Noncommutative Potential Theory 3

Fabio Cipriani

Dipartimento di Matematica
Politecnico di Milano

(joint works with *U. Franz, D. Guido, T. Isola, A. Kula, J.-L. Sauvageot*)

Villa Mondragone Frascati, 15-22 June 2014

Themes.

- Noncommutative potential theory: carré du champ, potentials, finite energy states, multipliers
- Dirac operator, Spectral triple on Lipschiz algebra of Dirichlet spaces
- Closable derivations on algebras of finite energy multipliers

References.

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One of the main subject of potential theory of Dirichlet spaces $(\mathcal{E},\mathcal{F})$ on C*-algebras with trace (A,τ), is the following class of functionals

Definition. (Carré du champ)

The carré du champ of $a\in\mathcal{F}$ is the positive functional $\Gamma[a]\in A^*_+$

$$\Gamma[a]:A\to\mathbb{C}\qquad \langle\Gamma[a],b\rangle := (\partial a|(\partial a)b)_{\mathcal{H}}\qquad b\in A$$

defined using the derivation $(\mathcal{B},\partial,\mathcal{H},\mathcal{J})$ representing $(\mathcal{E},\mathcal{F})$. Alternatively, whenever $a\in\mathcal{B}$ we can set

$$\langle\Gamma[a],b\rangle := \frac{1}{2}\{\mathcal{E}(ab^*|a)+\mathcal{E}(a|ab)-\mathcal{E}(a^*a|b)\}\qquad b\in\mathcal{B}\,.$$

When $\mathcal{E}[a]$ represents the energy of a configuration $a\in\mathcal{F}$ of a system, $\Gamma[a]$ may be interpreted as its energy distribution.

Example. In case of the Dirichlet integral on $\mathbb{R}^n$, the carré du champ are absolutely continuous with respect to the Lebesgue measure m and reduces to

$$\Gamma[a]=|\nabla a|^2\cdot m\qquad a\in H^1(\mathbb{R}^n)\,.$$

In general the energy distribution $\Gamma[a]$ is not comparable with the volume distribution represented by τ.

Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on (A,τ), $(\mathcal{F},\partial,\mathcal{H},\mathcal{J})$ its differential square root and $(\mathcal{F}^*,\partial^*,\mathcal{H},\mathcal{J})$ its adjoint. Recall that $(\mathcal{B},\partial,\mathcal{H},\mathcal{J})$ is a derivation.

Definition. (Dirac operator)

The Dirac operator $(D,\mathcal{H}_D)$ of the Dirichlet space is the densely defined, self-adjoint operator acting on $\mathcal{H}_D := L^2(A,\tau)\oplus\mathcal{H}$ as

$$D := \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix} \qquad \mathrm{dom}(D) := \mathcal{F}\oplus\mathcal{F}^* \subseteq \mathcal{H}_D$$

or more explicitly

$$D\begin{pmatrix} a \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}\begin{pmatrix} a \\ \xi \end{pmatrix} = \begin{pmatrix} \partial^*\xi \\ \partial a \end{pmatrix} \qquad \begin{pmatrix} a \\ \xi \end{pmatrix} \in \mathcal{F}\oplus\mathcal{F}^* .$$

By definition, the operator is anticommuting with involution $\gamma := \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$:

$$D\gamma + \gamma D = 0 .$$

Notice that $\quad D^2 = \begin{pmatrix} \partial^*\partial & 0 \\ 0 & \partial\partial^* \end{pmatrix} .$

Consider below $L^2(A,\tau)$, $\mathcal{H}$ and $\mathcal{H}_D$ as left A-modules.

Lemma. (Bounded commutators)

For $a \in \mathcal{B}$, the following properties are equivalent

- $[D,a]$ *is bounded on* $\mathcal{H}_D$
- $[\partial,a]$ *is bounded from* $L^2(A,\tau)$ *to* $\mathcal{H}$
- $\Gamma[a]$ *is absolutely continuous w.r.t.* τ *with bounded Radon-Nikodym derivative*

$$h_a \in L^\infty(A,\tau) \qquad \langle\Gamma[a],b\rangle = \tau(h_a b) \qquad b \in L^1(A,\tau)\,;$$

- *for* $a \in \mathcal{B} \cap \mathrm{dom}_{\mathcal{M}}(L)$, *these are also equivalent to* $a^*a \in \mathrm{dom}_{\mathcal{M}}(L)$.

Definition. (Lipschiz algebra)

The *-subalgebra $\mathcal{L}(\mathcal{F}) \subseteq \mathcal{B}$ of elements satisfying the first three properties above, is called the Lipschiz algebra of the Dirichlet space.

Example. In case of the Dirichlet integral $\mathcal{L}(H^1(\mathbb{R}^n))$ coincides with the algebra $\mathrm{Lip}(\mathbb{R}^n)$ of Lipschiz functions of the Euclidean metric.
Example. In a next lecture, we will see that on p.c.f. fractals, as a rule, the Lipschiz algebra reduces to constants functions.

Define the phase $F_D := D|D|^{-1}$ of the Dirac operator to be zero on $\ker(D)$.

Theorem. (Spectral triple and Fredholm module of $\mathcal{DS}$)

Assume the spectrum of $(\mathcal{E}, \mathcal{F})$ on $L^2(A, \tau)$ to be discrete. Then $(\mathcal{L}(\mathcal{F}), D, \mathcal{H}_D)$ is spectral triple in the sense

- $[D, a]$ *is bounded on* $\mathcal{H}_D$ *for all* $a \in \mathcal{L}(\mathcal{F})$
- $\mathrm{sp}(D)$ *is discrete away from zero.*

Moreover, setting $F := F_D + P_{\ker(D)}$*, then* $(\mathcal{L}(\mathcal{F}), F, \mathcal{H}_D)$ *is a Fredholm module*

- $F = F^*\,, \qquad F^2 = I$
- $[F, a]$ *is compact on* $\mathcal{H}_D$ *for all* $a \in \mathcal{L}(\mathcal{F})$.

- $H := -\Delta + V$ be a semibounded Hamiltonian with potential V on $L^2(\mathbb{R}^n, m)$
- assume the spectrum to be discrete $\mathrm{sp}(H) = \{E_0 < E_1 < \ldots\}$,
- $\psi_0 \in L^2(\mathbb{R}^n, m)$ the ground state with lowest eigenvalue E_0: $H\psi_0 = E_0\psi_0$
- $U : L^2(\mathbb{R}^n, m) \to L^2(\mathbb{R}^n, |\psi_0|^2 \cdot m)$ ground state transformation

$$U(f) := \psi_0^{-1} f \qquad f \in L^2(\mathbb{R}^n, m)$$

- H_{ϕ_0} the ground state representation of H: $H_{\phi_0} := U(H - E_0)U^{-1}$
- e^{-tH} positivity preserving on $L^2(\mathbb{R}^n, m) \;\Rightarrow\; e^{-tH_{\psi_0}}$ Markovian on $L^2(\mathbb{R}^n, |\psi_0|^2 \cdot m)$
- Dirichlet form on $L^2(\mathbb{R}^n, |\psi_0|^2 \cdot m)$

$$\mathcal{E}_{\psi_0}[a] = \|\sqrt{H_{\psi_0}a}\|_2^2 = \int_{\mathbb{R}^n} |\nabla a|^2 \cdot |\psi_0|^2 \cdot m \qquad a \in \mathcal{F}_{\psi_0}$$

- derivation $\partial : \mathcal{F}_{\psi_0} \to L^2(\mathbb{R}^n, m) \qquad \partial a = \nabla a$
- Lipschiz algebra $\mathcal{L}(\mathcal{F}_{\psi_0}) = \mathcal{L}(\mathbb{R}^n)$
- harmonic oscillator $V(x) := |x|^2$: spectral dimension of $(C_b(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n), D_{\psi_0}, L^2(\mathbb{R}^n, |\psi_0|^2 \cdot m) \oplus L^2(\mathbb{R}^n, |\psi_0|^2 \cdot m)) = 2n$

Potentials, Finite energy functionals

Finer properties of the differential calculus underlying a Dirichlet spaces rely on properties of the basic objects of the Potential Theory of Dirichlet forms.

Consider the Dirichlet space with its Hilbertian norm $\|a\|_{\mathcal{F}} := \sqrt{\mathcal{E}[a] + \|a\|^2_{L^2(A,\tau)}}$.

Definition. *Potentials, Finite Energy Functionals* (CS TAMS 2014)

- $p \in \mathcal{F}$ *is called a potential if*

$$(p|a)_{\mathcal{F}} \geq 0 \qquad a \in \mathcal{F}_+ := \mathcal{F} \cap L^2_+(A,\tau)$$

Denote by $\mathcal{P} \subset L^2(A,\tau)$ *the closed convex cone of potentials.*

- $\omega \in A^*_+$ *has finite energy if for some* $c_\omega \geq 0$

$$|\omega(a)| \leq c_\omega \cdot \|a\|_{\mathcal{F}} \qquad a \in \mathcal{F}\,.$$

Example. In a d-dimensional Riemannian manifold (V,g), the volume measure μ_W of a $(d-1)$-dimensional compact submanifold $W \subset V$ has finite energy.

Theorem. (CS TAMS 2014)

Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on (A,τ).

- *Potentials are positive: $\mathcal{P}\subset L^2_+(A,\tau)$*
- *Given a finite energy functional $\omega\in A^*_+$, there exists a unique potential*

$$G(\omega)\in\mathcal{P}\qquad \omega(a)=(G(\omega)|a)_{\mathcal{F}}\qquad a\in\mathcal{F}\,.$$

Example. If $h\in L^2_+(A,\tau)\cap L^1(A,\tau)$ then $\omega_h\in A^*_+$ defined by

$$\omega_h(a):=\tau(ha)\qquad a\in A$$

is a finite energy functional whose potential is given by $G(\omega_h)=(I+L)^{-1}h$.

Example. Let $\mathcal{E}_\ell$ be the Dirichlet form on $A:=C^*_r(\Gamma)$, associated to a negative definite function ℓ on a countable, discrete group Γ. Then ω is a finite energy functional iff

$$\sum_{t\in\Gamma}\frac{|\omega(\delta_s)|^2}{1+\ell(s)}<+\infty \quad\text{with potential}\quad G(\omega)(s)=\frac{\omega(\delta_s)}{1+\ell(s)}\qquad s\in\Gamma\,.$$

Since $\varphi_\ell := (1+\sqrt{\ell})^{-1}$ is a positive definite, normalized function, there exists a state $\omega_\ell \in A^*_+$ such that $\varphi_\ell(s) = \omega_\ell(\delta_s)$ for all $s \in \Gamma$. Thus ω has finite-energy iff

$$\sum_{s\in\Gamma} \frac{|\omega(\delta_s)|^2}{(1+\sqrt{\ell}(s))^2} = \sum_{s\in\Gamma} |\varphi_\ell(s)\cdot\varphi_\omega(s)|^2 < +\infty\,.$$

Notice that $\varphi_\ell \cdot \varphi_\omega$ is a coefficient of a sub-representation of the product $\pi_{\omega_\ell} \otimes \pi_\omega$ of the representations $(\pi_\ell, \mathcal{H}_\ell, \xi_\ell)$ and $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ associated to ω_ℓ and ω.
Hence if ω has finite-energy, $\pi_{\omega_\ell} \otimes \pi_\omega$ and λ_Γ are not disjoint.

Moreover, as ω has finite energy simultaneously with respect to $\mathcal{E}_\ell$ and $\mathcal{E}_{\lambda^{-2}\ell}$ for $\lambda > 0$, the family of normalized, positive definite functions

$$\varphi_\lambda(s) = \frac{\lambda}{\lambda + \sqrt{\ell(s)}} \cdot \varphi_\omega(s) \qquad s \in \Gamma\,,$$

generates a family of cyclic representations $\{\pi_\lambda : \lambda > 0\}$ contained in λ_Γ, deforming the cyclic representation π_ω associated to the finite energy state ω to the left regular representation λ_Γ. In fact

$$\lim_{\lambda\to 0^+} \varphi_\lambda = \delta_e\,, \qquad \lim_{\lambda\to+\infty} \varphi_\lambda = \varphi_\omega\,.$$

Theorem. *Deny's embedding* (CS TAMS 2014)

Let $\omega \in A^*_+$ be a finite energy functional with bounded potential

$$G(\omega) \in \mathcal{P} \cap L^\infty(A, \tau)\,.$$

Then

$$\omega(b^*b) \leq ||G(\omega)||_{\mathcal{M}}\, ||b||^2_{\mathcal{F}} \qquad b \in \mathcal{B}\,.$$

The embedding $\mathcal{F} \hookrightarrow L^1(A, \omega)$ is thus upgraded to an embedding $\mathcal{F} \hookrightarrow L^2(A, \omega)$.

Example. Let $\mathcal{E}_\ell$ be the Dirichlet form associated to a negative type function ℓ on a countable discrete group Γ. Deny's embedding applies whenever

- $\sum_s \frac{1}{1+\ell(s)} |\omega(\delta_s)|^2 < +\infty$ — ω has finite energy
- $\displaystyle\sum_s \frac{\omega(\delta_s)}{1+\ell(s)} \lambda(s) \in \lambda(\Gamma)''$ — ω has bounded potential.

It is possible, in concrete examples, to find ω which is a coefficient of $C^*(G)$, but not a coefficient of the regular representation (i.e. ω is singular with respect to τ).

Theorem. *Deny's inequality* (CS TAMS 2014)

For any finite energy functional $\omega \in A^*_+$ with potential $G(\omega) \in \mathcal{P}$, the following inequality holds true

$$\omega\Big(b^* \frac{1}{G(\omega)} b\Big) \leq ||b||^2_{\mathcal{F}} \qquad b \in \mathcal{F}\,.$$

In the noncommutative setting, since, in general, the finite energy functional ω is not a trace, the proof requires considerations of KMS-symmetric Dirichlet forms on standard forms of von Neumann algebras, illustrated in Lecture 1.

Theorem. (CS TAMS 2014)

Let $G \in \mathcal{P} \cap \mathcal{M}$ be a bounded potential. Then

- $\langle G, b^* b\rangle_{\mathcal{F}} \leq \|G\|_{\mathcal{M}} \cdot \|b\|^2_{\mathcal{F}} \qquad b \in \mathcal{B}$
- $\Gamma[G] \in A^*_+$ *is a finite finite energy functional.*

Multipliers of Dirichlet spaces

The following is another central subject of Potential Theory: its properties reveal geometrical aspects.

On the Dirichlet space $\mathcal{F}$ consider its Hilbertian norm $\|a\|_{\mathcal{F}} := \sqrt{\mathcal{E}[a] + \|a\|^2_{L^2(A,\tau)}}$.

Definition. (C-Sauvageot '12 arXiv:1207.3524)

An element of the von Neumann algebra $b \in L^\infty(A,\tau)$ is a multiplier of $\mathcal{F}$ if

$$b \cdot \mathcal{F} \subseteq \mathcal{F}, \qquad \mathcal{F} \cdot b \subseteq \mathcal{F}.$$

Denoting the algebra of multipliers by $\mathcal{M}(\mathcal{F})$, by the Closed Graph Theorem, multipliers are bounded operators on $\mathcal{F}$: $\mathcal{M}(\mathcal{F}) \subset \mathbb{B}(\mathcal{F})$.

Example. Let $\mathcal{F}_\ell$ be the Dirichlet space associated to a negative type function ℓ on a discrete group Γ. Then the unitaries $\delta_t \in \lambda(\Gamma)''$ are multipliers and

$$\|\delta_t\|_{\mathbb{B}(\mathcal{F}_\ell)} = \sup_{s\in\Gamma} \sqrt{\frac{1+\ell(st)}{1+\ell(s)}} \leq \sqrt{2}\sqrt{1+\ell(t)} \qquad t \in \Gamma.$$

Example. In case of the Dirichlet integral of a compact Riemannian manifold (V, g)

$$\mathcal{E}[a] = \int_V |\nabla a|^2 \, dm_g \qquad a \in H^{1,2}(V)\,,$$

from the Sobolev embedding

$$\|b\|^2_{\frac{2d}{d-2}} \leq c \cdot \|b\|^2_{\mathcal{F}} \qquad b \in H^{1,2}(V, g)\,,$$

one derives an embedding of the Sobolev algebra

$$H^{1,d}_\infty(V, g) := H^{1,d}(V, g) \cap L^\infty(V, m_g)$$

into the multipliers algebra

$$H^{1,d}_\infty(V, g) \hookrightarrow \mathcal{M}(\mathcal{F}) \qquad \|a\|_{\mathbb{B}(\mathcal{F})} \leq c \cdot \|a\|_{H^{1,d}_\infty(V,g)}\,.$$

Recall that the d-Dirichlet integral $\int_V |\nabla a|^d \, dm_g$ and the norm of the Sobolev algebra $H^{1,d}_\infty(V, g)$ are the conformal invariants of (V, g) (Gehering, Royden, J. LeLong-Ferrand, Mostow).

Theorem. *Existence and abundance of multipliers* (CS TAMS 2014)

Let $I(A,\tau) \subset L^\infty(A,\tau)$ be the norm closure of the ideal $L^1(A,\tau) \cap L^\infty(A,\tau)$. Then

- *$(I+L)^{-1}h$ is a multiplier for any $h \in I(A,\tau)$*

$$\|(I+L)^{-1}h\|_{\mathbb{B}(\mathcal{F})} \leq 2\sqrt{5}\|h\|_\infty \qquad h \in I(A,\tau)$$

- *bounded L^p-eigenvectors of the generator L, are multipliers*

$$h \in L^p(A,\tau) \cap L^\infty(A,\tau) \qquad Lh = \lambda h \quad \Rightarrow \quad \|h\|_{\mathbb{B}(\mathcal{F})} \leq 2\sqrt{5}(1+\lambda)\|h\|_\infty$$

- *the algebra of* **finite energy multipliers** *$\mathcal{M}(\mathcal{F}) \cap \mathcal{F}$ is a form core*
- *the Dirichlet form is regular on the C^*-algebra $\overline{\mathcal{M}(\mathcal{F}) \cap \mathcal{F}}$*
- *$\overline{\mathcal{M}(\mathcal{F}) \cap \mathcal{F}} = A$ provided the resolvent is strongly continuous on A*

$$\lim_{\varepsilon \downarrow 0} \|(I+\varepsilon L)^{-1}a - a\|_{\mathcal{M}} = 0 \qquad a \in A\,.$$

Remark. The definition of multiplier of a Dirichlet space $\mathcal{F}$ does not involve properties of the quadratic form $\mathcal{E}$ other than that to be closed. Proofs of existence and large supply of multipliers are based on the properties of potentials and finite energy states developed in noncommutative potential theory.

- How to replace the seminorm on the Lipschiz algebra of a Dirichlet space

$$\mathcal{L}(\mathcal{F}) \ni a \to \|[D,a]\|_{\mathcal{H}_D} = \|[\partial,a]\|_{L^2\to\mathcal{H}}$$

when the Lipschiz algebra is reduced or trivializes $\mathcal{L}(\mathcal{F}) \simeq \mathbb{C}$?

- Is there the possibility to define a distance when energy is distributed singularly w.r.t. volume, i.e. when an iconal equation is not more at hand?

Theorem. (CS TAMS 2014)

For elements of the Dirichlet algebra $a \in \mathcal{B} = A \cap \mathcal{F}$, we have equivalently

- $a \in \mathcal{M}(\mathcal{F}) \cap \mathcal{F}$ *(finite energy multiplier)*
- *the commutator* $[\partial, a]$ *is a bounded operator on from* $\mathcal{F}$ *to* $\mathcal{H}$
- $\|(\partial a)b\|_{\mathcal{H}} \le c_a \cdot \|b\|_{\mathcal{F}} \qquad b \in \mathcal{B}\,,$ *for some* $c_a \ge 0$
- $\mathcal{F} \hookrightarrow L^2(A, \Gamma[a])$

Definition. (CS TAMS 2014)

The multipliers subspace $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{H}$ is defined requiring its vectors satisfy

$$\|\xi b\|_{\mathcal{H}} \leq c_\xi \cdot \|b\|_{\mathcal{F}} \qquad b \in \mathcal{B}\,, \;\text{for some } c_\xi \geq 0$$

or, equivalently, that the following multiplication operator is bounded from $\mathcal{F}$ to $\mathcal{H}$

$$M_\xi : \mathcal{B} \to \mathcal{H} \qquad M_\xi(b) := \xi b \qquad b \in \mathcal{B}$$

and it is normed by $\|\xi\|_{\mathcal{M}(\mathcal{H})} := \|M_\xi\|_{\mathcal{F}\to\mathcal{H}}$.

Clearly, for $a \in \mathcal{B}$ ia multiplier, $a \in \mathcal{M}(\mathcal{F}) \cap \mathcal{F}$, if and only if $\partial a \in \mathcal{M}(\mathcal{H})$.

Theorem. (CS TAMS 2014)

Consider multipliers algebra $\mathcal{M}(\mathcal{F})$, multipliers subspace $\mathcal{M}(\mathcal{H})$, assume $1 \in \mathcal{F}$.

- *The Dirichlet space $\mathcal{F}$ is a $\mathcal{M}(\mathcal{F})$-bimodule*
- *$\mathcal{M}(\mathcal{H})$ is a Banach space embedded in $\mathcal{H}$:* $\quad \|\xi\|_{\mathcal{H}} \leq \|1\|_{\mathcal{F}} \cdot \|\xi\|_{\mathcal{M}(\mathcal{H})}$
- *$\mathcal{M}(\mathcal{H})$ is a $\mathcal{M}(\mathcal{F})$-bimodule*

Definition. (CS TAMS 2014)

Define the multiplier seminorm as

$$\|\partial a\|_{\mathcal{M}(\mathcal{H})} = \|M_{\partial a}\|_{\mathcal{F}\to\mathcal{H}} = \|[\partial, a]\|_{\mathcal{F}\to\mathcal{H}} \qquad a \in \mathcal{M}(\mathcal{F}) \cap \mathcal{F}\,.$$

Proposition. (CS TAMS 2014)

- *The derivation $\partial : \mathcal{M}(\mathcal{F}) \cap \mathcal{F} \to \mathcal{M}(\mathcal{H})$ is densely defined and closable from $\mathcal{F}$ to $\mathcal{M}(\mathcal{H})$*
- *its graph norm is equivalent to the multipliers norm*

$$\|a\|_{\mathcal{F}} + \|\partial a\|_{\mathcal{M}(\mathcal{H})} \asymp \|a\|_{\mathcal{M}(\mathcal{F})} \qquad a \in \mathcal{M}(\mathcal{F}) \cap \mathcal{F}$$

- *the derivation $\partial : \mathcal{M}(\mathcal{F}) \cap \mathcal{B} \to \mathcal{M}(\mathcal{H})$ is closable from A to $\mathcal{M}(\mathcal{H})$.*

Question. Which geometry underlies the automorphisms subgroup of A, leaving invariant the graph norm

$$\mathcal{M}(\mathcal{F}) \cap \mathcal{B} \ni a \to \|a\|_A + \|\partial a\|_{\mathcal{M}(\mathcal{H})}\,?$$

In a commutative setting $(C_0(X), m)$, the Choquet capacity associated to a Dirichlet forms $(\mathcal{E}, \mathcal{F})$ is defined as the following set function

$$\mathrm{Cap}(A) := \inf\{\|b\|_{\mathcal{F}} : b \in \mathcal{F}\,,\ b \geq 1_A\} \qquad A \subset X \text{ open}$$

$$\mathrm{Cap}(B) := \inf\{\mathrm{Cap}(A) : B \subset A \text{ open}\} \qquad B \subset X \text{ Borel}\,.$$

Proposition. (CS TAMS 2014)

Consider a Dirichlet form $(\mathcal{E}, \mathcal{F})$ in a commutative setting $(C_0(X), m)$. Then the multiplier seminorm of $a \in \mathcal{M}(\mathcal{F}) \cap \mathcal{F}$ is equivalent to

$$\|[\partial, a]\|_{\mathcal{F}\to\mathcal{H}} \asymp \sup_{B\subset X} \frac{\Gamma[a](B)}{\mathrm{Cap}(B)} \qquad \textit{isocapacitary inequality}\,.$$

Isocapacitary inequalities were considered by V. Maz'ya with respect to the Dirichlet integral on $\mathbb{R}^n$ and by M. Fukushima for Dirichlet spaces on locally compact spaces.

- On a Riemannian manifold (V,g), $n := \dim(V) \geq 3$, by Sobolev inequality

$$\|b\|^2_{\frac{2n}{n-2}} \leq c_S \cdot \|b\|^2_{\mathcal{F}}$$

we have the bound $\quad m(B)^{1-\frac{2}{n}} \leq c \cdot \mathrm{Cap}(B) \qquad B \subset X$ Borel
so that the algebra of finite energy multipliers contains the weak Sobolev-Marcinkiewic algebra and the Sobolev algebra

$$H^1_\infty(V,g) \subset H^1_{Mar,\infty}(V,g) \subset \mathcal{M}(\mathcal{F}) \cap \mathcal{F}$$

- on the other hand, as $\mathrm{Cap}(B_r) \leq c \cdot r^{n-2}$, B_r for all balls of radius $r > 0$, the algebra of finite energy multipliers is contained in the Sobolev-Morrey algebra and in the algebra of functions with bounded mean oscillations

$$\mathcal{M}(\mathcal{F}) \cap \mathcal{F} \subseteq H^1_{Mor,\infty}(V,g) \subseteq BMO(V,g)$$

- On the Euclidean space $\mathbb{R}^n$, one easily checks that the group of homeomorphisms leaving invariant the Sobolev seminorm

$$\|a\|_{H^1_\infty} := \int_{\mathbb{R}^n} |\nabla a|^n \, dm$$

coincides with conformal group $\mathrm{Co}(\mathbb{R}^n)$

- On the Euclidean space $\mathbb{R}^n$, it is much more difficult to see that the group of homeomorphisms leaving invariant the BMO seminorm

$$\|a\|_{\mathrm{BMO}} := \sup_{Q\subset\mathbb{R}^n} \frac{1}{m(Q)} \int_Q |a - a_Q| \, dm$$

still coincides with conformal group $\mathrm{Co}(\mathbb{R}^n)$

H.M. Reimann Comment. Math. Helv (49) 1974,

K. Astala Michigan Math. J. (30) 1983).

Proposition. (CS TAMS 2014)

The seminorm of the algebra $\mathcal{M}(H^1)\cap H^1$ *of finite energy multipliers of the*

$$\textit{Dirichlet integral} \qquad \mathcal{D}[f] := \int_{\mathbb{R}^n} |\nabla f|^2\, dm \qquad f\in H^1(\mathbb{R}^n)$$

is invariant under conformal group

$$\|[\nabla, a\circ\gamma]\|_{H^1\to L^2} = \|[\nabla, a]\|_{H^1\to L^2} \qquad a\in\mathcal{M}(H^1)\cap H^1\,, \quad \gamma\in \mathrm{Co}(\mathbb{R}^n)\,.$$

Steps of proof.

- $(\mathcal{D}, H^1(\mathbb{R}^n))$ is transient if and only if $n\geq 3$ so that $\|f\|_{\mathcal{D}} = \mathcal{D}[f]$ is a norm
- Green function $G(x,y) := c_n|x-y|^{2-n}$
- resolvent $G(f)(x) = (-\Delta^{-1}f)(x) = \int_{\mathbb{R}^n} G(x,y)f(y)\,dy$
- isometric actions of the conformal group $\mathrm{Co}(\mathbb{R}^n)$ on L^p-spaces

$$\gamma_r^* : L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n) \qquad \gamma_p^*(f)(y) := J_{\gamma^{-1}}^{1/p} f(\gamma^{-1}(y)) \qquad \gamma\in\mathrm{Co}(\mathbb{R}^n)$$

where $J_\gamma(x) := |\det(\gamma'(x))|$ is the Jacobian of the transformation $\gamma\in\mathrm{Co}(\mathbb{R}^n)$

- Hardy-Littlewood-Sobolev inequality for $0 < \lambda < n$, $p, q > 1$, $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{q} = 2$

$$I(f,h) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda}h(y)\,dx\,dy \le c\cdot\|f\|_p\cdot\|h\|_q \quad f \in L^p(\mathbb{R}^n)\,, \quad h \in L^q(\mathbb{R}^n)$$

- Riesz potentials

$$G_\lambda(f)(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{-\lambda}\,dy$$

bounded from $L^q(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ and from $L^p(\mathbb{R}^n)$ to $L^{q'}(\mathbb{R}^n)$

- resolvent boundedness $G : L^p(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ where $p = \frac{2n}{n+2}$, $r = \frac{2n}{n-2}$ (Sobolev exponent)
- resolvent conformal covariance

$$G(\gamma_p^*(f)) = \gamma_r^*(G(f)) \qquad f \in L^p(\mathbb{R}^n)$$

- conformal invariance of the Hardy-Littlewood-Sobolev functional

$$I(\gamma_p^*(f), \gamma_p^*(f)) = I(f,f) \qquad f \in L^p(\mathbb{R}^n) \qquad p = \frac{2n}{2n-\lambda}$$

- $\mathcal{D}[G(f)] = c_n \cdot I(f,f) \qquad f \in L^r(\mathbb{R}^n) \qquad r = \frac{2n}{n+2}$

- multipliers norm

$$\|a\|_{\mathcal{M}(H^1)} := \sup\{\|ab\|_{H^1} : \|b\|_{H^1} = 1\}$$

- and its conformal invariance

$$\|a\circ\gamma\|_{\mathcal{M}(H^1)} = \|a\|_{\mathcal{M}(H^1)} \qquad a \in \mathcal{M}(H^1) \qquad \gamma \in \mathrm{Co}(\mathbb{R}^n)$$

- multipliers seminorm

$$\|[\nabla, a]\|_{H^1\to L^2} := \sup\{\|[\nabla, a]b\|_{L^2} : \|b\|_{H^1} = 1\}$$

- and its conformal invariance

$$\|[\nabla, a\circ\gamma]\|_{H^1\to L^2} = \|[\nabla, a]\|_{H^1\to L^2} \qquad a \in \mathcal{M}(H^1) \qquad \gamma \in \mathrm{Co}(\mathbb{R}^n)\,.$$