- **Teorema** (Criterio di integrabilità)

Sia $f : [a, b] \to \mathbb{R}$ limitata.

$f \in \mathcal{R}(a,b) \iff \forall \varepsilon > 0 \ \exists D_\varepsilon$ part. di $[a,b]$ t.c. $S(D_\varepsilon) - s(D_\varepsilon) < \varepsilon \iff \exists$ succ. part. D'_n t.c. $S(D'_n) - s(D'_n) \to 0$

(i) (ii) (iii)

Dim.

$(i) \Rightarrow (ii)$

[Per def. sup/inf: $\forall \varepsilon > 0$ $\exists D'_\varepsilon : s(D'_\varepsilon) > \sup S_{inf} - \varepsilon/2$
$\exists D''_\varepsilon : S(D''_\varepsilon) < \inf S_{sup} + \frac{\varepsilon}{2}$]

$\Rightarrow$ Se $\sup S_{inf} = \inf S_{sup} =: S$, detto $D_\varepsilon = D'_\varepsilon \cup D''_\varepsilon$ risulta:

$$S - \frac{\varepsilon}{2} < s(D'_\varepsilon) \le s(D_\varepsilon) \le S(D_\varepsilon) \le S(D''_\varepsilon) < S + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad S(D_\varepsilon) - s(D_\varepsilon) < \left(S + \frac{\varepsilon}{2}\right) - \left(S - \varepsilon/2\right) = \varepsilon.$$

$(ii) \Rightarrow (iii)$

Basta scegliere $\varepsilon = \frac{1}{n} \rightsquigarrow D_n := D_{\frac{1}{n}}$ del pto precedente

$(iii) \Rightarrow (i)$

Risulta

$$0 \le \inf S_{sup} - \sup S_{inf} \le S(D'_n) - s(D'_n) \quad \forall n \in \mathbb{N}$$

quindi $\inf S_{sup} - \sup S_{inf} = 0$ per Th. carabinieri

$\boxed{\text{q.e.d.}}$