Differentiable absorption of Hilbert C*-modules

Inter Gravissimas...

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Hilbert C^*-modules

Setting

1. *X is a countably generated Hilbert C^*-module over a C^*-algebra A. Thus, exists a sequence $\{\xi_n\}$ such that*

$$\mathrm{span}\{\xi_n \cdot a \mid n \in \mathbb{N}\, , \ a \in A\}$$

is dense in X.

2. *H_A denotes the standard module over A. Thus, H_A consists of sequences $\{a_n\}$ with $\sum a_n^* a_n$ convergent in A.*

Kasparov's absorption theorem

Theorem (Kasparov)

There exists a bounded adjointable isometry $W : X \to H_A$.

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Corollary

X is unitarily isomorphic to PH_A, where $P := WW^* : H_A \to H_A$ is an orthogonal projection.

Unbounded derivations 1

Setting

1. *Y Hilbert C^*-module.*
2. *$\rho : A \to \mathscr{L}(Y)$ injective $*$-homomorphism.*
3. *$D : \mathscr{D}(D) \to Y$ unbounded selfadjoint and regular operator.*

Unbounded derivations 1

Setting

1. Y *Hilbert* C^**-module.*
2. $\rho : A \to \mathscr{L}(Y)$ *injective* $*$*-homomorphism.*
3. $D : \mathscr{D}(D) \to Y$ *unbounded selfadjoint and regular operator.*

Remark

$D : \mathscr{D}(D) \to Y$ *is selfadjoint and regular if and only if* $D : \mathscr{D}(D) \to Y$ *is selfadjoint and* $D \pm i : \mathscr{D}(D) \to Y$ *are surjective.*

Unbounded derivations 2

Assumption

There exists a dense $*$-subalgebra $\mathscr{A} \subseteq A$ such that

1. $\rho(a) : \mathscr{D}(D) \to \mathscr{D}(D)$.
2. $[D, \rho(a)] : \mathscr{D}(D) \to Y$ *extends to a bounded operator* $\delta(a) : Y \to Y$.

Unbounded derivations 2

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Remark

$\delta : \mathscr{A} \to \mathscr{L}(Y)$ *is a closed derivation with* $\delta(a^*) = -\delta(a)^*$

Questions

Question

1. *What can be said about the derivative* $\delta(P) = \{\delta(P_{ij})\}$*?*

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Remark

In general $\delta(P)$ *is not bounded. One counter example comes from the Hopf fibration* $S^3 \to S^2$*.*

Differentiable compact operators

Definition

The differentiable compact operators is the completion of $M_\infty(\mathscr{A})$ *in the norm*

$$\|\{a_{ij}\}\|_\delta := \|\{a_{ij}\}\| + \|\{\delta(a_{ij})\}\|$$

This Banach $*$*-algebra is denoted by* $\mathscr{K}(H_A)_\delta$.

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Remark

The differentiable compact operators form a dense $*$-subalgebra of the compact operators $\mathscr{K}(H_A)$ on the standard module.

Differentiability of generators

Assumption

There exists a sequence $\{\xi_n\}_{n=1}^\infty$ *in* X *such that*

1. $\{\xi_n\}$ *generates* X *as a Hilbert* C^**-module.*
2. $\langle \xi_n, \xi_m \rangle \in \mathscr{A}$.

Differentiable absorption theorem

Theorem (K.)

There exists a bounded adjointable isometry $W : X \to H_A$ and a positive selfadjoint bounded operator $K : H_A \to H_A$ such that

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4. $PK^2 \in \mathscr{K}(H_A)_\delta$.

Application: Graßmann connections

Definition

The continuous δ-forms is the smallest C^*-subalgebra of $\mathscr{L}(Y)$ which contains all $\delta(a)$ and $\rho(a)$. This C^*-algebra is denoted by $\Omega_\delta(A)$.

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Remark

There is a well-defined pairing

$$(\cdot,\cdot) : X \times X \widehat{\otimes}_A \Omega_\delta(A) \to \Omega_\delta(A)$$

such that $(\xi, \eta \otimes \omega) := \rho(\langle \xi, \eta \rangle) \cdot \omega$

Application: Graßmann connections 2

Theorem (K.)

There exists a dense $\mathscr{A}$-submodule $\mathscr{X} \subseteq X$ and a $\mathbb{C}$-linear map $\nabla_\delta : \mathscr{X} \to X \widehat{\otimes}_A \Omega_\delta(A)$ such that

Application: Graßmann connections 2

Theorem (K.)

There exists a dense $\mathscr{A}$*-submodule* $\mathscr{X} \subseteq X$ *and a* $\mathbb{C}$*-linear map* $\nabla_\delta : \mathscr{X} \to X \widehat{\otimes}_A \Omega_\delta(A)$ *such that*

1. $\nabla_\delta(\xi \cdot a) = \nabla_\delta(\xi) \cdot a + \xi \otimes \delta(a)$.

Application: Graßmann connections 2

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1. $\nabla_\delta(\xi \cdot a) = \nabla_\delta(\xi) \cdot a + \xi \otimes \delta(a)$.
2. $\delta(\langle \xi, \eta \rangle) = \big(\xi, \nabla_\delta(\eta)\big) - \big(\eta, \nabla_\delta(\xi)\big)^*$.

Differentiable absorption theorem

Theorem

There exists a bounded adjointable isometry $W : X \to H_A$ and a positive selfadjoint bounded operator $K : H_A \to H_A$ such that

1. $KP = PK$.
2. $W^* K W : X \to X$ *has dense image.*
3. $PK \in \mathscr{K}(H_A)$.
4. $PK^2 \in \mathscr{K}(H_A)_\delta$.

Symmetric lifts of unbounded operators 1

Remark

1. *The isometry* $W : X \to H_A$ *induces an isometry* $W : X \widehat{\otimes}_A Y \to H_Y$.

Symmetric lifts of unbounded operators 1

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1. *The isometry* $W : X \to H_A$ *induces an isometry* $W : X \widehat{\otimes}_A Y \to H_Y$.
2. *The unbounded diagonal operator*

$$\mathrm{diag}(D) : \mathscr{D}(\mathrm{diag}(D)) \to H_Y \qquad \{y_n\} \mapsto \{Dy_n\}$$

is selfadjoint and regular.

Symmetric lifts of unbounded operators 2

Definition

The symmetric lift of $D : \mathscr{D}(D) \to Y$ *is the composition*

$$W^* \mathrm{diag}(D) W : \mathscr{D}\big(\mathrm{diag}(D)W\big) \to X \widehat{\otimes}_A Y$$

The symmetric lift is denoted by $1 \otimes_\nabla D$.

Symmetric lifts of unbounded operators 2

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The symmetric lift of $D : \mathscr{D}(D) \to Y$ *is the composition*

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The symmetric lift is denoted by $1 \otimes_\nabla D$.

Proposition

$1 \otimes_\nabla D$ *is densely defined and symmetric. Furthermore,*

$$(1 \otimes_\nabla D)(x \otimes y) = \nabla_\delta(x)(y) + x \otimes D(y)$$

for all $x \in \mathscr{X}$ *and all* $y \in \mathscr{D}(D)$.

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1. *What can be said about the symmetric lift* $1 \otimes_\nabla D$*?*

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3. *Is it regular?*

Counterexample: Selfadjointness

Setting

1. *$X := C_0\big((0,\infty)\big)$ as a Hilbert C^*-module over $A := C_0(\mathbb{R})$.*
2. *$\rho : C_0(\mathbb{R}) \to \mathscr{L}\big(L^2(\mathbb{R})\big)$ given by pointwise multiplication.*
3. *$D := i\frac{d}{dt} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$*

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2. *$\rho : C_0(\mathbb{R}) \to \mathscr{L}\big(L^2(\mathbb{R})\big)$ given by pointwise multiplication.*
3. *$D := i\frac{d}{dt} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$*

Proposition

$1 \otimes_\nabla D$ is a symmetric extension of $i\frac{d}{dt} : C_c^\infty\big((0,\infty)\big) \to L^2\big((0,\infty)\big)$.

Selfadjoint and regular lifts

Remark

The selfadjoint and positive bounded operator

$$\Delta := W^* K^2 W \otimes 1 : X \widehat{\otimes}_A Y \to X \widehat{\otimes}_A Y$$

has dense image.

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has dense image.

Theorem (K.)

The unbounded operator

$$\Delta(1 \otimes_\nabla D)\Delta : \mathscr{D}((1 \otimes_\nabla D)\Delta) \to X \widehat{\otimes}_A Y$$

is densely defined and essentially selfadjoint and regular.