Short Distance Analysis in Algebraic Quantum Field Theory I: General Framework

Gerardo Morsella

Tor Vergata University, Roma

Erwin Schrödinger Institute, Vienna
June 27, 2012

Outline

Introduction 1/3

Standard approach to study short distance limit in QFT (of a bosonic field ϕ of mass m and coupling constant g):

- define rescaled field

$$\phi_\lambda(x) = Z_\lambda \phi(\lambda x), \qquad \lambda > 0$$

- renormalization constant Z_λ fixed by requiring that, e.g., $\langle \Omega, \phi_\lambda(x)\phi_\lambda(y)\Omega \rangle$ has finite limit for $\lambda \to 0$
- actual calculation of Z_λ can be performed via Renormalization Group (Callan-Symanzik) Equations, providing

$$Z_\lambda = \lambda \exp\left[-\int_g^{g_\lambda} \frac{\gamma(g')}{\beta(g')} dg' \right], \qquad \lambda \frac{dg_\lambda}{d\lambda} = -\beta(g_\lambda)$$

  β, γ functions obtained from coupling constant and field renormalization

Introduction 2/3

- with this choice ϕ_λ has mass λm and coupling g_λ
- if $\beta(g_\infty) = 0$ and $(g - g_\infty)\beta(g) < 0$ (ultraviolet fixed point)

$$g_\lambda \to g_\infty, \qquad Z_\lambda \sim \lambda^{1+\gamma(g_\infty)}$$
$$\langle \Omega, \phi_\lambda(x_1) \dots \phi_\lambda(x_n)\Omega\rangle \to \langle \Omega_0, \phi_0(x_1) \dots \phi_0(x_n)\Omega_0\rangle$$

  ϕ_0 massless field with coupling g_∞
- e.g. for QCD one finds, perturbatively,

$$\beta(g) = -\frac{7}{(4\pi)^2} g^3 + O(g^5)$$

  $\implies g_\infty = 0$ (asymptotic freedom, important in explaining experimental results of deep inelastic lepton-hadron scattering)
- several other important applications (confinement, operator product expansions, critical phenomena...)

Introduction 3/3

Drawbacks of the standard approach

- difficult to use when perturbation theory is not reliable
- model dependent
- Lagrangian may not be always present (e.g., 2d integrable models)
- basic fields in general not observable (fermionic, gauge fields...) $\Longrightarrow$ interpretation of results may be ambiguous (e.g., confinement, see talk II)

Scaling Algerbas [Buchholz-Verch '95]

Approach to RG and short distance limit in AQFT

- model independent
- fixed by the knowledge of the net of observables

Introduction 3/3

Drawbacks of the standard approach

- difficult to use when perturbation theory is not reliable
- model dependent
- Lagrangian may not be always present (e.g., 2d integrable models)
- basic fields in general not observable (fermionic, gauge fields...) $\Longrightarrow$ interpretation of results may be ambiguous (e.g., confinement, see talk II)

Scaling Algerbas [Buchholz-Verch '95]

Approach to RG and short distance limit in AQFT

- model independent
- fixed by the knowledge of the net of observables

Outline

Outline

Basic Assumptions 1/2

Standard framework of AQFT:

- $\gamma \in \Gamma \mapsto U(\gamma)$ unitary strongly continous representation of geometric symmetry group $\mathbb{R}^{s+1} \subset \Gamma \subset \mathscr{P}$ acting on vacuum Hilbert space $\mathscr{H}$ and with positive energy, $\alpha_\gamma := \mathrm{Ad}\, U(\gamma)$
- $\Omega \in \mathscr{H}$ vacuum vector, unique translation invariant vector: $U(x)\Omega = \Omega$, $x \in \mathbb{R}^{s+1}$
- $O \mapsto \mathscr{A}(O) \subset B(\mathscr{H})$ local net of von Neumann algebras indexed by open bounded sets $O \subset \mathbb{R}^{s+1}$, covariant w.r.t. U:

$$\alpha_\gamma(\mathscr{A}(O)) = \mathscr{A}(\gamma \cdot O), \qquad s \in \Gamma,$$

and with Ω cyclic: $\overline{\mathscr{A}\Omega} = \mathscr{H}$

Basic Assumptions 2/2

Further structures needed:

- C*-algebra of continuous elements w.r.t. Γ

$$\mathfrak{A}(O) := \{A \in \mathscr{A}(O) \; : \; \gamma \mapsto \alpha_\gamma(A) \text{ norm-continuous}\}$$

$\mathscr{A}(O) \subset \mathfrak{A}(O_1)^-$ if $\bar{O} \subset O_1$

- space of elements with 4-momentum transfer in compact $\hat{O} \subset \mathbb{R}^{s+1}$

$$\hat{\mathscr{A}}(\hat{O}) := \{A \in \mathfrak{A} \; : \; \alpha_f(A) = 0 \text{ for all } f \in L^1(\mathbb{R}^{s+1}), \operatorname{supp} \hat{f} \cap \hat{O} = \emptyset\}$$

$$\alpha_f(A) := \int_{\mathbb{R}^{s+1}} dx\, f(x)\alpha_x(A)$$

Renormalization Group and Scaling Algebras 1/2

Main issue in transferring RG to AQFT: absence of quantum fields used in the conventional approach to identify RG transformation

$$R_\lambda : \mathscr{A} \to \mathscr{A}, \qquad R_\lambda(\phi(x)) = \phi_\lambda(x), \qquad \lambda > 0$$

Key observation: RG scaling leaves fundamental constants c, $\hbar$ unchanged

Abstract RG transformation

- c fixed $\implies$ $R_\lambda(\mathscr{A}(O)) = \mathscr{A}(\lambda O)$
- $\hbar$ fixed $\implies$ $R_\lambda(\hat{\mathscr{A}}(\hat{O})) = \hat{\mathscr{A}}(\lambda^{-1}\hat{O})$
- definition of Z_λ $\implies$ $\sup_{\lambda>0} \|R_\lambda\| < \infty$

Accordingly, for $A \in \mathfrak{A}(O)$,

- $R_\lambda(A) \in \mathfrak{A}(\lambda O)$
- $\lim_{x\to 0} \sup_{\lambda>0} \|\alpha_{\lambda x}(R_\lambda(A)) - R_\lambda(A)\| = 0$

Renormalization Group and Scaling Algebras 1/2

Main issue in transferring RG to AQFT: absence of quantum fields used in the conventional approach to identify RG transformation

$$R_\lambda : \mathscr{A} \to \mathscr{A}, \qquad R_\lambda(\phi(x)) = \phi_\lambda(x), \qquad \lambda > 0$$

Key observation: RG scaling leaves fundamental constants c, $\hbar$ unchanged

Abstract RG transformation

- c fixed $\implies R_\lambda(\mathscr{A}(O)) = \mathscr{A}(\lambda O)$
- $\hbar$ fixed $\implies R_\lambda(\hat{\mathscr{A}}(\hat{O})) = \hat{\mathscr{A}}(\lambda^{-1}\hat{O})$
- definition of $Z_\lambda \implies \sup_{\lambda>0} \|R_\lambda\| < \infty$

Accordingly, for $A \in \mathfrak{A}(O)$,

- $R_\lambda(A) \in \mathfrak{A}(\lambda O)$
- $\lim_{x\to 0} \sup_{\lambda>0} \|\alpha_{\lambda x}(R_\lambda(A)) - R_\lambda(A)\| = 0$

Renormalization Group and Scaling Algebras 1/2

Main issue in transferring RG to AQFT: absence of quantum fields used in the conventional approach to identify RG transformation

$$R_\lambda : \mathscr{A} \to \mathscr{A}, \qquad R_\lambda(\phi(x)) = \phi_\lambda(x), \qquad \lambda > 0$$

Key observation: RG scaling leaves fundamental constants c, $\hbar$ unchanged

Abstract RG transformation

- c fixed $\implies R_\lambda(\mathscr{A}(O)) = \mathscr{A}(\lambda O)$
- $\hbar$ fixed $\implies R_\lambda(\hat{\mathscr{A}}(\hat{O})) = \hat{\mathscr{A}}(\lambda^{-1}\hat{O})$
- definition of $Z_\lambda \implies \sup_{\lambda>0} \|R_\lambda\| < \infty$

Accordingly, for $A \in \mathfrak{A}(O)$,

- $R_\lambda(A) \in \mathfrak{A}(\lambda O)$
- $\lim_{x\to 0} \sup_{\lambda>0} \|\alpha_{\lambda x}(R_\lambda(A)) - R_\lambda(A)\| = 0$

Renormalization Group and Scaling Algebras 2/2

Spirit of the algebraic approach: physical information is encoded in the net, not in individual elements $\implies$ all choices of $(R_\lambda)_{\lambda>0}$ should be equivalent $\implies$ we can consider them all the same time

Definition ([Buchholz, Verch '95])

In the C*-algebra $B(\mathbb{R}_+, \mathscr{A})$ of bounded functions $\lambda \in \mathbb{R}_+ \mapsto \underline{A}_\lambda \in \mathscr{A}$, with:

- pointwise defined operations
- norm $\|\underline{A}\| := \sup_{\lambda>0} \|\underline{A}_\lambda\|$
- Γ-action $\underline{\alpha}_\gamma(\underline{A})_\lambda := \alpha_{\gamma_\lambda}(\underline{A}_\lambda)$, $(x, \Lambda)_\lambda := (\lambda x, \Lambda)$

the local scaling algebra attached to the region O is

$$\underline{\mathfrak{A}}(O) := \{\underline{A} \in B(\mathbb{R}_+, \mathscr{A}) \, : \, \underline{A}_\lambda \in \mathscr{A}(\lambda O), \lim_{\gamma \to e} \|\underline{\alpha}_\gamma(\underline{A}) - \underline{A}\| = 0\}$$

$O \mapsto \underline{\mathfrak{A}}(O)$ is a local net, covariant w.r.t. Γ-action $\underline{\alpha}$

$\underline{\mathfrak{A}} := \overline{\bigcup_O \underline{\mathfrak{A}}(O)}$ scaling algebra

Renormalization Group and Scaling Algebras 2/2

Spirit of the algebraic approach: physical information is encoded in the net, not in individual elements $\implies$ all choices of $(R_\lambda)_{\lambda>0}$ should be equivalent $\implies$ we can consider them all the same time

Definition ([Buchholz, Verch '95])

In the C*-algebra $B(\mathbb{R}_+, \mathscr{A})$ of bounded functions $\lambda \in \mathbb{R}_+ \mapsto \underline{A}_\lambda \in \mathscr{A}$, with:

- pointwise defined operations
- norm $\|\underline{A}\| := \sup_{\lambda>0} \|\underline{A}_\lambda\|$
- Γ-action $\underline{\alpha}_\gamma(\underline{A})_\lambda := \alpha_{\gamma_\lambda}(\underline{A}_\lambda)$, $(x, \Lambda)_\lambda := (\lambda x, \Lambda)$

the local scaling algebra attached to the region O is

$$\underline{\mathfrak{A}}(O) := \{\underline{A} \in B(\mathbb{R}_+, \mathscr{A}) \ : \ \underline{A}_\lambda \in \mathscr{A}(\lambda O), \lim_{\gamma \to e} \|\underline{\alpha}_\gamma(\underline{A}) - \underline{A}\| = 0\}$$

$O \mapsto \underline{\mathfrak{A}}(O)$ is a local net, covariant w.r.t. Γ-action $\underline{\alpha}$

$\underline{\mathfrak{A}} := \overline{\bigcup_O \underline{\mathfrak{A}}(O)}$ scaling algebra

Scaling Limits 1/2

φ (locally normal) state on $\mathscr{A} \rightsquigarrow \underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}_\lambda)$ states on $\underline{\mathfrak{A}}$,

$$\mathrm{SL}^{\mathscr{A}}(\varphi) := \{\text{weak* limit points of } (\underline{\varphi}_\lambda)_{\lambda>0} \text{ for } \lambda \to 0\}.$$

Theorem ([Buchholz, Verch '95])

- $\mathrm{SL}^{\mathscr{A}}(\varphi) = (\underline{\omega}_{0,\iota})_{\iota \in I}$ *is independent of* φ.
- $\underline{\omega}_{0,\iota} \in \mathrm{SL}^{\mathscr{A}}$ *with GNS representation* $(\pi_{0,\iota}, \mathscr{H}_{0,\iota}, \Omega_{0,\iota})$. *Then* $\mathscr{A}_{0,\iota}(O) := \pi_{0,\iota}(\underline{\mathfrak{A}}(O))''$ *is a net of local algebras with* Γ *action defined by*

$$U_{0,\iota}(\gamma)\pi_{0,\iota}(\underline{A})\Omega_{0,\iota} = \pi_{0,\iota}(\underline{\alpha}_\gamma(\underline{A}))\Omega_{0,\iota}$$

(If $s = 1$ the vacuum $\Omega_{0,\iota}$ may be not unique, i.e. $\mathscr{A}_{0,\iota}$ not irreducible)

$O \mapsto \mathscr{A}_{0,\iota}(O)$ is the scaling limit net of $\mathscr{A}$.
Physical interpretation: $\mathscr{A}_{0,\iota}$ describes the short-distance (i.e. high-energy) behaviour of $\mathscr{A}$.

Scaling Limits 1/2

φ (locally normal) state on $\mathscr{A} \rightsquigarrow \underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}_\lambda)$ states on $\underline{\mathfrak{A}}$,

$$\mathrm{SL}^{\mathscr{A}}(\varphi) := \{\text{weak* limit points of } (\underline{\varphi}_\lambda)_{\lambda>0} \text{ for } \lambda \to 0\}.$$

Theorem ([Buchholz, Verch '95])

- $\mathrm{SL}^{\mathscr{A}}(\varphi) = (\underline{\omega}_{0,\iota})_{\iota \in I}$ *is independent of* φ.
- $\underline{\omega}_{0,\iota} \in \mathrm{SL}^{\mathscr{A}}$ *with GNS representation* $(\pi_{0,\iota}, \mathscr{H}_{0,\iota}, \Omega_{0,\iota})$. *Then* $\mathscr{A}_{0,\iota}(O) := \pi_{0,\iota}(\underline{\mathfrak{A}}(O))''$ *is a net of local algebras with* Γ *action defined by*

$$U_{0,\iota}(\gamma)\pi_{0,\iota}(\underline{A})\Omega_{0,\iota} = \pi_{0,\iota}(\underline{\alpha}_\gamma(\underline{A}))\Omega_{0,\iota}$$

(If $s = 1$ *the vacuum* $\Omega_{0,\iota}$ *may be not unique, i.e.* $\mathscr{A}_{0,\iota}$ *not irreducible)*

$O \mapsto \mathscr{A}_{0,\iota}(O)$ is the scaling limit net of $\mathscr{A}$.

Physical interpretation: $\mathscr{A}_{0,\iota}$ describes the short-distance (i.e. high-energy) behaviour of $\mathscr{A}$.

Scaling Limits 1/2

φ (locally normal) state on $\mathscr{A} \rightsquigarrow \underline{\varphi}_\lambda(\underline{A}) := \varphi(\underline{A}_\lambda)$ states on $\underline{\mathfrak{A}}$,

$$\mathrm{SL}^{\mathscr{A}}(\varphi) := \{\text{weak* limit points of } (\underline{\varphi}_\lambda)_{\lambda>0} \text{ for } \lambda \to 0\}.$$

Theorem ([Buchholz, Verch '95])

- $\mathrm{SL}^{\mathscr{A}}(\varphi) = (\underline{\omega}_{0,\iota})_{\iota\in I}$ *is independent of* φ.
- $\underline{\omega}_{0,\iota} \in \mathrm{SL}^{\mathscr{A}}$ *with GNS representation* $(\pi_{0,\iota}, \mathscr{H}_{0,\iota}, \Omega_{0,\iota})$. *Then* $\mathscr{A}_{0,\iota}(O) := \pi_{0,\iota}(\underline{\mathfrak{A}}(O))''$ *is a net of local algebras with* Γ *action defined by*

$$U_{0,\iota}(\gamma)\pi_{0,\iota}(\underline{A})\Omega_{0,\iota} = \pi_{0,\iota}(\underline{\alpha}_\gamma(\underline{A}))\Omega_{0,\iota}$$

(If $s = 1$ *the vacuum* $\Omega_{0,\iota}$ *may be not unique, i.e.* $\mathscr{A}_{0,\iota}$ *not irreducible)*

$O \mapsto \mathscr{A}_{0,\iota}(O)$ is the scaling limit net of $\mathscr{A}$.

Physical interpretation: $\mathscr{A}_{0,\iota}$ describes the short-distance (i.e. high-energy) behaviour of $\mathscr{A}$.

Scaling Limits 2/2

Classification of short-distance behavior:

- $\mathscr{A}$ has trivial scaling limit if $\mathscr{A}_{0,\iota} = \mathbb{C}\mathbb{1}$ for all $\underline{\omega}_{0,\iota}$
- $\mathscr{A}$ has unique quantum scaling limit if all nets $\mathscr{A}_{0,\iota}$ are isomorphic and $\neq \mathbb{C}\mathbb{1}$
- $\mathscr{A}$ has degenerate scaling limit otherwise

Theorem ([Buchholz, Verch '95])

If $\mathscr{A}$ has unique quantum scaling limit, then $\exists\, (\delta_\mu^{(0,\iota)})_{\mu>0}$ automorphisms of $\mathscr{A}_{0,\iota}$ s.t.

$$\delta_\mu^{(0,\iota)}(\mathscr{A}_{0,\iota}(O)) = \mathscr{A}_{0,\iota}(\mu O), \quad \delta_\mu^{(0,\iota)}\alpha_\gamma^{(0,\iota)} = \alpha_{\gamma_\mu}^{(0,\iota)}\delta_\mu^{(0,\iota)}$$

(In general $\mu \mapsto \delta_\mu^{(0,\iota)}$ not a representation of dilation group)

$\delta_\mu^{(0,\iota)}$ induced by the fact that if $\underline{\sigma}_\mu(\underline{A})_\lambda := \underline{A}_{\mu\lambda}$, $\underline{A} \in \underline{\mathfrak{A}}$, scaling limit states $\underline{\omega}_{0,\iota}$ and $\underline{\omega}_{0,\iota} \circ \underline{\sigma}_\mu$ give rise to isomorphic nets

Scaling Limits 2/2

Classification of short-distance behavior:

- $\mathscr{A}$ has trivial scaling limit if $\mathscr{A}_{0,\iota} = \mathbb{C}\mathbb{1}$ for all $\underline{\omega}_{0,\iota}$
- $\mathscr{A}$ has unique quantum scaling limit if all nets $\mathscr{A}_{0,\iota}$ are isomorphic and $\neq \mathbb{C}\mathbb{1}$
- $\mathscr{A}$ has degenerate scaling limit otherwise

Theorem ([Buchholz, Verch '95])

If $\mathscr{A}$ has unique quantum scaling limit, then $\exists\,(\delta_\mu^{(0,\iota)})_{\mu>0}$ automorphisms of $\mathscr{A}_{0,\iota}$ s.t.

$$\delta_\mu^{(0,\iota)}(\mathscr{A}_{0,\iota}(O)) = \mathscr{A}_{0,\iota}(\mu O), \quad \delta_\mu^{(0,\iota)}\alpha_\gamma^{(0,\iota)} = \alpha_{\gamma_\mu}^{(0,\iota)}\delta_\mu^{(0,\iota)}$$

(In general $\mu \mapsto \delta_\mu^{(0,\iota)}$ not a representation of dilation group)

$\delta_\mu^{(0,\iota)}$ induced by the fact that if $\underline{\sigma}_\mu(\underline{A})_\lambda := \underline{A}_{\mu\lambda}, \underline{A} \in \underline{\mathfrak{A}}$, scaling limit states $\underline{\omega}_{0,\iota}$ and $\underline{\omega}_{0,\iota} \circ \underline{\sigma}_\mu$ give rise to isomorphic nets

Outline

Dilation invariant theories

Local net $\mathscr{A}$:

- satisfies Haag-Swieca compactness if

$$A \in \mathscr{A}(O) \mapsto E(\hat{O})A\Omega \in \mathscr{H}$$

  is compact for all O, $\hat{O}$
- is dilation invariant if $\exists\, \mu \in \mathbb{R}^+ \mapsto \delta_\mu \in \mathrm{Aut}(\mathscr{A})$ representation of dilation group s.t.

$$\delta_\mu(\mathscr{A}(O)) = \mathscr{A}(\mu O), \quad \delta_\mu \alpha_\gamma = \alpha_{\gamma_\mu}\delta_\mu, \quad \omega\delta_\mu = \omega$$

Theorem ([Buchholz, Verch '95])

$\mathscr{A}$ dilation invariant and satisfying Haag-Swieca (e.g. massless free field [Buchholz, Jacobi '87]) $\implies$ $\mathscr{A}_{0,\iota} \simeq \mathscr{A}$ through

$$\phi(\pi_{0,\iota}(\underline{A})) = \text{w-}\lim_{\kappa} \delta_{\lambda_\kappa^{-1}}(\underline{A}_{\lambda_\kappa})$$

In particular $\mathscr{A}$ has unique quantum scaling limit

Dilation invariant theories

Local net $\mathscr{A}$:

- satisfies Haag-Swieca compactness if
$$A \in \mathscr{A}(O) \mapsto E(\hat{O})A\Omega \in \mathscr{H}$$
is compact for all O, $\hat{O}$
- is dilation invariant if $\exists\, \mu \in \mathbb{R}^+ \mapsto \delta_\mu \in \mathrm{Aut}(\mathscr{A})$ representation of dilation group s.t.
$$\delta_\mu(\mathscr{A}(O)) = \mathscr{A}(\mu O), \quad \delta_\mu \alpha_\gamma = \alpha_{\gamma_\mu}\delta_\mu, \quad \omega\delta_\mu = \omega$$

Theorem ([Buchholz, Verch '95])

$\mathscr{A}$ dilation invariant and satisfying Haag-Swieca (e.g. massless free field [Buchholz, Jacobi '87]) $\implies$ $\mathscr{A}_{0,\iota} \simeq \mathscr{A}$ through

$$\phi(\pi_{0,\iota}(\underline{A})) = \text{w-}\lim_{\kappa} \delta_{\lambda_\kappa^{-1}}(\underline{A}_{\lambda_\kappa})$$

In particular $\mathscr{A}$ has unique quantum scaling limit

Free scalar field for $s = 2, 3$

Theorem ([Buchholz, Verch '97])

$\mathscr{A}^{(m)}$ net generated by mass $m \geq 0$ free scalar field in $s = 2, 3$ spatial dimension $\Longrightarrow$ $\mathscr{A}^{(m)}_{0,\iota} \simeq \mathscr{A}^{(0)}$. In particular $\mathscr{A}^{(m)}$ has unique quantum scaling limit

Main ingredient: local normality of $\omega^{(m)}$ w.r.t. $\omega^{(0)}$ [Eckmann, Fröhlich '74]

$$\Downarrow$$

- existence of representation of $\mathscr{A}^{(m)}$ on $\mathscr{H}^{(0)}$ such that

$$\mathscr{A}^{(m)}(O_0) = \mathscr{A}^{(0)}(O_0)$$

for double cones O_0 based on $t = 0$ plane
- massless dilations δ_λ act on $\mathscr{A}^{(m)}$

Isomorphism $\mathscr{A}^{(m)}_{0,\iota} \simeq \mathscr{A}^{(0)}$ defined through

$$\phi(\pi_{0,\iota}(\underline{A})) := \text{w-}\lim_{\kappa} \delta_{\lambda_\kappa^{-1}}(\underline{A}_{\lambda_\kappa})$$

Free scalar field for $s = 2, 3$

Theorem ([Buchholz, Verch '97])

$\mathscr{A}^{(m)}$ net generated by mass $m \geq 0$ free scalar field in $s = 2, 3$ spatial dimension $\implies \mathscr{A}^{(m)}_{0,\iota} \simeq \mathscr{A}^{(0)}$. In particular $\mathscr{A}^{(m)}$ has unique quantum scaling limit

Main ingredient: local normality of $\omega^{(m)}$ w.r.t. $\omega^{(0)}$ [Eckmann, Fröhlich '74]

$$\Downarrow$$

- existence of representation of $\mathscr{A}^{(m)}$ on $\mathscr{H}^{(0)}$ such that

$$\mathscr{A}^{(m)}(O_0) = \mathscr{A}^{(0)}(O_0)$$

  for double cones O_0 based on $t = 0$ plane
- massless dilations δ_λ act on $\mathscr{A}^{(m)}$

Isomorphism $\mathscr{A}^{(m)}_{0,\iota} \simeq \mathscr{A}^{(0)}$ defined through

$$\phi(\pi_{0,\iota}(\underline{A})) := \text{w-}\lim_{\kappa} \delta_{\lambda_\kappa^{-1}}(\underline{A}_{\lambda_\kappa})$$

Lutz model

Local covariant net $\mathscr{A}_L$ defined by:

- ϕ generalized free scalar field with mass measure $d\rho(m) = dm$ in $s = 2, 3$
- $\mathscr{A}_L(O) := \{\exp(i\Box^{n(O)}\phi(f)) \,:\, \operatorname{supp} f \subset O\}''$ with $n(O) \to +\infty$ monotonically as "radius of O" $\to 0$

Morally: $\mathscr{A}_L(\lambda O)$ contains only observables with energy transfer growing faster than λ^{-1} (apart from $c\mathbb{1}$) $\implies$ RG orbits $\lambda \mapsto R_\lambda(A)$ converge to multiples of $\mathbb{1}$

Theorem ([Lutz '97])

$\mathscr{A}_L$ has trivial scaling limit

Lutz model

Local covariant net $\mathscr{A}_L$ defined by:

- ϕ generalized free scalar field with mass measure $d\rho(m) = dm$ in $s = 2, 3$
- $\mathscr{A}_L(O) := \{\exp(i\Box^{n(O)}\phi(f)) \,:\, \mathrm{supp}\, f \subset O\}''$ with $n(O) \to +\infty$ monotonically as “radius of O” $\to 0$

Morally: $\mathscr{A}_L(\lambda O)$ contains only observables with energy transfer growing faster than λ^{-1} (apart from $c\mathbb{1}$) $\implies$ RG orbits $\lambda \mapsto R_\lambda(A)$ converge to multiples of $\mathbb{1}$

Theorem ([Lutz '97])

$\mathscr{A}_L$ has trivial scaling limit

2d Models with factorizing S-matrix 1/4

$\mathscr{A}_m$ 2d local net with mass $m > 0$ and factorizing S-matrix S [Lechner '08]:

- $\mathscr{A}_m(W_{L/R})$ generated by Zamolodchikov-Schroer wedge-local fields ϕ_m, ϕ'_m
- non-trivial local algebras obtained as intersections of wedge algebras

Scaling limit of $\mathscr{A}_m$ difficult to compute: local operators not sufficiently explicit

Some information can be obtained through the scaling limit of wedge local fields

2d Models with factorizing S-matrix 1/4

$\mathscr{A}_m$ 2d local net with mass $m > 0$ and factorizing S-matrix S [Lechner '08]:

- $\mathscr{A}_m(W_{L/R})$ generated by Zamolodchikov-Schroer wedge-local fields ϕ_m, ϕ'_m
- non-trivial local algebras obtained as intersections of wedge algebras

Scaling limit of $\mathscr{A}_m$ difficult to compute: local operators not sufficiently explicit

Some information can be obtained through the scaling limit of wedge local fields

2d Models with factorizing S-matrix 2/4

Theorem ([Bostelmann, Lechner, M. '11])

If $f_j \in \mathscr{S}(\mathbb{R}^2)$ are derivatives of test functions and $f_{j,\lambda}(x) = \lambda^{-2} f_j(\lambda^{-1} x)$

$$\lim_{\lambda \to 0} \langle \Omega_m, \phi_m^{[l]}(f_{1,\lambda}) \dots \phi_m^{[l]}(f_{n,\lambda}) \Omega_m \rangle = \langle \Omega_0, \phi_0^{[l]}(f_1) \dots \phi_0^{[l]}(f_n) \Omega_0 \rangle$$

with ϕ_0 the Zamolodchikov-Schroer field associated to

$$S_0(p,q) := \lim_{\lambda \to 0^+} S_{\lambda m}(p,q) = \begin{cases} S(\infty) = \pm 1 & pq < 0 \\ S(0) & p = q = 0 \\ S\big(\log p - \log q\big) & p > 0, q > 0 \\ S\big(\log(-q) - \log(-p)\big) & p < 0, q < 0 \end{cases}$$

$$S_m(p,q) = S(\theta_m(p) - \theta_m(q))$$

Massless model defined by ϕ_0 is at least a subnet (tensor factor?) of the complete scaling limit and it is also interesting in its own right.

2d Models with factorizing S-matrix 2/4

Theorem ([Bostelmann, Lechner, M. '11])

If $f_j \in \mathscr{S}(\mathbb{R}^2)$ *are derivatives of test functions and* $f_{j,\lambda}(x) = \lambda^{-2} f_j(\lambda^{-1} x)$

$$\lim_{\lambda \to 0} \langle \Omega_m, \phi_m^{[l]}(f_{1,\lambda}) \dots \phi_m^{[l]}(f_{n,\lambda}) \Omega_m \rangle = \langle \Omega_0, \phi_0^{[l]}(f_1) \dots \phi_0^{[l]}(f_n) \Omega_0 \rangle$$

with ϕ_0 *the Zamolodchikov-Schroer field associated to*

$$S_0(p,q) := \lim_{\lambda \to 0^+} S_{\lambda m}(p,q) = \begin{cases} S(\infty) = \pm 1 & pq < 0 \\ S(0) & p = q = 0 \\ S\big(\log p - \log q\big) & p > 0, q > 0 \\ S\big(\log(-q) - \log(-p)\big) & p < 0, q < 0 \end{cases}$$

$$S_m(p,q) = S(\theta_m(p) - \theta_m(q))$$

Massless model defined by ϕ_0 is at least a subnet (tensor factor?) of the complete scaling limit and it is also interesting in its own right.

2d Models with factorizing S-matrix 3/4

$S_0(p,q) = \pm 1$ for $pq < 0 \implies \phi_0$ splits into (translation-dilation covariant) chiral fields on the real line

$$\varphi(x) = \int_0^{+\infty} dp \big(e^{-ipx} z(p) + e^{ipx} z^\dagger(p)\big), \qquad x \in \mathbb{R}$$

with z, $z^\dagger$ Zamolodchikov operators defined by S_0:

$$z(p)z(q) = S_0(p,q) z(q) z(p)$$
$$z(p)z^\dagger(q) = S_0(q,p) z^\dagger(q) z(p) + \frac{1}{p}\delta(p-q) \qquad p,q > 0$$

"Half-line" and interval algebras:

$$\mathscr{M}_+ := \{e^{i\varphi(f)} : f \in \mathscr{S}_\mathbb{R}(0,+\infty)\}'', \quad \mathscr{M}_- := \mathscr{M}_+'$$
$$\mathscr{A}(a,b) := \alpha_a(\mathscr{M}_+) \cap \alpha_b(\mathscr{M}_-)$$

$I \mapsto \mathscr{A}(I)$ local translation-dilation-reflection covariant net on $\mathbb{R}$

Question: are they non-trivial?

2d Models with factorizing S-matrix 3/4

$S_0(p,q) = \pm 1$ for $pq < 0 \implies \phi_0$ splits into (translation-dilation covariant) chiral fields on the real line

$$\varphi(x) = \int_0^{+\infty} dp \big(e^{-ipx} z(p) + e^{ipx} z^\dagger(p)\big), \qquad x \in \mathbb{R}$$

with z, $z^\dagger$ Zamolodchikov operators defined by S_0:

$$z(p)z(q) = S_0(p,q)z(q)z(p)$$
$$z(p)z^\dagger(q) = S_0(q,p)z^\dagger(q)z(p) + \frac{1}{p}\delta(p-q) \qquad p,q > 0$$

"Half-line" and interval algebras:

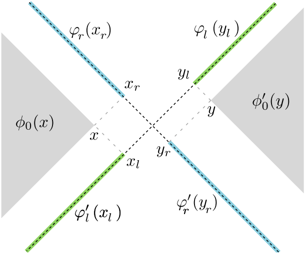
$$\mathscr{M}_+ := \{e^{i\varphi(f)} \,:\, f \in \mathscr{S}_\mathbb{R}(0,+\infty)\}'', \quad \mathscr{M}_- := \mathscr{M}_+'$$
$$\mathscr{A}(a,b) := \alpha_a(\mathscr{M}_+) \cap \alpha_b(\mathscr{M}_-)$$

$I \mapsto \mathscr{A}(I)$ local translation-dilation-reflection covariant net on $\mathbb{R}$

Question: are they non-trivial?

2d Models with factorizing S-matrix 4/4

$\varphi_{l/r}, \mathscr{A}_{l/r}$ copies of $\varphi, \mathscr{A}$ on left/right light ray



Theorem ([BLM '11])

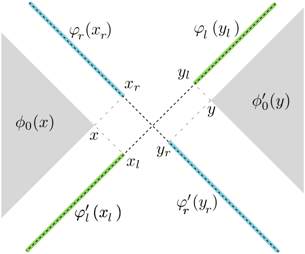
- $\phi_0(x) \cong \frac{1}{2\pi}\left(\varphi_l'(x_l) \otimes \mathbb{1} + S(\infty)^{N_l} \otimes \varphi_r(x_r)\right)$
- $\mathscr{A}_0(I \times J) \supset \mathscr{A}_l(I) \otimes \mathscr{A}_r(J)$

Examples of chiral nets:

- for $S = 1$ we have the free U(1) current:
  - Ω cyclic and separating for $\mathscr{A}(I)$
  - conformal symmetry with $c = 1$
- for $S = -1$ we have the critical Ising model
  - $\mathscr{H}_{\text{loc}} = \overline{\mathscr{A}(I)\Omega}$ states of even particle number
  - $\mathscr{A}(I)$ generted by energy density of free Fermi field
  - conformal symmetry on $\mathscr{H}_{\text{loc}}$ with $c = 1/2$

2d Models with factorizing S-matrix 4/4

$\varphi_{l/r}, \mathscr{A}_{l/r}$ copies of $\varphi, \mathscr{A}$ on left/right light ray



Theorem ([BLM '11])

- $\phi_0(x) \cong \frac{1}{2\pi}\left(\varphi_l'(x_l) \otimes \mathbb{1} + S(\infty)^{N_l} \otimes \varphi_r(x_r)\right)$
- $\mathscr{A}_0(I \times J) \supset \mathscr{A}_l(I) \otimes \mathscr{A}_r(J)$

Examples of chiral nets:

- for $S = 1$ we have the free U(1) current:
  - Ω cyclic and separating for $\mathscr{A}(I)$
  - conformal symmetry with $c = 1$
- for $S = -1$ we have the critical Ising model
  - $\mathscr{H}_{\text{loc}} = \overline{\mathscr{A}(I)\Omega}$ states of even particle number
  - $\mathscr{A}(I)$ generted by energy density of free Fermi field
  - conformal symmetry on $\mathscr{H}_{\text{loc}}$ with $c = 1/2$

Outline

Pointlike Fields from Local Algebras 1/3

Basic idea [Haag, Ojima '96]: assume

$$\Sigma_{E,r} = \{\sigma \upharpoonright \mathscr{A}(O_r) \ : \ \sigma \in P(E)B(\mathscr{H})_* P(E)\}$$

is compact and "does not change" for small r

$\implies$ "finite" number of states describe short distance behaviour

$\implies$ basis (ϕ_j) of $\Sigma^*_{E,r}$ are pointlike fields.

Quantitative version:

- $\Sigma = B(\mathscr{H})_*$, $C^\infty(\Sigma) = \cap_{\ell>0} R^\ell \Sigma R^\ell$, $R = (1+H)^{-1}$;
- $\|\sigma\|^{(\ell)} = \|R^{-\ell}\sigma R^{-\ell}\|$, $\sigma \in C^\infty(\Sigma)$;
- $\Xi : \sigma \in C^\infty(\Sigma) \to \sigma \in \Sigma$.

Definition ([Bostelmann '05])

$O \to \mathscr{A}(O)$ satisfies the microscopic phase space condition I if $\forall \gamma > 0$, $\exists \ell > 0$, $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that

$$\|\psi\|^{(\ell)} < \infty,$$

$$\|(\Xi - \psi)(\cdot) \upharpoonright \mathscr{A}(O_r)\|^{(\ell)} = o(r^\gamma).$$

Pointlike Fields from Local Algebras 1/3

Basic idea [Haag, Ojima '96]: assume

$$\Sigma_{E,r} = \{\sigma \restriction \mathscr{A}(O_r) \,:\, \sigma \in P(E)B(\mathscr{H})_* P(E)\}$$

is compact and "does not change" for small r

$\implies$ "finite" number of states describe short distance behaviour

$\implies$ basis (ϕ_j) of $\Sigma^*_{E,r}$ are pointlike fields.

Quantitative version:

- $\Sigma = B(\mathscr{H})_*$, $C^\infty(\Sigma) = \cap_{\ell>0} R^\ell \Sigma R^\ell$, $R = (1+H)^{-1}$;
- $\|\sigma\|^{(\ell)} = \|R^{-\ell}\sigma R^{-\ell}\|$, $\sigma \in C^\infty(\Sigma)$;
- $\Xi : \sigma \in C^\infty(\Sigma) \to \sigma \in \Sigma$.

Definition ([Bostelmann '05])

$O \to \mathscr{A}(O)$ satisfies the microscopic phase space condition I if $\forall\gamma > 0$, $\exists\ell > 0$, $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that

$$\|\psi\|^{(\ell)} < \infty,$$

$$\|(\Xi - \psi)(\cdot) \restriction \mathscr{A}(O_r)\|^{(\ell)} = o(r^\gamma).$$

Pointlike Fields from Local Algebras 2/3

rankψ minimal, $\psi = \sum_j \sigma_j \phi_j$, $\sigma_j \in \Sigma$, $\phi_j \in C^\infty(\Sigma)^*$.
Define $\Phi_\gamma := \text{span}\{\phi_j\}$. $\Phi_\gamma \subseteq \Phi_{\gamma'}$ if $\gamma < \gamma'$.

Theorem ([Bostelmann '05])

- Φ_γ *independent of* ψ;
- $\phi \in \Phi_\gamma \implies \exists A_r \in \mathscr{A}(O_r), \ell > 0$ *such that*

$$\|\phi - A_r\|^{(\ell)} = O(r).$$

$$\phi(f) = \int dx\, f(x) U(x) \phi U(x)^*, \qquad \phi \in \Phi_\gamma,$$

Wightman field on $C^\infty(H) = \cap_{\ell>0} R^\ell \mathscr{H}$, and $\phi(f) \eta \mathscr{A}(O)$.
ϕ free: $\Phi_0 = \mathbb{C}\mathbb{1}$, $\Phi_1 = \text{span}\{\mathbb{1}, \phi\}$, $\Phi_2 = \text{span}\{\Phi_1, \partial_\mu \phi, :\phi^2:\}$, ...

Pointlike Fields from Local Algebras 2/3

rankψ minimal, $\psi = \sum_j \sigma_j \phi_j$, $\sigma_j \in \Sigma$, $\phi_j \in C^\infty(\Sigma)^*$.
Define $\Phi_\gamma := \text{span}\{\phi_j\}$. $\Phi_\gamma \subseteq \Phi_{\gamma'}$ if $\gamma < \gamma'$.

Theorem ([Bostelmann '05])

- Φ_γ *independent of* ψ;
- $\phi \in \Phi_\gamma \implies \exists A_r \in \mathscr{A}(O_r), \ell > 0$ *such that*

$$\|\phi - A_r\|^{(\ell)} = O(r).$$

$$\phi(f) = \int dx\, f(x) U(x) \phi U(x)^*, \qquad \phi \in \Phi_\gamma,$$

Wightman field on $C^\infty(H) = \cap_{\ell>0} R^\ell \mathscr{H}$, and $\phi(f) \eta \mathscr{A}(O)$.

ϕ free: $\Phi_0 = \mathbb{C}\mathbb{1}$, $\Phi_1 = \text{span}\{\mathbb{1}, \phi\}$, $\Phi_2 = \text{span}\{\Phi_1, \partial_\mu \phi, :\phi^2:\}$, ...

Pointlike Fields from Local Algebras 2/3

rankψ minimal, $\psi = \sum_j \sigma_j \phi_j$, $\sigma_j \in \Sigma$, $\phi_j \in C^\infty(\Sigma)^*$.
Define $\Phi_\gamma := \text{span}\{\phi_j\}$. $\Phi_\gamma \subseteq \Phi_{\gamma'}$ if $\gamma < \gamma'$.

Theorem ([Bostelmann '05])

- Φ_γ *independent of* ψ;
- $\phi \in \Phi_\gamma \implies \exists A_r \in \mathscr{A}(O_r), \ell > 0$ *such that*

$$\|\phi - A_r\|^{(\ell)} = O(r).$$

$$\phi(f) = \int dx\, f(x) U(x) \phi U(x)^*, \qquad \phi \in \Phi_\gamma,$$

Wightman field on $C^\infty(H) = \cap_{\ell>0} R^\ell \mathscr{H}$, and $\phi(f) \eta \mathscr{A}(O)$.
ϕ free: $\Phi_0 = \mathbb{C}\mathbb{1}$, $\Phi_1 = \text{span}\{\mathbb{1}, \phi\}$, $\Phi_2 = \text{span}\{\Phi_1, \partial_\mu \phi, :\phi^2:\}$, ...

Pointlike Fields from Local Algebras 3/3

According to phase space condition, if $A \in \mathscr{A}(O_r)$:

$$A \sim \sum_j \sigma_j(A)\phi_j \qquad \text{as } r \to 0.$$

Can be generalized to local fields by $\varepsilon/3$ argument.

Theorem ([Bostelmann '05])

$\phi, \phi' \in \Phi_\gamma$. For all $\beta > 0$ exist $\sigma_j \in \Sigma$, $\phi_j \in \Phi_{\gamma'}$, $\ell > 0$ such that

$$\|\phi(f_d)\phi'(f'_d) - \sum_j \sigma_j(\phi(f_d)\phi'(f'_d))\phi_j\|^{(\ell)} = o(d^\beta),$$

where $f, f' \in \mathscr{S}$ and $f_d(x) = d^{-4}f(d^{-1}x)$.

Operator product expansion of $\phi(f)\phi'(f')$.

Pointlike Fields from Local Algebras 3/3

According to phase space condition, if $A \in \mathscr{A}(O_r)$:

$$A \sim \sum_j \sigma_j(A)\phi_j \qquad \text{as } r \to 0.$$

Can be generalized to local fields by $\varepsilon/3$ argument.

Theorem ([Bostelmann '05])

$\phi, \phi' \in \Phi_\gamma$. For all $\beta > 0$ exist $\sigma_j \in \Sigma$, $\phi_j \in \Phi_{\gamma'}$, $\ell > 0$ such that

$$\|\phi(f_d)\phi'(f'_d) - \sum_j \sigma_j(\phi(f_d)\phi'(f'_d))\phi_j\|^{(\ell)} = o(d^\beta),$$

where $f, f' \in \mathscr{S}$ and $f_d(x) = d^{-4} f(d^{-1}x)$.

Operator product expansion of $\phi(f)\phi'(f')$.

Renormalization of Pointlike Fields: Basic Idea

- Is the microscopic phase space condition valid for $\mathscr{A}_{0,\iota}$?
- Can we recover Z_λ such that $\phi_0(x) = \lim_{\lambda\to 0} Z_\lambda \phi(\lambda x)$?

$\psi : C^\infty(\Sigma) \to \Sigma$ as above of rank 1:

$$\psi = \sigma\phi, \qquad \sigma \in \Sigma, \phi \in \cup_{\gamma>0}\Phi_\gamma.$$

Typically $\|\sigma \restriction \mathscr{A}(\lambda O)\| \to 0$ as $\lambda \to 0$ (e.g. as $O(\lambda)$ for free fields).
Let $\underline{A} \in \underline{\mathfrak{A}}(O)$: $\psi^*(\underline{A}_\lambda) = \sigma(\underline{A}_\lambda)\phi$ should be thought as a field at scale λ
$\implies$ we can choose $Z_\lambda = \sigma(\underline{A}_\lambda) \sim \lambda$.
Message: maps ψ are the good scale independent objects.

Renormalization of Pointlike Fields: Basic Idea

- Is the microscopic phase space condition valid for $\mathscr{A}_{0,\iota}$?
- Can we recover Z_λ such that $\phi_0(x) = \lim_{\lambda\to 0} Z_\lambda \phi(\lambda x)$?

$\psi : C^\infty(\Sigma) \to \Sigma$ as above of rank 1:

$$\psi = \sigma\phi, \qquad \sigma \in \Sigma, \phi \in \cup_{\gamma>0}\Phi_\gamma.$$

Typically $\|\sigma \restriction \mathscr{A}(\lambda O)\| \to 0$ as $\lambda \to 0$ (e.g. as $O(\lambda)$ for free fields).
Let $\underline{A} \in \underline{\mathfrak{A}}(O)$: $\psi^*(\underline{A}_\lambda) = \sigma(\underline{A}_\lambda)\phi$ should be thought as a field at scale λ
$\implies$ we can choose $Z_\lambda = \sigma(\underline{A}_\lambda) \sim \lambda$.
Message: maps ψ are the good scale independent objects.

Renormalization of Pointlike Fields: Basic Idea

- Is the microscopic phase space condition valid for $\mathscr{A}_{0,\iota}$?
- Can we recover Z_λ such that $\phi_0(x) = \lim_{\lambda\to 0} Z_\lambda \phi(\lambda x)$?

$\psi : C^\infty(\Sigma) \to \Sigma$ as above of rank 1:

$$\psi = \sigma\phi, \qquad \sigma \in \Sigma, \phi \in \cup_{\gamma>0}\Phi_\gamma.$$

Typically $\|\sigma \restriction \mathscr{A}(\lambda O)\| \to 0$ as $\lambda \to 0$ (e.g. as $O(\lambda)$ for free fields).
Let $\underline{A} \in \underline{\mathfrak{A}}(O)$: $\psi^*(\underline{A}_\lambda) = \sigma(\underline{A}_\lambda)\phi$ should be thought as a field at scale λ
$\implies$ we can choose $Z_\lambda = \sigma(\underline{A}_\lambda) \sim \lambda$.
Message: maps ψ are the good scale independent objects.

Renormalization of Pointlike Fields: Basic Idea

- Is the microscopic phase space condition valid for $\mathscr{A}_{0,\iota}$?
- Can we recover Z_λ such that $\phi_0(x) = \lim_{\lambda\to 0} Z_\lambda \phi(\lambda x)$?

$\psi : C^\infty(\Sigma) \to \Sigma$ as above of rank 1:

$$\psi = \sigma\phi, \qquad \sigma \in \Sigma, \phi \in \cup_{\gamma>0}\Phi_\gamma.$$

Typically $\|\sigma \restriction \mathscr{A}(\lambda O)\| \to 0$ as $\lambda \to 0$ (e.g. as $O(\lambda)$ for free fields).
Let $\underline{A} \in \underline{\mathfrak{A}}(O)$: $\psi^*(\underline{A}_\lambda) = \sigma(\underline{A}_\lambda)\phi$ should be thought as a field at scale λ
$\implies$ we can choose $Z_\lambda = \sigma(\underline{A}_\lambda) \sim \lambda$.
Message: maps ψ are the good scale independent objects.

Phase Space and Scaling Limit 1/2

Scaling: $r \to \lambda r$, $E \to \lambda^{-1} E \implies$ phase space condition needs sharpening:

Definition

$O \to \mathscr{A}(O)$ satisfies the microscopic phase space condition II if $\forall \gamma > 0$, $\exists c, \varepsilon > 0$ and $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that for large E, small r,

$$\|\psi \restriction \Sigma_E, \mathscr{A}(O_r)\| \leq c(1 + Er)^\gamma,$$
$$\|(\Xi - \psi) \restriction \Sigma_E, \mathscr{A}(O_r)\| \leq c(Er)^{\gamma+\varepsilon}.$$

Satisfied by free fields in $s = 3$ [Bostelmann '00].

Reasonable for asymptotically free theories (logarithmic corrections to naive scaling).

Note: PSC II $\implies$ PSC I.

Phase Space and Scaling Limit 1/2

Scaling: $r \to \lambda r$, $E \to \lambda^{-1} E \implies$ phase space condition needs sharpening:

Definition

$O \to \mathscr{A}(O)$ satisfies the microscopic phase space condition II if $\forall \gamma > 0$, $\exists c, \varepsilon > 0$ and $\psi : C^\infty(\Sigma) \to \Sigma$ of finite rank such that for large E, small r,

$$\|\psi \upharpoonright \Sigma_E, \mathscr{A}(O_r)\| \leq c(1 + Er)^\gamma,$$
$$\|(\Xi - \psi) \upharpoonright \Sigma_E, \mathscr{A}(O_r)\| \leq c(Er)^{\gamma+\varepsilon}.$$

Satisfied by free fields in $s = 3$ [Bostelmann '00].
Reasonable for asymptotically free theories (logarithmic corrections to naive scaling).
Note: PSC II $\implies$ PSC I.

Phase Space and Scaling Limit 2/2

Orbits of pointlike fields under RG transformations $\underline{\phi}_\lambda = R_\lambda(\phi(0))$ are defined in analogy with the scaling algebra

$$\|\underline{\phi}\|^{(\ell)} := \sup_\lambda \|\underline{R}^\ell_\lambda \underline{\phi}_\lambda \underline{R}^\ell_\lambda\|, \quad \underline{R}_\lambda := (1 + \lambda H)^{-1}$$

$$\underline{\Phi} := \{\lambda \in \mathbb{R}_+ \mapsto \underline{\phi}_\lambda \in C^\infty(\Sigma)^* \ : \ \|\underline{\phi}\|^{(\ell)} < \infty, \lim_{\gamma \to e} \|\alpha_\gamma(\underline{\phi}) - \underline{\phi}\|^{(\ell)} = 0\}$$

Theorem ([Bostelmann, D'Antoni, M. '09])

Let $O \to \mathscr{A}(O)$ satisfy PSC II. Then:

- $\pi_{0,\iota}$ *extends to* $\underline{\Phi}$ *and* $\pi_{0,\iota}(\underline{\phi}) \in C^\infty(\Sigma_{0,\iota})^*$ *is a local field of* $\mathscr{A}_{0,\iota}$ *and* $\exists\, \underline{A}_r \in \underline{\mathfrak{A}}(O_r), \ell > 0$ *s.t.*

$$\|\underline{\phi} - \underline{A}_r\|^{(\ell)} = O(r)$$

- $O \mapsto \mathscr{A}_{0,\iota}(O)$ *satisfies PSC I;*
- $\dim\Phi_{0,\iota,\gamma} \leq \dim\Phi_\gamma$.

Phase Space and Scaling Limit 2/2

Orbits of pointlike fields under RG transformations $\underline{\phi}_\lambda = R_\lambda(\phi(0))$ are defined in analogy with the scaling algebra

$$\|\underline{\phi}\|^{(\ell)} := \sup_\lambda \|\underline{R}^\ell_\lambda \underline{\phi}_\lambda \underline{R}^\ell_\lambda\|, \quad \underline{R}_\lambda := (1+\lambda H)^{-1}$$

$$\underline{\Phi} := \{\lambda \in \mathbb{R}_+ \mapsto \underline{\phi}_\lambda \in C^\infty(\Sigma)^* \ : \ \|\underline{\phi}\|^{(\ell)} < \infty, \lim_{\gamma\to e} \|\alpha_\gamma(\underline{\phi}) - \underline{\phi}\|^{(\ell)} = 0\}$$

Theorem ([Bostelmann, D'Antoni, M. '09])

Let $O \to \mathscr{A}(O)$ satisfy PSC II. Then:

- $\pi_{0,\iota}$ *extends to* $\underline{\Phi}$ *and* $\pi_{0,\iota}(\underline{\phi}) \in C^\infty(\Sigma_{0,\iota})^*$ *is a local field of* $\mathscr{A}_{0,\iota}$ *and* $\exists\, \underline{A}_r \in \underline{\mathfrak{A}}(O_r), \ell > 0$ *s.t.*

$$\|\underline{\phi} - \underline{A}_r\|^{(\ell)} = O(r)$$

- $O \mapsto \mathscr{A}_{0,\iota}(O)$ *satisfies PSC I;*
- $\dim \Phi_{0,\iota,\gamma} \leq \dim \Phi_\gamma$.

Uniform Operator Product Expansion

Define $\underline{\alpha_f \phi} = \int dx\, f(x) \underline{\alpha_x}(\underline{\phi})$, unbounded operator $\forall\, \lambda > 0$.
Thanks to uniform approximation of $\underline{\alpha_f \phi}$ by $\underline{\alpha_f A_r}$,

$$\pi_{0,\iota}(\underline{\alpha_f \phi}\, \underline{\alpha_{f'} \phi'}) = \alpha_f^{(0,\iota)} \pi_{0,\iota}(\underline{\phi}) \alpha_{f'}^{(0,\iota)} \pi_{0,\iota}(\underline{\phi'}),$$

and furthermore:

Theorem ([BDM '09])

For all $\beta > 0$ exist finitely many $\sigma_{j,\lambda} \in \Sigma$, $\phi_j \in \Phi_{\gamma'}$ and $\ell > 0$ such that

$$\left\| \left(\lambda \mapsto \underline{\alpha_{f_d} \phi}_\lambda\, \underline{\alpha_{f'_d} \phi'}_\lambda - \sum_j \sigma_{j,\lambda}(\underline{\alpha_{f_d} \phi}_\lambda\, \underline{\alpha_{f'_d} \phi'}_\lambda) \phi_j \right) \right\|^{(\ell)} = o(d^\beta).$$

Therefore OPE terms converge to OPE terms.

Uniform Operator Product Expansion

Define $\underline{\alpha}_f \underline{\phi} = \int dx\, f(x) \underline{\alpha}_x(\underline{\phi})$, unbounded operator $\forall\, \lambda > 0$.
Thanks to uniform approximation of $\underline{\alpha}_f \underline{\phi}$ by $\underline{\alpha}_f \underline{A}_r$,

$$\pi_{0,\iota}(\underline{\alpha}_f \underline{\phi}\, \underline{\alpha}_{f'} \underline{\phi}') = \alpha_f^{(0,\iota)} \pi_{0,\iota}(\underline{\phi}) \alpha_{f'}^{(0,\iota)} \pi_{0,\iota}(\underline{\phi}'),$$

and furthermore:

Theorem ([BDM '09])

For all $\beta > 0$ exist finitely many $\sigma_{j,\lambda} \in \Sigma$, $\phi_j \in \Phi_{\gamma'}$ and $\ell > 0$ such that

$$\left\| \left(\lambda \mapsto \underline{\alpha}_{f_d} \underline{\phi}_\lambda\, \underline{\alpha}_{f'_d} \underline{\phi}'_\lambda - \sum_j \sigma_{j,\lambda}(\underline{\alpha}_{f_d} \underline{\phi}_\lambda\, \underline{\alpha}_{f'_d} \underline{\phi}'_\lambda) \phi_j \right) \right\|^{(\ell)} = o(d^\beta).$$

Therefore OPE terms converge to OPE terms.

Renormalization Group

Renormalization constants:

- $\underline{\phi}_\lambda = \sum_j \sigma_j(\underline{A}_\lambda)\phi_j$ has well-defined limit $\phi_{0,\iota} = \pi_{0,\iota}(\underline{\phi})$;
- therefore $Z_{j,\lambda} = \sigma_j(\underline{A}_\lambda)$ are renormalization constants.
- in particular for 2-point Wightman functions:

$$\langle \Omega_{0,\iota}, \phi_{0,\iota}(x)\phi'_{0,\iota}(x')\Omega_{0,\iota}\rangle = \lim_\kappa \sum_{j,k} Z_{j,\lambda_\kappa} Z'_{k,\lambda_\kappa}\langle \Omega, \phi_j(\lambda_\kappa x)\phi_k(\lambda_\kappa x')\Omega\rangle,$$

where $Z'_{k,\lambda} = \sigma_k(\underline{A}'_\lambda)$, $\phi'_{0,\iota} = \pi_{0,\iota}(\psi^*(\underline{A}'))$.

Scaling of OPE:

- no Lagrangian in AQFT $\implies$ flow of coupling constants not visible;
- OPE coefficients are the "structure constants" of the algebra of quantum fields;
- scaling changes OPE coefficients.

Renormalization Group

Renormalization constants:

- $\underline{\phi}_\lambda = \sum_j \sigma_j(\underline{A}_\lambda)\phi_j$ has well-defined limit $\phi_{0,\iota} = \pi_{0,\iota}(\underline{\phi})$;
- therefore $Z_{j,\lambda} = \sigma_j(\underline{A}_\lambda)$ are renormalization constants.
- in particular for 2-point Wightman functions:

$$\langle \Omega_{0,\iota}, \phi_{0,\iota}(x)\phi'_{0,\iota}(x')\Omega_{0,\iota}\rangle = \lim_\kappa \sum_{j,k} Z_{j,\lambda_\kappa} Z'_{k,\lambda_\kappa} \langle \Omega, \phi_j(\lambda_\kappa x)\phi_k(\lambda_\kappa x')\Omega\rangle,$$

  where $Z'_{k,\lambda} = \sigma_k(\underline{A}'_\lambda)$, $\phi'_{0,\iota} = \pi_{0,\iota}(\psi^*(\underline{A}'))$.

Scaling of OPE:

- no Lagrangian in AQFT $\implies$ flow of coupling constants not visible;
- OPE coefficients are the "structure constants" of the algebra of quantum fields;
- scaling changes OPE coefficients.

References I

D. Buchholz, R. Verch,
Rev. Math. Phys., **7** (1995) and **10** (1997)

M. Lutz,
Diploma Thesis, Göttingen University, 1997

H. Bostelmann, C. D'Antoni, G. M.,
Commun. Math. Phys., **286** (2009), arXiv:0711.4237

H. Bostelmann, G. Lechner, G. M.,
Rev. Math. Phys., **23** (2011), arXiv:1105.2781