On dimension and integration for spectral triples associated to quantum groups

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1. Spectral triples and non-commutative integration
2. The quantum group $SU_q(2)$
3. Quantum projective spaces

- Much information about Riemannian manifolds can be obtained by analyzing operators associated to them.
- For example take the Laplace-Beltrami operator.
- Consider a bounded region $\Omega \subset \mathbb{R}^n$. Denote by $N(\lambda)$ the number of (Dirichlet) eigenvalues which are $\leq \lambda$. Then we have (Weyl's law)

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{V_n}{(2\pi)^n} \mathrm{vol}(\Omega).$$

- Therefore given the list of eigenvalues $\{\lambda_i\}_{i=0}^{\infty}$ we can tell the dimension and the volume of the region in consideration.
- There is actually much more information than that in the list of eigenvalues (but still not enough to completely characterize the geometry, cf. "hearing the shape of a drum").

- Moreover a spectral approach is also what we need to describe the geometry of non-commutative spaces.
- As it is well known, non-commutative C^*-algebras provide a natural notion of non-commutative (topological) spaces.
- A C^*-algebra can be represented concretely as a norm closed subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.
- The idea is then to consider some (possibly unbounded) operator that should contain geometric information about the space.

- The notion of spectral triple provides the basis for non-commutative geometry in the sense of Connes.

Definition

A compact spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the data of a unital $*$-algebra $\mathcal{A}$, a faithful $*$-representation π on a Hilbert space $\mathcal{H}$, and a self-adjoint operator D such that

- $[D, \pi(a)]$ extends to a bounded operator for all $a \in \mathcal{A}$,
- $(D^2 + 1)^{-1/2}$ is a compact operator.

- For example consider a compact spin manifold M. Then if we take $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(S)$ and D the Dirac operator (with respect to a fixed metric) we get a spectral triple.
- The distance between two points $p, q \in M$ can be obtained as

$$d(p,q) = \sup\{|f(p) - f(q)| : f \in C^\infty(M), \|[D, f]\| \leq 1\}.$$

- The dimension of a manifold M can be recovered from the spectrum of the operator D. Indeed consider (assume D invertible)

$$|D|^{-z}, \quad z \in \mathbb{C}.$$

It is trace-class for all $\mathrm{Re}(z) > n$, where n is the dimension of M.
- Similarly, we can also recover the integral of a function $f \in C^\infty(M)$.
- One way is via the residue of the zeta function associated to this operator. We define the linear functional $\varphi : C^\infty(M) \to \mathbb{C}$ as

$$\varphi(f) = \operatorname*{Res}_{z=n} \mathrm{Tr}(f|D|^{-z}).$$

- It turns out that $\varphi(f)$ coincides with the integral of f (which includes the volume form), up to a multiplicative constant.

- These notions make sense also for non-commutative algebras.
- In this more general setting we will refer to them as spectral dimension and non-commutative integral.
- However, it turns out that for some quantum deformations they behave very differently from their commutative counterparts.
- For example let us consider the case of q-deformations. Suppose the spectrum of a classical operator is replaced by q-numbers

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

  Then we get completely different asymptotics.
- It is possible to construct spectral triples which are isospectral [Neshveyev, Tuset (2010)]. But then other notions of dimension will differ (for example the homological dimension).

- There are features of the non-commutative world which have no analogue in the commutative one.
- Consider for example the non-commutative integral φ. In the commutative case we have trivially $\varphi(ab) = \varphi(ba)$. Also true for a spectral triple, but is a non-trivial condition.
- For example for $SU_q(2)$ there is no faithful trace.
- In the case of compact quantum groups the Haar state satisfies $h(ab) = h(\vartheta(b)a)$, for a non-trivial modular group ϑ.
- Therefore we might want to take these features into account.

- Twisted spectral triples [Connes, Moscovici (2006)].
  Require $[D, f]_\sigma = Df - \sigma(f)D$ to be bounded, where σ is an automorphism of $\mathcal{A}$. The non-commutative integral then obeys

$$\varphi(fg) = \varphi\left(\sigma^n(g)f\right) \ .$$

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- Modular spectral triples [Carey, Phillips, Rennie (2010)].
  Use a weight Φ instead of the operator trace

$$\varphi(f) = \operatorname*{Res}_{z=n} \Phi\left(f|D|^{-z}\right) \ ,$$

  where $\Phi(\cdot) = \mathrm{Tr}(\Delta_\Phi \cdot)$. Then we have the modular property

$$\varphi(fg) = \varphi\left(\sigma_i^\Phi(g)f\right)$$

  where $\sigma_t^\Phi(f) = \Delta_\Phi^{it} f \Delta_\Phi^{-it}$ is the modular group of Φ.

- We can take both of these approaches into account to discuss integration for non-commutative spaces.
- The zeta function associated to D and Φ is defined by

$$\zeta(z) := \Phi(|D|^{-z}) = \mathrm{Tr}(\Delta_\Phi |D|^{-z}).$$

- If it exists, we define the spectral dimension to be the number

$$n := \inf\{s > 0 : \zeta(s) = \mathrm{Tr}(\Delta_\Phi |D|^{-s}) < \infty\}.$$

- More generally define $\zeta_x(z) := \Phi(x|D|^{-z})$. This is well defined if we assume that $\sigma^\Phi(x) \in \mathcal{A}$ for any $x \in \mathcal{A}$.
- Assume we have a simple pole. Then the non-commutative integral is the linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ defined by

$$\varphi(x) := \operatorname*{Res}_{z=n} \zeta_x(z) = \operatorname*{Res}_{z=n} \mathrm{Tr}(\Delta_\Phi x |D|^{-z}).$$

- We assume that $[D, x]_\sigma = Dx - \sigma(x)D$ is bounded for every $x \in \mathcal{A}$, for a fixed automorphism σ.
- We also assume that σ acts diagonally on the generators.

Theorem

Let φ be the non-commutative integral as before. Assume furthermore that D satisfies the following regularity property:

- *there exists some $0 < r \leq 1$ such that $|D|^r[|D|^s, x]_{\sigma^s}|D|^{-s}$ is a bounded operator, for every element $x \in \mathcal{A}$ and for all $s \geq n$.*

Then the modular group of φ is given by $\theta = \sigma^\Phi \circ \sigma^n$.

- This means that $\varphi(xy) = \varphi(\theta(y)x)$ for all $x, y \in \mathcal{A}$.

- Given an operator D, we can consider the following general question: are there any preferred choices for the weight Φ?
- In the realm of compact quantum groups a reasonable requirement is to recover the Haar state.
- We analyze this question for the quantum group $SU_q(2)$.
- We consider the Dirac operator D_q introduced in [Kaad, Senior (2012)]. It acts on the Hilbert space $\mathcal{H} = \mathcal{H}_h \oplus \mathcal{H}_h$, where $\mathcal{H}_h$ is the GNS space constructed using the Haar state.

- It is defined in terms of the generators of $U_q(\mathfrak{su}(2))$ as

$$D_q = \begin{pmatrix} (q^{-1}-q)^{-1}(qK^{-2}-1) & q^{-1/2}EK^{-1} \\ q^{1/2}FK^{-1} & (q^{-1}-q)^{-1}(1-q^{-1}K^{-2}) \end{pmatrix}.$$

- Upon restriction becomes the Dirac operator on the Podleś sphere.

Proposition (Kaad, Senior 2012)

For the Dirac operator D_q we have:

1. $[D_q, x]_{\sigma_L} = D_q x - \sigma_L(x) D_q$ *is bounded,*
2. $[|D_q|, x]_{\sigma_L}$ *is bounded (Lipschitz regularity),*
3. $D_q^2 = \chi^{-1}\Delta_L^{-1} C_q$*, where* C_q *is the Casimir and* $\chi = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$.

- Here $C_q t^l_{ij} = [l+1/2]_q^2 t^l_{ij}$ and $\Delta_L t^l_{ij} = q^{2j} t^l_{ij}$.

- There is an additional interesting feature of the operator D_q.
- Given a spectral triple, the Dirac operator D implements a differential calculus. The same is true in the twisted case, with the appropriate modifications.

Proposition

The operator D_q implements a left covariant differential calculus on $SU_q(2)$.

- In the context of twisted spectral triples, this particular calculus has been considered previously in the paper [Krähmer, Wagner (2011)], where it is given as an example of a more general framework.
- The operator D_q that we consider here, however, is slightly different from the one that appears in that paper.

- We now define a non-commutative integral in terms of D_q.
- In view of the requirement that $\sigma^{\Phi}(x) \in \mathcal{A}$ for every $x \in \mathcal{A}$, we choose Δ_Φ such that it implements an automorphism of $SU_q(2)$.
- It is known that the automorphisms that act diagonally on the generators can be parametrized by two numbers.
- In particular the modular group ϑ of the Haar state is of this form.
- Therefore we consider the family of weights given by

$$\Phi^{(a,b)}(\cdot) := \mathrm{Tr}(\Delta_L^{-a}\Delta_R^{b}\cdot), \qquad a, b \in \mathbb{R}.$$

- We now compute the corresponding spectral dimension.
- The relevant zeta function takes the form

$$\zeta^{(a,b)}(z) := \mathrm{Tr}(\Delta_L^{-a}\Delta_R^{b}|D_q|^{-z}).$$

Proposition

1. *$\zeta^{(a,b)}(z)$ is holomorphic for all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > a + |b|$, provided that $a \pm b > 0$,*
2. *in this case the corresponding spectral dimension is $n = a + |b|$,*
3. *$\zeta^{(a,b)}(z)$ has a meromorphic extension to the complex plane, with only simple poles if $b \neq 0$ and with only double poles if $b = 0$.*

- If these conditions are satisfied we can take the residue, so that the non-commutative integral makes sense.
- The conditions of the theorem previously shown apply to this case, so we can determine its modular group, which we denote by θ.
- We want to compare the non-commutative integral with the Haar state, which satisfies the property $h(xy) = h(\vartheta(y)x)$.
- A necessary condition to recover the Haar state from the non-commutative integral is that $\theta = \vartheta$.

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Proposition

We have $\theta = \vartheta$ if and only if $b = 1$.

- It is also possible to show that the non-commutative integral, once normalized, coincides with the Haar state h independently of a.

- Is there any way to fix the parameter a?
- Since the spectral dimension is given by $n = a + 1$, with the choice $a = 2$ we obtain the classical dimension.
- We will look at the heat kernel expansion. In the classical case for a second order operator of Laplace-type we have

$$\mathrm{Tr}(fe^{-tP}) \sim \sum_{k=0}^{\infty} t^{(k-n)/2} a_k(f, P) \ ,$$

- The heat kernel coefficients are related to the zeta function by

$$a_k(f, P) = \operatorname*{Res}_{z=(n-k)/2} \Gamma(z)\zeta(z, f, P).$$

- Locally the operator P can be written in the form

$$P = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E).$$

- In three dimensions the first two non-trivial coefficients are

$$a_0(f,P) = (4\pi)^{-n/2}\int_M f\sqrt{g}d^n x,$$
$$a_2(f,P) = (4\pi)^{-n/2}6^{-1}\int_M f(6E + R)\sqrt{g}d^n x.$$

- Consider the operator C obtained in the classical limit from C_q.
- For this operator we have that $a_2(C) = 0$ non-trivially. Indeed for the 3-sphere the scalar curvature is $R = 6$, but this is cancelled by E.

- In the non-commutative case we can ask for the analogue condition for D_q^2. Then the following residue should vanish

$$\operatorname*{Res}_{z=n-2} \Gamma(z)\zeta^{(a,1)}(z) = 0.$$

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Proposition

The residue of $\zeta^{(a,1)}(z)$ at $z = n-2$ is zero if and only if $a = 2$.

- Recall that the spectral dimension is given by $n = a+1$. Therefore for this value it coincides with the classical one.
- The parameters a and b control the behaviour of different coefficients of the heat kernel expansion. This property is not obvious from their definition.

- We consider the family of spectral triples for quantum projective spaces introduced in [D'Andrea, Dąbrowski (2010)].
- Their spectral dimension is zero. Here we want to reinterpret them in the sense of modular spectral triples.
- We will consider $\mathcal{A}(\mathbb{C}P_q^\ell)$ with $\ell \geq 2$, which can be constructed similarly to the quotient $S^{2\ell+1}/U(1)$ in the classical case.
- A particular element, denoted by $K_{2\rho}$, will play a central role in the following. One important property of this element is that it implements the square of the antipode, in the sense that $S^2(x) = K_{2\rho} h K_{2\rho}^{-1}$ for any $x \in U_q(\mathfrak{su}(\ell+1))$.
- More importantly for us, it also implements the modular group of the Haar state of $\mathcal{A}(SU_q(\ell+1))$.

- There is a non-degenerate pairing $\langle \cdot, \cdot \rangle$ between $U_q(\mathfrak{su}(\ell+1))$ and $\mathcal{A}(SU_q(\ell+1))$, which is used to define the canonical left and right actions as $x \triangleright a = a_{(1)} \langle x, a_{(2)} \rangle$ and $a \triangleleft x = \langle x, a_{(1)} \rangle a_{(2)}$.
- There is a faithful state on $\mathcal{A}(SU_q(\ell+1))$, called the Haar state and which we denote by h. It generalizes the properties of the Haar integral in the classical case.
- However the Haar state does not satisfy the trace property. In particular its modular group is implemented by the element $K_{2\rho}$ as

$$h(ab) = h(bK_{2\rho} \triangleright a \triangleleft K_{2\rho}).$$

- Passing to the quantum projective spaces it becomes

$$h(ab) = h(bK_{2\rho} \triangleright a).$$

- The Hilbert spaces H_N are the completion of $\bigoplus_{k=0}^{\ell} \Omega_N^k$. Here Ω_N^k are spaces of (twisted) forms, with $N \in \mathbb{Z}$.
- The spaces Ω_N^k can be decomposed into (vector spaces of) irreducible representations of $U_q(\mathfrak{su}(\ell+1))$, schematically as

$$\Omega_N^0 = \bigoplus_{m \in \mathbb{N}} V_{(m+c_1,0,\cdots,0,m+c_2)},$$

$$\Omega_N^k = \bigoplus_{m \in \mathbb{N}} V_{(m+c_3,0,\cdots,0,m+c_4)+e_k} \oplus V_{(m+c_5,0,\cdots,0,m+c_6)+e_{k+1}},$$

$$\Omega_N^\ell = \bigoplus_{m \in \mathbb{N}} V_{(m+c_7,0,\cdots,0,m+c_8)}.$$

- q-analogues of $\bar{\partial}$ and $\bar{\partial}^\dagger$ are defined. By taking suitable linear combinations we obtain a family of Dolbeault-Dirac operators D_N.

- On each Ω_N^k the square of D_N can be written in terms of the Casimir. Its eigenvalues grow like q^{-m}, with $m \in \mathbb{N}$.
- The spectral dimension of this spectral triple is zero. Indeed the eigenvalues grow like q^{-m} while the multiplicities grow polynomially.
- We now want to revisit this construction in the sense of modular spectral triples. Include the element $K_{2\rho}$ as

$$\varphi(a) = \operatorname*{Res}_{z=n} \operatorname{Tr}(K_{2\rho} a |D_N|^{-z})$$

- Under suitable assumptions we have

$$\varphi(ab) = \varphi(b K_{2\rho} \triangleright a),$$

which is the modular property of the Haar state.
- We need to determine the spectral dimension.

- Given a finite-dimensional irreducible representation T, its quantum dimension is defined as the number $\mathrm{Tr}(T(K_{2\rho}))$, where the trace is taken over the vector space that carries the representation T.
- In the classical case, that is for $q = 1$, the quantum dimension is simply the dimension of this vector space.
- For the vector space V_Λ of highest weight Λ we can use the Weyl dimension formula, which reads as

$$\dim V_\Lambda = \prod_{\alpha>0} \frac{(\Lambda+\rho,\alpha)}{(\rho,\alpha)},$$

where the product is over the positive roots and ρ is the Weyl vector, defined as the half-sum of the positive roots.

- There is also a q-analogue of this formula which allows to compute the quantum dimension. We have

$$\dim_q V_\Lambda = \prod_{\alpha > 0} \frac{[(\Lambda + \rho, \tilde{\alpha})]_q}{[(\rho, \tilde{\alpha})]_q},$$

  where $[x]_q$ is a q-number and $\tilde{\alpha}$ is a normalization of α.
- This quantity appears in the computation of the trace of $K_{2\rho}|D_N|^{-z}$. Indeed D_N^2 is a multiple of the identity on V_Λ.
- We compute it for the vector spaces appearing in the decomposition of the Hilbert spaces H_N.

Proposition

For $m \to \infty$, the quantum dimension of the vector space V_Λ with weight Λ is $\dim_q(V_\Lambda) = O(q^{-2\ell m})$.

Theorem

The operator $K_{2\rho}|D_N|^{-z}$ *is trace-class for* $\mathrm{Re}(z) > 2\ell$. *The residue at* $z = 2\ell$ *of its trace exists, so that the spectral dimension is* 2ℓ.

Theorem

The operator $K_{2\rho}|D_N|^{-z}$ is trace-class for $\mathrm{Re}(z) > 2\ell$. The residue at $z = 2\ell$ of its trace exists, so that the spectral dimension is 2ℓ.

- The results above remain valid if $K_{2\rho}$ is replaced by $K_{2\rho}^{-1}$, by a property of the quantum dimension.
- This implies that the functional on $\mathcal{A}(\mathbb{C}P_q^{\ell})^{\otimes(2\ell+1)}$ defined by

$$\tilde{\psi}(a_0, \cdots, a_{2\ell}) = \operatorname*{Res}_{z=2\ell} \mathrm{Tr}(K_{2\rho}^{-1} a_0 [D_N, a_1] \cdots [D_N, a_{2\ell}] |D_N|^{-z})$$

is a twisted cocycle with twist ϑ^{-1}.

- This kind of result also seems to hold for quantum Grassmannians (work in progress). This exhausts the class of quantum irreducible generalized flag manifolds corresponding to $G = SL(n+1)$, for which the results of [Krähmer (2003)] apply.
- In the setting of modular Fredholm modules similar results have been observed in some examples [Rennie, Sitarz, Yamashita (2013)]. This can be adapted to a larger class of spaces (work in progress).
- In a sense our discussion reproduces Weyl's law

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{V_n}{(2\pi)^n} \mathrm{vol}(\Omega)$$

when both sides are interpreted appropriately.

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Thank you for your attention!