EXERCISES ON DETERMINANTS, EIGENVALUES AND EIGENVECTORS

1. Determinants

Ex. 1.1. *Let* $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *(a) Compute* A^{-1}.
(b) Compute $\det(2A^2B)$, $\det(4A+B)$, $\det(2(A^3B^{-2}))$.

Ex. 1.2. *For* t *varying in* $\mathbb{R}$ *let us consider the matrix* $M_t = \begin{pmatrix} 1 & 0 & -1 & t \\ t & t+2 & 1 & t+2 \\ t+1 & 0 & 1 & 0 \\ 1 & 0 & 2 & t \end{pmatrix}$.
(a) Compute $\det M_t$ *as a function of* t *and find the values of* t *such that* $rk(M_t) = 4$.
(b) For t *varying in* $\mathbb{R}$, *let* L_t *and* R_t *be the lines of cartesian equations*

$$L_t : \begin{cases} x_1 - x_3 = t \\ tx_1 + (t+2)x_2 + x_3 = t+2 \end{cases} \qquad \text{and} \qquad R_t : \begin{cases} (t+1)x_1 + x_3 = 0 \\ x_1 + 2x_3 = t \end{cases} .$$

Find the values of t *such that the lines* L_t *and* R_t *meet at a point (hint: use (a)).*

Ex. 1.3. *For* t *varying in* $\mathbb{R}$ *let* $A_t = \begin{pmatrix} t & t & t & -t \\ 1 & 0 & t^2 - t & -1 \\ 1 & 0 & 0 & 1 \\ t^3 & t & t & -t \end{pmatrix}$.
(a) Compute $\det(A_t)$ *and find the values of* t *such that* $rk(A_t) = 4$.
(b) Find the values of t *such that the system of linear equations having* A_t *as* augmented *matrix has solutions (hint: use (a)) For such values of* t *solve the system.*

Ex. 1.4. *Let* $A \in \mathcal{M}_{4,4}$ *be a matrix whose rows are* $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$ *and suppose that* $\det A = -3$. *Let* $C \in \mathcal{M}_{4,4}$ *l be the matrix whose rows are* $\begin{pmatrix} 2A_1 \\ 3A_1 + 4A_2 \\ -2A_1 + 2A_2 + A_4 \\ A_1 + A_2 + 3A_3 \end{pmatrix}$. *Compute* $\det(C)$ *and* $\det(\frac{1}{2}C)$.

2. Eigenvalues and eigenvectors

Ex. 2.1. *Let* $A = \begin{pmatrix} 2 & 0 & 0 & 4 \\ 0 & 3 & 0 & 5 \\ 0 & 5 & -2 & 5 \\ 4 & 0 & 0 & 2 \end{pmatrix}$.
(a) Find the eigenvalues of A.
(b) Let $T_A : V_4 \to V_4$ *be the linear transformation defined by* $T_A(X) = A\,X$. *Is there a basis of* V_4 *composed by eigenvectors of* T_A*? If the answer is yes, produce explicitly such a basis.*
(c) Is there a matrix C *such that* $C^{-1}AC$ *is a diagonal matrix? If the answer is yes produce explicitlt such a matrix.*

Ex. 2.2. *Let* $\mathcal{B} = \{(0,1,-1),\ (0,1,1), (2,3,0)\}$.
(a) Prove that $\mathcal{B}$ *is a basis of* V_3.
Now Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ *and let* $T : V_3 \to V_3$ *be the linear transformation such that* $m^{\mathcal{B}}_{\mathcal{B}}(T) = A$.
(b) Find the eigenvalues of T.
(c) Find a basis of each eigenspace of T.
(d) Is there a basis of V_3 *composed of eigenvectors of* T*? If the answer is yes , produce explicitly such a basis.*
(e) Is there a matrix C *such that* $C^{-1}AC$ *is a diagonal matrix? If the answer is yes, produce explicitly such a matrix.*

Ex. 2.3. *Let* $A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -1 & -2 & 0 \end{pmatrix}$.
(a) Find the eigenvalues of A. *(b(Find a basis of each eigenspace of the transformation* $T_A : V_3 \to V_3$, *where, as susal,* $T_A(X) = AX$. *(c) Let* $v = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. *Compute* $A^{151}\,v$.

Ex. 2.4. *Let* $T : V_3 \to V_3$ *the linear transformation such that:* $T((1,1,0)) = (2,2,0)$, $T((0,-1,1)) = (0,-3,3)$ *e* $T((1,0,-1)) = (0,1,-1)$.
(a) Compute $rk(T)$ *and a basis of* $Im(V_3)$.
(b) A basis of $N(T)$. *(c) Eigenvalues and eigenvectors of* T. *(d)* $T((1,0,0))$.

Ex. 2.5. *Let* $T : V_4 \to V_4$ *be a linear transformation such that:*
(a) $T((1,0,-1,2)) = (2,0,-2,4)$, $T((0,1,1,1) = (0,-5,-5,-5)$, $T((1,2,-1,0) = (2,4,-2,0)$; *(b)* $(2,2,-2,2)$ *and* $(1,0,0,0)$ *are eigenvectors of* T; *(c)* $\det T = 100$.
(i) Find the eigenvalues of T *and a basis of each eigenspace of* T.
(ii) Is there a basis of V_4 *composed of eigenvectors of* T*? If the answer is yes, produce explicitly such a basis. (iv) Compute* $Tr(T)$.

Ex. 2.6. *(a) For t varying in* $\mathbb{R}$*, let* $A_t = \begin{pmatrix} 1 & t+2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$*. Let* $T_{A_t} : V_3 \to V_3$ *be the linear transformation defined, as usual,* $T)A_t(X) = A_t X$*. Find the values of* t *such that* T_{A_t} *is diagonalizable (that is, there is a basis of* $V_3(\mathbb{R})$ *composed by eigenvectors of* T_{A_t}*.*
(b) Same question for $B_t = \begin{pmatrix} 1 & t+2 \\ 2 & 3 \end{pmatrix}$*.*

Ex. 2.7. *Let* $V = L((1,0,1,1),(0,0,1,-1))$*. Let* $R_V : V^4 \to V^4$ *be the reflection with respect to* V *and let* $P_{V^\perp} : V^4 \to V^4$ *be the projection of* V^4 *on* $V^\perp$*. (a) Find the range and the eigenvalues of the linear transformation* $P_{V^\perp} \circ R_V$*. (b) Find a base of each eigenspace of* $P_{V^\perp} \circ R_V$*. Is* $P_{V^\perp} \circ R_V$ *diagonalizable?. (c) Is* $P_{V^\perp} \circ R_V$ *a* symmetric *linear transformation?*

Ex. 2.8. *For all the matrices* A *of the previous exercises, take the* diagonalizable *ones and compute their* $132th$ *power and their* $(-132)th$ *power.*

Ex. 2.9. *Let* $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 3 & -2 & 1 & 1 \\ -3 & -1 & -4 & -1 \\ -6 & -2 & -2 & -5 \end{pmatrix}$ *and let* $v = \begin{pmatrix} -2 \\ 1 \\ 5 \\ 0 \end{pmatrix}$*. (a) Verify that* -3 *is an eigenvalue of* A*.*
(b) Find all eigenvalues of A *(hint: use the previous point). Is* A *diagonalizable?*
(c) Let $C \in \mathcal{M}_4$ *be an invertible matrix. Find the eigenvalues of the matrix* $C^{34}\, A\, ((C^{34})^{-1})$*.*

Ex. 2.10. *Let* a, b, c, d, e *be real numbers, and let us consider the matrix* $A_{a,b,c,d,e} = \begin{pmatrix} a+e & b & c & d \\ a & b+e & c & d \\ a & b & c+e & d \\ a & b & c & d+e \end{pmatrix}$*. Find all eigenvalues of* $A_{a,b,c,d,e}$ *(in function of* a, b, c, d, e*).*

Ex. 2.11. *Let* $T : V_3 \to V_3$ *be a linear transformation with a double eigenvalue and a simple eigenvalue. Show that there is a basis* $\mathcal{B}$ *of* V_3 *such that the representative matrix* $m^{\mathcal{B}}_{\mathcal{B}}$ *is upper triangular.*

Ex. 2.12. *Let* $v = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ *and* $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *Find (if any) all matrices* $A \in \mathcal{M}_{3,3}(\mathbb{R})$ *such that:* $Tr(M) = 3$*,* $M\,v = 2v$*,* $M\,w = -w$ *and* A *has an eigenvector orthogonal both to* v *and* w*.*

Ex. 2.13. *Let* $T : V^3 \to V^3$ *be the linear transformation such that:* $T((1,1,0)) = (2,2,0)$*,* $T((0,-1,1)) = (0,-3,3)$ *e* $T((1,0,-1)) = (0,1,-1)$*.*
(a) Compute $rk(T)$ *and find a basis of* $T(V_3)$*. (b) Find a basis of* $N(T)$*.*
(c) Find the eigenvalues and eigenvectors of T*. (d) Compute* $T((1,0,0))$*.*

3. (Skew-)Hermitian and (skew-)symmetric transformations

Ex. 3.1. *(a) Let* $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *. Find (if any) all symmetric lmatrices* $A \in \mathcal{M}_{2,2}(\mathbb{R})$ *such that* -3 *is an eigenvalue of* A *and* $Av = 2v$.
(b) Let $w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. *Is there a symmetric matrix* $B \in \mathcal{M}_{2,2}(\mathbb{R})$ *such that* $Av = 2v$ *and* $Aw = 4w$*?*
(c) Is there a hermitian matrix $E \in \mathcal{M}_{2,2}(\mathbb{C})$ *such that* $Av = 2v$ *and* $Aw = 4w$*?*
(d) Is there a skew-symmetric matrix $C \in \mathcal{M}_{2,2}(\mathbb{R})$ *such that* $Cv = 2v$*?*
(e) is there a skew-hermitian matrix $D \in \mathcal{M}_{2,2}(\mathbb{C})$ *such that* $Dv = 2v$*? (d) Is there a skew-hermitian matrix* $F \in \mathcal{M}_{2,2}(\mathbb{C})$ *such that* $Fv = i\,v$*?*

Ex. 3.2. *Let* $v = \begin{pmatrix} 1+i \\ i \end{pmatrix}$.
(a) Find (if any) all hermitian matrices $A \in \mathcal{M}_{2,2}(\mathbb{C})$ *such that* $Av = 2v$.
(b) Find (if any) all skew-hermitian matrices $B \in \mathcal{M}_{2,2}(\mathbb{C})$ *such that* $Bv = 2v$.
(c) Find (if any) all hermitian matrices $C \in \mathcal{M}_{2,2}(\mathbb{C})$ *such that* $Cv = iv$.
(d) Find (if any) all skew-hermitian matrices $D \in \mathcal{M}_{2,2}(\mathbb{C})$ *such that* $Av = iv$.

Ex. 3.3. *(a) Is the set of all symmetric* $n \times n$ *real matrices a linear subspace of* $\mathcal{M}_{n,n}(\mathbb{R})$*? Let* V *be a real linear inner product space. Is the set of all symmetric linear transformations from* V *to* V *a linear subspace of* $\mathcal{L}(V,V)$*?*

Ex. 3.4. *Let* $A \in \mathcal{M}_{3,3}(\mathbb{R})$ *be a skew-symmetric matrix, such that* i *is a root of the characteristic polynomial* $P_A(\lambda)$.
(a) Find the other roots of $P_A(\lambda)$.
(b) Compute $rk(A)$, $Tr(A)$, $\det(A)$.

Ex. 3.5. *Let* $\mathcal{B} = \{A^1, A^2, A^3, A^4\}$ *be an orthogonal basis of* V_4 *such that* $\| A^1 \| = 3$, $\| A^2 \| = 5$, $\| A^3 \| = 2$, $\| A^4 \| = 2$. *Let* $A = (A^1\ A^2\ A^3\ A^4)$ *be the matrix whose columns are* A^1, A^2, A^3, A^4. *Compute* $|\det A|$.

Ex. 3.6. *Let* $\mathcal{B} = \{(1,2),(1,3)\}$. *Sia* $T : V_2 \to V_2$ *be the linear transformation such that* $m_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 9 & 13 \\ -6 & -9 \end{pmatrix}$. *Moreover, let* $S : \mathcal{V}_2 \to \mathcal{V}_2$ *be the linear transformation such that* $m_{\mathcal{B}}^{\mathcal{B}}(S) = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$. *Is* T *a symmetric linear transformation?. Is* S *a symmetric linear transformation?.*

Ex. 3.7. *Let* $v = (1,2,1)$. *Let us consider the function* $T : V_3 \to V_3$ *defined as follows:* $T(X) = X \times v$.
(a) Prove that T *is a linear transformation . (b) Is* T *symmetric?*
(b) Compute the matrix representing T *with respect to the canonical basis of* V_3 *and compare the result with the answer given to (a)*

(c) Find an orthonormal basis of V_3 $\mathcal{B} = \{v_1, v_2, v_3\}$ with v_1 parallel to v and compute $m^{\mathcal{B}}_{\mathcal{B}}(T)$.

4. Real quadratic forms. Conics

Ex. 4.1. *Let* $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. *Let us consider the real quadratic form* $Q(X) = XAX^T$.
(a) Reduce Q to diagonal form by means of an orthonormal basis (that is: find an orthonormal basis $\mathcal{B}$ of V_4 and real numbers λ_i, $i = 1, 2, 3, 4$ such that $Q(X) = \sum_{i=1}^4 \lambda_i {x'}^2$, where x'_i are the components of X with respect to the basis $\mathcal{B}$.
(b) Is Q a positive real quadratic form (this mean that $Q(X) \geq 0$ for all $X \in V_4$ and equality holds if and only if $X = O$).
(c) Compute the maximum and minimum value Q on the unite sphere S^3. Compute the points of maximum and the points of minimum.

Ex. 4.2. *For a, b, c varying in $\mathbb{R}$, let us consider the function $F_{a,b,c}$: $[0, 2\pi] \to \mathbf{R}$, $F_{a,b,c}(\theta) = a\cos^2(\theta) + b\cos(\theta)\sin(\theta) + c\sin^2(\theta)$. Assume that $\max F_{a,b,c} = 5$, $\min F_{a,b,c} = -2$ and that $\theta = \pi/3$ is a point of maximum. Compute a, b and c. (Hint: think F as $Q : S^1 \to \mathbb{R}$, where S^1 is the unit circle and Q is a quadratic form).*

Ex. 4.3. *Reduce the following real quadratic forms Q to diagonal form. Establish if they are positive, negative or what. Find explicitly (if any) non-zero vectors X such that $Q(X) > 0$ or $Q(X) < 0$ or $Q(X) = 0$. Find the maximum and minimum value of Q on the unit sphere and find the points of maximum and the points of minimum.*
(a) $Q(x, y) = x^2 - 3xy + y^2$.
(b) $Q(x, y, z) = -2x^2 - 5y^2 + 12yz + 7z^2$.
(c) $Q(x, y, z) = 4xy + 3y^2 + z^2$.
(d) $Q(x, y, z) = 25x^2 - 7y^2 + 48yz + 7z^2$.
(e) $Q(x, y, z) = \sqrt{2}(x^2 + y^2 + z^2) + 2x(y - z)$
(f) $Q(x, y, z) = 5x^2 - y^2 + z^2 + 4xy + 6xz$
(g) $Q(x, y, z) = 2(xy + yz + zx)$

Ex. 4.4. *Let* $Q(x_1, x_2, x_3, x_4, x_5) = \sum_{1=1}^5 x_i^2 + \sum_{i<j} x_i x_j$. *Is Q positive?*

Ex. 4.5. *Reduce the equations of teh following conics to canonical form, find the symmetry axes, the center, the foci.. (everything in coordinates with respect to the canonical basis).*
(a) $5x^2 + 5y^2 - 6xy + 16\sqrt{2}x + 38 = 0$.
(b) $5x^2 - 8xy + 5y^2 + 18x - 18y + 9 = 0$
(c) $2x^2 - 2\sqrt{3}xy + 2x + 2\sqrt{3}y - 5 = 0$

(d) $x^2 + 2xy + y^2 + 2x - 2y = 0$
(e) $5x^2 + 8xy + 5y^2 - 18x - 18y + 9 = 0$