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ON SOME SCHRÖDINGER EQUATIONS WITH NON REGULAR POTENTIAL AT INFINITY

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Dedicated to Professor Louis Nirenberg on the occasion of his 85th birthday

Abstract. In this paper we study the existence of solutions $u \in H^1(\mathbb{R}^N)$ for the problem $-\Delta u + a(x)u = |u|^{p-2}u$, where $N \geq 2$ and p is superlinear and subcritical. The potential $a(x) \in L^\infty(\mathbb{R}^N)$ is such that $a(x) \geq c > 0$ but is not assumed to have a limit at infinity. Considering different kinds of assumptions on the geometry of $a(x)$ we obtain two theorems stating the existence of positive solutions. Furthermore, we prove that there are no nontrivial solutions, when a direction exists along which the potential is increasing.

1. **Introduction.** In this paper we consider the problem

$$(P) \qquad \begin{cases} -\Delta u + a(x)u = |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

where $N \geq 2$, $p > 2$ and $p < 2^* := \frac{2N}{N-2}$ if $N \geq 3$. The potential $a(x)$ is a function such that

$$a \in L^\infty(\mathbb{R}^N), \quad \inf_{\mathbb{R}^N} a > 0, \tag{1}$$

but that is not required to have a limit at infinity.

Equations like (P), with

$$\lim_{|x|\to+\infty} a(x) = a_\infty \geq 0, \tag{2}$$

have been extensively studied, see e.g. [1, 2, 3, 6, 7, 11, 13] and references therein. The interest comes essentially from two reasons: the fact that such problems arise naturally in various branches of Mathematical Physics and the lack of compactness, challenging obstacle to the use of the variational methods in a standard way. Actually, (P) has a variational structure, its solutions correspond to the critical points of the energy functional

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$$E(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2)dx \qquad u \in H^1(\mathbb{R}^N)$$

constrained on the manifold

$$M = \left\{ u \in H^1(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} |u|^p = 1 \right\},$$

but M is not weakly closed in the $H^1(\mathbb{R}^N)$ topology, so minimization and minimax methods cannot be applied directly. It is well known that, when $a(x)$ is spherically symmetric (in particular when it is constant) the above difficulty can be overcome thanks to the compactness of the embedding in $L^p(\mathbb{R}^N)$ of the subspace of $H^1(\mathbb{R}^N)$ consisting of radially symmetric functions. On the other hand, when $a(x)$ does not enjoy of symmetry and $a_\infty > 0$ most of the proofs of the known existence results rely on representation theorems for the Palais-Smale sequences of E on M, see e.g. [6, 13]. Roughly speaking, those theorems show that the only obstacles to the compactness are the solutions of the limit problem $-\Delta u + a_\infty u = |u|^{p-2}u$. Furthermore, a uniqueness result for the positive solutions of the limit problem allows to say that, in the positive cone, the levels in which the Palais-Smale condition can fail are only a countable set.

When we drop (2), there is no more a limit problem and the compactness situation can be considerably different. For instance, if we suppose that, for all $\sigma \in S^{N-1}$, $\lim_{\rho\to+\infty} a(\rho\sigma)$ exists and, setting $\alpha(\sigma) := \lim_{\rho\to+\infty} a(\rho\sigma)$, that α is a noncostant continuous function, then it is not difficult to understand that, for any σ, a non compact (P-S) sequence for $E_{|M}$ can be obtained by translating along a sequence of points $(y_n)_n$, $|y_n| \to +\infty$, the positive solution of $-\Delta u + \alpha(\sigma)u = |u|^{p-2}u$, normalized in $L^p(\mathbb{R}^N)$. Hence it is clear that, in the positive cone too, the Palais-Smale condition can fail in a continuous set. As far as we know, situations of the type above described have not received much attention (we remark that a different problem where the Palais-Smale condition fails in a continuous set has been studied in [10]). The results we present in this paper are contribution to the study of this question.

The first theorem we state is a nonexistence result; we find, in fact, a large class of functions $a(x)$ for which (P) has no nontrivial solutions:

Theorem 1.1. *Let $a(x)$, satisfying (1), be such that $\frac{\partial a}{\partial \sigma}$ exists for some direction $\sigma \in S^{N-1}$ and $\frac{\partial a}{\partial \sigma} \geq 0$, $\frac{\partial a}{\partial \sigma} \not\equiv 0$, $\frac{\partial a}{\partial \sigma} \in L^\infty(\mathbb{R}^N)$. Then, if $u \in H^1(\mathbb{R}^N)$ solves (P), $u \equiv 0$.*

On the other hand, we have been able to describe two cases in which, under suitable assumptions on $a(x)$, the existence of positive solutions for (P) is ensured.

We consider first a simple situation in which (P) can be solved by minimization:

Theorem 1.2. *Let $a(x)$ satisfy (1). To any $\eta \in (0, \liminf_{|x|\to+\infty} a(x))$ there corresponds a (suitably large) radius $\rho_\eta > 0$ such that if*

$$\sup_{B(x_\eta,\rho_\eta)} a(x) \ \leq \ \liminf_{|x|\to+\infty} a(x) - \eta \ \ \text{for some } x_\eta \in \mathbb{R}^N, \tag{3}$$

then problem (P) has at least a positive solution.

The assumption of the above theorem can appear quite technical, but its meaning can be easily expressed saying that what is enough to prove the existence of a minimum of $E_{|M}$ is that the potential $a(x)$ is smaller than its liminf at infinity in a suitably large ball. When this is true, indeed, testing the functional E with the function ω that realizes the value $m := \min\left\{\int_{\mathbb{R}^N}(|\nabla u|^2 + lu^2)dx \ : \ u \in M\right\}$, where $l := \liminf_{|x|\to+\infty} a(x)$, it is possible to show that the infimum of $E_{|M}$ is smaller than m. Then a compactness argument in the spirit of the Concentration-Compactness principle (but not straightly descending from it, because there is not a limit problem at infinity) allows to conclude that the infimum is achieved. The situation of the above theorem is similar, in some sense, to that considered in [11] (see also [9] and references therein) for singularly perturbed Schrödinger equations of the type $-\epsilon^2\Delta u + a(x)u = f(u)$. Actually, it is not difficult to see, by a suitable scaling, that the condition $\liminf_{|x|\to+\infty} a(x) > \inf_{|x|\in\mathbb{R}^N} a(x)$, there used to obtain solutions for small ϵ, imply, when $f(u) = |u|^{p-2}u$, the fulfillment of the condition we impose in Theorem 1.2 to solve (P). Moreover, with respect to these papers, we stress the fact that, as mentioned before, our method to approach the problem is direct and needs neither any modification of the energy functional neither approximation techniques.

The main part of the paper is devoted to consider a more complicated topological case, in which (P) cannot be solved by minimization. The existence result we obtain is contained in the following Theorem 1.3:

Theorem 1.3. *Let $a(x)$ satisfy (1) and the following assumptions*

(H_1) $\lim_{\rho\to+\infty} a(\rho e_1) = \lim_{\rho\to+\infty} a(-\rho e_1) := \Theta,$ *where* $e_1 = (1, 0, \dots, 0)$,

(H_2) $\limsup_{\rho\to+\infty} a(\rho\sigma) \longrightarrow \Theta,$ *as* $\sigma \to e_1$ *and as* $\sigma \to -e_1$,

(H_3) $a(x_1, \dots, x_i, \dots, x_N) = a(x_1, \dots, -x_i, \dots, x_N)$ $\forall i \in \{2, \dots, N\},\ \forall x \in \mathbb{R}^N$,

(H_4) *i)* $a(x) \geq \Theta$, *ii)* $|a - \Theta|_{L^\infty(\mathbb{R}^N)} < (2^{1-2/p} - 1)\Theta$,

then problem (P) has at least a positive solution.

In addition, next Proposition 1 states explicitely the non existence, in the same situation, of a minimum of E, either on M, either on its submanifold consisting of the functions satisfying the symmetry condition (H_3):

Proposition 1. *Let $a(x)$ satisfy (1) and (H_1), (H_2), $(H_4)(i)$ and $a(x) \not\equiv \Theta$. Set*

$$M_s = \{u \in M \ : \ u(x_1, \dots, x_i, \dots, x_N) = u(x_1, \dots, -x_i, \dots, x_N),\\ i = 2, \dots, N,\ x \in \mathbb{R}^N\}. \tag{4}$$

Then

$$\inf_{M_s} E = \inf_M E = \min_{u\in M} \int_{\mathbb{R}^N} (|\nabla u|^2 + \Theta u^2)dx = m, \tag{5}$$

and the infima are not attained.

It is worth observing that, in spite of the symmetry assumption (H_3), the loss of compactness in the situation considered in Theorem 1.3 is severe, as Proposition 1 shows. Thus, facing (P) by variational methods, a critical level has to be searched above the infimum level of E on M. As a consequence, the first basic step is a careful analysis of the compactness in order to locate some energy interval in which the Palais-Smale condition holds. Then, a not trivial application of a linking theorem, involving also the use of a suitable barycenter map, allows to prove the existence of a solution and an estimate of its energy permits to conclude that it is positive.

Moreover, some remark about the assumptions of the above Theorem 1.3 is in order. First of all, we observe that, clearly, in (H_1) the unitary vector e_1 can be replaced by any other unitary vector, changing in accord to this (H_2) and (H_3) too.

But, mainly, we want point out that the assumption $(H_4)ii)$ has been set because it is easy to verify, but it imply a smallness condition on the range of $a(x)$ that is not necessary. Indeed, the oscillation of $a(x)$ can be arbitrarily large in $\mathbb{R}^N$, because to prove our result it is enough that $a(x)$ is suitably flat in a suitable cylinder around the x_1-axis: $(H_4)ii)$ can be replaced by the weaker, but more technical, following condition

$$(H_4')ii) \qquad \exists R > 0 \ \text{ s.t. } \quad \begin{cases} a) \ |\omega|^2_{L^2(\mathbb{R}^N \setminus C_R)} < \frac{(2^{-2/p} - 2^{-1})m}{|a-\Theta|_{L^\infty(\mathbb{R}^N)}}; \\ b) \ |a - \Theta|_{L^\infty(C_R)} < (2^{-2/p} - 2^{-1})\Theta \end{cases}$$

where $C_R = \left\{(x_1, x_2, \dots, x_N) \in \mathbb{R}^N \ : \ \sum_{i=2}^N x_i^2 < R^2\right\}$ and ω is the function realizing m.
In the proof of Theorem 1.3, in section 4, this fact will be clearly shown.

At last, completing the above results, we state a regularity result for the solutions of (P). We do not know whether a more general regularity theorem, including our cases, is available in the literature, but, since we have not been able to find a fitting reference and an estimate of the asymptotic behavior of the solutions of (P) is necessary to prove Theorem 1.1, we include a proof of it.

Theorem 1.4. *Let $a(x)$ satisfy (1) and let u be a solution of (P). Then $c, \delta \in \mathbb{R}$, $c > 0$, $\delta > 0$, exist so that*

$$|u(x)| \le c\, e^{-\delta|x|} \qquad \forall x \in \mathbb{R}^N, \tag{6}$$

$$|\nabla u| \in L^q(\mathbb{R}^N), \quad \forall q > 1, \quad \text{and} \quad \lim_{|x|\to+\infty} |\nabla u| = 0. \tag{7}$$

The paper is organized as follows: in section 2, after introducing notations and recalling some useful fact, the proof of the nonexistence Theorem 1.1 is given; section 3 contains the proof of Theorem 1.2, while Theorem 1.3 and Proposition 1 are proven in section 4; section 5 is devoted to the study of the regularity of the solutions of (P).

2. Useful facts and proof of the nonexistence result.

In what follows we set

$$\liminf_{|x|\to+\infty} a(x) = l \tag{8}$$

and we use the following notations:

- $H^1(\mathbb{R}^N)$ denotes the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm
$$\|u\| := \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + lu^2)dx\right]^{1/2}. \tag{9}$$
Moreover, considering in $\mathcal{C}_0^\infty(\mathbb{R}^N)$ the norm
$$\|u\|_a := \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2)dx\right]^{1/2}, \tag{10}$$
we observe that by (1) it is equivalent to (9). So, if we take the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_a$ instead of $\|u\|$, we obtain the same (topological) space $H^1(\mathbb{R}^N)$.
- $L^p(\mathbb{R}^N)$, $1 \le p < +\infty$, denotes the Lebesgue space; the norm in $L^p(\mathbb{R}^N)$ is denoted by $|\cdot|_p$.
- $B(x, r)$ denotes the open ball, of $\mathbb{R}^N$, having radius r and centered at x.

The following proposition is obtained collecting some well known results (see e.g. [6])

Proposition 2. *The infimum*

$$m := \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + l u^2) dx \ : \ u \in M \right\} \tag{11}$$

is attained by a positive function ω *that is unique, modulo translations, radially symmetric and verifies*

$$\lim_{|x|\to+\infty} |D^j \omega(x)||x|^{\frac{N-1}{2}} e^{\sqrt{l}|x|} = d_j > 0, \quad d_j \in \mathbb{R}, \ j = 0, 1. \tag{12}$$

Moreover, for any other critical point v *of* $\|\cdot\|^2$ *constrained on* M *the relation*

$$\|v\|^2 \geq 2^{1-2/p} m \tag{13}$$

holds.

Proof of Theorem 1.1. Without any loss of generality, in what follows we can assume $\sigma = e_1$.

Let u be a solution of (P): by Theorem 1.4 u is a smooth function exponentially decaying at infinity, $\frac{\partial u}{\partial x_1} \in L^q(\mathbb{R}^N)$, $\forall q > 1$, and $\frac{\partial u}{\partial x_1} \longrightarrow 0$, for $|x| \to +\infty$.

Considering (6) and (7), we get from (P)

$$-\Delta u_{x_1} + \frac{\partial a(x)}{\partial x_1} u + a(x) u_{x_1} = (p-1)|u|^{p-2} u_{x_1},$$

in weak sense, from which we deduce

$$\int_{\mathbb{R}^N} \nabla u \nabla u_{x_1} dx + \int_{\mathbb{R}^N} \frac{\partial a(x)}{\partial x_1} u^2 dx + \int_{\mathbb{R}^N} a(x) u_{x_1} u \, dx = (p-1) \int_{\mathbb{R}^N} |u|^{p-2} u \, u_{x_1} dx. \tag{14}$$

On the other hand, multiplying (P) by u_{x_1}, we have

$$\int_{\mathbb{R}^N} \nabla u \nabla u_{x_1} dx + \int_{\mathbb{R}^N} a(x) u \, u_{x_1} = \int_{\mathbb{R}^N} |u|^{p-2} u \, u_{x_1} dx$$

that, inserted in (14) brings to

$$\int_{\mathbb{R}^N} \frac{\partial a(x)}{\partial x_1} u^2 dx = (p-2) \int_{\mathbb{R}^N} |u|^{p-2} u \, u_{x_1} dx = \frac{p-2}{p} \int_{\mathbb{R}^N} \frac{\partial |u|^p}{\partial x_1} dx$$

but, because of the exponential decay of u, $\int_{\mathbb{R}^N} \frac{\partial |u|^p}{\partial x_1} dx = 0$, so we obtain $\int_{\mathbb{R}^N} \frac{\partial a(x)}{\partial x_1} u^2 dx = 0$. Taking into account that, by the Hopf's Lemma, if $u \not\equiv 0$, the Lebesgue measure $|\{x \in \mathbb{R}^N \ : \ u(x) = 0\}| = 0$ and that $0 \not\equiv \frac{\partial a(x)}{\partial x_1} \geq 0$, we infer $u \equiv 0$. ☐

3. **Proof of Theorem 1.2.** In what follows we set

$$m_a = \inf_{u \in M} E(u).$$

Proof of Theorem 1.2. Let us consider any $\eta \in (0, l)$. By (12), we can choose a radius $\rho_\eta > 0$ so that

$$\int_{B(0,\rho_\eta)} (|\nabla \omega|^2 + l\omega^2) dx + \int_{\mathbb{R}^N \setminus B(0,\rho_\eta)} (|\nabla \omega|^2 + |a|_\infty \omega^2) dx < m + \eta \int_{B(0,\rho_\eta)} \omega^2 dx. \tag{15}$$

Then, since (3) holds true, we deduce

$$\begin{aligned}
E(\omega(x-x_\eta)) &= \int_{\mathbb{R}^N}(|\nabla\omega(x-x_\eta)|^2+a(x)(\omega(x-x_\eta))^2)dx \\
&\le \int_{B(x_\eta,r_\eta)}|\nabla\omega(x-x_\eta)|^2+(l-\eta)(\omega(x-x_\eta))^2dx \\
&\quad +\int_{\mathbb{R}^N\setminus B(x_\eta,r_\eta)}(|\nabla\omega(x-x_\eta)|^2+|a|_\infty(\omega(x-x_\eta))^2)dx \\
&= \int_{B(0,r_\eta)}(|\nabla\omega|^2+l\omega^2)dx-\eta\int_{B(0,r_\eta)}\omega^2dx \\
&\quad +\int_{\mathbb{R}^N\setminus B(0,r_\eta)}(|\nabla\omega|^2+|a|_\infty\omega^2)dx \\
&< m. \qquad (16)
\end{aligned}$$

Hence

$$m_a<m. \qquad (17)$$

To show that m_a is attained, considering a minimizing sequence $(u_n)_n$ in M, i.e.

$$\begin{cases} a) \quad u_n\in H^1(\mathbb{R}^N), \quad |u_n|_p=1 \\ b) \quad E(u_n)=m_a+o(1), \end{cases} \qquad (18)$$

we must show that u_n is relatively compact. By the Ekeland's variational principle we can assume $\nabla E_{|_M}(u_n)=o(1)$, that is

$$\int_{\mathbb{R}^N}((\nabla u_n,\nabla w)+a(x)u_nw)dx-\mu_n\int_{\mathbb{R}^N}|u_n|^{p-2}u_nw\,dx=o(1)\|w\| \quad \forall w\in H^1(\mathbb{R}^N), \qquad (19)$$

for suitable $\mu_n\in\mathbb{R}$.

Since $\inf_{\mathbb{R}^N}a>0$, $E(u_n)\ge k\cdot\|u_n\|^2$, $k>0$ constant, so (18)(b) implies that $(u_n)_n$ is bounded in $H^1(\mathbb{R}^N)$. As a consequence, $u_0\in H^1(\mathbb{R}^N)$ exists so that, up to a subsequence,

$$\begin{array}{lll} a) & u_n \rightharpoonup u_0 & \text{weakly in } H^1(\mathbb{R}^N) \text{ and in } L^p(\mathbb{R}^N),\ 2\le p\le 2^* \\ b) & u_n(x)\longrightarrow u_0(x) & \text{a.e. in } \mathbb{R}^N \\ c) & u_n\longrightarrow u_0 & \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N). \end{array} \qquad (20)$$

Moreover, setting $w=u_n$ in (19) we get

$$m_a+o(1)=\int_{\mathbb{R}^N}(|\nabla u_n|^2+a(x)u_n^2)dx=\mu_n. \qquad (21)$$

(19), (20), (21) imply, then, u_0 is a weak solution of

$$-\Delta u+a(x)u=m_a|u|^{p-2}u \qquad \text{in } \mathbb{R}^N,$$

hence, if we show $|u_0|_p = 1$ we are done. We exclude, first, that $u_0 \equiv 0$. In this case, in fact, denoting by $\alpha(x) := 0 \wedge (a(x) - l)$,

$$\begin{aligned} E(u_n) &= \int_{\mathbb{R}^N} (|\nabla u_n|^2 + a(x)u_n^2)dx \\ &= \int_{\mathbb{R}^N} (|\nabla u_n|^2 + lu_n^2)dx + \int_{\mathbb{R}^N} (a(x) - l)u_n^2 dx \\ &\geq m + \int_{\mathbb{R}^N} \alpha(x)u_n^2 dx \\ &= m + \int_{B(0,\rho)} \alpha(x)u_n^2 dx + \int_{\mathbb{R}^N \setminus B(0,\rho)} \alpha(x)u_n^2 dx \end{aligned} \tag{22}$$

$\rho > 0$. So, using (20)(c), (2) and (17), we obtain

$$m_a + o(1) = E(u_n) \geq m + o(1) > m_a + o(1)$$

that is impossible.

Now, assume $0 < |u_0|_p < 1$ and set $v_n = u_n - u_0$. Because of (20) we have that $(v_n)_n$ converges weakly to 0 in $H^1(\mathbb{R}^N)$, in $L^p(\mathbb{R}^N)$, $2 \leq p \leq 2^*$, converges a.e. to 0 in $\mathbb{R}^N$ and converges to 0 in $L^p_{\text{loc}}(\mathbb{R}^N)$. Moreover, by the Brezis-Lieb lemma ([5])

$$|v_n|_p^p = |u_n|_p^p - |u_0|_p^p + o(1) = 1 - |u_0|_p^p + o(1). \tag{23}$$

Then, we deduce

$$\begin{aligned} E(u_n) &= E(u_0 + v_n) \\ &= E(u_0) + E(v_n) + 2\int_{\mathbb{R}^N} [(\nabla u_0 \cdot \nabla v_n) + a(x)u_0 v_n]dx \\ &\geq m_a|u_0|_p^2 + m|v_n|_p^2 + o(1) \\ &= (m - m_a)|v_n|_p^2 + m_a(|v_n|_p^2 + |u_0|_p^2) + o(1) \\ &\geq \left[(1 - |u_0|_p^p)^{2/p} + (|u_0|_p^p)^{2/p}\right] m_a + o(1) \\ &> m_a + o(1). \end{aligned}$$

Thus, again, we get a contradiction with $\lim_{n\to+\infty} E(u_n) = m_a$ and we can conclude that u_0 is a minimizer for E on M.

Let us see, at last, that a minimizing function u must have constant sign. Indeed, if u changes sign, then

$$E(u) = E(u^+) + E(u^-) \geq m_a(|u^+|_p^2 + |u^-|_p^2) = m_a[(|u^+|_p^p)^{2/p} + (1 - |u^+|_p^p)^{2/p}] > m_a.$$

Hence a minimizer, by the strong maximum principle, provides a positive solution of (P). □

4. **Proof of Proposition 1 and Theorem 1.3.** First of all let us remark that the assumptions (H_1) and $(H_4)(i)$ imply

$$l = \liminf_{|x|\to+\infty} a(x) = \Theta \tag{24}$$

moreover, let us set

$$L = \limsup_{|x|\to+\infty} a(x) \tag{25}$$

and

$$\tilde{m}_a = \inf_{u \in M_s} E(u).$$

We start with the

Proof of Proposition 1. The relation $\tilde{m}_a \geq m_a \geq m$ follows straightly from the definitions of $\tilde{m}_a$, m_a and $(H_4)(i)$. In order to prove the reverse inequality, first we observe that, by (H_2), a sequence $(\rho_n)_n$ exists so that $\rho_n \in \mathbb{R}^+ \setminus \{0\}$, $\rho_n \overset{n \to \infty}{\longrightarrow} +\infty$ and

$$\sup_{B(\rho_n e_1, n)} |a(x) - \Theta| < \frac{1}{2^n}. \tag{26}$$

Thus, setting $u_n(x) = \omega(x - \rho_n e_1)$, clearly $u_n(x) \in M_s$, so it is enough to show that

$$\lim_{n \to +\infty} E(u_n) = \lim_{n \to +\infty} E(\omega(x - \rho_n e_1)) = m. \tag{27}$$

Indeed

$$\begin{aligned} E(u_n) &= \int_{B(\rho_n e_1, n)} (|\nabla \omega(x - \rho_n e_1)|^2 + a(x)\omega^2(x - \rho_n e_1))dx \\ &\quad + \int_{\mathbb{R}^N \setminus B(\rho_n e_1, n)} (|\nabla \omega(x - \rho_n e_1)|^2 + a(x)\omega^2(x - \rho_n e_1))dx. \end{aligned}$$

Now, using (1) and (12), we deduce

$$\int_{\mathbb{R}^N \setminus B(\rho_n e_1, n)} (|\nabla \omega(x - \rho_n e_1)|^2 + a(x)\omega^2(x - \rho_n e_1))dx = o(1).$$

On the other hand, by (26)

$$\int_{B(\rho_n e_1, n)} (|\nabla \omega(x - \rho_n e_1)|^2 + a(x)\omega^2(x - \rho_n e_1))dx = \int_{B(0,n)} (|\nabla \omega(x)|^2 + \Theta\, \omega^2(x))dx + o(1).$$

Hence (27) follows and, in turn, $m = \tilde{m}_a = m_a$.

Finally, assume by contradiction that $\bar{u} \in M$ exists so that $E(\bar{u}) = m$. Then, in view of Proposition 2, $(H_4)(i)$, (24) and $a(x) \not\equiv \Theta$, we would deduce

$$m \leq \|\bar{u}\|^2 < E(\bar{u}) = m$$

a contradiction. □

To prove Theorem 1.3, in view of the symmetry assumption (H_3), we restrict our attention to the submanifold M_s and we look for critical points of E constrained on M_s. Then, the Palais principle of symmetric criticality guarantees that these critical points are also critical points of E constrained on M and, hence, provide solutions of (P).

The first step for proving Theorem 1.3 is, of course, a location of an energy interval in which some compactness is preserved. To this end, we prove the following

Proposition 3. *The functional E constrained on M_s satisfies the Palais-Smale condition in the interval $(m, 2^{1-2/p}m)$.*

Proof. Let $(u_n)_n$ be a sequence such that

$$\begin{cases} a) \quad |u_n|_p = 1,\ u_n(x_1,\dots,x_i,\dots,x_N) = u_n(x_1,\dots,-x_i,\dots,x_N) \quad i=2,\dots,N \\ b) \quad \lim_{n\to+\infty} E(u_n) = c \in (m, 2^{1-2/p}m) \\ c) \quad o(1)\|w\| = ((\nabla E_{|_M}(u_n), w)) \\ \qquad\qquad = \int_{\mathbb{R}^N}((\nabla u_n,\nabla w) + a(x)u_n w)dx - \mu_n \int_{\mathbb{R}^N}|u_n|^{p-2}u_n w\,dx \\ \qquad\qquad\qquad \forall w \in H^1(\mathbb{R}^N) \text{ and for suitable } \mu_n \in \mathbb{R}. \end{cases} \tag{28}$$

To prove the proposition we must show that $(u_n)_n$ is relatively compact.

Firstly, let us observe that, since $\inf_{\mathbb{R}^N} a > 0$, (28)$(b)$ implies $\|u_n\|$ bounded in $H^1(\mathbb{R}^N)$. Hence, there exists $u_0 \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$\begin{array}{lll} a) & u_n \rightharpoonup u_0 & \text{in } H^1(\mathbb{R}^N) \text{ and } L^p(\mathbb{R}^N),\ 2 \le p \le 2^* \\ b) & u_n(x) \longrightarrow u_0(x) & \text{a.e. in } \mathbb{R}^N \\ c) & u_n \longrightarrow u_0 & \text{in } L^p_{\text{loc}}(\mathbb{R}^N). \end{array} \tag{29}$$

Clearly, we are done if we show that $u_n \longrightarrow u_0$ in $H^1(\mathbb{R}^N)$. We carry out the proof of this fact in two steps: **Step 1**: if $u_0 \not\equiv 0$ then $u_n \longrightarrow u_0$ strongly in $H^1(\mathbb{R}^N)$. **Step 2**: $u_0 \not\equiv 0$.

Step 1. Let us suppose $u_0 \not\equiv 0$. Setting $w = u_n$ in (28)(c) and considering (28)(b), we obtain

$$c + o(1) = \int_{\mathbb{R}^N}(|\nabla u_n|^2 + a(x)u_n^2)dx = \mu_n \int_{\mathbb{R}^N}|u_n|^p dx + o(1)\|u_n\|$$

from which, in view of (28)(a),

$$\lim_{n\to+\infty}\mu_n = c \tag{30}$$

follows and then that u_0 is a weak nontrivial solution of

$$-\Delta u_0 + a(x)u_0 = c|u_0|^{p-2}u_0 \qquad \text{in } \mathbb{R}^{\rm N}. \tag{31}$$

Using the above relation together with (24) and (11) we deduce

$$m|u_0|_p^2 \le \|u_0\|^2 \le c|u_0|_p^p$$

so

$$|u_0|_p \ge \left(\frac{m}{c}\right)^{\frac{1}{p-2}}. \tag{32}$$

Let us, now, argue by contradiction and let us assume that $u_n \not\longrightarrow u_0$ strongly in $H^1(\mathbb{R}^N)$. We remark that, by the equivalence of $\|\cdot\|$ and $\|\cdot\|_a$, not only $\|u_n\| \not\longrightarrow \|u_0\|$, but also $\|u_n\|_a \not\longrightarrow \|u_0\|_a$. Setting $v_n := u_n - u_0$, obviously $\|v_n\| \not\longrightarrow 0$, so there exists $k_1 > 0$ such that, up to a subsequence, $\|v_n\| \ge k_1 > 0\ \forall n \in \mathbb{N}$. Moreover, a direct computation shows

$$\|v_n\|^2 = \|u_n\|^2 - \|u_0\|^2 + o(1) \tag{33}$$

and the Brezis-Lieb lemma [5] yields

$$|v_n|_p^p = |u_n|_p^p - |u_0|_p^p + o(1). \tag{34}$$

On the other hand (28)(c) combined with (30), and (31) give, respectively,

$$\begin{aligned} \|u_n\|_a^2 &= c|u_n|_p^p + o(1), \\ \|u_0\|_a^2 &= c|u_0|_p^p, \end{aligned}$$

hence, using (34), we deduce

$$|v_n|_p^p = \frac{1}{c}(\|u_n\|_a^2 - \|u_0\|_a^2) + o(1) \geq k_2 > 0 \quad k_2 \in \mathbb{R}. \tag{35}$$

Since (35) holds, setting

$$\delta = \limsup_{n\to+\infty} \sup_{y\in\mathbb{R}} \int_{B(y,1)} |v_n|^p dx$$

we can apply Lemma 1.21 of [13] and conclude $\delta > 0$. Thus, we may assume the existence of $y_n \in \mathbb{R}^N$, $n \in \mathbb{N}$, such that

$$\frac{\delta}{2} < \int_{B(y_n,1)} |v_n(x)|^p dx = \int_{B(0,1)} |v_n(x+y_n)|^p dx. \tag{36}$$

Moreover, $|y_n| \overset{n\to\infty}{\longrightarrow} +\infty$, because $v_n \overset{n\to\infty}{\longrightarrow} 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$. Defining $\tilde{v}_n(x) := v_n(x+y_n)$, and considering that $\tilde{v}_n$ is bounded in $H^1(\mathbb{R}^N)$, the existence follows of $v_0 \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v_0$ in $H^1(\mathbb{R}^N)$. Furthermore $v_0 \not\equiv 0$ because (36), together with the Rellich theorem, implies $\int_{B(0,1)} |v_0|^p \geq \delta/2$. We claim now that

$$|v_0|_p \geq \left(\frac{m}{c}\right)^{\frac{1}{p-2}}. \tag{37}$$

Indeed, once proven (37), it is easy to get a contradiction, because, using (32) and (37), we obtain

$$\begin{aligned} E(u_n) &\geq \|u_n\|^2 = \|u_0\|^2 + \|v_n\|^2 + o(1) \\ &\geq \|u_0\|^2 + \|v_0\|^2 + o(1) \\ &\geq m(|u_0|_p^2 + |v_0|_p^2) + o(1) \\ &\geq 2m\left(\frac{m}{c}\right)^{\frac{2}{p-2}} + o(1) \end{aligned}$$

and, letting $n \to +\infty$, we have

$$c = \lim_{n\to+\infty} E(u_n) \geq 2m\left(\frac{m}{c}\right)^{\frac{2}{p-2}}$$

that is $c \geq 2^{1-2/p}m$, contradicting (28)(b).

To prove (37), let us first show that $\lambda \in [\Theta, L]$ exists, so that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} a(x+y_n)u_n(x+y_n)v_0(x)dx = \lambda \int_{\mathbb{R}^N} v_0^2(x)dx. \tag{38}$$

To this end, taking into account $u_n(x+y_n) = \tilde{v}_n(x) + u_0(x+y_n)$ and $|y_n| \overset{n\to\infty}{\longrightarrow} +\infty$, we observe that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} (\nabla u_n(x+y_n), \nabla v_0(x))dx = \int_{\mathbb{R}^N} |\nabla v_0|^2 dx, \tag{39}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n(x+y_n)|^{p-2}u_n(x+y_n)v_0(x)dx = \int_{\mathbb{R}^N} |v_0|^p dx. \tag{40}$$

Then, from (28)(c) we infer

$$\int_{\mathbb{R}^N} (\nabla u_n(x+y_n), \nabla v_0(x))dx + \int_{\mathbb{R}^N} a(x+y_n)u_n(x+y_n)v_0(x)dx$$

$$= \mu_n \int_{\mathbb{R}^N} |u_n(x+y_n)|^{p-2}u_n(x+y_n)v_0(x)dx + o(1)\|v_0\|. \tag{41}$$

Thus, considering (30), (39) and (40), we deduce from (41) that the limit on the left hand side of (38) exists and then (38) comes as a consequence of

$$\Theta \int_{\mathbb{R}^N} v_0^2(x)dx + o(1) \le \int_{\mathbb{R}^N} a(x+y_n)u_n(x+y_n)v_0(x)dx \le L\int_{\mathbb{R}^N} v_0^2(x)dx + o(1).$$

Lastly, passing to the limit in (41) and using (38), (39), (40) and (30), we get

$$\|v_0\|^2 \le \int_{\mathbb{R}^N} (|\nabla v_0|^2 + \lambda v_0^2)dx = c\int_{\mathbb{R}^N} |v_0|^p dx$$

that, together with $\|v_0\|^2 > m|v_0|_p^2$, gives (37) and ends the proof of step 1.

<u>Step 2.</u> We argue by contradiction, so we assume $u_0 \equiv 0$. Obviously $\|u_n\| \not\to 0$ and $|u_n|_p \not\to 0$, hence we can repeat the argument developed in the previous step for the sequence $(v_n)_n$ to conclude that a sequence $(x_n)_n$, $x_n \in \mathbb{R}^N$, $|x_n| \overset{n\to\infty}{\longrightarrow} +\infty$, and a function $v_1 \in H^1(\mathbb{R}^N)$, $v_1 \not\equiv 0$ exist so that

$$\begin{array}{lll} a) & u_n(x+x_n) \rightharpoonup v_1(x) & \text{in } H^1(\mathbb{R}^N) \text{ and in } L^p(\mathbb{R}^N), \quad 2\le p\le 2^* \\ b) & u_n(x+x_n) \to v_1(x) & \text{a.e. in } \mathbb{R}^N \\ c) & u_n(x+x_n) \to v_1(x) & \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N). \end{array} \tag{42}$$

We remark also that, by (28)(c), $\forall w \in H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (\nabla u_n(x+x_n), \nabla w(x))dx + \int_{\mathbb{R}^N} a(x+x_n)u_n(x+x_n)w(x)dx$$
$$= \mu_n \int_{\mathbb{R}^N} |u_n(x+x_n)|^{p-2}u_n(x+x_n)w(x)dx + o(1)\|w\| \tag{43}$$

with $\lim_{n\to+\infty}\mu_n = c$, holds.

Setting

$$\frac{x_n}{|x_n|} := y_n$$

we can suppose that, up to a subsequence,

$$y_n \overset{n\to\infty}{\longrightarrow} \bar{y} \in \mathbb{R}^N \qquad |\bar{y}| = 1.$$

We distinguish the cases $\bar{y} = \pm e_1$ from the cases $\bar{y} \neq \pm e_1$.

Let $\bar{y}$ be equal to either e_1 or $-e_1$. Then, by (H_2)

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} a(x+x_n)u_n(x+x_n)w(x)dx = \Theta\int_{\mathbb{R}^N} v_1(x)w(x)\,dx$$

$\forall w \in H^1(\mathbb{R}^N)$, so from (43) we deduce that v_1 is a weak nontrivial solution of

$$-\Delta v_1 + \Theta v_1 = c|v_1|^{p-2}v_1 \tag{44}$$

and

$$m|v_1|_p^2 \le \|v_1\|^2 = c|v_1|_p^p. \tag{45}$$

Then we claim

$$u_n(x+x_n) \not\to v_1(x) \quad \text{strongly in } H^1(\mathbb{R}^N). \tag{46}$$

Otherwise, in fact, we would have also $u_n(x+x_n) \to v_1(x)$ strongly in $L^p(\mathbb{R}^N)$, and, then, $|v_1|_p = 1$. So, from (44), we would infer $\|v_1\|^2 = c$ and that v_1 is a critical point of $\|u\|^2$ constrained on M, hence, by Proposition 2, either $\|v_1\|^2 = m$ or $\|v_1\|^2 \ge 2^{1-2/p}m$, a contradiction.

Set now $w_n(x) := u_n(x+x_n) - v_1(x)$, by (46) $w_n \not\to 0$ strongly in $H^1(\mathbb{R}^N)$, so a constant $k_3 > 0$ exists so that, up to a subsequence, $\|w_n\| \ge k_3 > 0$, $\forall n \in \mathbb{N}$. By a direct computation and by an application of the Brezis-Lieb lemma, we obtain, respectively,

$$\|w_n\|^2 = \|u_n\|^2 - \|v_1\|^2 + o(1), \tag{47}$$

$$|w_n|_p^p = |u_n|_p^p - |v_1|_p^p + o(1). \tag{48}$$

Thus, using (28)(c)

$$\begin{aligned} |u_n|_p^p &= \frac{1}{\mu_n}\int_{\mathbb{R}^N}(|\nabla u_n|^2 + a(x)u_n^2)dx + o(1) \\ &\geq \frac{1}{\mu_n}\|u_n\|^2 + o(1) \\ &= \frac{1}{c}\,\|u_n\|^2 + o(1) \end{aligned}$$

from which, in view of (48), (45) and (46), we have

$$|w_n|_p^p \geq \frac{1}{c}(\|u_n\|^2 - \|v_1\|^2) + o(1) \geq k_4 > 0 \quad k_4 \in \mathbb{R}.$$

Again we can repeat the argument applied to $(v_n)_n$ in step 1 and get a sequence of points $(t_n)_n$, $t_n \in \mathbb{R}^N$, and a nonzero function $w_0 \in H^1(\mathbb{R}^N)$ such that $|t_n - x_n| \overset{n\to\infty}{\longrightarrow} +\infty$ and

$$w_n(x+t_n) \overset{n\to\infty}{\longrightarrow} w_0(x) \quad \text{weakly in } H^1(\mathbb{R}^N) \text{ and in } L^p(\mathbb{R}^N) \quad 2 \leq p \leq 2^*.$$

Furthermore, the same argument used to prove (37) applies in this case, allowing to obtain

$$|w_0|_p \geq \left(\frac{m}{c}\right)^{\frac{1}{p-2}}. \tag{49}$$

Lastly (47), (11), (45) and (49) give

$$\begin{aligned} E(u_n) &\geq \|u_n\|^2 = \|w_n\|^2 + \|v_1\|^2 + o(1) \\ &\geq m(|w_n|_p^2 + |v_1|_p^2) + o(1) \\ &\geq m(|w_0|_p^2 + |v_1|_p^2) + o(1) \\ &\geq 2m\left(\frac{m}{c}\right)^{\frac{2}{p-2}} + o(1) \end{aligned}$$

and, letting $n \to +\infty$, $c \geq 2^{1-2/p}m$, contradicting (28)(b).

Finally, consider $\bar{y} \neq \pm e_1$. Since $u_n \in M_s$, $\forall n \in \mathbb{N}$, and, then, satisfies the symmetry property in (28)(a), clearly u_n, besides (42), satisfies

$$\begin{array}{lll} a) & u_n(x+\hat{x}_n) \rightharpoonup v_1(x) & \text{in } H^1(\mathbb{R}^N) \text{ and in } L^p(\mathbb{R}^N), \quad 2 \leq p \leq 2^* \\ b) & u_n(x+\hat{x}_n) \to v_1(x) & \text{a.e. in } \mathbb{R}^N \\ c) & u_n(x+\hat{x}_n) \to v_1(x) & \text{in } L^p_{\text{loc}}(\mathbb{R}^N), \end{array}$$

with $\hat{x}_n = (x_{n,1}, -x_{n,2}, \ldots, -x_{n,N})$.

As a consequence, $u_n(x+x_n) \not\to v_1(x)$ strongly in $H^1(\mathbb{R}^N)$ and, setting $z_n(x) := u_n(x+x_n) - v_1(x)$ and considering $z_n(x+\hat{x}_n - x_n)$, again we have

$$\begin{array}{lll} a) & z_n(x+\hat{x}_n - x_n) \rightharpoonup v_1(x) & \text{in } H^1(\mathbb{R}^N) \text{ and in } L^p(\mathbb{R}^N), \quad 2 \leq p \leq 2^* \\ b) & z_n(x+\hat{x}_n - x_n) \to v_1(x) & \text{a.e. in } \mathbb{R}^N \\ c) & z_n(x+\hat{x}_n - x_n) \to v_1(x) & \text{in } L^p_{\text{loc}}(\mathbb{R}^N). \end{array}$$

Arguing as for proving (37) it is also easy to verify that

$$|v_1|_p \geq \left(\frac{m}{c}\right)^{\frac{1}{p-2}}$$

and, then, we deduce again

$$\begin{aligned} E(u_n) &\geq \|u_n\|^2 = \|z_n\|^2 + \|v_1\|^2 + o(1) \\ &\geq m(|v_1|_p^2 + |v_1|_p^2) + o(1) \\ &\geq 2m \left(\frac{m}{c}\right)^{\frac{2}{p-2}} + o(1) \end{aligned}$$

that, letting $n \to +\infty$, brings to the relation $c \geq 2^{1-2/p} m$, that contradicts (28)(b). ☐

Let us, now, recall the following definition of a barycenter type map

$$\beta : M_s \longrightarrow \mathbb{R}^N$$

given in [8]. For all $u \in M_s$ set

$$\tilde{u}(x) = \frac{1}{|B(x,1)|} \int_{B(x,1)} |u(y)|\, dy \qquad \forall x \in \mathbb{R}^N,$$

$|B(x,1)|$ denoting the Lebesgue measure of $B(x,1)$, consider

$$\hat{u}(x) = \left[\tilde{u}(x) - \frac{1}{2} \max_{\mathbb{R}^N} \tilde{u}(x)\right]^+ \qquad \forall x \in \mathbb{R}^N$$

and define

$$\beta(u) = \frac{1}{|\hat{u}|_p^p} \int_{\mathbb{R}^N} x(\hat{u}(x))^p dx.$$

We denote by β_1 the projection of β on the direction e_1, that is

$$\beta_1(u) := (\beta(u), e_1)$$

and we remark that, by (4), $\beta_1(u)$ is the only nonzero component of β. Moreover it is not difficult to verify that $\beta_1 : M_s \longrightarrow \mathbb{R}$ is well defined and continuous.

Let us set

$$\mathcal{B} = \inf\{E(u) \ : \ u \in M_s,\ \beta_1(u) = 0\}.$$

In the following three lemmas we assume $a(x) \not\equiv \Theta$.

Lemma 4.1. *Let the assumptions of Theorem 1.3 be satisfied, then*

$$\mathcal{B} > m. \tag{50}$$

Proof. By (5), $\mathcal{B} \geq m$. To prove (50) we argue by contradiction and we assume that a sequence $(u_n)_n$ exists so that

$$u_n \in M_s, \qquad \beta_1(u_n) = 0$$

and

$$m = \lim_{n\to+\infty} E(u_n) = \lim_{n\to+\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + a(x)u_n^2) dx. \tag{51}$$

Since $a(x) \geq \Theta$,

$$m \leq \|u_n\|^2 \leq E(u_n) = m + o(1)$$

then, by the uniqueness of the family of the functions realizing (11),

$$u_n(x) = \omega(x - x_n) + \phi_n(x)$$

where $x_n \in \mathbb{R}^N$, $\phi_n \in H^1(\mathbb{R}^N)$, $\lim_{n\to+\infty} \phi_n(x) = 0$ in $H^1(\mathbb{R}^N)$ and $(x_n)_n$ is unbounded, because, by Proposition 1, m is not achieved. We can also assume $x_n = \tau_n e_1$, where $\tau_n \in \mathbb{R}$ and $\tau_n \overset{n\to\infty}{\longrightarrow} +\infty$. By making a translation, we can then write

$$u_n(x + \tau_n e_1) = \omega(x) + \phi_n(x + \tau_n e_1).$$

Now, let us compute β_1 of both the terms: we have

$$\beta_1(u_n(x+\tau_n e_1)) = -\tau_n$$

and, by the continuity of the barycenter,

$$\beta_1(\omega(x)+\phi_n(x+\tau_n e_1)) = \beta_1(\omega(x)) + o(1) = o(1).$$

Since $\tau_n \longrightarrow +\infty$, we get a contradiction and (50) follows. $\square$

Let us define the operator $\Phi : \mathbb{R} \to M_s$ by

$$\Phi[\tau] = \omega(\cdot - \tau e_1).$$

Clearly Φ is continuous and

$$\beta_1 \circ \Phi[\tau] = \tau. \tag{52}$$

Lemma 4.2. *Let the assumptions of Theorem 1.3 be satisfied, then $\alpha > 0$, $\alpha \in \mathbb{R}$ exists so that*

$$\max\{E(\Phi[\alpha]), E(\Phi[-\alpha])\} < \mathcal{B} \tag{53}$$

and

$$\max\{E(\Phi[\tau]) \ : \ \tau \in [-\alpha, \alpha]\} < 2^{1-2/p} m. \tag{54}$$

Proof. Arguing as for proving (27) it is not difficult to verify that

$$\lim_{\tau \to \pm\infty} E(\Phi[\tau]) = m.$$

So, in view of (50), (53) follows.

On the other hand

$$E(\Phi[\tau]) \le m + |a - \Theta|_\infty \int_{\mathbb{R}^N} \omega^2 dx,$$

hence, the assumption $(H_4)(ii)$ yields

$$\max_{\mathbb{R}} E([\Phi[\tau]) < 2^{1-2/p} m \tag{55}$$

and, then, (54). $\square$

Lemma 4.3. *Let the assumptions of Theorem 1.3 be satisfied, with $(H_4')ii)$ instead of $(H_4)ii)$. Then the same claim of Lemma 4.2 follows.*

Proof. Clearly, the proof of the relation (53) does not depend on the assumption $(H_4)ii)$, so we must only show that (54) is still true. Indeed, we have for all R

$$\begin{aligned} E(\Phi[\tau]) &= m + \int_{\mathbb{R}^N \setminus C_R} (a(x) - \Theta)\omega^2(x - \tau e_1) dx + \int_{C_R} (a(x) - \Theta)\omega^2(x - \tau e_1) dx \\ &\le m + |a - \Theta|_{L^\infty(\mathbb{R}^N \setminus C_R)} \ |\omega|^2_{L^2(\mathbb{R}^N \setminus C_R)} + |a - \Theta|_{L^\infty(C_R)} \ |\omega|^2_{L^2(C_R)}; \end{aligned}$$

hence, the assumption $(H_4')(ii)$ yields (55) and, then (54). $\square$

In what follows we set, $\forall c \in \mathbb{R}$,

$$E^c = \{u \in M_s \ : \ E(u) \le c\}.$$

Proof of Theorem 1.3. First of all, we observe that the case $a(x) \equiv \Theta$ is treated in Proposition 2. So we assume $a(x) \not\equiv \Theta$. In order to prove the claim we want apply the Linking Theorem (see e.g. Theorem 8.22 in [1]). We consider

$$\Sigma = \Phi([-\alpha, \alpha]) \quad \text{and} \quad \Lambda = \{u \in M_s \ : \ \beta_1(u) = 0\}.$$

Clearly, by (52), $\partial\Sigma \cap \Lambda = \emptyset$. Thus, to prove that $\partial\Sigma$ and Λ link, we must show that

$$h(\Sigma) \cap \Lambda \neq \emptyset \qquad \forall h \in \mathcal{H} \tag{56}$$

where

$$\mathcal{H} = \{h \in \mathcal{C}(\Sigma, M_s) \ : \ h_{|\partial\Sigma} = \mathrm{Id}\}.$$

Given such a map h, we define

$$\mathcal{T}_h : [-\alpha, \alpha] \to \mathbb{R}, \quad \mathcal{T}_h(\tau) = \beta_1 \circ h \circ \Phi(\tau).$$

$\mathcal{T}_h$ is a continuous map and, since $\Phi(\pm\alpha) \in \partial\Sigma$, $\mathcal{T}_h(\alpha) = \alpha$, $\mathcal{T}_h(-\alpha) = -\alpha$. As a consequence, there exists $\bar\tau \in [-\alpha, \alpha]$ such that $\mathcal{T}_h(\bar\tau) = 0$ and this means $h(\Phi(\bar\tau)) \in \Lambda$, that is $h(\Sigma) \cap \Lambda \neq \emptyset$. Note, also, that (53) reads as $\max_{\partial\Sigma} E < \inf_\Lambda E$. Let us define

$$c := \inf_{h\in\mathcal{H}} \max_{\tau\in[-\alpha,\alpha]} E(h \circ \Phi[\tau]).$$

Taking $h = \mathrm{Id}$ and using (54) we obtain $c < 2^{1-2/p}m$ and, on the other hand, (56) yields $c \geq \mathcal{B} > m$. Since the Palais-Smale condition holds in $(m, 2^{1-2/p}m)$ we conclude that c is a critical value of Φ.

To complete the proof, we show that if u is a critical point of E at the level c, u has constant sign, and then by the maximum principle $u > 0$.

Arguing by contradiction, let $u = u^+ - u^-$, with $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. Since u solves

$$-\Delta u + a(x)u = c|u|^{p-2}u$$

we have $E(u^\pm) = c|u^\pm|_p^p$. On the other hand $E(u^\pm) \geq \|u^\pm\|^2 \geq m|u^\pm|_p^2$. Thus

$$|u^\pm|_p \geq \left(\frac{m}{c}\right)^{\frac{1}{p-2}},$$

from which we deduce

$$c = E(u) = E(u^+) + E(u^-) \geq \|u^+\|^2 + \|u^-\|^2 \geq m(|u^+|_p^2 + |u^-|_p^2) \geq 2m\left(\frac{m}{c}\right)^{\frac{2}{p-2}}$$

that implies $c \geq 2^{1-2/p}m$, a contradiction. □

Remark 1. The above proof makes clear that the assumption $(H_4)ii)$ can be replaced by $(H_4')ii)$, and that, moreover, also (H_2) can be replaced by the slightly weaker condition

$$\forall \varepsilon, r > 0 \quad \exists t^+ > 0,\ t^- < 0 \quad \text{such that}$$
$$|a(x) - \Theta|_{L^\infty(B(t^+e_1,r))} < \varepsilon \quad \text{and} \quad |a(x) - \Theta|_{L^\infty(B(t^-e_1,r))} < \varepsilon.$$

5. **Proof of Theorem 1.4.** The proof is carried out in two steps: first, we show that

$$u(x) \longrightarrow 0, \qquad \text{as} \quad |x| \to +\infty, \tag{57}$$

then, we prove the exponential decay and the sommability of the derivatives.

Step 1. Writing the equation in (P) as

$$-\Delta u + u = |u|^{p-2}u + (1 - a(x))u$$

we see that

$$u = u_1 + u_2$$

where u_1 and u_2 weakly solve, respectively

$$-\Delta u_1 + u_1 = (1 - a(x))u \tag{58}$$

$$-\Delta u_2 + u_2 = |u|^{p-2}u. \tag{59}$$

Let us consider, first, the case $N \geq 3$. Being $2 < p < 2^*$, we set $p = \frac{2N-\sigma}{N-2}$, with $\sigma \in (0, 4)$. Moreover, we use the notation $p_0 := 2^*$. Since $u \in L^2(\mathbb{R}^N) \cap L^{p_0}(\mathbb{R}^N)$, by a classical regularity result (see e.g. Proposition 4.3 [12]), we have

$$u_1 \in W^{2,2}(\mathbb{R}^N) \cap W^{2,p_0}(\mathbb{R}^N) \tag{60}$$

$$u_2 \in W^{2,q_1}(\mathbb{R}^N) \qquad q_1 = \frac{p_0}{p-1} < p_0. \tag{61}$$

If $2p_0 > 2q_1 > N$, then the Sobolev embedding theorem gives $u_1, u_2 \in L^\infty(\mathbb{R}^N)$ and

$$|u_i|_{L^\infty(\mathbb{R}^N \setminus \overline{B(0,r)})} \leq c_1 ||u_i||_{W^{2,q_1}(\mathbb{R}^N \setminus \overline{B(0,r)})}, \quad i = 1, 2, \quad r > 0, \tag{62}$$

where c_1 is a constant independent of the domain. Letting $r \to +\infty$ in (62), we deduce $\lim_{|x|\to+\infty} u_i(x) = 0$ and then (57).

Assume, now, $2q_1 \leq N$. We use a bootstrapping procedure to gain in smoothness. By the Sobolev embedding theorem, from (61) and (60) we obtain, respectively,

$$u_2 \in L^{p_1}(\mathbb{R}^N) \quad \text{where} \quad p_1 = \begin{cases} \frac{2N}{N-2-\sigma} (> p_0) & \text{if } 2q_1 < N \\ p_1 > p_0 \ \text{ so that } \ 2\frac{p_1}{p-1} > N & \text{if } 2q_1 = N \end{cases}$$

$$u_1 \in L^2(\mathbb{R}^N) \cap L^{s_1}(\mathbb{R}^N) \quad \text{where} \quad s_1 = \begin{cases} \frac{Np_0}{N-2p_0} (> p_1) & \text{if } 2p_0 < N \\ s_1 > p_1 & \text{if } 2p_0 \geq N. \end{cases}$$

Hence, by interpolation,

$$u \in L^2(\mathbb{R}^N) \cap L^{p_1}(\mathbb{R}^N). \tag{63}$$

Now, using (63), we get from the regularity theory applied to (58) and (59),

$$u_1 \in W^{2,2}(\mathbb{R}^N) \cap W^{2,p_1}(\mathbb{R}^N) \tag{64}$$

$$u_2 \in W^{2,q_2}(\mathbb{R}^N) \quad \text{where} \quad q_2 = \frac{p_1}{p-1} > q_1. \tag{65}$$

Clearly $p_1 > q_2$ so, if $2q_2 > N$, we are done, because we can get (62), and then (57), using the Sobolev embedding theorem. If $2q_2 \leq N$, arguing as in the previous step, and setting $\overline{p} = p - 1$, we have

$$u_2 \in L^{p_2}(\mathbb{R}^N) \quad \text{where} \quad p_2 = \begin{cases} \frac{2N}{(N-2)-(1+\overline{p})\sigma} (> p_1) & \text{if } \ 2q_2 < N \\ p_2 > p_1 \ \text{ so that } \ 2\frac{p_2}{\overline{p}} > N & \text{if } \ 2q_2 = N \end{cases}$$

$$u_1 \in L^2(\mathbb{R}^N) \cap L^{s_2}(\mathbb{R}^N) \quad \text{where} \quad s_2 = \begin{cases} \frac{Np_1}{N-2p_1} (> p_2) & \text{if } 2p_1 < N \\ s_2 > p_2 & \text{if } 2p_0 \geq N. \end{cases}$$

So $u \in L^2(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N)$.

Iterating the procedure, we find numbers q_k and p_k such that $q_k > q_{k-1}$, $p_k > p_{k-1}$, $q_{k+1} = \frac{p_k}{\overline{p}}$ (hence $p_k > q_{k+1}$) and

$$\begin{array}{rcl} u & \in & L^2(\mathbb{R}^N) \cap L^{p_k}(\mathbb{R}^N) \\ u_1 & \in & W^{2,2}(\mathbb{R}^N) \cap W^{2,p_k}(\mathbb{R}^N) \\ u_2 & \in & W^{2,q_{k+1}}(\mathbb{R}^N), \end{array}$$

$$\begin{aligned} p_k &= \frac{Nq_k}{N-2q_k} = \frac{2N}{(N-2)-(1+\overline{p}+\ldots+\overline{p}^{k-1})\sigma} \qquad (66) \\ q_{k+1} &= \frac{2N}{(N-2)-(1+\overline{p}+\ldots+\overline{p}^{k-1})\sigma} \cdot \frac{N-2}{N+2-\sigma}. \end{aligned}$$

In particular (66) implies

$$\operatorname{sign}(N-2q_k) = \operatorname{sign}[(N-2)-(1+\overline{p}+\ldots+\overline{p}^{k-1})\sigma]$$

from which, considering $\overline{p} > 1$, we deduce that for some $\overline{k}$, $N - 2q_{\overline{k}} \le 0$. This allows us to conclude that (62), and then (57), holds. The case $N = 2$ can be treated in a similar way, by using the embedding $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, for all $q \ge 2$.

Step 2. We argue on the positive part, u^+, of u. The argument for u^- is analogous.

The function u^+ solves

$$\begin{cases} -\Delta u + \frac{l}{2}u = u^{p-1} - \Big(a(x) - \frac{l}{2}\Big)u & \text{in } \Omega^+ \\ u \in H_0^1(\Omega^+) \end{cases} \qquad (67)$$

where $\Omega^+ = \{x \in \mathbb{R}^N : u(x) > 0\}$. If Ω^+ is bounded the claim is trivial. So we suppose Ω^+ unbounded. By step 1, $u^+(x) \overset{|x|\to+\infty}{\longrightarrow} 0$, hence a number $\overline{r} > 0$ exists so that

$$\Big(a(x) - \frac{l}{2}\Big)u^+ - (u^+)^{p-1} > 0 \quad \forall\, x \in \Omega^+ \cap \{x \in \mathbb{R}^N : |x| > \overline{r}\}.$$

Let us denote by $\gamma(x)$ the fundamental, radial solution of

$$\begin{cases} -\Delta v + \frac{l}{2}v = 0 & \text{in } \{x \in \mathbb{R}^N : |x| > \overline{r}\}, \\ v(\overline{r}) = \max_{|x|=\overline{r}} u^+. \end{cases} \qquad (68)$$

It is well known (see e.g. [4]) that

$$\gamma(r)|r|^{\frac{N-1}{2}} e^{\sqrt{\frac{l}{2}}r} \overset{|r|\to+\infty}{\longrightarrow} c_2 > 0. \qquad (69)$$

(67) and (68) imply that $w(x) := \gamma(x) - u^+(x)$ solves

$$-\Delta w + \frac{l}{2}w = \Big(a(x) - \frac{l}{2}\Big)u^+ - (u^+)^{p-1} \quad \text{in } \Omega^+ \cap \{x \in \mathbb{R}^N : |x| > \overline{r}\}.$$

Then, by the weak maximum principle,

$$\inf_{\Omega^+ \cap \{x\in\mathbb{R}^N:|x|>\overline{r}\}} w \ge \inf_{\partial(\Omega^+ \cap \{x\in\mathbb{R}^N:|x|>\overline{r}\})} w \ge 0$$

that is

$$0 \le u^+(x) \le \gamma(x) \qquad \text{when } |x| > r,$$

that together with (69) gives (6).

Finally, let us observe that u solves weakly

$$\begin{cases} -\Delta v + Lv = f \\ v \in H^1(\mathbb{R}^N) \end{cases}$$

where $f(x) = |u|^{p-2}u + (L - a(x))u$ and that $f \in L^q(\mathbb{R}^N)$, for all $q \in [1, +\infty)$, hence by classical regularity results $u \in W^{2,q}(\mathbb{R}^N)$ for all $q \in [1, +\infty)$, giving (7). $\square$

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