Analysis on singular spaces, Lie manifolds, and non-commutative geometry II

Pseudodifferential operators on groupoids

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Abstract of series

We study **Analysis and Index Theory** on singular and non-compact spaces.

Central: **exact sequence.**

$$0 \to I \to A \to Symb \to 0\,.$$

- A is a suitable **algebra of operators** that describes the analysis on a given (class of) singular space(s). **Will be constructed using Lie algebroids and Lie groupoids.**
- the **ideal** $I = A \cap \mathcal{K}$ of compact operators (to describe). **Will be determined using the representation theory of groupoids.**

The contents of the four talks

1. Motivation: Index Theory (a) Exact sequences and index theory (b) The Atiyah-Singer index theorem (c) Foliations (d) The Atiyah-Patodi-Singer index theorem (e) More singular examples. **No new results.**
2. Lie Manifolds: (a) Definition (b) The APS example (c) Lie algebroids (d) Metric and connection (e) Fredholm conditions (f) Examples :Lie manifolds and Fredholm c.
3. Pseudodifferential operators on groupoids: (a) Groupoids, (b) Pseudodifferential operators, (c) Principal symbol, (d) Indicial operators, (e) **Groupoid C^*-algebras** and Fredholm conditions, (f) The index problem and homology.
4. Applications: (a) Well posedness on polyhedral domains (L2), (b) Essential spectrum (L3), (c) An index theorem for Callias-type operators (L4).

Collaborators

- Bernd Ammann (Regensburg),
- Catarina Carvalho (Lisbon),
- Alexandru Ionescu (Princeton),
- Robert Lauter (Mainz ...),
- Bertrand Monthubert (Toulouse)

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Definition of Lie manifolds using Lie algebroids

Let $\mathcal{V} \subset \mathcal{V}_b := \{X \in \Gamma(T\overline{M}),\ X \text{ tangent to all faces of } \overline{M}\}$.

Definition. The pair $(\overline{M}, \mathcal{V})$ is a **Lie manifold** if, and only if, there is a vector bundle $A_{\mathcal{V}} \to \overline{M}$ such that:

- $\mathcal{V} \simeq \Gamma(A_{\mathcal{V}})$ ($\Leftrightarrow \mathcal{V}$ is $\mathcal{C}^\infty(\overline{M})$-projective).
- $\varrho : A_{\mathcal{V}} \to T\overline{M}$ is an **isomorphism** over $M := \overline{M} \smallsetminus \partial M$.
- $A_{\mathcal{V}}$ is a Lie algebroid ($\Leftrightarrow \mathcal{V}$ is a Lie algebra).

A metric on $A = A_{\mathcal{V}}$ gives a **metric** on $M : \overline{M} \smallsetminus \partial \overline{M}$.

IMPORTANT: $\ker(\varrho_x : A_x \to T_x\overline{M})$ is the **isotropy** of A at $x \in \overline{M}$. Can be shown to be a Lie algebra.

Definition of Lie manifolds

Definition. The pair $(\overline{M}, \mathcal{V})$ is a **Lie manifold** if, and only if, there is a vector bundle $A_{\mathcal{V}} \to T\overline{M}$ such that:

1. $\mathcal{V} \simeq \Gamma(A_{\mathcal{V}})$.
2. $A_{\mathcal{V}}$ **extends** TM to $\overline{M}$.
3. $A_{\mathcal{V}}$ is a Lie algebroid.

Old Definition. The pair $(\overline{M}, \mathcal{V})$ is a **Lie manifold** if, and only if,

1. $\mathcal{V}$ is a finitely-generated, projective $\mathcal{C}^\infty(\overline{M})$–module.
2. $\Gamma_c(T\overline{M}) \subset \mathcal{V}$.
3. $\mathcal{V}$ is closed under the Lie bracket $[\ ,\ ]$.

First example: cylindrical ends

- $\overline{M}$ = a manifold with **smooth boundary** with defining function x (so $\partial\overline{M} = \{x = 0\}$).
- $\mathcal{V} = \mathcal{V}_b$ the space of vector fields on $\overline{M}$ that are **tangent** to the boundary $\partial\overline{M}$.
- At the boundary $\partial\overline{M} = \{x = 0\}$, a local basis of $\mathcal{V}$ is given by $x\partial_x$, ∂_{y_2}, ... , ∂_{y_n}.

  ($y_2, \ldots, y_n$ are local coordinates on $\partial\overline{M}$.)

The Riemannian metric $(r^1 dr)^2 + g_{\partial\overline{M}}$, so a **manifold with cylindrical ends**.

APS, Debord-Lescure, Kondratiev, Melrose, Schulze.

Sobolev spaces

Let $(\overline{M}, \mathcal{V})$ be a Lie manifold and g a compatible metric on the interior M of $\overline{M}$. The space $L^2(M)$ is independent of the metric. .

We then define for $m \in \mathbb{Z}_+$

$$H^m(M) \ := \ \{u : M \to \mathbb{C},\ X_1 X_2 \ldots X_k u \in L^2(M),\ k \le m,\ X_j \in \mathcal{V}\}\,.$$

It turns out that $H^s(M)$ coincides with the domain of $\Delta_g^{s/2}$.

A choice of partition of unity with bounded derivatives (Albin, Gromov, Shubin) can also be used to define the spaces $H^s(M)$ (Ammann-Ionescu-V.N.).

Mapping properties

Recall that $\text{Diff}(\mathcal{V})$ is the algebra of differential operators generated by multiplication with functions in $\mathcal{C}^\infty(\overline{M})$ and by differentiation with vector fields in $\mathcal{V}$.

We are interested in the analytic properties of $P \in \text{Diff}(\mathcal{V})$. Given that

$$H^m(M) := \{u : M \to \mathbb{C},\ X_1 X_2 \ldots X_k u \in L^2(M),\ k \leq m,\ X_j \in \mathcal{V}\}.$$

We immediately see that

$$P \in \text{Diff}(\mathcal{V}),\ \text{ord}(P) \leq m \ \Rightarrow\ P : H^k(M) \to H^{k-m}(M)$$

is **bounded** for all k, m (we let $H^{-s}(M) := (H^s(M))^*$).

Fredholm operators

Assume $(\overline{M}, \mathcal{V})$ is "nice."

- Denote $(Z_\alpha)_{\alpha \in I}$ the family of **orbits** of $\mathcal{V}$ on $\partial\overline{M}$.
- Let G_α be a family of Lie groups that integrates the corresponding isotropies, $\alpha \in I$.

Theorem. (Lauter-Monthubert-V.N.) *We can choose the groups G_α and we can associate to each $P \in \mathrm{Diff}(\mathcal{V})$ a family of G_α-invariant operators P_α on $Z_\alpha \times G_\alpha$ such that:*

P is Fredholm $\Leftrightarrow$ P is elliptic and all P_α are invertible.

Comments

If $\overline{M}$ is compact without corners, then $I = \emptyset$, and we recover the usual result that states that **on a compact, smooth manifold, a differential operator is Fredholm if, and only if, it is elliptic.**

Each P_α is "of the same kind" as P (Laplace, Dirac, ...).

Questions on M are reduced to questions on P_α and G_α
$\Rightarrow$ **Harmonic analysis on various Lie groups.**
$\Rightarrow$ **inductive procedure to study geometric operators on M.**

Earlier results: **Kondratiev, Mazya, Plamenevski, Mazzeo, Melrose, Mendoza, Piazza, Schrohe, Schulze ...**

Reduction in the first example

We have defined the **indicial family** $\widehat{P}(\tau)$ of

$$P = \sum_{|\alpha|\leq m} a_\alpha(r, x')(r\partial_r)^{\alpha_1}\partial^{\alpha'} \quad \text{as}$$

$$\widehat{P}(\tau) := \sum_{|\alpha|\leq m} a_\alpha(0, x')(\imath\tau)^{\alpha_1}\partial^{\alpha'} \in \mathrm{Diff}(\partial\overline{M})\,.$$

We can define also $I(P) \in \Psi^m(\partial\overline{M} \times \mathbb{R})^{\mathbb{R}}$ by noticing that our operator P becomes *"more and more translation invariant"* as we approach the infinity. The limit is then

$$I(P) = \sum_{|\alpha|\leq m} a_\alpha(0, x')\partial_t^{\alpha_1}\partial^{\alpha'}.$$

Then $\widehat{P}(\tau)$ is the **Fourier transform** of $I(P)$, so τ is dual to t.

Groupoids and Fredholm conditions

The proof of the Fredholm conditions requires to **find a groupoid $\mathcal{G}$ with Lie algebroid $A(\mathcal{G})$ such that $A(\mathcal{G}) = A_{\mathcal{V}}$,.**

($\Gamma(A(\mathcal{G}))$ consists of **right invariant vector fields** on $\mathcal{G}$).

This amounts to a **Lie's third theorem for Lie algebroids**, which is not true in general, but **is true** in our cases (N., Crainic-Fernandez, Debord, Androulakis-Skandalis).

Then we use the representation theory of the groupoid C^*-algebra associated to $\mathcal{G}$ **(later).**

First Example Revisited

Recall $\mathcal{V} = \mathcal{V}_b :=$ the space of vector fields on $\overline{M}$ that are *tangent* to $\partial\overline{M}$. At the boundary $\partial\overline{M} = \{x = 0\}$, a local basis is given by $x\partial_x, \partial_{y_2}, \dots, \partial_{y_n}$.

The geometry is that of a **manifold with cylindrical ends**.

We have that Z_α are the connected components of the boundary and $G_\alpha = \mathbb{R}$.

P_α acts on $Z_\alpha \times \mathbb{R}$ and is $\mathbb{R}$ invariant. It coincides with the restriction of $I(P)$, who acts on $\partial\overline{M} \times \mathbb{R}$, to $Z_\alpha \times \mathbb{R}$.

(Melrose, Schulze, Atiyah-Patodi-Singer.)

Second example: asymptotically hyperbolic manifolds

- As before, $\overline{M}$ with smooth boundary $\partial\overline{M} = \{x = 0\}$.
- $\mathcal{V} = x\Gamma(T\overline{M}) =$ the space of vector fields on $\overline{M}$ that vanish on the boundary.
- At the boundary $\partial\overline{M} = \{x = 0\}$, a local basis is given by $x\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_n}$.
- No condition in the interior (all Lie manifolds).

Then the orbits Z_α are reduced to points, so $\alpha \in I := \partial\overline{M}$, and $G_\alpha = T_\alpha\partial\overline{M} \rtimes \mathbb{R}$.

Pseudodifferential calculus: Lauter, Mazzeo, Schulze.

Third example: asymptotically Euclidean manifolds

- As before, $\overline{M}$ with smooth boundary $\partial\overline{M} = \{x = 0\}$.
- $\mathcal{V} = x\mathcal{V}_b =$ the space of vector fields on $\overline{M}$ that vanish on the boundary $\partial\overline{M}$ and whose **normal covariant derivative to the boundary also vanishes.**
- At the boundary $\partial\overline{M} = \{x = 0\}$, a local basis is given by $x^2\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_n}$.

The resulting geometry for $\partial\overline{M} = S^{n-1}$ is that of an asymptotically Euclidean manifold.

Again the orbits Z_α are reduced to points, so $\alpha \in I := \partial\overline{M}$, but this time $M_\alpha = G_\alpha = T_\alpha\partial\overline{M} \times \mathbb{R}$ is **commutative. (Callias, fourth lecture)**

Fourth example: fibered boundaries

- As before, $\overline{M}$ with smooth boundary $\partial\overline{M} = \{x = 0\}$.
- We are given a fibration $\pi : \partial M \to B$.
- $\mathcal{V} =$ the space of vector fields on $\overline{M}$ that are tangent to the fibers of $\pi : \partial\overline{M} \to B$.
- A local basis is given by $x\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_k}$, $\partial_{y_{k+1}}, \ldots \partial_{y_n}$.

Geometry is related to that of locally symmetric spaces. Appears in the study of behaviour at the edge of boundary value problems.

Fredholm conditions: $I = \{\alpha\} = B$, $Z_\alpha = \pi^{-1}(\alpha)$, $G_\alpha = T_\alpha B \rtimes \mathbb{R}$ is a **solvable** Lie group. (Also Mazzeo, Lescure.)

Groupoids

Definition. A **groupoid** is a small category $\mathcal{G}$ all of whose morphisms are invertible.

More precisely: $\mathcal{G} = \mathcal{G}^{(1)} =$ the set of morphisms and $\overline{M} = \mathcal{G}^{(0)}$ is the set of units. (Right now $\overline{M}$ is an arbitrary set, but will be a compact manifold with corners in applications.)

Objects will typically be called *units* and the morphisms will be called *arrows*.

The structure of small category gives rise to **structural morphisms** d, r, u, μ, ι, as follows.

Structural morphisms

- $d, r : \mathcal{G} = \mathcal{G}^{(1)} \to \overline{M}$ are the **domain** and **range** maps.

  Two morphisms (or arrows) $g, h \in \mathcal{G}$ are **composable** if, and only if, $d(g) = r(h)$

- $u : \overline{M} = \mathcal{G}^{(0)} \to \mathcal{G}$, an injection that **identifies** an object with its identity morphism ($u = id$).

  Let $\mathcal{G}^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G}, d(g) = r(h)\}$, the set of composable arrows.

- $\mu : \mathcal{G}^{(2)} \to \mathcal{G}$ is the composition: $\mu(g, h) = gh$.

- $\iota : \mathcal{G} \ni g \to g^{-1} \in \mathcal{G}$ is the inversion.

Properties of structural morphisms

The structure of small category also gives rise to various **properties of the** structural morphisms d, r, u, μ, ι, as follows:

- If two arrows $g, h \in \mathcal{G}^{(1)}$ are composable ($d(g) = r(h)$), then $d(gh) = d(h)$ and $r(gh) = r(g)$.
- $u(r(g))g = r(g)g$, $gd(g) = g$, and $d(g) = r(g^{-1})$.
- The composition is required to be **associative.**
- $(gh)^{-1} = h^{-1}g^{-1}$, $(g^{-1})^{-1} = g$, $g^{-1}g = d(g)$.

Lie groupoids: definitions

Definition. A **submersion** $f : X \to Y$ between two manifolds with corners is a smooth map such that df_x is surjective for all $x \in X$ **and** $f^{-1}(y)$ **has no corners** for any $y \in Y$.

Definition. A **Lie groupoid** is a groupoid $\mathcal{G}$ such that $\overline{M} = \mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are manifolds with corners, $\overline{M}$ is Hausdorff, and

- $d, r : \mathcal{G} \to \overline{M}$,
- $u : \overline{M} \to \mathcal{G}$,
- $\iota : \mathcal{G} \ni g \to g^{-1} \in \mathcal{G}$, and
- $\mu : \mathcal{G}^{(2)} \ni (g, h) \to gh \in \mathcal{G}$

are smooth, and d (equivalently r) is a **submersion.**

Objects "on groupoids"

Recall that objects on, or associated to, Lie groups are *right invariant* quantitities. **Example:** the **Lie algebra** of a Lie group G is the set of *right invariant vector fields* on G.

The same idea applies to Lie groupoids, except that we need to be careful about what **right invariant** means.

We shall denote the set of arrows with the same domain x

$$\mathcal{G}_x = d^{-1}(x), \quad x \in \overline{M}\,.$$

The right multiplication by $g \in \mathcal{G}$ defines a diffeomorphism

$$\mathcal{G}_{r(g)} \ni h \to hg \in \mathcal{G}_{d(g)}\,.$$

The Lie algebroid of a Lie groupoid $\mathcal{G}$

Recall that all $\mathcal{G}_x$ have **no corners** and define

$$T_d\mathcal{G} = \cup T\mathcal{G}_x = \ker(d_* : T\mathcal{G} \to T\overline{M}),$$

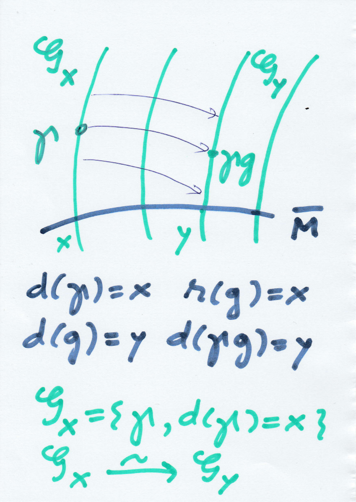
the ***vertical* tangent bundle.** Then

$$A(\mathcal{G}) := T_d\mathcal{G}|_{\overline{M}}.$$

Let $\mathcal{V}$ be the set of **right invariant vector fields on** $\mathcal{G}$ **that are tangent to the submanifolds** $\mathcal{G}_x$, $x \in \overline{M}$ (fibers of $d : \mathcal{G} \to \overline{M}$).

IMPORTANT: The sections of $A(\mathcal{G})$ identify with $\mathcal{V}$ (right invariant, vertical vector fields on $\mathcal{G}$), so $\Gamma(A(\mathcal{G}))$ has a natural Lie algebra structure: $A(\mathcal{G})$ is the **Lie algebroid** of $\mathcal{G}$.

Right invariance



Fibers of d

Pseudodifferential operators on $\mathcal{G}$

$\Psi^m(\mathcal{G})$ consists of families (P_x), $P_x \in \Psi^m(\mathcal{G}_x)$, satisfying

- right invariant,
- smooth,
- with support in a compact neighborhood of the units of $\mathcal{G}$.

For instance, $\Psi^{-\infty}(\mathcal{G}) = \mathcal{C}_c^\infty(\mathcal{G})$ with the convolution product.

We see that the **first order, differential** operators in $\Psi^\infty(\mathcal{G})$ coincide with $\mathcal{V}$ (right invariant, vertical vector fields on $\mathcal{G}$).

It follows that

$$\Psi^\infty(\mathcal{G}) \cap \text{Diff} = \text{Diff}(\mathcal{V})$$

Action on functions on M and $\overline{M}$

$\Psi^m(\mathcal{G})$ **acts** on $\mathcal{C}_c^\infty(M)$ as follows.

Let $P = (P_x) \in \Psi^m(\mathcal{G})$. Any right invariant function on $\mathcal{G}$ is of the form $f \circ r$. This gives $\pi(P) : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)$ by the formula

$$(\pi(P)f) \circ r|_{\mathcal{G}_x} = P_x f \circ r|_{\mathcal{G}_x}.$$

In particular, $\mathcal{V}$ and $\mathrm{Diff}(\mathcal{V})$ will act on $\mathcal{C}_c^\infty(M)$.

$\mathcal{V} \to \Gamma(T\overline{M})$ **is NOT injective, in general.** INJECTIVE for foliations, Lie manifolds.

Diff($\mathcal{V}$) and $\Psi^{(}\mathcal{G})$

Let $\mathcal{G}$ be a Lie groupoid with units $\overline{M}$ and with Lie algebroid $A(\mathcal{G})$.

Assume $(\overline{M}, A(\mathcal{G}))$ **defines a Lie manifold** ($\Leftrightarrow A(\mathcal{G})|_M \simeq TM$).

Then the differential operators in $\Psi^\infty(\mathcal{G})$, acting on $\mathcal{C}_c^\infty(M)$, identify with Diff($\mathcal{V}$).

Hence, the resolvents of operators in Diff($\mathcal{V}$) "are in" $\Psi^\infty(\mathcal{G})$.

Lie's third theorem for Lie groupoids

A major ingredient in the proof is then to establish **the existence** of a Lie groupoid $\mathcal{G}$ such that $\Gamma(A(\mathcal{G})) = \mathcal{V}$.

This amounts to a **Lie's third theorem** for $\mathcal{V}$ (or A).

The Lie's third theorem is **not valid** for any Lie algebroid, but **is valid** for those arising from Lie manifolds!

Androulakis-Skandalis, Crainic-Fernandez, Debord, V.N., Pradines.