = (1, 1, -1)$, $v = (1, 1, 0)$, $w - (1, -1, 1)$. For t varying in $\mathbb{R}$, let $S_t : V_3 \to V_3$ be the linear transformation defined by $S_t(u) = (1, 0, -1)$, $S_t(v) = (1, t + 1, 1)$, $S_t(w) = (1, 4, t + 2)$.
(a) Find the value of t such that S_t is injective.
(b) For $t = -5$ find a basis of $N(S_t)$, a basis of $S_t(V_3)$ and a a system of cartesian equations whose space of solutions is $N(S_t)$.
(c) Compute $m_{\mathcal{E}}^{\mathcal{B}}(S_t)$, where $\mathcal{B} = \{u, v, w\}$ and $\mathcal{E}$ is the canonical basis.

Ex. 6.4. Let $T : V_3 \to V_2$ be the linear transformation such that $T((1, 0, -1) = (2, -1)$, $T((0, 1, 1)) = (0, 1)$ and $T((0, 1, 0)) = (1, 0)$.
(a) Find $N(T)$.
(b) Find a line L in V_3 passing trough $P = (3, 4, 5)$ such that $T(L)$ is a point.
(c) Find a plane Π in V_3 passing trough $P = (3, 4, 5)$ such that $T(\Pi)$ is a line.

(d) Is there a plane M of V_3 such that $T(M)$ is a point?
(e) Let $\mathcal{B} = \{\{(1, 0, -1), (0, 1, 1), (0, 1, 0)\}$. Let $\mathcal{C} = \{(2, -1), (0, 1)\}$. Let $\mathcal{E}_3$ and $\mathcal{E}_2$ be the canonical base sof V_3 and V_2. Compute $m^{\mathcal{B}}_{\mathcal{E}_2}(T)$ and $m^{\mathcal{B}}_{\mathcal{C}}(T)$.

Ex. 6.5. Let us consider the linear transformation

$$R_{L((1,2))}R_{L((1,3))} : V_2 \to V_2,$$

where R_U denotes the reflection with respect to the linear subspace U. Compute its null-space and range.

Ex. 6.6. Let us consider the linear transformations

$$P_{L((1,2-1),(1,1,1))}R_{L((1,0,-1),(11,3))} : V_3 \to V_3$$

and

$$P_{L((1,2,-1),(1,1,1))}P_{L((1,0,-1),(1,1,3))} : V_3 \to V_3,$$

where, as above, R_U denotes the reflexion with respect to the linear subspace U, and P_V denotes the projection on the linear subspace V. Compute null-space and range of such transformations.

Ex. 6.7. For t varying in $\mathbb{R}$ let $u_t = (1, t + 1, 1)$, $v_t = (1, t + 2, 2)$ and $w_t = (2, 1, t + 1)$. Let $S_t : V_3 \to V_3$ be the linear transformation such that $S(E^1) = u_t$, $S(E_2) = v_t$, $S(E^3) = w_t$, where $\{E^1, E^2, E^3\}$ are the unit coordinate vectors.
(a) Find for which $t \in \mathbb{R}$ the transformation S_t is surjective.
(b) For all $t \in \mathbb{R}$ such that S_t is not surjective, find a basis of $S_t(V_3)$.
(c) Find, if possible, a vector $v \in V_3$ such that $S_{-1}(v) = (1, 0, 0)$.

Ex. 6.8. Let V be a linear space and let $\{v_1, v_2\} \subset V$ be a linearly independent set made of two elements. Let $T : V \to V$ be a linear transformation such that $T(v_1 + 2v_2) = 2v_1 - v_2$, and $T(v_1 - v_2) = v_1 + 3v_2$.
(a) Express $T(v_2)$ as linear combination of v_1 and v_2.
(b) Is there a $u \in V$ such that $T(u) = v_1$? If the answer is yes, find it.

Ex. 6.9. True/false? (Then explain all anwers)
(a) For a matrix $A \in \mathcal{M}_{5,6}$, $T(X) = AX$ defines a linear transformation $T : V_5 \to V_6$.
(b) Every linear transformation $T : V_6 \to V_4$ is surjective.
(c) Every linear transformation $T : V_4 \to V_6$ is injective.
(d) Every linear transformation $T : V_6 \to V_4$ such that $\dim N(T) = 2$ is surjective.
(e) Every linear transformation $T : V_4 \to V_6$ such that $\dim T(V_4) = 4$ is surjective.
(f) If $\dim V = \dim W$ a linear transformation $T :\to W$ is injective if and only if it is surjective.

Ex. 6.10. Let V and W be linear spaces and $T : V \to W$ a linear transformation. Let $v_1, ..., v_k \in V$.
(a) Prove that if $T(v_1), ..., T(v_k)$ are linearly independent then $v_1, ..., v_k$ are

linearly independent.
(b) Prove that if $v_1, ..., v_k$ are linearly independent and T is injective then $T(v_1), ..., T(v_k)$ are linearly independent.

Ex. 6.11. For t varying in $\mathbb{R}$, let us consider the linear system

$$\begin{cases} x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & + & 2x_2 & & & = & 0 \\ x_1 & + & x_2 & + & (t-1)x_3 & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & t \end{cases}$$

Find the values of t such that the system has solutions, and those such that the system has a unique solution. For such values of t, solve the system.

Ex. 6.12. Let us consider the lines of V_3 $L : \begin{cases} x + y = 0 \\ x + 2y + z = 0 \end{cases}$ and $M : \begin{cases} x - y = 1 \\ x + 5y + z = 0 \end{cases}$. What is the correct statement among the following: (a) they meet at a point; (b) they are parallel; (c) they don't meet but they are not parallel (in which case they are called *skew lines*).

Ex. 6.13. Solve the following systems (non necessarily with row elimination!):

(a) $\begin{cases} 2x_1 & - & x_2 & + & x_3 & = & 1 \\ 3x_1 & + & x_2 & - & x_3 & = & 3 \\ x_1 & + & 2x_2 & - & x_3 & = & -2 \end{cases}$

(b) $\begin{cases} 4x + y + z + 2v + 3w = 0 \\ 14x + 2y + 2z + 7v + 11w = 0 \\ 15x + 3y + 3z + 6v + 10w = 0 \end{cases}$

(c) $\begin{cases} 5x + 4y + 7z = 3 \\ x + 2y + 3z = 1 \\ x - y - z = 0 \\ 3x + 3y + 5z = 2 \end{cases}$ (d) $\begin{cases} 19x - y + 5z + t & = 3 \\ 18x + 5z + t & = 1 \\ 6x + 9y + t & = 1 \\ 12x + 18y + 3t & = 3 \end{cases}$

Ex. 6.14. Let us consider the homogeneous linear system $\begin{cases} x_1 + x_2 + x_4 = 0 \\ x_1 + 2x_3 + x_4 = 0 \\ x_2 - 2x_3 = 0 \end{cases}$.
(a) Find the dimension and a basis of the space of solutions;
(b) Find the dimension and a basis of the linear span of the columns of the linear system.
(c) Find the dimension and a basis of the linear span of the rows of the linear system.

Ex. 6.15. For which values of $t \in \mathbb{R}$ the system

$$\begin{cases} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ x_1 & + & (t+4)x_2 & - & 3x_3 & = & 1/2 \\ -2x_1 & + & (t-2)x_2 & + & (2t-6)x_3 & = & 5/2 \end{cases}$$

has respectively no solutions, a unique solution, infinitely many solutions?

Ex. 6.16. Find for which values of $t, a \in \ R$, the system

$$\begin{cases} x_1 + x_2 + tx_3 & = 1 \\ 2x_1 + tx_2 + x_3 & = -1 \\ 6x_1 + 7x_2 + 3x_3 & = a \end{cases}$$

has respectively no solution, a unique solution, infinitely many solutions. For this last case, describe the set of solutions.

Ex. 6.17. Do linear transformations with the below properties exist? If the answer is yes exhibit an example.
(a) $T : V_2 \to V_4$ such that $T(V_2) = L((1,0,1,0),(0,1,0,1),(1,0,0,0))$;
(b) $S : V_4 \to V_3$ suriettiva surjective and such that $N(S) = L((1,2,-1,1))$.

Ex. 6.18. For$t$ varying in $\mathbb{R}$, let $A_t = \begin{pmatrix} t & t+1 & t+3 \\ -1 & 0 & 2 \\ 2 & 0 & t+1 \end{pmatrix}$.
(a) For t varying in $\mathbb{R}$, compute $\dim(N(T_{A_t}))$ e $\dim(T_{A_t}(V_3))$.
(b) Exhibit a basis $\mathcal{B}$ of $T_{A_{-1}}(V_3)$ and find a basis of V^3 containing $\mathcal{B}$.

Ex. 6.19. (a) Let $T : V_n \to V_m$ be a linear transformation. Prove that, via T, the image of a parallelogram is either a parallelogram, or a segment or a point. For each case exhibit an example
(b) Describe the image of the unit square of V_2 (that is $[0,1] \times [0,1]$) via the linear transformations T_A, where:
(i) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, (ii) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, (iii) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, (iv) $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Ex. 6.20. Find three vectors $v_1, v_2, v_3 \in V_3$ such that $\begin{cases} v_2 + v_3 = (1,0,0) \\ v_1 + 2v_3 = (0,1,0) \\ v_1 + 2v_2 = (0,0,1) \end{cases}$

Is the solution unique?

Ex. 6.21. Let $A = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix}$. (a) Compute A^{-1}.
(b) Let $C = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$. Find all matrices B such that $AB = C$.

Ex. 6.22. Let $C = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & -3 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.
(a) Compute C^{-1}.
(b) Compute $(C^2)^{-1}$ (without computing C^2).

Ex. 6.23. Let $A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$.
(a) Compute the inverse of A.
(b) For $t \in \mathbb{R}$, let $B_t = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 2 & t & 0 \end{pmatrix}$. Find the values of t such hat there is a matrix $X \in \mathcal{M}_{3,3}$ such that $B_t X = A$.

Ex. 6.24. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
(a) Compute the inverses of A and B.
(b) Without computing AB and BA, compute the inverses of AB and BA.

Ex. 6.25. Let us consider the vectors $A_1 = (0, 0, 1, 1)$, $A_2 = (1, -2, 0, 1)$, $A_3 = (0, 1, 2, 1)$, $A_4 = (0, 0, 1, 0)$.
(a) Prove that $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$ is a basis of V_4.
(b) For all $(x, y, z, t) \in V_4$ find (in function of x, y, z, t) the components of (x, y, z, t) with respect to the basis $\mathcal{B}$.

Ex. 6.26. For $t \in \mathbb{R}$, let $A_t = \begin{pmatrix} t & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 2+t & 0 \\ t & 1 & 0 & t+3 \end{pmatrix}$.
(a) Find for which t the matrix A_t is invertible.
(b) Find for which t the system of linear equations of 4 equations in 3 unknowns having A_t as *augmented* matrix has solutions. For such t's, find explicitly the solutions.

Ex. 6.27. For the matrix A_t of the previous exercise find the values of t such that there exists a vector $B_t \in V_4$ such that the system $A_t X = B_t$ has no solutions.

Ex. 6.28. Let $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 2 & -1 & 3 \end{pmatrix}$.
(a) Find a matrix D such that $AD = B$.
(b) Find a matrix E such that $EA = B$ and a matrix F such that $FA = C$.

Ex. 6.29. Let $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix}$.
(a) Find A^{-1}. (b) Is there a matrix X such that $AX = B$? Is it unique? (c) Is there a matrix Y such that $YA^{-1} = B$? Is it unique? (d) Is there a matrix Z such that $BZ = A$? Is it unique? (e) Is there a matrix T such that $BT = C$? Is it unique?