# Structure and rigidity for Gaussian actions and their von Neumann algebras

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ENS Lyon

June 2014

# Setting

$\Gamma \curvearrowright (X, \mu)$ pmp dynamical system:

- $\Gamma$ discrete countable group;
- $(X, \mu)$ standard probability space;
- $\mu(g^{-1}A) = \mu(A)$, $A \subset X$, $g \in \Gamma$.

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**Definition**

The **Group-measure space construction** associated with an action $\sigma : \Gamma \curvearrowright (X, \mu)$ is $L^\infty(X, \mu) \rtimes \Gamma$ acting on $H = L^2(X, \mu) \otimes \ell^2(\Gamma)$, generated by

- $f \otimes 1$, $f \in L^\infty(X, \mu)$;
- unitaries $u_g = \sigma_g \otimes \lambda_g$, $g \in \Gamma$.

We have the relation $u_g f u_g^* = \sigma_g(f)$.

# Relate the action to its von Neumann algebra

In this pmp setting, $L^\infty(X,\mu) \rtimes \Gamma$ is a finite von Neumann algebra: it admits a trace $\tau$ satisfying

$$\tau(fu_g) = \int_X f d\mu \delta_{e,g}.$$

| Action $G \curvearrowright (X,\mu)$ | Algebra $M = L^\infty(X,\mu) \rtimes G$ |
|---|---|
| Free | $L^\infty(X,\mu)$ is maximal abelian |
| $G$-invariant set $Y \subset X$ | Central projection $\mathbf{1}_Y \in M' \cap M$ |
| Ergodic | Factor : $M' \cap M = \mathbb{C}1$ |

# Equivalence between actions

## Definition

Two pmp actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are said to be

- **conjugate** if there exists an isomorphism $\delta : \Gamma \to \Lambda$ and a bimeasurable bijection $T : (X,\mu) \to (Y,\nu)$ such that $T(g \cdot x) = \delta(g) \cdot T(x)$, for a.e. $x \in X$, $g \in \Gamma$;
- **orbit equivalent (OE)** if there exists bimeasurable bijection $T : (X,\mu) \to (Y,\nu)$ such that $T(\Gamma \cdot x) = \Lambda \cdot T(x)$ for a.e. $x \in X$;
- **W$^*$-equivalent** if the crossed product von Neumann algebras are isomorphic:

$$L^\infty(X,\mu) \rtimes \Gamma \simeq L^\infty(Y,\nu) \rtimes \Lambda.$$

Of course,

$$\text{Conjugacy} \Rightarrow \text{OE}$$

**Theorem (Singer 1955)**

*Two actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are OE if and only if there exists a pair isomorphism:*

$$(L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma) \simeq (L^\infty(Y,\nu) \subset L^\infty(Y,\nu) \rtimes \Lambda).$$

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When the action is free, $A := L^\infty(X, \mu)$ is a Cartan subalgebra :

- Maximal abelian ;
- Reguliar : $\{u \in \mathcal{U}(M), uAu^* = A\}'' = M$.

# Questions

- Can one recover the Cartan subalgebra inside $L^\infty(X,\mu) \rtimes G$?
- Can one deduce conjugacy of two actions from an orbit equivalence between them?
- What kind of concrete data on an action can be read on its von Neumann algebra?

**Examples (Amenable groups)**

- Finite and abelian groups are amenable;
- stable under taking subgroup, quotient, extensions, direct limits;
- Free groups are **not** amenable.

**Theorem (Connes 1976)**

*Let* $\Gamma \curvearrowright (X, \mu)$ *be any free ergodic pmp action.*
*Then* $L^\infty(X, \mu) \rtimes \Gamma$ *is isomorphic to the hyperfinite* $II_1$ *factor if and only if* $\Gamma$ *is amenable.*

$\rightsquigarrow$ $W^*$-equivalence is very poor in the amenable case.

# Gaussian actions
# &
# Ergodic properties

**Definition**

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If $\dim_{\mathbb{C}} K = \infty$, identify $K$ with a maximal Gaussian Hilbert space inside some $L^2(X, \mu)$, that is a subspace consisting of Gaussian random variables. For instance, use the CCR functor.
Any orthogonal transformation of $K$ comes from a unique pmp transformation of $(X, \mu)$. We get a $\Gamma$-action on $(X, \mu)$.

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### Example

Let $\Gamma \curvearrowright I$ and $\pi : \Gamma \to \mathcal{O}(\ell^2_{\mathbb{R}}(I))$ the corresponding shift representation. Then $\sigma_\pi$ is the generalized Bernoulli shift $G \curvearrowright (\mathbb{R}, \mu_0)^I$.

| Representation $\pi$ | Gaussian action $\sigma_\pi$ |
|---|---|
| Faithful | Free |
| Weakly mixing | Ergodic |
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**Definition**

An action $G \curvearrowright (X, \mu)$ is strongly ergodic if all sequences $(A_n)$ of almost invariant sets are trivial:

$$\left(\lim_n \mu(gA_n \Delta A_n) = 0,\ \forall g\right) \Rightarrow (\mu(A_n)(1 - \mu(A_n)) = 0)\,.$$

**Theorem (B.)**

*A Gaussian action $\sigma_\pi : G \curvearrowright (X, \mu)$ is strongly ergodic iff $\pi \otimes \pi$ has no almost invariant vectors.*

Solidity type result.

**Theorem (B. 2011)**

*Assume that $\pi$ is weakly contained in the regular representation and put $M := L^\infty(X) \rtimes_{\sigma_\pi} \Gamma$.*
*Then for any diffuse subalgebra $Q \subset L^\infty(X)$, we have that $Q' \cap M$ is amenable.*

- Chifan-Ioana 2008: Generalized Bernoulli shifts $\Gamma \curvearrowright [0,1]^I$ with $\mathrm{Stab}(i)$ amenable for all $i \in I$;
- Ozawa 2008: $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$.

The theorem applies to all Gaussian actions associated with representations of lattices in simple Lie groups with finite center.

**Corollary**

*Assume that $\pi$ is weakly contained in the regular representation. Consider the orbit equivalence relation $\mathcal{R}_\Gamma \subset X \times X$ induced by $\sigma_\pi$.*
*Then for any non-amenable subequivalence relation $\mathcal{R} \subset \mathcal{R}_\Gamma$, there exists an $\mathcal{R}$-invariant subset $Y \subset X$ with positive measure where $\mathcal{R}$ is ergodic.*

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### Theorem (B. 2011)

*Assume that $\pi$ is mixing and weakly contained in the regular rep., and consider an intermediate subalgebra $L^\infty(X) \subset Q \subset L^\infty(X) \rtimes_{\sigma_\pi} \Gamma$.*
*Then there exist central projections $(p_n)_{n\geq 0} \subset \mathcal{Z}(Q)$ such that $\sum_n p_n = 1$ and*

- *$p_0 Q$ is amenable;*
- *$p_n Q$ is a prime factor which does not have property Gamma, for all $n \geq 1$.*

## Strategy of proof of the solidity theorem

Put $M = L^\infty(X) \rtimes \Gamma$ and $\tilde{M} = L^\infty(X \times X) \rtimes \Gamma$.
Gaussian actions are **malleable**:

$$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \in \mathcal{O}(K \oplus K) \rightsquigarrow \alpha_t \curvearrowright L^\infty(X \times X)$$

$$\rightsquigarrow \alpha_t \in \mathrm{Aut}(\tilde{M}).$$

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**Proposition**

1. *(Spectral gap) If $P \subset M$ has no amenable direct summand, $\alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(P' \cap M)$ ;*
2. *If $\alpha_t \to \mathrm{id}$ uniformly on $\mathcal{U}(P)$ then $P \prec_M L\Gamma$ or $P' \cap M \prec_M L^\infty(X)$.*

# W*-rigidity

**Theorem (Popa 2006)**

*Take two ICC property (T) groups* $\Gamma$ *and* $\Lambda$*, e.g.* $\mathrm{SL}_n((\mathbb{Z})$ *and* $\mathrm{SL}_m(\mathbb{Z})$*. Then any W*-equivalence between their Bernoulli actions* $\Gamma \curvearrowright [0,1]^\Gamma$ *and* $\Lambda \curvearrowright [0,1]^\Lambda$*, comes from an isomorphism* $G \simeq H$ *together with a conjugacy of the actions.*

**Step 1 :** OE-rigidity results.
**Step 2 :** Conjugate Cartan subalgebras.

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**Step 1 :** OE-rigidity results.
**Step 2 :** Conjugate Cartan subalgebras.

**Question :** What about general Gaussian actions?

## Conjugate Cartan subalgebras

If $L^\infty([0,1]^\Gamma) \rtimes \Gamma = L^\infty([0,1]^\Lambda) \rtimes \Lambda$, then the deformation $(\alpha_t)$ converges uniformly on $\mathcal{U}(L\Lambda)$.
Hence $L\Lambda$ and $L\Gamma$ are unitarily conjugate.

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Now comes the key technical result.

**Theorem (Popa 2006)**

*Let $\Gamma \curvearrowright (X,\mu)$ be a Bernoulli action and put $M = L^\infty(X) \rtimes \Gamma$. Assume that $B \subset M$ is a Cartan subalgebra normalized by unitaries $u_n \in LG$ which go weakly to $0$.*
*Then $B$ is unitary conjugate to $L^\infty(X)$.*

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The proof relies on

- Deformation/rigidity;
- Algebraic structure of Bernoulli actions (cylinders);
- Very strong mixing properties of Bernoulli actions.

**Theorem (B. 2012)**

*Let* $\Gamma \curvearrowright (X, \mu)$ *be a* **mixing Gaussian action** *and put* $M = L^\infty(X) \rtimes \Gamma$*.*
*Assume that* $B \subset M$ *is a Cartan subalgebra normalized by unitaries* $u_n \in LG$ *which go weakly to* $0$*.*
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# Generalization to Gaussian actions

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**Corollary (B. 2012)**

*Any mixing Gaussian action $\sigma$ of an ICC property (T) group is $W^*$-superrigid:*
*If $\rho$ is* **any** *pmp action which is $W^*$-equivalent to $\sigma$ then it is conjugate to $\sigma$.*

Ioana 2010 : Bernoulli actions of the same groups.

## Other applications

- Trivial fundamental groups for crossed-product von Neumann algebras;
- Computation of outer automorphism groups.

**Theorem (B. 2012)**

*Take an ICC property (T) group $\Gamma$, and a mixing rep. $\pi$ which is not weakly contained in the regular representation.*
*Then the crossed-product by the associated Gaussian action is not stably isomorphic to a group factor, yet it has an anti-isomorphism.*

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*Proof.* If $L^\infty(X) \rtimes \Gamma \simeq LG$, then we prove that $G$ must be of the form $G \simeq H \rtimes \Gamma$, where $H$ is abelian and where the action $\Gamma \curvearrowright LH$ is conjugate to $\sigma_\pi$.
If $\pi$ is mixing, $\Gamma \curvearrowright H$ has finite stabilizers, and hence $\Gamma \curvearrowright LH$ is contained in the regular rep. □