Groupoids and Pseudodifferential calculus II

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Let's say that $0 \to I \to A \to A/I \to 0$ and $0 \to J \to B \to B/J \to 0$ are Morita equivalent exact sequences.

Statement

Let $G \rightrightarrows G^{(0)}$ be a smooth groupoid and denote by $\mathfrak{A}G$ its Lie algebroid. Claire defined exact sequences:

- Pseudo differential operators exact sequence

$$0 \to C^*(G) \longrightarrow \Psi_0^*(G) \xrightarrow{\sigma_0} C(S^*\mathfrak{A}G) \to 0 \qquad \text{(PDO)}$$

- Gauge adiabatic groupoid exact sequence :

$$0 \to C^*(G) \otimes \mathcal{K} \longrightarrow J(G) \rtimes \mathbb{R}_+^* \longrightarrow C(S^*\mathfrak{A}G) \otimes \mathcal{K} \to 0 \qquad \text{(GAG)}$$

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Moreover, the corresponding Morita equivalences of the ideals is the canonical one, as well as that of the quotients.

The bimodule $\mathcal{E}$

It is the closure of $\mathcal{J}(G)$ with respect to the $\Psi^*(G)$-valued "scalar" product

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For $f \in \mathcal{J}(G)$ and P order 0 (classical) pseudodifferential operator in G with compact support, $h = (f_t * P) \in \mathcal{J}(G)$

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To see this, we write $P = \int_0^\infty g_t \frac{dt}{t}$, and use the adiabatic groupoid of the adiabatic groupoid...

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$\mathcal{J}(G)$ is an ideal in $\mathcal{S}(G_{ad})$. We thus get an action of $\mathcal{S}(G_{ad})$ on $\mathcal{J}(G)$: $f = (f_t) \in \mathcal{S}(G_{ad})$ and $g \in \mathcal{J}(G)$: $(f \cdot g)_t = f_t * g_t$.

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Covariant representation

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Covariant representation: thus $\pi : C^*(G_{ad}) \rtimes \mathbb{R}_+^* = C^*(G_{ga}) \to \mathcal{L}(\mathcal{E})$.

Proof of Morita equivalence

Claim

$\pi(J(G) \rtimes \mathbb{R}_+^*) = \mathcal{K}(\mathcal{E})$.

Put $\mathcal{E}_0 = \mathcal{E}C^*(G)$. It is a closed submodule of $\mathcal{E}$. We prove that:

1. $\mathcal{E}_0 \simeq C^*(G) \otimes L^2(\mathbb{R}_+^*)$ and π induces isomorphism from $J_0(G) \rtimes \mathbb{R}_+^*$ onto $\mathcal{K}(\mathcal{E}_0)$ - natural Morita equivalence between $C^*(G) \otimes C_0(\mathbb{R}_+^*) \rtimes \mathbb{R}_+^*$ and $C^*(G)$.

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3. $\pi(J(G) \rtimes \mathbb{R}_+^*) \supset \mathcal{K}(\mathcal{E})$. Use again the adiabatic groupoid of the adiabatic groupoid...

The claim follows.

$\mathcal{E}$ is full

Since $\mathcal{E}_0$ is a full $C^*(G)$-module and and $\mathcal{E}/\mathcal{E}_0$ is a full $C(S^*\mathfrak{A}G)$-module, it follows that $\mathcal{E}$ is a full $\Psi^*(G)$-module.

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Pseudodifferential operators on G are smoothing operators on G_{ga}.

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Can we deduce that $J(G) \rtimes \mathbb{R}_+^* \simeq \Psi^*(G) \otimes \mathcal{K}$?

The stability of the ideal $J_0(G) \rtimes \mathbb{R}_+^*$ and of the quotient $J(G) \rtimes \mathbb{R}_+^* / C^*(G \times \mathbb{R}_+^*) \rtimes \mathbb{R}_+^*$ is not enough!

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Idea of proof. Invertible positive pseudodifferential operator D_1 of degree 1 on G unbounded multiplier of $C^*(G)$ (Vassout).
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D_1^{it} multiplier of $C^*(G)$. Therefore the restriction of α_t to $C^*(G)$ inner.

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D_1^{it} multiplier of $C^*(G)$. Therefore the restriction of α_t to $C^*(G)$ inner.

The identity at the quotient level.

Further topics: 1. Intrepretation of $C^*(G_{ad})$

We have interpreted $J(G) \subset C^*(G_{ad})$ as a crossed product.

Question

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Question

Can one interpret $C^*(G_{ad})$ in these terms?

Recall:

Construction (Baaj)

Let α be an action of a Lie group H act on a C^*-algebra A.
Pseudodifferential extension of $A \rtimes H$. (Lie algebra $\mathfrak{H}$).

$$0 \to A \rtimes H \longrightarrow \Psi_0^*(A, G) \xrightarrow{\sigma} C(S^*\mathfrak{H}) \otimes A \to 0.$$

Double pseudodifferential extension...

We prove:

Proposition

Commuting diagram, whose first line is Baaj's exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Psi^*(G) \rtimes \mathbb{R}_+^* & \longrightarrow & \Psi^*(\Psi^*(G), \mathbb{R}) & \xrightarrow{\sigma} & \Psi^*(G) \oplus \Psi^*(G) & \longrightarrow & 0 \\
 & & \pi \uparrow \simeq & & \uparrow & & \uparrow \mu_0 & & \\
0 & \longrightarrow & J(G) & \longrightarrow & C^*(G_{ad}) & \longrightarrow & C(G^{(0)}) & \longrightarrow & 0
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Where $\mu_0(f) = (\mu(f), 0)$

$\mu : C_0(G^{(0)}) \to \Psi^*(G)$ *inclusion by multiplication operators.*

2.a) Conic gauge groupoid

(This is related with work of Melo, Schick, Schrohe)
Our construction extends to groupoids with boundaries:

M compact manifold with boundary. Consider M as included in a manifold $\widetilde{M}$ without boundary such as its double - could be non compact.

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Let $\widetilde{G}$ be a smooth groupoid such that $\widetilde{G}^{(0)} = \widetilde{M}$. Assume ∂M transverse to $\widetilde{G}$ (*i.e.* $\mathfrak{A}G_x + T_x\partial M = T_x\widetilde{M}$ - for all $x \in \partial M$).

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Then the restriction G to ∂M of $\widetilde{G}$ is a smooth groupoid and has a neighborhood of the form $G \times (\mathbb{R} \times \mathbb{R})$ in $\widetilde{G}$. Can then form a "conic gauge groupoid" G_{cg}, with objects M by gluing G_{ga} with the restriction $\mathring{\widetilde{G}}$ of $\widetilde{G}$ to $\mathring{M}$.

2.b) Symbol algebra and index

The algebra $\Psi^*(G_{cg})$ contains as an ideal $C^*(\widetilde{\mathring{G}})$.

We then define:

- The full symbol algebra $\Sigma_f = \Psi^*(G_{cg})/C^*(\widetilde{\mathring{G}})$.
- The corresponding connecting map $ind_{cg} : K_*(\Sigma_f) \to K_{*+1}(C^*(\widetilde{\mathring{G}}))$.

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Let $\Psi_{\partial M}(M)$ be pseudodifferential operators on M that become scalar in ∂M (in other words $\Psi_{\partial M}(\widetilde{G}_M) = \Psi_0^*(\widetilde{\mathring{G}}) + C(M)$).

Corresponding symbols on M that become trivial on ∂M:

$$\Sigma_t = \Psi_{\partial M}(\widetilde{G}_M)/C^*(\widetilde{\mathring{G}}) = C(\partial M \cup S^*\mathfrak{A}G_{|\mathring{M}}).$$

2.b) Symbol algebra and index (2)

Theorem

1. *The inclusions* $\Psi_{\partial M}(\widetilde{G}_M) \subset \Psi^*(G_{cg})$ *and* $\Sigma_t \subset \Sigma_f$ *induce* KK*-equivalences.*

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Theorem

1. *The inclusions* $\Psi_{\partial M}(\widetilde{G}_M) \subset \Psi^*(G_{cg})$ *and* $\Sigma_t \subset \Sigma_f$ *induce* KK*-equivalences.*
2. *Natural exact sequence*
   $0 \to C_0(\mathfrak{A}^* G_{|\mathring{M}}) \to C(B^*\mathfrak{A} G_{|\mathring{M}} \cup \partial M) \to \Sigma_t \to 0.$

   Then ind_{cg} *is the composition of:*
   - *the* KK*-inverse of the inclusion* $\Sigma_t \subset \Sigma_f$*;*
   - *the connecting map* $\in KK^1(\Sigma_t, C_0(\mathfrak{A}^* G_{|\mathring{M}}))$ *of this exact sequence;*
   - *the index map* $\in KK(C_0(\mathfrak{A}^* G_{|\mathring{M}}), C^*(\widetilde{\mathring{G}}))$ *of the groupoid* $\widetilde{\mathring{G}}$*.*

2.c) Boutet de Monvel calculus. Recall...

As above: M compact manifold with boundary. Consider M as included in a manifold $\widetilde{M}$ without boundary.

Boutet de Monvel defines:

- Pseudodifferential operators on $\widetilde{M}$ with the transmitting property (acting on fonctions on M).

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  Let χ_M be the characteristic function of M.

Definition

A pseudodifferential operator P (with compact support) on $\widetilde{M}$ is said to have the transmitting property if for every smooth function $\tilde{f}$ on $\widetilde{M}$, then $P(\chi_M \tilde{f})$ coincides on $\mathring{M}$ with a smooth function on $\widetilde{M}$.

Can be described in terms of the (restriction to ∂M of the) total symbol of P.

Let $P_+(f)$ be the function on M which coincides with $P(\chi_M \tilde{f})$ on $\mathring{M}$.

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The difference: a singular Green operator.

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  They define a Morita equivalence between singular Green operators on M and ordinary pseudodifferential operators on its boundary.

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4. The algebra $\Psi^0_{BM}(G)$ of order zero Boutet de Monvel operators is identified as a subalgebra of $\Psi^*(G_{cg})$.
5. The inclusion $\Psi^0_{BM}(G) \subset \Psi^*(G_{cg})$ induces an isomorphism in K-theory, and (by 2.b) we recover the Boutet de Monvel index theorem.

Papers:

(All joint with Claire Debord)

[1] Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and Pseudodifferential calculus.
Adv. Math 2014
http://math.univ-bpclermont.fr/∼debord/

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Thank you!

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Recall $J(G) \subset C^*(G_{ad})$: the kernel of $C^*(G_{ad}) \to C(G^{(0)})$ given as a composition

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$C^*(G \times \mathbb{R}_+^*) \rtimes \mathbb{R}_+^* \simeq C^*(G) \otimes \mathcal{K}$ and
$C_0(S^*\mathfrak{A}G \times \mathbb{R}_+^*) \rtimes \mathbb{R}_+^* \simeq C_0(S^*\mathfrak{A}G) \otimes \mathcal{K}$. Back