# Operator algebraic properties for free wreath products by quantum permutation groups

-Conférence du GDRE "Noncommutative Geometry and Applications"-

François Lemeux
(part. joint work with Pierre Tarrago)

Université de Franche-Comté

*francois.lemeux@univ-fcomte.fr*

June 2014, 19th - Villa Mondragone

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## Definition (Woronowicz 80')

*$\mathbb{G}=(C(\mathbb{G}),\Delta)$ GQC : $C(\mathbb{G})$ Woronowicz $C^*$-algebra ; $C(\mathbb{G})$ unital, $\Delta : C(\mathbb{G})\to C(\mathbb{G})\otimes_{\min} C(\mathbb{G})$ s.t.*

1. $(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$,
2. $\{\Delta(a)(b\otimes 1) : a,b\in C(\mathbb{G})\}$ *et* $\{\Delta(a)(1\otimes b) : a,b\in C(\mathbb{G})\}$ *lin. dense in* $C(\mathbb{G})\otimes C(\mathbb{G})$.

Peter-Weyl theory : Corep. $u\in M_N(C(\mathbb{G}))\simeq M_N(\mathbb{C})\otimes C(\mathbb{G})$, $\Delta(u_{ij})=\sum_{k=1}^N u_{ik}\otimes u_{kj}$.

- $\mathrm{Hom}(u;v)=\{T\in M_{n_v,n_u}(\mathbb{C}) : v(T\otimes 1)=(T\otimes 1)u\}$,
- $u\sim v$, $\exists T$ invertible $T\in\mathrm{Hom}(u;v)$,
- $u$ is irreducible if $\mathrm{Hom}(u;u)=\mathbb{C}\mathrm{id}$.

## Theorem (Woronowicz)

*Let $\mathbb{G}=(C(\mathbb{G}),\Delta)$ be a GQC. The corepresentations of $C(\mathbb{G})$ decompose as direct sums of irreducibles.*

We consider the unital $C^*$-algebra defined by generators and relations:

$$C^*_{com} - \langle s_{ij} : 1 \leq i, j \leq N : (s_{ij}) \text{ magic unitary} \rangle \simeq C(S_N)$$

$$s_{ij} \mapsto (\sigma \in S_N \subset M_N(\mathbb{C}) \mapsto \sigma_{ij}).$$

Magic unitary: $(s_{ij})$ unitary matrix whose entries are projections which sum up to 1 on each row and column.

Removing the commutativity:

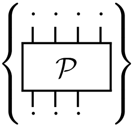
$$C(S_N^+) := C^* - \langle v_{ij} : 1 \leq i, j \leq N : (v_{ij}) \text{ magic unitary} \rangle,$$

one obtains a new $C^*$-algebra for $N \geq 4$. We have the coproduct on $C(S_N^+)$:

$$\Delta : C(S_N^+) \to C(S_N^+) \otimes C(S_N^+), \ \Delta(v_{ij}) = \sum_{k=1}^{N} v_{ik} \otimes v_{kj}.$$

$S_N^+ = (C(S_N^+), \Delta)$ is the quantum permutation group (Wang 98).

We denote by $NC(k,l)$ the set of non-crossing partitions on $k+l$ points:

$p =$ 

$\mathcal{P}$ non-crossing diagram.

## Theorem (Banica 99)

*$Hom_{S_N^+}(v^{\otimes k}; v^{\otimes l}) = span\{T_p : p \in NC(k,l)\}$, $T_p \in \mathcal{B}(\mathbb{C}^{N^{\otimes k}}; \mathbb{C}^{N^{\otimes l}})$ :*
*$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1,\dots,j_l} \delta_p(\underline{i},\underline{j}) e_{j_1} \otimes \cdots \otimes e_{j_l}$.*

## Corollaire (Banica 99)

*The irreducible corepresentations of $S_N^+$ can be labeled by $\mathbb{N}$ with*

- *$v^{(0)} = 1$ is the trivial representation and $v = 1 \oplus v^{(1)}$.*
- *$\overline{v^{(k)}} = (v_{ij}^{(k)*})$ is equivalent to $v^{(k)}, \forall k \in \mathbb{N}$.*
- *$\forall k, l \in \mathbb{N}, v^{(k)} \otimes v^{(l)} = \bigoplus_{r=0}^{2\min(k,l)} v^{(k+l-r)}$ (Clebsch-Gordan).*

## Definition (Bichon 00')

$H_N^+(\Gamma) := (C(H_N^+(\Gamma)), \Delta)$ *where* $C(H_N^+(\Gamma))$ *is the* $C^*$*-algebra generated by the elements* $a_{ij}(g), i, j = 1, \ldots, N$ *s.t.* $\forall g, h \in \Gamma$,

- $a_{ij}(g)a_{ik}(h) = \delta_{j,k} a_{ij}(gh), \quad a_{ji}(g)a_{ki}(h) = \delta_{j,k} a_{ji}(gh)$,
- $\sum_i a_{ij}(e) = 1 = \sum_j a_{ij}(e)$,
- $\Delta(a_{ij}(g)) = \sum_{k=1}^N a_{ik}(g) \otimes a_{kj}(g)$.

Bichon : $H_N^+(\Gamma) \simeq \widehat{\Gamma} \wr_* S_N^+$ where

$$C(\widehat{\Gamma} \wr_* S_N^+) := C^*(\Gamma)^{*N} * C(S_N^+)/\langle g^{(i)} v_{ij} - v_{ij} g^{(i)} = 0 \rangle$$

via $a_{ij}(g) \mapsto g^{(i)} v_{ij} = v_{ij} g^{(i)}$.

## Example

- $\Gamma = \{e\}$ *trivial :* $S_N^+$.
- $\Gamma = \mathbb{Z}/s\mathbb{Z}$ *: quantum reflection groups* $H_N^{s+}$.

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• **Fusion rules for quantum reflection groups**

Banica and Vergnioux obtained a combinatorial description of the intertwiner spaces for $H_N^{s+} = H_N^{+}(\mathbb{Z}/s\mathbb{Z})$ and then deduced the fusion rules:

**Theorem (Banica, Vergnioux 08)**

*The irreducible representations of $H_N^{s+}$ can be labelled by the worlds $(i_1, ..., i_k)$ whose letters are in $\mathbb{Z}/s\mathbb{Z}$, with involution $\overline{(i_1, \ldots, i_k)} = (-i_k, \ldots, -i_1)$ and the fusion rules:*

$$\begin{aligned}&(i_1, \ldots, i_k) \otimes (j_1, \ldots, j_l)\\ &= (i_1, \ldots, i_{k-1}, i_k, j_1, j_2, \ldots, j_l) \oplus (i_1, \ldots, i_{k-1}, i_k + j_1, j_2, \ldots, j_l)\\ &\oplus \delta_{i_k+j_1,0[s]}(i_1, \ldots, i_{k-1}) \otimes (j_2, \ldots, j_l)\end{aligned}$$

## • Operator algebraic properties for CQG

Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a GQC whose Haar state $h$ is a trace.

$$L^\infty(\mathbb{G}) := \overline{C_r(\mathbb{G})}^{\sigma w}, C_r(\mathbb{G}) = \pi_h(C(\mathbb{G})) \simeq C(\mathbb{G})/ker(\pi_h).$$

Notations:

- $Pol(\mathbb{G}) \subset C(\mathbb{G})$ sub-$*$-algebra (dense) generated by the coefficients of irreducible corepresentations,
- $C(\mathbb{G})_0 = C^* - \langle \sum_i U_{ii} : U \in Irr(\mathbb{G}) \rangle$ central algebra.

Some results:

- $C_r(U_N^+)$ is simple with unique trace, $N \geq 2$ (Banica 99).
- $C_r(O_N^+)$ is simple with unique trace, $L^\infty(O_N^+)$ is a full $II_1$ factor, $N \geq 3$ (Vaes and Vergnioux 07).
- $C_r(S_N^+)$ is simple with unique trace, $L^\infty(S_N^+)$ is a full $II_1$ factor, $N \geq 8$ (Brannan 13).
- $L^\infty(O_N^+), L^\infty(U_N^+)$, $L^\infty(S_N^+)$ have the Haagerup property, $N \geq 2$ (Brannan 12, 13).

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Intertwiner spaces in $H_N^+(\Gamma)$:
Strategy: Find a CQG $\mathbb{G} = (C(\mathbb{G}), \Delta)$,

- s.t. we have a surjective morphisme $\pi : C(\mathbb{G}) \twoheadrightarrow C(H_N^+(\Gamma))$,

$$\rightarrow \text{ If } \Gamma = \langle S \rangle, |S| = p, \ \mathbb{G} = *_{i=1}^{p}(H_N^{\infty+})$$

- s.t. the intertwiner spaces in $\mathbb{G}$ have a combinatorial description,
- s.t. the kernel of $\pi$ admits a combinatorial description.

$\Rightarrow$ The intertwiners in $H_N^+(\Gamma)$ are given by the intertwiners in $\mathbb{G} = *_{i=1}^{p}(H_N^{\infty+})$ and by the relations in the kernel.

$\rightsquigarrow$ Combinatorial description of the intertwiner spaces for tensor products of the corepresentations $a(g) := (a_{ij}(g))_{1 \leq i,j \leq N}$, $g \in \Gamma$.

## Theorem (L.)

*Let $\Gamma$ be discrete $N \geq 4$.*

$$Hom_{H_N^+(\Gamma)}(a(g_1) \otimes \cdots \otimes a(g_k); a(h_1) \otimes \cdots \otimes a(h_l)) = span\{T_p : p \in NC_\Gamma(g_1, \dots, g_k; h_1, \dots, h_l)\}$$

$NC_\Gamma(g_1, \dots, g_k; h_1, \dots, h_l)$ *: NC part. s.t. in each bloc* $\prod g_i = \prod h_j$.

## Theorem (L.)

*The irreducible corepresentations of $H_N^+(\Gamma)$ can be indexed by the words $(g_1, \dots, g_k)$, $g_i \in \Gamma$, with involution $\overline{(g_1, \dots, g_k)} = (g_k^{-1}, \dots, g_1^{-1})$ and fusion rules:*

$$\begin{aligned}&(g_1, \dots, g_k) \otimes (h_1, \dots, h_l)\\ &= (g_1, \dots, g_{k-1}, g_k, h_1, h_2, \dots, h_l) \oplus (g_1, \dots, g_{k-1}, g_k h_1, h_2, \dots, h_l)\\ &\oplus \delta_{g_k h_1, e}(g_1, \dots, g_{k-1}) \otimes (h_2, \dots, h_l).\end{aligned}$$

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Irreducible + fusion rules for $H_N^+(\Gamma)$: allow to prove several interesting properties for the associated operator algebras.

**Theorem (L.)**

*The von Neumann algebras $L^\infty(H_N^+(\Gamma))$ have the Haagerup property for all $N \geq 4$ and all finite groups $\Gamma$.*

Strategy:

- Construct convolution operators on $L^\infty(H_N^+(\Gamma))$ from states on the central algebra $C(H_N^+(\Gamma))_0$ (Brannan 12),
- Understand $\pi : C(H_N^+(\Gamma))_0 \to C(S_N^+)_0 \simeq C([0, N])$,
- Consider the states on $C(H_N^+(\Gamma))_0$, given by $ev_x \circ \pi$ + estimates on Tchebytchev polynomials.

## Theorem (L.)

*The reduced $C^*$-algebra $C_r(H_N^+(\Gamma))$ is simple with unique trace for all $N \geq 8$ and all discrete groups $\Gamma$.*

Strategy:

- Adapt Powers methods ($C_r^*(F_N)$ is simple),
- Conditional expectation $P : C_r(H_N^+(\Gamma)) \twoheadrightarrow C_r(S_N^+)$,
- Simplicity of $C_r(S_N^+)$ (for $N \geq 8$, Brannan).

## Theorem (L.)

*$L^\infty(H_N^+(\Gamma))$ is a full $II_1$ factor for all $N \geq 8$ and all discrete groups $\Gamma$.*

Strategy:

- Adapt "14-$\epsilon$" (Murray-von Neumann $L(F_N)$ does not have prop $\Gamma$),
- $L^\infty(H_N^+(\Gamma)) = M \oplus N$, $M \simeq L^\infty(S_N^+)$,
- $L^\infty(S_N^+)$ is full ($N \geq 8$, Brannan).

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Let $\mathbb{G}$ be a GQCM of Kac type generated by a unitary $u = (u_{kl})_{kl}$. Let $v = (v_{ij})$ be a magic unitary generating $C(S_N^+)$, $N \geq 4$:

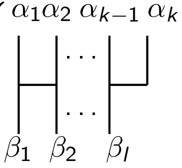
$$C(\mathbb{G}) *_w C(S_N^+) := C(\mathbb{G})^{*N} * C(S_N^+)/\langle [u_{kl}^{(i)}, v_{ij}] = 0 \rangle$$

Bichon proved:

- $\mathbb{G} \wr_* S_N^+ = (C(\mathbb{G}) *_w C(S_N^+), \Delta)$ is a GQCM of Kac type with a coproduct $\Delta$.
- From representations $\alpha \in Rep(\mathbb{G})$, one can construct representations $r(\alpha) = \left( v_{ij} \alpha_{kl}^{(i)} \right)_{1 \leq i,j \leq N}^{1 \leq k,l \leq d_\alpha}$ of $\mathbb{G} \wr_* S_N^+$.

Description of intertwiners:

$$R_p \in \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha_1) \otimes \cdots \otimes r(\alpha_k); r(\beta_1) \otimes \cdots \otimes r(\beta_l)) \subset$$
$$\subset \mathcal{B}\left( (\mathbb{C}^N \otimes H^{\alpha_1}) \otimes \cdots \otimes (\mathbb{C}^N \otimes H^{\alpha_k}); (\mathbb{C}^N \otimes H^{\beta_1}) \otimes \cdots \otimes (\mathbb{C}^N \otimes H^{\beta_l}) \right)?$$

$R_p$ associated to $p = \left\{\right.$ 

$\left.\right\} \in NC_{\mathbb{G}}((\alpha_1, \ldots, \alpha_k); (\beta_1, \ldots, \beta_l))$

where $NC_{\mathbb{G}}$ is the set of non-crossing partitions $p$ s.t.

- the points of $p$ are decorated by the representations of $\mathbb{G}$,
- the blocks of $p \in NC_{\mathbb{G}}$ are decorated by the morphisms of $\mathbb{G}$.

Projet: Monoidal equivalence $\mathbb{G} \wr_* S_N^+ \simeq_{mon} \mathbb{H}$ with $\mathbb{H} = (C(\mathbb{H}), \Delta)$:

- $C(\mathbb{H}) \subset C(\mathbb{G}) * C(SU_q(2))$ generated by the coefficients of $s(\alpha) = b \otimes \alpha \otimes b$,
- $\alpha \in Rep(\mathbb{G})$, $q + q^{-1} = \sqrt{N}$, $0 < q \leq 1$, $b$ if the fundamental representation of $SU_q(2)$.

$\Rightarrow$ This monoidal equivalence and the work of De Commer, Freslon, Yamashita (13) imply in particular that $L^\infty(\mathbb{G} \wr_* S_N^+)$ has the Haagerup property if and only if $L^\infty(\mathbb{G})$ has the Haagerup property.