Representations of Conformal Nets and Noncommutative Geometry

Sebastiano Carpi

Università di Chieti-Pescara

Frascati, June 17, 2014

Introduction

Conformal nets = description of (chiral) 2D CFT by means of operator algebras.

Noncommutative geometry = study of operator algebras from a geometric point of view and of geometry from an operator algebraic point of view.

The theory of conformal nets is deeply related with various branches of the theory of operator algebras and in particular with subfactor theory and Tomita-Takesaki modular theory.

Until recently, the possible relations between conformal nets and noncommutative geometry have not been investigated.

Aim of this talk: illustration of some recent ideas and results in this direction: "noncommutative geometrization program" for CFT through conformal nets and their representations (theory of superselection sectors).

Remark: The central idea is to look at the CFT observables as "functions on the corresponding noncommutative infinite-dimensional phase space of the theory" and consider them from the point of view of noncommutative geometry. On the other hand space-time will remain classical and hence commutative.

This talk is mainly based on

S. Carpi, R. Hillier, R. Longo: arXiv:1304.4062, to appear in J. Noncommut. Geom.

S. Carpi, R. Hillier, R. Longo, Y. Kawahigashi, F. Xu: arXiv:1207.2398

Graded-local conformal nets on S^1

- Two-dimensional CFT $\equiv$ quantum field theories on the two-dimensional Minkowski space-time with scaling invariance $\Rightarrow$ certain relevant fields (the chiral fields) depend only on $x - t$ (right-moving fields) or on $x + t$ (left-moving fields).
- Chiral CFT $\equiv$ CFT generated by left-moving (or right-moving) fields only. Chiral CFTs can be considered as QFTs on $\mathbb{R}$ and by conformal symmetry on its compactification S^1. Hence we can consider quantum fields on the unit circle $\Phi(z)$, $z \in S^1$ and the corresponding smeared field operators $\Phi(f)$, $f \in C^\infty(S^1)$.
- The operators $\Phi(f)$ generate graded-local conformal nets of von Neumann algebras on S^1 $\mathcal{A} : I \mapsto \mathcal{A}(I)$, $I \in \mathcal{I}$ ($\mathcal{I} \equiv$ family of open nonempty nondense intervals of S^1), acting on a separable Hilbert space $\mathcal{H}$ (the vacuum Hilbert space).
- Graded local-conformal nets on on S^1 can be defined axiomatically.

Axioms for graded-local conformal nets on S^1

- **A.** Isotony. $I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$
- **C.** Conformal covariance. There exists a projective unitary rep. U of the universal covering group $\widetilde{\mathrm{Diff}(S^1)}$ of $\mathrm{Diff}(S^1)$ on $\mathcal{H}$ such that

$$U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\gamma I)$$

and

$$(\gamma(z) = z \text{ for all } z \in I') \Rightarrow U(\gamma) \in \mathcal{A}(I); \quad I' \equiv \text{interior of } \mathrm{S}^1 \smallsetminus \mathrm{I}$$

- **D.** Positivity of the energy. U is a positive energy representation, i.e. the self-adjoint generator L_0 of the rotation subgroup of U (conformal Hamiltonian) has nonnegative spectrum.
- **E.** Vacuum. $\mathrm{Ker}(L_0) = \mathbb{C}\Omega$, where Ω (the vacuum vector) is a unit vector in $\mathcal{H}$ cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.
- **F** Graded locality. There exists a self-adjoint unitary Γ (the grading operator) on $\mathcal{H}$ satisfying $\Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)$ for all $I \in \mathcal{I}$ and $\Gamma\Omega = \Omega$ and such that

$$\mathcal{A}(I') \subset Z\mathcal{A}(I)'Z^*, \quad I \in \mathcal{I}, \quad Z := \frac{\mathbf{1} - \mathrm{i}\Gamma}{1 - \mathrm{i}}.$$

Local conformal nets

- A local conformal net is a graded-local conformal net with trivial grading $\Gamma = \mathbf{1}$.
- The even subnet of a graded-local conformal net $\mathcal{A}$ is defined as the fixed point subnet $\mathcal{A}^\gamma$, with grading gauge automorphism $\gamma = \mathrm{Ad}\Gamma$.
- The restriction of $\mathcal{A}^\gamma$ to the Γ-invariant subspace $\mathcal{H}^\Gamma$ of $\mathcal{H}$ gives rise in a natural way to a local conformal net.

Virasoro algebra

- The projective unitary representation U of $\widetilde{\mathrm{Diff}(S^1)}$ gives rise to a representation of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{-n,m}\mathbf{1} \quad n, m \in \mathbb{Z}$$

  with central charge $c \in \mathbb{R}$ on the dense subspace $\mathcal{H}^{\mathrm{fin}} \subset \mathcal{H}$ spanned by the eigenvectors of L_0.
- The Virasoro field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is the chiral energy-momentum tensor of the theory.
- If the representation of the Virasoro algebra on $\mathcal{H}^{\mathrm{fin}}$ is irreducible then the net $\mathcal{A}$ is generated by the field $L(z)$ and it is called the Virasoro net with central charge c. The Virasoro nets give examples of local conformal nets for all the values of c corresponding to the unitary representations of the Virasoro algebra.

Super-Virasoro algebras

The Virasoro algebra admits supersymmetric extensions (super-Virasoro algebras) $\Rightarrow$ superconformal symmetry.

The Neveu-Schwarz super-Virasoro algebra is the super Lie algebra generated by even L_n, $n \in \mathcal{N}$, odd G_r, $r \in \frac{1}{2} + \mathbb{Z}$, and a central even element $\hat{c}$, satisfying the relations

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
[G_r, G_s] &= 2L_{r+s} + \frac{\hat{c}}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.
\end{aligned}
\tag{1}
$$

The Ramond super-Virasoro algebra is defined analogously but with $r \in \mathbb{Z}$.

One can further extend these super Lie algebras and obtain the so called $N = 2$ super-Virasoro algebras (Neveu-Schwarz and Ramond)

Superconformal nets

- If the representation of the Virasoro algebra associated with a graded-local conformal net $\mathcal{A}$ extends to a representation of Neveu-Schwarz super-Virasoro algebra which is, in a natural sense, compatible with the net structure, then $\mathcal{A}$ is said to be a superconformal net.
- If the representation of the Neveu-Schwarz super-Virasoro algebra on $\mathcal{H}^{\mathrm{fin}}$ is irreducible then the superconformal net $\mathcal{A}$ is generated by the super-Virasoro fields $L(z)$ and $G(z)$ and it is called the super-Virasoro net with central charge c. The the super-Virasoro nets give examples of local conformal nets for all the values of c corresponding to the unitary representations of the Neveu-Schwarz super-Virasoro algebra.
- In a simlar way one can define the $N = 2$ superconformal nets and the $N = 2$ super-Virasoro net with central charge c. Every $N = 2$ superconformal net is also a superconformal net.

Representations of graded-local conformal nets

- A representation of a graded-local conformal net $\mathcal{A}$ on S^1 is a family $\pi = \{\pi_I : I \in \mathcal{I}\}$ of representations π_I of $\mathcal{A}(I)$ on a common Hilbert space $\mathcal{H}_\pi$ such that $I_1 \subset I_2 \Rightarrow \pi_{I_2|\mathcal{A}(I_1)} = \pi_{I_1}$.
- When $\mathcal{A}$ is a local conformal net, the equivalence class $[\pi]$ of an irreducible representation on a separable $\mathcal{H}_\pi$ is called a sector.
- The identical representation π_0 of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$ is called the vacuum representation and the corresponding sector $[\pi_0]$ the vacuum sector.
- π is said to be localized in a given interval I_0 if $\mathcal{H}_\pi = \mathcal{H}$ and $\pi_{I_1}(x) = x$ whenever $I_1 \cap I_0 = \varnothing$ and $x \in \mathcal{A}(I_1)$. Then, if $\mathcal{A}$ is a local conformal net, it can been shown that $\pi_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$ for all I containing I_0, namely π_I is an endomorphism of $\mathcal{A}(I)$ for all $I \supset I_0$.

Universal algebras and DHR endomorphisms

The universal C*-algebra of a local conformal net $\mathcal{A}$ can be defined as the unique (up to isomorphism) unital C*-algebra $C^*(\mathcal{A})$ such that

- there are unital embeddings $\iota_I : \mathcal{A}(I) \to C^*(\mathcal{A}),\ I \in \mathcal{I}$ such that $\iota_{I_2|\mathcal{A}(I_1)} = \iota_{I_1}$ if $I_1 \subset I_2$, and all $\iota_I(\mathcal{A}(I)) \subset C^*(\mathcal{A})$ together generate $C^*(\mathcal{A})$ as a C*-algebra;
- for every representation π of $\mathcal{A}$ on $\mathcal{H}_\pi$, there is a unique representation (denoted by the same symbol) $\pi : C^*(\mathcal{A}) \to B(\mathcal{H}_\pi)$ such that $\pi_I = \pi \circ \iota_I, \quad I \in \mathcal{I}$.

The universal von Neumann algebra $W^*(\mathcal{A})$ of local conformal net $\mathcal{A}$ is the enveloping von Neumann algebra of $C^*(A)$.

DHR endomorphisms

There is a canonical correspondence between localized representations of a local conformal net $\mathcal{A}$ and DHR (localized and transportable) endomorphisms of $C^*(\mathcal{A})$. If π is a representation of $\mathcal{A}$ localized in $I \in \mathcal{I}$ then the corresponding DHR endomorphism ρ satisfies

$$\pi = \pi_0 \circ \rho$$

The DHR endomorphism corresponding to π_0 is the identical endomorphism id.

Every DHR endomorphism ρ of $C^*(\mathcal{A})$ uniquely extends to a normal endomorphism of $W^*(\mathcal{A})$ denoted again by ρ.

Neveu-Schwarz and Ramond and representations

- Let $\mathcal{I}_{-1} = \{I \in \mathcal{I} : -1 \notin I\}$.
- A soliton of a graded-local conformal net $\mathcal{A}$ on S^1 is a family $\pi = \{\pi_I : I \in \mathcal{I}_{-1}\}$ of representations π_I of $\mathcal{A}(I)$ on a common separable Hilbert space $\mathcal{H}_\pi$ such that $I_1 \subset I_2 \Rightarrow \pi_{I_2 | \mathcal{A}(I_1)} = \pi_{I_1}$.
- Given a representation π of the graded-local conformal net $\mathcal{A}$ on a separable $\mathcal{H}_\pi$ one obtains a soliton by considering only the representations π_I with $I \in \mathcal{I}_{-1}$ but not every soliton arises in this way.
- A general soliton π of $\mathcal{A}$ is a soliton whose restriction to the even subnet $\mathcal{A}^\gamma$ comes from a representation of $\mathcal{A}^\gamma$.
- Let π be an irreducible general soliton of $\mathcal{A}$. If π comes from a representation of $\mathcal{A}$ then it is said to be an irreducible Neveu-Schwarz representation. If this is not the case π is said to be an irreducible Ramond representation
- The Neveu-Schwarz representations of a superconformal net $\mathcal{A}$ give rise to representations of the Neveu-Schwarz super-Virasoro algebra while Ramond representations give rise to representations of the Ramond super-Virasoro algebra.

Spectral triples

Spectral triples: $(\mathfrak{A}, (\pi, \mathcal{H}), D)$ also called K-cycles.

- $\mathfrak{A}$ unital $*$-algebra.
- π representation of $\mathfrak{A}$ on the Hilbert space $\mathcal{H}$.
- D selfadjoint operator on $\mathcal{H}$ with compact resolvent, with domain $\mathrm{dom}(D) \subset \mathcal{H}$ (the Dirac operator) such that, $[\pi(A), D]$ is defined and bounded on $\mathrm{dom}(D)$ for all $A \in \mathfrak{A}$.

The spectral triple is said to be even if there is selfadjoint operator Γ (grading operator) such that $\Gamma^2 = \mathbf{1}$, $\Gamma D \Gamma = -D$ and $[\Gamma, \pi(\mathfrak{A})] = \{0\}$.

The spectral triple is said to be θ-summable if $\mathrm{Tr}(e^{-\beta D^2}) < +\infty$ for all $\beta > 0$.

JLO cocycle and index pairing

- $\mathfrak{A} \equiv$ locally convex unital *-algebra
- $(\mathfrak{A}, (\pi, \mathcal{H}), D) \equiv$ even θ-summable spectral triple with grading Γ such that $A \mapsto \pi(A)$, $A \mapsto [D, \pi(A)]$ are continuous maps $: \mathfrak{A} \to B(\mathcal{H})$.
- $(\mathfrak{A}, (\pi, \mathcal{H}), D) \mapsto \tau$. τ is the even JLO cocycle defining an entire cyclic cohomology class $[\tau]$.
- $e \in \mathfrak{A}$ idempotent $\mathcal{H}_\pm := \ker(\Gamma \mp \mathbf{1})$ then the Fredholm index

$$\begin{aligned} \tau(e) \quad &:= \quad \dim\ker\left((\pi(e)D\pi(e))|_{\pi(e)\mathcal{H}_+}\right) \\ &- \quad \dim\ker\left((\pi(e)^* D\pi(e)^*)|_{\pi(e)\mathcal{H}_-}\right) \in \mathbb{Z} \end{aligned}$$

only depends on the entire cohomology class of τ and on the K-theory class of e in $K_0(\mathfrak{A})$ (index pairing).

Spectral triples from Ramond representations

.

- $\mathcal{A}$ superconformal net
- π irreducible graded Ramond representation of $\mathcal{A} \Rightarrow$ representation of the Ramond super-Virasoro algebra on $\mathcal{H}_\pi^{\mathrm{fin}}$ by operators L_n^π, G_r^π, $n, r \in \mathbb{Z}$ and central charge $c \geq 0$.
- In particular ${G_0^\pi}^2 = L_0^\pi - c/24$
- By considering $c/24 - L_0^\pi$ as the analogous of the Laplacian $D_\pi := G_0^\pi$ is a natural candidate for a Dirac operator.
- In typical examples we also have $\mathrm{Tr}(e^{-\beta D_\pi^2}) = \mathrm{Tr}(e^{-\beta(L_0^\pi - c/24)}) < +\infty$ for all $\beta > 0$ (theta-summability).
- We can then look for a suitable subalgebra $\mathfrak{A} \subset W^*(\mathcal{A}^\gamma)$ such that $(\mathfrak{A}, (\pi, \mathcal{H}), D_\pi)$ is a spectral triple. Then we can define the corresponding even JLO cocycle τ_π
- Question: does the entire cyclic cohomology class of τ_π depends on the unitary equivalence class of π?

We considered two strategies.

Strategy **1**.

- $\Delta_R \equiv$ family mutually inequivalent irreducible graded Ramond representations of the superconformal net $\mathcal{A}$ such that $\mathrm{Tr}(e^{-\beta(L_0^\pi - c/24)}) < +\infty$, for all $\pi \in \Delta_R$.
- $\mathfrak{A}_{\Delta_R} \equiv \{A \in W^*(\mathcal{A}^\gamma) : [D_\pi, \pi(A)]$ bounded on $\mathrm{dom}(D_\pi)$ $\forall \pi \in \Delta_R\}$
- Natural locally convex topology on $\mathfrak{A}_{\Delta_R}$.
- $(\mathfrak{A}_{\Delta_R}, (\pi, \mathcal{H}_\pi), D_\pi)$ θ-summable even spectral triple with the right continuity properties for all $\pi \in \Delta_R$ $\Rightarrow$ JLO cocycle τ_π for all $\pi \in \Delta_R$.

Accordingly one can try to study the maps $\Delta_R \ni \pi \mapsto \tau_\pi$ in models for superconformal nets.

The simpler class of models is given by the super-Virasoro nets but in each of these examples we have at most one irreducible graded Ramond representation. Hence we have relevant examples of spectral triples and JLO cocycles in CFT but the map $\Delta_R \ni \pi \mapsto \tau_\pi$ is not interesting.

The situation is different if one considers the $N = 2$ super-Virasoro nets. In this case every irreducible Ramond representation is graded. Then we have the following

Theorem (Carpi, Hillier, Kawahigashi, Longo, Xu)

Let $\mathcal{A}_c$ be the $N = 2$ super-Virasoro net with central charge c and let Δ_R be a maximal family of mutually inequivalent irreducible Ramond representations of $\mathcal{A}_c$. Then there exist projections $p_\pi \in \mathfrak{A}_{\Delta_R}$, $\pi \in \Delta_R$ such that $\tau_{\pi_1}(p_{\pi_2}) = \delta_{\pi_1,\pi_2}$ for all $\pi_1, \pi_2 \in \Delta_R$. In particular if $\pi_1 \neq \pi_2$ then $[\tau_{\pi_1}] \neq [\tau_{\pi_1}]$.

Strategy **2**.

- Let $\mathcal{A}$ be a superconformal net, let π be a fixed irreducible graded Ramond representation of $\mathcal{A}$ and assume that $\mathrm{Tr}(e^{-\beta(L_0^\pi - c/24)}) < +\infty$.
- Consider a family Δ of DHR endomorphisms of $C^*(\mathcal{A}^\gamma)$ containing the identity endomorphism id.
- $\mathfrak{A}_\Delta \equiv \{A \in W^*(\mathcal{A}^\gamma) : [D_\pi, \pi \circ \rho(A)]$ bounded on $\mathrm{dom}(D_\pi)$ $\forall \rho \in \Delta\}$
- Natural locally convex topology on $\mathfrak{A}_\Delta$.
- $(\mathfrak{A}_\Delta, (\pi \circ \rho, \mathcal{H}_\pi), D_\pi)$ θ-summable even spectral triple with the right continuity properties for all $\rho \in \Delta$ $\Rightarrow$ JLO cocycle τ_ρ for all $\rho \in \Delta$.

Again one can study the maps $\Delta \ni \rho \mapsto \tau_\rho$. Stronger results can be obtained if $\Delta \subset \Delta^1$, where Δ^1 is a suitably defined family of differentiably transportable DHR endomorphisms.

Theorem (Carpi, Hillier, Longo)

The JLO cocycles τ_ρ, $\rho \in \Delta$ have the following properties:

(1) *Suppose $\tau_{\mathrm{id}}(\mathbf{1}) \neq 0$ and that, for fixed $\sigma \in \Delta$ and all $\rho \in \Delta$ with $[\pi_0 \circ \rho] \neq [\pi_0 \circ \sigma]$, $\pi \circ \rho$ and $\pi \circ \sigma$ are disjoint. Then, for all $\rho \in \Delta$ with $[\pi_0 \circ \rho] \neq [\pi_0 \circ \sigma]$, we have $[\tau_\rho] \neq [\tau_\sigma]$.*

(2) *Suppose that, for fixed automorphism $\sigma \in \Delta$ and all $\rho \in \Delta$ with $\rho \neq \sigma$, $\pi \circ \rho$ and $\pi \circ \sigma$ are disjoint. Then for every $\rho \in \Delta$ with $\rho \neq \sigma$, we have $[\tau_\rho] \neq [\tau_\sigma]$.*

(3) *Suppose $\Delta \subset \Delta^1$ and that, for fixed automorphism $\sigma \in \Delta$ and all $\rho \in \Delta$ with $[\pi_0 \circ \rho] \neq [\pi_0 \circ \sigma]$, $\pi \circ \rho$ and $\pi \circ \sigma$ are disjoint. Then for every $\rho \in \Delta$, we have*

$$[\pi_0 \circ \rho] = [\pi_0 \circ \sigma] \quad \textit{iff} \quad [\tau_\rho] = [\tau_\sigma].$$

In either case, the two non-equivalent cocycles are separated by pairing them with a suitable element from $K_0(\mathfrak{A}_\Delta)$.

This theorem can be applied to various models including the $(N = 1)$ super-Virasoro nets and supersymmetric loop group models.

THANK YOU VERY MUCH!