First Name: Last Name:

Linear Algebra and Geometry, Midterm exam, 02.16.2011

NOTE: you must give reasons for every solution/assertions. Final solutions, without adequate explanations, will not be evaluated. In practice, for each exercise you must (briefly) explain the steps of the reasoning.

1. Let $\{A, B, C\}$ be an orthogonal basis of $\mathcal{V}_3$ such that $\parallel A \parallel = 2$, $\parallel B \parallel = \sqrt{2}$, $\parallel C \parallel = \sqrt{3}$. Let $V = A + B + C$ and let respectively θ_A, θ_B, θ_C be the angles between V and A, B, C. (a) Compute $\cos\theta_A$, $\cos\theta_B$ and $\cos\theta_C$. (b) Compute $\parallel A \times V \parallel$ and $\parallel A \times B \parallel$. (c) Compute $|V \cdot (A \times B)|$.

Solution. (a) $\parallel V \parallel^2 = V \cdot V$. Using the fact that the basis is ortogonal, $V \cdot V = A \cdot A + B \cdot B + C \cdot C = 4 + 2 + 3 = 9$. In conclusion: $\parallel V \parallel = 3$. Therefore

$$\cos\theta_A = \frac{V \cdot A}{\parallel V \parallel \parallel A \parallel} = \frac{\parallel A \parallel}{\parallel V \parallel} = \frac{2}{3}$$

In the same way: $\cos\Theta_B = \sqrt{2}/3$, $\cos\Theta_C = \sqrt{3}/3$.
(b) $\parallel A \times V \parallel$ is the area of the parallelogram defined by A and V. This is equal to
$\parallel A \parallel \parallel V \parallel \sin\theta_A = 6\sqrt{5/9} = 2\sqrt{5}$. Moreover, since A and B are orthogonal, $\parallel A \times B \parallel = \parallel A \parallel \parallel B \parallel = 2\sqrt{2}$.
(c) $|V \cdot (A \times B)|$ is the volume of the parallelepiped defined by V, A and B. This is equal to
$\parallel A \times B \parallel \parallel V \parallel \cos\theta_C = 2\sqrt{2}\sqrt{3}$.

2. Let A, B, C be three vectors in $\mathcal{V}_3$. Prove or disprove the following assertions:
(a) If A, B and C are linearly independent then the vectors $A+2B$, $A+B-C$, $A+B$ are linearly independent.
(b) The vectors $A + 2B$, $A + B - C$, $A + B$ can be linearly independent even if A, B and C are linearly dependent
(c) The vectors $A+2B$, $A+B-C$, $-A+2C$ are always linearly dependent, regardless of the linear dependence or independence of A, B, C.

Solution. (a) is correct. Let $x, y, z \in \mathbf{R}$ such that $O = x(A + 2B) + y(A + B - C) + z(A + B) = (x + y + z)A + (2x + y + z)B - yC$. Since A, B and C are independent, this means that $\begin{cases} x + y + z = 0 \\ 2x + y + z = 0 \\ y = 0 \end{cases}$.
It follows easily that $x = y = z = 0$.
(b) is false, because the fact that A, B and C are dependent means that they are contained in a plane trough the origin. Hence also all linear combinations of A, B and C are contained in that plane.
(c) is correct. We look for $x, y, z \in \mathbf{R}$ such that $O = x(A + 2B) + y(A + B - C) + z(-A + 2C) = (x + y - z)A + (2x + y)B + (-y + z)C$. One finds easily $x = -z$, $y = 2z$. For example: $x = -1$, $y = 2$, $z = 1$. This means that $A + 2B = 2(A + B - C) + (-A + 2C)$. This equality is always true.

3. In $\mathcal{V}_3$, let us consider the plane $M = \{(0, 0, 1) + t(1, 0, 1) + s(1, -1, 0)\}$, and the line $L = \{t(1, 1, 1)\}$. (a) Find all points in L such that their distance from M is equal to $\sqrt{3}$. (b) For each such point P find the cartesian equation of the plane parallel to M containing P.

Solution. (a) A normal vector to the plane is $(1, 0, 1) \times (1, -1, 0) = (1, 1, -1)$. A point of the line is of the form $t(1, 1, 1) = (t, t, t)$. Its distance form the plane is

$$\frac{|((t, t, t) - (0, 0, 1)) \cdot (1, 1, -1)|}{\parallel (1, 1, -1) \parallel} = \frac{|t + 1|}{\sqrt{3}}|$$

Hence $|t + 1|\sqrt{3} = \sqrt{3}$, that is $|t + 1| = 3$, which has the two solutions $t = 2$ and $t = -4$. The requested points are $(2, 2, 2)$ and $(-4, -4, -4)$.
(b) A cartesian equation of a plane parallel to M is of the form $x + y - z = d$. Replacing $(2, 2, 2)$ we find $d = 2$ and replacing $(-4, -4, -4)$ we find $d = -4$.

4. Let $\mathcal{C}$ be a parabola with vertical directrix and the focus at the origin. Suppose that the point P of polar coordinates $(\rho, \theta) = (4, \pi/3)$ belongs to $\mathcal{C}$. (a) Find: the directrix, the vertex, the polar equation and the cartesian equation of $\mathcal{C}$ if the directrix lies in the half-plane $x > 0$. (b) Find: the directrix, the vertex, the cartesian equation of $\mathcal{C}$ if the directrix lies in the half-plane $x < 0$.

Solution. (a) $P = 4(1/2, \sqrt{3}/2) = (2, 2\sqrt{3})$. The distance between P and the directrix has to be equal to 4. Since the directrix is a vertical line in the half-plane $x > 0$, it must be the line $x = 6$. Hence the vertex is $(3, 0)$. Polar equation: $\rho = 6/(\cos\theta + 1)$. Cartesian equation: $\sqrt{x^2 + y^2} = 6 - x$. Squaring one obtains $y^2 = 12(x - 3)$.
(b) $P = 4(1/2, \sqrt{3}/2) = (2, 2\sqrt{3})$. The distance between P and the directrix has to be equal to 4. Since the directrix is a vertical line in the half-plane $x < 0$, it must be the line $x = -2$. Hence the vertex is $(-1, 0)$. Cartesian equation: $\sqrt{x^2 + y^2} = 2 + x$. Squaring one obtains $y^2 = 4(x + 1)$.

5. A curve in $\mathcal{V}_3$ is such that the velocity vector makes a constant angle with a fixed unit vector $\mathbf{c} \in \mathcal{V}_3$. (a) Suppose that the curve lies in a plane containing $\mathbf{c}$. Describe the curve. (b) Give an example of such a curve which is not contained in a plane.

Solution. (a) In this case the unit tangent vector is constant (it is contained in fixed plane for all t and it makes a constant angle with a fixed vector of that plane). This implies that the curve is (a portion of) a line. Indeed: $T(t) = \mathbf{v}(t)/v(t) = C$, that is $\mathbf{v}(t) = Cv(t)$. Integrating $\mathbf{x}(t) = C(\int v(t)) + B$.
(b) The helix $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$. Indeed $\mathbf{v}(t) = (-a\sin t, a\cos t, b)$. We have that $v(t)$ is constant: $v(t) = \sqrt{a^2 + b^2}$. Since also $\mathbf{v}(t) \cdot \mathbf{k} = b$ is constant, the angle with $\mathbf{k} = (0, 0, 1)$ is constant.