KK-theory for C^*-precosheaves

and
holonomy-equivariance

Mondragone (Frascati), June 2014

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Contents

C^*-precosheaves.

A C^*-*precosheaf* over a poset Δ is a pair $\mathcal{A} = (A, \jmath)_\Delta$, where $A = \{A_a\}_{a \in \Delta}$ is a family of C^*-algebras and $\jmath$ is a family of *-monomorphisms fulfilling

$$\jmath_{a'a} : A_a \to A_{a'} \quad : \quad \jmath_{a''a} = \jmath_{a''a'} \circ \jmath_{a'a} \ , \ a \leq a' \leq a'' \ .$$

Net bundles: any $\jmath_{a'a}$ is a *-isomorphism.

Hilbert precosheaves, *group precosheaves*...

C^*-precosheaves naturally arise in several contexts:
$\rightsquigarrow$ non-simple C^*-algebras, C^*-inductive limits
$\rightsquigarrow$ Algebraic quantum field theory

Ideals. A C^*-algebra, $\Delta \subseteq \tau A$:= the set of closed proper two-sided ideals of A $\rightsquigarrow$ a C^*-precosheaf $\mathcal{A}$,

$$A_I := I \quad , \quad \jmath_{I'I} := A_I \subseteq A_{I'} \quad , \quad I, I' \in \Delta \ .$$

$\rightsquigarrow$ Ex.: X lc Hausdorff space with base Δ,
A a $C_0(X)$-*algebra* $\rightsquigarrow$ $A_Y := C_0(Y)A$, $\forall Y \in \Delta$.

QFT. M spacetime, H Hilbert space (the *vacuum*) $\rightsquigarrow$ A *causal net* of C^*-algebras $\mathcal{A}$,

$$A_o \subseteq A_{o'} \subseteq B(H) \quad , \quad A_e \subseteq A'_o \quad , \quad e \perp o \subseteq o' \in \Delta \ .$$

$\rightsquigarrow$ Ex.: a *field* $\phi = \phi^* : \mathcal{S}(M) \to \mathcal{L}(D)$, $D \subset H$ dense,

$$A_o := C^* \{ e^{i\phi(f)} : \mathrm{supp}(f) \subseteq o \} \, , \, o \in \Delta.$$

Geometry of posets, QFT and holonomy.

(Algebraic) quantum field theory $\rightsquigarrow$
a *geometry* where points are replaced by elements of
a base Δ of the spacetime, ordered under inclusion $\rightsquigarrow$
the structure of relevance is the one of poset.

A key result in AQFT: a causal net $\mathcal{A}$ determines the field net $\mathcal{F}$ and the gauge group $G \rightsquigarrow \mathcal{F}^G = \mathcal{A}$ (Doplicher-Roberts).

These results can be interpreted in terms of a 1–cohomology with coefficients in the unitary precosheaf of $\mathcal{A}$ (Roberts).

The simplicial set. Δ: a poset with order relation $\leq$.

The set of "points"

$$C_0(\Delta) := \Delta \ .$$

The set of "lines"

$$C_1(\Delta) := \{b = (\partial_0 b, \partial_1 b \leq |b|) \in \Delta\} \quad \rightsquigarrow \quad b : \partial_1 b \to \partial_0 b \ .$$

The set of "triangles"

$$C_2(\Delta) := \{c = (\partial_0 c, \partial_1 c, \partial_2 c \in C_1(\Delta), \partial_{hk} \ldots ; |c| \in \Delta)\} \ .$$

$\ldots$

The fundamental group. A *path* is a sequence

$$p = b_n * \ldots * b_1 \quad : \quad \partial_0 b_k = \partial_1 b_{k+1} \rightsquigarrow$$

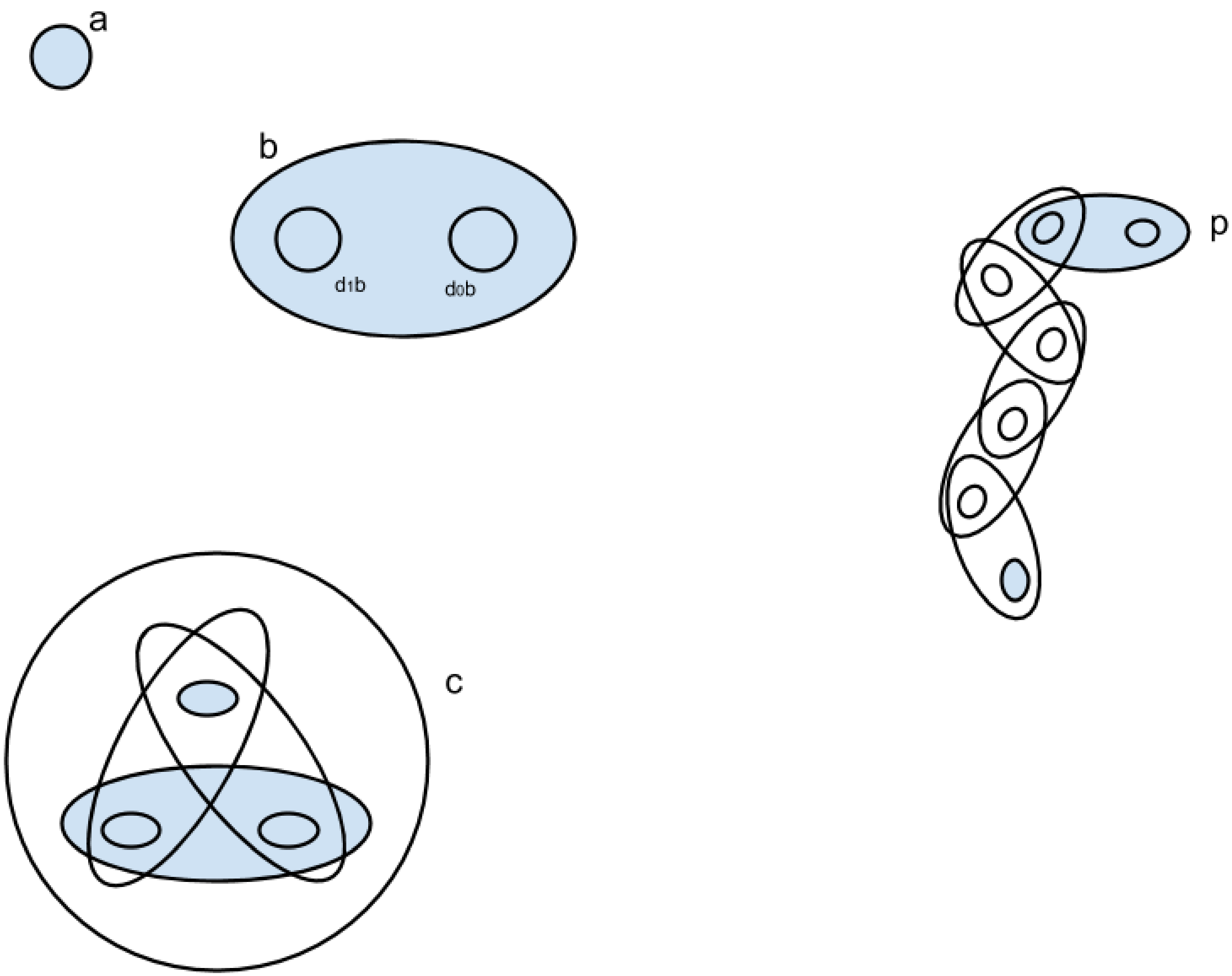
$$p : \partial_1 b_1 \ \rightarrow \ \partial_0 b_n \ .$$

$\rightsquigarrow$ *Loops*: $\partial_0 b_n = \partial_1 b_1$.

There is a notion of *homotopy* for paths $\rightsquigarrow$
$\pi_1(\Delta)$: homotopy classes of loops based on $a \in \Delta$.

Ruzzi RMP 17(2005): Δ a base of arcwise and simply connected open sets generating the topology of X $\rightsquigarrow$

$$\pi_1(\Delta) \ \simeq \ \pi_1(X) \ .$$


a
b
d_1b
d_0b
c
p

The homotopy morphism.

Path frame with pole $a \in \Delta$: a family

$$p_a = \{ p_{ao} : o \to a \} \ .$$

Define $N_1(\Delta) := \{ (o'o) \ : \ o \leq o' \in \Delta \}$. $\rightsquigarrow$

$$\gamma : N_1(\Delta) \to \pi_1(\Delta) \quad , \quad \gamma(o'o) := [p_{ao'} * (o'o) * p_{oa}] \rightsquigarrow$$

$$\gamma(o''o) = \gamma(o''o') * \gamma(o'o) \ , \ o \leq o' \leq o'' \ .$$

$\rightsquigarrow$ γ is independent of p_a up to conjugation.

$\rightsquigarrow$ $\pi_1(\Delta)$ encodes properties of the order structure.

Cohomology and QFT.

$\mathcal{U} = (U, \imath)_\Delta$: a group precosheaf. *1–cocycles*: families

$$z \in \prod_b U_{|b|} \quad : \quad z_{\partial_0 c} z_{\partial_2 c} \; = \; z_{\partial_1 c} \quad , \quad \forall c \in C_2(\Delta)$$

(we write $z_{\partial_k c} \equiv i_{|c|\partial_k c}(z_{\partial_k c})$ for simplicity).

$\rightsquigarrow$ $Z^1(\mathcal{U})$ the set of cocycles.

$\rightsquigarrow$ $H^1(\mathcal{U}) := Z^1(\mathcal{U})/\simeq$ the cohomology set.

$\rightsquigarrow$ $\imath_{o'o} = id_G$, $\forall o \subseteq o'$ $\rightsquigarrow$ we write $H^1(\Delta, G)$.

$\mathcal{A}$ an observable net over the base Δ of *doublecones* of the Minkowski space ($A_o = A_o''$, $\forall o \in \Delta$) $\rightsquigarrow$

Doplicher, Roberts: $Z^1(\mathcal{UA})$ yields a symmetric tensor category with conjugates (*DR-category*) isomorphic to the dual of the gauge group G.

X a generic spacetime: $Z^1(\mathcal{UA})$ is still a DR-category, but the question of the gauge group is open.

Question: Which are the topological properties of X encoded by the geometry of Δ? (Roberts-R.-V.)

$H^1(\Delta, G) \simeq \mathrm{hom}^{\mathrm{ad}}(\pi_1(X), G) \rightsquigarrow$
$H^1(\Delta, Z) \simeq H^1(X, Z)$, Z abelian.

$\mathbf{net\,bun}(\Delta, F) \simeq \mathbf{lc}(X, F) \rightsquigarrow$
$\mathbf{lc}(X, F) \rightsquigarrow$ **obj**: F-bundles with *locally constant* transition maps; **arr**: *locally constant* bundle morphisms.

RV: $\mathcal{A} = (A, \jmath)_\Delta$ C^*-precosheaf $\rightsquigarrow$
A canonical $C_0(X)$-algebra $\mathscr{A}$ $\rightsquigarrow$
Elements of A_Y, $Y \in \Delta$, are local sections of $\mathscr{A}$ $\rightsquigarrow$
$\mathscr{A}$ is universal for "morphisms" from $\mathcal{A}$ to $C_0(X)$-algebras:

$$\eta : \mathcal{A} \to \mathscr{B}_{sheaf} \quad \rightsquigarrow \quad \eta_* : \mathscr{A} \to \mathscr{B} \ .$$

Representation theory and K-homology.

Elliptic operators appear in (algebraic) QFT as *super-charges*, that is, odd square roots of the Hamiltonian (Connes, JLO, Longo, ...).

Conformal field theory: the supercharge is a *net of θ-summable Fredholm modules* (CHKL CMP 295 (2010)).

The previous result is in the setting of *Hilbert space reps* of the observable net.

But C^*-precosheaves in general *cannot* be represented on a Hilbert space (RV).

Representation theory.

$\mathcal{A} = (A, \jmath)_\Delta$ C^*-precosheaf, $\mathcal{H} = (H, U)_\Delta$ Hilbert net bundle. A *representation* π of $\mathcal{A}$ on $\mathcal{H}$ $\rightsquigarrow$

$$\pi := \{\pi_o : A_o \to B(H_o)\} \quad : \quad \pi_{o'} \circ \jmath_{o'o} = \mathrm{ad} U_{o'o} \circ \pi_o \ , \ o \leq o' \ .$$

Motivation: any $z \in Z^1(\mathcal{U}\mathcal{A})$ defines a rep (*sector*) π^z of the observable net $\mathcal{A}$ (Brunetti-Ruzzi CMP 287(2009)).

Hilbert space reps: $U_{o'o} \equiv id_H$.
Examples in which they do not exist (RV):
- $C^*\Pi$-net bundles with $\Pi := \pi_1(\Delta)$ not amenable
- Some C^*-precosheaves over Δ finite, ...

Net bundles vs C^*-dynamical systems. $\mathcal{B} = (B, \jmath)_\Delta$ a net bundle $\rightsquigarrow$ we define the *parallel displacement*

$$\left\{ \begin{array}{l} \jmath_p : B_o \stackrel{\simeq}{\rightarrow} B_a \quad , \quad p = b_n * \ldots * b_1 : o \rightarrow a \ , \\ \\ \jmath_p := \jmath_{a|b_n|}^{-1} \circ \ldots \circ \jmath_{|b_1|o} \ . \end{array} \right.$$

In particular, $o = a$ $\rightsquigarrow$ the *holonomy C^*-system*

$$\alpha : \pi_1(\Delta) \rightarrow \mathbf{aut} B_* \quad , \quad B_* := B_a \ .$$

$\rightsquigarrow$ $\mathcal{B} \mapsto (B_*, \alpha)$ is a functor, indeed an equivalence.

Enveloping net bundles. $\mathcal{A} = (A, \jmath)_\Delta$ C^*-precosheaf $\rightsquigarrow$ $\mathcal{A}$ can be embedded in a C^*-net bundle/C^*-dynamical system encoding its rep theory:

Pick $a \in \Delta$ and define the $*$-algebra A_* generated by

$$(p, t) \quad : \quad p : \partial_1 p \to a \ , \ t \in A_{\partial_1 p} \ ,$$

with algebraic relations

$$(p, t')(p, t'') + z(p, t)^* = (p, t't'' + zt^*) \ ,$$

geometric relations

$$\begin{cases} (p, t) = (p', t) \quad , \quad p \sim p' \ , \\ (p * (o'o), t) = (p, \jmath_{o'o}(t)) \quad , \quad o \leq o' \ , \end{cases}$$

and $\pi_1(\Delta)$-action

$$\alpha : \pi_1(\Delta) \to \mathbf{aut} A_* \quad , \quad \alpha_q(p, t) := (q * p, t) \ , \ q : a \to a \ .$$

$\rightsquigarrow$ (A_*, α): the $\pi_1(\Delta)$-C^*-dynamical system with the C^*-norm induced by covariant reps.

There is an inclusion morphism

$I_o : A_o \to A_*$, $I_o(t) := (p_{ao}, t)$, $o \in \Delta$ $\Rightarrow$

$I_{o'} \circ \jmath_{o'o} = \alpha_{\gamma(o'o)} \circ I_o$ (recall $\gamma : N_1(\Delta) \to \pi_1(\Delta)$) $\Rightarrow$

Defining $A_{*,o} := I_o(A_o)$ we obtain the *filtration*

$$\alpha_{\gamma(o'o)}(A_{*,o}) \subseteq A_{*,o'} \subseteq A_* \ , \forall o \leq o'.$$

Covariant reps (π_*, U_*) of (A_*, α) $\leftrightarrow$ reps π of $\mathcal{A}$:

$$\pi_* \circ I_o = I_o \circ \pi_o \quad , \quad \forall o \leq o' \ .$$

Theorem (RV) Δ poset with $\Pi := \pi_1(\Delta)$.
There are functors

$$\mathbf{C^*\,pcoshf}(\Delta) \ \stackrel{*}{\rightarrow} \ \mathbf{C^*\,dyn}(\Pi) \ \stackrel{\vec{}}{\rightarrow} \ \mathbf{C^*\,alg}$$
$$\mathcal{A} \mapsto (A_*, \alpha) \mapsto \vec{A} \ .$$

The functor $*$ is an equivalence when restricted to the subcategory of C^*-net bundles.

(A_*, α) is universal for reps of $\mathcal{A}$.

$\vec{A}$ (Fredenhagen algebra): the quotient of A_* under *invariant reps* $\leftrightarrow$ Hilbert space reps of $\mathcal{A}$:

$$\pi : A_* \to BH \ : \ \pi \circ \alpha_q = \pi \ , \ \forall q \in \Pi \ .$$

K-homology. *Fredholm $\mathcal{A}$-module based on $a \in \Delta$:*

$(\pi, F) \rightsquigarrow$ • π a rep of $\mathcal{A}$ over $\mathcal{H} = (H, U)_\Delta$;

• $F = F^* \in B(H_a) \rightsquigarrow$

$F_p := \mathrm{ad}U_p(F) \in B(H_o)$, $p : a \to o$.

$$\begin{cases} F_p - F_{\overline{p}} \in K(H_o) \ , \ \forall p, \overline{p} : a \to o \\ \\ (F_p^2 - 1_o)\pi_o(t) \, , \, [F_p, \pi_o(t)] \in K(H_o) \end{cases}$$

- $\mathscr{F}_a(\mathcal{A})$: F.mods based on a: $\mathscr{F}_a(\mathcal{A}) \leftrightarrow \mathscr{F}_e(\mathcal{A})$.
- F *global* $\dot{\Leftrightarrow}$ $F_q = F$, $\forall q : a \to a \Rightarrow F_p = F_o \rightsquigarrow \mathscr{F}(\mathcal{A})$
- $\mathscr{F}_G(B, \beta)$: G-F.mods over (B, β).

Theorem (RV). Given $a \in \Delta$ and $\Pi := \pi_1(\Delta)$. Then

$$\mathscr{F}_a(\mathcal{A}) \leftrightarrow \mathscr{F}_\Pi(A_*, \alpha).$$

Index. By the previous theorem we have the map

$$\text{index} : \mathscr{F}_a(\mathcal{A}) \to R(\Pi) := KK^{\Pi}(\mathbb{C}, \mathbb{C}) \ .$$

In particular: $(\pi, F) \in \mathscr{F}(\mathcal{A}) \rightsquigarrow$
index$(\pi, F) \in$ the ring of u.f.d. reps $\rightsquigarrow$

But any u.f.d. rep u defines the lc Hermitian bundle $\mathscr{E} \to X$, $\mathscr{E} := \hat{X} \times_u H_u \rightsquigarrow$

CCS: $c_k(u) \in H^{2k-1}(X, \mathbb{R}/\mathbb{Z})$, $k \in \mathbb{N} \rightsquigarrow$

$$ccs : \mathscr{F}(\mathcal{A}) \to \mathbb{Z} \oplus H^{odd}(X, \mathbb{R}/\mathbb{Q}) \ , \ \ ccs(\cdot) = \sum_k c_k(\cdot)_{\mathbb{Q}} \frac{(-1)^k}{k!} \ .$$

Holonomy-equivariant KK-theory.

Aim: construct a KK-bifunctor for C^*-precosheaves.

We need the right notion of *Hilbert module* with coefficients in $\mathcal{B} = (B, \imath)_\triangle$.

Remark. B a $C_0(X)$-algebra, H a right B-module $\rightsquigarrow$ $H_Y := HB_Y$, $B_Y := C_0(Y)B \rightsquigarrow$

$$U_{Y'Y} : H_Y \to H_{Y'} \quad , \quad Y \subseteq Y' \ .$$

$\rightsquigarrow$ The maps $U_{Y'Y}$ are not adjointable.

To define the right notion we take the idea from the filtration $(B_{*,\bullet}, \alpha_\gamma)_\triangle$ of the holonomy C^*-system (B_*, α).

A *Hilbert $\mathcal{B}$-precosheaf* is a Π-Hilbert (B_*, α)-module (H, u) with a family $H_o \subseteq H$, $o \in \Delta$, such that:

$$\begin{cases} (H_o, H_o) = B_{*,o} \ , \ \ H_o B_{*,o} = H_o \ , \ \ \forall o \in \Delta \ , \\ \\ u_{\gamma(eo)}(H_o) B_{*,e} = H_e B_o^e \ , \ \forall o \leq e \ \ \Rightarrow \ \ u_{\gamma(eo)}(H_o) \subseteq H_e \ , \\ \\ \forall a \in \Delta \Rightarrow H = \mathrm{span}\{ u_{\gamma(p)}(H_o), p : o \to a \} \ , \end{cases}$$

where $B_o^e \triangleleft B_{*,e}$ is the ideal generated by $\alpha_{\gamma(eo)}(B_{*,o})$.

Example. $H = \ell^2 \otimes L^2\Pi \otimes B_*$, $u_q = id \otimes \lambda_q \otimes \alpha_q$, $q \in \Pi$. $\mathcal{B}$ ideal C^*-precosheaf, or with non-degenerate structure morphisms. $H_o := \ell^2 \otimes L^2\Pi \otimes B_{*,o}$, $\forall o \in \Delta$.

Remark. $\mathcal{B}$ C^*-net bundle $\Rightarrow$ $\mathbf{Hilb}(\mathcal{B}) \simeq \mathbf{Hilb}_\Pi B_*$

Kasparov $\mathcal{A}$-$\mathcal{B}$-module based on $a \in \Delta$: a pair $(\pi, F) \rightsquigarrow$

$$\begin{cases} \pi_o : A_o \to B(H) \ , \ \mathrm{ad}u_{\gamma(o'o)} \circ \pi_o = \pi_{o'} \circ \jmath_{o'o} \ , \ \pi_o(A_o)H_o \subseteq H_o \\ F = F^* \in B(H) \ , \ F_p H_o \subseteq H_o \ , \ F_p - F_{\overline{p}} \in K(H_o) \\ (F_p^2 - 1)\pi_o(t) \ , \ [F_p, \pi_o(t)] \in K(H_o) \end{cases}$$

$\forall o \leq o' \in \Delta$, $p, \overline{p} : a \to o$, $t \in A_o$.

$\rightsquigarrow$ Operator homotopy, degenerate modules, $\oplus$ ...
$\rightsquigarrow$ The *holonomy-equivariant group* $KK^{\Delta}(\mathcal{A}, \mathcal{B})$.
$\rightsquigarrow$ Evaluations maps

$$KK^{\Delta}(\mathcal{A}, \mathcal{B}) \to KK(A_o, B_o) \quad , \quad [\pi, F] \mapsto [\pi_o, F_p|_{H_o}] \ .$$

$\underline{\mathcal{B}\ C^*\text{-net bundle}}$ $\Rightarrow$ the filtration condition is redundant:

$$KK^{\Delta}(\mathcal{A},\mathcal{B}) \simeq KK^{\sqcap}(A_*,B_*) \, .$$

$\mathcal{B} = \mathbb{C} \rightsquigarrow KK_i^{\Delta}(\mathcal{A},\mathbb{C}) \simeq K^i_{\sqcap}(A_*)$, $i = 0,1$.
$\mathcal{A} = \mathbb{C} \rightsquigarrow \sqcap$ discrete $\rightsquigarrow KK_0^{\Delta}(\mathbb{C},\mathbb{C}) \simeq K^0(C^*\sqcap)$.

Generic $\mathcal{B}$: the forgetful functor $\mathbf{Hilb}(\mathcal{B}) \to \mathbf{Hilb}_{\sqcap} B_*$ induces a natural transformation $KK^{\Delta} \to KK^{\sqcap} \rightsquigarrow$

$$KK^{\Delta}(\mathcal{A},\mathcal{B}) \to KK^{\sqcap}(A_*,B_*) \quad , \quad (\pi,F) \mapsto (\pi_*,F) \ .$$

$\underline{\Delta \text{ directed}} \rightsquigarrow \sqcap = \mathbf{0} \rightsquigarrow A_* \simeq \vec{A},\ B_* \simeq \vec{B} \rightsquigarrow$

$$KK^{\Delta}(\mathcal{A},\mathcal{B}) \ \to \ KK(\vec{A},\vec{B}) \ ,$$

not injective because of the filtration condition.

Non-simple C^*-algebras.

A a C^*-algebra $\rightsquigarrow$
Prim(A): spectrum with the *Jacobson topology* τA $\rightsquigarrow$
$\tau A \leftrightarrow$ the set of closed ideals of A.

Kirchberg, Meyer-Nest, ...: X a space $\rightsquigarrow$
X-*algebra*: a C^*-algebra A with a continuous map

$$\eta : \mathrm{Prim}(A) \to X \ .$$

$\rightsquigarrow$ $\mathbf{C^*\,alg}(X)$.

- X lc Hausdorff $\Rightarrow$ A $C_0(X)$-algebra (Dauns-Hofmann);
- η open $\Rightarrow$ A continuous C^*-bundle.

Δ: a base of *proper* subsets of $X \rightsquigarrow$
An order preserving map

$$\eta^* : \Delta \to \tau A \ , \ \eta^*(Y) := \eta^{-1}(Y) \ .$$

$\rightsquigarrow$ A_Y, $Y \in \Delta$: the closed ideal of A defined by $\eta^*(Y)$
$\rightsquigarrow$ $\mathcal{A} = (A, \jmath)_\Delta$ is a C^*-precosheaf

Given $\Pi := \pi_1(\Delta)$ we have the functor

$$\begin{cases} \mathbf{C^* alg}(X) \ \to \ \mathbf{C^* dyn}(\Pi) \ , \\ A \mapsto \mathcal{A} \mapsto (A_*, \alpha) \ . \end{cases}$$

(A_*, α) universal for reps π of the type

$$\pi_Y : A_Y \to BH \quad : \quad \pi_{Y'} | A_Y = \mathrm{ad} U_{\gamma(Y'Y)} \circ \pi_Y \ .$$

A, B X-algebras $\rightsquigarrow$ the *holonomy-equivariant* KK*-theory*

$$KK^{\Delta}(A,B) \ := \ KK^{\Delta}(\mathcal{A},\mathcal{B}) \, .$$

<u>Δ *directed*</u> $\rightsquigarrow$ a natural transformation $KK_X \to KK^{\Delta}$ where KK_X is the Kirchberg-Kasparov KK-bifunctor $\rightsquigarrow$

$$KK_X(A,B) \ \to \ KK^{\Delta}(A,B) \, .$$

<u>*In general*</u>: we can consider

$$KK^{\Delta}(\mathcal{A},\mathcal{B}) \ \simeq \ KK^{\sqcap}(A_*,B_*) \ ,$$

where A is an X-algebra and $\mathcal{B}$ *is a* C^**-net bundle* (KK_X is difficult to compute).

Concluding remarks.

An application to QFT: a generalized *statistical dimension* for sectors in curved spacetimes: $\mathcal{F}$, $\mathcal{A} = \mathcal{F}^G$

$\rightsquigarrow$

$$Z^1(\mathcal{UA}) \to KK^{\beta\Pi\times G}(\mathbb{C},\mathbb{C}) \to \mathbb{Z} \oplus H^{odd}(M,\mathbb{R}/\mathbb{Q})$$

(!) No supercharges are needed.

Work in progress: The Kasparov product for KK^{Δ}.

$\rightsquigarrow$ KK-equivalences

$\rightsquigarrow$ Classification results for C^*-precosheaves and X-algebras.

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