## LINEAR ALGEBRA AND GEOMETRY SUPPLEMENTARY EXERCISES ABOUT CHAPTERS 12 AND 13

**1**. Let A = (1, 1, -1), B = (1, 2, 0), C = (1, 1, 1). (a) Do A, B, C form a basis of  $\mathcal{V}_3$ ? (b) If the answer to (a) is in the affirmative, compute the coordinates of the vector (0, 1, 1) with rescret to the basis  $\{A, B, C\}$ .

**2.** Let A = (1, 1, -1), B = (1, 2, 0), C = (-2, -5, -1). (a) Is  $\{A, B, C\}$  a basis of  $\mathcal{V}_3$ ? (b) If the answer to (a) is in the negative, express a A, B or C as a linear combination of the remaining two vectors.

**3.** Let  $\{A, B, C\}$  be an orthogonal basis of  $\mathcal{V}_3$ . Moreover let V = A - 2B + C and W = -A + B + C. If ||V|| = 12, ||W|| = 6 and the cosine of the angle between V and W is  $-3\sqrt{3}/2$ , compute ||A||, ||B|| and ||C||.

4. Let  $\{A, B, C\}$  be a orthogonal basis such hat ||A|| = ||B|| = 2 and ||C|| = 1. Moreover let V = A + B + C and W = -A + B - C. Find the cosine of the angle between V and W.

**5**. Let P = (2, 3, -4), A = (1/2, 1, 1), B = (2, 2, 1). Find two points, Q and R such that the following three condition hold: (1) Q - P has the same direction of A and R - P has the same direction of B (We recall that two non-zero vectors A and B are said to have the same direction if B = cA with c > 0); (2) ||Q - P|| = ||R - P||; (3) the area of the triangle whose vertices are P, Q, R is 9/2.

6. (a) Let us consider the lines of  $\mathcal{V}_2$ : L: 2x - y + 1 = 0 and  $R = \{(3,1) + t(4,3)\}$ . Describe and find cartesian equations for the locus of points  $X \in \mathcal{V}_2$  such that d(X,L) = d(X,R).

(b) Let  $L = \{(1,2) + t(1,3)\}$  and  $R = \{(-2,5) + t(2,6)\}$ . Describe and find cartesian equations for the locus of points  $X \in \mathcal{V}_{\in}$  such that d(X, L) = d(X, R).

(c) Let L and R be two non-parallel lines in  $\mathcal{V}_2$ . What is the locus of points  $X \in \mathcal{V}_2$  such that d(X, L) = d(X, R)? What is the locus of points  $X \in \mathcal{V}_2$  such that d(X, L) = d(X, R), if L and R are parallel?

(d) For L and R as in (a), describe and find equations for the locus of points  $X \in \mathcal{V}_2$  such that d(X, L) = 2d(X, R).

7. (a) Find the intersection of the lines  $\{(1,1) + t(1,2)\}$  and  $\{(1,1) + t(2,1)\}$ . (b) Same question with the two lines  $\{(2,-1,2) + t(1,1,-1)\}$  and  $\{(2,0,0) + t(1,0,1)\}$ .

8. Let us consider the lines of  $\mathcal{V}_2$ : L = (1, -1) + t(1, 1) and R : x + 2y = 1. How many are the points  $X \in \mathcal{V}_{\in}$  such that  $\begin{cases} d(X, L) = \sqrt{2} \\ d(X, R) = \sqrt{3} \end{cases}$  Find one of them.

**9.** Let A = (1, -2) and P = (1, -1). Find both a parametric and a cartesian equation of the line passing through P and whose angle with A is  $\theta = \pi/3$ .

11. Find the line containing (1, -1, 1), parallel to the plane M : x - 3y = 2, and perpendicular to A = (2, 1, 1).

12. Let  $L = \{(1, -1, 1) + t(1, 1, 1)\}$  and Q = (1, 1, -1). Compute d(Q, L) and the point of L which is nearest to Q.

**13.** Let  $L = \{(1,1) + t(1,-2)\}$  and let Q = (1,0). (a) Compute d(Q,L) and find the point  $\overline{P}$  of L which is nearest to Q. (b) Find all points  $S \in L$  such that the triangle whose vertices are  $Q, \overline{P}$  and S is isosceles (this means that two edges have same length). Compute the area of these triangles.

14. Let A = (1, -2, 2) and B = (1, 2, 0). Find a vector  $C \in L(A, B)$  such that the angle between A and C is  $\pi/4$  and the area of the parallelogram determined by A and C is 1.

**15.** Let A = (1, 1, 1) and B = (1, -1, 1). (a) Find a orthonormal basis C, D, E of  $\mathcal{V}_3$  such that C is parallel to A and D belongs to L(A, B). (b) Find the vector  $V \in \mathcal{V}_3$  such that  $C \cdot V = \sqrt{3}$ ,  $D \cdot V = \sqrt{2}$ ,  $E \cdot V = \sqrt{2}$ .

16. Let L be the line  $\{(1,0,1) + t(1,0,2)\}$ , and let M be the plane  $\{(0,1,0) + t(1,1,-1) + s(1,0,1)\}$ . Moreover let Q = (1, 0, 0).

(a) Find all points X belonging to L such that d(X, M) = 1.

(b) Find all points X belonging to L such that d(X,Q) = 1.

17. Let L be the intersection of the two planes  $M_1: x - y + z = 1$  and  $M_2: y - 2z = -1$ . Let moreover  $\Pi$ be the plane  $\{(1,2,1) + t(1,-1,-) + s(-1,0,2)\}$ . Find (if any) a plane containing L and perpendicular to  $\Pi$ . Is such plane unique?

18. Let A = (1, 2, -2) and P = (1, -5, -2). Find two planes orthogonal to A such that their distance from Q is 6.

**19.** Let  $M = \{(3, 1, 1) + t(-2, 1, 0) + s(2, 1, 2)\}$ . Find the plane parallel to M such that their distance from (1,1,1) is 2. For each of such planes, find the nearest point to (1,1,1).

**20.** Let  $M_1: x - 3y + 2z = 0$  and  $M_2: 2x - 3y + z = 3$ . Let moreover A = (1, 0, -1).

(a) Find the plane M containing the line  $L = M_1 \cap M_2$  and parallel to A.

(b) Let Q = (0, -1, 1). Find the line R which is perpendicular to M and contains Q.

(c) Find  $R \cap M$ .

**21.** let P = (0, -1, 0), Q = (1, 0, -1), R = (1, -1, 1).

(a) Find the area of the triangle whose vertices are P, Q and R.

(b) Let S = (1, -1, -2). Compute the volume of the parallepiped determined by the vectors Q - P, R - Pand S - P.

(c) Find the points X, collinear with P and S such that the volume of the parallepiped determined by the vectors Q - P, R - P and X - P is 2.

**22.** For t varying in **R**, let  $A_t = (t+1, 1, 1)$ ,  $B_t = (1, t+2, 1)$ ,  $C_t = (1, 1, t+2)$ .

(a) Find all  $t \in \mathbf{R}$  such that  $A_t, B_t, C_t$  form an independent set. For such t compute (as a function of t) the volume of the parallelepiped determined by  $A_t, B_t, C_t$ . In particular, compute such volume for t = -2.

(b) Compute, for each  $h \in \mathbf{R}$ , the solutions of the system (depending on h):  $\begin{cases} hx + y + z = 0\\ x + hy + z = 0\\ x + y + hz = 0 \end{cases}$ 

**23.** For t varying in **R**, let A(t) = (t+1, t+2, 2t),  $B(t) = (1, t^2+1, 1)$ , C(t) = (1, 2, 1). (a) Find all t such that A(t), B(t) and C(t) do not form a basis of  $\mathcal{V}_3$ . (b) For all  $t, s \in$  let us consider the system

$$xA(t) + yB(t) + zC(t) = (s, 2s, 1).$$

For which values of t and s there is no solution? For which values of t and s there are more than one solution?

24. For t, a varying in **R**, let us consider the system  $\begin{cases}
x + y + z = 1 \\
2x + ty + z = -1.
\end{cases}$ Find for which values of t and a 6x + 7y + 3z = a

there is a unique solution, no solution, infinitely many solutions.

**25.** Let A = (3, 4). Describe and find parametric equations of the locus of all vectors  $X \in \mathcal{V}_2$  such that the angle between A and X is  $\pi/4$ . (Suggestions: use a orthonormal basis of  $\mathcal{V}_2$  such that one of its vectors is parallel to A.)

**26.** Let A = (2, -1, 2) and B = (1, 0, -1) (note that  $A \cdot B = 0$ ). Describe and find parametric equations of the locus of vectors  $X \in \mathcal{V}_3$  which are orthogonal to B and such that the angle between A and X is  $\pi/4$ . (Suggestion: use a suitable orthonormal basis of  $\mathcal{V}_3$ ).

**27.** Let C be the parabola of focus F = (0,0) and directrix L : x + 2y = 1. Find the coordinates of the vertex of  $\mathcal{C}$ .

**28.** Let  $\mathcal{C}$  be a conic such that  $(3,4) \in \mathcal{C}$ , having a focus at the origin and such that the corresponding directrix is x - y = 5. (a) Find the eccentricity, the vertices, the center, the other focus and the other directrix. (b) Find the polar equation of  $\mathcal{C}$ .