Influence of quantum matter fluctuations on the expansion parameter of timelike geodesics

Nicola Pinamonti

Dipartimento di Matematica
Università di Genova

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joint work with N. Drago, arXiv.1402.4265

Motivations

- At short distance the spacetime should be **non-commutative**.
- This feature should be encoded in the **"Quantum Gravity"**

No satisfactory description.

- We can get information about such a theory analyzing **particular regimes** *[Hawking]*.
- Gravity classically Matter by quantum theory.

$$G_{ab}(x) = 8\pi\langle T_{ab}(x)\rangle_\omega$$

- *Doplicher, Fredenhagen and Roberts 95* use this to obtain **uncertainty relations** for the coordinates on a **flat** quantum space.

Motivations

- **Semiclassical equations**: Quantum fields as source for classical ones, like:

$$G_{ab}(x) = \langle T_{ab}(x) \rangle \ .$$

- Fluctuations of $T_{ab}(x)$ **diverge**. Cannot be renormalized.

- **Smearing** is needed: $T_{ab}(f)$, $\langle T_{ab}(f)^n \rangle$ give the **probability dist**.

- However, smearing **brakes covariance**.

Solution: quantize the full theory.

- Intermediate step: **Langevin equation** (like Brownian motion). (**Passive** influence of the right side on the left one).

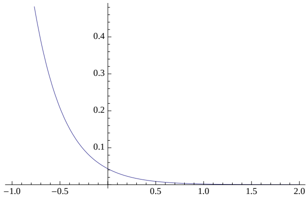
$$G_{ab} = T_{ab}$$

Two-dimensional model

- Carlip, Mosna and Pitelli **PRL** (2011)
  "Vacuum Fluctuations and the Small Scale Structure of Spacetime".
  - **Effective 2d dilatonic** model for gravity.
  - Analyze the probability of a geodesic collapse at small scales.
  - Expansion parameter of null geodesics.

$$\dot{\theta} + \frac{1}{2}\theta^2 = -T$$

- **Probability distribution** for a energy density in a 2d CFT.
  [Fewster Ford Roman 2010]
  - Mean value vanishes.
  - It is bounded from below.
  - There is a long positive tail.
  - Negative energies are more likely.



Motivations

The Raychaudhuri equation for **timelike** geodesics provides a simplified model:

$$\underbrace{\dot{\theta} + \frac{1}{3}\theta^2}_{geometry} = \ldots - (\underbrace{T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}}_{matter})\xi^\mu\xi^\nu$$

- It can be seen as a one-dimensional **non-linear** field theory.
- **Test of the ideas** in a simplified setting.
- Might provide hints on the underlying quantum gravity.

Plan of the talk

- Restriction of matter fields on timelike curves.
- Perturbative analysis of Raychaudhuri equation.
- Probability of focusing and some final comments on the arising probability distribution.
- Bounds for uncertainty of quantum coordinates.

This talk is based on

- N. Drago, NP, [arXiv.1402.4265] (2014).

- C.J. Fewster, L.H. Ford, T.A. Roman **PRD** (2010).
- S.Carlip, R.A.Mosna and J.P.M.Pitelli **PRL** (2011).
- S. Doplicher, G. Morsella, NP **JGP** (2013).

Matter fields - Restriction on timelike curves

- Massless minimally coupled scalar quantum field.

$$-\Box\varphi = 0$$

- The quantization is very well under control.

- The $*-$algebra generated by linear fields $\varphi(f)$, implementing:

$$\varphi^*(f) = \varphi(\overline{f}) \,, \qquad [\varphi(f), \varphi(h)] = i\Delta(f,h) \,, \qquad \varphi(\Box f) = 0 \,.$$

- Assign to every spacetime *[Brunetti Fredenhagen Verch]*

$$M \mapsto \mathcal{A}(M)$$

- Local non linear fields can be added to the algebra. *[Hollands Wald]*

Extended algebra of fields

Following *[Brunetti Fredenhagen Duetsch]*, $\mathcal{A}(M)$ algebra of functionals over smooth field configurations.

After deforming $\mathcal{A}(M)$ $\Delta \to -2iH$ it can be extended trivially.

$$\mathcal{F}(M) := \{F : \mathcal{E}(M) \to \mathbb{C} |\ F \text{ inf. diff. with compact support}, \\ WF(F^{(n)}) \cap (\overline{V}^n_+ \cup \overline{V}^n_-) = \emptyset\},$$

where the product is

$$F \star_H G := \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}, H^{\otimes n} G^{(n)} \rangle$$

H is an Hadamard parametrix, enjoying the microlocal spectrum condition.

Fields on timelike curves

- Let be $\gamma \subset M$ a smooth timelike curve.
- Not every element of $\mathcal{F}(M)$ can be restricted on γ:

$$\mathcal{F}(M) \ni F(\varphi) \to \int \varphi\delta(\gamma) f d\mu, \qquad F(\delta(\gamma)\varphi) \text{ diverges.}$$

- We can define fields intrinsically on γ

$$\mathcal{F}(\gamma) := \{F : \mathcal{E}(\gamma) \to \mathbb{C} | \ F \text{ inf. diff. with compact support,}$$
$$WF(F^{(n)}) \cap (\mathbb{R}^n_+ \cup \mathbb{R}^n_-) = \emptyset\},$$

$$F \star_h G := \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}, h^{\otimes n} G^{(n)} \rangle$$

being h a two-point function with $WF(h) \subset \mathbb{R}_+ \times \mathbb{R}_-$.

Connection with the spacetime theory

Question

Can we imbed $\mathcal{F}(\gamma)$ into $\mathcal{F}(M)$?

Yes because we can restrict

$$h = H \circ (\gamma \otimes \gamma) = H \cdot \delta(\gamma \otimes \gamma)$$

$WF(\delta(\gamma \otimes \gamma))$ contains only spatial directions.

Theorem

Let $\imath_\gamma : \mathcal{E}(M) \to \mathcal{E}(\gamma)$ *defined by* $\imath_\gamma \varphi := \varphi \circ \gamma$ *realizing the* **restriction of field configurations on** γ

Its **pullback** *imbed* $\mathcal{F}(\gamma) \subset \mathcal{F}(M)$: $\imath_\gamma^* \mathcal{F}(\gamma) \subseteq \mathcal{F}(M)$.

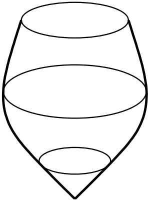
$$\imath_\gamma^* F \star_H \imath_\gamma^* G = \imath_\gamma^* (F \star_h G),$$

- It does **not work** on **light like** curves.

Raychaudhuri equation

- Consider a congruence of timelike geodesic $\mathcal{C}$.

  The **expansion parameter** θ measures the **rate of change** of $\frac{4}{3}\pi r^3$ along $\mathcal{C}$



  - $\theta > 0$ expansion
  - $\theta = 0$ parallel motion
  - $\theta < 0$ contraction

- Its evolution is governed by the **Raychaudhuri** equation

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu}\xi^\mu\xi^\nu,$$

$\omega_{\mu\nu}$: angular velocity of the geodesics;
$\sigma_{\mu\nu}$: deformation parameter;
ξ^μ : tangent vector of the geodesic.

Raychaudhuri equation - an example in cosmology

- Einstein equation can be used to evaluate $R_{\mu\nu}$.

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$$

- In the case of an expanding **flat FRW spacetime**

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2 \;, \qquad \theta(t) = 3H(t)$$

Raychaudhuri equation

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \xi^\mu \xi^\nu,$$

is equivalent to **Friedmann equations** (up to an initial condition).

Question

Can we treat fluctuations of the expansion parameter as fields in the matter algebra?

- The equation for ψ ($\theta = 3\dot{\psi}/\psi$) defined up to a scale.

$$\ddot{\psi} + \underbrace{\frac{1}{3}\left(\sigma_{\mu\nu}\sigma^{\mu\nu} - \omega^{\mu\nu}\omega_{\mu\nu} + T_{\mathrm{cl}}\right)}_{:=V}\psi + \frac{1}{3}\dot{\varphi}^2\psi = 0,$$

  We are interested in the fluctuations of ψ induced by the ones of φ.
- We shall use **perturbation theory** and test if ψ vanishes
  1. The fluctuations of $\omega_{\mu\nu}, \sigma_{\mu\nu}$ are negligible;
  2. The influence of ψ on φ is negligible.
- It is a one dimensional problem. It is a field theory on a line.

Retarded propagator of the theory

A **poor man** interacting quantum field theory.

$$\ddot{\psi} + V\psi + \frac{1}{3}\dot{\varphi}^2\psi = 0,$$

The solution is formally

$$\psi = \psi_0 + R_V(\dot{\varphi}^2\psi),$$

$R_V : \mathcal{D}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ the retarded propagator of $P_\gamma = -\frac{d^2}{dt^2} - V$ i.e.

$$R_V P_\gamma(f) = P_\gamma R_V(f) = f, \quad \mathrm{supp}(R_V f) \subseteq J^+(\mathrm{supp}(f)).$$

The integral kernel of R_V has the form

$$R_V(x,y) = \underbrace{S(x,y)}_{\in\mathcal{E}(\mathbb{R}^2)}\vartheta(x-y), \quad (R_V f)(x) = \int R_V(x,y)f(y)dy.$$

We look for a recursive solution.

Perturbative analysis: Yang-Feldman method

Solution as a **formal power series** in λ around a **free classical solution** ψ_0.

$$\psi(f) = \psi_0(f) + \lambda\psi_1(f) + \lambda^2\psi_2(f) + \dots$$

[Epstain, Glaser, Steinmann, Hollands, Wald, Brunetti, Duetsch, Fredenhagen] Choose $\lambda \in C_0^\infty(\gamma)$

$$\psi_n(f) = R_V(\lambda\dot{\varphi}^2\psi_{n-1})(f) \quad n = 1, 2, \dots$$

$$\psi_n(f) = \int f_R(x_{n-1})S(x_{n-1}, x_{n-2})\dots S(x_1, x_0)\lambda(x_{n-1})\dots\lambda(x_0)\cdot$$

$$\underbrace{\cdot\vartheta(x_{n-1} - x_{n-2})\dots\vartheta(x_1 - x_0)\dot{\varphi}^2(x_{n-1}) \star_h \dots \star_h \dot{\varphi}^2(x_0)}_{:=r(x_{n-1},\dots,x_0)}$$

- To solve it we need to consider ill defined $R_V(x, y) \cdot h(x, y)$.
- We want r for every possible $V \Longrightarrow$ we leave S out of r.
- **Small problem**, S is not symmetric $\Longrightarrow$ slightly modify the standard construction.

Construction of $r(x_n, \dots, x_0)$ in $\mathcal{F}(\gamma)$

The $r(x_n, \dots, x_0)$ are distributions with values in $\mathcal{F}(\gamma)$

1. **retardation 1:** if $x_n > \dots > x_0$ then
$$r(x_n, \dots, x_0) = \dot{\varphi}^2(x_n) \star_h \dots \star_h \dot{\varphi}^2(x_0);$$
2. **retardation 2:** if it does not hold that $x_n \geq \dots \geq x_0$ then
$$r(x_n, \dots, x_0) = 0;$$
3. **factorization**: if $x_n \geq \dots \geq x_0$ and $x_{m+1} > x_m, \ m \in \{1, \dots, n\}$, then
$$r(x_n, \dots, x_0) = r(x_n, \dots, x_{m+1}) \star_h r(x_m, \dots, x_0);$$
4. **initial element**: $r(x_0) = \dot{\varphi}^2(x_0)$.

Solution

The construction of r is an application of the recently developed **pAQFT**. *[Epstain, Glaser, Steinmann, Hollands, Wald, Brunetti, Duetsch, Fredenhagen, Rejzner]*

Inductive construction of r *[Epstain Glaser]* uses the previous general properties.

- We have the **initial element**.
- Suppose that you have all rs with $n-1$ entries then
  1. Construct $r(x_n, \ldots, x_0)$ outside the full diagonal $x_n = \ldots = x_0$ with the **factorization property**.
  2. Extend it to the full diagonal by means of **Steinmann** scaling degree tecniques *[Brunetti Fredenhagen]*.

In the last step there is the usual renormalization freedom expressed by a certain number of constants.

Adiabatic limit

- With those r we can obtain $\psi_n(f) \in \mathcal{F}(\gamma)$ for every n.
- The **last step** is the analysis of the limit $\lambda \to 1$ (in $\mathcal{F}(\gamma)$).
- It can be performed in $\mathcal{F}(\gamma)$ because the equation for ψ is linear in ψ and we smear ψ with compactly supported smooth function f.
- Formally we can split $\psi = \psi^+ + \psi^-$

$$\ddot{\psi}^\pm + V\psi^\pm + \frac{1}{3}\dot{\varphi}^2\psi^\pm = \pm b,$$

  - b smooth and supported in the past of f.
  - $\text{supp}(\psi^\pm)$ in the future/past of $\text{supp}(b)$.

For ψ^+ with $\lambda = 1$ the retarded integral are compact.

With those r we can obtain $\psi_n(f)$ for every n in the limit $\lambda = 1$.

Question

What kind of fields are $\psi_n(f)$?

Theorem

$\psi_n(f)$ are functionals over matter field configuration. They are elements of $\mathcal{F}(\gamma)$ $\forall n$.

- The perturbative analysis of the moments of ψ can be put on firm mathematical grounds.
- If we have a state ω for the matter fields, we can construct the probability distribution for $\psi(f)$.

Application in Minkowski

- Estimate the focusing probability of a family of **timelike parallel geodesics** on Minkowski within the interval of time I.

(collapse condition, realize ψ with negative values.)

$$\psi_0(t) = \psi_0, \qquad \ddot{\psi} = \psi_0 + R_V(\lambda\dot{\varphi}^2\psi), \quad R_V(t,s) = -(t-s)\vartheta(t-s).$$

A **second order** estimate on the Minkowski vacuum gives

$$\begin{aligned} \omega(\psi(f)) &\approx \psi_0, \\ \varsigma^2(f) &\approx \omega(\psi_1(f) \star_\omega \psi_1(f)) = \frac{\psi_0^2}{\pi^2 7!}\int_0^{+\infty} dp\ p^3 \overline{\widehat{f}(p)}\widehat{f}(p). \end{aligned}$$

- f is a smooth approximation of the characteristic function of the time interval I.
- The **smaller** the support, the **larger** the variance.

Decay probability

The **probability** density of ψ is approximated by a Gaussian distribution

$$\mathbb{P}(\psi(f_\tau) \leq 0) \approx \mathcal{N}\left(-\psi_0, 0, 1\right), \quad f_\tau(s) := f(s-\tau).$$

Consider a sequence $\{X_n\}_n$ of random variables such that

$$X_n \sim \psi(f_\tau) \quad \forall n,$$

- Focusing occures.
- **Time of the first collapse** is distributed as an exponential of parameter $\lambda_\tau := \mathbb{P}(\psi(f_\tau) \leq 0)$.
- The result is **qualitatively** similar to the one obtained by Carlip et all.
- The **larger** the support of f the **smaller** the collapse probability due to quantum fluctuations.

Towards quantum spacetime?

- In *[DFR 95]* the authors find the commutation rules among the coordinates

$$[q^\mu, q^\nu] = iQ^{\mu\nu}$$

compatible with the following uncertainty relations

$$\Delta x_0 \left(\Delta x_1 + \Delta x_2 + \Delta x_3\right) \geq \lambda_P^2,$$

$$\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_3 \Delta x_1 \geq \lambda_P^2$$

which are obtained using the following:

Minimal Principle:

We cannot create a singularity just observing a system.

- Together with the **Heisenberg principle (HP)** (valid in Minkowski).
- The uncertainties are tailored to the flat spacetime.

- In *[Doplicher Morsella np 2013]* the semiclassical equation in connection with that principle was used to obtain a minimal length scale in spherically symmetric spacetimes.
- A model for a measuring apparatus was discussed and the preparation of the system was considered $\rightarrow$ kinematical point of view.
- In the semiclassical approximation, the matter fluctuations can induce the formation of singularities.
- They can be made small smearing over long time intervals.
- **Open task:** Obtain bounds for the coordinate uncertainties relations without studying the measuring apparatus.

Summary

- Algebra of matter fields on timelike geodesics can be considered.
- Passive influence of matter fluctuation on expansion parameter can be studied within **pAQFT**.
- Bounds for uncertainty relations among spacetime coordinates can be studied.

Open Questions

- Can we get bounds for the validity of semiclassical equations?
- Can we do better then perturbation theory?
- Can we address intrinsic fluctuation of the expansion parameter?
- What about their influence on the matter?
- Quantum gravity solves those issues?

Thanks a lot for your attention!