# Fondamenti della Programmazione: Metodi Evoluti

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## Lezione 5: Logica

# Reminder: contracts

Associated with an individual feature:

- Pre-conditions (must be true BEFORE feature execution)
- Post-conditions (must be true AFTER feature execution)

Associated with a class:

- Class invariants (expresses consistency requirements between queries of a class)

# Logic



How to express conditions in contracts?

We need a mathematical notation since conditions have to be automatically checked

Logic is the answer!

# Reasoning and programming



Logic is the basis of

- Mathematics: proofs are only valid if they follow the rules of logic.

- Software development:
  - Conditions in program actions: "If *m* is positive, then execute this instruction"

  - Conditions in contracts:
    "*x* must not be zero, so that
    we can calculate $\frac{x+7}{x}$ "

# Boolean expressions

A condition is expressed as a boolean expression.

It consists of

- Boolean variables (identifiers denoting boolean values)
- Boolean operators (**not**, **or**, **and**, **=**, **implies**)

and represents possible

- Boolean values (truth values, either **True** or **False**)

# Examples



Examples of boolean expressions
(with *rain_today* and *cuckoo_sang_last_night* as boolean variables):

- *rain_today*
  (a boolean variable is a boolean expression)
- **not** *rain_today*
- (**not** *cuckoo_sang_last_night*) **implies** *rain_today*

(Parentheses group sub-expressions)

# Negation (not)



| *a* | **not** *a* |
| --- | --- |
| **True** | **False** |
| **False** | **True** |

For any boolean expression *e* and any values of variables:

- Exactly one of *e* and **not** *e* has value **True**
- Exactly one of *e* and **not** *e* has value **False**
- One of *e* and **not** *e* has value **True** (Principle of the Excluded Middle)
- Not both of *e* and **not** *e* have value **True** (Principle of Non-Contradiction)

# Disjunction (or)



| *a* | *b* | *a* **or** *b* |
|---|---|---|
| **True** | **True** | **True** |
| **True** | **False** | **True** |
| **False** | **True** | **True** |
| **False** | **False** | **False** |

**or** operator is commutative

**or** operator is associative:

- *a* **or** (*b* **or** *c*) = (*a* **or** *b*) **or** *c*

**Disjunction principle**:

- An **or** disjunction has value **True** except if both operands have value **False**

**NB:** differently from 'or' in common language **or** is non-exclusive

# Conjunction (and)



| *a* | *b* | *a* **and** *b* |
|---|---|---|
| **True** | **True** | **True** |
| **True** | **False** | **False** |
| **False** | **True** | **False** |
| **False** | **False** | **False** |

**and** operator is commutative

**and** operator is associative

- *a* **and** (*b* **and** *c*) = (*a* **and** *b*) **and** *c*

**Conjunction principle**:

- An **and** conjunction has value **False** except if both operands have value **True**

# Truth assignment and truth table

Truth assignment for a set of variables: particular choice of values (**True** or **False**), for every variable

A truth assignment satisfies an expression if the value for the expression is **True**

A truth table for an expression with $n$ variables has

- $n + 1$ columns
- $2^n$ rows

# Combined truth table for basic operators

| *a* | *b* | **not** *a* | *a* **or** *b* | *a* **and** *b* |
|---|---|---|---|---|
| **True** | **True** | **False** | **True** | **True** |
| **True** | **False** | | **True** | **False** |
| **False** | **True** | **True** | **True** | **False** |
| **False** | **False** | | **False** | **False** |

# Tautologies



**Tautology**: a boolean expression that has value **True** for every possible truth assignment

Examples:

- *a* **or** (**not** *a*)
- **not** (*a* **and** (**not** *a*))
- (*a* **and** *b*) **or** ((**not** *a*) **or** (**not** *b*))

# Contradictions



Contradiction: a boolean expression that has value **False** for every possible truth assignment

Examples:

- *a* **and** (**not** *a*)
- **not** (*a* **or** (**not** *a*))

Satisfiable: for at least one truth assignment the expression yields **True**

- Any tautology is satisfiable
- No contradiction is satisfiable.

# Equivalence (=)



| $a$ | $b$ | $a = b$ |
|---|---|---|
| **True** | **True** | **True** |
| **True** | **False** | **False** |
| **False** | **True** | **False** |
| **False** | **False** | **True** |

= operator is commutative ($a = b$ has same value as $b = a$)

= operator is reflexive ($a = a$ is a tautology for any $a$)

Substitution:

- For any expressions $u$, $v$ and $e$, if $u = v$ is a tautology and $e'$ is the expression obtained from $e$ by replacing every occurrence of $u$ by $v$, then $e = e'$ is a tautology

# Types of propositions



*Tautology*

- **True** for all truth assignments
  - P **or** (**not** P)
  - **not** (P **and** (**not** P))
  - (P **and** Q) **or** ((**not** P) **or** (**not** Q))

*Contradiction*

- **False** for all truth assignments
  - P **and** (**not** P)

*Satisfiable*

- **True** for at least one truth assignment

*Equivalent*

- φ and χ are equivalent if they are satisfied under exactly the same truth assignments, or if φ = χ is a tautology

# De Morgan's laws

They show the duality between **and** and **or**: negating an expression is equivalent to negating variables and swapping **and** and **or**

Tautologies

- **not** (*a* **or** *b*) = (**not** *a*) **and** (**not** *b*)
- **not** (*a* **and** *b*) = (**not** *a*) **or** (**not** *b*)

- *a* **or** *b* = **not** (**not** *a*) **and** (**not** *b*)
- *a* **and** *b* = **not** (**not** *a*) **or** (**not** *b*)

More tautologies (distributivity):

- (*a* **and** (*b* **or** *c*)) = ((*a* **and** *b*) **or** (*a* **and** *c*))
- (*a* **or** (*b* **and** *c*)) = ((*a* **or** *b*) **and** (*a* **or** *c*))

# Syntax convention: binding of operators



Order of binding (starting with tightest binding): **not**, **and**, **or**, **implies** (to be introduced), **=** .

Style rules:

When writing a boolean expression, drop the parentheses:

- Around the expressions of each side of "=" if whole expression is an equivalence.
- Around successive elementary terms if they are separated by the same associative operators.

# Implication (implies)



| *a* | *b* | *a* **implies** *b* |
|---|---|---|
| **True** | **True** | **True** |
| **True** | **False** | **False** |
| **False** | **True** | **True** |
| **False** | **False** | **True** |

*a* **implies** *b*, for any *a* and *b*, is the value of (**not** *a*) **or** *b*

In *a* **implies** *b*: *a* is antecedent, *b* consequent

Implication principle:

- An implication has value **True** except if its antecedent has value **True** and its consequent has value **False**
- In particular, always **True** if antecedent is **False**

# Implication in ordinary language



**implies** in ordinary language often means causation, as in "if … then …"

- "*If the weather stays like this, skiing will be great this week-end*"
- "*If you put this stuff in your hand luggage, they won't let you through.*"

# Misunderstanding implication



Whenever $a$ is **False**,
$a$ **implies** $b$ is **True**, regardless of $b$:

- "Today is Wednesday implies 2+2=5."
- "2+2=5 implies today is Wednesday."

Both of the above implications are **True**

Cases in which $a$ is **False** tell us nothing about the truth of the consequent

# Reversing implications (1)



It is not generally true that

~~*a* **implies** *b* = (**not** *a*) **implies** (**not** *b*)~~

Example (wrong!):

- *"All the people in Rome who live near Spanish Steps are rich. I do not live near Spanish Steps, so I am not rich."*

*live_near_spanish_steps* **implies** *rich* [1]

~~(**not** *live_near_spanish_steps* ) **implies** (**not** *rich* )~~ [2]

# Reversing implications (2)



Correct:

*a* **implies** *b* **=** **(not** *b***) implies (not** *a***)**

Example:

- *"All the people who live near Spanish Steps are rich. She is not rich, so she can't be living near Spanish Steps"*

*live_near_spanish_steps* **implies** *rich* **=**

**(not** *rich***) implies (not** *live_near_spanish_steps***)**

# Semistrict boolean operators (1)

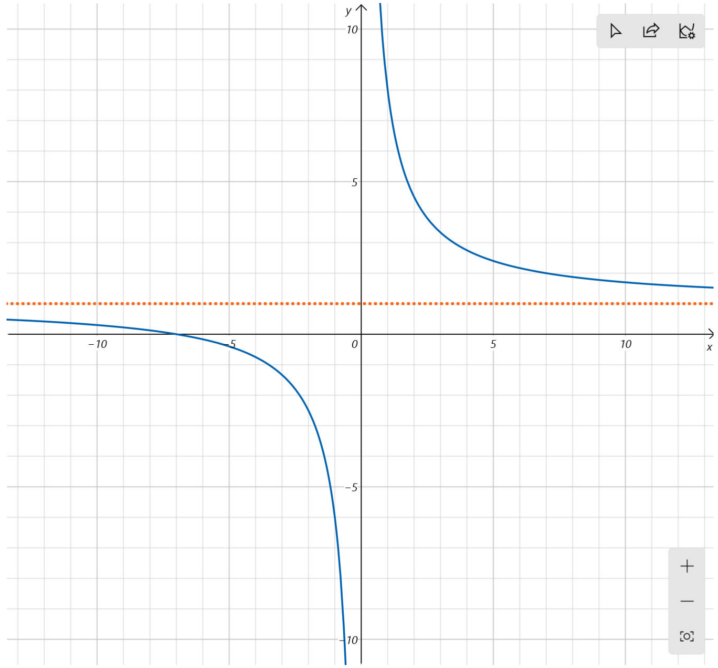
Example boolean-valued expression (*x* is an integer):

$$\frac{x+7}{x} > 0$$

True for $x < -7$ or $x > 0$

False for $x \geq -7$ and $x < 0$

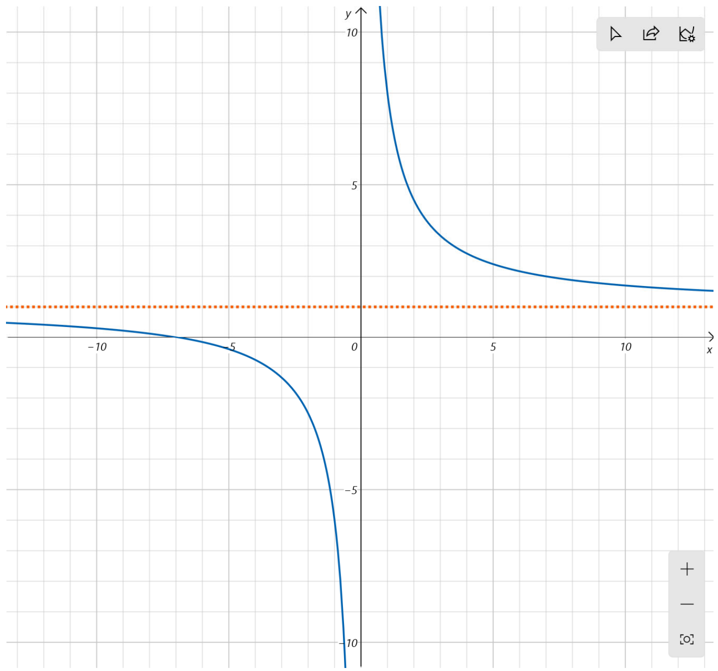
Undefined for $x = 0$



# Semistrict boolean operators (2)

- Avoid division by zero:

  $(x \mathbin{/=} 0)$ **and** $(((x + 7) \,/\, x) > 0)$




True for $x < -7$ or $x > 0$

False for $x \geq -7$ and $x \leq 0$

Always defined

# Semistrict boolean operators (3)



BUT:

- What happens when evaluating it?
- Program would crash during evaluation of division

We need a non-commutative version of **and** (and **or**):

Semistrict boolean operators

# Semistrict operators (and then, or else)



*a* **and then** *b*: has same value as *a* **and** *b* if *a* and *b* are both defined, and has **False** whenever *a* has value **False** even if b is undefined

*a* **or else** *b*: has same value as *a* **or** *b* if *a* and *b* are both defined, and has **True** whenever *a* has value **True** even if b is undefined

($x$ /= 0) **and then** ((($x$ + 7) / $x$) > 0)

Semistrict operators allow us to define an order of expression evaluation (left to right).

- Important for programming when undefined objects may cause program crashes

# Ordinary vs. Semistrict boolean operators



Use

- Ordinary boolean operators (**and** and **or**) if you can guarantee that both operands are defined
- **and then** if a condition only makes sense when another is true
- **or else** if a condition only makes sense when another is false

Example:

- "If you are not single, then your spouse must sign the contract"

*is_single* **or else** *spouse_must_sign*

**not** *is_single* **and then** *spouse_must_sign*

# Semistrict implication



Example with implies:

- "If you are not single, then your spouse must sign the contract."

(**not** *is_single*) **implies** *spouse_must_sign*

Definition of **implies**: in our case, always semistrict!

- *a* **implies** *b* = (**not** *a*) **or else** *b*

# Strict or semi-strict?



Hands-On

- $a = 0$ **or** $b = 0$
- $a \mathrel{/=} 0$ **and** $b \mathbin{//} a \mathrel{/=} 0$
- $a \mathrel{/=}$ **Void and** $b \mathrel{/=}$ **Void**
- $a < 0$ **or** $sqrt\,(a) > 2$
- $(a = b$ **and** $b \mathrel{/=}$ **Void**$)$ **and** $a.age = 0$

# Programming language notation for boolean operators



| Eiffel keyword | Common mathematical symbol |
|---|---|
| **not** | $\sim$ or $\neg$ |
| **or** | $\vee$ |
| **and** | $\wedge$ |
| **=** | $\Leftrightarrow$ |
| **implies** | $\Rightarrow$ |

# Propositional and predicate calculus



Propositional calculus:

property *p* holds for a single object

Predicate calculus:

property *p* holds for several objects

# Generalizing or



*G* : group of objects, *p* : property

Generalization of **or**:

Does *at least one* of the objects in *G* satisfy *p* ?

Is at least one station of Line 8 an exchange?

*Station_Balard.is_exchange* **or** *Station_Lourmel.is_exchange* **or** *Station_Boucicaut.is_exchange* **or**

... (all stations of Line 8)

Existential quantifier: *exists*, or ∃

∃ *s* : *Stations_8* | *s.is_exchange*

"There exists an *s* in *Stations_8* such that *s.is_exchange* is true"

# Generalizing and

Generalization of **and**:

Does *every* object in *G* satisfy p?

Are all stations of Tram 8 exchanges?

*Station_Balard.is_exchange* **and** *Station_Lourmel.is_exchange* **and** *Station_Boucicaut.is_exchange* **and** …

(all stations of Line 8)

Universal quantifier: *for_all*, or ∀

∀ *s: Stations_8* | *s.is_exchange*

"For all *s* in *Stations8* | *s.is_exchange* is true"

# Existentially quantified expression



Boolean expression:

$\exists\ s : SOME\_SET \mid s.some\_property$

- *True* if and only if at least one member of *SOME_SET* satisfies property *some_property*

Proving

- **True**: Find one element of *SOME_SET* that satisfies the property
- **False**: Prove that no element of *SOME_SET* satisfies the property (test all elements)

# Universally quantified expression



Boolean expression:

$\forall$ *s: SOME_SET* | *s.some_property*

- *True* if and only if every member of *SOME_SET* satisfies property *some_property*

Proving

- **True**: Prove that every element of *SOME_SET* satisfies the property (test all elements)
- **False**: Find one element of *SOME_SET* that does not satisfies the property

# Duality



Generalization of DeMorgan's laws:

**not** ($\exists$ *s* : *SOME_SET* | *P* ) = $\forall$ *s* : *SOME_SET* | **not** *P*

**not** ($\forall$ *s* : *SOME_SET* | *P* ) = $\exists$ *s* : *SOME_SET* | **not** *P*

# Empty sets



$\exists s : SOME\_SET \mid some\_property$

If *SOME_SET* is empty: always False

$\forall s : SOME\_SET \mid some\_property$

If *SOME_SET* is empty: always True

## Tautology / contradiction / satisfiable?



Hands-On

Let the range of variables be INTEGER

x < 0 **or** x >= 0

tautology

x > 0 **implies** x > 1

satisfiable

∀x | x > 0 **implies** x > 1

contradiction

∀x | x∗y = y

satisfiable

∃y | ∀x | x∗y = y

tautology