Quantum symmetry groups of Hilbert C*-modules equipped with orthogonal filtrations

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- D. Goswami - *Quantum group of isometries in classical and noncommutative geometry.*
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  Comm. Math. Phys., 285(1):141–160 (2009)
- T. Banica & A. Skalski - *Quantum symmetry groups of C^*-algebras equipped with orthogonal filtrations.*
  Proc. Amer. Math. Soc., 106(5):980–1004 (2013)

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In other words: The symmetry group $\mathrm{Sym}(X)$ of a given space X is the group satisfying the universal property: for each group G acting on X, there exists a unique morphism $G \to \mathrm{Sym}(X)$.
It is given by:

$$\begin{array}{rcl} G & \to & \mathrm{Sym}(X) \\ g & \mapsto & (x \mapsto x.g) \end{array}$$

Analogously, the quantum symmetry group of a given space is defined as the final object in the category of quantum groups acting on this space.

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- Define the category of its quantum transformation groups.

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Steps for defining the quantum symmetry group of a given object:

- Define the category of its quantum transformation groups.
- Check that this category admits a final object.

Definition

Let A be a C^*-algebra and let E be a Hilbert A-module. An *orthogonal filtration* $(\tau, (V_i)_{i\in\mathcal{I}}, J, W)$ of E consists of:

- a faithful state τ on A,
- a family $(V_i)_{i\in\mathcal{I}}$ of finite-dimensional subspaces of E such that:
  1. for all $i, j \in \mathcal{I}$ with $i \neq j$, $\forall \xi \in V_i$ and $\forall \eta \in V_j$, $\tau(\langle \xi | \eta \rangle_A) = 0$,
  2. the space $\mathcal{E}_0 = \sum_{i\in\mathcal{I}} V_i$ is dense in $(E, \| \cdot \|_A)$,
- a one-to-one antilinear operator $J : \mathcal{E}_0 \to \mathcal{E}_0$,
- a finite-dimensional subspace W of E.

Example

Let M be a compact Riemannian manifold. The space of continuous sections of the bundle of exterior forms on M, $\Gamma(\Lambda^* M)$, is a Hilbert $C(M)$-module. A natural orthogonal filtration of $\Gamma(\Lambda^* M)$ is given by:

- $(V_i)_{i\in\mathbb{N}}$ *is the family of eigenspaces of the de Rham operator* $D = \overline{\mathrm{d} + \mathrm{d}^*}$,
- $\tau = \int \cdot \, \mathrm{d}vol$,
- $W = \mathbb{C}.(m \mapsto 1_{\Lambda^*_m M})$,
- $J : \Gamma(\Lambda^* M) \to \Gamma(\Lambda^* M)$ *is the canonical involution.*

Definition

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- $(\mathcal{A}, \mathcal{H}, D)$ is said to be *regular* if for all $a \in \mathcal{A}$ and all $n \in \mathbb{N}$, a and $[D, a]$ are in the domain of the unbounded operator δ^n on $\mathcal{L}(\mathcal{H})$, where $\delta = [|D|, \cdot\,]$.

Definition

$(\mathcal{A}, \mathcal{H}, D)$ *satisfies the finiteness and absolute continuity condition* if furthermore, the space $\mathcal{H}^\infty = \bigcap_{k \in \mathbb{N}} \mathrm{dom}(D^k)$ is a finitely generated projective left $\mathcal{A}$-module, and if there exists $q \in \mathcal{M}_n(\mathcal{A})$ with $q = q^2 = q^*$ such that:

1. $\mathcal{H}^\infty \cong \mathcal{A}^n q$,
2. the left $\mathcal{A}$-scalar product ${}_{\mathcal{A}}\langle \cdot | \cdot \rangle$ induced on $\mathcal{H}^\infty$ by the previous isomorphism satisfies:

$$\frac{Tr_\omega({}_{\mathcal{A}}\langle \xi | \eta \rangle |D|^{-p})}{Tr_\omega(|D|^{-p})} = (\eta | \xi)_{\mathcal{H}}.$$

Example

Setting:

- $A =$ *closure of* $\mathcal{A}$ *in* $\mathcal{L}(\mathcal{H})$,
- $E =$ *completion of* $\mathcal{H}^\infty$ *(for the* A*-norm)*,
- $(V_i)_{i\in\mathbb{N}} =$ *eigenspaces of* D,
- $\tau = a \mapsto \dfrac{Tr_\omega(a|D|^{-p})}{Tr_\omega(|D|^{-p})}$

If we assume furthermore that τ *is faithful and* $\mathcal{E}_0 = \sum_{i\in\mathcal{I}} V_i$ *is dense in* E*, we get an orthogonal filtration of* E *(with* $J : \mathcal{E}_0 \to \mathcal{E}_0$ *any one-to-one antilinear map and e.g.* $W = (0)$*).*

Definition

A *Woronowicz C^*-algebra* is a couple $(C(\mathbb{G}), \Delta)$, where $C(\mathbb{G})$ is a C^*-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a $*$-morphism such that:

- $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$,
- the spaces $\mathrm{span}\{\Delta(C(\mathbb{G})).(C(\mathbb{G}) \otimes 1)\}$ and $\mathrm{span}\{\Delta(C(\mathbb{G})).(1 \otimes C(\mathbb{G}))\}$ are both dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$.

Definition

Let $(C(\mathbb{G}), \Delta)$ be a Woronowicz C^*-algebra, and A be a C^*-algebra. A *coaction of* $C(\mathbb{G})$ *on* A is a $*$-morphism $\alpha : A \to A \otimes C(\mathbb{G})$ satisfying:

- $(\alpha \otimes id_{C(\mathbb{G})}) \circ \alpha = (id_A \otimes \Delta) \circ \alpha$
- $\alpha(A).(1 \otimes C(\mathbb{G}))$ is dense in $A \otimes C(\mathbb{G})$.

Definition

Let $(C(\mathbb{G}), \Delta, \alpha)$ be a Woronowicz C^*-algebra coacting on a C^*-algebra A, and let E be a Hilbert A-module. A *coaction of* $C(\mathbb{G})$ *on* E is a linear map $\beta : E \to E \otimes C(\mathbb{G})$ satisfying:

- $(\beta \otimes id) \circ \beta = (id \otimes \Delta) \circ \beta$
- $\beta(E).(A \otimes C(\mathbb{G}))$ is dense in $E \otimes C(\mathbb{G})$
- $\forall \xi, \eta \in E, \langle \beta(\xi) | \beta(\eta) \rangle_{A \otimes C(\mathbb{G})} = \alpha(\langle \xi | \eta \rangle_A)$
- $\forall \xi \in E, \forall a \in A, \beta(\xi.a) = \beta(\xi).\alpha(a)$

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- $\forall \xi, \eta \in E, \langle \beta(\xi) | \beta(\eta) \rangle_{A \otimes C(\mathbb{G})} = \alpha(\langle \xi | \eta \rangle_A)$
- $\forall \xi \in E, \forall a \in A, \beta(\xi.a) = \beta(\xi).\alpha(a)$

We say that the coaction (α, β) of $C(\mathbb{G})$ on E is *faithful* if there exists no nontrivial Woronowicz C^*-subalgebra $C(\mathbb{H})$ of $C(\mathbb{G})$ such that $\beta(E) \subset E \otimes C(\mathbb{H})$.

Definition

Let E be a Hilbert A-module endowed with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$. A *filtration-preserving coaction* of a Woronowicz C^*-algebra $C(\mathbb{G})$ on E is a coaction (α, β) of $C(\mathbb{G})$ on E satisfying:

- $(\tau \otimes id) \circ \alpha = \tau(\cdot) 1_{C(\mathbb{G})}$,
- $\forall i \in \mathcal{I}, \beta(V_i) \subset V_i \odot C(\mathbb{G})$,
- $(J \otimes *) \circ \beta = \beta \circ J$ on $\mathcal{E}_0$,
- $\forall \xi \in W, \beta(\xi) = \xi \otimes 1_{C(\mathbb{G})}$.

Definition

- We say that a Hilbert A-module E is *full* if the space $\langle E|E\rangle_A = \mathrm{span}\{\langle\xi|\eta\rangle_A \ ; \ \xi, \eta \in E\}$ is dense in A.
- If $(\alpha_{\mathbb{G}}, \beta_{\mathbb{G}})$ and $(\alpha_{\mathbb{H}}, \beta_{\mathbb{H}})$ are filtration preserving coactions of Woronowicz C^*-algebras $C(\mathbb{G})$ and $C(\mathbb{H})$ on E, then a morphism from $C(\mathbb{G})$ to $C(\mathbb{H})$ is a morphism of Woronowicz C^*-algebras $\mu : C(\mathbb{G}) \to C(\mathbb{H})$ satisfying:

$$\alpha_{\mathbb{H}} = (id_A \otimes \mu) \circ \alpha_{\mathbb{G}} \quad \text{and} \quad \beta_{\mathbb{H}} = (id_E \otimes \mu) \circ \beta_{\mathbb{G}}.$$

Theorem

Let A be a C^*-algebra and let E be a full Hilbert A-module endowed with an orthogonal filtration $(\tau, (V_i)_{i \in \mathcal{I}}, J, W)$. There exists a universal Woronowicz C^*-algebra coacting on E in a filtration-preserving way. The quantum group corresponding to that universal object is called the ***quantum symmetry group*** *of $(E, \tau, (V_i)_{i \in \mathcal{I}}, J, W)$.*

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This generalizes and unifies the universal objects constructed by Banica-Skalski and Goswami.

Objective: Find a Woronowicz C^*-algebra $(C(\mathbb{G}_u), \Delta_u, \alpha_u, \beta_u)$ coacting on E in a filtration preserving way, and such that for each $(C(\mathbb{G}), \Delta, \alpha, \beta)$ coacting on E in a filtration preserving way, there exists a unique morphism $C(\mathbb{G}_u) \to C(\mathbb{G})$.

We define on $\mathcal{E}_0$ a right and a left scalar product by:

$$(\xi|\eta)_\tau = \tau(\langle \xi|\eta\rangle_A)$$
$$_\tau(\xi|\eta) = \tau(\langle J(\xi)|J(\eta)\rangle_A).$$

For each $i \in \mathcal{I}$ we fix:

- an orthonormal basis $(e_{ij})_{1\leqslant j\leqslant d_i}$ of V_i for the right scalar product $(\cdot|\cdot)_\tau$,
- an orthonormal basis $(f_{ij})_{1\leqslant j\leqslant d_i}$ of V_i for the left scalar product $_\tau(\cdot|\cdot)$.

We denote by $p^{(i)} \in GL_{d_i}(\mathbb{C})$ the change of basis matrix from (f_{ij}) to the basis (e_{ij}) of V_i and we set $s^{(i)} = p^{(i)t}\overline{p^{(i)}}$.

Lemma

Let (α, β) be a filtration preserving coaction of a Woronowicz C^*-algebra $C(\mathbb{G})$ on E. For all $i \in \mathcal{I}$, since $\beta(V_i) \subset V_i \odot C(\mathbb{G})$, there exists a multiplicative matrix $v^{(i)} = (v_{kj}^{(i)})_{1 \leqslant k,j \leqslant d_i}$ such that:

$$\forall j, \beta(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes v_{kj}^{(i)}.$$

Then the matrix $v^{(i)}$ is unitary and

$$v^{(i)t} s^{(i)} \overline{v^{(i)}} \left(s^{(i)}\right)^{-1} = s^{(i)} \overline{v^{(i)}} \left(s^{(i)}\right)^{-1} v^{(i)t} = I_{d_i}$$

For all $i \in \mathcal{I}$, we consider $\mathcal{A}_u(s^{(i)})$ the universal Woronowicz C^*-algebra generated by a multiplicative and unitary matrix $u^{(i)} = (u_{kj}^{(i)})_{1 \leqslant k,j \leqslant d_i}$, satisfying the following relations:

$$u^{(i)t} s^{(i)} \overline{u^{(i)}} (s^{(i)})^{-1} = s^{(i)} \overline{u^{(i)}} (s^{(i)})^{-1} u^{(i)t} = I_{d_i}.$$

We set $\mathcal{U} = \mathop{\ast}\limits_{i \in \mathcal{I}} \mathcal{A}_u(s^{(i)})$ and $\beta_u : \mathcal{E}_0 \to \mathcal{E}_0 \odot \mathcal{U}$ the linear map given by:

$$\beta_u(e_{ij}) = \sum_{k=1}^{d_i} e_{ik} \otimes u_{kj}^{(i)}.$$

Lemma

Let (α, β) be a faithful filtration preserving coaction of a Woronowicz C^*-algebra $C(\mathbb{G})$ on E. There exists a Woronowicz C^*-ideal $I \subset \mathcal{U}$ and a faithful filtration preserving coaction (α_I, β_I) of $\mathcal{U}/I$ on E such that:

- $\mathcal{U}/I \cong C(\mathbb{G})$,
- β_I *extends* $(id \otimes \pi_I) \circ \beta_u$.

If $C(\mathbb{G}) \cong \mathcal{U}/I$ and $C(\mathbb{H}) \cong \mathcal{U}/J$ with $I \subset J$, then there exists a unique morphism $C(\mathbb{G}) \to C(\mathbb{H})$.

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We have to set $C(\mathbb{G}_u) = \mathcal{U}/I$ where I is the smallest C^*-ideal such that there exists a filtration preserving coaction (α_I, β_I) of $\mathcal{U}/I$ on E.

Example ($\mathbb{C}^n$)

The quantum symmetry group of the Hilbert $\mathbb{C}$*-module* $\mathbb{C}^n$ *equipped with the orthogonal filtration* $(id_{\mathbb{C}}, (\mathbb{C}^n), J, (0))$ *where* $J : \mathbb{C}^n \to \mathbb{C}^n$ *is any invertible antilinear map, is* $\mathcal{A}_o(J) = < (u_{ij})_{1 \leqslant i,j \leqslant n}$ *unitary* $;\ u = J\overline{u}J^{-1} >$.

Example

We set $I = [0,1]$ *and we denote by* $\delta_+ : L^2(I) \to L^2(I)$ *the operator* $\frac{d}{dx}$ *with domain:*

$$dom(\delta_+) = \{f \in H^1(I) \ ; \ f(0) = f(1) = 0\}.$$

Its adjoint operator is $\delta_- = -\frac{d}{dx}$ *with domain* $H^1(I)$*.*
We define $D_0 : L^2(\Lambda^*(I)) \to L^2(\Lambda^*(I)) \cong L^2(I) \oplus L^2(I)$ *by:*

$$D_0 = \begin{pmatrix} 0 & \delta_- \\ \delta_+ & 0 \end{pmatrix}.$$

The quantum symmetry group of the Hilbert module associated with (A, H, D) *where* $A = C(I)^n \cong C([0,1] \times \{1, \ldots, n\})$*,* $H = L^2(\Lambda^*(I))^n$ *and* $D = diag(D_0, \ldots, D_0)$*, is the hyperoctahedral quantum group* $\mathcal{A}_h(n) = C(H_n^+)$*.*