Noncommutative Potential Theory 2

Fabio Cipriani

Dipartimento di Matematica
Politecnico di Milano

(**joint works with *U. Franz, D. Guido, T. Isola, A. Kula, J.-L. Sauvageot***)

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Themes.

- Dirichlet forms on C*-algebras
- Canonical Differential calculus in Dirichlet spaces
- Dirichlet form of the Dirac operator and Curvature of Riemannian manifolds
- Dirichlet foms in group C*-algebras, Free Probability, Derivations and Rigidity

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In this lecture we focus the attention on a C*-algebra with semifinite, faithful, lower semicontinuous, positive trace (A,τ) and denote $\mathcal{M} = L^\infty(A,\tau)$ the von Neumann algebra acting on the space $L^2(A,\tau)$, generated by the GNS representation of (A,τ).

Definition. (Dirichlet form on C*-algebras with trace)

A Dirichlet form $(\mathcal{E},\mathcal{F})$ on (A,τ) is a d.d.l.s.c., quadratic form on $L^2(A,\tau)$ such that

- $\mathcal{E}[a^*] = \mathcal{E}[a]$ — *real*
- $\mathcal{E}[a \wedge 1_{A^{**}}] \le \mathcal{E}[a] \qquad a^* = a$ — *Markovian*
- $(\mathcal{E},\mathcal{F})$ *is a complete Dirichlet form if its matrix expansions for* $n \ge 1$

$$\mathcal{E}_n[(a_{ij})_{ij}] := \sum_{ij} \mathcal{E}[a_{ij}]$$

are Dirichlet forms on $\mathcal{M} \otimes \mathbb{M}_n(\mathbb{C})$ *(tacitly assumed since now on)*

- $A \cap \mathcal{F}$ *is form core, dense in the* C^**-algebra* A — *regular*

The domain $\mathcal{F}$ *is called Dirichlet space when endowed with the graph norm*

$$\|a\|_{\mathcal{F}} := \sqrt{\mathcal{E}[a] + \|a\|^2_{L^2(\mathcal{M})}}\,.$$

- Regularity holds true if the Dirichlet form comes from a C_0-semigroup on the C^*-algebra A. In general it is a weaker property that should be viewed as a vestige of that strong Feller property.
- In the commutative setting it has been introduced by Beurling-Deny to develop a potential theory on locally compact spaces.
- In the same framework it has been used by Fukushima to construct an associated Hunt process.
- We will first develop on it the differential calculus in Dirichlet spaces,

the first step being the following observation

Lemma. *Dirichlet algebra* (Beurling-Deny, Lindsay-Davies)

- $\mathcal{B} := \mathcal{F} \cap A$ *is a norm dense* **-subalgebra of* A *called* *Dirichlet algebra*
- $\mathcal{B}_e := \mathcal{F} \cap \mathcal{M}$ *is a* w^**-dense* **-subalgebra of* $\mathcal{M}$ *called* *weak Dirichlet algebra*

Proof. By convexity, l.s.c. and Markovianity, for $a = a^* \in \mathcal{B}_e$, $\|a\| = 1$, we have

$$\frac{a^2}{2} = a - \int_0^1 dt\, a \wedge t \quad \Rightarrow \quad \mathcal{E}\Big[\int_0^1 dt\, a \wedge t\Big] \leq \int_0^1 dt\, \mathcal{E}[a \wedge t] \leq \mathcal{E}[a]$$

so that $a^2 \in \mathcal{B}_e$, and by scaling, the same it is true for all $a = a^* \in \mathcal{B}_e$.
Hence, if $b = b^*$, $c = c^* \in \mathcal{B}_e$, then $(b+c) = (b+c)^*$, $(b-c) = (b-c)^*$ so that

$$bc+cb = (b+c)^2 - b^2 - c^2 \in \mathcal{B}_e \qquad b^2 - c^2 = \frac{(b+c)(b-c)+(b-c)(b+c)}{2} \in \mathcal{B}_e\,,$$

$$(b+ic)^2 = (b^2-c^2) + i(bc+cb) \in \mathcal{B}_e\,.$$

Decomposing a generic $a \in \mathcal{B}_e$ as $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$ we conclude that $a^2 \in \mathcal{B}_e$.
If $a, b \in \mathcal{B}_e$, considering the matrix

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in M_2(\mathcal{B}_e)\,,$$

and applying the above result to the extension of $\mathcal{E}$ on $M_2(A)$, we obtain

$$\begin{bmatrix} ab & 0 \\ 0 & ba \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^2 \in M_2(\mathcal{B}_e)$$

so that $ab \in \mathcal{B}_e$.

Definition. *Derivations* (CS 2003)

A derivation $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ on (A, τ) *is made of*

- *a norm dense* **-subalgebra* $\mathcal{B} \subseteq A \cap L^2(A, \tau)$
- *a symmetric, Hilbert A-bimodule* $(\mathcal{H}, \mathcal{J})$

$$\mathcal{J} : \mathcal{H} \to \mathcal{H} \quad \text{anti-unitary} \quad \mathcal{J}(a\xi b) = b^* \mathcal{J}(\xi) a^* \qquad a, b \in \mathcal{B}\,, \;\; \xi \in \mathcal{H}$$

- *a linear, symmetric map* $\partial : \mathcal{B} \to \mathcal{H}$

$$\partial(a^*) = \mathcal{J}(\partial a) \qquad a \in \mathcal{B}$$

satisfying the Leibniz rule

$$\partial(ab) = (\partial a)b + a(\partial b) \qquad a, b \in \mathcal{B}\,.$$

Example. Let (V, g) be a Riemannian manifold without boundary, $A := C_0(V)$, $C_c^\infty(V)$, $\mathcal{H} := L^2(T^{\mathbb{C}}V)$ the Hilbert space of square integrable sections of the complexified tangent bundle $T^{\mathbb{C}}V := TV \otimes \mathbb{C}$ acted on by continuous functions by pointwise multiplication, $\partial := \nabla^{\mathbb{C}}$ the complexified gradient operator and involution $\mathcal{J}(\xi \otimes z) := \xi \otimes \bar{z}$ for $\xi \otimes z \in TV \otimes \mathbb{C}$.

For $a = a^* \in A$, consider the representations L_a, R_a of $\mathrm{C}(sp(a))$, uniquely defined for $f \in C(sp(a))$, $\xi \in \mathcal{H}$ by

$$L_a(f)\xi = \begin{cases} f(a)\xi & \text{if} \quad f(0) = 0 \\ \xi & \text{if} \quad f \equiv 1 \end{cases} \qquad R_a(f)\xi = \begin{cases} \xi f(a) & \text{if} \quad f(0) = 0 \\ \xi & \text{if} \quad f \equiv 1\,, \end{cases}$$

and the representation $L_a \otimes R_a$ of $C(sp(a)) \otimes C(sp(a)) = C(sp(a) \times sp(a))$.

Let $I \subseteq \mathbb{R}$ be a closed interval and for $f \in \mathrm{C}^1(I)$, denote by $\tilde{f} \in \mathrm{C}(I \times I)$ its

$$\tilde{f}(s,t) = \begin{cases} \frac{f(s)-f(t)}{s-t} & \text{if} \quad s \neq t \\ f'(s) & \text{if} \quad s = t. \end{cases} \qquad \text{difference quotient}\,.$$

Derivation satisfy not only Leibniz property but also the following

Lemma. *Chain rule for derivations* (CS 2003)

Let $(D(\partial), \partial, \mathcal{H}, \mathcal{J})$ be a norm closed derivation, densely defined in A. Then for $a = a^* \in D(\partial)$, a closed interval $sp(a) \subseteq I, f \in C^1(I)$ such that $f(0) = 0$ we have

$$f(\partial) \in D(\partial)\,, \qquad \partial(f(a)) = (L_a \otimes R_a)(\tilde{f})\,\partial(a)\,,$$

which implies

$$\|\partial(f(a))\|_{\mathcal{H}} \leq \|f'\|_{C(I)} \cdot \|\partial(a)\|_{\mathcal{H}}\,.$$

Approximating the *unit contraction* $\mathbb{R} \ni t \mapsto t \wedge 1 \in \mathbb{R}$, one obtains

Theorem. (CS 2003)

Let $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ be a derivation on (A, τ) closable on $L^2(A, \tau)$. Then a Dirichlet form is obtained as closure of the quadratic form $(\mathcal{E}, \mathcal{B})$ given by

$$\mathcal{E}[\xi] := \|\partial a\|^2_{\mathcal{H}} \qquad a \in \mathcal{B}\,.$$

Theorem. (CS 2003)

Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(A,\tau)$ and $\mathcal{B}:=A\cap\mathcal{F}$ its Dirichlet algebra. The there exists a derivation $(\mathcal{B},\partial,\mathcal{H},\mathcal{J})$ such that

$$\mathcal{E}[\xi]:=\|\partial a\|^2_{\mathcal{H}}\qquad a\in\mathcal{B}\,.$$

- A sesquilinear from on $\mathcal{B}\otimes\mathcal{B}$ is defined by the sesquilinear form $\mathcal{E}$

$$(c\otimes d|a\otimes b):=\frac{1}{2}\big(\mathcal{E}(c,abd^*)+\mathcal{E}(cdb^*,a)-\mathcal{E}(db^*,c^*a)\big)$$

- by the Stinspring representation of the resolvent

$$(I+\varepsilon L)^{-1}(a)=W^*_\varepsilon\pi_\varepsilon(a)W_\varepsilon\qquad a\in\mathcal{M}$$

- it is shown to be positive definite by the following identity

$$(c\otimes d|a\otimes b)=\lim_{\varepsilon\to 0}\frac{1}{2}\tau\Big(d^*\frac{L}{I+\varepsilon L}(c^*)ab+d^*c^*\frac{L}{I+\varepsilon L}(a)b-d^*\frac{L}{I+\varepsilon L}(c^*a)b\Big)$$
$$=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\tau\Big(d^*(W_\varepsilon c-\pi_\varepsilon(c)W_\varepsilon)^*(W_\varepsilon a-\pi_\varepsilon(a)W_\varepsilon)b+d^*c^*(I-W^*_\varepsilon W_\varepsilon)ab\Big)$$

- denote $\mathcal{H}_0$ the Hilbert space obtained from $\mathcal{B}\otimes\mathcal{B}$ by separation and completion
- prove the bound $\|a\otimes b\|^2_{\mathcal{H}_0}\le\|b\|^2_A\cdot\mathcal{E}[a]$ for $a,b\in\mathcal{F}$
- a right A-module structure on $\mathcal{H}_0$ is obtained setting

$$(a\otimes b)c := a\otimes bc \qquad a,b,c\in\mathcal{B}$$

- a left A-module structure on $\mathcal{H}_0$ is obtained setting

$$d(a\otimes b) := da\otimes b - d\otimes ab \qquad a,b,c,d\in\mathcal{B}$$

- a derivation $\partial_0:\mathcal{B}\to\mathcal{H}_0$ is obtained setting

$$(\partial_0(a)|b\otimes c) := \frac{1}{2}(\mathcal{E}(a,bc)+\mathcal{E}(b^*,ca^*)-\mathcal{E}(b^*a,c)) \qquad a,b,c\in\mathcal{B}$$

- obtaining the identity

$$\mathcal{E}[a]-\|\partial_0(a)\|^2_{\mathcal{H}_0}=\lim_{\varepsilon\to0}\frac{1}{2}\tau\Big(\frac{L}{I+\varepsilon L}(a^*a)\Big)$$

  which proves the result in the conservative case $T_t1_{\mathcal{M}}=1_{\mathcal{M}}$, $t\ge0$
- a long detour based on the norm closability of ∂_0 is needed to handle the left hand side of the above identity to prove the general case.

Theorem (CS 2003, C 2006)

- *The form domain $\mathcal{F}$ is closed under Lipschiz functional calculus*

$$a = a^* \in \mathcal{F} \text{ and } f \in \mathrm{Lip}_0(\mathbb{R}) \quad \Rightarrow \quad f(a) \in \mathcal{F} \text{ and } \mathcal{E}[f(a)] \leq \|f\|^2_{\mathrm{Lip}_0(\mathbb{R})} \cdot \mathcal{E}[a]$$

- *Dirichlet algebra and C^* -algebra have equivalent K-groups:* $K_*(\mathcal{B}) = K_*(A)$
- *in the finite trace case, the Dirichlet algebra $\mathcal{B}$ is a semisimple Banach algebra when endowed with the norm* $\|a\|_{\mathcal{B}} := \|a\|_{\mathcal{M}} + \sqrt{\mathcal{E}[a]} \qquad a \in \mathcal{B}$, *in particular, it a unique Banach algebra topology*
- *in the commutative, finite trace case,, conservative case where $A = C(X)$, finite dimensional, locally trivial vector bundles $E \to X$ acquire a canonical Dirichlet structure (a class of compatible atlas with transition matrices have entries in $\mathcal{B}$)*
- *the space of sections $\mathcal{B}(E, X)$ of the Dirichlet structure has a canonical Banach module structure over $\mathcal{B}$*
- *capacity of bundles can be defined, providing a valuation of non triviality.*

- (V, g) Riemannian manifold
- $(Cl(T_xV), \tau_x)$ complexified Clifford algebras with traces at $x \in V$
- $Cl(V, g)$ complexified Clifford bundle
- $C_0^*(V, g) := C_0(Cl(V, g))$ Clifford C*-algebra of continuous sections
- $\tau = \int_V \tau_x \, dx$ trace on $C_0^*(V, g)$
- $\nabla : C^\infty(Cl(V, g)) \to C^\infty(Cl(V, g) \otimes T^*V)$ Levi-Civita connection

As an application of the above results, we get a shorter proof of the following

Theorem. (Davies-Rothaus (1989))

The closure of the quadratic form given by the Bochner integral

$$\mathcal{E}_B[\sigma] := \int_V |\nabla \sigma(x)|^2 \, dx \qquad \sigma \in C^\infty(Cl(V, g))$$

is Dirichlet form on $L^2(C_0^*(V), \tau)$.

- The Hilbert space $L^2(Cl(V,g)\otimes T^*V)$ is a $C_0^*(V,g)$-bimodule

$$\sigma_1\cdot(\sigma_2\otimes\omega)\cdot\sigma_3 := (\sigma_1\cdot\sigma_2\cdot\sigma_3)\otimes\omega$$

  where the "$\cdot$" on the r.h.s. denote the Clifford product among sections of the Clifford bundle
- a symmetry $\mathcal{J} : L^2(Cl(V,g)\otimes T^*V)\to L^2(Cl(V,g)\otimes T^*V)$ is defined by

$$\mathcal{J}(\sigma\otimes\omega) := \sigma^*\otimes\bar{\omega}$$

  where σ^* denotes the involution of the Clifford algebra and $\bar{\omega}$ the natural involution of complexified 1-forms
- by definition the Levi-Civita connection satisfies the metric property: for any vector field X

$$\nabla(f\sigma)=\sigma\otimes df+f\nabla\sigma \qquad X(\sigma|\sigma)=(\nabla_X\sigma|\sigma)+(\sigma|\nabla_X\sigma)$$

- since the contraction i_X commutes with the actions of the Clifford algebra and $\nabla_X=i_X\circ\nabla$ we have

$$i_X(\nabla(\sigma\cdot\sigma))=i_X((\nabla\sigma)\cdot\sigma+\sigma(\nabla\sigma))$$

- as this is true for any vector field X we have

$$\nabla(\sigma\cdot\sigma)=(\nabla\sigma)\cdot\sigma+\sigma(\nabla\sigma)$$

  from which the Leibniz property follows by polarization.

A similar result for the Dirac Laplacian depends upon the sign of the curvature.

- Denote by $D\sigma := \sum_{i=1}^n e_i \cdot \nabla_{e_i}\sigma$ the Dirac operator of (V, g).

Theorem. (CS 2003)

The following properties are equivalent:

- *the quadratic form of the Dirac Laplacian*

$$\mathcal{E}_D[\sigma] := \|D\sigma\|^2_{L^2(C_0^*(V),\tau)} = \int_V |D\sigma(x)|^2\, dx \qquad \sigma \in H^1(Cl(V, g))$$

  is Dirichlet form on $L^2(C_0^*(V), \tau)$
- *the heat semigroup* e^{-tD^2} *is a Markovian,* C_0*-semigroup on* $C_0^*(V, g)$
- *the curvature operator is nonnegative:* $\widehat{R} \geq 0$.

Example. On a compact, connected, orientable surface $\sum$, there exists a metric g such that $\mathcal{E}_D$ is a Dirichlet form if and only if $\sum$ is homeomorphic to the sphere S^2 or to the torus T^2.

- The curvature endomorphisms $R_x(v_1, v_2) : T_xV \to T_xV$

$$R_x(v_1, v_2)v := -(\nabla_{v_1}\nabla_{v_2}v - \nabla_{v_2}\nabla_{v_1}v - \nabla_{[v_1,v_2]}v)(x) \qquad v, v_1, v_2 \in C_c^\infty(TV), \quad x \in V$$

- define the curvature tensor $R_x \in \bigotimes^4 T_x^* V$

$$R_x(v_1, v_2, v_3, v_4) = (R_x(v_1, v_2)v_3|v_4)_{T_xV} \qquad v_1, v_2.v_3, v_4 \in C_c^\infty(TV), \quad x \in V$$

- and the curvature operators $\widehat{R}_x : \bigwedge_x^2 V \to \bigwedge_x^2 V$

$$(\widehat{R}_x v_1 \wedge v_2 | v_3 \wedge v_4)_{\bigwedge_x^2 V} = R_x(v_1, v_2, v_3, v_4) \qquad v_1, v_2.v_3, v_4 \in C_c^\infty(TV), \quad x \in V$$

- the proof uses Bochner Identity $D^2 = \Delta_B + \frac{1}{4}\Theta_R$ in terms of quadratic forms

$$\mathcal{E}_D = \mathcal{E}_B + \frac{1}{4}Q_R\,, \qquad Q_R[\sigma] = \int_V Q_R(x)[\sigma_x]\,dx$$

- where the curvature part can be written

$$Q_R(x)[\sigma_x] = \sum_{\alpha=1}^{n(n-1)/2} \mu_\alpha \|[\eta_\alpha, \sigma_x]\|_2^2$$

in terms of a basis of orthonormal eigenvectors $\{\eta_\alpha : \alpha = 1, \dots, n(n-1)/2\}$ of $\widehat{R}_x$ corresponding to its eigenvalues $\{\mu_\alpha : \alpha = 1, \dots, n(n-1)/2\}$

Thus, as commutators are bounded derivations and $\mathcal{E}_B$ is a Dirichlet form, if $\widehat{R} \geq 0$, all eigenvalues are nonnegative $\mu_\alpha \geq 0$ and Q_R as well $\mathcal{E}_D$ result (superposition of) Dirichlet forms.

In the opposite direction, the strategy is in to moves

- prove that, given a Euclidean space E with orthonormal base $\{e_i\}_{i=1}^{n=\dim E}$ and a symmetric operator $T : \bigwedge^2(E) \to \bigwedge^2(E)$, a form on $L^2(Cl(E), \tau)$ of type

$$Q_T(x)[\xi] = \sum_{1\leq i<j\leq n} \sum_{1\leq k<l\leq n} \langle e_k \wedge e_l | T(e_i \wedge e_j)\rangle \langle [e_k \cdot e_l, \xi] | [e_i \cdot e_j, \xi]\rangle$$

  is Dirichlet if and only if $(\xi | T\xi)_{\bigwedge^2(E)} \geq 0$ for all $\xi \in \bigwedge^2(E)$. This part uses again the correspondence Dirichlet form/derivations and the ideal structure of Clifford algebras (depending on parity of $\dim E$);
- disentangle the role of connection and curvature in the Bochner identity $\mathcal{E}_D = \mathcal{E}_B + \frac{1}{4} Q_R$ and prove that if $\mathcal{E}_D$ is a Dirichlet form a fortiori Q_R has to be a Dirichlet form too.

The decoupling will be realized according to the following general decomposition of derivations and Dirichlet forms.

Definition. *Derivations splitting* (CS (2003))

Consider a derivation $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ on A and the von Neumann algebra $\mathcal{L}_{A-A}(\mathcal{H})$ of operators commuting with both left and right actions of A.

- *$T \in \mathcal{L}_{A-A}(\mathcal{H})$ is ∂- bounded if $\mathcal{B} \ni b \mapsto T(\partial b) \in \mathcal{H}$ is bounded from A to $\mathcal{H}$*
- *a projection $p \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ is approximately ∂- bounded if increasing limit $p = \lim_\alpha p_\alpha$ of a net of ∂-bounded projections $p_\alpha \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$*
- *equivalently, $p \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ is approximately ∂- bounded if the A-bimodule $p\mathcal{H}$ splits as direct sum $\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ and the derivation $p \circ \partial$ decomposes as a direct sum $\bigoplus_{n\in\mathbb{N}} \partial_n$ of bounded derivations*
- *a projection $p \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ is completely ∂-unbounded if 0 is the only ∂-bounded projection smaller than p*
- *a projection $p \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ is bounded, approximately bounded, completely unbounded if the identity $1_\mathcal{H}$ is a ∂-bounded, approximately ∂-bounded, completely ∂-unbounded projection.*

Lemma. (CS 2003)

There exists a greatest approximately ∂-bounded projection $P_j \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$. Any ∂-bounded operator $T \in \mathcal{L}_{A-A}(\mathcal{H})$ satisfies $TP_j = P_j$.

Definition. *Decomposition of Derivations and Dirichlet forms* (CS 2003)

Any derivation $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ on A decomposes canonically as

$$\partial = \partial_c \oplus \partial_j : \mathcal{B} \to \mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_j$$

- $\partial_c := (I - P_j) \circ \partial, \quad \mathcal{H}_c := P_j\mathcal{H}$ *is the completely unbounded part*
- $\partial_j := P \circ \partial, \quad \mathcal{H}_j := P_j\mathcal{H}$ *is the approximately bounded part.*
- *Dirichlet forms decompose accordingly as $\mathcal{E} = \mathcal{E}_c + \mathcal{E}_j$.*

Conclusion of the proof. Using repeatedly the above decomposition, one realizes that $(\mathcal{E}_D)_j = \frac{1}{4}Q_R$ so that the curvature part Q_R in the Bochner identity is a Dirichlet form and then the curvature operator has to be nonnegative $\widehat{R} \geq 0$.

- let G be a locally compact, unimodular group with identity $e \in G$
- $\lambda, \rho : G \to \mathcal{B}(L^2(G))$ left, right regular representations
- $C_r^*(G)$ reduced group C^*-algebra with trace $\qquad \tau(a) = a(e) \qquad a \in C_c(G)$ and GNS space $L^2(A, \tau) \simeq L^2(G)$
- for any continuous, negative definite function $\ell : G \to [0, +\infty)$

$$\mathcal{E}_\ell[a] = \int_G \ell(g)|a(g)|^2\, dg \qquad a \in L^2(G)$$

is a Dirichlet form,

$$(T_t a)(t) = e^{-t\ell(g)} a(g) \qquad (La)(g) = \ell(g)a(g) \qquad a \in C_c(G)$$

are the associated Markovian semigroup and its associated generator.

The associated derivation can be realized using

- the orthogonal representation $\pi : G \to B(\mathcal{K})$ and the 1-cocycle

$$c : G \to \mathcal{K} \qquad c(gh) = c(g) + \pi(g)c(h) \qquad g, h \in G$$

representing $\ell(g) = \|c(g)\|^2_{\mathcal{K}}$
- the Hilbert $C_r^*(G)$-bimodule is given by $L^2(G, \mathcal{K}_{\mathbb{C}})$ acted on the left by $\lambda \otimes \pi$ and on the right by $id \otimes \rho$
- the derivation is given by

$$\partial : C_c(G) \to L^2(G, \mathcal{K}_{\mathbb{C}}) \qquad (\partial a)(g) = c(g)a(g) \qquad g \in G\,.$$

Application: NC-Hilbert's transform, Free Information, Factoriality in Free Probability

Let (M, τ) be a nc-probability space and consider

- $1 \in B \subset M$ a *-subalgebra
- $X := \{X_1, \cdots, X_n\} \in M$ nc-random variables, algebraically free w.r.t. B
- $B[X] \subset M$ *-subalgebra generated by X and B
  (regarded as nc-polynomials in the variables X with coefficients in B
- $W \subset M$ the von Neumann subalgebra generated by $B[X]$.

Theorem. (Voiculescu '00)

There exists unique derivations $\partial_{X_i} : B[X] \to HS(L^2(W, \tau))$ *such that*

- $\partial_{X_i} X_j = \delta_{ij} 1 \otimes 1\,, \qquad i, j = 1, \cdots, n;$
- $\partial_{X_i} b = 0 \qquad i = 1, \cdots, n, \quad b \in B\,.$

Under the assumption $1 \otimes 1 \in \mathrm{dom}\,(\partial^*_{X_i})$ *for all* $i = 1, \cdots, n$, *it follows that*

- $(\partial_{X_i}, B[X])$ *is closable in* $L^2(W, \tau)$ *for all* $i = 1, \cdots, n$,
- *the closure of* $\mathcal{E}_X[a] := \sum_{i=1}^n \|\partial_{X_i} a\|^2$ *is a Dirichlet form.*

Definition. (Voiculescu 1998)

Under the assumption $1 \otimes 1 \in \mathrm{dom}\,(\partial^*_{X_i}) \quad i = 1, \cdots, n$, *define*

- $\mathcal{J}(X : B) := \left(\sum_{i=1}^n \partial^*_{X_i}\partial_{X_i}\right)(X_1 + \cdots + X_n) \in L^2(W, \tau)$
  nc-Hilbert Transform of X w.r.t. B
- $\Phi(X : B) := \|\mathcal{J}(X : B)\|^2$
  relative free Fisher information of X w.r.t. B.

In case $M = L^\infty(\mathbb{R}, m)$, $B = \mathbb{C}$, $X \in M$ has distribution μ_X one has $W = L^\infty(\mathbb{R}, \mu_X)$, $\mathbb{C}[X]$ is the algebra of polynomials on $\mathbb{R}$ and $\partial_X f$ coincides with the difference quotient. In case $p := \frac{d\mu_X}{dm} \in L^3(\mathbb{R}, m)$, then $\mathcal{J}(X : B)$ is the usual Hilbert transform

$$Hp(t) := \mathrm{p.v.}\ \frac{1}{\pi}\int_{\mathbb{R}} \frac{p(s)}{t-s}\,ds\,.$$

Theorem. (Voiculescu 1998)

The Free Poincaré inequality or Spectral Gap holds true for some $c > 0$

- $\|Y - \tau(Y)\|_2^2 \leq c \cdot \mathcal{E}_X[Y] \qquad Y \in \mathcal{F} := \bigcap_{i=1}^n dom\,(\partial_{X_i})\,.$

Theorem. (Y. Dabrowski 2009)

If the free Fisher information is finite $\Phi(X:\mathbb{C}) < +\infty$ then W is a factor.

"Proof": a consequence of the assumption is that X_j are diffuse operators.

If $Z \in W \cap W'$ is also in the domain of the Dirichlet form $Z \in \mathcal{F}$, then

$$0 = \partial_{X_i}([Z, X_j]) = [\partial_{X_i}(Z), X_j] \qquad i \neq j\,.$$

Thus each $\partial_{X_i}(Z) \in HS(L^2(W,\tau))$ is a compact operator commuting with a diffuse operator X_j and then vanishes $\partial_{X_i}(Z) = 0$.

The Free Poincaré inequality allows to conclude that Z is a multiple of the identity.

A standard resolvent regularization allows to remove the extra assumption $Z \in \mathcal{F}$.

Definition. (J. Peterson 2009)

Let $N \subset M$ be finite von Neumann algebras with common n.f. finite trace τ.

- *An L^2-deformation of N is a Markovian semigroup T_t on $L^2(M,\tau)$;*
- *an inclusion $B \subset N$ is said to be L^2-rigid if any L^2-deformation for N converges uniformly on the unit ball of B*

$$\lim_{t\to\infty} \sup_{\|b\|_B=1} \|b - T_t b\|_2 = 0\,.$$

Example 4.1.1. Let $\Lambda \subset \Gamma$ be countable discrete groups and $T_t = e^{-t\ell}$ the L^2-deformation of Λ given by for some function $\ell : \Gamma \to [0,+\infty)$ of negative type.

Then the deformation converges uniformly on the unit ball of $L(\Lambda)$ iff ℓ is inner, i.e.

$$\ell(t) = \|\xi - \pi(t)\xi\|^2_{\mathcal{K}} \qquad t \in \Gamma$$

for some orthogonal representation $\pi : \Gamma \to \mathbb{B}(\mathcal{K})$ and a unit vector $\xi \in \mathcal{K}$.

Theorem. (J. Peterson 2009)

Let N be a finite von Neumann algebras with normal, finite, faithful trace τ.

- *if $B \subset N$ is a subalgebra with no non-zero amenable summands then the inclusion $B' \cap N$ is L^2-rigid;*
- *if N is a non-amenable II_1 factor which is non-prime or has property Γ, then N is L^2-rigid;*
- *(Ozawa Theorem) if a countable discrete group Γ has a proper cocycle $c : \Gamma \to (\ell(\Gamma))^{\oplus\infty}$, then $L(\Gamma)$ is solid,*
  i.e. $B' \cap L(\Gamma)$ is amenable for any diffuse subalgebra $B \subset L(\Gamma)$.

Last result applies in particular to free group factors $\Gamma = \mathbb{F}_n$, $\quad 2 \leq n \leq +\infty$.