Groupoids and Pseudodifferential calculus I.

D. & Skandalis - Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and Pseudodifferential calculus - Adv. Math 2014

http://math.univ-bpclermont.fr/∼debord/

NGA - Frascati

2014

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- From Geometry : The *Gauge adiabatic groupoid* short exact sequence :

$$0 \to C^*(G) \otimes \mathcal{K} \longrightarrow J(G) \rtimes \mathbb{R}_+^* \longrightarrow C(S^*\mathfrak{A}G) \otimes \mathcal{K} \to 0 \qquad \text{(GAG)}$$

  Where $J(G) \subset C^*(G_{ad})$ is an ideal of the C*-algebra of the adiabatic groupoid G_{ad} of G, and the natural action of $\mathbb{R}_+^*$ on G_{ad} is considered.

Main Results

Theorem (D. & Skandalis)

There is an ideal $\mathcal{J}(G) \subset C_c^\infty(G_{ad})$ such that :

- ★ *The order 0 pseudo differential operators on G are multipliers of $C_c^\infty(G)$ of the form $\int_0^\infty f_t \frac{dt}{t}$ where $f = (f_t)_{t\in\mathbb{R}_+} \in \mathcal{J}(G)$.*

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- Describe the short exact sequence (PDO).
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- Describe the ideal $\mathcal{J}(G)$ and give a precise statement of ★.

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An exponential map $\theta : V' \to V$ for G is a diffeomorphism where $G^{(0)} \subset V' \subset \mathfrak{A}G$, $G^{(0)} \subset V \subset G$, V and V' being open and such that :

- $\theta|_{G^{(0)}} = Id$ and $r \circ \theta = \pi$,
- For $x \in G^{(0)}$, $d\theta(x, 0)$ is the "identity" on the normal direction of the inclusion of $G^{(0)}$: $\mathfrak{A}G_x \simeq T_{(x,0)}\mathfrak{A}G/T_x G^{(0)} \to \mathfrak{A}G_x$.

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  i.e. for any N the function $\varphi - \sum_{k=0}^{N} a_{m-k} \in \mathcal{S}^{m-N}(\mathfrak{A}^*G)$, grows less fast at ∞ then an order $m-N$ polynomial in $|\xi|$, as well as all its derivatives.

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Where there is a polyhomogeneous symbol $\varphi \in \mathcal{S}^m(\mathfrak{A}^* G)$ such that P_0 is the limit in $\mathcal{M}(C_c^\infty(G))$ of P_0^R when $R \to \infty$ where :

$$P_0^R(\eta) = \int_{\substack{\xi \in \mathfrak{A}^* G_{r(\eta)} \\ \|\xi\| \leq R}} e^{i<\theta^{-1}(\eta),\xi>} \varphi(r(\eta), \xi) d\xi$$

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moreover it gives the short exact sequence :

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Choose an exponential map $\theta : V' \subset \mathfrak{A}G \xrightarrow{\simeq} V \subset G$ for G.

The adiabatic groupoid is $G_{ad} = G \times \mathbb{R}_+^* \cup \mathfrak{A}G \times \{0\} \rightrightarrows G^{(0)} \times \mathbb{R}_+$

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The Gauge adiabatic groupoid is then $G_{ga} = G_{ad} \rtimes \mathbb{R}_+^* \rightrightarrows G^{(0)} \times \mathbb{R}_+$.

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The evaluation map at 0 gives the exact sequence :

$$0 \to \begin{array}{c} C^*(G_{ad}|_{\mathbb{R}_+^*}) \\ \simeq C^*(G) \otimes C_0(\mathbb{R}_+^*) \end{array} \longrightarrow C^*(G_{ad}) \xrightarrow{ev_0} \begin{array}{c} C^*(\mathfrak{A}G) \\ \simeq C_0(\mathfrak{A}^*G) \end{array} \to 0$$

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Look at the ideal $C_0(\mathfrak{A}^*G \setminus G^{(0)}) \subset C_0(\mathfrak{A}^*G)$ and set $J(G) = ev_0^{-1}(C_0(\mathfrak{A}^*G \setminus G^{(0)}))$.

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Which is equivariant under the action of $\mathbb{R}_+^*$ and leads to

$$0 \to \underset{\simeq C^*(G)\otimes\mathcal{K}}{\left(C^*(G)\otimes C_0(\mathbb{R}_+^*)\right) \rtimes \mathbb{R}_+^*} \to \underset{\subset C^*(G_{ga})}{J(G)\rtimes\mathbb{R}_+^*} \to \underset{\simeq C(S^*\mathfrak{A}G)\otimes\mathcal{K}}{C_0(\mathfrak{A}^*G \setminus G^{(0)})\rtimes\mathbb{R}_+^*} \to 0 \qquad \text{(GAG)}$$

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The aim now is to take a fresh look on $\Psi_0^*(G)$ with the gauge adiabatic groupoid in mind.

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- The schwartz algebra $S_c(G_{ad})$:

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For all $k,\ l \in \mathbb{N}^n,\ j,\ m \in \mathbb{N}$:

$$sup\Big((\|X\|^2 + t^2)^{\frac{m}{2}}\Big|\frac{\partial^{|k|+|l|+j}}{\partial x^k \partial X^l \partial t^j} g \circ \Theta(x, X, t)\Big|\Big) < +\infty$$

Recall that $\Theta : W' \subset \mathfrak{A}G \times \mathbb{R}_+ \xrightarrow{\simeq} W \subset G_{ad}$ is given by $\Theta(x, X, t) = (\theta(x, tX), t)$ for $t \neq 0$ and $\Theta(x, X, 0) = (\theta(x, X), 0)$.

The ideal $\mathcal{J}(G)$

Definition-Proposition

$\mathcal{J}(G) \subset S_c(G_{ad})$ is the ideal of functions $f = (f_t)_{t \in \mathbb{R}_+}$ which satisfies the following equivalent conditions :

The ideal $\mathcal{J}(G)$

Definition-Proposition

$\mathcal{J}(G) \subset S_c(G_{ad})$ is the ideal of functions $f = (f_t)_{t\in\mathbb{R}_+}$ which satisfies the following equivalent conditions :

1. *For any $g \in C_c^\infty(\mathfrak{A}G)$ the map*

$$(x,t) \in G^{(0)} \times \mathbb{R}_+ \mapsto \int_{\mathfrak{A}G_x} g(x,X)\chi \cdot f_t \circ \theta(x,X)dX$$

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2. *The map*

$$(x,\xi,t) \in \mathfrak{A}^*G \times \mathbb{R}_+ \mapsto \widehat{\chi\cdot f_t \circ \theta}(x,\xi)dX$$

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$\chi \in C_c^\infty(V)$ is equal to 1 near $G^{(0)} \subset G$.

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4. $f = h + g$ *where* $h \in \mathcal{J}_0(G)$ *and* $g \in C^\infty(W)$ *satisfies :*
   For all $k,\ l \in \mathbb{N}^n$, $j \in \mathbb{N}$ *and* $m \in \mathbb{Z}$ *:*

$$sup\left((\|\xi\|^2 + t^2)^{\frac{m}{2}}\left|\frac{\partial^{|k|+|l|+j}}{\partial x^k \partial \xi^l \partial t^j}\widehat{g\circ\Theta}(x,\xi,t)\right|\right) < +\infty$$

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Remark : Condition 3 bellow reassures us : the definition of $\mathcal{J}(G)$ do not depends on the choice of the exponential map θ.

$\diamondsuit$

$\mathcal{J}(G)$ and pseudodifferential operators on G

Theorem (D. & Skandalis)

For $f = (f_t)_{t\in\mathbb{R}_+} \in \mathcal{J}(G)$ *and* $m \in \mathbb{N}$ *let*

$$P = \int_0^{+\infty} t^m f_t \frac{dt}{t} \quad \text{and} \quad \sigma : (x,\xi) \in \mathfrak{A}^* G \mapsto \int_0^{+\infty} t^m \widehat{f}(x, t\xi, 0) \frac{dt}{t}$$

Then P *belongs to* $\mathcal{P}_{-m}(G)$ *and its principal symbol is* σ.

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What does it mean : There exists a pseudodifferential operator $P \in \mathcal{P}_{-m}(G)$ with principal symbol σ such that if $g \in C_c^\infty(G)$:

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Remark : Moreover any $P \in \mathcal{P}_{-m}(G)$ is a $P_f = \displaystyle\int_0^{+\infty} t^m f_t \frac{dt}{t}$ for some $f = (f_t)_{t\in\mathbb{R}_+} \in \mathcal{J}(G)$.

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- If $f \in C^\infty(W)$ with $\widehat{f \circ \Theta}$ flat on $G^{(0)} \times \{0\} \subset \mathfrak{A}^* G \times \mathbb{R}_+$ then $f(\gamma, t) = t^{-n}\chi(\gamma)\chi'(t)\varphi(\frac{\theta^{-1}(\gamma)}{t}, t)$ where $\varphi \in C_c^\infty(\mathfrak{A}G \times \mathbb{R}_+)$ and $\hat{\varphi}$ vanishes as well as all its derivatives on $G^{(0)} \times \{0\}$.

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$$f_t(\gamma) = (2\pi)^{-n}\chi(\gamma)\chi'(t)\int e^{i<\theta^{-1}(\gamma),\xi>}\hat{\varphi}(x, t\xi, t)d\xi$$

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In the multiplier algebra of $C_c^\infty(G)$ we have

$$\int_0^{+\infty} t^m f_t \frac{dt}{t} = (2\pi)^{-n} \chi(\gamma) \int e^{i<\theta^{-1}(\gamma),\xi>} a(x, \xi) d\xi$$

where

$$a(x, \xi) = \int_0^{+\infty} t^m \chi'(t) \hat{\varphi}(x, t\xi, t) \frac{dt}{t}$$

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For ξ big enough we get

$$a(x,\xi) \sim \sum_{k=0}^{\infty} a_{k+m}(x,\xi)$$

where

$$a_{k+m}(x,\xi) = \int_0^{\infty} b_k(x,t\xi) t^{k+m} \frac{dt}{t}$$

is homogeneous in ξ of degree $-k-m$. □