Functorial Rieffel deformations and tame smooth generalized crossed products

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Introduction

Joint work in progress with R. Meyer.

- Rieffel deformations: $G \curvearrowright \mathscr{A} \rightsquigarrow \mathscr{A}^{\phi}$.

Recent papers in this direction:

- Lechner and Waldmann (2011)
  - $G = \mathbb{R}^n$;
  - oscillatory integrals;
  - deformation of both algebras and modules.
- Brain, Landi and van Suijlkom (2013)
  - $G = T^N$;
  - functorial deformations.

Our aims:

- Functorial deformations.
- Actions of loc. compact Abelian group G.
- Avoid oscillatory integrals.

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Functorial deformation and categories

In this talk, we take G, loc. compact Abelian *Lie* group and

- $\mathscr{V}$, $\mathscr{W}$, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{M}$ are Fréchet spaces/algebras/modules.

Functorial deformations: certain morphisms are preserved.

If $\mathscr{A} \xrightarrow{\psi} \mathscr{B}$ in "good category", then $\mathscr{A}^\phi \xrightarrow{\psi^\phi} \mathscr{B}^\phi$.

- Cat. *smooth tempered rep.* of G, denoted ST-$\mathfrak{Rep}(G)$.
  $\rightsquigarrow$ char. property ST-$\mathfrak{Rep}$: $\mathscr{S}(G)\hat{\otimes}_{\mathscr{S}(G)}\mathscr{V} \simeq \mathscr{V}$.

Tensor prod. $\mathscr{V}_1\hat{\otimes}\mathscr{V}_2$ with diag. rep. of G is in ST-$\mathfrak{Rep}(G)$.

- ST-$\mathfrak{Rep}(G)$, monoidal category, for proj. tensor prod. $\hat{\otimes}$.
  $\rightsquigarrow$ notions of *algebras* $\mathscr{A}\hat{\otimes}\mathscr{A} \xrightarrow{m} \mathscr{A}$ and *modules*.

In our case: G-tempered algebras and G-tempered modules.

1. Natural transf. $\Phi^{\mathscr{A},\mathscr{A}}$
2. with associativity

$\rightsquigarrow$ On algebra $\mathscr{A}$, new prod. m^ϕ
$m^\phi := m \circ \Phi^{\mathscr{A},\mathscr{A}}$ $\rightsquigarrow \mathscr{A}^\phi$.

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Natural transformations and applications

Natural transformation: family of $\Phi^{\mathscr{V}} : \mathscr{V} \to \mathscr{V}$ s.t.

$\forall \mathscr{V}, \mathscr{W}$, this diagram commutes:

$$\begin{array}{ccc} \mathscr{V} & \longrightarrow & \mathscr{W} \\ {\scriptstyle \Phi^{\mathscr{V}}} \downarrow & & \downarrow {\scriptstyle \Phi^{\mathscr{W}}} \\ \mathscr{V} & \longrightarrow & \mathscr{W} \end{array}$$

Proposition

The natural transformations $\mathscr{V}_1 \hat{\otimes} \mathscr{V}_2 \to \mathscr{V}_1 \hat{\otimes} \mathscr{V}_2$ are in bijection with the multiplier algebra of $\mathscr{S}(G \times G)$.

Proof: next 2 slides.

Skip

$$\mu((f_0 \otimes f_0') * (f_1 \otimes f_1')) = (f_0 \otimes f_0') * \mu(f_1 \otimes f_1'). \quad \text{(Multiplier)}$$

Multiplier μ_ϕ: take $\phi \in \hat{G} \times \hat{G} \to \mathbb{C}$ "regular" and $f \in \mathscr{S}(G^2)$,

$$\mu_\phi(f) := \mathcal{F}^{-1}(\phi \cdot \mathcal{F}(f)) = \mathcal{F}^{-1}(\phi) * f,$$

then μ_ϕ is a multiplier of $\mathscr{S}(G \times G)$.

Natural transformations: proof – 1

From natural transformation to multiplier μ on $\mathscr{S}(G \times G)$.

$$\mu((f_0 \otimes f_0') * (f_1 \otimes f_1')) = (f_0 \otimes f_0') * \mu(f_1 \otimes f_1'). \quad \text{(Multiplier)}$$

Steps:

- Specialize: $\mathscr{V}_j = \mathscr{W}_j = \mathscr{S}(G)$,
- $m_{f_0 \otimes f_0'}(f_1 \otimes f_1') = (f_0 \otimes f_0') * (f_1 \otimes f_1')$.
- Set $\mu := \Phi^{\mathscr{S}(G), \mathscr{S}(G)}$.

$$\begin{array}{ccc}
\mathscr{S}(G) \otimes \mathscr{S}(G) & \xrightarrow{m_{f_0 \otimes f_0'}} & \mathscr{S}(G) \hat{\otimes} \mathscr{S}(G) \\
\mu \downarrow & & \downarrow \mu \\
\mathscr{S}(G) \otimes \mathscr{S}(G) & \xrightarrow{m_{f_0 \otimes f_0'}} & \mathscr{S}(G) \hat{\otimes} \mathscr{S}(G).
\end{array}$$

Outcome: μ is a multiplier.

Natural transformations: proof – 2

From multiplier μ on $\mathscr{S}(G \times G)$ to natural transformations.

We set $\Phi^{\mathscr{V}_1,\mathscr{V}_2} := \mu \otimes \mathrm{Id}_{\mathscr{V}_1 \hat{\otimes} \mathscr{V}_2}$,

$$\text{on } \mathscr{S}(G \times G)\hat{\otimes}_{\mathscr{S}(G\times G)}\mathscr{V}_1\hat{\otimes}\mathscr{V}_2 \simeq \mathscr{V}_1\hat{\otimes}\mathscr{V}_2.$$

With these definitions, using that μ is a multiplier:

- $\Phi^{\mathscr{V}_1,\mathscr{V}_2}$ is well-defined on the $\mathscr{S}(G^2)\hat{\otimes}_{\mathscr{S}(G^2)}\mathscr{V}_1\hat{\otimes}\mathscr{V}_2$.
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Outcome: natural transformation defined by family $\Phi^{\mathscr{V}_1,\mathscr{V}_2}$.

NB: nat. transfo. construction replaces oscillatory integrals!

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Equivalence of monoidal categories

Coherence conditions for equivalence of monoidal cat.

1. $\Phi^{\mathbb{1},\mathscr{V}}: \mathbb{1}\hat{\otimes}\mathscr{V} \to \mathbb{1}\hat{\otimes}\mathscr{V}$ and $\Phi^{\mathscr{V},\mathbb{1}}: \mathscr{V}\hat{\otimes}\mathbb{1} \to \mathscr{V}\hat{\otimes}\mathbb{1}$ are Id.
2. For all $\mathscr{V}_1, \mathscr{V}_2$ and $\mathscr{V}_3$, the following commutes:

$$\begin{array}{ccc}
\mathscr{V}_1\hat{\otimes}\mathscr{V}_2\hat{\otimes}\mathscr{V}_3 & \xrightarrow{\Phi^{\mathscr{V}_1\hat{\otimes}\mathscr{V}_2,\mathscr{V}_3}} & \mathscr{V}_1\hat{\otimes}\mathscr{V}_2\hat{\otimes}\mathscr{V}_3 \\
\Phi^{\mathscr{V}_1,\mathscr{V}_2\hat{\otimes}\mathscr{V}_3}\downarrow & & \downarrow\Phi^{\mathscr{V}_1,\mathscr{V}_2}\hat{\otimes}\,\mathrm{Id}_{\mathscr{V}_3} \\
\mathscr{V}_1\hat{\otimes}\mathscr{V}_2\hat{\otimes}\mathscr{V}_3 & \xrightarrow{\mathrm{Id}_{\mathscr{V}_1}\hat{\otimes}\Phi^{\mathscr{V}_2,\mathscr{V}_3}} & \mathscr{V}_1\hat{\otimes}\mathscr{V}_2\hat{\otimes}\mathscr{V}_3
\end{array}$$

If the gen. multiplier is μ_ϕ, above imposes

ϕ is a *normalised cocycle i.e.*

1. $\phi(1,\hat{\gamma}) = 1$ et $\phi(\hat{\gamma},1) = 1$ (Normalisation)
2. $\phi(\hat{\gamma}_1\hat{\gamma}_2,\hat{\gamma}_3)\phi(\hat{\gamma}_1,\hat{\gamma}_2) = \phi(\hat{\gamma}_1,\hat{\gamma}_2\hat{\gamma}_3)\phi(\hat{\gamma}_2,\hat{\gamma}_3)$ (Cocycle)
3. $\mathcal{F}(\phi) * \mathscr{S}(G\times G) \subseteq \mathscr{S}(G\times G)$ (Regularity)

Summary of the construction

Starting point:

- *tame smooth* rep. of G, *i.e.* $\mathscr{S}(G)\hat{\otimes}_{\mathscr{S}(G)}\mathscr{V}_j \simeq \mathscr{V}_j$.
- $\phi\colon \hat{G} \times \hat{G} \to \mathbb{C}$ a *normalised cocycle*, *i.e.*
  1. $\phi(1, \hat{\gamma}) = 1$ et $\phi(\hat{\gamma}, 1) = 1$
  2. $\phi(\hat{\gamma}_1\hat{\gamma}_2, \hat{\gamma}_3)\phi(\hat{\gamma}_1, \hat{\gamma}_2) = \phi(\hat{\gamma}_1, \hat{\gamma}_2\hat{\gamma}_3)\phi(\hat{\gamma}_2, \hat{\gamma}_3)$
  3. $\mathcal{F}(\phi) * \mathscr{S}(G \times G) \subseteq \mathscr{S}(G \times G)$

Outcome:

- G-algebra $\mathscr{A}$: new product $m^\phi := m \circ \Phi^{\mathscr{A},\mathscr{A}} \rightsquigarrow \mathscr{A}^\phi$.
- G-module $\mathscr{M}$: new action $\alpha^\phi := \alpha \circ \Phi^{\mathscr{A},\mathscr{M}} \rightsquigarrow \mathscr{M}^\phi$.

Construction possible for general loc. compact Ab. groups G!

Case of $G = S^1 \times \mathbb{Z}$

For $G = S^1 \times \mathbb{Z}$, we have $\hat{G} = \mathbb{Z} \times S^1$ and consider:

$$\phi((n_1, \omega_1), (n_2, \omega_2)) = \omega_1^{n_2}.$$

Let G act on a C^*-algebra A by $t \mapsto \alpha_t(a)$ and $\sigma \colon A \to A$. Define the Fréchet subalgebra $\mathscr{A}$ by:

$$\mathscr{A} := \{a \in A : t \mapsto \alpha_t(a) \text{ is smooth}\}.$$

- The representation of G on $\mathscr{A}$ is smooth tempered.
- Set $\mathscr{A}_n := \{a \in \mathscr{A} | \alpha_t(a) = e^{i2\pi nt} a\}$.
- For $a_n \in \mathscr{A}_n$ and $a_m \in \mathscr{A}_m$, deformed product $\times^\phi$:

$$a_n \times^\phi a_m = \sigma^{-m}(a_n) a_m.$$

Particular case: $A = C(T^2)$, $\alpha_{t,n}(a)(x, y) = a(x + t, y - n\theta)$.

$\rightsquigarrow$ Recover the NC torus in this way!

Representation and deformation of the involution

In the case of $G = S^1 \times \mathbb{Z}$ and ϕ as before.

Enriching the setting: "involutive" Fréchet algebras $\mathscr{A}$,
with involution inherited from G-cov. rep. π on $\mathscr{H}$.

- Introduce $\mathscr{V} := \mathscr{H}^\infty$ – it has a smooth temperated G-rep.
- Deform π into $\pi^\phi(a)\xi_m := \pi(\sigma^{-m}(a))\xi_m$.
- Given a gauge-homogeneous element a with gauge k,

$$a^{*_\phi} := \sigma^k(a^*).$$

Outline

Tame smooth Generalized Crossed Products (GCP)

Involutive Fréchet alg. $\mathscr{A}$, with smooth σ and grading $\mathscr{A}_k$.

1. All bimodules $\mathscr{A}_k$ admit *frames* $(\eta_j^{(k)})_j$ *i.e.*

$$\sum_j \eta_j^{(k)} (\eta_j^{(k)})^* = 1.$$

Existence of such frames: principal $U(1)$-bundle.

2. The size of frames is uniformely bounded for $k \in \mathbb{N}$.
3. Ordering the frames in lexicographical order of (j, k), for all continuous seminorm p, the (real) sequences $p(\xi_\ell)$ and $p\Big((\xi_\ell)^*\Big)$ have polynomial growth (in ℓ).

Under these conditions, $\mathscr{A}$ is a *tame smooth* ppal $U(1)$-bundle.

Deformation

Tame smooth GCP: main result

Theorem (G. & Grensing – 2011)

Given a smooth tame GCP (previous slide),

1. Exact sequence in HP^*, with $\mathscr{B} := \mathscr{A}_0$:

$$\begin{array}{ccccc} HP^0(\mathscr{B}) & \longleftarrow & HP^0(\mathscr{B}) & \longleftarrow & HP^0(\mathscr{A}) \\ \downarrow \# & & & & \# \uparrow \\ HP^1(\mathscr{A}) & \longrightarrow & HP^1(\mathscr{B}) & \longrightarrow & HP^1(\mathscr{B}). \end{array}$$

2. Transfer formula: $\forall K \in K_j(\mathscr{A})$, $\forall \varphi \in HP^{j+1}(\mathscr{B})$,

$$\langle [K], \#\varphi \rangle = 2i\pi \langle \partial [K], \varphi \rangle.$$

Remarks:

1. is a generalization of results by Nest (1988).
2. relies on previous results by Nistor (1997).

Outline

Invariance of tame smooth GCP under deform.

Proposition

If
- $\mathscr{A}$ is a smooth tame GCP,
- with involution inherited from a G-cov. rep.
- σ, polynomial bound for all fixed seminorm p.

then the deformed algebra $\mathscr{A}^\phi$ is also a smooth tame GCP.

The gauge action of $\mathscr{A}^\phi$ is unchanged.

1. Given frames $(\eta_j^{(k)})$ for $\mathscr{A}$, we have:

$$\sum \eta_j^{(k)} \times^\phi (\eta_j^{(k)})^{*_\phi} = \sum \sigma(\eta_j^{(k)})\sigma(\eta_j^{(k)})^* = 1.$$

2. The size of frames remains the same.
3. The growth condition is preserved,
   since the Fréchet structure is unaltered and σ isom.

Example: quantum Heisenberg manifolds.

Def. of tame smooth GCP

Tame smooth GCP: stability of HP^*

Theorem (G. & Meyer – 2014)

If $\mathscr{A}$ is a tame smooth GCP

- with involution inherited from a G-cov. rep.
- σ, polynomial bound for all fixed seminorm p,
- there is a path σ_u, $u \in [0,1]$ with $\sigma_0 = \mathrm{Id}$, $\sigma_1 = \sigma$ and $\forall a \in \mathscr{A}$, $u \mapsto \sigma_u^{-1}(a)$ smooth map,

then $HP^j(\mathscr{A}^\phi) = HP^j(\mathscr{A})$.

Steps of the proof:

1. Both $\mathscr{A}$ and $\mathscr{A}^\phi$ are tame smooth GCP $\rightsquigarrow$ PV-sequences for HP.
2. Gauge-invariant subalgebra $\mathscr{B}$ left undeformed. $\rightsquigarrow$ To prove: $HP^j(\mathscr{B}) \to HP^j(\mathscr{B})$, same in both diag.
3. Conclude using quasi-homomorphisms and diffeotopy inv.

Outline

Tame smooth GCP: commutative case

Consider

- a compact manifold X, $B := C(X)$,
- a Hermitian line bundle $\mathcal{L} \to X$.

Write $P \to X$ for the assoc. ppal $U(1)$-bundle, $A := C(P)$.

Proposition (G.)

The smooth elements $\mathscr{A}$ of A form a *tame smooth* GCP for an explicit family $\left(v_\omega^{(\ell)}\right)_{\omega,\ell}$ of frames.

- The index ω correspond to a (finite) trivialisation of $\mathcal{L}$.
- The Fréchet structure is given by seminorms:

$$p(a) = \|\partial_1 \cdots \partial_N a\|$$

for derivations ∂_ι on P.

More details

Tame smooth GCP: main result – on the proof

Steps of the proof:

1. Define Toeplitz extension of Fréchet algebras:

$$0 \to \mathscr{C} \to \mathscr{T}_{\mathscr{A}} \to \mathscr{A} \to 0.$$

2. Deduce that there is a six-term exact sequence with $HP^j(\mathscr{C})$, $HP^j(\mathscr{T}_{\mathscr{A}})$ and $HP^j(\mathscr{A})$.
3. Prove Morita equivalence $\mathscr{C} \overset{M}{\simeq} \mathscr{B} \rightsquigarrow HP^j(\mathscr{C}) \simeq HP^j(\mathscr{B})$.
4. Use a *quasi-homomorphism* $\mathscr{T}_{\mathscr{A}} \to \mathscr{C}$ and $\mathscr{C} \overset{M}{\simeq} \mathscr{T}_{\mathscr{A}}$ to show $HP^j(\mathscr{T}_{\mathscr{A}}) \simeq HP^j(\mathscr{B})$.

End result:

a six-term exact sequence with $HP^j(\mathscr{B})$ and $HP^j(\mathscr{A})$.

Tame smooth GCP: stability of HP^*

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If $\mathscr{A}$ is a tame smooth GCP

- with involution inherited from a G-cov. rep.
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Toeplitz extensions and diagrams

- In Toeplitz extension, G-equiv. maps thus preserved:

$$0 \to \mathscr{C}^{\phi} \to \mathscr{T}_{\mathscr{A}^{\phi}} \to \mathscr{A}^{\phi} \to 0$$

Difference between $\mathscr{A}$ and $\mathscr{A}^{\phi}$:

$$\begin{array}{ccccc}
HP^0(\mathscr{B}) & \longleftarrow & HP^0(\mathscr{B}) & & \\
\big\downarrow \simeq & & \simeq \big\uparrow & & \\
HP^0(\mathscr{C}) & \longleftarrow & HP^0(\mathscr{T}_{\mathscr{A}}) & \longleftarrow & HP^0(\mathscr{A}) \\
\big\downarrow & & & & \big\uparrow \\
HP^1(\mathscr{A}) & \longrightarrow & HP^1(\mathscr{T}_{\mathscr{A}}) & \longrightarrow & HP^1(\mathscr{C}) \\
 & & \big\downarrow \simeq & & \simeq \big\uparrow \\
 & & HP^1(\mathscr{B}) & \longrightarrow & HP^1(\mathscr{B}).
\end{array}$$

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\end{array}$$

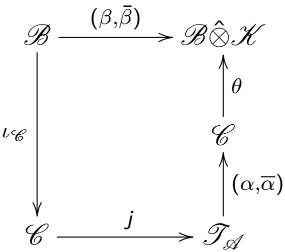
Maps $HP^j(\mathscr{B}) \to HP^j(\mathscr{B})$ – lifting in q-hom.

Quasi-homomorphism (Cuntz, '83) noted $(\alpha, \bar{\alpha})$: $\mathscr{A} \rightrightarrows \hat{\mathscr{B}} \trianglerighteq \mathscr{B}$

- α and $\bar{\alpha}$ homomorphisms from $\mathscr{A}$ to $\hat{\mathscr{B}}$, containing $\mathscr{B}$.
- $(\alpha - \bar{\alpha})(\mathscr{A}) \subseteq \mathscr{B}$, $\alpha(\mathscr{A})\mathscr{B} \subseteq \mathscr{B}$, $\mathscr{B}\alpha(\mathscr{A}) \subseteq \mathscr{B}$.

Such q-hom. induce maps $HP^j(\mathscr{A}) \to HP^j(\mathscr{B})$, $j = 0, 1$.

Original proof PV-sequence in HP (G. & Grensing, '11):



Applying HP^j, recover square.

- j canonical inclusion.
- $\iota_{\mathscr{C}}$ induce isom. in HP.
- $(\alpha, \bar{\alpha})$ explicit q-hom.
- θ induce isom. in HP.

$(\beta, \bar{\beta})$ defined by composition.

Explicitly, $\beta(b) = b \otimes e_{00}$ and $\bar{\beta}(b) = (\xi_i^* b \xi_j) \otimes e_{ij}$.

Maps $HP^j(\mathscr{B}) \to HP^j(\mathscr{B})$ – invariance under def.

β and $\bar\beta$ computed using ambiant algebra $\mathscr{A}^\phi$: $\beta(b) = b \otimes e_{00}$

$$\bar\beta(b) = (\xi_i^{*_\phi}) \times^\phi b \times^\phi \xi_j \otimes e_{ij} = \xi_i^* \sigma^{-1}(b)\xi_j \otimes e_{ij}$$

where ξ_j, $j = 1, \dots, N$ frame of $\mathscr{A}_1$.

- Given a path σ_u, $u \in [0,1]$ with $\sigma_0 = \mathrm{Id}$, $\sigma_1 = \sigma$ and $\forall a \in \mathscr{A}$, $u \mapsto \sigma_u^{-1}(a)$ smooth map,
  $\rightsquigarrow$ define q-hom. $(Z\beta, Z\bar\beta)\colon \mathscr{B} \rightrightarrows Z\mathscr{B}\hat\otimes\mathscr{K} \trianglerighteq Z\mathscr{B}\hat\otimes\mathscr{K}$.
- Diffeotopy invariance of HP:
  $(\beta_0, \bar\beta_0)$ and $(\beta_1, \bar\beta_1)$ induce the same map in HP.
- By construction, $(\beta_0, \bar\beta_0)$ and $(\beta_1, \bar\beta_1)$ factorise *via* $\mathscr{C} \to \mathscr{T}_{\mathscr{A}}$ and $\mathscr{C}^\phi \to \mathscr{T}_{\mathscr{A}^\phi}$, respectively.

Consequences:

- $HP^j(\mathscr{B}) \to HP^j(\mathscr{B})$, invariant under def. $\mathscr{A} \rightsquigarrow \mathscr{A}^\phi$.
- $HP^j(\mathscr{A})$ and $HP^j(\mathscr{A}^\phi)$ have the same dimension.

Outline

Outline

Conclusion

Summary:

- Functorial deformation for G, locally compact Abelian.
- Application to smooth tame GCP:
  - Construction of such GCP by deformation.
  - Stability of HP^* under deformation

O. G. and R. Meyer
Functorial Rieffel deformations for tempered actions on bornological algebras
In preparation.

Thank you for your attention!

...

Proof: Definition of Frames $v_\omega^{(l)}$

- Trivialise $\mathcal{L}$ by open sets $(U_\omega)_{\omega\in\Omega}$, Ω finite. Fix $\omega \in \Omega$.
- Pick a section u_ω of $\mathcal{L}$ s.t. $(u_\omega|u_\omega)(x) \leqslant 1$ for all $x \in X$ and $\forall x \in U_\omega$, $(u_\omega|u_\omega)(x) = 1$.
- Choose (χ_ω) associated to U_ω s.t. $\sum_\omega \chi_\omega^2 = 1$ and set

$$v_\omega^{(\ell)} := \chi_\omega u_\omega \otimes u_\omega \otimes \cdots \otimes u_\omega.$$

For any fixed ℓ, the family $(v_\omega^{(\ell)})_\omega$ is a frame, *i.e.*

$$\sum_\omega {}_B\langle v_\omega^{(\ell)}, v_\omega^{(\ell)}\rangle = 1.$$

First two conditions of “tame smoothness” are satisfied! ◂ Hypo.

Growth: suffices to consider each $\omega \in \Omega$ separately.
Hence, fix ω and write $\chi := \chi_\omega$, $u := u_\omega$, ...

Proof: Strategy of Evaluation

Aim: evaluate growth in ℓ of $w_\ell = \|\partial_1 \cdots \partial_N v^{(\ell)}\|$.

Strategy: separate dependencies on N and on ℓ.

- Since ∂_ι are derivations, we can expand

$$\partial_1 \cdots \partial_N \left(\chi u \otimes u \otimes \cdots \otimes u\right)$$

as sum over maps $j\colon \{1, \ldots, N\} \to \{0, 1, \ldots, \ell\}$. ▸ More

- j induces a partition $\mathcal{P}_j$ of $\{1, \ldots, N\}$ (independant of ℓ).
- Among j' with $\mathcal{P}_j$, j is charact. by values $k_1, \ldots, k_{|\mathcal{P}_j|}$.

$$w_l \leqslant \sum_j \|T(j)\| \leqslant C \sum_{\mathcal{P}} \sum_{k_1 \neq k_2 \neq \cdots \neq k_{|\mathcal{P}|}} \prod_{p \in \mathcal{P}} \|F_{k_p,p}\|$$

$$\leqslant C \sum_{\mathcal{P}} \sum_{k_1=0}^{\ell} \sum_{k_2=0}^{\ell} \cdots \sum_{k_{|\mathcal{P}|}=0}^{\ell} \prod_{p \in \mathcal{P}} \|F_{k_p,p}\|.$$

Choice of $\mathcal{P}$: finite set, independent of ℓ, thus

if for all p, $\|F_{k,p}\|$ doesn't depend on k, we are ◂ done.

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Indexation by maps

$$\partial_1(\chi\sigma u \otimes u) = \partial_1(\chi)u \otimes u + \chi\partial_1(u) \otimes u + \chi u \otimes \partial_1(u)$$

.

Starting from $v^{(\ell)}$, recover ℓ terms.

All these terms are characterised by the "position" of the ∂_j, *i.e.* by a map $j\colon \{1, \ldots, N\} \to \{0, 1, \ldots, \ell\}$.

Back

Terms $T(j)$

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$$\partial_2\partial_1(\chi\sigma u \otimes u) =$$
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$$\partial_1(\chi)\partial_2(u) \otimes u + \chi\partial_2\partial_1(u) \otimes u + \chi\partial_2(u) \otimes \partial_1(u)$$
$$\partial_1(\chi)u \otimes \partial_2(u) + \chi\partial_1(u) \otimes \partial_2(u) + \chi u \otimes \partial_2\partial_1(u).$$

Starting from $v^{(\ell)}$, recover $\ell \times \ell$ terms.

All these terms are characterised by the “position” of the ∂_j, *i.e.* by a map $j\colon \{1, \dots, N\} \to \{0, 1, \dots, \ell\}$.

◂ Back

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$$
\begin{aligned}
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◂ Back

▸ Terms $T(j)$

Proof: Term $T(j)$

Since the ∂_j are derivations, for $v^{(\ell)}$ with ℓ terms,

$$\partial_{s_1} \cdots \partial_{s_N} (\chi u \otimes u \otimes \cdots \otimes u)$$

is a sum of terms indexed by the "positions" of the derivations ∂_j, *i.e.* by $j \colon \{s_1, \ldots, s_N\} \to \{0, 1, \ldots, \ell\}$.
Given $1 \leqslant k \leqslant \ell$, if $j^{-1}(k) = \{\iota_1, \ldots, \iota_\beta\}$, we write:

$$F_{k,j^{-1}(k)} := \partial_{j^{-1}(k)} u = \partial_{s_{\iota_1}} \partial_{s_{\iota_2}} \cdots \partial_{s_{\iota_\beta}} u$$

Of course, if $j^{-1}(k) = \emptyset$, $\partial_{j^{-1}(k)} = \mathrm{Id}$.

For a given j, the associated term $T(j)$ is

$$T(j) := \partial_{j^{-1}(0)}(\chi)\partial_{j^{-1}(1)}(u) \otimes \cdots \otimes \partial_{j^{-1}(\ell)}(u).$$

Since $\|u\| \leqslant 1$, $\|T(j)\| \leqslant C \prod_{k=1}^{\ell} \|F_{k,j^{-1}(k)}\|$
where C bounds the term $\partial_{j^{-1}(0)}\chi$.

Finally, $F_{k,p}$ doesn't depend on k, so the result follows. Back

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Since $\|u\| \leqslant 1$, $\|T(j)\| \leqslant C \prod_{k=1}^{\ell} \|F_{k,j^{-1}(k)}\|$
where C bounds the term $\partial_{j^{-1}(0)}\chi$.

Finally, $F_{k,p}$ doesn't depend on k, so the result follows. ◂ Back

Setting: general groups and monoidal categories

In this talk,

- G is a *locally compact Abelian* group;
- $\mathscr{V}$, $\mathscr{W}$, $\mathscr{A}$, $\mathscr{B}$ are bornological spaces (algebras).

Work of Bruhat ('61):

Schwartz alg. $\mathscr{S}(G)$ and Fourier transf. $\mathscr{S}(G) \xrightarrow{\mathcal{F}} \mathscr{S}(\hat{G})$.

- Central object:

  smooth tempered rep. of G, denoted ST-$\mathfrak{Rep}(G)$.

Characteristic property of ST-$\mathfrak{Rep}$: $\mathscr{S}(G)\hat{\otimes}_{\mathscr{S}(G)}\mathscr{V} \simeq \mathscr{V}$.

- ST-$\mathfrak{Rep}(G)$ is a symmetric monoidal category, using projective tensor product $\hat{\otimes}$.

Tensor prod. $\mathscr{V}_1\hat{\otimes}\mathscr{V}_2$ with diag. rep. of G is in ST-$\mathfrak{Rep}(G)$.