Invariant theory, quantum groups and free probability

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Partitions in (free) probability

If $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ centered *gaussian,* then $m_{2n+1}(f) = 0$ and

$$m_{2n}(f) = \int f(x)^{2n}dx = \frac{(2n)!}{2^n(n!)} = |P_2(2n)|.$$

Here P_2 is the set of *pair partitions* of $2n$ elements.

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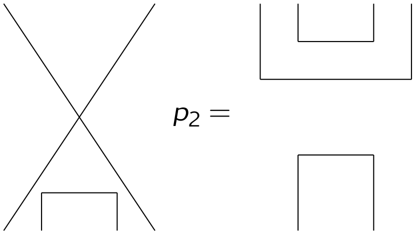

Partitions in (free) probability

If $g(x) = \frac{1}{2\pi}\sqrt{4-x^2}$ centered *semicircular*, then $m_{2n+1}(g) = 0$ and

$$m_{2n}(g) = \int g(x)^{2n} dx = \frac{1}{n+1}\binom{2n}{n} = |NC_2(2n)|.$$

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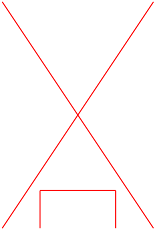
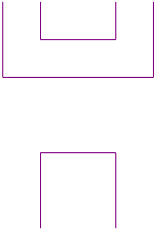
$p_1 =$  $p_2 =$ 

Invariant theory for O_N

Consider $O_N \circlearrowleft V = \mathbb{C}^N$, what are invariant polynomials on $V^{\otimes n}$?

- $J : x \otimes y \mapsto \langle x, y \rangle$ invariant of $V^{\otimes 2}$.
- For $p \in P_2(2n)$,

$$J_p : x_1 \otimes \cdots \otimes x_{2n} \mapsto \prod_{a,b \in p} \langle x_a, x_b \rangle$$

invariant of $V^{\otimes 2n}$.

Theorem (FFT)

The algebra of invariants $\mathbb{C}[V^{\otimes n}]^{O_N}$ is linearly spanned by the polynomials $(J_p)_{p \in P_2(n)}$.

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Schur-Weyl duality for O_N

What about $\mathrm{Hom}_{O_N}(V^{\otimes k}, V^{\otimes l})$? Consider basis (e_i), (e_j)

- Brauer : $T \in \mathrm{Hom}_{O_N}(V^{\otimes k}, V^{\otimes l})$ if and only if there is an invariant J such that $\langle e_i, T(e_j)\rangle = J(e_i \otimes e_j)$.
- If T_p corresponds to J_p, then

$$\mathrm{Hom}_{O_N}(V^{\otimes k}, V^{\otimes l}) = \mathrm{Span}\{T_p, p \in P_2(k+l)\}.$$

Corollary

Let $\chi \in C(O_N)$ be the character of V. Then,

$$\int \chi^{2n} d\mu_{Haar} = \dim(\mathrm{Hom}_{O_N}(\mathbb{C}, V^{\otimes 2n})) = |P_2(2n)|$$

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Quantum orthogonal group

Find G such that $\mathrm{Hom}_G(V^{\otimes k}, V^{\otimes l}) = \mathrm{Span}\{T_p, p \in NC_2(k+l)\}$?

- $C(O_N) = C^*_{ab}\Big((u_{ij})_{1\leqslant i,j\leqslant N}, u = [u_{ij}] \text{ orthogonal }\Big)$
- $C(O_N^+) = C^*\Big((u_{ij})_{1\leqslant i,j\leqslant N}, u = [u_{ij}] \text{ orthogonal }\Big)$ [Wang]

$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \rightsquigarrow$ free orthogonal quantum group.

Theorem (Banica)

$\mathrm{Hom}_{O_N^+}(V^{\otimes k}, V^{\otimes l}) = \mathrm{Span}\{T_p, p \in NC_2(k+l)\}$

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If χ is the character of V, then $h(\chi^{2n}) = |NC_2(2n)|$.

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Banica-Speicher : $\mathrm{Hom}_{\mathbb{G}}(V^{\otimes k}, V^{\otimes l}) = \mathrm{Span}\{T_p, p \in \mathcal{C}\}$?

Definition

A *category of partitions* $\mathcal{C} = (\mathcal{C}(k,l))_{k,l}$ is stable by :

- Vertical concatenation
- Horizontal concatenation
- Horizontal symmetry
- Rotation

Examples : O_N, O_N^+, S_N, S_N^+, H_N, H_N^+.

$\rightsquigarrow$ Generalize to *colored* partitions !

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Theorem (Tannaka-Krein)

$\mathcal{A}$ color set, $\mathcal{C}$ category of partitions, N integer. There exists a CQG $\mathbb{G}$ with representations $(u^x)_{x\in\mathcal{A}}$ of dimension N such that

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F-Weber : If $\mathcal{C}$ is *noncrossing* and $N \geqslant 4$, complete description of the representation theory.

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Free fusion semirings

A *fusion set* is given by

- A set S
- A *conjugation* $x \mapsto \overline{x}$, with $\overline{\overline{x}} = x$
- A *fusion operation* $x, y \mapsto x * y \in S \cup \{\emptyset\}$.

Extend to the abelian semigroup $\mathbb{N}[F(S)]$

- $\overline{w_1 \dots w_n} = \overline{w_n} \dots \overline{w_1}$
- $(w_1 \dots w_n) * (w_1' \dots w_k') = \delta_{w_n * w_1' \neq \emptyset} \, w_1 \dots (w_n * w_1') \dots w_k'$

Free fusion *semiring* $(R^+(S), +, \otimes)$ with

$$w \otimes w' = \sum_{w = az, w' = \overline{z}b} ab + a * b.$$

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- $S = \{x, \overline{x}\}, x * \overline{x} = \overline{x} * x = x * x = \overline{x} * \overline{x} = \emptyset : U_N^+$
- $S = \Gamma$ discrete group : $\widehat{\Gamma} \wr_* S_N^+$
- $S = \{x, \overline{x}, x * \overline{x}, \overline{x} * x\} : \widetilde{H}_N^+$

Theorem (F)

$\mathcal{C}$ category of noncrossing partitions. The following are equivalent :

1. *$R^+(\mathbb{G})$ is free*
2. *$\mathbb{G}$ has no non-trivial one-dimensional representation*
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An admissible fusion set is given by :

- *Integers $n_{\mathcal{O}}$ and $n_{\mathcal{U}}$*
- *A family of groups $(\Gamma_i)_{i\in I}$ with integers n_i.*

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Let $\mathbb{G}$ be a free partition quantum group. Then, $\mathbb{G}$ is a free product of O_N^+, U_N^+ and $Z_N^+(\Gamma_i, n_i)$.

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Applications

- Operator algebras : simplicity, uniqueness of trace and factoriality , approximation properties .
- Classical invariants : classification of all groups with combinatorial Schur-Weyl duality.
- Free probability : models for noncommutative distributions , De Finetti theorems .