Some C^*-algebras which are Coronas of Non-C^*-Banach Algebras

Dan-Virgil Voiculescu
UC Berkeley

Some Old Sources

- machinery for the study of normed ideal perturbations of Hilbert space operators based on adaptation of non-commutative Weyl-v. Neumann thm to normed-ideals
- problems about mod Hilbert-Schmidt.

  BDF type thm. for operators with trace-class self-commutator using the Pincus g-function instead of index

Some Recent Stimuli (loosely)

- trace-class commutators also occur in the non-microstates approach to free entropy

- Connes' technical use of some of the normed ideal perturbation results in work on characterization of non-commutative manifolds.

Certain commutants mod a normed ideal of systems of operators are the framework to turn old problems about BDF-thm. analogue for operators with trace-class self-commutator into K-theory questions. The Banach algebras and C^*-algebras which appear may be of interest in connection with extending KK-theory beyond C^*-algebras, non-commutative potential theory and as exotic examples of C^*-algebras similar to Coronas.

Normed Ideal Perturbations Machinery

$\mathcal{H}$, $\mathcal{R}(\mathcal{H})$ finite rank, $\mathcal{K}(\mathcal{H})$ compact, $\mathcal{B}(\mathcal{H})$

$\Phi(s_1 \geq s_2 \geq \dots \geq 0)$ norming function

$X \in \mathcal{K}(\mathcal{H})$, $s_1 \geq s_2 \geq \dots$ eigenvalues $(X^*X)^{1/2}$

$|X|_\Phi = \Phi(s_1 \geq s_2 \geq \dots)$, $\mathfrak{S}_\Phi = \{X \in \mathcal{K}(\mathcal{H}) \mid |X|_\Phi < \infty\}$

$\mathfrak{S}_\Phi^{(0)}$ closure $\mathcal{R}(\mathcal{H})$ in $\mathfrak{S}_\Phi(\mathcal{H})$.

$\mathcal{C}_p$ $\quad \Phi_p = (\sum s_j^p)^{1/p}$, $|X|_p$

$\mathcal{C}_p^-$ $\quad \sum s_j j^{-1+1/p}$, $\mathcal{C}_p^- \subset \mathcal{C}_p$, $\mathcal{C}_1^- = \mathcal{C}_1$.

$\tau = (T_1, \ldots, T_n) \in (\mathcal{B}(\mathcal{H}))^n$

$$k_\Phi(\tau) = \liminf_{A \in \mathcal{R}_1^+(\mathcal{H})} |[A, \tau]|_\Phi$$

$\mathcal{R}_1^+(\mathcal{H})$ $\quad 0 \le A \le I$ in $\mathcal{R}(\mathcal{H})$, $[A, \tau] = ([A, T_j])_{1 \le j \le n}$

$$|(X_j)_{1 \le j \le n}|_\Phi = \max_{1 \le j \le n} |X_j|_\Phi$$

$k_\Phi(\tau) = 0 \iff \exists\, A_n \uparrow I,\ |[A_n, \tau]|_\Phi \to 0$
$A_n \in \mathcal{R}_1^+(\mathcal{H})$.

$\tau = (T_j)_{1 \le j \le n}$, $T_j = T_j^*$, $[T_j, T_k] = 0 \quad 1 \le j, k \le n$

$k_p(\tau) = 0$ if $p > n$

$k_n^-(\tau) > 0 \iff \tau$ has non-empty
n-dim Lebesgue abs. cont. spectrum

$k_\infty^-(\tau) < \infty$ always, $k_\infty^-(\text{Cuntz-n-tuple}) > 0$

$\mathfrak{S}_{\underline{\Phi}}^{(0)} \not\supseteq \mathcal{C}_\infty^- \Longrightarrow k_{\underline{\Phi}}^-(\tau) = 0, \ \forall \tau$

$\tau = \tau^*, \ k_{\underline{\Phi}}^-(\tau) > 0$

$\Updownarrow$

$\exists \ Y_j = Y_j^* \in \left(\mathfrak{S}_{\underline{\Phi}}^{(0)}\right)^{\text{dual}}$

$\sum_j i \, [T_j, Y_j] = Z \in \mathcal{C}_1 + (\mathcal{B}(\mathcal{H}))_+$

$\operatorname{Tr} Z > 0.$

Adapted noncommutative Weyl-v. Neumann type Thm.

A unital C^*-alg., $\omega = (X_j)_{j\in\mathbb{N}} \subset A$ generator, $X_j = X_j^*$

$\rho_j : A \longrightarrow \mathcal{B}(\mathcal{H})$ unital $*$ rep. $j = 1,2$

$\rho_j^{-1}(\mathcal{K}(\mathcal{H})) = 0 \quad j = 1,2$

$k_\Phi\left(\left(\rho_j(X_h)\right)_{1\le h\le n}\right) = 0,\ j = 1,2,\ n\in\mathbb{N}$

Then: $\exists\, U$ unitary

$$U\rho_1(X_k) - \rho_2(X_k)U \in \mathfrak{S}_\Phi^{(0)}$$

$k\in\mathbb{N}$.

Corollary

$\tau = (T_1, \dots T_n)$, $T_j = T_j^*$ commuting $1 \le j \le n$

$k_\Phi(\tau) = 0$

$\Uparrow$

$\exists\ (D_1, \dots, D_n)$ commuting, $D_j = D_j^*$

diagonal in some ONB

$|T_j - D_j|_\Phi < \varepsilon \qquad 1 \le j \le n$

$T_j - D_j \in \mathfrak{S}_\Phi^{(0)}$.

The Banach algebras $\mathcal{E}(\tau;\mathcal{J})$

$\tau = \tau^* = (T_j)_{1\le j\le n} \subset \mathcal{B}(\mathcal{H})$

$(\mathcal{J}, |\ |_{\mathcal{J}})$ normed ideal

$$\mathcal{E}(\tau;\mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X, T_j] \in \mathcal{J},\ 1\le j\le n\}$$

Banach $*$-algebra with isometric involution with the norm

$$|||X||| = \|X\| + |[X,\tau]|_{\mathcal{J}} .$$

$$\mathcal{K}(\tau;\mathfrak{J}) = \mathcal{E}(\tau;\mathfrak{J}) \cap \mathcal{K}(\mathcal{H})$$

closed 2-sided ideal in $\mathcal{E}(\tau;\mathfrak{J})$

<u>Fact</u> Assume $\mathfrak{J} = \mathfrak{S}_{\Phi}^{(0)}$, $k_{\mathfrak{J}}(\tau) = 0$.

If $X_1, \dots, X_m \in \mathcal{E}(\tau;\mathfrak{J})$, then

$\exists\ A_s \in \mathcal{R}_1^+(\mathcal{H})$, $A_s \uparrow I$ $s \to \infty$

$|||A_s||| \to 1$, $|||(I - A_s)K||| \to 0$ if $K \in \mathcal{K}(\mathcal{H})$

$$|||[X_p, A_s]||| \to 0,\ s \to \infty.$$

$$p : B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H}) = B/K(\mathcal{H})$$

$$\mathcal{E}(\tau;\mathcal{J})/K(\tau,\mathcal{J}) = \mathcal{E}/K(\tau;\mathcal{J})$$

$\mathcal{E}/K(\tau;\mathcal{J}) \subset B/K(\mathcal{H})$ algebraically.

<u>Fact</u> Assume $\mathcal{J} = \mathfrak{S}_{\Phi}^{(0)}$, $k_{\mathcal{J}}(\tau) = 0$.

Then, algebraic embedding

$$\mathcal{E}/K(\tau;\mathcal{J}) \subset B/K(\mathcal{H})$$

is isometric and

$\mathcal{E}/K(\tau;\mathcal{J})$ is a C^*-subalgebra of $B/K(\mathcal{H})$.

Multipliers and Duality

Fact Assume $k_\Phi(\tau) = 0$. Then the algebra of bounded multipliers

$$\mathcal{M}(\mathcal{K}(\tau; \mathfrak{S}_\Phi^{(0)})) = \mathcal{E}(\tau; \mathfrak{S}_\Phi).$$

If Φ is a norming function, there is a dual norming function Φ^* so that $(\mathfrak{S}_\Phi^{(0)}, |\ |_\Phi)^{\text{dual}} = (\mathfrak{S}_{\Phi^*}, |\ |_{\Phi^*})$ with $(X, Y) \longrightarrow \operatorname{Tr} XY$ duality map.

Fact Assume $k_\Phi(\tau)=0$, $\mathfrak{S}_\Phi^{(0)}=\mathfrak{S}_\Phi$, $\mathfrak{S}_{\Phi^*}^{(0)}=\mathfrak{S}_{\Phi^*}$.

Endow $\mathscr{C}_1\times(\mathfrak{S}_{\Phi^*})^n$ with $\|(x,(y_j)_{1\le j\le n})\|=$

$=\max\left(|x|_1,\ \sum_{1\le j\le n}|y_j|_{\Phi^*}\right)$ and let

$$N=\left\{\left(\sum_{1\le j\le n}[T_j,y_j],(y_j)_{1\le j\le n}\right)\in\mathscr{C}_1\times(\mathfrak{S}_{\Phi^*})^n \,\middle|\, (y_j)_{1\le j\le n}\in(\mathfrak{S}_{\Phi^*})^n \text{ so that } \sum_j[T_j,y_j]\in\mathscr{C}_1\right\}$$

Then $(\mathcal{K}(\tau;\mathfrak{S}_\Phi))^{dual}\sim(\mathscr{C}_1\times(\mathfrak{S}_{\Phi^*})^n)/N$

and $((\mathscr{C}_1\times(\mathfrak{S}_{\Phi^*})^n)/N)^{dual}\sim\mathcal{E}(\tau;\mathfrak{S}_\Phi)$.

To understand more about $\mathcal{E}(\tau; \mathcal{J})$ would need to understand their K-theory, for some very basic τ and appropriate $\mathcal{J}$ at least.

Some K-theory facts are known for the "two-dimensional" case of τ a pair of commuting hermitian operators and $\mathcal{J} = \mathcal{C}_2$.

$N = X + iY$ normal operator

$\sigma(N) = \{z \in \mathbb{C} \mid |z| \leq 1\} = \overline{\mathbb{D}}$

$\mathcal{E}((X,Y); \mathcal{C}_2)$ does not depend on choice of N, will denote $E\Delta(\overline{\mathbb{D}})$.

(N_1, N_2 such normal operators $\exists U$ unitary, $UN_1U^* - N_2 \in \mathcal{C}_2$).

$K_0(E\Delta(\overline{\mathbb{D}}))$ related to questions about almost normal operators.

Almost Normal Operators

$[T^*, T] \in \mathcal{C}_1$

Pincus Principal Function

$g_T \in L^1_{rc}(\mathbb{C}, \lambda)$

If $z \in \mathbb{C} \setminus \sigma_e(T)$, $g_T(z) = \text{index}(T - zI)$

g_T defined a.e. on $\mathbb{C}$, may be $\neq 0$ on $\sigma_e(T)$.

L.G. Brown connected g_T to algebraic K-theory

J.W. Helton – R. Howe $T = X + iY$ almost normal

$$\mathrm{Tr}\,[P(X,Y), Q(X,Y)] =$$

$$= \frac{1}{2\pi} \iint \left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x} \right) g_T \, dx\,dy$$

A. Connes $g_T\, dx\, dy$ from cyclic cohomology.

$$\alpha \in K_0(E\Lambda(\bar{\mathbb{D}})) \Rightarrow \exists\, P = P^* = P^2 \in E\Lambda(\bar{\mathbb{D}})$$
$$[P]_0 = \alpha$$

$$\begin{tikzcd}
K_0(E\Lambda(\bar{\mathbb{D}})) \ni \alpha \arrow[r, squiggly] & {P = P^* = P^2,\ [P]_0 = \alpha} \arrow[d, squiggly] \\
L^1_{rc}(\mathbb{D},\lambda) \ni g_{PNP} & PNP \text{ almost normal in } P\mathcal{H} \arrow[l, squiggly]
\end{tikzcd}$$

r: real c: compact supp

Map $K_0 \longrightarrow L^1_{rc}$

well-defined homomorphism.

$$K_0(E\Lambda(\bar{\mathbb{D}})) \longrightarrow K_0(E/\mathcal{K}\Lambda(\bar{\mathbb{D}}))$$

isomorphism

$$E/\mathcal{K}\Lambda(\bar{\mathbb{D}}) = \mathcal{E}/\mathcal{K}((X,Y),\mathcal{C}_2)$$

C^*-algebra

Is $K_0(E\Lambda(\bar{\mathbb{D}})) \longrightarrow L^1_{nc}(\mathbb{D})$

an isomorphism?

[Surjective? maybe some integrality result]

Related to BDF mod $\mathcal{C}_2$ problems for almost normal operators.

- $\mathcal{E}/\mathcal{K}((X,Y);\mathcal{C}_2) \subset \{a \in \mathcal{B}/\mathcal{K}(\mathcal{H}) \mid [a, p(N)] = 0\}$

  $\mathcal{E}/\mathcal{K}$ has non-trivial K_0, while Calkin commutant has trivial K_0.
  So $\mathcal{E}/\mathcal{K}$ is not like a smooth subalgebra.

- $\mathcal{K}((X,Y);\mathcal{C}_2)$ is a Dirichlet algebra. Dirichlet form
  $|[X,a]|_2^2 + |[Y,a]|_2^2$.

Centre $\mathcal{E}/\mathcal{K}\,\Lambda(\overline{\mathbb{D}})$.

More generally

Fact (Bourgain-V.)

$\tau = (X_1, \dots, X_n)$, $X_j = X_j^*$, $[X_j, X_k] = 0$ $1 \le j, k \le n$

$k_{\mathfrak{J}}(\tau) = 0$, $\mathfrak{J} = \mathfrak{S}_{\Phi}^{(0)}$.

$\sigma(X_1, \dots, X_n)$ perfect set

Then $\text{Centre}(\mathcal{E}/\mathcal{K}(\tau; \mathfrak{J})) =$

$= p(C^*(\tau))$.

[Cor. Centre $\mathcal{E}/\mathcal{K}\,\Lambda(\overline{\mathbb{D}}) \sim C(\overline{\mathbb{D}})$]

Countable Degree-1 Saturation

Def. (Farah – Hart)

M C^*-alg., X_m, X_m^*, $m \in \mathbb{N}$ non-commuting indeterminates. Degree 1 $*$-polynomial linear combination of a, aX_mb, aX_m^*b $a, b \in M$. M is <u>countably degree-1 saturated</u> if for every sequence of degree-1 $*$-polynomials P_n and compact sets K_n, $n \in \mathbb{N}$, TFAE:

(i) there are $b_m \in M$, $m \in \mathbb{N}$ such that $\|b_m\| \leq 1$ and $\|P_n(b)\| \in K_n$ for all $n \in \mathbb{N}$, where $b = (b_1, b_2, \dots)$

(ii) for every $N \in \mathbb{N}$, there are $b_m \in M$, $\|b_m\| \leq 1$, $m \in \mathbb{N}$ such that

$$\operatorname{dist}(\|P_n(b)\|, K_n) \leq 1/N$$

for all $n \in \mathbb{N}$.

Fact Assume $k_{\mathfrak{J}}(\tau) = 0$ and $\mathfrak{J} = \mho_{\mathfrak{J}}^{[0]}$.
Then $\mathcal{E}/\mathcal{K}(\tau; \mathfrak{J})$ is countably degree-1 saturated.

(Ref 1

1°. Almost Normal Operators mod Hilbert-Schmidt and the K-theory of the Algebras $E\Lambda(\Omega)$.

arXiv: 1112.4930 v2

to appear JNCG

2°. (with J. Bourgain) The essential Centre of the mod a diagonalization ideal Commutant of an n-tuple of commuting hermitian operators

arXiv: 1309.1686 v.1.

(Ref2

3. Countable degree-1 Saturation of certain C^*-algebras which are Coronas of Banach algebras.

arXiv: 1310.4862

to appear in Groups, Geometry and Dynamics.