Noncommutative Potential Theory 1

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Themes.

- Review of Classical Potential Theory CPT
- Dirichlet forms on Standard Forms of von Neumann algebras
- KMS symmetric semigroups on C^*-algebras
- Approach to equilibria in Quantum Spin Systems
- Quantum Lévy Processes on Compact Quantum Groups
- Characterization of Haagerup Approximation Property by Dirichlet forms

References.

- L. Gross, *Existence and uniqueness of physical ground states*, J.Funct. Anal. 10 (1972).
- L. Gross, *Hypercontractivity and logarithmic Sobolev inequalities for the Clifford-Dirichlet form*, Duke Math. J. 42 (1975).
- S. Albeverio - R. Hoegh-Krohn, *Dirichlet Forms and Markovian semigroups on C*-algebras*, Comm. Math. Phys. 56 (1977).
- J.-L. Sauvageot, *Semi-groupe de la chaleur transverse sur la C^*-algebre d'un feuilletage riemannien*, C. R. Acad. Sci. Paris, Ser. I 310 (1990).
- F. Cipriani, *Dirichlet Forms and Markovian Semigroups on Standard Forms of von Neumann Algebras*, 147 (1997) - PhD Thesis S.I.S.S.A. (1992)
- F. Cipriani, *Dirichlet forms on Noncommutative Spaces*, L.N.M. 1954 (2008)
- E. B. Davies - J. M. Lindsay, *Noncommutative symmetric Markov semigroups*, Math. Z. 210 (1992).
- S. Goldstein and J. M. Lindsay, *Beurling-Deny conditions for KMS-symmetric dynamical semigroups*, C. R. Acad. Sci. Paris, Ser. I 317 (1993).
- D. Guido, T. Isola, and S. Scarlatti, *Non-symmetric Dirichlet forms on semifinite von Neumann algebras*, J. Funct. Anal. 135 (1996).

References.

- A. Majewski, B. Zegarlinski, *On quantum stochastic dynamics and noncommutative Lp spaces*, Lett. Math. Phys. 36 (1996)
- Y.M. Park, *Construction of Dirichlet forms on standard forms of von Neumann algebras*, IDAQP 3 (2000)
- F.Cipriani, U. Franz, A, Kula, *Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory*, J. Funct. Anal. 266 (2014).
- R. Okayasu, R. Tomatsu, *Haagerup approximation property for arbitrary von Neumann algebras*, arXiv:1312.1033
- M. Caspers, A. Skalski, *The Haagerup approximation property for von Neumann algebras via quantum Markov semigroups and Dirichlet forms*, arXiv:1404.6214

Classical Potential Theory concerns properties of the Dirichlet integral

$$\mathcal{D} : L^2(\mathbb{R}^d, m) \to [0, +\infty] \qquad \mathcal{D}[u] := \int_{\mathbb{R}^d} |\nabla u|^2 \, dm :$$

- lower semicontinuous quadratic form on the Hilbert space $L^2(\mathbb{R}^d, m)$
- finite on the Sobolev space $H^1(\mathbb{R}^d)$
- closed form of the Laplace operator

$$\Delta = -\sum_{k=1}^{d} \partial_k^2 \qquad \mathcal{D}[u] = \|\sqrt{\Delta} u\|_2^2$$

- generator of the heat semigroup $e^{-t\Delta} : L^2(\mathbb{R}^d, m) \to L^2(\mathbb{R}^d, m)$
- whose heat kernel

$$e^{-t\Delta}(x, y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}$$

is the fundamental solution of the heat equation $\partial_t u + \Delta u = 0$

The contraction property or Markovianity $\mathcal{D}[u \wedge 1] \leq \mathcal{D}[u]$ is responsible for

- Maximum Principle for solution of the Laplace equation $\Delta u = 0$
- Maximum Principle for solutions of the heat equation $\partial_t u + \Delta u = 0$
- contractivity, positivity preserving and continuity properties of the heat semigroup $e^{-t\Delta}$ on the spaces $L^2(\mathbb{R}^d, m)$, $L^\infty(\mathbb{R}^d, m)$, $L^1(\mathbb{R}^d, m)$.

The Brownian motion (Ω, P_x, X_t) is the stochastic processes on $\mathbb{R}^d$ associated to $\mathcal{D}$

$$(e^{-t\Delta}u)(x) = P_x(u \circ X_t)$$

whose polar sets B (avoided by the processes) are the $\mathrm{Cap}(B) = 0$ sets for the electrostatic capacity associated to $\mathcal{D}$.

- The above properties are proved by the knowledge of the Green function

$$\Delta^{-1}u(x) = \int_{\mathbb{R}^d} G(x,y)u(y)\, m(dy) \qquad G(x,y) = |x-y|^{2-d} \qquad d \geq 3\,.$$

- Beurling and Deny (late '50) developed a kernel free potential theory generalizing the notion of Dirichlet integral to locally compact spaces.
- Fukushima (middle '60) achieved the construction of the associated Hunt process.

Let $(\mathcal{M}, L^2(\mathcal{M}), L^2_+(\mathcal{M}), J)$ be a standard form of a von Neumann algebra $\mathcal{M}$.
Let $\xi_0 \in L^2_+(\mathcal{M})$ be a fixed cyclic and separating vector and $\xi \wedge \xi_0 \in L^2_+(\mathcal{M})$ be the projection of a real vector $\xi = J\xi \in L^2(\mathcal{M})$ onto the positive cone $L^2_+(\mathcal{M})$.

Definition. (Dirichlet form)

A Dirichlet form $\mathcal{E} : L^2(\mathcal{M}) \to (-\infty, +\infty]$ is a l.s.c., quadratic form such that

- *the domain $\mathcal{F} := \{\xi \in L^2(\mathcal{M}) : \mathcal{E}[\xi] < +\infty\}$ is dense in $L^2(\mathcal{M})$*
- $\mathcal{E}[J\xi] = \mathcal{E}[\xi]$ — *real*
- $\mathcal{E}[\xi \wedge \xi_0] \leq \mathcal{E}[\xi]$ — *Markovian*
- $(\mathcal{E}, \mathcal{F})$ *is a complete Dirichlet form if its matrix expansions for $n \geq 1$*

$$\mathcal{E}_n[(\xi_{ij})_{ij}] := \sum_{ij} \mathcal{E}[\xi_{ij}]$$

are Dirichlet forms on $\mathcal{M} \otimes \mathbb{M}_n(\mathbb{C})$ (tacitly assumed since now on)

The domain $\mathcal{F}$ is called Dirichlet space when endowed with the graph norm

$$\|\xi\|_{\mathcal{F}} := \sqrt{\mathcal{E}[\xi] + \|\xi\|^2_{L^2(\mathcal{M})}}\,.$$

Definition. (Markovian semigroup)

A self-adjoint C_0-semigroup $\{T_t : t \geq 0\}$ on $L^2(\mathcal{M})$ is Markovian if

- $T_t J = J T_t \qquad t \geq 0$
- $\xi \leq \xi_0 \quad \Rightarrow \quad T_t \xi \leq \xi_0 \qquad t \geq 0$
- $\{T_t : t \geq 0\}$ *on* $L^2(\mathcal{M})$ *is completely Markovian if its matrix expansions*

$$T_t^n([\xi_{ij}]_{ij}) := [T_t \xi_{ij}]_{ij}$$

are Markovian semigroups on $L^2(\mathcal{M} \otimes \mathbb{M}_n(\mathbb{C}))$ *(tacitly assumed since now on)*

Consider the symmetric embedding $\quad i_0 : \mathcal{M} \to L^2(\mathcal{M}) \qquad i_0(x) := \Delta_{\xi_0}^{1/4} x \xi_0$
and the faithful, normal state $\quad \omega_0 : \mathcal{M} \to \mathbb{C} \qquad \omega_0(x) := (\xi_0 | x \xi_0)_2$.

Theorem. (Modular ω_0-symmetry)

Markovian semigroups are in 1:1 correspondence with C_0^*-continuous, positively preserving, contractive semigroups $\{S_t : t \geq 0\}$ on $\mathcal{M}$ which are ω_0-symmetric

$$\omega_0(S_t(x)\sigma^{\omega_0}_{-i/2}(y)) = \omega_0(\sigma^{\omega_0}_{-i/2}(x)S_t(y)) \qquad x, y \in \mathcal{M}_{\sigma^{\omega_0}}, \quad t > 0$$

through $\qquad i_0(S_t(x)) = T_t(i_0(x)) \qquad x \in \mathcal{M}$.

Theorem. (Generalized Beurling-Deny correspondence)

Dirichlet forms are in 1:1 correspondence with Markovian semigroups by

$$\mathcal{E}[\xi] = \lim_{t\to 0} \frac{1}{t}(\xi|a - T_t\xi) \qquad a \in \mathcal{F}$$

or through the self-adjoint generator $(L, dom\,(L))$

$$T_t = e^{-tL} \qquad \mathcal{E}[a] = \|\sqrt{L}a\|^2_{L^2(A,\tau)} \quad a \in \mathcal{F} = dom\,(\sqrt{L})\,.$$

In particular, Dirichlet forms are nonnegative $\mathcal{E} \geq 0$ *and Markovian semigroups are positivity preserving and contractive.*

- Extending Markovian semigroups from $\mathcal{M}$ to $L^2(\mathcal{M})$ via non symmetric embeddings

$$i_\alpha(x) := \Delta^\alpha_{\xi_0} x\xi_0 \qquad \alpha \in [0, 1/2] \qquad \alpha \neq 1/4\,,$$

  produces semigroups on $L^2(\mathcal{M})$ which automatically commute with Δ_{ξ_0}.
- By duality and interpolation, Markovian semigroups extend to C_0-semigroups on noncommutative $L^p(\mathcal{M})$ spaces, $p \in [1, +\infty)$.

Theorem. *(Ergodic Markovian semigroups)*

The following properties are equivalent:

- *the Markovian semigroup* $\{T_t : t \geq 0\}$ *on* $L^2(\mathcal{M}, \omega)$ *is ergodic:*
  for $\xi, \eta \in L^2_+(\mathcal{M}, \omega)$ *there exists* $t > 0$ *such that* $(\xi | T_t \eta)_2 > 0$
- *the Markovian semigroup* $\{T_t : t \geq 0\}$ *on* $L^2(\mathcal{M}, \omega)$ *is indecomposable:*
  for some $t > 0$, T_t *leaves invariant no proper face of the cone* $L^2_+(\mathcal{M}, \omega)$
- $\lambda := \inf\{\mathcal{E}[\xi] : \|\xi\|_2 = 1\}$ *is a Perron-Frobenius eigenvalue:*
  it is a simple eigenvalue with cyclic eigenvector $\xi_\lambda \in L^2_+(\mathcal{M}, \omega)$.

Faces F of the self-polar cone $L^2_+(\mathcal{M}, \omega)$ are in 1:1 correspondence with Peirce projections $P_e = eJeJ$ associated to projections $e \in \mathrm{Proj}\,(\mathcal{M})$

$$F = P_e(L^2_+(\mathcal{M}, \omega))\,.$$

In the trace case, the above equivalences were established by L. Gross in his paper *Existence and uniqueness of physical ground states*, J. Funct. Anal. 10 (1972).

Let $\{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous automorphisms group on the C*-algebra A, A_α the algebra of its analytic elements and let $\omega \in A^*_+$ be a KMS$_\beta$-state for $\beta \in \mathbb{R}$.

Definition. (KMS symmetric semigroups on C*-algebras)

A C_0-semigroup $\{S_t : t \geq 0\}$ on A is KMS$_\beta$ symmetric with respect to ω if

$$\omega(bS_t(a)) = \omega(\alpha_{-\frac{i\beta}{2}}(a)S_t(\alpha_{+\frac{i\beta}{2}}(b))) \qquad a, b \in B$$

for some dense, α-invariant, *-subalgebra $B \subseteq A_\alpha$.

- equivalently $\omega(\alpha_{-\frac{i\beta}{2}}(b)S_t(a)) = \omega(\alpha_{-\frac{i\beta}{2}}(a)S_t(b)) \qquad a, b \in B$
- KMS symmetry is a deformation of the KMS condition, in fact for $t = 0$ we get

$$\omega(ba) = \omega(\alpha_{-\frac{i\beta}{2}}(a)\alpha_{+\frac{i\beta}{2}}(b)) = \omega(a\alpha_{+i\beta}(b)) \qquad a, b \in B\,.$$

- In case $\{\alpha_t : t \in \mathbb{R}\}$ and $\{S_t : t \geq 0\}$ commute, KMS symmetry reduces to

$$\omega(bS_t(a)) = \omega(S_t(b)a) \qquad \textit{GNS symmetry}$$

also referred to as detailed balance.

Proposition.

The following conditions are equivalent

- *a* C_0*-semigroup* $\{S_t : t \geq 0\}$ *on* A *is* KMS$_\beta$ *symmetric with respect to* ω
- *for any* $a, b \in A$ *and on the KMS-strip* $D_\beta \subset \mathbb{C}$ *there exists a bounded continuous function* $F_{a,b} : \overline{D_\beta} \to A$*, analytic in* D_β *such that for* $s \in \mathbb{R}$, $t \geq 0$

$$F_{a,b}(s) = \omega(\alpha_{-s}(a)S_t(\alpha_{+s}(b)))\,, \qquad F_{a,b}(s+i\beta) = \omega(\alpha_{+s}(b)S_t(\alpha_{-s}(a)))\,.$$

Let $\omega_0 \in A^*_+$ be a KMS$_\beta$-state for $\{\alpha_t : t \in \mathbb{R}\} \subset \mathrm{Aut}\,(A)$ and consider

- the cyclic GNS representation $(\pi_{\omega_0}, \mathcal{H}_{\omega_0}, \xi_0)$ of A
- the von Neumann algebra $\mathcal{M} := \pi_{\omega_0}(A)''$ acting on the space
- $L^2(\mathcal{M}, \omega_0) \simeq \mathcal{H}_{\omega_0}$ carrying
- the standard form determined by $L^2_+(\mathcal{M}, \omega_0) = \overline{\{\Delta^{1/4}_{\xi_0} \pi_{\omega_0}(A_+)\xi_0\}}$
- the normal extension of ω_0 to $\mathcal{M}$ given by $\omega_0(x) := (\xi_0|\xi_0 x)_2\,, \qquad x \in \mathcal{M}$
- the modular automorphisms group $\{\sigma^{\omega_0}_t : t \in \mathbb{R}\}$ of $\mathcal{M}$.

Proposition.

A KMS$_\beta$ symmetric, C_0-semigroup $\{S_t : t \geq 0\}$ on A

- *leaves globally invariant the kernel of the cyclic representation:* $S_t(\ker(\pi_{\omega_0})) \subseteq \ker(\pi_{\omega_0})$
- *extends to a ω_0-symmetric, C^*_0-semigroup $\{T_t : t \geq 0\}$ on the von Neumann algebra $\mathcal{M}$ by $T_t \circ \pi_{\omega_0} = \pi_{\omega_0} \circ S_t$*
- *extends to a Markovian semigroup on $L^2(\mathcal{M}, \omega_0)$*
- *determines a Dirichlet form on the standard form $(\mathcal{M}, L^2(\mathcal{M}, \omega_0), L^2_+(\mathcal{M}, \omega_0))$*

Example (C. JFA 147 (1997)).

On a standard form $(\mathcal{M}, L^2(\mathcal{M}), L^2_*(\mathcal{M}), J)$ consider $j(x) := JxJ$ for $x \in \mathcal{M}$ and

- finite subsets $\{a_k : k = 1, \dots, n\} \subset \mathcal{M}$, $\{\mu_k, \nu_k : k = 1, \dots, n\} \subset (0, +\infty)$
- operators $d_k : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ defined by $d_k := i(\mu_k a_k - \nu_k j(a_k^*))$
- quadratic form on $L^2(\mathcal{M})$ given by $\mathcal{E}[\xi] := \sum_{k=1}^n \|d_k \xi\|^2_{L^2(\mathcal{M})}$

Then $\mathcal{E}$ is

- J-real iff $\sum_{k=1}^n [\mu_k^2 a_k^* a_k - \nu_k^2 a_k a_k^*] \in \mu \cap \mathcal{M}'$
- Markovian if moreover $\sum_{k=1}^n [\mu_k^2 a_k^* a_k - \mu_k \nu_k (a_k j(a_k) + a_k^* j(a_k^*)) + \nu_k^2 a_k a_k^*]\xi_0 \geq 0$;
- the associated Markovian semigroup is conservative, $T_t \xi_0 = \xi_0$ for all $t \geq 0$, if moreover the numbers $(\mu_k/\nu_k)^2$ are eigenvalues of the modular operator Δ_{ξ_0} corresponding to eigenvectors $a_k \xi_0$
- the generator has the form

$$L = \sum_{k=1}^n [\mu_k^2 a_k^* a_k - \mu_k \nu_k (a_k j(a_k) + a_k^* j(a_k^*)) + \nu_k^2 a_k a_k^*] .$$

Example (C.-Fagnola-Lindsay CMP 210 (2000)).

- Consider the canonical base $\{e_k : k \in \mathbb{N}\}$ of Hilbert space $h := l^2(\mathbb{N})$
- the C*-algebra of compact operators $\mathcal{K}(h)$
- the von Neumann algebra of bounded operators $\mathcal{B}(h)$
- the Hilbert-Schmidt standard form $(\mathcal{B}(h), \mathcal{L}^2(h), \mathcal{L}^2_+(h), J)$
- fix parameters $\mu > \lambda > 0$ and set $\nu := (\lambda/\mu)^2$
- the state $\omega_\nu(x) := (1-\nu)\sum_{k\geq 0}\nu^k |e_k >< e_k|$
- the cyclic vector $\xi_\nu := (1-\nu)^{1/2}\sum_{k\geq 0}\nu^{k/2}|e_k >< e_k|$
- creation/annihilation operators
  $a^*(e_k) := \sqrt{k+1}e_{k+1} \quad a(e_k) := \sqrt{k}e_{k-1} \quad a(e_0) = 0$
  satisfying the *Canonical Commutation Relation*: $\qquad aa^* - a^*a = I.$

Then the closure of the quadratic form $\mathcal{E} : \mathcal{L}^2(h) \to [0, +\infty)$

$$\mathcal{E}[\xi] := \|\mu a\xi - \lambda\xi a^*\|^2 + \|\mu a\xi^* - \lambda\xi^* a^*\|^2 \qquad \mathcal{F} := \text{linear span}\{|e_k >< e_l| : k, l \in \mathbb{N}\}$$

is a Dirichlet form and the associated Markovian semigroup reduces to an ergodic, Markovian, C$_0$-semigroup on $\mathcal{K}(h)$ leaving the state ω_ν invariant.

- Consider the lattice $\mathbb{Z}^d$ and the class $\mathcal{L}$ of its finite subsets
- denote $A_X := \bigotimes_{x\in X} \mathbb{M}_2(\mathbb{C})$ the algebra of observables in $X \in \mathcal{L}$
- denote $A_0 = \bigcup_{X\in\mathcal{L}} A_X$ the normed algebra of local observables
- $A := \overline{A_0}$ the C*-algebra of quasi-local observabes
- consider an interaction $\Phi := \{\Phi_X = \Phi_X^* \in A_X : X \in \mathcal{L}\}$

Then if $\lambda > 0$ is such that $\|\Phi\|_\lambda := \sup_{x\in\mathbb{Z}^d} \sum_{x\in X\in\mathcal{L}} |X|4^{|X|}e^{\lambda \mathrm{diam}\,(X)}\|\Phi_X\| < +\infty$, a norm closable derivation A is defined on by

$$D(\delta) := A_0 \qquad \delta(a) := \sum_{X\cap Y\neq\emptyset} i[\Phi_Y, a] \qquad a \in A_X \quad X \in \mathcal{L}$$

and the automorphisms group $\{\alpha_t^\Phi : t \in \mathbb{R}\}$ generated by its closure satisfies

- **analiticity**: the evolution $\mathbb{R} \ni t \to \alpha_t^\Phi(a)$ of local observables $a \in A_0$, extends analytically to the strip D_{β_λ}, $\beta_\lambda := \frac{\lambda}{2\|\Phi\|_\lambda}$
- **finite group velocity**: for $a \in A_{\{x\}}, \quad b \in A_X, \quad t \in \mathbb{R}$ we have

$$\|[\alpha_t^\Phi(a), b]\| \leq 2\|a\| \cdot \|b\| \cdot |X| \cdot e^{-(\lambda \mathrm{dist}(x,X))-2|t|\|\Phi\|_\lambda)}\,.$$

Isotropic, anisotropic Heisenberg and Ising models correspond to different Φ.

- Let $\omega \in A^*_+$ be a KMS$_\beta$ state of the automorphisms group $\{\alpha^\Phi_t : t \in \mathbb{R}\}$
- consider the standard form $(\mathcal{M}, L^2(\mathcal{M},\omega), L^2(\mathcal{M},\omega))$ of $\mathcal{M} := \pi_\omega(A)''$ generated by the cyclic representation $(\pi_\omega, L^2(\mathcal{M},\omega), \xi_\omega)$
- consider the Pauli matrices $\{\sigma^x_j \in A_{\{x\}} : j = 0,1,2,3\}$ at sites $x \in \mathbb{Z}^d$
- their images $a^x_j := \pi_\omega(\sigma^x_j) \in \mathcal{M}$
- denote $f_0 : \mathbb{R} \to \mathbb{R}$ the function $f_0(t) := (\cosh(2\pi t))^{-1}$

Theorem. *(Y.M. Park, IDAQP Rel. Top. 3, (2000))*

At sufficiently high temperature $\beta < \frac{\lambda}{\|\Phi\|_\lambda}$*, the form* $\mathcal{E} : L^2(\mathcal{M},\omega) \to [0,+\infty]$

$$\mathcal{E}[\xi] := \sum_{x\in\mathbb{Z}^d}\sum_{j=0}^{3}\mathcal{E}_{x,j}[\xi] \qquad \mathcal{E}_{x,j}[\xi] := \int_{\mathbb{R}} \|[\sigma_{t-i/4}(a^x_j) - j(\sigma_{t-i/4}(a^x_j))]\xi\|^2 f_0(t)dt$$

is a Dirichlet form with respect to the cyclic vector $\xi_\omega \in L^2_+(\mathcal{M},\omega)$.

Proof: combines i) stability of the Markovian property and lower semicontinuity under superposition ii) the condition on the temperature implies that a^x_j are analytic elements for the modular group so that the forms $\mathcal{E}_{x,j}$ are well defined and also provide a dense domain in $L^2(\mathcal{M},\omega)$ where $\mathcal{E}$ is finite.

Theorem. *(Y.M. Park J. Math. Physics 46 (2005))*

The following properties are equivalent

- ω *is an extremal* KMS_β *state for the automorphisms group* $\{\alpha_t^\Phi : t \in \mathbb{R}\}$
- ω *is a factor state*
- *the Markovian semigroup* $\{T_t : t \geq 0\}$ *on* $L^2(\mathcal{M}, \omega)$ *is ergodic.*

"Proof": by construction, $\{a_j^x : x \in \mathbb{Z}^d\ ,\ j = 0, 1, 2, 3\}$ generates $\mathcal{M}$ and one gets

$$\{\xi \in L^2(\mathcal{M}, \omega) : T_t\xi = \xi\ ,\ t > 0\} = \overline{(\mathcal{M} \cap \mathcal{M}')\xi_0}\,.$$

- Let $\omega \in A^*_+$ be a KMS$_\beta$ state of the automorphisms group $\{\alpha^\Phi_t : t \in \mathbb{R}\}$
- consider the standard form $(\mathcal{M}, L^2(\mathcal{M},\omega), L^2(\mathcal{M},\omega))$ of $\mathcal{M} := \pi_\omega(A)''$
- consider the partial traces $\mathrm{Tr}_X : A \to A$ corresponding to $X \in \mathcal{L}$

Theorem. *(A. Majewski-B. Zegarlinski Lett. Math. Phys. 36 (1996))*

There exist $\lambda > 0$ such that $\|\Phi\|_\lambda < +\infty$ and $\beta > 0$ such that

- *There exist $\gamma_X \in A_X$ normalized and rapidly decaying*

$$\mathrm{Tr}_X(\gamma_X^*\gamma_X) = 1 \qquad \|\gamma_{X+j} - \mathrm{Tr}_i(\gamma_{X+j})\|_A \le c \cdot (1 + |i-j|)^{-(2d+\varepsilon)}$$

- *such that the generalized conditional expectation $E_X(a) := \mathrm{Tr}_X(\gamma_X^* a \gamma_X)$ are completely positive, unital and KMS$_\beta$ symmetric*
- *$L_X(a) := a - E_X(a)$ is a bounded generators of a completely positive, unital, KMS$_\beta$ symmetric semigroup*
- *a bounded Dirichlet form is given by*

$$\mathcal{E}^X : L^2(\mathcal{M},\omega) \to [0,+\infty) \qquad \mathcal{E}^X[i_\omega(\pi_\omega(a))] := \omega(\alpha^\Phi_{-\frac{i\beta}{4}}(a))\alpha^\Phi_{-\frac{i\beta}{4}}(L_X a))$$

- *the quadratic form* $\quad \mathcal{E} : L^2(\mathcal{M},\omega) \to [0,+\infty] \qquad \mathcal{E} := \sum_{j\in\mathbb{Z}^d} \mathcal{E}^{X+j}$
is densely defined, closable, Markovian and its closure is a Dirichlet form.

A Compact Quantum Group $\mathbb{G} = (A, \Delta)$ is a unital C*-algebra $A =: C(\mathbb{G})$ and

- a *coproduct* $\Delta : A \to A \otimes A$, a unital, $*$-homomorphism which is
- *coassociative* $(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta$ and satisfies
- *cancelation rules* $\overline{\mathrm{Lin}}((1 \otimes A)\Delta(A)) = \overline{\mathrm{Lin}}((A \otimes 1)\Delta(A)) = A \otimes A$.

A unitary corepresentation of $\mathbb{G}$ is a unitary matrix $U = (u_{jk}) \in M_n(A)$ such that

- $\Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk} \qquad j, k = 1, \ldots, n\,.$

Theorem. (Woronowicz (1987))

Let $\{U^s : s \in \widehat{\mathbb{G}}\}$ be a complete family of inequivalent irr. unitary corepr. of $\mathbb{G}$. Then the algebra of polynomials, defined by

$$\mathrm{Pol}(\mathbb{G}) := \mathrm{Span}\{u^s_{jk};\, s \in \widehat{\mathbb{G}}, 1 \le j, k \le n_s\}$$

is a dense Hopf $*$-algebra with counit $\varepsilon(u^s_{jk}) := \delta_{jk}$ and antipode $S(u^s_{jk}) := (u^s_{kj})^*$ satisfying (m_A being the product in A)

$$(\varepsilon \otimes \mathrm{id})\Delta(a) = a \quad (\mathrm{id} \otimes \varepsilon)\Delta(a) = a \quad m_A(S \otimes \mathrm{id})\Delta(a) = \varepsilon(a)I = m_A(\mathrm{id}\otimes)\Delta(a)\,.$$

- Convolution $\xi \star \xi' \in A^*$ of functionals $\xi\,,\xi' \in A^*$ is defined by

$$\xi \star \xi' := (\xi \otimes \xi') \circ \Delta\,;$$

- convolution $\xi \star a \in A$ of a functional $\xi \in A'$ and an element $a \in A$ is defined by

$$\xi \star a := (\mathrm{id} \otimes \xi)(\Delta a) \qquad a \star \xi := (\xi \otimes \mathrm{id}\,)(\Delta a)$$

Theorem. (Woronowicz (1987))

On a CQG $\mathbb{G} = (A, \Delta)$ there exists a unique (Haar) state $h \in A^*_+$ such that

$$h \star a = a \star h = h(a)1_A \qquad a \in A\,.$$

It is a $(\sigma\,, -1)$-KMS state with respect to a suitable $*$-automorphisms group of A

$$\{\sigma_t : t \in \mathbb{R}\} \qquad h(ab) = h(\sigma_{-i}(b)a) \qquad a, b \in \mathcal{A}\,.$$

Notice that, in general, the Haar state is not a trace.

Theorem. (Woronowicz (1987))

The antipode S is closable and its closure $\overline{S}$ admits the polar decomposition:

$$\overline{S} = R \circ \tau_{\frac{i}{2}},$$

- $\tau_{\frac{i}{2}}$ *generates a $*$-automorphisms group $\{\tau_t : t \in \mathbb{R}\}$ of the C^*-algebra A*
- *R is a linear, anti-multiplicative, norm preserving involution on A such that $\tau_t \circ R = R \circ \tau_t$ for all $t \in \mathbb{R}$, called unitary antipode.*

Example: $SU_q(N)$

- The compact quantum group $SU_q(2) = (A, \Delta)$, $0 < q \le 1$, is given by the universal C^*-algebra A generated by the coefficients of the matrix

$$U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

with relations on α and γ that ensuring unitarity $UU^* = U^*U = 1$

- *comultiplication* $\Delta(\alpha) := \alpha \otimes \alpha + \gamma \otimes \gamma, \quad \Delta(\gamma) := \gamma \otimes \alpha + \alpha^* \otimes \gamma$
- *counit* $\varepsilon(\alpha) = 1 \qquad \varepsilon(\gamma) = 0$
- *antipode* $S(\alpha) := \alpha^*\,, \qquad S(\gamma) := -q\gamma\,, \qquad S(u^s_{jk}) = (-q)^{(j-k)} u^s_{-k,-j}$
- *Haar state* $h(u^s_{jk}) = \delta_{s,0}$
- *automorphisms group* $\sigma_z(u^s_{jk}) = q^{2iz(j+k)} u^s_{jk} \qquad z \in \mathbb{C}$
- *unitary antipode* $R(u^s_{jk}) = q^{k-j}(u^s_{kj})^*$

Let $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$ and $(\mathcal{P}, \Phi)$ a noncommutative probability space.

- **Random variable** on $\mathcal{A}$ is a $*$-algebra homomorphism $j : \mathcal{A} \to \mathcal{P}$
- **distribution** of the random variable $j : \mathcal{A} \to \mathcal{P}$ is the state $\varphi_j = \Phi \circ j$
- **convolution** of the random variables $j_1, j_2 : \mathcal{A} \to \mathcal{P}$ is the random variable

$$j_1 \star j_2 = m_{\mathcal{P}} \circ (j_1 \otimes j_2) \circ \Delta\,.$$

A **Quantum Stochastic Process** is a family of random variables $(j_{s,t})_{0 \leq s \leq t}$

- $j_{rs} \star j_{st} = j_{rt}$ for all $0 \leq r \leq s \leq t \leq T$ **increment property** and $j_{tt} = \varepsilon \mathbf{1}_{\mathcal{P}}$
- j_{st} converges to j_{ss} in distribution for $t \searrow s$ **weak continuity**.

- A QSP is called a **Lévy Process** if has
- **independent** increments, i.e. for disjoint intervals $(t_i, s_i]$

$$\Phi\left(j_{s_1 t_1}(a_1)\ldots j_{s_n t_n}(a_n)\right) = \Phi\left(j_{s_1 t_1}(a_1)\right)\ldots\Phi\left(j_{s_n t_n}(a_n)\right)$$

  and $[j_{s_i,t_i}(a_1), j_{s_j,t_j}(a_2)] = 0$ for $i \neq j$,
- **stationary** increments, i.e. $\varphi_{st} = \Phi \circ j_{st}$ depends only on $t - s$,

Theorem. (CFK 2011)

Lévy process $(j_{st})_{0\le s\le t}$ on a $\mathcal{A}$ are in 1:1 correspondence with Markov semigroup (T_t) on A which are translation invariant

$$\Delta\circ T_t = (\mathrm{id}\otimes T_t)\circ\Delta \qquad t\geq 0\,.$$

"Proof". Distributions $\varphi_t := \varphi_{0,t} = \Phi\circ j_{0,t}$ form a continuous convolution semigroup of states on $\mathcal{A}$:

$$\varphi_0=\varepsilon \qquad \varphi_s\star\varphi_t=\varphi_{s+t} \qquad \lim_{t\to 0}\varphi_t(b)=\varepsilon(b) \qquad b\in\mathcal{A}$$

whose **generating functional** $\varphi_t=\exp_\star tG$ is defined as $G=\frac{d}{dt}\varphi_t\big|_{t=0}$.
A **semigroup** $T_t:\mathcal{A}\to\mathcal{A}$ is defined by the convolution

$$T_t=(\mathrm{id}\otimes\varphi_t)\circ\Delta=\varphi_t\star a, \quad t\geq 0$$

and its **infinitesimal generator** $L:\mathcal{A}\to\mathcal{A}$ results as the convolution operator associated to the generating functional

$$L(a)=(\mathrm{id}\otimes G)\circ\Delta(a)=G\star a\,.$$

The semigroup extends to a translation invariant, Markov semigroup (T_t) on A and its generator is the closure of G. Moreover, has the relations

$$G=\varepsilon\circ L\,, \qquad \varphi_t=\varepsilon\circ T_t \qquad t>0\,.$$

Theorem. *(C-Franz-Kula JFA 266 (2014))*

Let $T_t = e^{-tL}$ be a Lévy semigroup on $\mathcal{A}$ with generating functional $G = \varepsilon \circ L$. The following properties are then equivalent

- *the semigroup is KMS_{-1} symmetric with respect to the Haar state*
- *the generator is KMS_{-1} symmetric with respect to the Haar state*
- *the generating functional is invariant by the action of the unitary antipode R*

$$G = G \circ R \quad \text{on the Hopf algebra} \quad \mathcal{A} = \mathrm{Pol}(\mathbb{G})\,.$$

Proposition. *(C-Franz-Kula JFA 266 (2014))*

$L^2(A,h)$ decomposes as orthogonal sum of the finite dimensional subspaces

$$L^2(A,h) = \bigoplus_{s\in\widehat{\mathbb{G}}} E_s \qquad E_s := \mathrm{Span}\,\{u^s_{jk}\xi_h : j,k = 1,\cdots,n_s\} \qquad s \in \widehat{\mathbb{G}}\,.$$

L decomposes as a direct sum $L = \bigoplus_{s\in\widehat{\mathbb{G}}} L_s$ of its restrictions to the E_s subspaces. Its spectrum thus coincides with $\sigma(L) = \overline{\bigcup_{s\in\widehat{\mathbb{G}}} \sigma(L_s)}$.

- The C*-algebra $C_u(O_N^+)$ is generated by $\{v_{jk} = v_{jk}^* : i,k = 1,\cdots,N\}$ subject to

$$\sum_{l=1}^N v_{lj}v_{lk} = \delta_{jk} = \sum_{l=1}^N v_{jl}v_{kl} \qquad \Delta v_{jk} = \sum_{l=1}^N v_{lj} \otimes v_{lk}$$

- classes of irreducible, unitary corepresentations $\widehat{O_N^+} \cong \mathbb{N}$
- the Haar h state is a trace, faithful on $\mathrm{Pol}(O_N^+)$ but not on $C_u(O_N^+)$
- the Lévy semigroup e^{-tL} is constructed on the reduced C*-algebra $C_r(O_N^+)$
- denote $U_s \in \mathrm{Pol}[-N,N]$ the Chebyshev polynomial of the second kind

$$U_0(x) = 1, \quad U_1(x) = x, \quad U_s(x) = xU_{s-1}(x) - U_{s-1}(x), \qquad x \in [-N,N], \quad s \in \mathbb{N}$$

- generating functional $G(u_{jk}^{(s)}) := \delta_{jk}\frac{U_s'(N)}{U_s(N)}, \quad s \in \mathbb{N}, \quad j,k = 1,\cdots,U_s(N)$
- the generator has discrete spectrum, eigenvalues and multiplicities are given by

$$\lambda_s = \frac{U_s'(N)}{U_s(N)}, \qquad m_s = (U_s(N))^2$$

- spectral dimensions: $\quad d_N = 3$ for $N = 2$, $\quad d_N = +\infty$ for $N \geq 3$.

A second countable, locally compact group G has the Haagerup Approximation property HAP if there exists a sequence of normalized, positive definite functions $\varphi_n \in C_0(G)$, converging to the constant function 1 uniformly on compact subsets.

Equivalently, G has the HAP if there exists a proper, continuous, negative definite function on G.

By a result of U. Haagerup, the free groups $\mathbb{F}_n$ have the HAP as their length functions are negative definite.

A long research (Connes-Jones, Choda, Jolissaint, Boca, Popa) culminated with various definitions of the HAP valid in general von Neumann algebras.

Let us consider the following one.

Definition. (Okayasu-Tomatsu 2014)

A von Neuman algebra $\mathcal{M}$ has the HAP if there exists a standard form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, \mathcal{J})$ and a sequence of contractive, completely positive operators $T_n : \mathcal{H} \to \mathcal{H}$ such that $\|\xi - T_n\xi\|_{\mathcal{H}} \to 0$ as $n \to +\infty$, for all $\xi \in \mathcal{H}$.

Recently the above HAP has been found to be equivalent to others involving Markovian semigroups and Dirichlet forms.

Theorem. (Caspers-Skalski 2014)

The following properties are equivalent

- *The von Neumann algebra $\mathcal{M}$ has the HAP*
- *there exists a Markovian semigroup $\{T_t : t \geq 0\}$ w.r.t. a cyclic and separating vector $\xi_0 \in \mathcal{P}$, such that T_t is compact for all $t > 0$*
- *there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ w.r.t. a cyclic and separating vector $\xi_0 \in \mathcal{P}$, such that its spectrum is discrete.*

As an application one can prove the following result.

Corollary. (Brannan 2012)

The von Neumann algebras $L^\infty(C_r(O_N^+), h)$ of the free orthogonal quantum groups O_N^+ in the cyclic representation of the Haar state h on $L^2(C_r(O_N^+), h)$, have Haagerup approximation property.

Proof. The result follows from the Caspers-Skalski equivalence and the construction of a Dirichlet form with discrete spectrum illustrated above.