ADDENDUM TO SECTIONS 15.15 AND 15.16.

1. Properties of the orthogonal complement

Let S be a subset (finite or infinite) of a euclidean linear space V. According to the book, we can define $S^\perp$ as the set of elements in V perpendicular to each element of S. In symbols $S^\perp = \{x \in V \mid \ < s, x >= 0 \ \forall s \in S\}$. Then:

Proposition 1.1. $S^\perp = L(S)^\perp$

Proof. Since $S \subseteq L(S)$, clearly $L(S)^\perp \subseteq S^\perp$. To prove the other inclusion, we have to prove that if $x \in S^\perp$ then $x \in L(S)^\perp$, that is that $< \sum c_i s_i, x >= 0$ for all *finite* linear combinations $\sum c_i s_i$ of elements of S. But $< \sum c_i s_i, x >= \sum c_i < s_i, x >= 0$ □

Example 1.2. *In $V = V_n$ with the dot product, let $S = \{w_1, \ldots, w_k\} \subset V_n$. The previous proposition says that*

$$L(w_1, \ldots, w_k) = \{X \in V_n \mid \begin{cases} w_1 \cdot X = 0 \\ \ldots \\ w_r \cdot X = 0 \end{cases} \}$$

Note that the above is a (homogeneous) system of linear equations since if $w_i = (a_{i1}, \ldots, a_{in})$ then $w_i \cdot X = a_{i1}x_1 + \cdots + a_{in}x_n$.
For example, $L((1,2,1,-1),(3,1,5,-1))^\perp$ is the subspace of V_4 formed by the solutions of the system

$$\begin{cases} x_1 + 2x_2 + x_3 - x_4 = 0 \\ 3x_1 + x_2 + 5x_3 - x_4 = 0 \end{cases}$$

The next proposition tells us that, in the finite-dimensional case, the dimension of the orthogonal complement is what it should be (for example, we already now that the orthogobal complement of a line in V_2 is a line and that the orthogonal compement of a plane in V_3 is a line).

Proposition 1.3. *Let V be a finite-dimensional euclidean space, and let W be a linear suspace of V. Then:*
(a) $\dim W^\perp = \dim V - \dim W$
(b) $(W^\perp)^\perp = W$.

Proof. (a) Let us denote k and h the dimensions of W and $W^\perp$ respectively. Let $E = \{e_1, \ldots, e_k\}$ and $F = \{f_1, \ldots, f_h\}$ be orthogonal bases of W and $W^\perp$ respectively (they exist by virtue of Theorem 15.14 of the book). Then $E \cup F$ is an orthogonal set of $k + h$ non-zero vectors, and therefore it is

an independent set (Theorem 15.10). By the Orthogonal Decomposition Theorem (Theorem 15.15) $E \cup F$ spans V: indeed Th. 15.15 tells us that every element of V is the sum of an element of W (a linear combination of the elements of E) and of an element of $W^\perp$ (a linear combination of the elements of F). Hence $E \cup F$ is a basis of V. Therefore $k + h = \dim V$.
(b) Clearly

$$W \subseteq (W^\perp)^\perp \tag{1}$$

But, by (a),

$$\dim(W^\perp)^\perp = \dim V - \dim W^\perp = \dim V - (\dim V - \dim W) = \dim W.$$

Therefore the inclusion in (1) is an equality. □

Remark 1.4. Note that part (b) tells us something we already knew in particular cases: in V_2 a vector orthogonal to a vector which is perpendicular to a vector v is a vector parallel to v. In V^3 we know that, given two (non-parallel) vectors v and w, $L(v, w) = (v \times w)^\perp$ (Theorem 13.15 with $P = O$). Since
$L(v \times w) = L(v, w)^\perp$ (Theorem 13.13(b)), this means precisely that $(L(v.w)^\perp)^\perp = L(v, w)$.

Example 1.5. *Continuing with Example 1.2, part (a) of the above proposition tells us the following: let us consider a linear system*

$$\begin{cases} w_1 \cdot X = 0 \\ ... \\ w_k \cdot X = 0 \end{cases} \Leftrightarrow \begin{cases} a_{11}x_1 + \cdots a_{1n}x_n = 0 \\ ... \\ a_{k1}x_1 + \cdots + a_{kn}x_n = 0 \end{cases} \tag{2}$$

Then the dimension of the space of the solutions, that is $W^\perp$, equals $n - \dim W$, where, as we know from Theorem 15.7(b), $\dim W$ is the maximal number of independent vectors among $\{w_1, \ldots, w_k\}$. In other words, the dimension of the space of solutions is the number of unknowns minus the number of independent equations.

Example 1.6. *In the setting of Examples 1.2 and 1.5, (b) of the previous proposition tells the following: let $W = L(w_1, \ldots w_h)$ be a subspace of V_n. Solving the system (2) one finds the space of solutions $W^\perp = L(u_1, \ldots, u_h)$ (where $h = n - \dim W$). Then W is the set of solutions of the system*

$$\begin{cases} u_1 \cdot X = 0 \\ ... \\ u_h \cdot X = 0 \end{cases}$$

In other words given a basis or, more generally, a spanning set $\{w_1, \ldots, w_k\}$ of W, that is a parametric equation of W, we have found a linear system whose space of solutions is W (a system of "cartesian equations" for W).

For example, let $W = L((1,2,1,-1),(3,1,5,-1))$. *Then* $W^\perp$ *is the subspace of* V_4 *formed by the solutions of the system*

$$\begin{cases} x_1 + 2x_2 + x_3 - x_4 = 0 \\ 3x_1 + x_2 + 5x_3 - x_4 = 0 \end{cases}$$

We solve the system by multiplying the first equation by 3 and subtracting it from the second one. We arrive to the equivalent system

$$\begin{cases} x_1 + 2x_2 + x_3 - x_4 = 0 \\ -5x_2 + 3x_3 + x_4 = 0 \end{cases}$$

Then $x_4 = 5x_2 - 3x_3$ *and* $x_1 = -2x_2 - x_3 + x_4 = 3x_2 - 4x_3$. *Therefore the space of solutions* $W^\perp$ *is formed by the 4-tuples*

$$(3x_2 - 4x_3, x_2, x_3, 5x_2 - 3x_3) = x_2(3,1,0,5) + x_3(-4,0,1,-3).$$

Therefore $W^\perp = L((3,1,0,5),(-4,0,1,-3))$. *By the above* W *is the space of solutions of the system*

$$\begin{cases} 3x_1 + x_2 + 5x_4 = 0 \\ -4x_1 + x_3 - 3x_4 = 0 \end{cases}$$

(check!).

Finally, note that all this works also in the complex case, as well.

2. Exercises

Ex. 2.1. *In* $V_4(\mathbb{R})$, *let* $W = L((1,2,1,-1),(1,1,-1,-1),(-1,1,5,1)$. *Find dimension and a basis of* W. *Find dimension and a basis of* $W^\perp$.

Ex. 2.2. *Find a basis of* $V_4(\mathbb{R})$ *containing the vectors* $(1,1,-1,1)$ *and* $(1,2,0,1)$.

Ex. 2.3. *In* $V_4(\mathbb{R})$, *with usual dot product. (a) Find a orthonormal basis* $\{e_1,e_2,e_3,e_4\}$ *of* $V_4(\mathbb{R})$ *such that* e_1 *is parallel to* $(0,3,4,0)$. *(b) Compute the distance between* $(1,0,0,0)$ *and* $L((0,3,4,0))^\perp$.

Ex. 2.4. *In* $V_4(\mathbb{R})$, *with the usual dot product. Let* $u_1 = (1,2,-2,3)$ *and* $u_2 = (2,3,-1,-1)$. *(a) Find a orthogonal basis* $\{v_1,v_2,v_3,v_4)$ *of* $V_4(\mathbb{R})$ *such that* $L(v_1,v_2) = L(u_1,u_2)$. *(b) Find an element* $u \in L(u_1,u_2)$ *and an element* $w \in L(u_1,u_2)^\perp$ *such that* $(1,0,0,0) = u + w$. *(c) Compute* $d((1,0,0,0),L(u_1,u_2))$ *and the point of* $L(u_1,u_2)$ *nearest to* $(1,0,0,0)$.

Ex. 2.5. *In* $V_n(\mathbb{R})$, *with the usual dot product. (a) In* V_3, *find a system of cartesian equations for the line* $L((1,2,-1)) = \{t(1,2,-1) \mid t \in \mathbb{R}\}$. *(b) In* $V_4(\mathbb{R})$ *find a system of cartesian equations for the two-dimensional subspace* $W = L((1,1,-1,0),(1,2,1,1)) = \{t(1,1,-1,0) + s(1,2,1,1) \mid t,s \in \mathbb{R}\}$.

Ex. 2.6. *In* $V_2(\mathbb{R})$ *define an inner product by*

$$< (x_1, x_2), (y_1, y_2) >= 3x_1y_1 - 2x_1y_2 - 2x_2y_1 + 3x_2y_2.$$

(a) Verifiy that this is in fact an inner product. (b) Find the orthogonal complement of $L((1,1))$ *with respect to this inner product. (c) Find the distance between* $(1,0)$ *and* $L((1,1))$ *with respect to this inner product. (d) Find an orthonormal basis of* $V_2(\mathbb{R})$ *with respect to this inner product.*

Ex. 2.7. *Are the vectors of* $V_2(\mathbb{C})$ $(i+1, 3-2i)$ *and* $(5, 5-5i)$ *parallel?*

Ex. 2.8. *Are the vectors of* $V_3(\mathbb{C})$ $A = (1+i, 2, 2-3i)$, $B = (2-i, 1-i, -1-i)$ *and* $C = (1, 1+i, 3)$ *dependent or independent?*

Ex. 2.9. *In* $V_n(\mathbb{C})$, *with the usual dot product* $(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = \sum_{i=1}^{n} x_i \bar{y}_i$.
(a) In $V_2(\mathbb{C})$, *find* $L((i+1, 2-5i))^{\perp}$.
In $V_3(\mathbb{C})$ *find* $L((1+i, i, 1+2i), (2+i, 2, 2-i))^{\perp}$.