Noncommutative Friedmann-Walker spacetimes, quantum field theory and the Einstein equations

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Introduction

- The DFR model: spacetime and fields
- Friedmann expanding backgrounds
- Friedmann expanding n.c. spacetimes
- Quantum fields and Friedmann equations

The DFR proposal

DFR (1995): "Grav. stability under localization experiments": Determining the localization of a quantum field theoretic observable needs concentration of energy in a region of the size of the uncertainty; extreme precision should cause the formation of a black hole.

The following program was outlined:

- Derive physically meaningful uncertainty relations between coordinates of spacetime events from gravitational stability under localization experiments.
- Promote these coordinates to the status of operators and find commutation relations among them from which the uncertainty relations follow.
- Construct quantum fields over the resulting noncommutative spacetime.

Starting point: fixed classical background, to be recovered by some $L_P \to 0$ procedure.

Noncommutative Minkowski space (DFR)

Spacetime uncertainty relations (STUR) derived by the linear approximation:

$$c\Delta t\left(\Delta x^1+\Delta x^2+\Delta x^3\right)\geq L_P^2$$

$$\Delta x^1\Delta x^2+\Delta x^1\Delta x^3+\Delta x^2\Delta x^3\geq L_P^2$$

From here, commutation relations (in principle, highly non unique!):

$$[x_\mu,x_\nu]=iL_P^2 Q_{\mu\nu},\qquad x_\mu=x_\mu^*$$

One can show that the STUR are satisfied using the "Quantum conditions"

$$[x_\mu,Q_{\nu\rho}]=0,\qquad Q^{\mu\nu}Q_{\mu\nu}=0,\qquad (Q^{\mu\nu}(*Q)_{\mu\nu})^2=16I.$$

The x_μ generate a C^*-algebra E, (some of) its states are our n.c. Minkowski. Covariance is granted by the following action of the (full) Poincaré group P:

$$\alpha_{(\Lambda,a)}(x_\mu)=\Lambda_\mu^\nu x_\nu+a_\mu I,\qquad \alpha_{(\Lambda,a)}(Q_{\mu\nu})=\Lambda_\mu^{\mu'}\Lambda_\nu^{\nu'}Q_{\mu'\nu'}.$$

Quantum fields on n.c. Minkowski space

A quantum field Φ on the quantum spacetime is defined by

$$\Phi(x) = \int_{R^4} dk\, e^{ikx} \otimes \hat{\Phi}(k).$$

It is a map from states on E to smeared field operators,

$$\omega \to \Phi(\omega) = \langle \omega \otimes I, \Phi(x) \rangle = \int_{R^4} dx\, \Phi(x)\psi_\omega(x).$$

The r.h.s. is a quantum field on the ordinary spacetime, smeared with ψ_ω defined by $\hat{\psi}_\omega(k) = \langle \omega, e^{ikx} \rangle$. If products of fields are evaluated in a state, the r.h.s. will in general involve non-local expressions. One has

$$[\Phi(\omega), \Phi(\omega')] = i \int d^4x d^4y\, \Delta(x-y)\psi_\omega(x)\psi_{\omega'}(y).$$

Thus (smeared) non commutative quantum fields are functions from a quantum spacetime to a C^*-algebra F (analogue to the one generated by ordinary fields) and are described by elements affiliated to $E \otimes F$. Knowledge of the classical commutator entails knowledge of its n.c. spacetime counterpart one.

Curved spacetimes and n.c. Einstein's equations

Problem: generalise the above construction to curved spacetimes. Big problem: make sense of "n.c. Einstein equations"

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}(\Phi) \qquad F(\Phi) = 0,$$
$$[x_\mu, x_\nu] = iQ_{\mu\nu}(g).$$

Friedmann flat expanding spacetimes with metric (comoving coordinates)

$$ds^2 = dt^2 - a(t)^2(dx_1^2 + dx_2^2 + dx_3^2).$$

Combination of mathematical simplicity (due to symmetry) and physical relevance (cosmological models).

Uncertainty relations for FFE spacetimes (comoving coordinates)

- Black holes do not form if the (positive) excess of proper mass-energy δE inside a two-surface S of proper area ΔA contained in a slice of constant universal time t_0 satisfies the inequality:

$$\sqrt{\Delta A}\left(\frac{1}{4\sqrt{\pi}}+\frac{H\sqrt{\Delta A}}{4\pi c}\right)\geq\frac{G}{c^4}\delta E.$$

where $H(t)=a'(t)/a(t)$ is the Hubble parameter $(a, a'>0)$.

For a box-like localisation region with comoving edges $\Delta x_1^c, \Delta x_2^c, \Delta x_3^c$,

$$\Delta A=a^2(t)(\Delta x_1^c\Delta x_2^c+\Delta x_1^c\Delta x_3^c+\Delta x_2^c\Delta x_3)=a^2(t)\Delta A^c.$$

Estimate δE making use of Heisenberg's uncertainty relations and get

$$a^2(t)\Delta A_c\left(\frac{1}{4\sqrt{3}}+\frac{a'(t)\sqrt{\Delta A_c}}{12c}\right)\geq\frac{\lambda_P^2}{2},$$

$$c\Delta t\cdot\sqrt{\Delta A_c}\min_{t\in\Delta t}\left\{a(t)\left(\frac{1}{4\sqrt{3}}+\frac{a'(t)\sqrt{\Delta A_c}}{12c}\right)\right\}\geq\frac{\lambda_P^2}{2}.$$

Solve the first inequality with respect to the comoving area ΔA_c gives

$$\Delta A_c \geq f\left(a(t), a'(t)\right),$$

with $f_1 = (x_0 - c\sqrt{3}a/a')^2$ and x_0 is the greatest solution of a certain cubic equation from which one has

$$c\Delta t \cdot \sqrt{\Delta A_c} \geq \frac{\lambda_P^2}{2} \max_{t \in \Delta t} \left\{a(t)\Delta A_c\right\} \geq \frac{\lambda_P^2}{2} \max_{t \in \Delta t} \left\{a(t) f\left(a(t), a'(t)\right)\right\}.$$

The corresponding *quantum uncertainty relations* are:

$$\Delta_\omega A_c \geq \frac{\lambda_P^2}{2} |\omega(f)|,$$

$$c\Delta_\omega t\left(\Delta_\omega x_1 + \Delta_\omega x_2 + \Delta_\omega x_3\right) \geq \frac{\lambda_P^2}{2} |\omega(af)|.$$

Uncertainty relations for FFE spacetimes (conformal coordinates)

If we work in conformal coordinates, an analogous procedure and the approximation

$$\Delta t = \frac{\Delta t}{\Delta \tau} \Delta \tau \simeq a(t(\tau)) \Delta \tau$$

gives

$$\Delta_\omega A_c \geq \frac{\lambda_P^2}{2} |\omega(f)|, \tag{1}$$

$$c \Delta_\omega \tau \left(\Delta_\omega x_1 + \Delta_\omega x_2 + \Delta_\omega x_3 \right) \geq \frac{\lambda_P^2}{2} |\omega(f)|. \tag{2}$$

where the function f is the same as before.

Building n.c. FFE spacetimes

Definition. A C^*-algebra E of operators with (self adjoint?) generators x_μ, $\mu = 0, \ldots, 3$ affiliated to it, is said to be a concrete covariant realisation of the n.c. spacetime M corresponding to the (classical) spacetime M with global isometry group G if:

1) the relevant STUR are satisfied;
2) there is a (strongly continuous) unitary representation of the global isometry group G under which the operators η transform as their classical counterparts (covariance);
3) there is some reasonable classical limit procedure for $L_P \to 0$ such that the η_μ become in an appropriate sense commutative coordinates on some space containing the manifold M as a factor.

For FFE (De Sitter exluded) $G = SO(3) \ltimes R^3$. By isotropy and homogeneity, we restrict attention to c.r. of the form ($x_0 = t$,or $x_0 = \tau$, and $\iota = 1, \ldots, n$)

$$[x_\mu, x_\nu] = Q(t, X^\iota)_{\mu\nu}, \quad [x_\mu, X^\iota] = 0, \quad [X^\iota, X^\iota] = 0.$$

4) The generators of the Friedmann spacetime algebra have commutation relations of the form above.
5) We should in some suitable sense recover the DFR model in the limit $a \to 1$.

Items 3) and 5) will not be addressed.

The assumption that the Q's only depend on t (or τ), combined with covariance, has far reaching consequences.

Proposition. Let the generators $t, \mathbf{x}, X^\iota$ satisfy 1) and 3) and the components of the two-tensor $Q(t, X^\iota)$ be regular functions of the comm. variables (t, X^ι). Then the corresponding commutation relations are of the form

$$[t, \mathbf{x}] = g_1(t)\mathbf{e}(X^\iota), \qquad [\mathbf{x}, \mathbf{x}] = \mathbf{m}(X^\iota) + \mathbf{m}_\perp(t, X^\iota), \tag{3}$$

with $\mathbf{m}_\perp(t, X^\iota) \cdot \mathbf{e}(X^\iota) = 0$ and some regular function g. Moreover, the operators $\mathbf{e}(X^\iota), \mathbf{m}(X^\iota), \mathbf{m}_\perp(t, X^\iota)$ transform as vectors under the action of the automorphism α_R, $R \in SO(3)$.

We set $\mathbf{m}_\perp(t, X^\iota) = g_2(t)\mathbf{m}_\perp$ and are left with two arbitrary (regular) functions g_1, g_2 and nine central generators assembled in a triple $\mathbf{e}, \mathbf{m}, \mathbf{m}_\perp$ of three-vectors, plus one orthogonality condition.

Proposition. Let $x_\mu, \mathbf{e}, \mathbf{m}, \mathbf{m}_\perp$ be as above. Suppose the *Quantum Conditions*

$$\mathbf{m} = 0, \quad \mathbf{e}^2 = I, \quad \mathbf{m}_\perp^2 = I,$$

are satisfied and let f the function on the right hand side of the conformal STUR above. Then, for any state $\omega \in E^*$ in the domain of $\tau, \mathbf{x}, \mathbf{e}, \mathbf{m}_\perp$, requirement 2) is met if $g_1(\tau) = g_2(\tau) = f(\tau)$.

Sketch of Proof. The operators $\mathbf{e}, \mathbf{m}_\perp$ being central with joint spectrum Σ, we perform the corresponding central decomposition of E^*. The proof relies on the inequalities

$$\Delta_\omega(x_\mu)\Delta_\omega(x_\nu) \geq \int_\Sigma \Delta_{\omega_\sigma}(x_\mu)\Delta_{\omega_\sigma}(x_\nu) d\mu_\omega(\sigma) \geq \frac{1}{2}\int_\Sigma |\omega_\sigma\left(Q_{\mu\nu}\right)| d\mu_\omega(\sigma). \tag{4}$$

which entail, for example,

$$\int_\Sigma \sum_{k=1}^{3} \Delta_{\omega_\sigma}(t)\Delta_{\omega_\sigma}(x_i) d\mu_\omega(\sigma) \geq \int_\Sigma \sqrt{\sum_{k=1}^{3} \Delta_{\omega_\sigma}(t)^2 \Delta_{\omega_\sigma}(x_i)^2 d\mu_\omega(\sigma)} \geq$$
$$\geq \frac{1}{2}\int_\Sigma ||\omega_\sigma(g_1(t)\mathbf{e})|| d\mu_\omega(\sigma) = \frac{1}{2}\int_\Sigma ||\mathbf{e}(\sigma)|| \cdot |\omega_\sigma(g_1(t))| d\mu_\omega(\sigma) \geq$$
$$\geq \frac{1}{2}\left|\int_\Sigma \omega_\sigma(g_1(t)) d\mu_\omega(\sigma)\right| = \frac{1}{2}|\omega(g_1(\tau))|.$$

Existence of covariant representations

To construct irrep. for the conf. coord., set $\sigma_0 = (e, m_\perp)$ with $e = (1, 0, 0), m_\perp = (0, 0, 1)$. Thus

$$Q(\tau, \sigma_0) = f\sigma_0, \quad Q(\tau, \sigma^{\mathrm{std}}) = f\sigma^{\mathrm{std}} = f \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $Q(\tau, \sigma^{\mathrm{st}d}) = AQ(\tau, \sigma_0)A^T$ for some invertible matrix A. Setting $\overline{x} = (\tau, \mathbf{x})$, we have $\left[(A\overline{x}^{\sigma_0})_\mu, (A\overline{x}^{\sigma_0})_\nu\right] = i\lambda^2 f(\overline{x}_0^{\sigma_0})\sigma_{\mu\nu}^{\mathrm{st}d}$. An irrep. is thus a suitable combination of two Schrödinger operators p, q on $L^2(R)$ and two central operators. Fixing a diffeomorphism $\gamma : R \to \mathrm{sp}(\tau)$, we must have

$$\overline{x}^{\sigma^{\mathrm{st}d}} \;=\; \left[[\gamma'(q)^{-1} f(\gamma(q)), p]_+, \gamma(q), aI, bI\right].$$

To obtain a covariant rep., set (as operators on $C_c^\infty(R^3)$)

$$\overline{x}_0^{\sigma_0} = \gamma_1(q_1), \quad \overline{x}_2^{\sigma_0} = \gamma_1(q_1) + p_2, \quad \overline{x}_3^{\sigma_0} = p_3,$$
$$\overline{x}_1^{\sigma_0} = \frac{1}{2}\left[\gamma_1'(q_1)^{-1} f(\gamma_1(q_1)), p_1\right]_+ .$$

and

$$\tilde{P}_1^{\sigma_0} = \left(\int \frac{\gamma_1'(s)\, ds}{f(\gamma_1(s))}\right)(q_1), \quad \tilde{P}_2^{\sigma_0} = q_2, \quad \tilde{P}_3^{\sigma_0} = q_3.$$

Consider the complex conjugate Hilbert space $\underline{K}$, elements $\underline{\phi} \in \underline{K}$ and linear operators $\underline{A}$ such that $\underline{A\phi} = \underline{A}\,\underline{\phi}$. On the Hilbert space $K \otimes \underline{K}$, set

$$\tau^\sigma = \tau^{\sigma_0}, \quad \overline{\mathbf{x}}^\sigma = R_\sigma \overline{\mathbf{x}}^{\sigma_0} \otimes I, \quad \overline{\mathbf{P}}^\sigma = R_\sigma^{-1}\tilde{\mathbf{P}} \otimes I + I \otimes R_\sigma^{-1}\underline{\tilde{\mathbf{P}}}, \qquad (5)$$

with $R_\sigma \in G = SO(3)$ such that $R_\sigma \sigma_0 = \sigma$. We can now define the Hilbert space $H = \int_G^\oplus K \otimes \underline{K} d\mu(R)$, with $d\mu(R)$ the Haar measure on G, and the operators

$$\overline{x}_\mu = \int_G^\oplus \overline{x}_\mu^\sigma d\mu(R), \quad \overline{\mathbf{P}} = \int_G^\oplus \overline{\mathbf{P}}^\sigma d\mu(R). \qquad (6)$$

Combined with the unitaries $(U(0,R)\phi)^\sigma = \phi^{R^{-1}\sigma}$ (τ is invariant):

$$U(\mathbf{a}, R)\overline{\mathbf{x}}U(\mathbf{a}, R)^{-1} = R\overline{\mathbf{x}} + \mathbf{a}, \quad U(\mathbf{a}, R) = U(\mathbf{a}, 0)U(0, R).$$

The C^*-algebra of the model

- According to noncommutative geometry, C^*-algebras describe *topological* noncommutative spaces.
- The topological space underlying the class of spacetimes under consideration is always R^4.

It is thus natural to assume that our C^*-algebra be the one of flat spacetime. Consider the Banach $*$-algebra $E_0 = C_0(\Sigma, L^1(R^4, d^4\alpha))$ with involution $f^*(\sigma, \alpha) = \overline{f(\sigma, \alpha')}$, norm $\sup_{\sigma \in \Sigma} ||f(\sigma, \cdot)||$ and product

$$(f \times g)(\sigma, \alpha) \doteq \int f(\sigma, \alpha')g(\sigma, \alpha - \alpha')e^{i\sigma(\alpha, \alpha')}d^4\alpha'.$$

Definition. The C^*-algebra E of the noncommutative Friedmann spacetimes is the C^*-closure of the Banach $*$-algebra E_0 with respect to its max. C^*-seminorm.
Proposition (Perini). The C^*-algebra E is isomorphic to $C_0(\Sigma \times R^2, \mathcal{K})$, where $\mathcal{K}$ is the C^*-algebra of the compact operators on a separable Hilbert space.
Remark.: any symmetric densely defined operator is affiliated to $\mathcal{K}$, thus our coordinates too. Some "moyalology" gives their expression for Weyl quantisation:

$$\overline{x}_\mu(q) = \int h(k)_\mu e^{ikq}.$$

for suitable $h(k)_\mu$, where the q's indicate the Heisenberg generators of E.

Quantum fields and n.c. Friedmann equations

We do not really know how to make sense of "n.c. Einstein equations", but for homogeneous isotropic spacetimes these reduce (with respect to the universal time t and $T_{\mu\mu} = (\rho, P, P, P)$) to

$$H = 8\pi\rho, \qquad 3(H' + H^2) = -4\pi(\rho + 3P)$$

Conservation of energy implies we can solve $-R = 8\pi T$ and consider the first equation as an initial condition. We thus rewrite the second equation as

$$R = 8\pi I \otimes \Omega(T_{00})$$

where Ω is a suitable state. But now the energy-momentum tensor explicitly depends on $a(t)$ through the commutation relations. Adding the equation

$$[x_\mu, x_\nu] = iL_P^2 Q_{\mu\nu}(a(t)),$$

we obtain a closed system of equations to solve for a. But how to define a quantum field?

Problem: we cannot use the naive definition for Minkowski case.
Reason: lack of time-translation invariance $\to$ no natural time Fourier transform.
The preceding discussion leads us to the following prescription: consider the commutative version of the $\overline{x}(q)$ as a coordinate transformation and define $\hat{\Phi}$ as the (ordinary, commutative) Fourier transform of the ordinary field Φ **with respect to the** q**'s** and take

$$\Phi(\tau, \overline{\mathbf{x}}) = \int_{R^4} d^4 k \, e^{ikq} \otimes \hat{\Phi}(k).$$

where the $\overline{x}$ are the DFR operators. This gives of course no creation/destruction operators but a bona fide object affiliated to $E \otimes F$.
Example: the two-point function of the free, scalar, conformally coupled field in suitable pure, homogeneous, quasi-free states is ($\bar{S}_k, S_k$ are known functions)

$$\Omega((\tau, \overline{\mathbf{x}}); (\tau', \overline{\mathbf{x}}')) = \frac{1}{8\pi^3} \int_{R^3} \frac{\bar{S}_{|\mathbf{k}|}(\tau)}{a(\tau)} \frac{S_{|\mathbf{k}|}(\tau')}{a(\tau')} e^{i\mathbf{k}\cdot(\overline{\mathbf{x}} - \overline{\mathbf{x}}')} d\mathbf{k}$$

We thus define (the q's are operators on the left hand side of the tensor product)

$$\Phi(\tau, \bar{\boldsymbol{x}}) = \frac{1}{4\pi^2} \int_{R^4} d^4k e^{ikq} \otimes \left[\int_{R^4} d^4 q' e^{-ikq'} \left(\int_{R^3} d^3 \boldsymbol{k}' a(\boldsymbol{k}') \frac{S_{|\boldsymbol{k}'|}(\tau(q'))}{a(\tau(q'))} e^{-i\boldsymbol{k}' \bar{\boldsymbol{x}}(q')} + \text{h.c.} \right) \right]$$

Remark. Outrageously preliminary calculations in a suitably chosen KMS state (we switch back to universal time) give $a(t) \simeq e^{ct}$ for small times!!

Short bibliography

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