$$\left(\mathbb{1} + \frac{d}{dz}\right)^{-1}$$

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Abstract

We investigate the structure of fully non-linear P.D.E.'s in holomorphic functions, with emphasis on the functorial generalisation of so called "irregular" O.D.E.'s. Highlights are an implicit function theorem removing the perturbation conditions of Nash-Moser type, best possible existence results when the singularity of the linearised P.D.E. is at worst bi-dimensional, and various, again optimal, corollaries on existence of centre manifolds and conjugation to normal form of 3-dimensional vector fields.

Introduction

Resolution of singularities of vector fields is a very different problem from resolution of singularities of varieties. A toy, but key, [Kol07] example in the latter is an irreducible hypersurface singularity of degree d in characteristic zero. By the preparation theorem we can write this locally as,

$$f(\underline{x}, y) = y^d + a_1(\underline{x})y^{d-1} + \ldots + a_d(\underline{x}) = 0$$

for a_i some functions of some variables $\underline{x} = (x_1, \ldots, x_n) \in X$. This determines y as an implicit function of $\underline{x}$, and if we were to restrict our attention to local uniformisation about a valuation v, of, say, rational rank 1, since this tends to be the hard case, we could even expand y, and indeed $x_i, i > 1$ as a power series in $x = x_1$. i.e.

$$y = \sum_{q \in \mathbb{Q}_+} c_q x^q$$

for coefficients in the residue field of the valuation and q increasing. By induction one may suppose that the corresponding series for the x_i occur with bounded denominators, so, without any essential loss of generality, we may as well say integers. The question of local uniformisation is wholly equivalent to understanding y_m as an implicit function of x, where, $y = y_m + p_m(x)$, and $p_m(x)$ is the first m terms in the above power series. A modicum of thought, cf. [CRS08], reveals that for $m \gg 0$,

$$\frac{\partial}{\partial y_m} f(x, y_m)$$

is a unit so Hensel's lemma (a.k.a. the implicit function theorem) applies to conclude that the series in y also has bounded denominators.

Already at this level the situation for vector fields is much more subtle, since, for a start, again rational rank 1 to fix ideas, a similar toy example does not exist except in dimension 2. Nevertheless, one can gain some inkling for the difficulty by supposing that y is an implicit function of x determined by an O.D.E. of degree k,

$$f(x, y, Dy, \ldots, D^k y) = 0, \quad D = x\frac{\partial}{\partial x}$$

At which point, it should be plain that Hensel's lemma will never apply since the functional derivative in y will be a linear O.D.E.,

$$y \longmapsto f_0(x)y + f_1(x)Dy + \ldots + f_k(x)D^k y$$

and the best that one might hope for after the above substitutions $y \mapsto y_m$, is that some of the f_i become units. Nevertheless, even when this does happen, one still may not be able to say anything meaningful since there may be resonances amongst the $f_i(0)$, i.e. they may well fail to be linearly independent over $\mathbb{Q}$, which becomes a problem many times compounded as one makes examples closer to the truth on replacing an O.D.E. by a P.D.E., and forming systems of such.

There is also a mutual enrichment between the question of resolutions of singularities for vector fields, or, indeed differential operators, and the question of the very existence of solutions to singular P.D.E.'s which is not really present at the level of resolution of varieties. Indeed, plainly, our original hypersurface had a solution y as an implicit function of $\underline{x}$ over any algebraically closed field, while even a system of r O.D.E.'s in r unknowns,

$$\dot{z}_i = a_i(z_1, \ldots, z_r), \quad z_i = z_i(t),\ 1 \leqslant i \leqslant r$$

may well fail, already for $r = 3$, to have any solution if the a_i are too singular, [GML92]. Formally, however, this is easily decidable (by way of Jordan form) as soon as we can resolve the vector filed,

$$a_1 \frac{\partial}{\partial z_1} + \ldots + a_n \frac{\partial}{\partial z_n}$$

At which point, we come across the most useful algebraic analogy in the form of Artin's large scale generalisation, of Hensel's lemma which asserts that anything defined in an algebraic way which has solutions formally, enjoys solutions in the étale topology, or, equivalently the coarsest Grothendieck topology in which (the usual easy version of) Hensel's lemma holds, so, trivially, one has convergence of the said solutions in the classical topology.

Naturally, therefore, one is lead to enquire as to whether there may be a similar Grothendieck topology for P.D.E.'s. Already for 1st order O.D.E.'s, the classical topology is insufficient, and even the linear case should only really be considered understood when it corresponds to a linear system, i.e. a connection ∇ on a bundle E of rank r satisfying the Leibniz rule with respect to,

$$D = \begin{cases} x\dfrac{d}{dx} & \text{“regular” singularity,} \\ x^{r+1}\dfrac{d}{dx} & \text{“irregular” singularity, } r \in \mathbb{N} \end{cases}$$

which in turn defines the system of O.D.E.'s,

$$\nabla(\underline{f}) = \underline{g},\ \underline{g} \in E$$

subject, at least in the “irregular” case, to the non-degeneracy condition,

$$\nabla(e_i) = \sum_j a_{ij} e_j,\ \mathrm{Det}(a_{ij})(0) \neq 0$$

In either case, functorially with respect to the ideas, we have a canonical singularity (strictly log-canonical in rare cases of the former, [MPa] III.i.2) whence the regular/irregular terminology while appropriate when it was introduced, [Del70], now risks creating a certain confusion since both cases are wholly natural, and neither may be improved in any way by blowing up. Plainly the first of these analytically continues to the surface of the logarithm in an obvious way, and is an example of what we will call *logarithmically flat*. On the other hand one cannot necessarily do any better, so plainly $\exp : H \to \Delta$ defined on a half plane H will have to be in our topology, as a neighbourhood of 0 even though this fails to be in the image of the exponential. Equally, this is just a round about way to state the obvious, whereas the so called "irregular" case is rather more interesting. The classical example is Euler's equation,

$$E - z^2 \frac{d}{dz} E = z$$

which has the formal solution,

$$E(z) = \sum_{n=0}^{\infty} n! z^{n+1}$$

and, appearances notwithstanding, an analytic continuation to the surface of the logarithm, $\zeta = \log z$, where the above formula is actually the asymptotic expansion in neighbourhoods $|\operatorname{Im}(\zeta)| < 3\pi, \operatorname{Re}(\zeta) \to -\infty$, after which, unlike it's flat counterpart, the two are unrelated, i.e. we find a Stokes' phenomenon. Nevertheless it has a related functorial description, i.e. E is the exponential, cf. §I.2, in the convolution algebra $\mathrm{H}^1_c(\mathbb{G}_a, \omega_{\mathbb{G}_a})$ under the more or less canonical identification, [Köt69], §27.4, of this group with germs of functions vanishing at ∞, about which, and perhaps a little confusingly, one should view the above z as the local coordinate. As a result, [Éca85], all local existence results for such systems in the "irregular" case are subordinate to the much richer theory of the function E itself.

In either case, or, more accurately up to some issues of resonance among eigenvalues in the log-flat case, the non-degeneracy condition on ∇ guarantees that the problem is formally un-obstructed, while adding sectors $S \to \Delta$, i.e. restricting the argument of z, to the classical topology and viewing them as neighbourhoods of 0 is more than sufficient to obtain solutions everywhere. At which point, e.g. [Mal91] §IV, one can define an appropriate topus, and construct a rich body of theory. At its most basic level this topus is a sub-category of sheaves on the real blow up of the disc in the origin (whence a manifold with boundary, and not unrelated to the twin facts that the essential of [MPa] is the case of real manifold with boundary, [Pan06], while, slightly incorrectly, many algebraic stacks may be thought of as cone manifolds) and, irrespectively of any questions of patching and analytic continuation, one has solutions in a neighbourhood of every point on the real blow up. This leads us to pose,

Principle Question *Could it be that given an arbitrary singular analytic and fully non-linear P.D.E. one first resolves (in a sense to be made precise, but for the sake of argument imagine a best possible final situation stable under blowing up) its singularities, and modulo some possibility of some wholly computable formal obstruction on the resolution, finds solutions on a real blow up supported in the total exceptional divisor ?*

Plainly an affirmative answer to the question defines a Grothendieck topology on which every formally un-obstructed P.D.E. becomes soluble, and, equally plainly, there is no smaller extension of the classical topology that could work.

Unfortunately, as we shall see, III.3.4 & IV.2.6, this is false, albeit not hopelessly so, whence, for the moment, let us take it as a guiding principle. Obviously, in the first instance,

Sub-Question *What about the linear case of the question ?*

This sub-question should then be sub-divided according to what constitutes resolution, and a demonstration of existence. Similarly,

Related Question *What is the relation between the linear case and the non-linear one ?*

The related question can at times be answered by the Nash-Moser implicit function theorem, albeit [Zeh75] is better adapted to the analytic situation. It is, however, slightly the wrong way to think of the problem. More precisely, §I.2, an analytic fully non-linear partial differential operator between vector bundles E and F is exactly the same thing as an analytic mapping between the implied sheaves of Fréchet spaces in the topology of compact convergence. In particular, linearisation has sense under the weaker hypothesis that this map is simply differentiable, while the derivative itself is a continuous linear map between sheaves of Fréchet spaces, which, in turn, I.2.2, is the same thing as being a linear differential operator. Here, as we will expand upon momentarily, it should be emphasised this is not exactly Peetre's theorem, [Pee60] & [Pee59], since functorially with respect to the ideas, I.2.1, one must respect the definition of differential operator,[EGA], i.e. holomorphic differential operators can have infinite order. It follows that the Nash-Moser conditions are not fully exploiting the underlying geometry, and that the right condition is to seek an inverse to the linearisation which itself is a map of sheaves. Unfortunately, it equally follows, that in the strict sense this is impossible unless the operator has order 0, a.k.a. a matrix of invertible functions. It is not, however, excluded that partial sheafification of an inverse is possible, which, indeed is what one usually does in practice, e.g. integration from a base point is well defined on many, but certainly not all, open sets. The precise meaning of partial sheafication, and a related technical condition of "Holder continuity" of the functional derivative are the contents of I.3.2(a)-(c). Furthermore sheaves of analytic functions are sheaves of Fréchet spaces in a particularly simple way, i.e. inverse limits of Banach spaces on larger and larger compacts, so,

Fact (I.3.5) *There is an implicit function theorem for fully non-linear analytic P.D.E.'s (more generally $C^{1,\alpha}$ maps between sheaves of Fréchet spaces of sections of holomorphic vector bundles) which is every bit as easy to use as the implicit function theorem for Banach spaces. In particular, there are no perturbation conditions of Nash-Moser type, and it is only ever necessary to invert (in a way that partially sheafifies) the linearisation of the given P.D.E. of interest.*

The precise statement is in §I.3, and is presented as above to emphasise its salient features. Much of the set up is demonstrably optimal up to a universal constant, I.1.4, which is an observation of independent interest. In practice the conditions of the implicit function theorem imply that any finite combination of integration and differentiation to construct an inverse will always work. Infinite combinations are allowed too, but here one should read the small print. By way of an example of the latter one has,

Example (I.4.4) *Let $f \mapsto P(f)$ be an analytic fully non-linear differential operator of finite order with logarithmically flat singularities satisfying a Siegel condition (e.g. defined over $\bar{\mathbb{Q}}$) then the equation in holomorphic functions,*

$$P(f) = g$$

may be solved if and only if it is formally un-obstructed.

Of course in certain P.D.E.'s one can do better than a Siegel condition for a fully analytic solution, but these are rather particular, and in general the above may well be optimal. This point is discussed in more detail in I.4.5, and elaborated by way of the example I.5.

As should be clear from the preceeding remarks on linear systems, fully analytic solutions of singular P.D.E.'s are a rarity, and what one should take from the above discussion is that the related question is answered, and we are reduced to the linear sub-question. As far as the sub-division of the same is concerned the only relevant results in which the existence of an appropriate bi-rational modification are known are resolution of vector field singularities for surfaces, [Sei68], and 3-folds, [MPa], which imply various special cases such as "irregular" 1st order O.D.E.'s where the aforesaid non-degeneracy fails, 2nd order O.D.E.'s, and bi-dimensional first order P.D.E.'s, and leads us to,

Test Question *Can we answer the question for 1st order linear P.D.E.'s on a surface (and so, by the implicit function theorem any P.D.E. in any dimension when the functional derivative has order at most -1 with at worst a surface singularity).*

Here some things are known, which, grosso modo, may be summarised as conjugation of saturated plane fields to normal forms. Ignoring, momentarily, some extremely subtle results such as [Éca94], although strictly speaking such conjugations are solutions to P.D.E.'s they may, by way of power series expansions, be reduced to the solution of O.D.E.'s. In particular, and, a priori rather encouragingly, the question is known to be true in such cases, and, indeed, in a highly structured way, [Éca85]. An evident lacuna here is the hypothesis of saturation of the field, and correcting this, involves some work, e.g. I.5.2 & IV.1.3. Nevertheless, we are effectively dealing with a foliation, and so the linearised P.D.E.'s in question may be described as follows: U is some bi-dimensional domain fibred by s over some base B with simply connected fibres embedded in $\mathbb{C}$, i.e.

$$\begin{tikzcd} U \arrow[r, "s\times\xi"'] \arrow[d, "s"] & B \times \mathbb{C} \\ B & \end{tikzcd}$$

and we restrict our attention to the functorial generalisation of the so called "irregular" case, whence there will never be any formal obstructions, and our P.D.E. in functions on U looks rather easy, viz:

$$\left(\mathbb{1} + \frac{\partial}{\partial \xi}\right)(f) = g$$

so, where is the difficulty ? Well a moments reflection reveals the following,

(a) The implicit function theorem does a lot, but it doesn't do miracles, so we need a bound on the size of f in terms of g. This bound is allowed to blow up on the boundary of the fibres U_s in $\mathbb{C}$ in accordance with the conditions of the theorem, but not on a boundary point of U_s in $\mathbb{P}^1_{\mathbb{C}}$, so any old rubbish will not work.

(b) Quite conceivably, fibre by fibre one can solve with appropriate bounds, but this is not enough since the solution must vary holomorphically with the base.

Our previous considerations on differential operators offer some further illumination, we have the power series,

$$\left(\mathbb{1} + \frac{\partial}{\partial \xi}\right)^{-1} \text{``}=\text{''} \sum_{n=0}^{\infty} (-1)^n \frac{\partial^n}{\partial \xi^n}$$

while, quite generally, I.2.2, the condition for a formal series,

$$\sum_{n=0}^{\infty} c_n \frac{\partial^n}{\partial \xi^n}$$

to be a differential operator on any open in $\mathbb{C}$ is that its Borel transform,

$$z \longmapsto \sum_{n=0}^{\infty} n! c_n z^n$$

is entire in z. In our current context, up to some irrelevant normalisation, this amounts, not by coincidence, to the solution $E(z)$ of Euler's equation being entire. Although this is false, it's also remarkably close to being true, and one might hope that there is a (sheaf) arrow,

$$\mathscr{D}\!\mathit{iff}_{\mathbb{C}}^{-\infty} \longrightarrow \mathscr{D}\!\mathit{iff}_{\mathbb{C}}^{-\infty}\Big[\big(\mathbb{1} + \frac{\partial}{\partial \xi}\big)^{-1}\Big]$$

or, better, arrows, with similar properties, e.g. Stokes' phenomenon, to,

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow[E]{} & \mathbb{C} \\ \big\downarrow{\scriptstyle \exp} & & \\ \mathbb{C} & & \end{array}$$

and, whence, a universal way to invert $\mathbb{1} + \frac{\partial}{\partial \xi}$, which, being universal would, obviously vary holomorphically from fibre to fibre. Such considerations are taken from the first author's limited understanding of [Éca85], but try as he might, he cannot implement the programme, and, worse, II.1.5, III.3.4, IV.2.6 suggest that a wholly necessary condition is that in each fibre there are paths where

$\mathrm{Re}(\xi) \to -\infty$. Consequently, for §II-III we have to undertake a case by case analysis of the possibilities for $s : U \to B$ that are presented by resolution of foliation singularities.

Amongst the dozen cases that must be considered essentially two distinct types of behaviour emerge: the fibres U_s have $-\infty$ in their limit and solutions do indeed enjoy many similarities with the function E- albeit in IV.3 the behaviour is much more complicated, e.g. one has "Stokes' curves", IV.3.5, as opposed to "Stokes' lines", or the fibres U_s are bounded, albeit, not uniformly in s. The first case, II.1.1, that we encounter has exactly this latter form, and the basic technique for dealing with the aforesaid problems (a) & (b) in constructing an inverse is taken from [Was85]. At first glance this may appear to be sad rubbish. In reality, II.1.5, it transpires to be a finely tuned instrument which cannot really be improved as far as the linear equation is concerned, although for nonlinear equations the implicit function theorem gives a very big improvement II.1.4. Subsequently, therefore, whenever we find that $-\infty$ is not in our fibre we employ variations of increasing difficulty on the same theme. Sooner, III.3, rather than later, IV.2, we find that it fails to answer our question. More precisely, associated to a plane canonical foliation singularity there are typically two invariant branches, only one of which may, in general, be supposed to be the exceptional divisor, and in our variations on a theme we find ourselves taking logarithms in both. The construction, however, is not to blame, since such additional logarithms are demonstrably necessary, III.3.4, IV.2.6 so that, as posed, the question is false, and strikingly so in the latter case, i.e. the question cannot even be solved for the further logarithm in a full half plane. Let us therefore make,

Summary *The Test Question in the so called "irregular" case is almost answered in the affirmative in §II-III, but there are counterexamples. In fact, with the exceptions of II.1.1, & II.3.1 every case where the fibres of $s : U \to B$ are bounded is a candidate for such. It is possible, however, to answer affirmatively a modified question in which we permit in solutions not only the logarithm of the exceptional divisor but also that of other functions intrinsically associated to the geometry of the singularity. On the domains of definition of such functions, the implicit function theorem then solves a whole slew of fully non-linear P.D.E.'s for free.*

That new phenomenon should emerge in the "irregular" case of the test question is, perhaps, not surprising since it effectively governs the major new feature in the local dynamics of canonical foliation singularities in dimension 3. More precisely, the condition of being log-canonical is equivalent to each singular point of the foliation enjoying a generator ∂ such that the associated linear endomorphism,

$$\partial : \frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2}$$

is non-nilpotent. This, and the slightly stronger property of being canonical, is a functorial property, and it is equivalent to the existence of a non-trivial formal centre manifold in each point, i.e. the invariant formal subscheme defined by the vanishing of the eigenfunctions of ∂ viewed as a linear endomorphism of the completion of the local ring. One may, V.1, of course, engage in further blowing up with a view to some further *convenient*, if not necessarily functorial, improvement in the singularities. After which, the case where the centre manifold has dimension 1 amounts either to an identity of the same with the singular locus, and our calculations (which are not included) indicate behaviour consistent with [Éca85], or an isolated singularity whose local dynamics, again consistent with [Éca85], are much as one might intuit from a saddle node on a surface. The fundamentally new case, however, occurs in the presence of a 2-dimensional centre manifold around a non-isolated singular locus, i.e. there is, with multiplicity, only 1 eigenvalue. In particular, one has several notions of formal, or, more accurately sub-schemes of the singular locus in which one can complete. As such, even formally, [McQ], §1.5, & post II.2.2, the centre manifold may fail to exist in completing along the union of 2 components close to a point where the number of eigenvalues jumps from 1 to 2. Otherwise, around *central* components where there is one eigenvalue at the generic point, it is well defined after completion in the same, and we apply what we have learned to address,

Central Question *As suggested by the principle question, having performed sufficient a priori blowing up, does every point in some real blow up supported in the exceptional divisor admit a neighbourhood in which the centre manifold converges.*

This is the subject of §V, and it should be clear that III.3.4 & IV.2.6 already imply that it cannot be answered any better than the qualified way in which we have responded to the test question for P.D.E.'s on surfaces, *i.e.*

Central Answer (V.3.3) *As stated the central question is false. An invariant manifold as tangent to the formal central surface as one pleases exists, however, on taking further logarithms of invariant "divisors", other than the exceptional one. This can only happen at points where the induced foliation in the central surface is singular and leaves the central components invariant. The word "divisor" has been placed in inverted commas, because while well defined formally already they do not necessarily converge without first taking logarithms in the exceptional divisor. The singularities that may require such additional logarithms, correspond to the P.D.E.'s III.3, III.4, III.5, IV.2, IV.3.6(c), and close to some small nuisance region in IV.3, all of which is best possible.*

The central difficulty in answering the central question post §II-III is, modulo suitable preparation which should always be done modulo large powers of the ideal of the central components and not ideals at points, is the complication occasioned by invariant branches through the singularities in the induced foliation in the formal surface. As indicated above these may have a purely formal existence. They are, however, of dimension 1, so their existence, V.2.3, respectively that of the related invariant divisor, V.2.5 has many similarities with the centre manifold of a node on a surface, respectively its conjugation to normal form. The further fact that the logarithms of the functions in question are indeed invariant by the foliation, and intrinsic to its geometry, is tied to,

Final Question *What can be said about existence domains for conjugation to normal forms of the singularities around the formal central manifold.*

Obviously logarithms about the exceptional divisor is the ideal. Obviously the ideal will be false, and we'll need further logarithms. We will also have an essential preparation for addressing the principle question in dimension 3. Normal form should, of course, be understood by way of completion in the central components, so, even this stage, VI.2.1, is not as trivial as one might think. The basic extra difficulty, however, is one of preparation, i.e. typically achieving the normal form modulo the square of the centre manifold, and this tends to involve a certain loss of domain, basically the aperture of the sectors encountered in §II-III, equivalently where the centre manifold exists, will shrink. Indeed, the only case where the loss of domain is more serious are when the formal invariants occur in some highly improbable combinations. Such combinations need not be a resonance as it is usually understood albeit it ought to be considered such, and the "usual understanding" ought to be considered mistaken. There are, in fact, two such combinations where the loss of domain is worse than a mere loss of aperture, of which one is a "usual" resonance, VI.4.8, whereas the more complicated one, IV.2.6(c), is not, and actually requires a further logarithm in an invariant "divisor" which was not required in finding the central manifold. A further, and extremely important, feature of these existence domains is their structure at the generic points of the singular components. These naturally separate into those which are invariant by the induced foliation in the formal central manifold, and those which are everywhere transverse to it. The latter exhibit all the good properties of, and are extremely similar to, the domains for conjugation to normal form of a 2 dimensional saddle node. The latter, where the normal form is,

$$z\frac{\partial}{\partial z} + x^p\frac{\partial}{\partial y}$$

behave exactly as our principle question anticipates, i.e. the existence domain is a disc in y and z, while the exceptional divisor, $x = 0$, has its argument constrained to a sector S. This sector, however, is small, i.e. $\pi/p - \varepsilon$, and II.1.5(a)-(d), this is best possible whether for conjugation to normal form, or, even the existence of the centre manifold. Whence, for example, there are many invariant surfaces in such sectors asymptotic to the centre manifold, and the actual dynamics in a neighbourhood of a point is potentially much much more complicated than when the central component is transverse.

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Notation Let $\Delta^\times$ be a punctured disc. The exponential affords a canonical isomorphism,

$$a : \pi_1(\Delta^\times) \xrightarrow{\sim} \mathbb{Z}(1) = \mathbb{Z}2\pi\sqrt{-1}$$

and, for γ an oriented loop we define,

$$\oint_\gamma := \frac{1}{a(\gamma)}\int_\gamma$$

which, inter alia, does not depend on the choice of the square root of -1.

Contents

I Differential Operators

I.1 Canonical Norms

Ultimately, we will require norms on differential operators, so, in the first instance vector fields. The following, cf. [Kob98], §3-§4 are standard,

I.1.1 Definition Let $x \in X$, $\partial \in T_X(x)$, X a smooth complex space, then the Kobayashi, respectively Carathéodory, pseudo-metric is defined by,

$$\|\partial\|_X^{\mathrm{Kob}}(x) = \inf_f \left\{ \frac{1}{R} \; : \; f_*\Big(\frac{\partial}{\partial z}\Big) = R\partial \right\}, \quad \text{respectively,}$$
$$\|\partial\|_X^{\mathrm{Cat}}(x) = \sup_f \left\{ D \; : \; f_*\left(\partial\right) = D\frac{\partial}{\partial z} \right\},$$

where for z a standard coordinate on the unit disc Δ, the infimum, respectively supremum, is taken over pointed maps $f : (\Delta, 0) \to (X, x)$, respectively $f : (X, x) \to (\Delta, 0)$.

From the point of view of local analysis the good definition should be that of Carathéodory, but it appears to be rather difficult to obtain natural upper, rather than lower, bounds in contradistinction to that of Kobayashi. Consequently we shall avail ourselves of,

I.1.2 Triviality Let X be a smooth complex space, f a function, and ∂ a vector field, on X, respectively, then, for all $x \in X$,

$$|\partial f(x)| \leqslant \|f\|_X \|\partial\|_X^{\mathrm{Cat}}(x) \leqslant \|f\|_X \|\partial\|_X^{\mathrm{Kob}}(x)$$

where $\|f\|_X$ is the sup norm on X, possibly infinite, peu importe.

Proof. The first inequality is the definition of the Carathéodory norm up to a conformal mapping, while the general inequality $\| \;\|^{\mathrm{Cat}} \leqslant \| \;\|^{\mathrm{Kob}}$ follows from the Schwarz lemma, in fact the easy undergraduate version. ▫

In so much as we will be doing local analysis, product domains will play an important role. Consequently, even though much coarser bounds would be sufficient, it's convenient to recall,

I.1.3 Fact Let $X \times Y$ be any product of smooth complex spaces with ξ, η the projections to X and Y respectively, then for $\star$ either Kob or Cat,

$$\|\_\|_{X\times Y}^{\star} = \max\{\|\xi_*\_\|_X^{\star}, \|\eta_*\_\|_Y^{\star}\}.$$

Proof. This is a good illustration of how much more tricky Cat is than Kob. Specifically, as ever Schwarz implies that in either case the right hand side is bounded by the left. For Kob the converse is trivial, i.e. given discs, $f : \Delta \to X$, $g : \Delta \to Y$, the discs,

$$\Delta \longrightarrow X \times Y \; : \; z \to f(\lambda z) \times g(z)$$

or perhaps $f(z) \times g(\lambda z)$, depending on which disc is bigger, for an appropriate multiplier of modulus at most 1 gives the bound. One reduces the Cat case to this case by way of a highly non-trivial theorem of Lempert [Lem82], that Cat & Kob coincide on affinely convex domains. The discussion in [Kob98], §4.9.1, is for a slightly different, viz. not necessarily inner, definition of Cat, so we'll quickly adapt/plagiarise it. More precisely, approximate a function f on $X \times Y$ to Δ by tensors, $\sum_{i=1}^{n} \xi^* a_i \otimes \eta^* b_i$, and supposing without loss of generality that a_i, respectively b_i, are bounded on X, respectively Y, for all i, put:

$$U = \{\underline{s} \in \mathbb{C}^n \ : \ \left|s_i\right| < \left\|a_i\right\|_X, \ \left|\underline{s} \cdot \underline{b}(y)\right| < 1, \ y \in Y\}$$
$$V = \{\underline{t} \in \mathbb{C}^n \ : \ \left|t_i\right| < \left\|b_i\right\|_Y, \ \left|\underline{a}(x) \cdot \underline{t}\right| < 1, \ x \in Y\}$$

where $\underline{\sigma} \cdot \underline{\tau} = \sum_i \sigma_i \tau_i$ is the standard dot product.

Now, supposing, as we may, that our function f and the approximating tensor, T, vanish at the same point of interest $x \times y$, we may decompose a vector ∂ at the same as $\partial_X \amalg \partial_Y$ and apply Lempert's theorem to conclude,

$$\left\|T_* \partial\right\|_{\Delta}^{\mathrm{Cat}}(x \times y) \leqslant \max_{i,j}\{\left\|(a_i)_* \partial_X\right\|_U^{\mathrm{Cat}}, \left\|(b_j)_* \partial_Y\right\|_V^{\mathrm{Cat}}\}$$

while the ubiquitous Schwarz lemma, in the guise of distance decreasing for Cat, bounds the right hand side as required. ▫

Again, in the light of the use of product domains, the following complement on the uniformisation theorem will be of some utility,

I.1.4 Fact Let $\Omega \subseteq \mathbb{C}$ be a proper simply connected sub-domain then,

$$\frac{e^{-2}}{d(p, \partial\Omega)} \leqslant \left\|\frac{\partial}{\partial z}\right\|_{\Omega}^{\mathrm{Kob}}(p) = \left\|\frac{\partial}{\partial z}\right\|_{\Omega}^{\mathrm{Cat}}(p) \leqslant \frac{1}{d(p, \partial\Omega)}$$

where d is the Euclidean distance from $\partial\Omega$, and z a standard coordinate.

Proof. The middle identity is the uniformisation theorem, and the right hand inequality is trivial. We first prove the left hand equality for domains where the uniformisation theorem extends C^0 up to the boundary, so let $f : \Delta \to \Omega$ be such a uniformisation, and appeal to translation invariance of the Euclidean distance to suppose that $p = 0$. As such, for d the distance of 0 to $\partial\Omega$,

$$-\log d \left\|\frac{\partial}{\partial z}\right\|_{\Omega}^{\mathrm{Kob}}(0) = \oint_{\zeta \in \partial\Delta} \log\left(\frac{|f|}{d}\right) \frac{d\zeta}{\zeta}$$

The afore-noted trivial inequality is the positivity of the integrand on the right. To achieve an upper bound for the integral let $\rho \in (0, 1)$ be some radius to be chosen, then for q a point on the disc of radius ρ,

$$\log \frac{|f|}{d}(q) - \log \rho = (1 - \rho^2) \oint_{\zeta \in \partial\Delta} \log \frac{|f|}{d} \, |\zeta - q|^{-2} \frac{d\zeta}{\zeta}$$

The distance to the boundary of $f(\Delta_\rho)$ is at most d, so there is some q for which the left hand side is bounded by $-\log\rho$, and whence,

$$\oint_{\zeta\in\partial\Delta} \log\left(\frac{|f|}{d}\right)\frac{d\zeta}{\zeta} \leqslant \frac{(1+\rho)}{(1-\rho)}|\log\rho|$$

Since ρ was arbitrary, the infimum of the right hand side for $\rho \in (0,1)$ will do, *i.e.* $\rho \to 1$, and the general case follows by way of an exhaustion (e.g. by sub-discs via the uniformisation theorem) $\Omega_t \to \Omega$ which is continuous with respect to either side of the inequality. ▫

This can be combined into some rather general estimates, according to an inductive principle that we will apply repeatedly. To wit:

I.1.5 Product Set up Let $P = P_1 \times \cdots \times P_n$ be a product of simply connected proper subdomains of $\mathbb{C}$, with $\partial_1, \ldots, \partial_n$ a basis, or equivalently, an everywhere invertible $\mathcal{O}_P$-linear endomorphism,

$$A : T_P \longrightarrow T_P$$

so, for $\partial/\partial z_i$ a standard field, $\partial_i = A(\partial/\partial z_i)$. Punctually, this mapping admits the operator norm $\|A\|(x)$, for $T_P(x)$ normed in the Carathéodory metric, and we denote by $\|A\|$ its supremum over P, possibly infinite, again, peu importe. Subsequently we introduce a function,

$$d_P(p) = \min_i \operatorname{dist}(p_i, \partial P_i)$$

for p_i the projections of $p \in P$, and dist denoting the Euclidean distance. Finally, for $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geqslant 0}$ a multi-index, let $|a| = a_1 + \cdots + a_n$, and observe:

I.1.6 Fact Let things be as above, with f any function on P, then,

$$|\partial_1^{a_1} \ldots \partial_n^{a_n} f(p)| \leqslant a_1! \ldots a_n! \ \frac{e^{|a|-1}\|A\|^{|a|}}{d_P(p)^{|a|}} \ \|f\|_P$$

with $\|f\|$ the sup-norm of f over P, and $|a| > 0$.

Proof. By induction on $|a|$. The case $|a| = 1$ is immediate from the set up, and the trivial direction of I.1.4. Next let, $(a_1+1, a_2, \ldots, a_n)$ be a multi-index of weight 1 beyond what we suppose known. For each $1 \leqslant i \leqslant n$, let $d_i = \operatorname{dist}(p_i, \partial P_i)$ and for $t \in (0,1)$ to be chosen, introduce the product domain $Q_t = \prod_i Q_{it}$, where,

$$Q_{it} = \{q \in P_i \ : \ \operatorname{dist}(q, \partial P_i) \geqslant t d_i\}$$

Then, $\inf_{q\in Q_t} d_P(q) \geqslant t d_P(p)$ and $d_{Q_t}(p) \geqslant (1-t) d_P(p)$, so that, if $d = d_P(p)$,

$$|\partial_1 \partial^{\underline{a}} f(p)| \leqslant \frac{\|\partial^{\underline{a}} f\|_{Q_t}}{(1-t)d} \ \|A\| \leqslant a_1! \ldots a_n! \ \frac{e^{|a|-1}\|A\|^{|a|+1}}{d^{|a|+1}(1-t)t^{|a|}} \ \|f\|_P$$

The right hand side is optimised for $t = \dfrac{|a|}{1+|a|}$, while the supremum of $(1 + 1/x)^x$, $x \geqslant 1$ is e. ▫

If we use the trickier part of I.1.4 and profit from the product structure to proceed one factor at a time, the we have a better formula, viz.

I.1.7 Remark Let things be as above, but with A the identity, then for some absolute constant,

$$|\partial_1^{a_1} \dots \partial_n^{a_n} f(p)| \leqslant a_1! \dots a_n!\; C^{|a|} \|\partial_1\|_{P_1}^{a_1} \dots \|\partial_n\|_{P_n}^{a_n}\; \|f\|_P$$

and $\|\_\|$ the Carathéodory/Kobayashi norm.

Again, the non-intervention of the non-trivial direction of I.1.4 in the proof of I.1.6, but rather the compatibility of the basis with the Euclidean distance suggests that the truly practical metric to work with is neither that of Kobayashi nor Carathéodory, but:

I.1.8 Definition/General Set Up Let $\Omega \subseteq \mathbb{C}^n$ be a not necessarily bounded domain, and denote by $d_\Omega : \Omega \to \mathbb{R} \cup \{\infty\}$ (or just d if there is no risk of confusion) the Euclidean distance to the boundary in the norm $\|z\| = \max_i |z_i|$, then we say that Ω is Cauchy hyperbolic if $d_\Omega(x) \neq \infty$, $\forall x \in \Omega$ and metricise the tangent space by,

$$\left\| \frac{\partial}{\partial z} \right\|_\Omega^{\text{Cauchy}} (x) \doteqdot \frac{1}{d_\Omega(x)}$$

so that, trivially, $\|\frac{\partial}{\partial z}\|_\Omega^{\text{Cauchy}} \geqslant \|\frac{\partial}{\partial z}\|_\Omega^{\text{Kob}}$, but not conversely, e.g. product of a disc with $\mathbb{C}$. Furthermore, for $\partial_1, \dots, \partial_n$ a basis of the tangent space defined as per I.1.5 by a linear endomorphism, $\partial_i = A(\partial/\partial z_i)$, for $\partial/\partial z_i$ the standard fields, let $\|A\|(x)$ be its pointwise norm in the Cauchy metric, and $\|A\|$ the supremum of the same over Ω.

With this long winded set up out of the way we can directly appeal to I.1.6 to conclude,

I.1.9 Further Remark Let things be as above then there is an absolute constant C such that for f any function on a Cauchy hyperbolic domain,

$$|\partial_1^{a_1} \dots \partial_n^{a_n} f(p)| \leqslant a_1! \dots a_n!\; \frac{C^{|a|} \|A\|^{|a|}}{d_P(p)^{|a|}}\; \|f\|_\Omega$$

I.2 Peetre's Theorem

Recall that Peetre's Theorem, [Pee59], [Pee60], asserts that any $\mathbb{C}$-linear map of locally free sheaves of differentiable functions is a finite order linear operator. Stated thus it is false for holomorphic functions. However, it is an actual consequence of the theorem that C^∞ differential operators are of finite order, as such with the right definition of differential operator it remains true. To further investigate this let,

$$L : \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{X}}$$

be a $\mathbb{C}$-linear map of the structure sheaf on whatever one's favourite poison $\mathcal{X}$ may be, e.g. stacks in the canonical site of analytic spaces in the maximal generality. Irrespectively the structure of L is a local question, so we could say $\mathcal{X}$ a polydisc, but let's keep with the previous discussion and say a product domain P. The sheaf $\mathcal{O}_P$ is naturally a sheaf of Fréchet spaces for the topology of convergence on compact subsets, and we insist that L is continuous for this topology. Following [Köt69], §27.3, we embed P in a product of $\mathbb{P} = (\mathbb{P}^1)^n$ of projective spaces in the obvious way, and observe that for every $\underline{t} \in \mathbb{P}$, including infinities,

$$(z_1 - t_1)^{-1} \dots (z_n - t_n)^{-1} \in \Gamma\Big(\prod_{i=1}^{n} P_i - \{t_i\}, \mathcal{O}_P \Big)$$

where, as ever, z_i is a natural coordinate function on the i^{th} copy of $\mathbb{C}$. Unsurprisingly we put,

$$K(s,t) = L\left\{ (z_1 - t_1)^{-1} \dots (z_n - t_n)^{-1} \right\}.$$

By hypothesis, K varies holomorphically in the variable s. It is also analytic in the variable t, since, supposing no t_i infinite for convenience,

$$\frac{K(s, t+\tau_i) - K(s,t)}{\tau_i} = L\Big\{ \prod_{j \neq i} (z_j - t_j)^{-1} (z_i - t_i)^{-1} (z_i - (t_i + \tau_i))^{-1} \Big\}$$

holds in $\Gamma\Big(\prod_{j\neq i} P_j \backslash \{t_j\} \times P \backslash \Delta \Big)$, for Δ a small disc around t_i and $t_i + \tau_i$. Better still, the function to which L is applied converges on compact subsets ($\tau_i \to 0$) of $\prod_{i=1}^{n} P_i \backslash \{t_i\}$, so:

$$\frac{\partial K}{\partial t_i} = L\Big\{ (z_i - t_i)^{-2} \prod_{j \neq i} (z_j - t_j)^{-1} \Big\}$$

Consequently K is an analytic function on $P \times \mathbb{P}$ off the diagonals, $\Delta_i \doteqdot \{s_i = t_i\}$. Furthermore for any open $U \subseteq P$, of product type with each factor enjoying a boundary a simple closed curve, and $f \in \Gamma(U)$,

$$f(z) = \oint_{\gamma_1 \times \dots \times \gamma_n} f(t) \frac{dt}{(z_1 - t_1) \dots (z_n - t_n)}$$

Interchanging L with the contour integral is perfectly justified, e.g. op.cit. gives a proof for L a linear functional, which is valid mutatis mutandis for any Banach, whence any Fréchet space, and so,

$$(Lf)(s) = \oint_{\gamma_1 \times \dots \times \gamma_n} f(t) K(s,t) dt$$

Now change coordinates, viz.: $\tau_i = (t_i - s_i)^{-1}$, so that we have a holomorphic function on $P \times \mathbb{C}^n$, which, bearing in mind that K vanishes on each divisor at infinity, we may expand as a power series in τ,

$$K(s,\tau) = \left(\tau_1 \cdots \tau_n\right) \sum_{\underline{a}} k_{\underline{a}}(s) \tau^{\underline{a}}$$

where the sum is over multi-indices $\underline{a} \in \mathbb{Z}^n_{\geqslant 0}$, and is absolutely convergent for any value of τ. Consequently,

$$(Lf)(s) = \sum_{\underline{a}} \frac{k_{\underline{a}}(s)}{a_1! \dots a_n!} \frac{\partial^{a_1 + \dots + a_n}}{\partial s_1^{a_1} \dots \partial s_n^{a_n}} f(s)$$

which certainly merits the name of differential operator - the sum in question is absolutely convergent, as it had to be, by I.1.7, and functorially with respect to the ideas this is the right definition.

To clarify this recall from[EGA], that on a scheme X, $\mathscr{P}^{\infty}_X$ is functions on the completion $X \times X$ in the diagonal viewed as an $\mathcal{O}_X$-module by the way of the first projection. As such, algebraically speaking,

$$\mathscr{D}\!\mathit{iff}^{-\infty}_X \doteqdot \mathrm{Hom}_{\mathcal{O}_X} \left(\varprojlim \mathscr{P}^n_X, \mathcal{O}_X \right) = \varinjlim \mathscr{D}\!\mathit{iff}^{(-n)}_X$$

is the direct limit of finite order operators, equivalently $K(s,t)$ should be polynomial in the variable τ. In the analytic topology, however, one should make,

I.2.1 Definition For $\mathcal{X}$ one's favourite poison (more general than stacks in the standard topology doesn't make sufficient sense, i.e. $\mathscr{D}\!\mathit{iff}$ is undefined), $\mathscr{P}^{\infty}_X$ is the germ of functions in a neighbourhood of the diagonal of $\mathcal{X} \times \mathcal{X}$ viewed as an $\mathcal{O}_{\mathcal{X}}$-module under the 1^{st} projection. Consequently unlike its algebraic counterpart it is not a Fréchet space (supposing that our scheme was over $\mathbb{C}$) but a nuclear DFS, separated in the natural direct limit of Banach spaces that comes from its realisation as a germ - cf. [Köt69], §27.4, where the specifics are about germs around compact subsets of the plane, but the discussion applies verbatim in any dimension, whence the sheaf of differential operators,

$$\mathscr{D}\!\mathit{iff}^{-\infty}_{\mathcal{X}} \doteqdot \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}} \left(\mathscr{P}^{\infty}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}} \right)$$

is functorially a (nuclear) Fréchet space.

The necessary extension to sheaves of locally free $\mathcal{O}_{\mathcal{X}}$-modules, or even for operators with values in a co-coherent sheaf, being a triviality, we may avoid complicating our notation, and clarify our discussion by way of Borel/Laplace/Fourier transforms.

More precisely the algebra A of differential operators with constant coefficients in the standard fields $\partial/\partial z_i$, say ∂_i for brevity, is a perfectly good commutative algebra. It's maximal ideals are derivations of exponentials evaluated in zero, so that it's Gelfand representation is:

$$A \longrightarrow \Gamma(\mathbb{C}^n) \ : \ D \longmapsto \oint_{\gamma_0} D(e^{z_1\zeta_1 + \dots + z_n\zeta_n}) \frac{d\zeta_1}{\zeta_1} \dots \frac{d\zeta_n}{\zeta_n}$$

where the notations means, view $D(*)$ as an operator in ζ, and take the contour integral in ζ around the origin on a product of loops, to get an entire function of z. This is not, however, as one sees from the presence of factorials the entire function that we are considering, but rather its image under the Borel transform of a rather special type of element in the convolution algebra $H^n_c(\omega_{\mathbb{G}^n_a})$, where we identify $\mathbb{G}^n_a$ with $\mathbb{C}^n$ and make a certain confusion between functions and functionals via the Haar measure (in a holomorphic sense) $dz_1 \cdots dz_n$. This leads, in the aforementioned (s,τ) coordinates, to an alternative formula,

$$K(s,\tau) = \int_{\sigma\in\mathbb{R}^n_+} e^{-\left(\frac{\sigma_1}{\tau_1}+\cdots+\frac{\sigma_n}{\tau_n}\right)} d\sigma_1 \ldots d\sigma_n$$
$$\cdot \oint_{\gamma_1\times\cdots\times\gamma_n} L\Big\{e^{\sigma_1(z_1-s_1)+\cdots+\sigma_n(z_n-s_n)}\Big\} \frac{dz_1}{(z_1-s_1)} \cdots \frac{dz_n}{(z_n-s_n)}$$

which is probably the opposite of help, since the equivalent,

$$K(s,\tau) = \big(\tau_1 \ldots \tau_n\big) \oint_{\gamma_1\times\cdots\times\gamma_n} L\Big\{\prod_{i=1}^{n}(1-\tau_i(z_i-s_i))^{-1}\Big\} \frac{dz_1}{(z_1-s_1)} \cdots \frac{dz_n}{(z_n-s_n)}$$

is clearer, where in either case the contour is a product of loops around the s_i, and we call this the Laplace transform, λ, of the operator. Such considerations do however help for the inverse map,

$$\beta : \Gamma\big(P\times\mathbb{C}^n, \mathcal{O}_{P\times\mathbb{C}^n}(-0_1\cdots-0_n)\big) \longrightarrow \Gamma(P, \mathscr{D}\mathit{iff}^{-\infty}_P)$$
$$K(s,\tau) \longmapsto \oint_\gamma K(s, \frac{1}{z_i}) \exp(z_1\partial_1+\cdots+z_n\partial_n) dz_1\cdots dz_n$$

where the contour is taken around a product of loops at the origin in the z variable, or at ∞ if one prefers the τ variable. Irrespectively this is a Borel (albeit depending on one's prejudices Fourier-Laplace are perfectly legitimate alternatives) transform of a function into a differential operator, and we may summarise our discussion by the way of,

I.2.2 Fact For $\mathcal{X}$ one's favourite poison (say smooth for convenience) a map $L : \mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{X}$ is a $\mathbb{C}$-linear map of Fréchet spaces iff it is an element of $\Gamma(\mathcal{X}, \mathscr{D}\mathit{iff}^{-\infty}_\mathcal{X})$. Furthermore the Borel and Laplace transforms on any open $U \to \mathcal{X}$ isomorphic to a product domain in $\mathbb{C}^n$ yield mutually inverse isomorphisms of Fréchet spaces,

$$\begin{tikzcd} \Gamma\big(U, \mathscr{D}\mathit{iff}^{-\infty}_\mathcal{X}\big) \arrow[r, bend left, "\lambda"] & \Gamma\big(P\times\mathbb{C}^n, \mathcal{O}_{P\times\mathbb{C}^n}(-0_1\cdots-0_n)\big) \arrow[l, bend left, "\beta"] \end{tikzcd}$$

where the former has the topology of operator norms on compact sets, and the latter sup-norms on compact sets.

All of which is quite tidy modulo the choice of the isomorphism with a product domain. Certainly by I.1.6 the Borel transform is well defined after the less demanding choice of a basis of $\Gamma(U, T_{\mathcal{X}})$, and at the price of shrinking U, one could then define the Laplace transform. Such subterfuge is probably un-necessary, albeit we ignore the question.

I.3 An Implicit Function Theorem

Again let $\mathcal{X}$ be one's favourite poison, but,

$$P : E \longrightarrow F$$

an arbitrary continuous map of sheaves of Fréchet spaces, for E, F locally free sheaves of $\mathcal{O}_x$-modules in the topology of convergence on compact sets. Consequently for $U \to \mathcal{X}$ open, and $f \in \Gamma(U, E)$ we may apply I.2.2 to conclude,

I.3.1 Triviality Suppose $P : \Gamma(U, E) \to \Gamma(U, F)$ is differentiable at f then,

$$P'(f) \in \Gamma\big(U, \mathscr{D}\!\mathit{iff}_{\mathcal{X}}^{-\infty}(E, F)\big)$$

so, up to trivialisation of E and F, a matrix of differentiable operators.

Consequently it is ridiculous to imagine that we can invert, be it on the left or the right, the derivative at the sheaf level. Indeed, again by I.2.2, the inverse would be a differential operator in the sense of I.2.1, and the only such operators appear to be invertible matrices of functions - for example this may easily be reduced to Liouville's theorem for operators with constant coefficients, albeit the general case appears to be rather more fastidious. We can, however, reasonably suppose that our inverse constitutes the resolution of a Cauchy problem in the following sense:

I.3.2(a) Set up Let things be as above, and let $\underline{\delta} = (\delta_1, \dots, \delta_p) \in \big(\mathbb{R}_{>0} \cup \{\infty\}\big)^p$ be given, and put $I = \prod_{i=1}^p [0, \delta_i]$, $I^* = \prod_{i=1}^p (0, \delta_i]$. Suppose further that for $\underline{d} \in I$, there are domains $U(d) \subset U$ such that for the partial ordering $\underline{d} \leqslant \underline{e}$ iff $e_i \geqslant d_i$, $\forall i$, $U(\underline{e}) \subset U(\underline{d})$, and $U(\underline{0}) = U$. Then we say that a family of continuous linear operators:

$$K_{\underline{d}} : \Gamma(U(\underline{d}), F) \longrightarrow \Gamma(U(\underline{d}), E)$$

solves a Cauchy problem if it is a right inverse for a differential operator $D \in \Gamma\big(U, \mathscr{D}\!\mathit{iff}_{\mathcal{X}}^{-\infty}(E, F)\big)$ such that for every $\underline{\delta} \geqslant \underline{d} \geqslant \underline{e} \geqslant 0$ the diagram,

$$\begin{array}{ccc} \Gamma(U(\underline{e}), F) & \xrightarrow{K_{\underline{e}}(\underline{\delta})} & \Gamma(U(\underline{e}), F) \\ {\scriptstyle \mathrm{RES}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{RES}} \\ \Gamma(U(\underline{d}), F) & \xrightarrow{K_{\underline{d}}(\underline{\delta})} & \Gamma(U(\underline{d}), F) \end{array}$$

with vertical arrows the natural restrictions, commutes. Whence, there is some partial sheafication for domains between $U(\underline{\delta})$ and U, so, should there be no possibility of confusion we'll just write $K(\underline{\delta})$, or even K.

The basic example that one wants to have in mind is product domains as per I.1.5 where the $U(\underline{d})$ are as per the proof of I.1.6. The set up is, however, far more general than this, and is really adapted to an arbitrary basis of fields $\partial_1, \dots, \partial_n$ where one moves away from the boundary a distance d_i along geodesic discs in the i^{th} direction. Whence, in practice, it is unlikely that one can take the p of I.3.2(a) strictly bigger than the dimension of U, but this is irrelevant to the structure of the estimates, so its harmless to take p arbitrary.

Let us further observe the generality implicit in I.2.2. Suppose for example that D has order 1, then one might imagine that K is an integral operator over paths based at some point, or sub-variety, $\mathfrak{b}$. In good cases $\mathfrak{b}$ (which can, by the way, be taken at infinity where appropriate) won't depend on $\underline{\delta}$. In bad cases, which will happen, we need to take $\mathfrak{b}$ on the boundary, so we'll have a very limited freedom over the domains V for which K is under control, in fact, only those between $U(\underline{\delta})$ and U. Furthermore while $K_{\underline{d}}(\underline{\delta})$ might be defined up to the boundary of $U(\underline{d})$, we don't insist on this, so there's space to differentiate as well as integrate thanks to I.1.6.

This said, manifestly our interest is to invert our given $P : E \to F$ on the right around the section $P(f)$. Translating on the left and right we may, more conveniently, suppose $f = P(f) = 0$, and we add:

I.3.2(b) More Setting up We will suppose that the continuous map of sheaves of Fréchet spaces, P, is not only differentiable at f, but that it is uniformly $C^{1,\alpha}$, to wit, after translating to $f = P(f) = 0$:

There exits $\alpha, \varepsilon > 0$, and a function $\phi : \mathbb{R}^p_{>0} \to \mathbb{R}^p_{>1}$ decreasing in its arguments such that for all $U(\underline{e})$ between $U(\underline{\delta})$ and U, and sections, h, on the same with $\|h\|_{U(\underline{e})} < \varepsilon$,

$$\left\|Ph - P'(0)h\right\|_{U(\underline{d})} \leqslant \left\|h\right\|^{1+\alpha}_{U(\underline{e})} \; \phi(\underline{d} - \underline{e})$$

where of course, we take supremum norms over the appropriate domains of pointwise norms between Euclidean spaces, and $\underline{e} < \underline{d}$, i.e. $e_i < d_i$, $\forall i$.

Again, let us observe the generality of our set up. For example take $U \subseteq \mathbb{C}^n$, $p = n$, $\partial_1, \dots, \partial_n$ the standard fields, with W the vector space of constant coefficient operators in the same, but of bounded order. Consequently, for every $d_i > 0$ we can define,

$$\begin{aligned} U(d_i) &= \left\{(z_i, z^i) \;:\; \mathrm{dist}(z_i, \partial U_{z^i}) > d_i\right\} \\ U(\underline{d}) &= U(d_1) \cap \cdots \cap U(d_n) \end{aligned}$$

where z^i is the projection onto $\mathbb{C}^{n-1}$ which omits the i^{th} factor, and dist is the Euclidean distance in $\mathbb{C}$. Consequently if $z \in U(\underline{d})$, and Δ_{d_i} is the disc of radius d_i in the i^{th} direction, then $z + \Delta_{d_i} \subset U$, which, inter alia, does not imply that $z + \Delta_{d_1} \times \cdots \times \Delta_{d_n} \subset U$, but, nevertheless:

I.3.3 Intermission Let things be as above, then for all multi-indices $\underline{a} = (a_1, \dots, a_n)$ and functions f on U,

$$\left\|\partial_1^{a_1} \dots \partial_n^{a_n} f\right\|_{U(\underline{d})} \leqslant \frac{a_1! \cdots a_n!}{d_1^{a_1} \cdots d_n^{a_n}} \, e^{|a|} \, \|f\|_U$$

Proof. As per I.1.6, by induction on $|a|$, the case $|a| = 1$ being trivial since the Kobayashi metric in the i^{th} direction is at most $1/d_i$. Now, for some suitable $t \in (0,1)$ to be chosen, and $(a_1+1, a_2, \ldots, a_n)$ an unknown multi-index:

$$\left\|\partial_1 \partial^{\underline{a}} f\right\|_{U(\underline{d})} \leqslant \frac{1}{(1-t)d_1} \left\|\partial^{\underline{a}} f\right\|_{U(td_1, d_2, \ldots, d_n)} \leqslant \frac{a_1! \cdots a_n!}{d_1^{a_1+1} d_2^{a_2} \cdots d_n^{a_n}} \; \frac{e^{|a|}}{(1-t)t^{|a|}} \; \left\|f\right\|_U$$

while again the supremum of $(1+1/x)^x$, $x \geqslant 1$ is e. ▫

Now suppose U sufficiently small that we may identify E and F with trivial bundles of rank r and s respectively, then a very large class of operators may be defined as those of the form:

$$(P\underline{h})(x) = \Phi(x, h_j, \partial_1^{i_1} \cdots \partial_n^{i_n} h_j)$$

for $\Phi : U \times \Delta^r \times W^r \to \mathbb{C}^s$ analytic, Δ a disc of radius at least ε, for ε as per I.3.2(b) and polynomial in W. As such for P mapping the origin to the origin,

$$P\underline{h} - P'(0)\underline{h} = \Psi(x, h_j, \partial_I h_j)$$

where Ψ enjoys all the properties of Φ enunciated above, except that it is at least quadratic in the h_j. Whence an immediate application of I.3.3 yields,

I.3.4 Remark/Triviality Suppose P is of the form discussed above then it is uniformly $C^{1,\alpha}$ for some $\alpha \in \mathbb{N}$, with ϕ at worst reciprocal polynomial in its arguments.

It therefore remains to consider the shape of the right inverse K. Plainly we'll want Kh to be sufficiently small so as to apply our hypothesis of uniformly $C^{1,\alpha}$, and whence:

$$\left\|PKh - h\right\|_{U(\underline{d})} \leqslant \left\|Kh\right\|_{U(\underline{e})}^{1+\alpha} \; \phi(\underline{d} - \underline{e})$$

Having already, and wholly reasonably, hypothesised that K is linear we may conclude our set up with:

I.3.2(c) End of Set up The solution K of the Cauchy problem will be supposed to be *not ludicrous*, i.e. for $K(\underline{\delta})$ a family of operators on domains $U(\underline{d})$ between $U(\underline{\delta})$ and U it is required that there is a bound,

$$\left\|Kh\right\|_{U(\underline{d})} \leqslant \left\|h\right\|_{U(\underline{e})} \; \psi(\underline{d} - \underline{e})$$

for $\psi : \mathbb{R}^p_{>0} \to \mathbb{R}^p_{>1}$ decreasing in its arguments such that the logarithm of the function, $t \mapsto \theta(t\underline{d}) \doteqdot \phi(t\underline{d})\psi(t\underline{d})^{1+\alpha}$, $t \in \mathbb{R}_{>0}$ is absolutely integrable at $t = 0$. Plainly reciprocal polynomial bounds for ψ of the type encountered for ϕ in I.3.4 lead to a solution which isn't ludicrous. Putting all of this together, we assert:

I.3.5 Claim Let the set-up be as in I.3.2(a)-(c), i.e. P is uniformly $C^{1,\alpha}$ and there is some $\underline{\delta} > 0$ such that $K(\underline{\delta})$ solves a Cauchy problem for the derivative of P in a way which isn't ludicrous, and for convenience ε of I.3.2(b) less than 1, then for g a section of F over U the equation,

$$P(f) = g$$

has a solution on any $U(\underline{d})$ between $U(\underline{\delta})$ and U provided,

$$\|g\|_U < \max\left\{1, \Theta(\underline{d})^{-\frac{(1+\alpha)}{\alpha^2}}\right\}\varepsilon$$

where

$$\Theta(\underline{d}) = \exp\left(\int_0^1 \log\theta(t\underline{d})dt\right)$$

Better still, under such conditions the norm of f is no worse than,

$$\|f\|_{U(\underline{d})} \leqslant \left\{\|g\|_U + \frac{\rho}{1-\rho}\right\} \psi(\underline{d})$$

for $\rho = \left(\|g\|_U \Theta(\underline{d})^{\frac{1+\alpha}{\alpha^2}}\right)^{e\log(1+\alpha)}$. In particular, the equation has a solution on $U(\underline{\delta})$, and if this is all one is interested in one may dispense with the condition of absolute integrability in I.3.2(c) for all $\underline{d}$ other than $\underline{\delta}$.

Proof. Write $PK = \mathbb{1} + Q$, then for any section h on any neighbourhood $U(\underline{d})$ between $U(\underline{\delta})$ and U we have the estimate,

$$\|Qh\|_{U(\underline{d})} \leqslant \|h\|_{U(\underline{e})}^{1+\alpha}\ \theta(\underline{d} - \underline{e})$$

provided $\underline{d} \geqslant \underline{e}$, and $\|h\|_{U(\underline{e})} < \varepsilon$. Now consider, $0 = t_{n+1} < t_n < \cdots < t_1 < t_0 = 1$ where, for $m \leqslant n$:

$$1 - t_{(n+1)-m} = \frac{1}{\sigma_n}\sum_{i=m}^{n}\frac{(i-m+1)}{(1+\alpha)^i}, \quad \sigma_n = \sum_{i=0}^{n}\frac{(1+i)}{(1+\alpha)^i}$$

and for any fixed $\underline{\delta} \geqslant \underline{d} > 0$, apply the basic estimate on Q successively on $U(t_i\underline{d}) \subset U(t_{i+1}\underline{d})$ to obtain,

$$\log\|Q^{n+1}h\|_{U(\underline{d})} \leqslant (1+\alpha)^{n+1}\log\|h\|_U + \sum_{i=0}^{n}(1+\alpha)^i\log\theta\big((t_i - t_{i+1})\underline{d}\big)$$

$$= (1+\alpha)^{n+1}\log\|h\|_U + (1+\alpha)^n\sum_{i=0}^{n}\frac{1}{(1+\alpha)^{n-i}}\log\theta\Big(\underline{d}\frac{1}{\sigma_n}\sum_{j=n-i}^{n}\frac{1}{(1+\alpha)^j}\Big)$$

Since θ was supposed decreasing in its arguments, the latter sum is bounded by

$$\sigma_n(1+\alpha)^n\int_0^{s_n/\sigma_n}\log\theta(t\underline{d})dt\ , \quad s_n = \sum_{j=0}^{n}\frac{1}{(1+\alpha)^j}$$

so that,

$$\log \left\| Q^{n+1}h \right\|_{U(\underline{d})} \leqslant (1+\alpha)^{n+1} \left\{ \log \left\| h \right\|_{U} + \frac{(1+\alpha)}{\alpha^2} \log \Theta(\underline{d}) \right\}$$

provided that $\|h\|_U$ was taken sufficiently small, i.e. right hand side bounded by $\log \varepsilon$ for all n. Armed with this bound we can now iterate in the obvious way, viz: we look for a fixed point of,

$$Tf \doteqdot g - Qf$$

by the way of the iteration: $f_{n+1} = Tf_n$, $f_0 = 0$. Thus, for $\underline{d}$ as above, and $n \in \mathbb{N}$,

$$\left\| f_{n+1} - f_n \right\|_{U(\underline{d})} \leqslant \left\{ \left\| g \right\|_U \ \Theta(\underline{d})^{\frac{1+\alpha}{\alpha^2}} \right\}^{(1+\alpha)^n}$$

provided, as before, that $\|g\|_U$ was taken sufficiently small, i.e.

$$\left\| g \right\|_U < \max \left\{ 1, \Theta(\underline{d})^{-\frac{(1+\alpha)}{\alpha^2}} \right\} \varepsilon$$

under which conditions the sequence f_n is uniformly Cauchy on the domain $U(\underline{d})$, even up to the boundary for $\underline{d} > \underline{0}$, and converge to a section f with a supremum on the same at worst:

$$\left\| f \right\|_{U(\underline{d})} \leqslant \left\| g \right\|_U + \frac{\rho}{1-\rho}, \quad \rho = \left(\left\| g \right\|_U \Theta(\underline{d})^{\frac{1+\alpha}{\alpha^2}} \right)^{e \log(1+\alpha)} \qquad \square$$

From the point of view of practicality, and verification let us make,

I.3.6 Remarks The condition $\alpha > 0$ is essential. The curious reader may usefully follow through the proof in the case $\alpha = 0$, and will see that with the given partition $t_0, \ldots, t_{n+1}$ as chosen, $\|Q^n h\|$ blows up as a power of $n!$, which is what should happen.

Furthermore, as indicated post I.3.3, a form of θ which will often occur in practice is,

$$\theta(d_1, \ldots, d_p) = C(d_1 \cdots d_p)^{-N}$$

for some suitable constants C and N. Consequently, $\Theta(\underline{d})$ is of exactly the same form, albeit for a possibly different constant C. Whence to have a solution on $U(\underline{d})$, $\left\| g \right\|_U$ must satisfy,

$$\left\| g \right\|_U < \varepsilon C (d_1 \cdots d_p)^N$$

again for possibly different constants C and N. To achieve this estimate may require some preparation. For example, consider $p = 1$, U a product of polydiscs of radius R, and $U(d)$ a product of polydiscs of radius $R - d$, $d < R$, and suppose furthermore that K is unchanged as we vary R to r (a, so to speak, good case such as an integral operator based at the origin, or power series solutions). In this scenario, we may usefully imagine that g vanishes to a certain order M at the origin, so that on $U(R-r)$ we will have an estimate of the form,

$$\|g\|_{U(R-r)} \leqslant \frac{Cr^M}{(R-r)^M}\|g\|_U$$

for some other constants C and M. As such we'll get a solution on $U(R-r+d)$ as soon as,

$$d^N > \frac{C}{\varepsilon}\frac{r^M}{(R-r)^M}\|g\|_U$$

and since $d < r$ there are limitations. Nevertheless, $d = r/2$, say, r sufficiently small, and $M > N$ will work fine.

I.4 An Example

While fully analytic solutions are quite far from what one may reasonably anticipate for the general singular analytic PDE, they nevertheless provide a useful example of how the conditions of the implicit function theorem apply in practice. To this end it is often both useful, and even necessary, to prepare the situation somewhat by way of a polynomial approximation, and so we observe:

I.4.1 Triviality Let F be a complete topological vector space, and $P : E \to F$ a continuous map from another t.v.s. E such that:

(a) P sends $0 \mapsto 0$ and is differentiable at the origin.

(b) The functional derivative at zero has a continuous right inverse K.

(c) There is a co-final system of open linear sub-spaces F^p of $0 \in F$, $p \in \mathbb{N}$, such that for some $\alpha > 0$ the operator, $Q \doteqdot PK - \mathbb{1} : F \to F$ satisfies $Q(F^p) \subset F^{p+\alpha}$ for all $p \in \mathbb{N}$.

Then under these hypothesis, for $g \in F^1$, the equation,

$$P(f) = g$$

has a solution $f \in E$.

Proof. As ever, we seek a fixed point of the operator:

$$Tf \doteqdot g - Qf$$

by the way of the iteration scheme, $f_{n+1} = Tf_n$, $f_0 = 0$. Whence by hypothesis, $f_1 = g \in F^1$, and,

$$f_{n+1} - f_n = (-1)^n Q^n (f_1 - f_0) \in F^{1+n\alpha}$$

Consequently, since the F^p are linear, f_n is Cauchy with some limit f_∞, and $f = Kf_\infty$ is the desired solution. □

An obvious case to which we may directly apply this observation is to local rings $\mathcal{O}$ complete in the $\mathfrak{m}$-adic topology of its maximal ideal, with $\dim_k \mathfrak{m}/\mathfrak{m}^2 < \infty$, where the residue field k is of any characteristic. In such situation a differential operator must necessarily be of finite order, so, say, for simplicity order 1 and polynomial in its derived arguments. Whence for $\underline{P} : \mathcal{O}^{\oplus n} \to \mathcal{O}^{\oplus m}$ we may write,

$$\underline{P}(f_j) = \sum_I \underline{P}_I(f_j)\,(\partial_p f_q)^I$$

for $\underline{P}_I(f)$ some vectors in formal functions, I some finite number of multi-indices (i_{pq}), $1 \leqslant q \leqslant n$, and ∂_p some finite number of generators in $\mathrm{Der}(\mathcal{O})$. Consequently the functional derivative in 0 is,

$$\underline{P}'(0)(\underline{\rho}) = \sum_{j=1}^{n} \frac{\partial \underline{P}_0}{\partial f_j}(0)\rho_j + \sum_{p,q} P_{pq}(0)\partial_p \rho_q$$

and of course, we hypothesise that a continuous right inverse K exists. In practice K will preserve the $\mathfrak{m}$-adic filtration, but let's hypothesise that it's a little bit worse, e.g. $K(\mathfrak{m}^N \mathcal{O}^{\oplus m}) \subseteq \mathfrak{m}^{N-\beta}\mathcal{O}^{\oplus n}$, for all $N \geqslant \beta$. Observe furthermore that,

$$f_q \in \mathfrak{m}^N \Longrightarrow \partial_p f_q \in \mathfrak{m}^{N-1} \Longrightarrow (\partial_p f_q)^I \in \mathfrak{m}^{(N-1)|I|}$$

for $|I|$ the sum of the i_{pq}'s, and $(N-\beta-1)|I| \geqslant N+1$ for all $N \geqslant 2\beta+3$ provided $|I| \geqslant 2$. The terms in 0 and 1 not inside the functional derivative satisfy slightly better estimates ($N \geqslant 2\beta+1$, and $2\beta+2$ respectively), whence for $Q = PK - \mathbb{1}$, and $N \geqslant 2\beta+3$, $Q(\mathfrak{m}^N \mathcal{O}^{\oplus n}) \subseteq \mathfrak{m}^{N+1}\mathcal{O}^{\oplus n}$. Consequently we have formal solutions of the equation $\underline{P}(f) = g$ as soon as $g \in \mathfrak{m}^{2\beta+3}\mathcal{O}^{\oplus m}$. Whether or not we can do better, simply depends on the equation. It is, however, a finite dimensional obstruction, accessible to finite dimensional linear algebra, whose solution is a fortiori necessary for the existence of analytic solutions. Consequently let us summarise this discussion by the way of,

I.4.2 Definition/Summary Let $\underline{P}$ be a differential operator between analytic vector bundles of the shape envisaged in I.3.4, then we say that the equation $\underline{P}(f) = g$ is formally un-obstructed at a point where $\underline{P}$ sends $0 \mapsto 0$, if after completion in the maximal ideal $\mathfrak{m}$ there is a formal solution f. Furthermore under the hypothesis that the functional derivative admits a right inverse satisfying $K(\mathfrak{m}^N F) \subseteq \mathfrak{m}^{N-\beta}E$, $\beta > 0$ for all N (and actually less stringent conditions continue to work), this is true as soon as $g \in \mathfrak{m}^{2\beta+3}F$, at worst. As such, the obstruction to the existence of formal solutions for an arbitrary $g \in F$ is a finite problem in finite dimensional linear algebra, viz: existence of a fixed point of

$$T : F/\mathfrak{m}^{2\beta+3} \longrightarrow F/\mathfrak{m}^{2\beta+3} \;:\; f \longmapsto g - Qf.$$

Indeed when we have such a point, say f_0, then we start the sequence occurring in the proof of I.4.1 at f_0, and this gives, $f_1 - f_0 \in \mathfrak{m}^{2\beta+3}$, so, we again get formal solutions. Similarly, if the right inverse K is actually the completion of that envisaged in I.3.5, then again we can start the proof at f_0, or some even higher iterate, so that $f_1 - f_0 \in \mathfrak{m}^N$ for N as big as we like. Consequently the conditions of I.3.5 for $\|g\|$ to be appropriately small can be weakened to a smallness condition for $f_1 - f_0$, which will be achievable, as explained in I.3.6, by the simple expedient of taking N sufficiently large and restricting to an appropriately small neighbourhood.

With these preliminaries, we need a further definition before getting to an application, viz:

I.4.3 Definition/Revision Let ∂ be a holomorphic vector field vanishing at a point p. Then Leibniz's rule yields a linear mapping

$$\partial \in \mathrm{End}\left(\frac{\mathfrak{m}(p)}{\mathfrak{m}(p)^2}\right)$$

and we say that ∂ is without resonance if for $\lambda_1, \ldots, \lambda_n$ the eigenvalues and $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n_{\geqslant 0}$, $\sum j_k > 0$ or $j_i = -1$, $j_k \in \mathbb{Z}_{\geqslant 0}$, $k \neq i$, $\sum j_k + j_i > 1$, we have:

$$J \cdot \Lambda \doteqdot \sum j_k \lambda_k \neq 0$$

We say further that the Λ satisfy the Siegel's condition if,

$$|J \cdot \Lambda| \geqslant C\,|J|^{-N}$$

for $|J| = \sum |j_k|$ and constants C, N independent of J.

Now let's consider a scalar valued differential operator of finite order $f \mapsto P(f)$, with the polynomial in the derived variables restriction of I.3.4. While the order is arbitrary, we'll suppose its functional derivative has order at most 1, with -1 part not just any vector field, but one satisfying Siegel's condition. Under this condition it is known (in fact it can be proved by the implicit function theorem, but more is known, so we'll come back to this) that there are analytic coordinates in which ∂ may be expressed as

$$\partial = \lambda_i x_i \frac{\partial}{\partial x_i}$$

where we suppose the ambient space smooth and of dimension n, and employ the summation convention Consequently after multiplying $P(f)$ by a unit, we can suppose that the functional derivative is constant linear, viz:

$$P'(0)\rho = \chi\rho + \partial\rho$$

and we extend Siegel's condition in the obvious way, i.e. for $J \in \mathbb{Z}^n_{\geqslant 0}$ we require,

$$|\chi + J \cdot \Lambda| \geqslant C\,|J|^{-N}$$

Consequently, $P'(0)$ has a right inverse by power series, K, defined on all functions if $\chi \neq 0$, or only on the maximal ideal $\mathfrak{m}$ at the origin if $\chi = 0$. In either case K preserves powers of the maximal ideal ($\beta = 0$ in I.4.2) and is equally good for applying I.3.5 and I.4.1 at the same time. Consequently we assert,

I.4.4 Claim Let $f \mapsto P(f)$ be a finite order differential operator with the minor caveats of I.3.4, and suppose the functional derivative in 0 (as ever $P(0) = 0$) is of 1^{st} order with both the said derivative and its -1 part satisfying Siegel's condition, then if the equation $P(f) = g$ is formally un-obstructed, it has an analytic solution f on a sufficiently small neighbourhood of the origin.

Proof. If $h = \sum h_J x^J$ is the Taylor expansion of an arbitrary h, then

$$Kh = \sum \frac{h_J}{(\chi + J \cdot \Lambda)} x^J$$

$h \in \mathfrak{m}$, if $\chi = 0$. Whence, in the notations of I.3, supposing our coordinates x_i defined on unit polydisc, Δ^n, for $\underline{d} > \underline{e}$,

$$\|Kh\|_{\Delta^n(\underline{d})} \leqslant C \left(\prod_{i=1}^{n} \frac{1 - e_i}{d_i - e_i} \right)^N \|h\|_{\Delta^n(\underline{e})}$$

for N as per the Siegel condition, and an appropriate constant C. The shape of ϕ is as per I.3.4, ψ is as above, so a little better than ϕ since the $1 - e_i$ help, but ultimately we cannot say that θ is any better than the shape discussed in I.3.6. As per op.cit. we simply take all the d_i, resp. e_i equal, i.e. reduce to the case $p = 1$ in the notations of I.3.2, and as envisaged in I.3.6, and discussed in I.4.2, start the recurrence in the implicit function theorem at some f_0 such that $f_0 = (\mathbb{1} - Q) f_0$ to a sufficiently large power of the maximal ideal, viz: $nN + C$, for an appropriate C determined by the order of derivatives in P, e.g. 3 for order 1. ▫

One would suspect that in this generality the Siegel condition is best possible. On the other hand it is known, [Brj71], that vector fields can be linearised under a weaker Diophantine condition, which calls for:

I.4.5 Scholion (Linearisation of fields) This is almost I.4.4, but actually the vector valued version with a functional derivative $A + \partial$, where A is some constant matrix, and the non-linearity has no derivatives. In this case one can get a better result by profiting from the $1 - e_i$ term that appeared (and was approximated by 1) in the proof of I.4.4. To keep the notations of I.3.2 one should therefore work with the logarithm of the distance to the boundary, and introduce a function,

$$b(t) = \sum_Q e^{-t\,|Q|} (\Lambda \cdot Q)^{-1}$$

or $\Lambda \cdot Q - \lambda_i$ as appropriate, $t > 0$, so that,

$$\|Kf\|_{\Delta(d)} \leqslant b(d - e) \, \|f\|_{\Delta(e)}$$

for K a suitable right inverse. The function ϕ can be taken to be 1, so,

$$\log \Theta(d) = \int_0^1 \log b(td) dt$$

and appropriate smallness is $\Theta(d)$ no worse than $(1-e^{-d})^N$ for some N. Exactly what the relation between this condition and that of Brjuno may be is far from clear, but it's certainly better than Siegel's. Unfortunately, however, the linearisation of fields problem doesn't immediately reduce to this equation, and requires some preparation. More precisely one wants to find an automorphism that conjugates a field D with appropriate eigenvalues into the standard field ∂. To ensure that solving some PDE in functions yields an automorphism requires (proceeding by way of an inductive statement on the dimension for the said linearisation) preparation, e.g. finding a smooth invariant surface, and the equation for doing this has derivatives in its non-linearity, i.e. in principle one is in the situation of I.4.4 rather than the very particular operator that we introduced at the start of I.4.5. Furthermore, even once one does this, the functional derivative of the resulting conjugation equation is of the form $A + \partial$ for A a matrix of functions, rather than a constant matrix. This has to be conjugated to a constant matrix, and again, the equation for doing this (whose functional derivative is of the form $q \mapsto [A(0), q] + \partial$ in gl_n) has derivatives in the non-linearity, so we still cannot profit from equations of the special form where we can do better than Siegel.

Consequently, in brief: the implicit function theorem I.3.6 yields linearisation of fields under Siegel's condition. It could do better if one could arrange a series of steps that allow the linearisation via a series of equations of the special form introduced above. A priori a series of such steps is not so clear, and perhaps, this problem with its special symmetry is best studied directly.

I.5 Monomialisation

By way of a further example, or an example within an example, let us examine a more general problem than I.4.5, i.e. final forms for vector fields rather than foliations. More precisely, in the presence of resolution of singularities, an arbitrary field may eventually be resolved to one where the implied foliation has canonical singularities , and, should the field be non-saturated, vanishing along a simple normal crossing divisor. A particular instance is, therefore,

I.5.1 Set up (a) Let ∂ be a vector field at the origin in $\mathbb{C}^n$ of the form,

$$u\, x_1^{p_1} \cdots x_n^{p_n} \Big(\sum_i \lambda_i x_i \frac{\partial}{\partial x_i} \Big), \qquad p_i \in \mathbb{Z}_{\geqslant 0}$$

Conditions for linearisation, holomorphic, or formal, have been discussed in I.4.5, we'll certainly, therefore, suppose that there are no resonances, so, the discussion is the shape of the unit. To investigate it, let us consider attempting to eliminate it by way of a coordinate change, $\xi_i = e^{f_i} x_i$, so we'd need to solve,

$$e^{-p_j f_j} \big(\lambda_i + D f_i \big) = \lambda_i u^{-1}$$

where we apply the summation convention, $D = \lambda_i x_i \frac{\partial}{\partial x_i}$, and we harmlessly suppose that $u(0) = 1$. An appropriated vector valued operator is, therefore,

$$\mathcal{L}(f)_i = e^{-p_j f_j} \big(\lambda_i + D f_i \big) - \lambda_i$$

which has functional derivative,

$$\big(Lf\big)_i = Df_i - \lambda_i\big(p_j f_j\big)$$

Now the matrix $\lambda_i p_j$ has rank 1, whence $n-1$ zero eigenvalues, all of which admit a corresponding eigenvector. Under the no-resonance condition, there is also a non-zero eigenvalue $\lambda_j p_j$, and this matrix is diagonalisable. Consequently, in notation of I.4.3, we encounter obstructions to the invertibility of L as soon as $J \cdot \Lambda = P \cdot \Lambda$, $P = (p_1, \ldots, p_n)$. So certainly, $J = P$ is an obstruction and we suppose,

I.5.1 Set up (b) Suppose the only obstruction is at P, viz: notations as above and as per I.4.3, $J \cdot \Lambda = P \cdot \Lambda \Rightarrow J = P$.

The operator $\mathcal{L}$ preserves the maximal ideal, as does the obvious right inverse K by power series provided it is defined, so we can conclude after a finite number of iterations à la I.4.1 that,

$$u = (1 - \nu x_1^{p_1} \cdots x_n^{p_n}) \qquad \mathrm{mod}\ \mathfrak{m}^{|P|+1}$$

for some $\nu \in \mathbb{C}$. After which the operator is no longer obstructed, so let's aim to solve,

$$P(f)_i = \lambda_i(u^{-1} - 1 - \nu x_1^{p_1} \cdots x_n^{p_n})(\lambda_i + Df_i) - \lambda_i(1 + \nu x_1^{p_1} \cdots x_n^{p_n})$$

which respects the filtration by the maximal ideal, and has functional derivative,

$$\big(Lf\big)_i = (1 + \nu x_1^{p_1} \cdots x_n^{p_n})Df_i - \lambda_i(p_j f_j)$$

If we have Siegel or Bryuno's conditions then we can find analytic functions $y_1, \ldots, y_n$ such that,

$$(1 + \nu x_1^{p_1} \cdots x_n^{p_n})D = \lambda_i y_i \frac{\partial}{\partial y_i}$$

so we get a right inverse on $\mathfrak{m}^{|P|+1}\mathcal{O}^{\oplus n}$, K, by power series, admitting bounds of the shape considered in I.4.4 under, say, Siegel's condition, and $Q = PK - \mathbb{1}$ respects the filtration by the maximal ideal. Now we're in the situation of I.4.2, where starting the induction for finding a fixed point, $f \mapsto \lambda_i(u^{-1} - 1 - \nu x_1^{p_1} \cdots x_n^{p_n}) - (Qf)_i \doteqdot Tf$ at $T_0 \in \mathfrak{m}^{|P|+1}\mathcal{O}^{\oplus n}$, and the iterates stay in the un-obstructed space, whence:

I.5.2 Summary/Fact/Example Suppose the set up I.5.1(a)-(b), then there is a formal change of coordinates bringing ∂ into the form:

$$\frac{x_1^{p_1} \cdots x_n^{p_n}}{1 + \nu x_1^{p_1} \cdots x_n^{p_n}} \Big(\lambda_i x_i \frac{\partial}{\partial x_i}\Big), \quad \nu \in \mathbb{C}$$

Moreover, in the presence of Siegel's condition this can even be done analytically.

II Integrable Forms

II.1 Smooth Integral

We'll proceed to consider a series of more and more difficult cases, which serve equally as a series of examples in the use of the implicit function theorem. Consequently we begin with the easiest possible case of a functional derivative of order 1 (as will be the case throughout this chapter) whose -1 part has a smooth first integral, and an invariant divisor, i.e.

II.1.1 Set Up Let x_i, $1 \leqslant i \leqslant n$, y be standard coordinates in some polydisc $\Delta^n \times \Delta$, and introduce fields $D_i = x_i \frac{\partial}{\partial x_i}$, $\partial = x_1^{p_1} \cdots x_n^{p_n} \frac{\partial}{\partial y}$, $p_i \in \mathbb{N}$. Now consider a differential operator $f \mapsto P(f)$ of finite order with the proviso that all the differentials that occur are monomials in D_i and ∂, and, as per I.3.4, it is polynomial in the same. Finally suppose that the operator sends $0 \mapsto 0$ and the functional derivative in 0 is,

$$D \doteqdot P'(0) \ : f \longmapsto (\mathbb{1} + \partial)(f)$$

Rather special cases of this may be found in the literature, essentially, $(\mathbb{1} + \partial)(f) = a(f)$, for a function of f alone, c.f. [Was85]. However, for essentially the same methodology, the implicit function theorem gives much more. To this end fix sectors S_i in the x_i-variables such that the total width of the sector S, image of: $S_1 \times \cdots \times S_n \mapsto x_1^{p_1} \cdots x_n^{p_n} \in S$ has an aperture $\sigma < \pi$, and consider a domain in the y-variable of the form shown in figure II.1.1

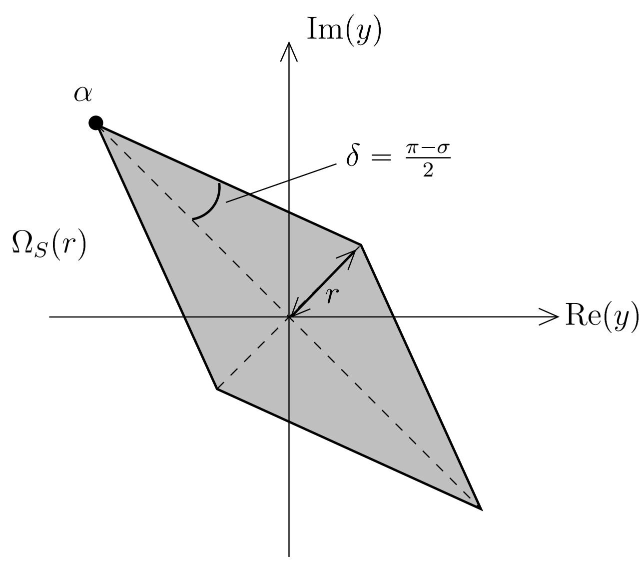

Figure II.1.1

where $|\alpha| = 1$, and $r > 0$ are chosen such that for $x \in S$,

$$\min\left\{\mathrm{Re}\left(\frac{y}{x}\right) \ : \ y \in \Omega_S(r)\right\} = \mathrm{Re}\left(\frac{\alpha}{x}\right)$$

and this minimum is realised uniquely at α/x.

Plainly we are in the situation envisaged by I.3.3, where the natural coordinates are $\eta = yx_1^{-p_1}\cdots x_n^{-p_n}$, $e^{\xi_i} = x_i$, ξ_i belonging to some appropriate strips, $\mathrm{Re}(\xi_i) < 0$, $\mathrm{Im}(\xi_i) \in I_i$, which we continue to denote by S_i. We therefore have a domain $\pi : U \to S_1 \times \cdots \times S_n$, and $\Omega_S(r)$ has been chosen to ensure the following properties:

II.1.2 Remarks For any $x_i \in S_i$, there is a constant $C > 0$, depending only on r such that $\alpha x_1^{-p_1}\cdots x_n^{-p_n}$ (which, by the way, is a holomorphic section of π) can be joined by a path $\gamma : [0,1] \to U_{\underline{x}}$ in the fibre over $\underline{x}$ satisfying,

$$\|\dot{\gamma}(t)\| \leqslant C\frac{\partial}{\partial t}\mathrm{Re}\big(\gamma(t)\big)$$

where in an abus de langage, we identify the fibre with its conformal image under η in $\mathbb{C}$. Furthermore, this remains true with the same constant, under any scaling $\Omega_S(r) \to \lambda\Omega_S(r)$, $\lambda \in \mathbb{R}_{>0}$. Finally, and manifestly, $U \supseteq \Delta_y \times S_1 \times \cdots \times S_n$, for Δ_y a sufficiently small disc in the original coordinates.

Having thus introduced our domains we observe:

II.1.3 Triviality The operator D restricted to $\Gamma(U)$ has a bounded right inverse K satisfying,

$$|Kg|(p) \leqslant C\,\|g\|_U$$

for any $p \in U$, and C the constant appearing in II.1.2.

Proof. Given a bounded function g on U define, over a fibre $\underline{x}$,

$$(Kg)(\underline{x},\eta) = e^{-\eta}\int_{\alpha x_1^{-p_1}\cdots x_n^{-p_n}}^{\eta} e^{\rho}g(\underline{x},\rho)\,d\rho$$

and apply the properties of U enunciated in II.1.2. ▫

The triviality of II.1.3 not withstanding, there are a couple of rather complicated issues lurking under the surface. In the first place K is defined using a section of $\pi : U \to S_1 \times \cdots \times S_n$ with values in the boundary, whence it does not solve a Cauchy problem in the sense of I.3.2(a), and the implicit function theorem cannot be immediately applied. However the operator P was taken not with arbitrary occurrences of derivatives in the y variable, but only in powers of ∂. Consequently if we consider the domain $U(\underline{\delta}) \in \mathbb{C}^{n+1}$, as per I.3.3, in the ξ_i and η variables, with the $(n+1)^{th}$-entry in η, we can take $\delta_{n+1} > 0$ and use the base points $\alpha x_1^{-p_1}\cdots x_n^{-p_n} - \delta_{n+1}$. Of course this may fail to be in U as it is actually defined, but we can remedy this by attaching a small disc around $\alpha x_1^{-p_1}\cdots x_n^{-p_n}$ in each fibre of radius $2\delta_{n+1}$, and by another abus de langage we'll continue to denote this domain by U (or possibly U_R, R the radius of the disc if there is a risk of confusion). In any case we now have a solution to a Cauchy problem for all domains $U(\underline{d})$ between $U(\underline{\delta})$ and U with a worse constant than II.1.3, viz.: as before plus 1 plus e^{2R}. Now we can apply the implicit function theorem to obtain:

II.1.4 Corollary Let $f \mapsto P(f)$ be as per the set up II.1.1, U_R, R fixed $> \delta_{n+1}$ as above, then for some absolute constant ε depending only on $\underline{\delta}, \sigma, r, R$, whence, in particular independent of shrinking the radii of the initial polydiscs in the x_i (and even y), the equation

$$P(f) = g$$

has a solution on $U_R(\underline{\delta})$, for $\|g\|_{U_R} < \varepsilon$. In particular, if g on our original $\Delta^n \times \Delta$ vanishes at the origin, then on a domain of the form $S_1' \times \cdots \times S_n' \times \Delta'$ for S_i' of small radii but aperture as close to S_i as we like, and Δ' a sufficiently small disc in y, we have a solution.

Before progressing, let us observe,

II.1.4 (bis) Remark One can, of course, for essentially no extra cost introduce extra variables z_k in some polydisc, together with fields $\frac{\partial}{\partial z_k}$, $1 \leqslant k \leqslant m$, and more generally $f \mapsto P(f)$ with monomials in the D_i, ∂, $\frac{\partial}{\partial z_k}$ and polynomial in the same. The only thing that changes is that ε is no longer absolute but becomes $\varepsilon C(d_1 \cdots d_m)^N$, for some constants C and N. Consequently one falls into the situation envisaged in I.3.6, and one should hypothesise again that g vanishes at the origin, then shrink the radii in the x_i, y (in practice the x_i alone would do, since there's usually some analytic blow up that's permitted) at a suitably quick rate to compensate for the lack of smallness occasioned by the new variables.

Beyond this, the matter of not being able to go beyond π in aperture is an extreme misfortune, but unfortunately it appears to be best possible. To consider the matter in more detail, let us suppose for convenience that $p_1 = 1$, $n = 1$, and change coordinate $x \mapsto -x$. In the first instance we consider the formal solution of:

$$\left(\mathbb{1} - x\frac{\partial}{\partial y}\right)(f) = g$$

in the completion of $\mathcal{O}$ in the ideal $x = 0$, viz:

$$f = \sum_{n=0}^{\infty} x^n \frac{\partial^n g}{\partial y^n}$$

Now, if for simplicity we suppose that $g = xh(y)$, then the Borel transform of f is,

$$\widehat{f}(\xi, y) = \oint_\gamma e^{\xi z} f\left(\frac{1}{z}, y\right) dz$$

where the contour is around a neighbourhood of ∞, so that:

$$\widehat{f}(\xi, y) = h(y + \xi)$$

Whence for h enjoying a natural boundary on say a disc, the domain of convergence of $\hat{f}$ is rather small. As such,

II.1.5 (a) Fact The equation $Df = g$ can not in general be solved analytically in both x and y, i.e. we certainly cannot replace a sector in II.1.3 by a disc.

Proof. Otherwise $\hat{f}(\xi, y)$ is entire in ξ, and even of exponential growth. □

However the situation is even worse than this, since:

II.1.5 (b) Nastier fact The equation $Df = g$ can not in general be solved in a domain of the form $\{|x - \rho| < |\rho|\} \times \Delta_y$ (so a bit smaller than a sector of width π) while satisfying the estimates:

$$f(x, y) = \sum_{m=0}^{n-1} a_m(y)x^m + R_m(x, y); \quad |R(x, y)| \leqslant C^n n!|x|^n$$

for every n, where $a_m(y)$; $R(x, y)$ are analytic and C is some constant.

Proof. This is Watson's theorem, or more accurately a strengthening thereof by Alan Sokal, [Sok80], which says that under these conditions $\hat{f}(\xi, y)$ has analytic continuation to $V \times \Delta_y$ where V is a small neighbourhood of $\mathbb{R}_+\rho$. □

Now, while it might be objected that such an asymptotic expansion will not occur, this is by no means so. Indeed:

II.1.5 (c) Worse still Let Ω be a domain of the form $\{|x - \rho| < |\rho|\} \times \Delta_y$ then there is no bounded operator from the space of holomorphic functions on Ω in the supremum norm which affords a right inverse to D.

Proof. Suppose otherwise, and let K be such an operator. Now put $\partial = x\frac{\partial}{\partial y}$, and consider solving the equation,

$$Df = (\mathbb{1} - \partial)(f) = g$$

by the way of an expansion,

$$f_n = \sum_{m=0}^{n-1} (\partial^m g) + x^n R_n$$

Then we require to solve:

$$(\mathbb{1} - \partial)R_n = \frac{\partial^n g}{\partial y^n}$$

so by I.1.6, on a suitably small disc in y, independent of n, the right hand side is certainly of the form $C^n n!\|g\|$, for some norm taken on some slightly bigger disc on which g is hypothetically defined. Consequently if K exists, we'll contradict II.1.5(b) as soon as we know that the solutions f_n do not depend on n. This is, however, the case since if f_n, f_m are two different solutions so constructed, then their difference satisfies

$$|f_n - f_m| \leqslant C_{mn}$$

for some constant depending on m and n, while,

$$f_m - f_n = e^{-y/x} F_{mn}(x)$$

for $F_{mn}(x)$ analytic in $\{|x-\rho| < |\rho|\}$. Since this holds for all y in a disc,

$$|F_{mn}(x)| \leqslant C_{mn} e^{-c/|x|}$$

and the domain of x is large enough to imply that F_{mn} is zero, c.f. [Mal91], IX.4.5. ▫

II.1.5 (d) Hopeless situation For a domain Ω of the form $\{|x-\rho| < |\rho|\} \times \Delta_y$, or $S \times \Delta_y$ a sector of aperture beyond π, there is no adequate right inverse to D to which the implicit function theorem can be applied.

Proof. The hypothesis of the theorem require that at the very worst there are subdomains $\Omega' \subset \Omega'' \subset \Omega$ of the same form such that the right inverse is a bounded map of Banach spaces,

$$\Gamma_\infty(\Omega'') \longrightarrow \Gamma_\infty(\Omega')$$

of bounded holomorphic functions. A condition which is hardly unreasonable, but impossible to fulfil by the way of the trivial change of the domain of definition of f_n in II.1.5(c) from Ω to Ω'. ▫

Regrettably, therefore, II.1.3-4 is really best possible, and the solutions occur in a region where they are highly non-unique because its aperture is too small. Consequently if one were thinking of these domains as neighbourhoods of a real blow up in the divisor $x = 0$, then one has to abandon all hope of patching as one moves along the divisor in the y-direction.

II.2 Almost smooth

We continue our progression by small steps, and so consider a case not far from the previous one, viz:

II.2.1 Set Up Again let x_i, $1 \leqslant i \leqslant n$, y be standard coordinates in some polydisc, and introduce fields $D_i = x_i \frac{\partial}{\partial x_i}$ $\partial = x_1^{p_1} \cdots x_n^{p_n} y \frac{\partial}{\partial y}$, $p_i \in \mathbb{N}$. Now consider a differential operator $f \mapsto P(f)$ of finite order with all derivatives monomial in ∂ and the D_i, and as per I.3.4, polynomial in the same. Finally suppose P sends $0 \mapsto 0$ and the functional derivative is,

$$D \doteqdot P'(0) \ : f \longmapsto (\mathbb{1} + \partial)(f)$$

Here we can usefully exploit the freedom that the implicit function theorem provides in the construction of a right inverse. Specifically let $S_1, \ldots, S_n$ be any sectors in x_i such that the image S of $x_1^{p_1} \cdots x_n^{p_n}$ does not contain the negative real axis, and g a bounded function analytic on $S_1 \times \cdots \times S_n \times \Delta$ for Δ a disc in y. As such, instead of trying to solve,

$$Df = g$$

we can aim to solve,

$$Df = -\partial g$$

which has the trivial marginal cost of obliging us to restrict to discs $\Delta(d) \subset \Delta$ of points in discs a distance d from the boundary on which,

$$\|\partial g\|_{S_1\times\cdots\times S_n\times\Delta(d)} \leqslant d^{-1}\|g\|_{S_1\times\cdots\times S_n\times\Delta}$$

Similarly writing $-\partial g = x_1^{p_1}\cdots x_n^{p_n} h$, it's even true that

$$\|h\|_{S_1\times\cdots\times S_n\times\Delta(d)} \leqslant d^{-1}\|g\|_{S_1\times\cdots\times S_n\times\Delta}$$

and we can expand h in power series,

$$\sum_{k=1}^{\infty} y^k h_k(\underline{x})$$

with coefficients bounded by,

$$\|h_k(\underline{x})\|_{S_1\times\cdots\times S_n\times\Delta(d)} \leqslant r^{-k} d^{-1}\|g\|_{S_1\times\cdots\times S_n\times\Delta}$$

for r the radius of Δ. Consequently we must solve the equations,

$$(\xi + k) f_k(\underline{x}) = h_k(\underline{x})$$

for $\xi = x_1^{-p_1}\cdots x_n^{-p_n}$ belonging to an unbounded region contained $+\infty$ of the form shown in figure II.2.1; which is trivially possible up to a constant C depending on the aperture of the sector, which gives a solution of the original equation $Df = g$, by the way of an operator $g \mapsto Kg$ satisfying,

$$\|Kg\|_{S_1\times\cdots\times S_n\times\Delta(d)} \leqslant C\frac{(r-d)}{d^2}\,\|g\|_{S_1\times\cdots\times S_n\times\Delta}$$

So more generally for any $\delta > 0$ (on which K does not depend), and $\delta \geqslant d > e \geqslant 0$, we have a solution to a Cauchy problem in the sense of I.3.2(a) which satisfies,

$$\|Kg\|_{S_1\times\cdots\times S_n\times\Delta(d)} \leqslant C\frac{(r-d)}{(d-e)^2}\,\|g\|_{S_1\times\cdots\times S_n\times\Delta(e)}$$

and, better still, going away from the boundary of the S_i only improve this bound, whence:

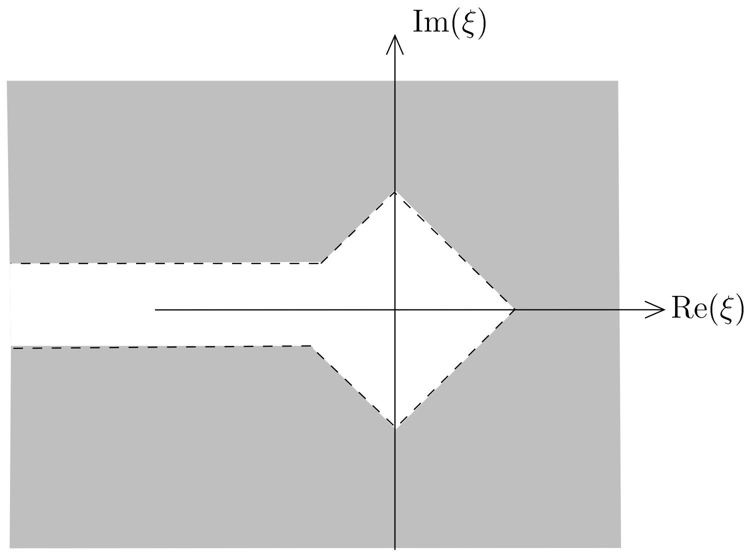

Figure II.2.1

II.2.2 Fact Let $f \mapsto P(f)$ be as per the set up II.2.1, with $U = S_1 \times \cdots \times S_n \times \Delta$, then for some constants C and N, the equation,

$$P(f) = g$$

has a solution on $U(d)$ provided,

$$\|g\|_U \leqslant \min\left\{1, C(d_1 \cdots d_{n+1})^N\right\}$$

Consequently for g on the original polydisc $\Delta^n \times \Delta$, either vanishing on some divisor $x_i = 0$ (as will be the case in applications) or to sufficiently high order in y (so that we can apply the considerations of I.3.6) we find a solution in a region of the form $S_1' \times \cdots \times S_n' \times \Delta'$ with S_i' sectors of sufficiently small radii but aperture as close to that of S_i as we like, and Δ' an appropriately small disc in y.

Furthermore, the discussion of II.1.4(bis) applies mutatis mutandis on introducing further variables z_k, $1 \leqslant k \leqslant m$, and fields $\frac{\partial}{\partial z_k}$. Indeed it's even true under our immediate restriction that the image S of $S_1 \times \cdots \times S_n$ does not contain $x_1^{p_1} \cdots x_n^{p_n}$ negative real that one can replace the field ∂ by $\frac{\partial}{\partial y}$ provided, and wholly necessarily, the functional derivative of P remains,

$$f \longmapsto f + x_1^{p_1} \cdots x_n^{p_n} y \frac{\partial}{\partial y} f$$

Unfortunately, but manifestly, we can not do better if we wish to insist on full analyticity in y. In fact, this difficulty has notable geometric content. For example, consider the 3-dimensional vector field,

$$\partial \doteqdot \left(z - \frac{y}{1-y}\right)\frac{\partial}{\partial z} - xy\frac{\partial}{\partial y}$$

Then under completion in $x = 0$, this field has a formal centre manifold. Under completion in $y = 0$, however, this situation is much more subtle. If we seek a centre manifold as a graph, $z - \zeta(x, y)$, then we see that we require to solve the equation,

$$(\mathbb{1} + xy\frac{\partial}{\partial y})(\zeta) = \frac{y}{1-y}$$

which we know to be obstructed for $x = -k^{-1}$, $k \in \mathbb{N}$, and writing $x = -(t + 1/k)/k$ for t a local coordinate at $-1/k$, we see that the semi-simple part of the Jordan form at $(-k^{-1}, 0, 0)$ has eigenvalues 1 and $1/k$, and for some suitable coordinates Z, Y normal to the singular locus the normal form is,

$$\partial = \left(Z + a_k(t)Y^k\right)\frac{\partial}{\partial Z} + \frac{Y}{k}(1+t)\frac{\partial}{\partial Y}$$

and for $a_k(0) \neq 0$, indeed it is 1 in the above example, the equation of the centre manifold is

$$a_k(t)Y^k = Zt$$

which is a quotient singularity of order k. Consequently there does not exist a centre manifold as a formal scheme in the completion along $y = 0$. Evidently this is the same phenomenon that requires the exclusion of the axis in II.2.2, and the existence of an obstruction in this case was not unknown, cf. [vS79]. However, op.cit. is missing the point since the obstruction exists even formally, cf. [McQ], §1.5.

This said let us profit from II.2.2 in order to suppose that our sectors S_i are such that the image ξ of $x_1^{p_1} \cdots x_n^{p_n}$ lies in a sector S close to the negative real axis, and put $y = \exp(\eta)$, for η in some half plane $\mathrm{Re}(\eta) < -R$, so that our equation becomes,

$$\left(\mathbb{1} + \xi\frac{\partial}{\partial \eta}\right)(f) = g$$

Which we could content ourselves to solve in strips in the η-plane, but it costs nothing to solve in a so called *spiralling region*, viz: a domain of the form shown in the figure II.2.2, where the lines not perpendicular to the real axis are chosen so that their argument remains strictly between π and $-\pi$ on multiplication by ξ as ξ varies in S. Consequently our problem is exactly as in II.1 where we're obliged to take our base point on the boundary, so again for the Euclidean distance in η/ξ, by which we view the fibres of $\Omega_S(R) \times S_1 \times \cdots \times S_n \to S_1 \times \cdots \times S_n$ as embedded in $\mathbb{C}$, we add a small disc of radius r around the base point in each fibre to get a domain $\pi : U_r \to S_1 \times \cdots \times S_n$ in which we may apply the implicit function theorem in the same way, to obtain:

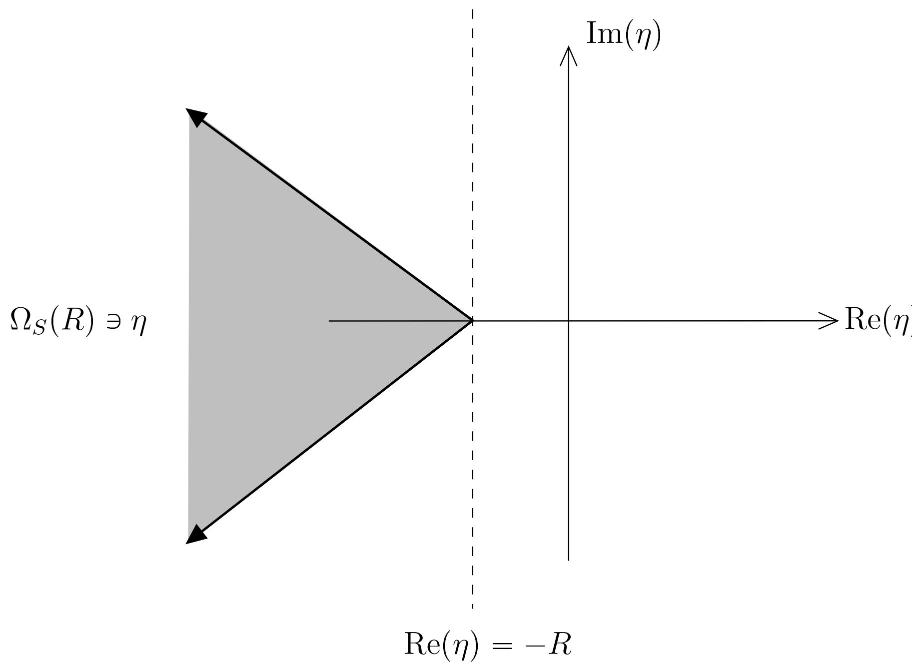

Figure II.2.2

II.2.3 Fact Let $f \mapsto P(f)$ be as per the set up II.2.1, U_r as above, $r > \delta_{n+1}$ (distance from the boundary in the η/ξ variable) then for some absolute constant ε depending only on $\underline{\delta}, \sigma, r, R$, and whence, in particular independent of shrinking the radii of the original polydiscs whether in x_i or y, the equation:

$$P(f) = g$$

has a solution in $U_r(\underline{\delta})$, for $\|g\|_{U_r} < \varepsilon$. In particular, if g on our original $\Delta^n \times \Delta$ vanishes at the origin, then on a domain $\big(S'_1 \times \cdots \times S'_n \times \Omega_S(R')\big)(\delta)$ for $S'_1 \times \cdots \times S'_n$ of smaller radii but with image as close to the original aperture of S as we like, and possibly $R' \ll R$, we have a solution.

Furthermore as per II.1.4(bis), or indeed II.2.2, we can add extra variables z_k, $1 \leqslant k \leqslant m$, and fields $\frac{\partial}{\partial z_k}$ for next to no additional cost beyond shrinking the above regions $\big(S'_1 \times \cdots \times S'_n \times \Omega_S(R')\big)(\delta)$ suitably in the radii of the sectors, or, possibly R'. Unlike II.2.2, however, ∂ cannot be replaced by $\frac{\partial}{\partial y}$ even if the functional derivative remains unchanged.

II.3 Singularly Integrable

The trickiest case to consider in this section brings together all our previous difficulties, and some more, so we'll restrict our attention to dimension 2, viz:

II.3.1 Set Up Let x, y belong to some bi-disc $\Delta \times \Delta$, and in the sense of I.3.4, as already provided for by example in II.1.1 and II.1.2 let $f \mapsto P(f)$ be a differential operator polynomial in the fields $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$. As ever we suppose $0 \mapsto 0$, and we consider the situation in which the functional derivative in 0 is,

$$D \doteqdot P'(0) \ : f \longmapsto \left\{(1+\varepsilon)\mathbb{1} + \frac{(x^p y^q)^r}{q}\Big(qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}\Big)\right\}(f)$$

where ε is a small function of x and y and $p, q \in \mathbb{N}$. By no means trivially the functional derivative has this form for analytic coordinates x and y iff it has such a form formally, c.f. [Brj71]. In any case we denote the -1 part by ∂, and put $s = x^p y^q$.

We first consider the case $\varepsilon = 0$, so that in s, x coordinates our operator is,

$$D = \mathbb{1} + s^r x\frac{\partial}{\partial x}$$

Consequently passing to $x = e^{\xi}$, $y = e^{\eta}$, $\mathrm{Re}(\xi) < -R_X$, $\mathrm{Re}(\eta) < -R_Y$, with $X \ni \xi$, $Y \ni \eta$ the said half spaces, $s : U(= X \times Y) \to B$ fibres over s as shown in figure II.3.1.

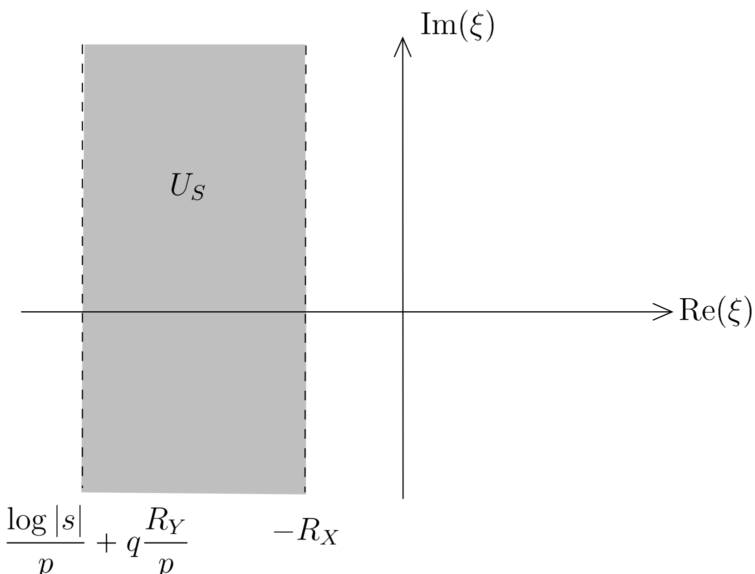

Figure II.3.1

As ever we propose to solve our equation $Df = g$, by the way of the integral operator:

$$(Kg)(s,\xi) = e^{-\xi/s^r}\int_{\mathrm{pt}}^{\xi} e^{t/s^r} g(s,t)\frac{dt}{s^r}$$

where pt is some suitable base point (more accurately section of the fibration s) which must be chosen as a function of the argument of s. Consequently, in the first instance, let $S \ni s$ be a sector of aperture strictly less than π/r which does not contain a r^{th} root of either the positive or negative real axis. Under this hypothesis in the fibrewise variable $\rho = \xi/s^r$, U_s is a region enclosed between two parallel straight lines with slope bounded strictly (as a function of the aperture of S) away from purely imaginary. As such the base point at $-\infty$ (i.e. $\mathrm{Re}(\rho) \to -\infty$) is the good choice, and, for $U_S = \coprod_s U_s$,

$$\|Kg\|_{U_S} \leqslant C(S)\, \|g\|_{U_S}$$

for some constant - basically $\Big(\pi/r - |S|\Big)^{-1}$ - where $|S|$ is the aperture. Unfortunately the definition of $-\infty$ is not continuous as one passes through s^r positive or negative real. More precisely take $S \ni s$ to be a sector around a r^{th} root of the positive or negative real axis of aperture strictly less than π/r, and take $\xi \in X'$, $\eta \in Y'$ to be spiralling domains as encountered already in II.2 then the fibre of $U' = X' \times Y'$ over S in $\rho = \xi/s^r$ takes the form shown in figure II.3.1(b), or alternatively, a similar figure on the right for s^r close to negative

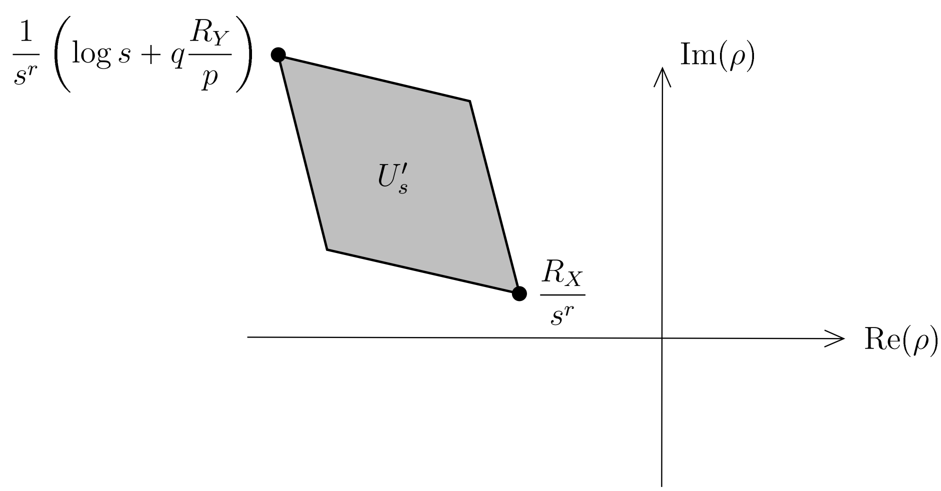

Figure II.3.1(b)

real, where, depending on the point of view the aperture is sufficiently small to allow for large spiralling, or the spiralling is small to permit large aperture. Irrespectively there is a constant $C(U'_S)$ depending on these parameters, and even independent of positive rescaling in ρ, such that throughout $U'_S = \coprod_s U'_s$, if we take as base point the unique point with $\mathrm{Re}(\rho)$ minimal in each fibre (which by the way has holomorphic variation in s),

$$\|Kg\|_{U'_S} \leqslant C(U'_S)\, \|g\|_{U'_S}$$

Unlike the previous case, however, the said base point will change as we shrink U'_S be Euclidean distance as per I.3.3 in the embedding in $\mathbb{C}^2$ by the way of, say, (ρ, η) coordinates, so as per II.1, the thing to do is put a small disc in ρ around the base point. Regardless K is bounded, so for ε of II.3.1 sufficiently small,

$$(\mathbb{1} + \varepsilon K)$$

is invertible, and we have our right inverse to D. Observe furthermore that the mapping $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$: $(z, w) \mapsto (z^p w^q)^r$ is continuous, so the domains on which we have constructed the said inverse, and on which we apply the implicit function theorem, contain open sectors about any values of $\arg(x)$ and $\arg(y)$, i.e.

II.3.2 Corollary Let the operator $f \mapsto P(f)$ be of the form found in the set up II.3.1 then for any values of $\arg(x)$, $\arg(y)$ there are open sectors S, T around the same such that for any sufficiently small holomorphic function on a bi-disc (in fact the domains U_S, or U'_S discussed above would suffice) the equation,

$$P(f) = g$$

has a solution in $S \times T$. Further the definition of sufficiently small does not depend on the radii of the sectors, so if g vanishes at the origin, we necessarily have a solution on sectors of the same shape, but of smaller radii. Again, as per II.1.4(bis), this remains true if P is polynomial not just in $x\frac{\partial}{\partial x}$, $y\frac{\partial}{\partial y}$ but even in some further fields $\frac{\partial}{\partial z_k}$, $1 \leqslant k \leqslant m$, corresponding to additional variables z_k, and, of course, an un-changed functional derivative, and appropriately adjusted definition of sufficiently small radii.

II.4 Integrable and Transverse

By far the most trivial case that we'll have to deal with is the example par excellence that may be found in the literature, but we'll require it uniformly in parameters, viz:

II.4.1 (a) Set up Let $x \in \Delta$, $y \in \Delta^n$, be variables in discs, and $f \mapsto P(f)$ an operator polynomial in the fields $x^{p+1}\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$. As ever, $0 \mapsto 0$ and we suppose that the functional derivative in 0 has the form

$$D \doteqdot P'(0) \ : f \longmapsto \left\{ (1+\varepsilon)\mathbb{1} - \frac{x^{p+1}}{p}\frac{\partial}{\partial x} \right\} (f)$$

for ε a small function of x and y.

As per II.3 we first treat the case $\varepsilon = 0$. To this end take S to be any sector of width $2\pi/p$ which under the conformal mapping $\xi = x^{-p}$ branches on the right, as in figure II.4.1, where the branch b_S has an argument bounded strictly away from purely imaginary. Under such hypothesis we trivially have a bounded right inverse to D (for $\varepsilon = 0$), given by,

$$\big(Kg\big)(\xi, y) = e^{-\xi} \int_{-\infty}^{\xi} e^t g(t, y)\, dt$$

Let us furthermore observe:

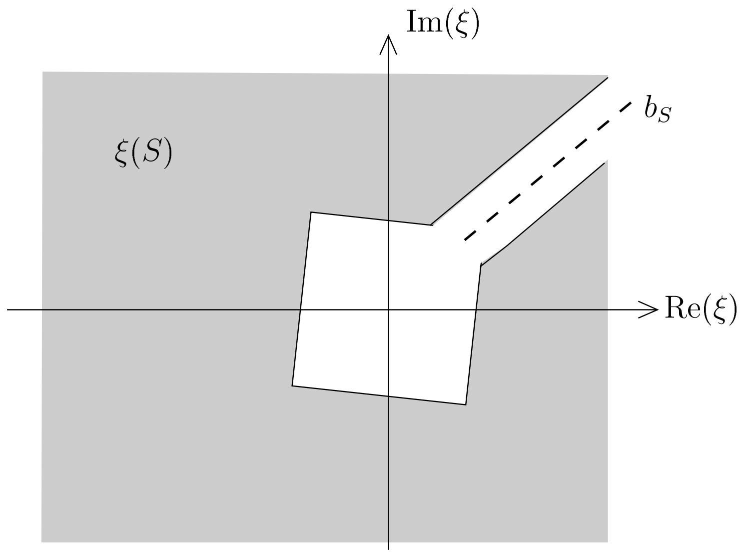

Figure II.4.1

II.4.2 Fact These operators actually patch together on sectors of width $< 3\pi/p$ (interpreted on a branched cover if $p = 1$), again to bounded operators. At $3\pi/p$, however, they may explode and develop *Stokes' lines*.

Proof. The sector depends on the branch, so, more correctly, let us call it S_b, with S_0 that branched around the real axis. For b in the right half plane, S_b intersects S_0 in a sector which contains rays along which $\mathrm{Re}(\xi) \to -\infty$. On the other hand if f_1, f_2 are two bounded solutions of our linear equation, with h their difference, then:

$$\big(\mathbb{1} + \frac{\partial}{\partial \xi}\big)(h) \ = \ 0$$

whence $e^{\xi} h$ is a function of y alone, while $S_b \cap S_0$ contains rays on which $\mathrm{Re}(\xi) \to -\infty$, and h is bounded, so this function is zero. ▫

We therefore have a bounded operator, again denoted K, defined on the domain $V \times \Delta^n$, where V is the domain of aperture up to 3π obtained from gluing the domains U^b. Necessarily the operator,

$$(\mathbb{1} + \varepsilon K)$$

is invertible for ε sufficiently small, so we deduce:

II.4.3 Corollary Let $f \mapsto P(f)$ be as per the set up II.4.1(a), then every value of the argument of x admits an open sector of aperture anything up to $3\pi/p$ such that if g is holomorphic on $S \times \Delta^n$ then the equation,

$$P(f) = g$$

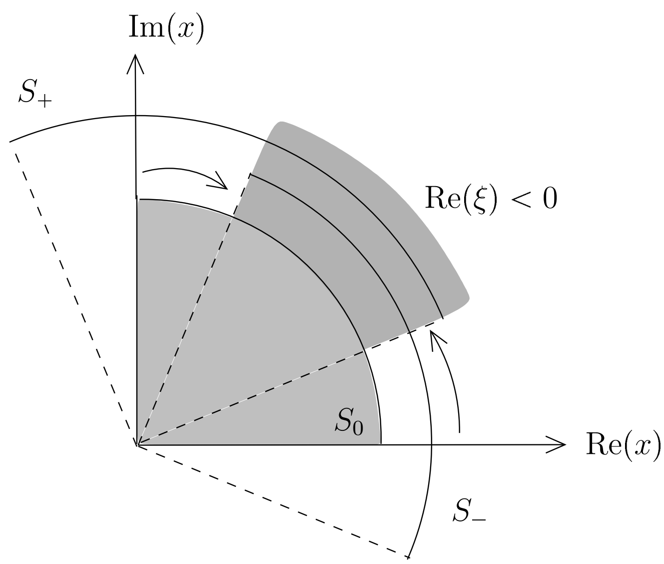

Figure II.4.2

has a solution on $S' \times (\Delta')^n$, for S', Δ' strictly inside S and Δ respectively. More precisely putting $U = V \times \Delta^n$, for V as above, and taking $\xi \times y$ as an embedding in $\mathbb{C}^{n+1}$ with respect to which we compute the Euclidean distance, then the condition for a solution f on $U(\underline{d})$ to exist is,

$$\|g\|_U \leqslant \max\left\{1, C\,(d_0 d_1 \cdots d_n)^N\right\}$$

for suitable constants C and N, with d_0 the distance in the ξ variable. In particular if g (as it will in applications) vanishes along the divisor $x = 0$ in the original polydisc, then the lack of dependence of C and N on the radii, provides a solution by the simple expedient of shrinking in the x-variable.

To which it might usefully be added,

II.4.4 Remark At least for $n = 0$ this is wholly classical. The solutions are even Borel summable for a determination of the argument of x uniform in y (something which is trivially possible if x is a meromorphic function on some variety), and the solutions even patch for x in a fixed sector and y varying along some divisor because the sectors are large, $\pi/p + \varepsilon$ would do, but we have even more. The previous sections II.1-3 should, however, make it clear that this is very far from the general picture.

Being rather easy, it's not difficult to extend to the general case:

II.4.1 (b) Set Up As before but now with an exceptional divisor smooth non-reduced in the transverse direction and invariant otherwise, i.e. a functional derivative,

$$D \doteqdot P'(0)\ : f \longmapsto \left\{(1+\varepsilon)\mathbb{1} - \frac{x^{p+1}h}{p}\frac{\partial}{\partial x}\right\}(f)$$

for h a function of y alone.

The only thing that changes is the definition of branched on the right, which actually means within $\pi/2$ of the argument of h under the conformal mapping $\xi = x^{-p}$. Under which conditions, and $\varepsilon = 0$,

$$(Kg)(\xi, y) = e^{-\xi/h} \int_{-\infty}^{\xi/h} g(t, y) e^{t/h} \frac{dt}{h}$$

works as before for $-\infty$ in an appropriate half space determined by the argument of h. There is of course a game between the branch and h, i.e. if, as one should expect from II.1, h varies through π or more, there are competing infinities, similarly if ξ varies too much then the argument of h cannot vary at all, and, arguing as per II.4.2, we see that the general condition is:

$$p\{\text{aperture in } \xi\} + \{\text{aperture in } h\} < 3\pi$$

Furthermore K is bounded, $(\mathbb{1} + \varepsilon K)$ is invertible for ε sufficiently small, while the base point at $-\infty$ is wholly fixed irrespectively of changes in the Euclidean distance and whence,

II.4.5 Corollary Suppose that the above h is a simple normal crossing divisor $y_1^{a_1} \cdots y_n^{a_n}$, and let $z_1, \ldots, z_m$ be some further variables, with $f \mapsto P(f)$, $0 \mapsto 0$, a differential operator polynomial in the fields $x\frac{\partial}{\partial x}$, $y_i\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial z_j}$. Then for a functional derivative as per II.4.1(b) every value of the arguments of x and y_i admit open sectors $S \times \prod_i S_i$ satisfying the above condition on the aperture, so that if $U = S \times \prod_i S_i \times \Delta^m$ then we have a solution to the equation

$$P(f) = g$$

on $U(\underline{d})$ (logarithmic coordinates in the y_i variables) provided $\|g\|_U$ is sufficiently small. As ever the appropriate lack of dependence on the radii whether of S or S_i, implies that if g vanishes along either x or some y_i in our original polydisc Δ^{1+n+m}, then by the simple expedient of decreasing the radius, be it in x or y_i, we will obtain a solution.

III Linearisable Singularities

III.1 Base at infinity

In the same spirit as the previous chapter we will progress from the easy to the more difficult, and so begin with:

III.1.1 Set Up Let $(x, y) \in \Delta^2$ belong to a bi-disc, and consider differential operators $f \mapsto P(f)$ polynomial in the fields $x^p\Big(x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}\Big)$, $x^N\frac{\partial}{\partial y}$ (with $N \geqslant \mathrm{Re}\,|\lambda|$) with a functional derivative of the form

$$D \doteqdot P'(0): \; f \mapsto \left\{(1+\varepsilon)\mathbb{1} - \frac{x^p}{p}\Big(x\frac{\partial}{\partial y} + \lambda y\frac{\partial}{\partial y}\Big)\right\}(f)$$

where ε is a small function of x and y, $p \in \mathbb{N}$, and the constant $\lambda \neq 0$ has $\mathrm{Re}(\lambda) \geqslant 0$.

Plainly we restrict x to a sector S of width $2\pi/p$, such that the image of the branch b under the conformal mapping $x \mapsto \xi = x^{-p}$ is to be found on the right, i.e. $\mathrm{Re}(\xi) \geqslant 0$ and the argument bounded strictly away from purely imaginary. We further introduce the first integral $s = yx^{-\lambda} = y\xi^{\lambda/p}$, and restrict our initial attention to $\varepsilon = 0$, so that:

$$D = \mathbb{1} + \frac{\partial}{\partial \xi}$$

Now consider the domain $U^b \subset \mathbb{C}^2$, taken with (ξ, s) coordinates such that the fibre over s ($\in \mathbb{C}$ if $\mathrm{Re}(\lambda) > 0$, or a disc otherwise) is as shown in figure III.1.1, where the implied excluded area is what we will term a *rhombus adapted to the branch*, and the size of the long axis R is,

$$\max\{C_1|s|^{1/\alpha}, C_2\}$$

for some suitable constants C_1, C_2, $\alpha = \mathrm{Re}\,(\lambda/p) > 0$, otherwise just take $R = C_2$ if $\alpha = 0$. Observe that this domain has the usual convenient properties, viz:

III.1.2 Facts The domain U^b satisfies,

(a) There is a sector $S \ni x$ and a disc $\Delta \in y$, with each having appropriately small radii and S of aperture $2\pi/p$, such that $S \times \Delta \subset U^b$. Conversely, $U^b \subset S' \times \Delta'$, for S', Δ' of appropriately large radii.

(b) For every $s \in B$, the base of the fibration, there is a path $\gamma_s : [0, 1] \to U^b_s$ from $-\infty$ (i.e. $\mathrm{Re}(\xi) \to -\infty$) such that,

$$\|\dot{\gamma}_s(t)\| \leqslant C\frac{\partial}{\partial t}\,\mathrm{Re}\,\big(\gamma_s(t)\big)$$

for some suitable constant C, depending on the branch b.

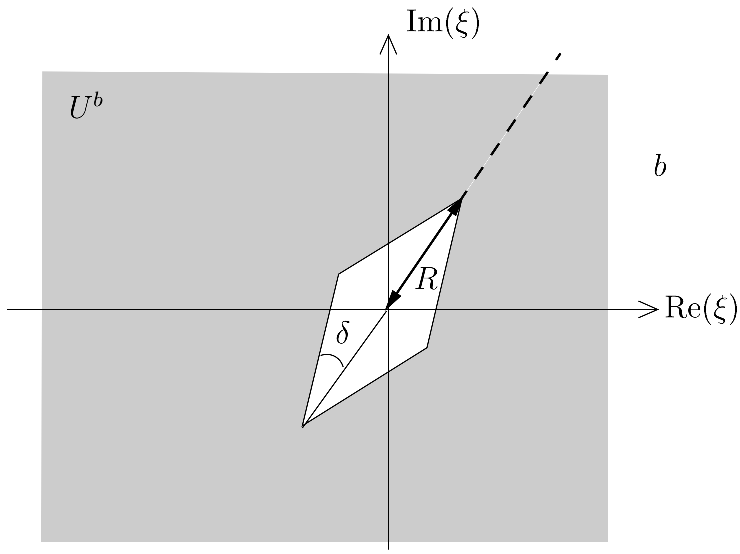

Figure III.1.1

Now in U^b we can construct a right inverse to D in the usual way, viz:

$$(Kg)(\xi, s) = e^{-\xi} \int_{-\infty}^{\xi} e^{\rho} g(\rho, s) d\rho$$

which by II.4.2 can be patched to an operator, again denoted K, on a domain U of aperture up to $3\pi/p$ formed by gluing the U^b. Provided the branches b are bounded away from purely imaginary, and $U(\underline{d})$ is understood in the s variable on one factor and a metric glued from the Euclidean distance in the ξ variable on the other, there is a constant such that,

$$\|Kg\|_{U(\underline{d})} \leqslant C \ \|g\|_{U(\underline{d})}$$

with $U(\underline{d})$ as per I.3.3. Now in so much as we're avoiding the question of how the fibres U_s change if we move in from the boundary in the original x (better $\log x$) and y coordinates we're obliged to take operators which are polynomial in $\frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial s}$, which is exactly the restriction that we have made in II.1.1, albeit with a bit more work, one could reasonably hope do to better. In any case the perturbation in ε poses no problem since,

$$DK = \mathbb{1} + \varepsilon K$$

and K is uniformly bounded on the $U(\underline{d})$, so the obvious infinite series works. Whence to summarise,

III.1.3 Fact Let $f \mapsto P(f)$, $0 \mapsto 0$ be an operator as per III.1.1 with U as above, then the equation,

$$P(f) = g$$

has a solution on $U(\underline{d})$ as soon as,

$$\|g\|_U \leqslant C \ \min\{1, (d_1 d_2)^N\}$$

for some constants C and N implied by III.1.2(b), and the higher derivatives, or powers of linear ones in P, but not on the radius of the sector in the original x variable. Consequently if g vanishes along the divisor $x = 0$, then every value of $\arg(x)$ is contained in an open sector S of width up to $3\pi/p$ and of sufficiently small radius, such that for some fixed (i.e. independent of x) sufficiently small disc $\Delta \ni y$, the equation has a solution on $S \times \Delta$. Furthermore as per II.1.4(bis), and similar, none of this is changed by the addition of further variables z_k, $1 \leqslant k \leqslant m$, with a polynomial operator in $\frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial s}$ and the $\frac{\partial}{\partial z_i}$, provided the functional derivative remains the same, so that for g vanishing on $x = 0$, we again get solutions on $S \times \Delta_y \times \Delta^m$ provided that the radius is sufficiently small in x.

In so much as this has been an introductory case, we can usefully note,

III.1.4 Remark Again this is very much the best possible scenario, with obvious similarities to §II.4, and for exactly the same reason as II.4.2, Stokes' line for K may appear at $3\pi/p$.

III.2 Still at infinity

As noted in the previous case is as good as one might reasonably expect, being even Borel summable and so forth. There remains a variant on the same, which continues to preserve these salient features, viz:

III.2.1 Set Up Let $(x, y) \in \Delta^2$ belong to a bi-disc, $p, q \in \mathbb{N}$, and $f \mapsto P(f)$ be a differential operator polynomial in the fields

$$x^p y^q \partial, \quad \chi = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}$$

where $\partial = x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$ and $\lambda \in \mathbb{C}^*$ is subject to the restrictions

$$p + q\lambda \neq 0, \quad \mathrm{Re}\,(p + q\lambda) \geqslant 0, \quad \mathrm{Re}\left(\frac{p}{\lambda} + q\right) \geqslant 0$$

and we suppose that $P(0) = 0$, and that it has functional derivative;

$$D \doteqdot P'(0) \ : \ f \longmapsto \left\{(1+\varepsilon)\mathbb{1} - \frac{x^p y^q}{p + q\lambda}\partial\right\}(f)$$

for ε a small function in x and y. Observe that under these restrictions $\lambda \notin \mathbb{R}_{<0}$ so these hypothesis hold for analytic coordinates x, y iff they hold for formal coordinates.

As ever we start with $\varepsilon = 0$, and we consider the function $\xi = (x^p y^q)^{-1}$, and restrict ourselves to sectors S in ξ which have aperture 2π, and branch, b, in the half plane $\mathrm{Re}(\xi) > 0$ bounded strictly away from purely imaginary. Similarly, we suppose $s = yx^{-\lambda}$ is defined, and varies in a domain B with a well defined argument up to 2π, or indeed we may well take its logarithm, $e^\sigma = s$. In any case, $s^{1/(p+q\lambda)}$ is supposed defined and,

$$x = s^{-q/(p+q\lambda)}\xi^{-1/(p+q\lambda)} \; ; \; y = s^{p/(p+q\lambda)}\xi^{-\lambda/(p+q\lambda)}$$

Now put $\alpha = \mathrm{Re}(\lambda/(p+q\lambda))$, $\beta = \mathrm{Re}(1/(p+q\lambda))$ and consider a domain $U^b \to B$, fibred in the s-variable of exactly the same shape as before in III.1.1, but where the axis R has length,

$$\log R = \max\left\{\frac{1}{\beta}\,\mathrm{Re}\left(-\frac{q\sigma}{p+q\lambda}\right) + C_1, \frac{1}{\alpha}\,\mathrm{Re}\left(\frac{p\sigma}{p+q\lambda}\right) + C_2, C_3\right\}$$

for some suitable constants C_1, C_2, C_3 and $\alpha\beta \neq 0$. Otherwise if α, respectively β, is zero, which cannot, by the way, happen simultaneously, one simply omits the term in α, respectively β. The domain B of the variable $\sigma = \log s$, is all of $\mathbb{C}$ for $\alpha\beta \neq 0$, and the half plane $\mathrm{Re}\left(\sigma/(p+q\lambda)\right) < 0$, for $\alpha = 0$, respectively, $\mathrm{Re}\left(\sigma/(p+q\lambda)\right) > 0$, for $\beta = 0$. Plainly the function $s^{1/(p+q\lambda)}$ is to be understood as $\exp(\sigma/(p+q\lambda))$, which yields domains U^b in $\mathbb{C}^2$ as specified by the prescription à la III.1 on the fibres U^b_σ, and whence their features are very much akin to III.1.2, viz:

III.2.2 Facts For appropriate values of C_1, C_2, C_3,

(a) The function (x, y) maps U^b to our original bi-disc Δ^2.

(b) For all values of $\arg(x)$ and $\arg(y)$, modulo the prescription that $\arg(x^p y^q)$ is not equal to that of the branch in ξ, there are open sectors $X \ni x$, $Y \ni y$ of appropriately small radii on the same, and determinations of $\log x$ and $\log y$ (any determination for $\alpha\beta \neq 0$, otherwise many but not all), together with a map $(\xi, \eta) : X \times Y \to U^b$ whose composition with (x, y) is the identity.

(c) For every $\sigma \in B$, there is a path $\gamma_\sigma : [0,1] \to U^b_\sigma$ from $-\infty$, i.e. $\mathrm{Re}(\gamma_\sigma(t)) \to -\infty$ as $t \to 0$, to any point such that,

$$\|\dot{\gamma}_\sigma(t)\| \leqslant C\,\frac{\partial}{\partial t}\,\mathrm{Re}\left(\gamma_\sigma(t)\right)$$

for a constant C depending on the branch.

Thus while we don't formalise it, the most convenient way to think of this is that for the various branches in ξ, the $U^b \to \Delta^2$ define a Grothendieck topology which by (b) is exactly the same as that corresponding to sectorial neighbourhoods.

In any case, we plainly get solutions on sectors as soon as we get solutions on the $U^b(\underline{d})$, for $\underline{d}$ Euclidean distances in ξ and σ and everything is as III.1, to wit,

(1) $K : g \mapsto e^{-\xi}\int_{-\infty}^{\xi} e^\rho g(\rho, \sigma) d\rho$ is a right inverse to D, uniformly bounded (as $\underline{d}$ varies), for the sup-norm on $U^b(\underline{d})$, since the fibres $U^b(\underline{d})_\sigma$ preserve

the shape indicated in III.1, i.e. the implied constant in III.2.2(c) is the same for all $U^b(\underline{d})$, since only the length of the axis changes as we move by the Euclidean distance in σ.

(2) Arguing as in II.4.2 we can glue to a bounded right inverse, again denoted K, on a domain U of aperture up to 3π in ξ containing all of the U^b for b bounded away from purely imaginary. The resulting $U(\underline{d})$ being equivalently the gluing of the $U^b(\underline{d})$, or understood with respect to the Euclidean distance in σ on one factor, and a metric comparable to the Euclidean distance in ξ on the other.

(3) For $\varepsilon \neq 0$, $DK = \mathbb{1} + \varepsilon K$, and by uniform boundedness on the $U(\underline{d})$, we can do this by power series in ε.

(3) The field $\frac{\partial}{\partial \xi}$ is, of course, $x^p y^q \partial$, up to a constant, while $\frac{\partial}{\partial \sigma}$ is χ, in fact, up to the same constant.

(4) Nothing changes on adding in new variables z_j, $1 \leqslant j \leqslant m$, fields $\frac{\partial}{\partial z_j}$ and going to operators polynomial in $\frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial \sigma}$, and $\frac{\partial}{\partial z_j}$.

Thus to summarise,

III.2.3 Fact Let $f \mapsto P(f)$ be as in III.2.1, or even as above with extra variables z_j provided the functional derivative is conserved, then for U as above (or $U \times \Delta^m$ should there be extra variables) the equation

$$P(f) = g$$

has a solution on $U(\underline{d})$ provided $\|g\|_U$ is sufficiently small, with smallness understood as per III.1.3. In particular if g vanishes on either the divisor $x = 0$ or $y = 0$, then for every value of $\arg(x)$, $\arg(y)$ there are open sectors X, Y about the same of sufficiently small radii such that we have a solution on $X \times Y \times \Delta^m$ for some smallish disc in the additional variables.

Again let us note,

III.2.4 Remark Again this is very much a good case in the spirit of II.4.4, i.e. despite the singular nature of ∂ we still haven't encountered any of the less desirable phenomenon detailed in II.1.5, notwithstanding the apparently more straightforward set up II.1.1.

III.3 The Bad Case

We now proceed to the case that will bring together all the difficulties to date, and add new phenomenon so far not encountered. To begin with let us introduce,

III.3.1 (a) Set Up Let $(x, y) \in \Delta^2$ and let ∂ be a vector field singular at the origin, such that for $\mathfrak{m}$ the maximal ideal of the same,

$$\partial : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2$$

is semi-simple with ratio of eigenvalues, $\lambda \in \mathbb{C}^*$ enjoying $\mathrm{Re}(\lambda) < 0$. As such the coordinates x, y are chosen such that the divisor $x = 0$ and $y = 0$ are invariant by ∂ (this can always be done analytically) and,

$$\partial = x\frac{\partial}{\partial x} + \lambda c(x,y) y \frac{\partial}{\partial y}$$

and $c(x,y) \equiv 1 \pmod{\mathfrak{m}^2}$.

The fact that ∂ cannot necessarily be linearised is, somewhat surprisingly, neither here nor there. All of the bad phenomenon already occur in the fully analytic setting where $\lambda = -1$, and xy is a first integral. Nevertheless we avail ourselves of Écalle's spiralling normalisation:

III.3.1 (b) Revision (cf. [Éca94]) For $e^{\log x} = x$, a priori $\log x$ in a left half plane, there are constants C and N such that on a domain of the form:

$$\left\{ |x| \, |\log x|^N \leqslant C \; ; \; |y| \leqslant C \right\}$$

the field ∂ may be conjugated to a linear one. In particular for S a sector of width 2π in x and Δ a disc in y, of respectively sufficiently small radii we have coordinates x, y on $S \times \Delta$ such that,

$$\partial = x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$$

with $y = a(x, y_{\mathrm{old}})\, y_{\mathrm{old}}$, $(x, y_{\mathrm{old}}) \to (x, y)$; y_{old} the original coordinate, invertible in $S \times \Delta$, and taking arguments in y_{old} is the same thing as taking arguments in y.

In light of this discussion, and indeed the generality that will be subsequently necessary, we therefore introduce,

III.3.1 (c) Final Set Up Let (x, y) be coordinates in a domain $S \times \Delta$, with S a sector of aperture $2\pi/p$ in x, and y varying in a disc Δ. Denote by ∂ the field $x\frac{\partial}{\partial x} - p\nu y \frac{\partial}{\partial y}$, $\mathrm{Re}(\nu) > 0$, $p \in \mathbb{N}$, and let $f \mapsto P(f)$, $0 \mapsto 0$, be a differential operator polynomial in $x^p \partial$ and $y\frac{\partial}{\partial y}$ with the functional derivative of the form:

$$D \doteqdot P'(0) : f \longmapsto \left\{ (1+\varepsilon)\mathbb{1} - \frac{x^p}{p}\partial \right\} (f)$$

for ε a small function in x and y.

Plainly we first consider $\varepsilon = 0$, put $\xi = x^{-p}$ for x in a sector of aperture $2\pi/p$ such that ξ branches on the right, i.e. $\mathrm{Re}(\xi) > 0$, along a slit bounded away from strictly imaginary. We shall subsequently see that a solution analytic in y, even if we were in the stronger hypothesis of III.3.1(a) is simply impossible, so we don't waste time, and introduce logarithms, $e^X = \xi$, $e^Y = y$, for X, Y in appropriate right, respectively left, half planes. A convenient variable is $\sigma = \frac{1}{\nu} Y - X$; it's domain is, a priori, either all of $\mathbb{C}$, for $\nu \notin \mathbb{R}$, or a left half plane for $\nu \in \mathbb{R}$. Now consider the function Y/ν, which has domain a left half plane rotated through $1/\nu$, and intersect this rotated half plane with the half plane $\mathrm{Re}(Y/\nu) < 0$ so that the exponential takes values in the disc. Within the disc take a domain $\Omega_\alpha(r)$ of the type encountered in II.1, as shown in figure III.3.1, for some appropriate $|\alpha| = 1$, and r sufficiently small, to be specified

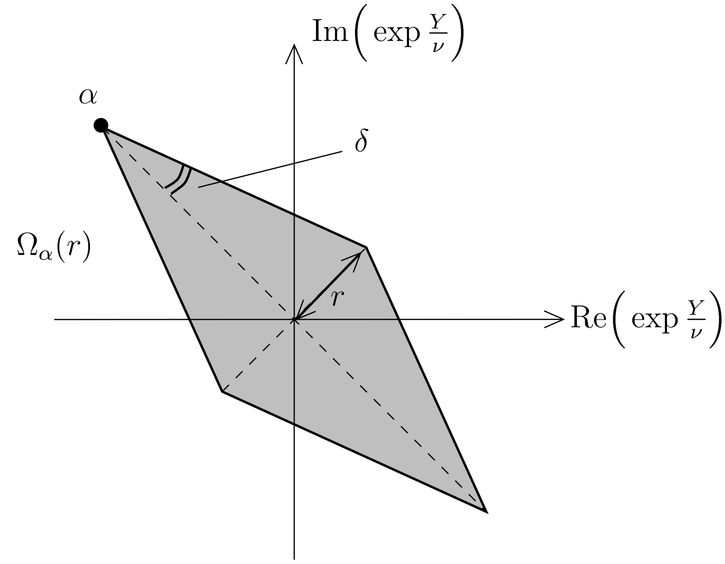

Figure III.3.1

so that rotating through a fixed sector of aperture $< \pi$ keeps some other $\rho\alpha$, for $|\rho| = 1$, ρ to be specified, as the unique point in the domain with minimal $\mathrm{Re}(\rho\alpha)$.

Now, take the pre-image, $\omega_\alpha(r)$ of this domain under the exponential in the intersection of the two half planes that constitute the domain of Y/ν (or just one half plane if $\nu \in \mathbb{R}$). Next take the image of $\omega_r(\alpha) \times H$, H domain of X, under $(Y/\nu, X) \mapsto (\sigma, X)$, limit σ to a strip, i.e. $\mathrm{Im}(\sigma) \in I$ for some interval I of width less than π, and make an appropriate choice of α, so that as σ varies in the strip, $e^{-\sigma}\alpha$ is the unique point in the image U_I of this domain under $\mathrm{id} \times \exp$ in the fibre of α where $\mathrm{Re}(\xi)$ has its minimum, and this is strictly bounded away (i.e. r sufficiently small) from the other corners of the rhombus $\Omega_\alpha(r)$ where a minimum might occur. Unsurprisingly, the domains U_I have all the properties that we require, viz:

III.3.2 Fact Supposing the sector S of III.3.1(c) satisfies the branching on the right condition for ξ, take $x = \xi^{-1/p}$, and $y = \xi^\nu e^{\sigma\nu}$ for an appropriate determination of the arguments in ξ (a.k.a. strip in X),

(a) (x, y) maps U_I to our original domain $S \times \Delta$ in x and y.

(b) For any $J \subset I$ sufficiently small (where sufficiently small depends only on ν), (x, y) on U_J is Schlict, and U_I is covered by finitely many U_J.

(c) Varying I we can cover, with finitely many I, a region containing $S' \times \Delta' \backslash \{0\}$, for $S' \subset S$ and $\Delta' \subset \Delta$ of sufficiently small radii, and, $S' \times \{0\}$ is in the closure of every U_I.

(d) The points $p(\sigma) = e^{-\sigma}\alpha$ in the fibres $U_{I,\sigma}$ are a holomorphic section of the fibration in σ, and every point $\xi \in U_{I,\sigma}$, any σ, can be joined by a path $\gamma_\sigma : [0,1] \to U_{I,\sigma}$, $\gamma(0) = p(\sigma)$, to $p(\sigma)$ such that,

$$\|\dot{\gamma}_\sigma(t)\| \leqslant C \frac{\partial}{\partial t} \operatorname{Re}\left(\gamma_\sigma(t)\right)$$

for C some suitable constant depending on I, or more accurately its width alone.

This gives therefore a domain $U_I \subseteq \mathbb{C}^2$ with fibres over σ as shown in figure III.3.2, with σ varying in a strip so that $p(\sigma)$ not only stays on the left, but

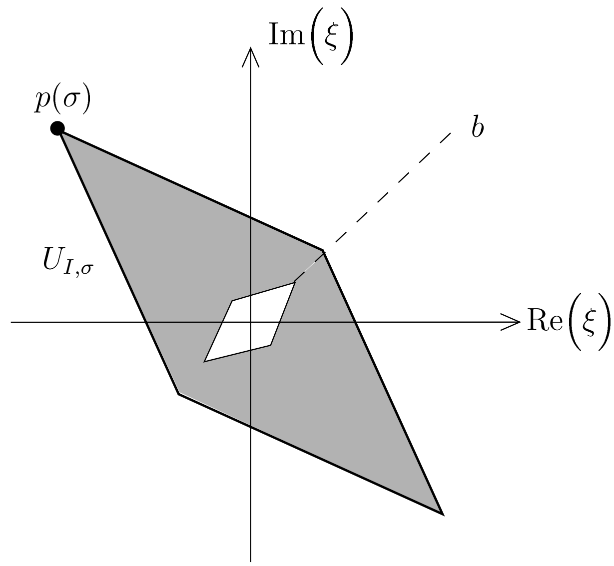

Figure III.3.2

the lines emanating from it are bounded away from purely imaginary. For convenience, we change the domain of the original x variable a little so that the inner boundary is a rhombus adapted to the branch, and in each fibre add a disc, à la III.1, around $p(\sigma)$ of radius $\kappa > 0$, and call the resulting domain $U_{I,\kappa}$. All of the properties (1)-(4) enunciated prior to III.2.3 remain valid, with the obvious change to the integral operator from the base point at ∞ to $p(\sigma)$, and $\frac{\partial}{\partial \xi}$, $\frac{\partial}{\partial \sigma}$ being $x^p \partial$ and $y\frac{\partial}{\partial y}$ up to some constants, so that:

III.3.3 Fact Suppose in addition to our set up III.3.1(c) the resulting sector in $\xi = x^{-p}$ branches on the right, and that in the presence of further variables, the functional derivative is conserved, then for U_I as above (or $U_I \times \Delta^m$ should there be extra variables) the equation

$$P(f) = g$$

has a solution in $U_I(\underline{d})$ provided $\|g\|_U$ is sufficiently small. The smallness condition can be guaranteed by the simple expedient of shrinking the radius of the sector in the x-variable provided g satisfies a condition of type

$$\|g\|_{S_r \times \Delta} \leqslant C r^{\alpha}$$

on sectors in x of radius r, and $\alpha > 0$, C anything.

Bearing in mind that this is conceivably rather far from what one might expect in terms of an intuition that comes from good cases such as II.4.1 and II.2.1-2, let us prolong this investigation by way of,

III.3.4 Scholion Again, although not formalised, the best way to consider the $U_I \to S \times \Delta$ are as a Grothendieck topology, which is the natural one for solutions, as we will see shortly. The kind of spiralling sectors U_J, J sufficiently small, being open in the classical topology, whence, solutions on such sectors really spiral for $\nu \notin \mathbb{R}$. Of course, one could take sectors in x and y alone if $\nu \in \mathbb{R}$, but this seems rather pointless. Indeed, for the applications envisaged in [McQ] sectors in y are actually worse, and moreover the divisor $y = 0$ has, in general, no algebraic description if the field ∂ arose whether by resolution of singularities on a projective surface, or a formal surface inside an ambient projective variety. Consequently, there's no sane sense in which sectors in x and y should be considered a better topology than the U_I. One can, of course, as per post II.4.1(b), fiddle with the construction somewhat, i.e. if $|S|$ is the aperture in ξ, and $|I|$ the width in σ, do anything with,

$$|S| + |I| < 3\pi$$

albeit say $|I| < \pi$ to be on the safe side. Thus, $|S|$, $|I|$ are never simultaneously big enough to be able to conclude to the uniqueness of solutions, and attempt patching to a wider aperture. Furthermore full analyticity in y is simply impossible. Indeed, suppose otherwise, and consider the equation,

$$\big(\mathbb{1} - \frac{x^p}{p}\partial\big)(f) = g$$

which is hypothesised to have a solution f analytic in $S \times \Delta$, whenever g is. As such, if $g_k(\xi)$, $k \in \mathbb{N} \cup \{0\}$ are the Taylor coefficients in y^k of g, then those of f, say $f_k(\xi)$, must satisfy,

$$\left(\mathbb{1} + \frac{k\nu}{\xi}\right) f_k + \frac{\partial}{\partial \xi} f_k = g_k$$

Now, consider values of ξ in a sector S, of small aperture, around the negative real axis, then on S bounded solutions of this equation are unique. Consequently for $\xi \in S$,

$$f_k(\xi) = \xi^{-k\nu} e^{-\xi} \int_{-\infty}^{\xi} s^{k\nu} e^s ds$$

which indeed doesn't depend on the branch $\xi \mapsto \xi^\nu$, since different choices cancel. Restricting our attention to $\xi = -s$, $s \in \mathbb{R}_{>0}$ sufficiently large, we can write this as

$$f_k(-s) = \int_0^\infty e^{-t} \Big(1 + \frac{t}{s}\Big)^{k\nu} g_k(-s-t) dt$$

Plainly we wish to impose that the solutions f are bounded on $S \times \Delta$, so we can interchange limits as $s \to \infty$, restrict our attention to $g_k = c_k$, constant, and conclude,

$$\lim_{s \to \infty} f_k(-s) = c_k \, \Gamma(k\nu + 1)$$

subsequencing in s if necessary. This is, however, manifestly absurd, since the left hand side is bounded by C^k for some constant C. Consequently for g analytic on Δ, a function of y, with c_k decreasing not too rapidly, there are no bounded solutions f analytic on $S \times \Delta'$, no matter how small Δ', so taking finite iterates such as,

$$\sum_{n=0}^{N} \big(\frac{x^p}{p} \partial\big)^n g$$

yields many other functions without solutions, but which according to our habitual condition of sufficient smallness, are as small as we like. Manifestly the problem is akin to II.1.5(a), but much worse, since even meromorphic in y behaves very badly, e.g. if $g = (1-y)^{-1}$, then there is a " resurgent " type solution, say $\nu = 1$ for simplicity,

$$f(x,y) = \int_0^\infty \frac{e^{-t}}{1 - y(1 - tx^p)} dt$$

however here a path from 0 to ∞ must be chosen, and this path must avoid $(y-1)y^{-1}x^{-p}$, from which the necessity of restricting not just the argument of x but also that of yx^p.

III.4 Intermission

Before proceeding to the evident extension of III.3 to a divisor supported along $x = 0$ and $y = 0$, let us first address the case of a purely real eigenvalue, since this will require the spiralling linearisation of Écalle as per III.3.1(b), so say:

III.4.1 Set Up Let $(x, y) \in \Delta^2$ be coordinates in a polydisc, ∂ as per III.3.1(a), but with $\lambda \in \mathbb{R}_{<0}$. Further let $f \mapsto P(f)$, $0 \mapsto 0$, be a differential operator polynomial in the fields,

$$x^p y^q \partial, \quad \chi = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}$$

$(p, q \in \mathbb{N})$ enjoying a functional derivative of the form,

$$D \doteqdot P'(0) : f \longmapsto \left\{ (1+\varepsilon)\mathbb{1} - \frac{1}{p+q\lambda} x^p y^q \partial \right\} (f)$$

for ε a small function of x and y.

Multiplying, as necessary P by a unit, we begin with $\varepsilon = 0$, and,

$$x^p y^q \partial = x^p y^q \left(x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \right)$$

for a domain of the type indicated in III.3.1(b). Consequently, we already have a logarithm X of x, and we further take one Y of y, defined by $y = e^Y$, Y belonging to a left half plane. Exactly one of $p/\lambda + q$, respectively $p + \lambda q$ may be negative, and for consistency with III.3 we suppose that it is the former. Slightly more conveniently, therefore we put $-\nu = \lambda/(p+\lambda q)$, $\nu \in \mathbb{R}_{>0}$, and take as an invariant function $\sigma = (1+q\nu)/p\,(Y - \lambda X)$, with domain of definition a left half plane. Thus for $\xi = (x^p y^q)^{-1}$, branched on the right, as ever strictly bounded away from purely imaginary, and ζ its logarithm, our operator becomes,

$$D = \mathbb{1} + \frac{\partial}{\partial \xi}$$

in σ, ξ coordinates, with,

$$X = -\left(\frac{1}{p} + \frac{q\nu}{p}\right)\zeta - q\sigma, \quad Y = \nu\zeta + p\sigma$$

Now take Y/ν to belong to some appropriate $\omega_\alpha(r)$, α, r to be chosen, as per III.3.1(c). Furthermore for suitable constants C_1, C_2 to be chosen define $R(\sigma)$ by,

$$\log R(\sigma) = \max\left\{ \frac{-pq}{1+q\nu} \operatorname{Re}(\sigma) + C_1, C_2 \right\}$$

Finally confine σ to a strip, $\operatorname{Im}(\sigma) \in I$, according to which α, r will be chosen, and for convenience suppose $\operatorname{Re}(\sigma) < C_3$, sufficiently negative. This allows us to define a $U_I \subseteq \mathbb{C}^2$ in ξ, σ variables by way of,

$$\xi \in e^{-p\sigma}\omega_\alpha(r) \cap E_{R(\sigma)}$$

where $E_{R(\sigma)}$, cf. III.3.2, is the exterior of a rhombus adapted to the branch. Consequently the fibres of U_I have the form shown in figure III.4.1, so that, modulo the augmentation of $R(\sigma)$ with σ, nothing has changed from III.3.2, and actually it's quite a bit better. The important points being that we have a holomorphic section $p(\sigma) = \alpha e^{-p\sigma/\nu}$, and that as $\sigma \in I$ varies, the inner (which is in fact fixed) and outer sides of the appropriate rhombi remain bounded away from purely imaginary. As such the U_I posses the salient features to which we have become accustomed, viz:

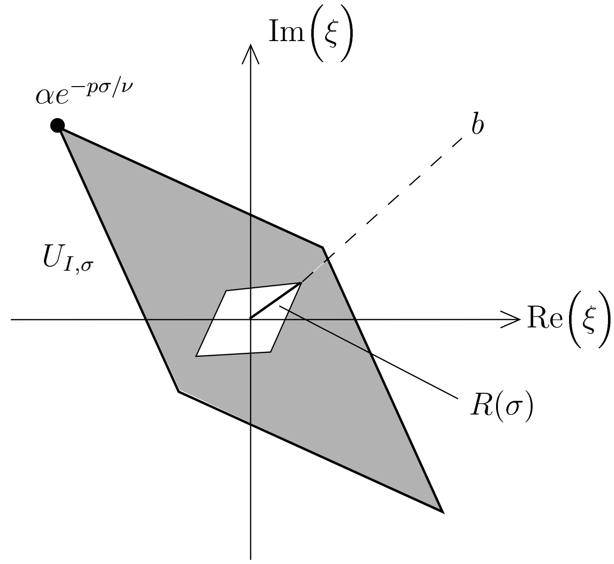

Figure III.4.1

III.4.2 Fact Choose an appropriate branch ζ of the logarithm of ξ, and put $x = e^X$, $y = e^Y$ for X, Y functions of ζ and σ as above, then,

(a) (x, y) maps U_I to our original domain $\Delta \times \Delta$.

(b) For any values of $\arg(x)$, $\arg(y)$, there are open sectors $S \ni \arg(x)$, $T \ni \arg(y)$ of sufficiently small radii such that for an appropriate choice of I and the branch b, we have a map $S \times T \to U_I$ whose composition with (x, y) is the identity.

(c) The holomorphic section $p(\sigma) = \alpha e^{-p\sigma/\nu}$ can be joined to any other point of the fibre $U_{I,\sigma}$ by a path $\gamma_\sigma : [0, 1] \to U_{I,\sigma}$ satisfying

$$\|\dot{\gamma}_\sigma(t)\| \leqslant C \frac{\partial}{\partial t} \operatorname{Re}\big(\gamma_\sigma(t)\big)$$

for C a constant depending only on the width of I, provided $\operatorname{Re}(\sigma)$ is sufficiently negative.

Now, we have (1)-(4) as per III.1 or III.2 after adjoining in each fibre a disc of some fixed radius κ, and of course, modulo the usual caveats, adjoining further variables $z_1, \ldots, z_m$ in some polydisc Δ^m, so to summarise,

III.4.3 Fact Let things be as in III.4.1, then for U_I (more correctly $U_{I,\kappa}$) as above (or $U_{I,\kappa} \times \Delta^m$ in the presence of further variables), the equation,

$$P(f) = g$$

has a solution in $U_I(\underline{d})$ provided $\|g\|_U$ is sufficiently small. The smallness condition may be guaranteed by the simple expedient of shrinking the radii in x or y sufficiently provided g vanishes on $x = 0$ or $y = 0$. In particular every value of $\arg(x)$, $\arg(y)$ admit, under this condition, open sectors S,T of sufficiently small radii such that we have a solution in $S \times T \times (\Delta')^m$, and Δ' a smaller disc in the additional variables.

To conclude, let us note:

III.4.4 Remark Availing ourselves of Écalle's spiralling linearisation, we can equally dispense, in the above notations, with the case $\lambda = -p/q$. Indeed the only substantive change that one should make in this case is to the actual set up II.3.1, and replace the field $qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}$ occurring there by ∂, so that after dividing P by a unit (in the domain of the linearisation as above), one may proceed exactly as in §II.3.

III.5 Worst Case

Actually to what extent this remaining case is really worse than III.3 is a matter of debate. It is, however, certainly the most fastidious, whence the separate treatment of a real eigenvalue in III.4, so as to lighten the complication and allow us to reduce to,

III.5.1 Set Up Let $(x, y) \in \Delta^2$ be coordinates in a polydisc, and $f \mapsto P(f)$, $0 \mapsto 0$, be a differential operator polynomial in the fields,

$$x^p y^q \partial, \ \left(\partial = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}\right), \quad \chi = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}$$

$\lambda \notin \mathbb{R}$, $p, q \in \mathbb{N}$, $\mathrm{Re}(p/\lambda + q) < 0$, and functional derivative of the form,

$$D \doteqdot P'(0) : f \longmapsto \left\{(1+\varepsilon)\mathbb{1} - \frac{1}{p+q\lambda}x^p y^q \partial\right\}(f)$$

for ε a small function of x and y.

As before we first treat $\varepsilon = 0$, pass to logarithmic coordinates $e^X = x$, $e^Y = y$, both in appropriate left half planes, and take ξ, ζ, σ, ν as post III.4.1, observing, however, that the a priori domain of σ is all of $\mathbb{C}$, and $\mathrm{Re}(\nu) > 0$, $\nu \notin \mathbb{R}$. As such a better invariant function is $\tau = p\sigma/\nu$, and it is this variable whose imaginary part should be constrained to some interval I, according to which α, r are chosen by the prescription $Y/\nu \in \omega_\alpha(r)$, otherwise, the definition of U_I is, modulo $\mathrm{Re}(\tau) \ll 0$ (depending on I), exactly as before in §III.4. The properties III.4.2(a) & (c) remain valid, while (b) gets replaced by the Schlictness of $U_J \to \Delta^2$ for $J \subset I$ of sufficiently small width, but now with a further bound on $\arg(\xi)$, together with the finite covering considerations of III.3.2(b)-(c), i.e.

III.5.2 Fact Define (x, y) as in III.4, then,

(a) (x, y) maps U_I to Δ^2, our original domain.

(b) For $J \subset I$ sufficiently small, and $\arg(\xi)$ confined, or better $\mathrm{Im}(\zeta)$ in a sufficiently small interval K, the restriction of (x, y) to $U_J(K)$ is Schlict, and U_I is covered by finitely many such $U_J(K)$.

(c) Varying I we may cover some smaller bi-disc $(\Delta')^2$ punctured in $xy = 0$, by finitely many U_I, which in turn have in their closure (in fact naturally extend) to every point of the real blow up in $x = 0$, followed by the same in $y = 0$.

(d) The holomorphic section $p(\tau) = e^{-\tau}\alpha$ can be joined to any other point of the fibre $U_{I,\tau}$ by a path $\gamma_\tau : [0, 1] \to U_{I,\tau}$ satisfying,

$$\|\dot{\gamma}_\tau(t)\| \leqslant C \frac{\partial}{\partial t} \mathrm{Re}\left(\gamma_\tau(t)\right)$$

for C a constant depending only on I, provided $\mathrm{Re}(\tau)$ is sufficiently negative.

The conditions (1)-(4) of III.1/2 after adjoining in each fibre a disc around the base point of radius κ carry through verbatim, and we let Δ^m be a polydisc in any additional variables we may wish to add provided the functional derivative is unchanged, and whence,

III.5.3 Fact Let the set up be as III.5.1 then for $U_{I,\kappa}$ as above (or for that matter $U_{I,\kappa} \times \Delta^m$ in the presence of further variables), the equation,

$$P(f) = g$$

has a solution in $U_I(\underline{d})$ provided $\|g\|_U$ is sufficiently small, where this smallness condition is guaranteed by the simple expedient of shrinking radii if g vanishes on $x = 0$ or $y = 0$. In particular on the finite covering $U_J(K)$ of a sufficiently small punctured bi-disc by Δ^m, envisaged in III.5.2(b)-(c), we have solutions to our equation.

As such, it only remains to close this chapter by way of,

III.5.4 Remark Of course it could be that there are, in this case, solutions in open sectors $S \times T$ in $\arg(x)$, $\arg(y)$ as encountered in III.4.3, since, plainly, this is a different question to that addressed in III.3.4. On the other hand III.3.4 is consistent with, and identical to, all the phenomena encountered in II.1.5 and various subsequent manifestations of the same, that the “right” variables in which one should take arguments are fibrewise, and on the base of the fibration. Consequently, such an improvement would be rather surprising.

IV Saddle-Nodes

IV.1 Normal Forms

As has been said, in addressing the difficulties of nodes, we will be applying the implicit function theorem towards the limit of what is possible. Consequently we must be precise about the form of the operator to be investigated, beginning with:

IV.1.1 Set Up Let D be a germ of a vector field on a bi-disc $\Delta \times \Delta$, such that the induced foliation, defined by the vector field ∂, has a saddle node at the origin, and D vanishes to order p along the strong branch. More or less consistently $x = 0$ will be the equation for the strong branch, while y, another coordinate, will vary from being tangent to the weak branch, whence genuinely analytic, or the weak branch itself, so possibly formal, or analytic if x is restricted to a sector. In any case, we always take D so that, $D = x^p \partial$.

While normal forms for $p = 0$, and conjugation to them in sectors have been studied extensively, c.f. [MR82], [Éca92], this doesn't provide that much of a shortcut to the $p \neq 0$ case, so we may as well just do things from scratch. The formal situations is as follows:

IV.1.2 Fact There are formal coordinates x, y in which D may be written as:

$$x^p \left\{ R(x) y \frac{\partial}{\partial y} + \frac{x^{r+1}}{1 + \nu x^{r+p}} \frac{\partial}{\partial x} \right\}$$

where $r \in \mathbb{N}$, $R(x)$ has degree at most r, $R(0) \neq 0$, and $\nu \in \mathbb{C}$.

Proof. The weak branch, $y = 0$, exists formally, and the restriction of D to it is a 1-dimensional vector field vanishing to some order, $p + r + 1$, by definition of r, at the origin. Changing x as necessary we may suppose,

$$D\big|_{y=0} = \frac{x^{p+r+1}}{1 + \nu x^{r+p}} \frac{\partial}{\partial x}$$

Similarly, $\frac{Dy}{y}$ is a function which on restriction to $y = 0$, is of the form $x^p \times$ unit, which by a change of y coordinate we may suppose is a polynomial $R(x)$ of degree $\leqslant r$. This suffices to identify the formal invariants, and even as preparation to apply I.4.1. However, to avoid repetition in treating the analytic case we further simplify the behaviour of Dx by studying it mod y^2, and from a little linear algebra it emerges that the previous expression for $Dx\big|_{y=0}$ is actually, after a suitable coordinate change valid mod y^2. Consequently we may suppose that our field has the form,

$$D = x^p \left\{ \Big(R(x) + y\, a(x, y) \Big) y \frac{\partial}{\partial y} + \Big(\frac{x^r}{1 + \nu x^{r+p}} + y^2\, b(x, y) \Big) x \frac{\partial}{\partial x} \right\}$$

and we seek a formal conjugation of this to the normal form IV.1.2 in ξ, η of the form,

$$x = e^{f\eta^2} \xi, \quad y = e^{g\eta} \eta$$

for some formal functions f and g. This amounts to solving the following system of PDE's:

$$\eta^{-2}\left\{e^{-pf\eta^2}\frac{\partial\xi}{\xi}-\frac{\xi^r e^{rf\eta^2}}{1+\nu e^{f(r+p)\eta^2}\xi^{r+p}}\right\}+$$
$$e^{-pf\eta^2}\left(\partial f+2f\frac{\partial\eta}{\eta}\right)-e^{2g\eta}\,b\big(e^{f\eta^2}\xi,e^{g\eta}\eta\big)=0$$
$$\eta^{-1}\left\{e^{-pf\eta^2}\frac{\partial\eta}{\eta}-R\big(e^{f\eta^2}\xi\big)\right\}+e^{-pf\eta^2}\left(g\frac{\partial\eta}{\eta}+\partial g\right)-e^{g\eta}\,a\big(e^{f\eta^2}\xi,e^{g\eta}\eta\big)=0$$

where the terms on the extreme left are formal functions, not just meromorphic. Re-arranging in the obvious way, there is a vector valued operator $\underline{P}(f,g)$ sending $0\mapsto 0$ for which we require to solve the equations,

$$\underline{P}(f,g)=\begin{bmatrix}a\\ b\end{bmatrix}$$

Now the functional derivative of $\underline{P}$ has the form,

$$\underline{P}'(0):\begin{bmatrix}f\\ g\end{bmatrix}\longmapsto(A+\mathbb{1}\partial)\begin{bmatrix}f\\ g\end{bmatrix}$$

where A is a matrix of functions such that,

$$A(0)=R(0)\begin{bmatrix}2&0\\0&1\end{bmatrix}$$

The operator $A(0)+\mathbb{1}\partial$, operates on $\xi^i\eta^j$ as $(2+j)R(0)$, or $(1+j)R(0)$, according to the row, up to a topologically nilpotent operator N. Whence the same is true for $A+\mathbb{1}\partial$, so $A+\mathbb{1}\partial$ has a right inverse, and we conclude by I.4.1. ▫

Now if this seems a little heavy handed as an approach to the formal normal form, it's because it will, fortunately, be applicable mutatis mutandis analytically. In the first place, we can, and will suppose without warning that we start with a situation as close to IV.1.2 as we please in the $\mathfrak{m}$-adic topology, for $\mathfrak{m}$ the maximal ideal at the origin. On the other hand the weak branch may not exist analytically. It does, however, exist on domains of the form $S\times\Delta$ for S a sector in the x variable of width up to $3\pi/r$ "patched from sectors branched on the right" (e.g. apply II.4.3 with $n=1$), and again we may suppose the asymptotics with respect to the formal solution in IV.1.2 as good as we like. Now, since we don't really care about a full asymptotic expansion we can suppose,

$$D\Big|_{y=0,\,x\in S}=x^{p+r+1}\frac{\partial}{\partial x}$$

This can be done for a coordinate $x\in S$, as close to, $z(1+\nu z^{p+r}\log z)^{-1}$ as we like, z extending to an analytic coordinate in the whole disc. Or, more precisely,

$$x=z\left(1+\nu z^{p+r}(\log z+f(z))\right)^{-1}$$

for $f(z)$ bounded up to the boundary of S, so x is Schlict. Again, we also prepare the y variable so that,

$$\frac{\partial y}{y x^p}\bigg|_{y=0,\, x\in S} = x^p R(x)$$

This can be done via a change of coordinates,

$$(x, y) \longmapsto \left(x, y\, e^{g(x)}\right)$$

$g(x) = O(|x|^N)$, for N as big as we like, so, once more, it's Schlict. Finally there is the preparation,

$$Dx\big|_{x\in S} = x^{p+r+1} \pmod{y^2}$$

where by construction we already have,

$$\frac{Dx}{x^{p+1}} = x^r + y a(x) \pmod{y^2}$$

and $a(x) = O(|x|^N)$. Obviously we seek a new variable X in the form $X = x(1 + f(x)y)$, and this amounts to solving the linear equation:

$$R(x)f + x^{r+1}\frac{\partial f}{\partial x} - x^r(p+r)f = -a(x)$$

Now this is trivially possible, c.f. II.4.1(a) et sequel, on a sector of width $3\pi/r$, but there is a conflict between the branching here and that for producing the weak branch. More precisely, the latter is done by patching solutions in sectors of width $2\pi/r$ such that the conformal mapping $x \mapsto x^r$ has a branch within $\pi/2$ (be it above or below) $R(0)$, while the former must be branched, again above or below, within $\pi/2$ of $-R(0)$, so at the end of the day we can only actually do both equations for sectors of width $2\pi/r - \varepsilon$, and the branch of the resulting conformal mapping $x \mapsto x^r$ is "on the left", i.e. within $\pi/2$ of $-R(0)$. This concludes the preparation, and again we seek a change of coordinates of the form,

$$x = e^{f\eta^2}\xi, \quad y = e^{g\eta}\eta$$

which conjugate our vector field to that of IV.1.2 with $\nu = 0$. The resulting system of PDE's being exactly the same as in the proof of op.cit. So dividing through by $R(\xi)$ the functional derivative is a small bounded perturbation of,

$$\mathbb{1}\left(\eta\frac{\partial}{\partial \eta} + \frac{\xi^{r+1}}{R(\xi)}\frac{\partial}{\partial \xi}\right) + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

under a change of coordinates of the form,

$$\xi \longmapsto \xi\left(1 + c_1\xi^{r-1} + \cdots + c_r \xi \log \xi\right)$$

ξ sufficiently small to guarantee that it's Schlict, the -1 part of this formula becomes,

$$\eta\frac{\partial}{\partial \eta} + \frac{\xi^{r+1}}{R(0)}\frac{\partial}{\partial \xi}$$

and we make the evident change of coordinates,

$$\zeta = \frac{R(0)}{\xi^r}, \quad s = \eta \exp\left(\frac{1}{r}\zeta\right)$$

so that the functional derivative becomes, on dividing through by r,

$$\frac{1}{r}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \mathbb{1}\,\frac{\partial}{\partial \xi}$$

where $s \in \mathbb{C}$, and, without loss of generality, $\mathrm{Re}(\zeta) \geqslant \log|s|$. This admits a bounded right inverse,

$$K_0 \underline{f}(\zeta, \eta) = \exp(B\zeta) \int_\zeta^\infty \exp(-Bz) \underline{f}(z, \eta) dz, \quad \underline{f} = \begin{bmatrix} f \\ g \end{bmatrix}$$

and, rather evidently, B is the matrix occurring on the previous line. Notice, in particular, the branching is correct, i.e. our previous restrictions imposed ξ^r within $\pi/2$ of $-R(0)$, which amounts to ζ being branched in $\mathrm{Re}(\zeta) < 0$, so one chooses the boundary in the original x variable so that the leaves have the shape shown in figure IV.1.2, or some appropriate variation thereof for the branch b.

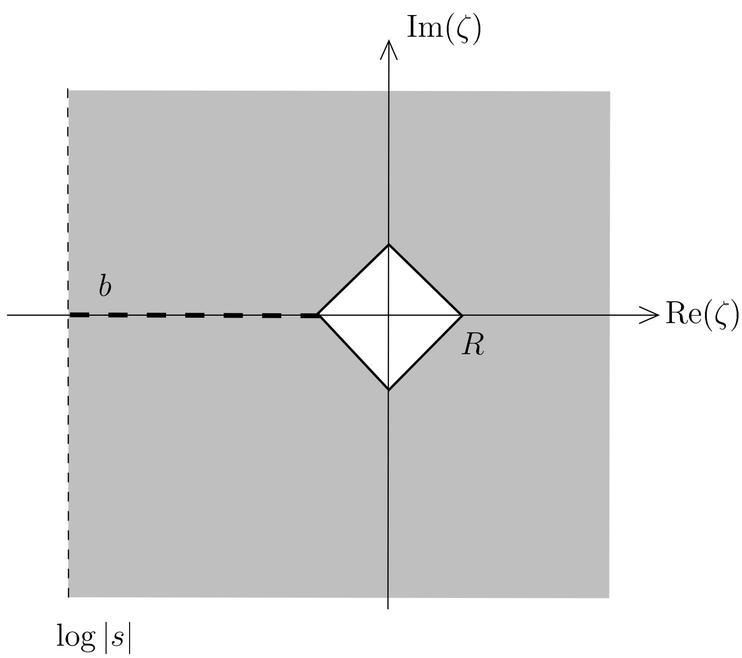

Figure IV.1.2

Consequently,

$$\underline{P}'(0) K_0 = \mathbb{1} + \varepsilon K_0$$

for ε as small as we like a function, whence $\underline{P}'(0)$ has a bounded right inverse K. Plainly we arrange that the a, b occurring in the proof of IV.1.2 are in as large a power of $\mathfrak{m}$ as we a priori please - actually 1 would be enough if they were divisible by x, which could always be arranged via blowing up, otherwise we must take care at the y boundary and proceed as in I.3.6. As such shrinking the radii in x and y as appropriate we arrive to,

IV.1.3 Fact Let D be as in the set up IV.1.1, then there is a Schlict mapping $(\xi, \eta) \mapsto (x, y)$ from a domain of the form $S_\xi \times \Delta_\eta$, Δ_η a disc, S_ξ a sector of width $2\pi/r - \varepsilon$ with $\xi \mapsto \xi^r$ branching within $\pi/2$ of $-R(0)$ which conjugates D to the field,

$$\xi^p \left\{ R(\xi)\eta \frac{\partial}{\partial \eta} + \xi^{r+1} \frac{\partial}{\partial \xi} \right\}$$

Proof. It remains to prove that the mapping constructed above is Schlict. To this end, apply the conformal mapping $\zeta : \xi \mapsto \xi^{-r}$. Then the derivatives in ζ and η of f and g are bounded on sub-domains a finite Euclidean distance from the boundary, so for $\underline{z}(\zeta, \eta)$ our mapping,

$$\underline{z} = \mathrm{id} + \eta \underline{B}(\zeta, \eta)$$

with the derivatives of $\underline{B}$ bounded as indicated, so at the price of a small shrinking of radii and aperture, the mapping is Schlict. ▫

To which we may usefully add,

IV.1.4 Remark Since we're dealing with fields, not foliations, even for $p = 0$, the factor $R(\xi)$ cannot be removed, nor can it's degree be decreased.

IV.2 Bounded Sectors

We can begin to solve some PDE's whose functional derivatives are nodes, i.e.

IV.2.1 Set Up Let $(x, y) \in \Delta^2$ be coordinates in a bi-disc, and $f \mapsto P(f)$ a differential operator polynomial in the fields

$$D = x^p \partial, \quad y \frac{\partial}{\partial y}$$

for D, ∂ as per IV.1.1, and with functional derivative in 0,

$$P'(0) : f \longmapsto \left(\mathbb{1} - \frac{D}{p + r} \right)(f)$$

As the name bounded sectors suggests we don't really need the full detail of IV.1.3 since we'll be constructing bounded right inverses. Nevertheless we'll use it since it will allow us to treat with an uniform notation the bounded and unbounded cases. Consequently, let us immediately restrict to $S \times \Delta$ a domain as per op.cit. in which we may write,

$$\partial = R(x) y \frac{\partial}{\partial y} + x^{r+1} \frac{\partial}{\partial x}$$

Take $-z(x)$ to be a primitive of $x^{-(r+1)}R(x)$ in the obvious way, i.e. avoid adding in any constants, having determined a branch of $\log x$, which is holomorphic in S. This leads to a first integral $s = y\exp(z)$ where the domain of s is potentially all of $\mathbb{C}$. Plainly z is unbounded, and as ever it's convenient to work in neighbourhoods of ∞, so $\zeta = 1/x$, and, of course,

IV.2.2 Triviality Denoting, equally, by S the corresponding sector in ζ of width up to $2\pi/r$ the mapping $\zeta \mapsto z(\zeta)$ is conformal onto neighbourhoods of infinity branched on the left, and of a slightly smaller aperture.

In terms of the mapping $s : S \times \Delta \to \mathbb{C}$ the fibres are rather easy to describe, viz: $\mathrm{Re}(z) \geqslant \log|s|$, for y in the disc of radius 1. The appropriate variable, however, against which we should attempt to integrate is $\xi = \zeta^{p+r}$, where we only have conformality on sectors of width $2\pi/(p+r)$, or $2\pi/(1+p/r)$ in z space, and p is arbitrary. Consequently for $p > 3r$ there are necessarily sectors where the fibres are bounded, and $-\infty$ cannot be used as a base point. By IV.2.2 we can, for $|\zeta| \gg 0$ and slightly shrinking the aperture reasonably confuse sectors in ζ, z and ξ. The most convenient confusion is to take sectors, Σ, in the z variable, so, implicitly, never bigger than $2\pi/(1+p/r)$. Nevertheless the variable ζ is the governing variable, so Σ really means one of the connected components of $\zeta^{-1}\Sigma$. There are also certain critical values c of the argument where there is need for caution, viz:

IV.2.3 Caveat The values of the argument of ξ where say a half line $\zeta \in \mathbb{R}_+ c$ has $\mathrm{Re}(c^{p+r}) = 0$, i.e. $\xi(c)$ purely imaginary, will require caution. Plainly these occur at intervals of $\pi/(p+r)$, and away from them any other ray $\mathbb{R}_+\gamma$ is asymptotic ($|\zeta| \gg 0$, depending on the distance between γ and the ξ imaginary direction) to a straight line in ξ space bounded away from purely imaginary.

Things are fastidious enough without worrying about the optimal size of sectors, so we'll content ourselves with neighbourhoods thereof, beginning with;

IV.2.4 (a) Easy case The ray through ζ_0, has $\mathrm{Re}(\zeta^{p+r}) \to +\infty$ as it goes to infinity, and $\mathrm{Re}\left(z(\zeta_0)\right) < 0$.

Take some small sector Σ around this direction which doesn't cross a ξ-imaginary direction. Furthermore choose a straight line $\mathrm{Im}(z/\overline{\nu}) = 0$, so that for λ in the projection of Σ onto $\mathbb{S}^1$ in the ζ plane, $\mathrm{Re}\left(\lambda^p \overline{R(0)}/\overline{\nu}\right)$ is never zero. This latter condition means: the level curves $\mathrm{Im}(z/\nu) = \text{const}$, are uniformly asymptotic to straight lines in the ξ-plane for $|\zeta| \gg 0$, as ζ varies in Σ. Plainly if the aperture of Σ is sufficiently small, there are many such ν. Now define the domain of ζ, Z say, by it's manifestation in the z-plane, as shown in figure IV.2.4, where we further refine the choice of ν so that $\mathrm{Re}\,\xi(p) < \mathrm{Re}\,\xi(q)$, e.g. argument of ζ_0. Now take η in the left half plane H to be the logarithm of y, and σ the logarithm of s, then the domain of σ is $H + Z = H + p$. This give a fibring, $\sigma : \mathcal{L} \to H + p$ with fibres,

$$\mathcal{L}_\sigma = \{z \in Z \ : \ \mathrm{Re}(z) > \mathrm{Re}(\sigma)\}$$

On any $\mathcal{L}_\sigma$ the harmonic function $\mathrm{Re}(\xi)$ has it's minimum on the boundary. On the other hand, $\mathrm{Re}(\xi) \to \infty$ as $z \in Z$ goes to infinity along any fixed argument,

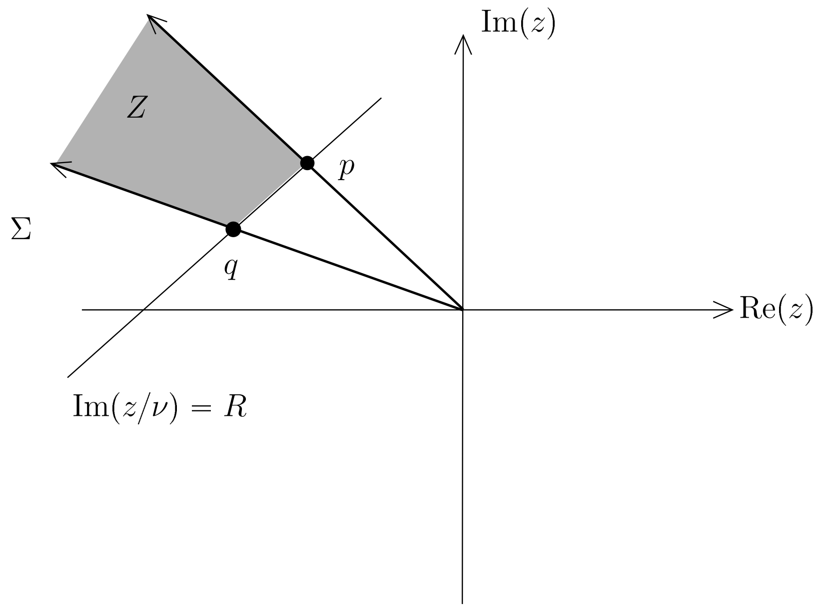

Figure IV.2.4

while by construction the derivative of $\mathrm{Re}(\xi)$ is never zero on any ray bounding Σ ($|\zeta| \gg 0$), so for $\mathrm{Re}(\sigma) \ll 0$, the minimum occurs at p, and strictly so. Whence,

$$V = \left\{\sigma \ : \ \mathrm{Re}(\xi)\big|_{L_\sigma} \text{ has a strict minimum at p } \right\}$$

is open and non-empty. It's also closed since if $\sigma_n \to \sigma \in H + p$, p is at worst a not necessarily strict minimum. However p may be joined to any point in $\mathcal{L}_\sigma$ by a path that follows $\mathrm{Im}(z/\nu) = \text{const}$, followed by a straight line in the z plane with constant argument, along which the derivative of $\mathrm{Re}(\xi)$ is piecewise never zero, and since $\mathrm{Re}(\xi) \to +\infty$ along lines of constant argument, it's actually piecewise increasing. Consequently V is $H + p$, p is the good choice of base point, and for γ the path described above,

$$\left|d\gamma^* \xi\right| \leqslant C \ d\,\mathrm{Re}(\gamma^* \xi)$$

for C depending on the aperture and R in the definition of Z. Consequently after adjoining a small Euclidean disc in the ξ space around p we have a right inverse to $P'(0)$, viz:

$$Kg(\xi, \sigma) = e^{-\xi} \int_p^{\xi} e^t g(t, \sigma) dt$$

a priori after base changing our original fibration in s by the logarithm σ, but manifestly periodic in σ, whence a right inverse in the domain of the s variable.

IV.2.4 (b) Trickier Case The ray ζ_0 has $\mathrm{Re}(\zeta^{p+r}) \to -\infty$ as it goes to infinity, and $\mathrm{Re}\,\big(z(\zeta_0)\big) < 0$.

Again take the domain $Z \ni \zeta$ to be defined via its manifestation in z space, viz: a small sector around ζ_0 cut by a line $\mathrm{Im}(z/\nu) = R$ to be specified. Now restrict $\eta = \log y$ to a spiralling neighbourhood, viz: $\eta \in N$, where N is a cone in the left half plane contained in Σ and the cone $p + C$ whose intersection with Σ defined Z. Then rather conveniently the domain of $\sigma = \log s$ is also Z. This leads to a fibration $\sigma : \mathcal{L} \to Z$ with fibres as shown in figure IV.2.4. Now we can take the cone N as close to parallel to one of the bounding rays

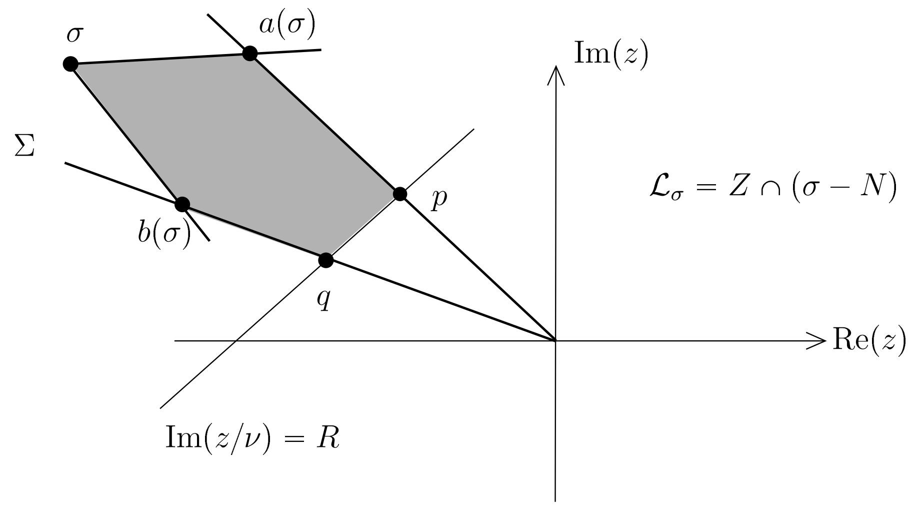

Figure IV.2.4

for Σ as we please, so for $\mathrm{Re}(\sigma)$ very large, $\min \mathrm{Re}(\xi)$ in $\mathcal{L}_\sigma$ will be attained at σ. Whence in the first instance, if $W \subset Z$ is the open where $\mathrm{Re}\, a(\sigma) < \mathrm{Re}\, p$, $\mathrm{Re}\, b(\sigma) < \mathrm{Re}\, q$ - notation as in the figure IV.2.4 - then the open subset V of W where $\mathrm{Re}(\xi)$ has a strict minimum at σ (for all of $\mathcal{L}_\sigma$) is closed and non-empty since the derivative of $\mathrm{Re}(\sigma)$ is non-zero on all of the lines bounding $\mathcal{L}_\sigma$. As such this even remains true in $\overline{W}$, and σ continues to be the strict minimum on a neighbourhood of $\overline{W}$, so a connectedness argument again yields that σ is the strict minimum for $\mathrm{Re}(\xi)$ for all $\sigma \in Z$. Consequently σ is our candidate for a base point with holomorphic variation and we consider the operator:

$$Kg(\xi, \sigma) = e^{-\xi} \int_{\xi(\sigma)}^{\xi} e^t g(t, \sigma) dt$$

For any σ the maximum of $(Kg)(\xi, \sigma)$ is attained on the boundary. Any point on the boundary of $\mathcal{L}_\sigma$ may, however, be reached from σ by a path in the boundary along which $\mathrm{Re}(\xi)$ is not only increasing (exactly how depends on the "sign" of the derivative on $\mathrm{Im}(z/\nu) = R$ between p and q) but for $R \gg 0$ is as close to a piecewise linear path in the ξ space bounded away from strictly imaginary as we please. As such by the usual procedure of adding a small Euclidean disc of fixed radius in ξ space around $\xi(\sigma)$ we have a right inverse for $P'(0)$ to which we may apply the implicit function theorem. Whence there remains to discuss,

IV.2.4 (c) Need for caution case The ray through ζ_0 is asymptotically purely imaginary, $\mathrm{Re}\big(z(\zeta_0)\big) < 0$.

We again take Σ to be a small sector around ζ_0, cut by a line $\mathrm{Im}(z/\nu) = R$ to be chosen, so that the domain Z of ζ is as before. Without loss of generality we may suppose that Σ is bounded by rays λ_-, λ_+ on which $\mathrm{Re}(\xi)$ goes to $-\infty$, respectively $+\infty$ and the argument of ζ increases from λ_- to λ_+ - otherwise change the choice of the root of -1. Now denote by θ the increment from λ_- to a general point in Σ, and ϕ the increment to the ξ-imaginary value. Then the condition that the level curves $\mathrm{Im}\Big(z/\big(R(0)\lambda_-^r\big)\Big) =$ constant are bounded away from strictly imaginary in the ξ-plane is: $(p+r)\lambda_- + p\theta$ bounded strictly away from $\pi/2 \bmod \pi\mathbb{Z}$. The ξ-imaginary value however occurs at $(p+r)\phi$, so this can be achieved provided λ_+ is strictly within $(r/p)\,\phi$ of the ξ-imaginary value. Consequently we take ν between λ_- and λ_+, but very close to λ_- so that $\mathrm{Im}(z/\nu) =$ constant is bounded away from strictly imaginary throughout Σ, all of this being understood for $|\zeta|$ sufficiently large. Similarly again we restrict $\eta = \log y$ to a cone N, but with bounding rays μ_-, μ_+ between ν and λ_- and ultimately as close to λ_- as we please.

The shape of the leaves is, therefore, as per IV.2.4(b), with all boundary lines bounded away from purely imaginary, Z continuing to be the domain of σ, and with the further proviso that $\mathrm{Re}(\xi)$ is negative on the λ_- boundary, positive on the λ_+ boundary, and for appropriate choices of μ_-, μ_+, the set where $\mathrm{Re}(\xi)$ has a strict minimum on $\mathcal{L}_\sigma$ at σ is non-empty. Consequently a slightly easier version of the previous connectedness argument applies to conclude that $\mathrm{Re}(\xi)$ always has a strict minimum at σ as σ varies through Z, and the right inverse K to $P'(0)$ is given by the same formula. We may, therefore, summarise our conclusions in the usual way, viz:

IV.2.5 Fact Let $f \to P(f)$ be as in IV.2.1, or even with extra variables z_j provided the functional derivative is conserved and the operator is also polynomial in the $\frac{\partial}{\partial z_j}$, then for any value θ of the argument of x such that $\mathrm{Re}\left(R(0)e^{ir\theta}\right) < 0$ there is an open sector S around x, and a cone N centred on 0 in the domain of $\log y$ (supposed, without loss of generality a left half plane) such that for Δ^m a domain for the extra variables, the equation

$$P(f) = g$$

has a solution in $U(\underline{d})$, $U = S \times N \times \Delta^m$, provided $\|g\|_U$ is sufficiently small. In the cases IV.2.4(b)-(c), i.e. $\mathrm{Re}(x^{r+p}) \leqslant 0$, N is a proper sub-cone of H and the sufficiently smallness condition is guaranteed as soon as g vanishes at the origin by the simple expedient of shrinking the radii. In case IV.2.4(a), where we can take $N = H$, and actually obtain analytic solutions in y, we can allow the operator to be polynomial in $\frac{\partial}{\partial y}$, but g should vanish to a sufficiently high order (determined by P) at the origin to permit us to reason as per I.3.6.

It will transpire that for many values of x, essentially those admitting a sector of width $2\pi/(p+r)$ which extends into $\mathrm{Re}(z) > 0$ we can do better than this. In general, however, full analyticity in y is not possible, which we'll discuss by way of,

IV.2.6 Scholion The appearance of $\log y$, and whence spiralling rather than full neighbourhoods of y is not a result of stupidity but rather of an intrinsic problem that the latter is impossible. The discussion III.3.4 adapts itself easily to this case. For convenience we'll suppose p even, and $r = 1$, although this is extremely far from being necessary. Putting $R(0) = 1$, we assert,

IV.2.6 (bis) Claim There are no fully analytic solutions in y in any open neighbourhood of the negative real axis of the equation

$$\big(\mathbb{1} - D\big)(f) = \frac{1}{1-y}$$

Proof. Suppose otherwise, and let $\sum_{k=0}^{\infty} f_k(x) y^k$ be the Taylor expansion of f, then the f_k must satisfy,

$$f_k\,(1 - kx^p) - x^{p+2}\frac{\partial f_k}{\partial x} = 1$$

Changing to $\zeta = 1/x$, so ζ and z planes coincide, this becomes,

$$f_k\,(\zeta^p - k) + \frac{\partial f_k}{\partial \zeta} = \zeta^p$$

If this equation has a solution in $\mathbb{R}_{<0}$, then it's unique, more correctly the bounded solution is unique, so:

$$f_k(\zeta) = \int_{-\infty}^{\zeta} e^{Q_k(t) - Q_k(\zeta)} t^p dt$$

where $Q_k(\zeta) = \frac{\zeta}{p+1}\big(\zeta^p - k(p+1)\big)$, so that,

$$f_k(\zeta) = 1 + ke^{-Q_k(\zeta)}\int_{-\infty}^{\zeta} e^{Q_k(t)} dt$$

Now consider the critical value of $Q_k(\zeta)$ at $\gamma_k = -k^{1/p}$ and the behaviour on $(\gamma_k - 1, \gamma_k)$, which is of the form

$$\frac{p\,k^{1+1/p}}{p+1}\Big(1 + O(k^{-1/p})\Big)$$

so rather comfortably for $k \gg 0$,

$$Q_k(\zeta)\big|_{(\gamma_k - 1, \gamma_k)} \geqslant \frac{k^{1+1/p}}{1+1/p}(1-\varepsilon)$$

whence no matter the value of ζ, eventually $f_k(\zeta)$ admits a lower bound of the form,

$$|f_k(\zeta)| = f_k(\zeta) \geqslant \exp\left(\frac{1}{2}\ \frac{k^{1+1/p}}{1+1/p}\right)$$

for $k \gg 0$, which is impossible. ▫

Obviously one can make much more brutal examples, but just as in III.3.4 even meromorphicity doesn't improve the situation, and we can even look at the solution of "resurgent type", viz.

$$\int_0^\infty \left\{1 - y\exp\Big(-\frac{\big(1-t(p+1)x^{p+1}\big)^{1/(p+1)}+1}{x}\Big)\right\}^{-1} e^{-t}dt$$

so there is a problem as soon as,

$$t = 1 - \frac{\Big(1 + x\left(\log y + n\right)\Big)^{p+1}}{(p+1)x^{p+1}}$$

for $\log y$ a fixed branch of the logarithm and $n \in \mathbb{Z}(1)$, which constitutes a barrier to giving any sense to this formula without imposing further restrictions in $\log y$.

IV.3 Unbounded Sectors

We retain the set up, definitions, and notations of IV.2, but pass to consider values where $\mathrm{Re}\big(z(\zeta)\big) > 0$. Consequently ζ belongs to some sector to be specified and a convenient shape of the domain Z of ζ will be of the form shown in figure IV.3. As such the domain of s is $\mathbb{C}$. The parameter R is to be specified, and we observe,

IV.3.1 Triviality For $R \gg 0$, all level curves $(\mathrm{Re}(z) = \text{ const}) \cap Z$ are uniformly asymptotic to straight lines bounded away from purely imaginary in the ξ-plane provided for $\zeta \in \Sigma$ of module 1, $\zeta^p \neq \pm R(0)/|R(0)|$. The values of the argument of ζ for which this occurs will be referred to as *critical*.

We will be able to take $-\infty$ as a base point, whence for consistent definitions of $-\infty$ we will be able to patch à la II.4.2 from small to large aperture, so in the first instance we won't worry about this too much. Consequently let the argument of ζ_0 be given and suppose that it's not critical, then,

IV.3.2 Very Easy Case $\mathrm{Re}(\zeta^{p+r}) \to -\infty$ along the ray through ζ_0.

In this case all rays in Z, for Σ sufficiently small are uniformly asymptotic to straight lines in the ξ-space bounded away from purely imaginary. Whence they provide paths γ in the ξ-space from $-\infty$ to any point satisfying,

$$\left|d\gamma^*\xi\right| \leqslant C\ d\,\mathrm{Re}(\gamma^*\xi)$$

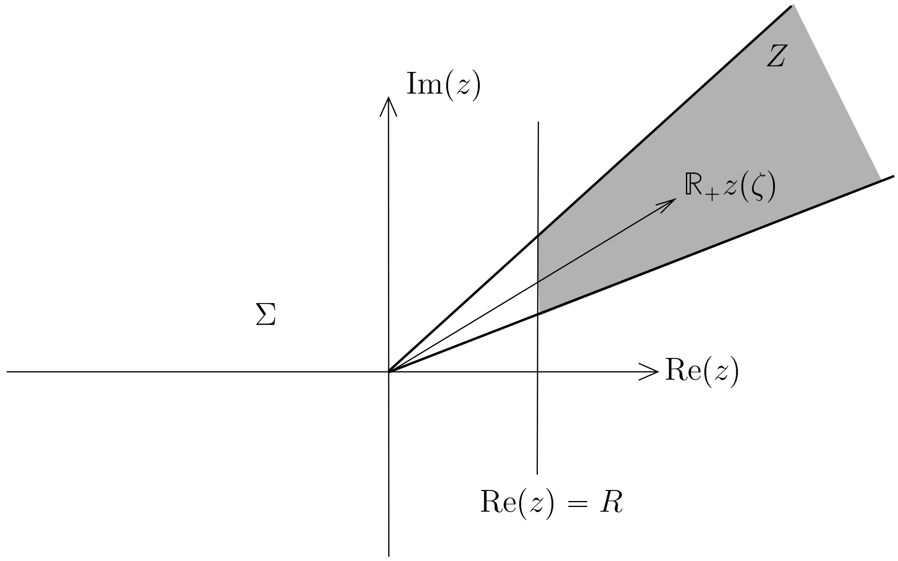

Figure IV.3

whence, as ever, a suitable right inverse for $P'(0)$ is provided by,

$$(Kg)(\xi, s) = e^{-\xi} \int_{-\infty}^{\xi} e^t g(t, s) dt$$

Otherwise in the limit, the ray through ζ_0 has non-negative argument. Now suppose ζ_0 is between two critical arguments which continue to lie in $\mathrm{Re}\left(z(\zeta)\right) \geqslant 0$, then we have two possibilities for how to put a sector Σ around ζ_0 which has rays ζ_- with $\mathrm{Re}(\zeta_-^{p+r}) \to -\infty$, and one of them must be possible, i.e.

IV.3.2 (b) Another Easy Case Not in IV.3.2(a) and ζ_0 is between two critical values c_-, c_+ of the argument, $\mathrm{Re}(z(c_\pm)) \geqslant 0$, then there is an open sector $\Sigma \ni \zeta_0$ not containing critical values of the argument, with bounding rays σ_-, σ_+ uniformly asymptotic in the ξ-space to straight lines, on the latter $\mathrm{Re}(\xi) \to +\infty$, and on the former to $-\infty$.

So here we can go from $-\infty$ to any point by first travelling along σ_- then straight down a level curve $\mathrm{Re}(z) =$ constant . For $R \gg 0$, $\mathrm{Re}(\xi) < 0$ on σ_-, positive on σ_+, and we are away from critical, so this path is uniformly asymptotic to a path in the ξ-space consisting of straight lines bounded away from purely imaginary and increasing $\mathrm{Re}(\xi)$, i.e. integration from $-\infty$ works again.

Now suppose, $z(\zeta_0)$ is between exactly one critical point and the imaginary axis in the z-plane. If the critical point itself has $\mathrm{Re}(c^{p+r}) \to -\infty$, then we'll be able to find Σ exactly as in IV.3.2(b). Otherwise the situation is as follows,

$$c^p = R(0), \quad c^{p+r} = \lambda, \ \mathrm{Re}(\lambda) \geqslant 0, \quad R(0) j^r = \pm\sqrt{-1}$$

where c is the critical argument, and the z-imaginary axis j, which, if say we look in the upper half plane of z, may be written $j = ce^{i\theta}$, $\theta > 0$, so:

$$i = \lambda e^{ir\theta}, \quad j^{p+r} = \lambda e^{i(p+r)\theta} = ie^{ip\theta}$$

and $\theta < \pi/p$ by hypothesis. Consequently we again encounter rays bounded strictly away from purely imaginary in the ξ-plane, asymptotic to straight lines and $\mathrm{Re}(\xi) \to -\infty$, whence;

IV.3.2 (c) Still Easy Case Not cases (a) or (b), $\mathrm{Re}\,\big(z(\zeta_0)\big) > 0$, ζ_0 not critical, then there is a sector $\Sigma \subset \{\mathrm{Re}(z) > 0\}$ with exactly the same properties encountered in IV.3.2(b).

It therefore remains to treat the critical points, and the z-imaginary points. In this respect, observe that there are no critical points with c^{p+r} purely imaginary, $\mathrm{Re}(R(0)c^r) > 0$. So, away from the imaginary axis, we have,

IV.3.2 (d) Last Easy Case Not (a)-(c), whence ζ_0 critical, but $\mathrm{Re}(\zeta_0^{p+r}) \to -\infty$, then we can take our sector exactly as IV.3.2(a) with the same paths.

Whence there remains to treat the case $\mathrm{Re}(c^{p+r}) > 0$. Irrespectively of the position of c, by the calculation preceding IV.3.2(c) we can find sectors Σ_+, $\Sigma_- \subset \{\mathrm{Re}(z) > 0\}$ starting at c, the former being above and the latter below, which eventually contains rays with $\mathrm{Re}(\zeta^{p+r}) \to -\infty$. Plainly, however, it is impossible to use $-\infty$ as a base point without permitting paths which are somewhere tangent to imaginary in the ξ-plane. We must, therefore, be rather carefull beginning with an analysis of the critical condition, which is really only an approximation to the truth, viz:

$$d\,\mathrm{Re}(\xi)\,d\,\mathrm{Re}(z) = 4\pi(p+r)\,|\zeta|^{2(r-1)}\ \mathrm{Im}\left(\ \zeta^p\overline{R\left(\frac{1}{\zeta}\right)}\ \right) dd^c|\zeta|^2.$$

Now consider $\zeta = tce^{i\varepsilon}$ of argument close to that of c, and in an abuse of notation, put $R(1/\zeta) = R(0)R(1/\zeta)$ so that we have $R(0) = 1$ and we're examining the vanishing of the real analytic function

$$\mathrm{Im}\left\{e^{ip\varepsilon}\ \overline{R\left(\frac{1}{tc}e^{-i\varepsilon}\right)}\right\}$$

In the degenerate case that $R(1/\zeta)$ has zero imaginary part on the ray through c, this goes like $\varepsilon\,(1 + O(1/t))$, so for $t \gg 0$, it only vanishes on $\varepsilon = 0$. Otherwise there is $1 \leqslant k \leqslant r$, and a real analytic unit $u(r, \varepsilon)$ such that the function may be written, after multiplication by a suitable unit as,

$$\varepsilon \pm \frac{1}{t^k}u\big(\frac{1}{t}, \varepsilon\big)$$

where we permit the fixed ambiguity $\pm$ so that u takes values in $\mathbb{R}_{>0}$ - the ambiguity, of course, being fixed since the (ε, t) space is connected. Thus in terms of the coordinate $\tau = t^{-1}$ at ∞, there is a real analytic change of coordinate $\tau \mapsto \rho$ taking positive to positive, such that the equation becomes,

$$\varepsilon \pm \rho^k$$

Whence it has exactly one solution, and we summarise this by way of,

IV.3.3 Fact For $t \gg 0$, there is a real analytic function $t \mapsto cb(t) \in \mathbb{S}^1$, such that $tb(t)c$ is the unique point in the ζ-plane where the level curves $\mathrm{Re}(\xi) = \text{const}$, $\mathrm{Re}(z) = \text{const}$ have a tangent on $|\zeta| = t$ in any open sector around c which does not contain any other critical points. The function $b(t)$ satisfies the estimate $b(t) = 1 + O(1/t)$.

In a slight abuse of notation, therefore, we replace our sectors Σ_+, Σ_- by domains with the same respective bounding rays in the z-plane on which $\mathrm{Re}(\zeta) \to \infty$, but replace the common boundary $\mathbb{R}_+ c$ by the image of $cb(t)$ in z-plane, while continuing to use the same symbols Σ_+, Σ_-, albeit Σ_+^b, Σ_-^b if there is risk of confusion. The derivative of $\mathrm{Re}(\xi)$ along this common boundary may be conveniently calculated in the ζ plane, and it is, unsurprisingly, asymptotic to a straight line in ξ space bounded away from purely imaginary with $\mathrm{Re}(\xi) \to +\infty$. Consequently in the ξ plane our domain has the form shown in figure IV.3.3; in which the salient features are that $\mathrm{Re}(\xi)$ increases along

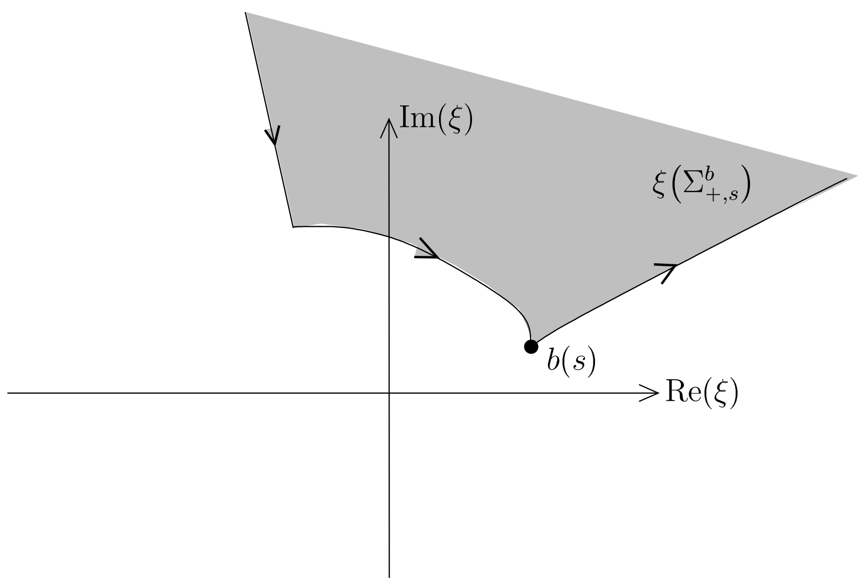

Figure IV.3.3

the boundary which has a unique imaginary tangent at $b(s)$, for b the implicit function of s defined by IV.3.3, and, of course, $|\xi| \gg 0$.

Now define domains L_+, L_- by way of a definition of their fibres over $s \in \mathbb{C}$, viz: rather than taking the lower, respectively upper, boundary to be the curve b, one takes from $b(s)$ a straight line of fixed gradient in the z-plane very close to the tangent to the ray through c, but slightly below, or above, as appropriate. Whence the fibres $L_{*,s}$ conserve the salient features of $\Sigma_{*,s}^b$ observed above. If one is not too greedy in the choice of the ray bounding L_* on which $\mathrm{Re}(\xi)$ goes to $-\infty$, the level curves parallel to the straight line that we have added are uniformly asymptotic to straight lines in ξ-space bounded away from purely imaginary. Observe furthermore that the $L_{*,s}$ are unchanged under rotations in the y-variable, whence if d_2 is the Euclidean distance in y there is a well defined domain L_{*,d_2} with exactly the same properties which corresponds to the effect of shrinking the domain of definition of y to a disc of radius d_2 less. With this in mind, we intend to employ the implicit function theorem on the domains,

$$U_*(d_1, d_2) \doteqdot L_*(d_1, \widetilde{d}_2) \cap L_{*,d_2}$$

with $L_*(d_1, \widetilde{d}_2) \subset \mathbb{C}^2$ in (s, z) coordinates as per pre I.3.3, and $\widetilde{d}_2$ the function,

$$-\log\left(1 - \frac{d_2}{Y}\right)$$

for Y the radius of the disc in y, so that for $e_2 < d_2$,

$$\widetilde{d}_2 - \widetilde{e}_2 \geqslant \frac{d_2 - e_2}{Y - e_2} \geqslant \frac{d_2 - e_2}{Y}$$

Consequently for a suitable constant C determined by the straight line to which the curve b is uniformly asymptotic, any $n \in \mathbb{N}$, and any function g,

$$\left\|\partial^n g\right\|_{U_*(\underline{d})} \leqslant n!\, C^n \, \frac{Y^n}{(d_2 - e_2)^n} \left\|g\right\|_{U_*(\underline{e})}$$

Let us employ this to invert the operator $\mathbb{1} - D = \mathbb{1} - x^p\partial$ as follows: Observe that D^n is an operator of the form,

$$\sum_{i=1}^{n} c_{i,n} \zeta^{-i(n)} \partial^i$$

where $i(n) \geqslant pn$, whence,

$$\left\|\zeta^{pn} D^n g\right\|_{U_*(\underline{d})} \leqslant C_n \, \frac{Y^n}{(d_2 - e_2)^n} \left\|g\right\|_{U_*(\underline{e})}$$

The constant C_n depending only on ∂ and n. Now for n sufficiently large to be chosen, put

$$Tg = \sum_{i=0}^{n-1} D^i g$$

and,

$$\widetilde{K}g(\xi, s) = e^{-\xi} \int_{-\infty}^{\xi} e^t (D^n g)(t, s) dt$$

where for paths we come in along the ray with $\mathrm{Re}(\xi) \to -\infty$, then in the ξ-plane go parallel to the real axis, and if necessary (and evidently around b it is) go straight down the imaginary axis. This operation admits a bound,

$$\left\|\widetilde{K}g\right\|_{U_*(\underline{d})} \leqslant C\left\|D^n g\right\|_{U_*(\underline{d})} + 2\left\|\zeta^{pn} D^n g\right\|_{U_*(\underline{d})} \int_1^{\infty} \frac{dt}{t^{np/(p+r)}}$$

so for n large enough, i.e. $np > p + r$, $K = T + \widetilde{K}$ yields a right inverse to $P'(0)$ satisfying the bound,

$$\left\| Kg \right\|_{U_*(\underline{d})} \leqslant C_n \frac{Y^n}{(d_2 - e_2)^n} \left\| g \right\|_{U_*(\underline{e})}$$

for some constant C_n depending only on ∂ and n. The domains $U_*(\underline{d})$ are a little complicated but they do contain a $L_{*,\delta}$ for δ a bit bigger than d, and $|\zeta| \gg 0$ provided the lower, respectively upper, boundary line is take sufficiently close to parallel to the boundary b, so, in fact an open neighbourhood of Σ_*^b for some slightly smaller disc. Let us summarise all this by way of,

IV.3.4 Fact Let $f \mapsto P(f)$ be not just an operator as IV.2.1 but actually polynomial in ∂ and $\frac{\partial}{\partial y}$, and even possibly with extra variables z_j in which it is also polynomial in the the $\frac{\partial}{\partial z_j}$, provided that the functional derivative is conserved, then for any value of the argument of x such that $\mathrm{Re}\left(R(0)x^p/|x^p|\right) > 0$ and not critical with $\mathrm{Re}\left(x^{p+r}/|x^{p+r}|\right) > 0$ there is an open sector S around x, and a smaller (in fact any strictly smaller than the original) disc $\Delta \ni y$ such that for Δ^m a domain for the extra variables the equation

$$P(f) = g$$

has a solution in $U(\underline{d})$, $U = S \times \Delta \times \Delta^m$, provided $\left\| g \right\|_U$ is sufficiently small. As per I.3.6 the sufficient smallness condition can be guaranteed by the simple expedient of shrinking radii, provided g vanishes to a sufficiently high order at the origin determined by P. In the case that the argument is critical, say c, there are open neighbourhoods $S_+(c)$, $S_-(c)$ of the domains Σ_+, Σ_- in the leaf space as discussed above in which all the above remains true on $S_*(c) \times \Delta^m$. The domains $S_*(c)$ have the same tangent space at the origin as sectors with a ray on c, and the other above or below as appropriate, but one of them may very well fail to contain even a closed sector containing c.

To which let us adjoin,

IV.3.5 Remark The curve of critical points is a fairly serious obstruction. Locally at $b(s)$ in ξ space we have a situation like in figure IV.3.5, where in a poor approximation we take b to be $\mathbb{R}_+ c$, i.e. the Euclidean distance in ξ space from b to this line can grow polynomially and c_+, c_- are rays above and below intersecting the boundary curve in b_+ and b_-. Irrespectively the Euclidean distance between b_* and b not only grows polynomially, but on the face of it any attempt to find a right inverse to $\mathbb{1} + \frac{\partial}{\partial \xi}$ which is bounded at b_+ and b_- appears doomed to failure due to the change in the derivative of $\mathrm{Re}(\xi)$ at b, which seems to preclude any estimate for $|d\gamma|$ in terms of $d\,\mathrm{Re}(\gamma)$ along any path.

There remains to discuss the purely imaginary cases; to this end, as pre IV.3.2(c) denote by j a solution of $R(0)j^r = \pm i$, then we have the following possibilities,

IV.3.6 (a) Easy Imaginary Case $\mathrm{Re}(j^{p+r}) < 0$, exactly as per IV.3.2(a), any small sector around j does, and ultimately conclude IV.3.4 with a sector around j.

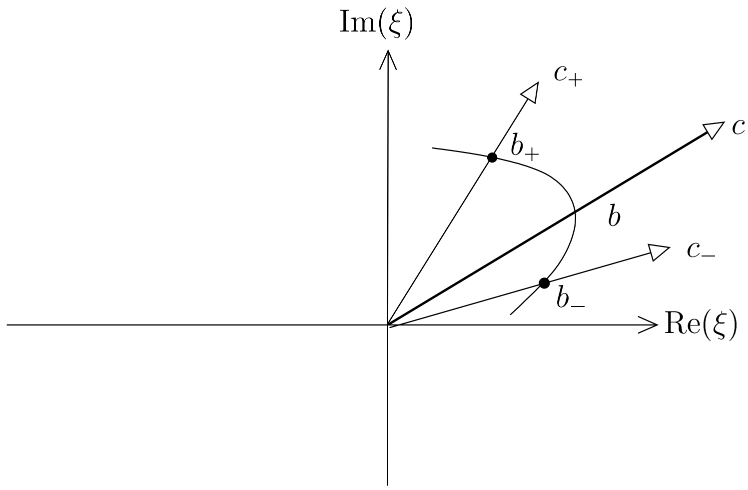

Figure IV.3.5

IV.3.6 (b) Still Easy Imaginary Case $\mathrm{Re}(j^{p+r}) > 0$. Here by the observations pre-ceding IV.3.2(c) the nearest critical point c in the $\mathrm{Re}(z) > 0$ plane has $\mathrm{Re}(c^{p+r}) \to -\infty$, so we can find a sector à la IV.3.2(b)/(c) all but ε of which is in the $\mathrm{Re}(z) > 0$ plane, and conclude IV.3.4 with a sector around j.

So now we have the rare, but not impossible, $j^{p+r} = \pm i$, i.e. imaginary z going to imaginary ξ, at least asymptotically, and it is necessarily a critical value. Around j there are regions in $\mathrm{Re}(z) < 0$ and $\mathrm{Re}(z) > 0$ and we have further sub-distinctions,

IV.3.6 (c) "Bounded" Case For arguments k close to j, and $\mathrm{Re}(R(0)k^r) < 0$, $\mathrm{Re}(k^{p+r}) < 0$. Here IV.2.4(c) is valid verbatim, i.e. we can find a sector S around j, and a cone in the logarithm of y, and we have a conclusion as per IV.2.5. We could have attempted this fully analytically using the boundary curve b, and $\pm i\infty$ in one of the components, but it's not worth the trouble.

IV.3.6 (d) "Unbounded" Case As IV.3.6(c) but $\mathrm{Re}(k^{p+r}) > 0$. One can just do this from $-\infty$, even with a bounded right inverse K, viz: as paths one can come in along the ray which goes to $-\infty$, and then horizontally straight across in the ξ-plane, since the bounding curves $\mathrm{Re}(z) = \sigma$ are always to the right of the ray. Whence the conclusion is as per IV.3.4 with an actual sector around j.

IV.4 Last Case

There remains to discuss the possibility defined by,

IV.4.1 Set Up Let $(x, y) \in \Delta^2$ be coordinates in a bi-disc such that $x = 0$, respectively, $y = 0$, is the strong, respectively weak branch of a node with saturated generator ∂ and r as per IV.2.1. Let $q \in \mathbb{N}$, $p \in \mathbb{Z}_{\geqslant 0}$ be given and

$f \mapsto P(f)$ a differential operator sending $0 \mapsto 0$ polynomial in the fields,

$$x^p y^q \partial, \quad x^r \left(qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y} \right)$$

with functional derivative of the form

$$(\mathbb{1} + \varepsilon) - D$$

where ε is a function vanishing at the origin, and the -1 part D is a field of the form (unit) $\times\, x^p y^q \partial$, where the unit is chosen so that $D(x^p y^q) = \big(x^p y^q\big)^2$. Up to a homothety in y anything of which the -1 part of the functional derivative is parallel to a node while vanishing to order p, respectively q, along the strong, respectively weak, branch, and nowhere else, has exactly this form, after scaling by a unit.

Now proceed exactly as before, restricting to a sector $S \times \Delta$, introduce the normal form for ∂ and the functions ζ, $z(\zeta), s$ as before. A slightly more convenient rescaling is to only rotate in the y variable so that $D(x^p y^q) = \lambda (x^p y^q)^2$, $\lambda \in \mathbb{R}_{>0}$ fixed, so that we can shrink $|y|$ as we please. Whence if $\xi = (x^p y^q)^{-1}$, then in ξ, s coordinates the functional derivative is

$$(\mathbb{1} + \varepsilon) + \lambda \frac{\partial}{\partial \xi}$$

and we first treat the case $\varepsilon = 0$. Now observe that, $s^q = \xi^{-1} \exp\big(\widetilde{z}(\zeta)\big)$ where $\widetilde{z}(\zeta) = z(\zeta) + p \log \zeta$ and by construction $z(\zeta)$ has the form,

$$\frac{R(0)}{r} \zeta^r \Big(1 + c_1 \zeta^{-1} + \cdots + c_r \zeta^{-r} \log \zeta \Big)$$

Whence there is no practical difference between $\widetilde{z}(\zeta)$ and $z(\zeta)$, i.e. $\zeta \mapsto \widetilde{z}(\zeta)$ is still a conformal mapping in neighbourhoods of ∞ on sectors of aperture up to $2\pi/r$. Now for a suitably large R consider the regions $\mathrm{Im}(\widetilde{z}) \geqslant R$, respectively $\mathrm{Im}(\widetilde{z}) \leqslant -R$. The conjugacy which brings us into the normal form IV.1.3 in the $\widetilde{z}$-plane is branched within $\pi/2$ of -1, so whichever of these half planes we wish to study we may suppose that the branch is in the other. Consequently let's say we're in the upper half plane H_R^+, and with no-branching there in. Choose a branch of the logarithm η of ξ in a strip domain T with $\mathrm{Im}(\eta) < 0$, branched on the right (i.e. $\mathrm{Re}(\xi) > 0$), and suppose further that $\xi(T)$ is a rotated rhombus adapted to the branch, cf. figure III.1.1, around which, either on the left or right, one can go from the point $\mathrm{Re}(\xi)$ minimum to any other with $\mathrm{Re}(\xi)$ strictly increasing, i.e. adjust the left boundary of T a little.

With these prescriptions, the domain of the logarithm σ of s^q is precisely H_R^+, and we have a fibring $\sigma : L_R \to H_R^+$ of the total space by the leaves of ∂, so, fibres: $(\sigma + T) \cap H_R^+$. Observe that, $L_R\big|_{H^+_{R+2\pi}} \supseteq L_{R+2\pi}$, so it will suffice to solve our equation on the domains $\widetilde{L}_R \doteqdot L_R\big|_{H^+_{R+2\pi}}$. This is rather convenient

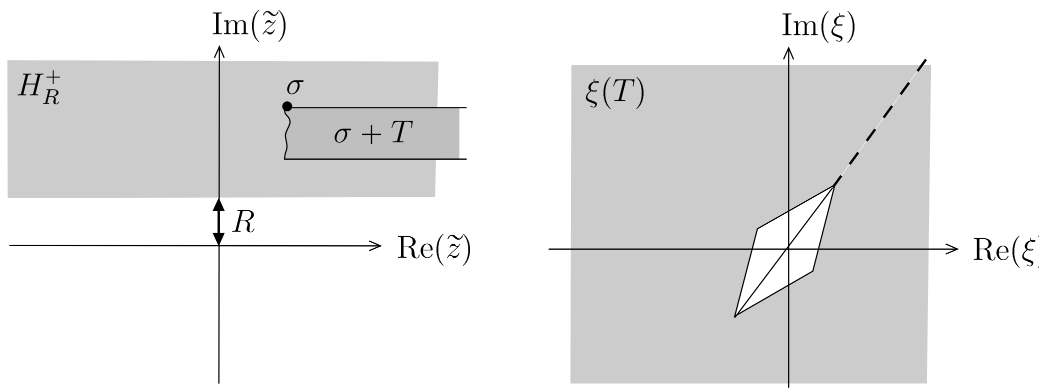

Figure IV.4.1

since (ξ, σ) embeds $\widetilde{L}_R$ in $\mathbb{C}^2$ with fibre over σ exactly the complement of our rhombus adapted to the branch. Whence,

$$(K_0 g)(\xi, \sigma) = e^{-\xi/\lambda} \int_{-\infty}^{\xi} e^{t/\lambda} g(t, \sigma) \frac{dt}{\sigma}$$

is a left inverse for $\varepsilon = 0$ which is absolutely bounded irrespective of any shrinking in x or y. While this can be analytically continued to the surface of the logarithm of s, it can not be done in a bounded way. We are, however, exactly in the situation of II.4.2, so, again we can glue the $\widetilde{L}_R$ as the branch varies to obtain a domain U_R^+ of aperture up to 3π in the ξ variable, together with a bounded right inverse on the same, which we continue to denote by K_0.

The situation is marginally more complicated in the right half planes $\mathrm{Re}(\widetilde{z}) \geqslant R$. Taking T to be a strip domain implies that the domain of the logarithm σ of s^q is all of $\mathbb{C}$. This gives a fibring $\sigma : \Lambda_R \to \mathbb{C}$ of the leaf space of ∂ with fibres: $\{\mathrm{Re}(\widetilde{z}) > R\} \cap (\sigma + T)$, so viewed under the conformal mapping ξ,

$$\sigma\big(L_\sigma\big) = \begin{cases} |\xi| \geqslant e^{R - \mathrm{Re}(\sigma)} & \mathrm{Re}(\sigma) < R,\ \xi \notin b \\ |\xi| \geqslant 1 & \mathrm{Re}(\sigma) \geqslant R,\ \xi \notin b \end{cases}$$

Supposing, after appropriate homothety that the domain of ξ is the unit disc minus the branch b. A priori, this cannot quite be done even for a branch on the real axis, due to the imaginary tangent on the boundary, and the augmentation of this problem as $\mathrm{Re}(\sigma)$ increases. This is, however, little different from the problem already encountered in III.1.1, and we may simply adopt the expedient of defining a domain $V_R^b \subseteq \Lambda_R$ fibre by fibre, in such a way that the excluded region in the plane of ξ is always a rhombus adapted to the branch, and, of course $V_R^b \supseteq \Lambda_{R+C}$, for some constant depending on the branch. The difference, of course, between our current situation and say III.1.1 is that such surgery has no relation to our base point, which is always $-\infty$, so that in V_R^b we have an absolutely bounded right inverse ($\varepsilon = 0$) irrespective of any homotheties or shrinking given by,

$$(K_0 g)(\xi, \sigma) = e^{-\xi} \int_{-\infty}^{\xi} e^t g(t, \sigma) dt$$

and again these patch to a bounded right inverse, again denoted K_0, on a domain V_R of aperture up to, but never equal to, 3π.

As such, if we took the branching in $\tilde{z}$ to be along the negative real axis, then we have right inverses K_0 in domains U_R^+, U_R^-, and V_R, with a minor lack of symmetry between the first two depending on whether the strip domain $T \ni \eta$ is in the upper or lower half plane. Irrespectively, they glue to a region U_R which certainly contains,

$$\left\{ \tilde{z} \ : \ |\operatorname{Im}(\tilde{z})| > R + 3\pi, \operatorname{Re}(\tilde{z}) > R + C \right\} \times \left\{ \log(\xi) : \log(\xi) \in T \right\}$$

for C a constant depending on how close we push the imaginary width of T to 3π. Plainly the regions U_R^+ and U_R^- do not meet, but where either of these meets V_R we find that the difference, E, say, between the operators we have constructed on them is a bounded operator such that

$$\left(\mathbb{1} + \frac{\partial}{\partial \xi} \right) (E) = 0$$

so $e^{\xi} E$ takes values in the base of the fibration and is bounded by,

$$\inf_{\xi \in L_s} e^{\operatorname{Re}(\xi)} \|E\|$$

A priori, perhaps not every leaf in the intersection has paths on which $\operatorname{Re}(\xi) \to -\infty$, but many do, e.g. $|\operatorname{Im}(\sigma)| > R + 3\pi$, and the base of the intersection is connected, so, in fact $E = 0$. Whence the K_0 patch to a right inverse in all of U_R. We can then take care of the perturbation in the usual way, i.e.

$$P'(0) K_0 = \mathbb{1} + \varepsilon K_0$$

with everything bounded irrespective of any shrinking in x or y, so, after such ε is sufficiently small and the right hand side becomes invertible.

At this point, however, we have an unfortunate last minute complication vis-à-vis the behaviour around $\tilde{z} \in \mathbb{R}_{<0}$. Outwith the good fortune that our normal form is analytic in the original bi-disc, which would, indeed be very fortunate since it's true with probability zero, we necessarily have a branch in the left half plane of $\tilde{z}$, so we'll content ourselves to investigate a domain for $\tilde{z}$ of the form:

$$Z^+_{R_1, R_2} = \left\{ \tilde{z} \ : \ \operatorname{Re}(\tilde{z}) \leqslant -R_2, \operatorname{Im}(\tilde{z}) \geqslant -R_1 \right\}$$

or the reflection in the real axis, $Z^-_{R_1, R_2}$ thereof. Taking our strip T in the lower, respectively upper, half plane, we again have the convenience that the domains of σ and $\tilde{z}$ coincide. Whence the leaf space for ∂ expresses itself as a fibring $\sigma : L^*_{R_1,R_2} \to Z^*_{R_1,R_2}$ with fibres, $(\sigma + T) \cap Z^*_{R_1,R_2}$. Observe that $L^*_{R_1,R_2}\big|_{Z_{R_1,R_2-2\pi}} \supset L^*_{R_1,R_2-2\pi}$, so it will suffice to work on the sub-domains,

$$M^*_{R_1,R_2} = L^*_{R_1,R_2}\big|_{Z^*_{R_1,R_2-2\pi}}$$

Under the conformal mapping ξ, the fibres become annuli with a fixed inner circle, and outer circle $\exp(-\operatorname{Re}(\sigma) - R_1)$ cut along the branch b. Whence, in fact, this descends to an actual fibring over the punctured disc (of radius $\exp(-R_1)$) with coordinate s^q, i.e. $M^*_{R_1,R_2}$ is the base change of this to the surface of the logarithm. Nevertheless this is something that we know, since we first encountered it in II.1 even without the inner circle, that this is something we can't do.

Consequently we have to accept the solution of the form already explained in §III.3. More precisely, for a given value ψ of the argument of s^q, replace the right boundary of $Z^*_{R_1,R_2}$ by the pre-image under the exponential of an appropriately large rhombus adapted to the argument of s^q, to form a domain Z^*_{ψ,R_2}.

Again $L^*_{R_1,R_2}\big|_{Z_{R,R_2}} \supseteq L^*_{R_1,R_2}$ for $R > R_2$, so there's no loss of generality in supposing that the domain of σ is some Z^*_{R,R_2}, $R \gg R_1$, strictly to the left of Z^*_{ψ,R_2}, and our leaf space, again restricting to $R_2 - 2\pi$, may be taken, after descent, to be a fibring, $s^q : L^*_{\psi,R_2-2\pi} \to \Delta^*$ over a punctured disc with fibres the rhombus multiplied, whence stretched and rotated, by s^{-q}. This therefore gives a domain like that post III.3.2 with a holomorphic base point, $p(s)$, where $\operatorname{Re}(\xi)$ is minimal for aperture up to π around s. The desired right inverse is therefore,

$$\left(K_0 g\right)(s,\xi) = e^{-\xi} \int_{p(s)}^{\xi} e^t g(s,t) dt$$

which is absolutely bounded by the various angles, wholly independent of any scaling, so, modulo adjoining a small Euclidean disc in ξ around the base point, and taking the usual power series to deal with the $\mathbb{1} + \varepsilon$ term, we have a right inverse K to $P'(0)$ which comfortably satisfies the conditions of the implicit function theorem, albeit at the price of arguments in s, as already encountered in III.3. Whence it's here that the polynomiality in the fields IV.4.1is really required - otherwise one could do much better. In any case, we have,

IV.4.2 Fact Let $f \mapsto P(f)$ be a differential operator of the form IV.4.1, even with extra variables z_j, $1 \leqslant j \leqslant m$, with polynomiality in the $\frac{\partial}{\partial z_j}$, and even polynomial in $x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}$ if we're away from $\tilde{z} \in \mathbb{R}_{<0}$, provided, as ever, that the functional derivative is conserved. Then there is a domain U_R as described above (so, modulo conditions on the radius better than an open sector excluding the values of x for which $R(0)x^r/|x^r| \in \mathbb{R}_{<0}$ in the x variable and of argument up to 3π in $x^p y^q$) such that on $U_R \times \Delta^m$ the equation,

$$P(f) = g$$

has a solution in $U_R((\underline{d}))$ provided $\|g\|_{U_R}$ is sufficiently small. The smallness criteria can simply be guaranteed by vanishing of g at the origin, and diminishing the original radii in x and y, possibly at the price of loosing a little aperture in S, and of course, very small shrinking of the extra domain Δ^m. Otherwise, i.e. $R(0)x^r/|x^r| \in \mathbb{R}_{<0}$, we can take a domain U^* (precise form as above) which contains closed connected sectors above, $* = +$, respectively below, $* = -$, x and the other boundary the ray through x, for which every value of the argument of $x^p y^q$ admits an open sector S of aperture up to π, such that on replacing U_R by U^* all of the above conclusions continue to hold.

V 3-D Centre Manifolds

V.1 Algebraic Reductions

V.1.1 Set Up (Characteristic 0) Let ∂ be a germ of a singular vector field around a point, where germ may be understood as being in any of Zariski, étale or analytic topologies, according to the pleasure of the reader. Denoting the maximal ideal of the point by $\mathfrak{m}$ we further suppose:

(a) The induced linear map

$$\partial : \frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2}$$

is non-nilpotent with one non-zero eigenvalue, counted with multiplicity, which, for convenience, we'll suppose to be 1.

(b) ∂ is non-zero in co-dimension one.

Although the discussion will always be local, we wish to make further simplification by way of blowing up, so we'll think of this as a foliation $X \to [X/\mathcal{F}]$ (equivalently, a rank one saturated sub-sheaf of the ambient, and always smooth, tangent sheaf) with singularity as prescribed above. The given singularity is canonical in the sense of [MPb], so, a fortiori has the following property

(c) For any sequence of blow ups in smooth invariant centres,

$$\pi : \widetilde{X} = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

i.e. the centre of $X_i \to X_{i-1}$ is invariant by the induced foliation on X_{i-1}, $\pi^*\partial$ is everywhere defined and continues to satisfy (b), equivalently generates the induced foliation on $\widetilde{X}$.

(d) For any point $\tilde{x}$ of the induced singular locus of the induced foliation on $\widetilde{X}$, the corresponding linearisation of $\pi^*\partial$ on the Zariski tangent space at $\tilde{x}$ is non-nilpotent.

With this rather laborious introduction out of the way, observe that in the completion $\widehat{\mathcal{O}}$ of the local ring in the maximal ideal, the Jordan decomposition may, in formal coordinates x, y, z be written as,

$$\partial = \partial_S + \partial_N, \qquad \partial_S = z\frac{\partial}{\partial z}, \quad \partial_N = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y)z\frac{\partial}{\partial z}$$

Our goal, in a sense to be made precise is "the convergence of z", and before we can do this it will be necessary to simplify the situation by way of blowing-up. By way of illustration, consider the case where our original singularity is isolated, and denote by $\pi : \widetilde{X} \to X$ the blowing up in this point with E the exceptional divisor. Off the proper transform $\widetilde{S}$ of the formal invariant hypersurface $S : (z = 0)$, the induced foliation is smooth except at one point, p, say $(= [0, 0, 1]$ for $E \xrightarrow{\sim} \mathbb{P}^2$ in homogeneous x, y, z coordinates). As it happens at p there are 3 non-zero eigenvalues, $(1, -1, -1)$, but there is a better functorial property that occurs. More precisely, we are supposing that our original foliation $\mathcal{F}$ has the form:

$$0 \longrightarrow \Omega^1_{[X/\mathcal{F}]} \longrightarrow \Omega^1_X \longrightarrow K_{\mathcal{F}} \cdot I_Z \longrightarrow 0$$

where the aforesaid saturation condition defines the left and right hand sides of this sequence, albeit there are more functorial ways to make the definition which justify the notation, and in the present discussion $I_Z = (z, a, b)$. On $\widetilde{X}$, however, there is an exact sequence,

$$0 \longrightarrow \Omega^1_{[\widetilde{X}/\mathcal{F}]}(\log E) \longrightarrow \Omega^1_{\widetilde{X}}(\log E) \longrightarrow \pi^* K_{\mathcal{F}} \cdot I_W \longrightarrow 0$$

Again the left and right hand terms being defined by the saturation requirement but here for $T_{\widetilde{X}}(-\log E)$. In any case, off $\widetilde{S}$ the ideal I_W has no support, and we formalise this as follows:

V.1.2 Definition Let $X \to [X/\mathcal{F}]$ be a foliation with canonical singularities, and, as per V.1.1(c),

$$\pi : \widetilde{X} = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

a sequence of blow ups in invariant centres lying over the foliation singular locus in X, with E the total exceptional divisor. Then the locus of *non-log flat* points is the sub-scheme defined by I_W, where I_W is the ideal in the short exact sequence,

$$0 \longrightarrow \Omega^1_{[\widetilde{X}/\mathcal{F}]}(\log E) \longrightarrow \Omega^1_{\widetilde{X}}(\log E) \longrightarrow \pi^* K_{\mathcal{F}} \cdot I_W \longrightarrow 0$$

Any point not in the support of this ideal will be referred to as log-flat.

To clarify this definition, let us offer,

V.1.3 Remarks: (a) Suppose more generally $\mathcal{X} \to [\mathcal{X}/\mathcal{F}]$ is a foliated Deligne-Mumford stack with $\mathcal{X}$ smooth, or even just foliated Gorenstein with canonical singularities, and otherwise arbitrary. Then it follows from [BM97] that the singular locus Z of $\mathcal{F}$ can be monomialised by a sequence of invariant blow ups for the action of $\mathcal{F}$. A priori this may not be same as the sense of V.1.1(c) since the pull-back of $T_{\mathcal{F}}$ $(= K_{\mathcal{F}}^\vee)$ along a blow up need not be saturated. However, the singularities are canonical, so, in fact, there is no such problem. Consequently, whenever the singularities are canonical there are sequences such as that of V.1.2 which are as global as one could probably wish (i.e. with respect to étale patching) and yield an exceptional divisor E on $\widetilde{\mathcal{X}}$ which is both invariant, and contains the induced singular locus.

(b) Log flatness is conserved by invariant blowing up. Actually it's also conserved by the only other operation that is guaranteed to preserve the canonicity of the singular locus, viz: blow up in an everywhere transverse centre, but this is less relevant.

(c) Our ultimate goal is to understand "centre manifolds", i.e. the a priori purely formal sub-scheme defined by the vanishing of all the eigenvectors with non-zero eigenvalue in the Jordan decomposition at a point. By hypothesis, log-flat points have an invariant algebraic (i.e. component of the exceptional divisor) through them with non-zero eigenvalue. Whence in such a study, the dimension has been reduced by one in the strongest possible way as soon as log-flatness occurs.

Returning to our initial example of an isolated singularity in dimension 3, we conclude that we can safely ignore everything that occurs outside $\widetilde{S}$. In the case where $a, b \in \mathfrak{m}^2$, the trace of $\widetilde{S}$ (necessarily $E \cap \widetilde{S}$ by the way) is a connected component of the singular locus, Z, say, by abuse of notation and $\widetilde{S}$ is a formal sub-scheme of the completion of X in Z. In a bad case where $a \in \mathfrak{m}$, but $b \in \mathfrak{m}^2$, the foliation on $\widetilde{S}$ is smooth except at one point q, say, and the ambient foliation still has an isolated singularity of the type being discussed. Most of the time the situation on $\widetilde{X}$ will already be that of the previous good case, but in very bad cases we may have to blow up once more, this time in q, to obtain the good case. Regardless, we see that the isolated situation quickly reduces to a non-isolated singularity which we summarise by way of,

V.1.4 Triviality Suppose our singularity is as per V.1.1, but isolated, then there is a sequence of blow ups in singular points (of length at most 3) such that every point of the induced foliation $\widetilde{X} \to [\widetilde{X}/\mathcal{F}]$ is either,

(a) Log-flat.

(b) A singularity of type V.1.1 with a non-isolated (in fact isomorphic to $\mathbb{P}^1$) singular locus, which is also the non-log flat locus.

The above possibility V.1.4(b) is actually a somewhat special case of the general situation since we must distinguish between components of the singular locus according to the number of eigenvalues at their generic point. In the general non-isolated case at hand we may write,

$$\partial_N = f_1(x,y)^{n_1} \cdots f_k(x,y)^{n_k} \Big(\alpha(x,y)\frac{\partial}{\partial x} + \beta(x,y)\frac{\partial}{\partial y}\Big) + c(x,y)z\frac{\partial}{\partial z}$$

in the completion of the local ring, with f_i irreducible and α, β (possibly units) relatively prime and all the f_i vanishing at our point 0 under consideration. In particular all the components of the singular locus are L.C.I., and we distinguish:

V.1.5 Definition (dimension 3) A central component C of the singular locus is a non-isolated one where the generic (whence everywhere if the singularities are canonical, and/or in the germ sense) number of non-zero eigenvalues counted with multiplicity is 1.

Let I be the ideal of the union of the central components. This is L.C.I, so ∂ descends to a linear map,

$$\partial : \frac{I}{I^2} \longrightarrow \frac{I}{I^2}$$

of the (locally free) co-normal bundle, which is rank 1-everywhere, so the kernel and co-kernel are locally free $\mathcal{O}/I$-modules of rank 1. The latter is naturally the kernel of $\mathbb{1}-\partial$, i.e. even if we were global rather than local, $\mathrm{Tr}(\partial)\otimes\partial^\vee$ defines a no-where vanishing section of $\mathcal{O}(K_{\mathcal{F}})/I$, which can thus be identified with $\mathcal{O}/I$ by the same so that locally the map is,

$$f \longmapsto \frac{\partial(f)}{\mathrm{Tr}(\partial)} \bmod I^2$$

In any case, linear algebra gives,

V.1.6 Triviality In the completion $\hat{X}$ of X in I (and whence globally if that were the context) there is a smooth formal hypersurface S such that for any point $c \in C$, the completion S_c of S at c is cut out by the unique eigenvector of the Jordan decomposition of ∂ at c which has non-zero eigenvalue and defines a smooth formal sub-scheme of the completion $\hat{X}_c$ of X in c, i.e. the centre manifold is a well defined formal sub-scheme of $\hat{X}$.

Again, let us clarify what is going on,

V.1.7 Remark As we've seen in II.2, and again let us emphasise [McQ] §1.5 as a source of further details, this is false at non-central components, i.e. even for something with Jordan decomposition at a point,

$$z\frac{\partial}{\partial z} + xy\frac{\partial}{\partial y}$$

one can make examples, even algebraic ones, where the failure to converge in the completion of $z = y = 0$ is as bad as one likes. This is the phenomenon of the *beast*. Nevertheless V.1.6 defines a larger object than that given by Jordan decomposition and it is this object that is to be understood as the formal central sub-variety, so, by definition, it's trace is the union of the central components.

This said any blow up in any point in any central component, is log-flat everywhere off the proper transform of S. Inside S such operations make the induced vector field improve in the usual way, i.e. after a finite number of steps the induced foliation in S will have canonical singularities and not just the central, but also the non-central components form a simple normal crossing divisor in S. At this point we may blow-up in the central components themselves to obtain that every central component of the proper transform $\tilde{S}$ of S is the intersection of a component of the exceptional divisor with $\tilde{S}$. In particular $\tilde{S}$ is even a formal sub-scheme of the completion of X in these divisors. Plainly, this entire discussion is étale local, i.e. globalises, and is applicable globally, were that the context, to any connected component of the singular locus where the singularities are of the form V.1.1. Globally, if the singularities are canonical on a non-isolated connected component, the only other possibility is that generically there are 2 eigenvalues. Where the number of eigenvectors (with multiplicity) goes to one, we have already smoothed such components, and elsewhere they're already smooth for co-dimension reasons. Consequently we can blow-up in these too. Now, while S is not a formal sub-scheme around such a component, it does determine a formal sub-scheme in the completion of the fibre in the exceptional divisor afforded by this operation, with trace the point where the proper transform of the neighbouring central component meets the divisor. Away from this point everything is log-flat, which is a Zariski open condition, so around this fibre an explicit calculation via Jordan decomposition implies that Zariski locally, all non log-flat points lie on the proper transform of the central components. Whence, we have achieved:

V.1.8 Fact Let $X \to [X/\mathcal{F}]$ be as in V.1.1, then there is a sequence of blow ups,

$$\pi : \widetilde{X} = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X$$

in smooth centres of the induced singular locus such that the proper transform $\widetilde{S}$ of the formal central sub-variety satisfies,

(a) Off the trace of $\widetilde{S}$ (equivalently the induced central components), $\widetilde{X} \to [\widetilde{X}/\mathcal{F}]$ is log-flat.

(b) Every component of the singular locus meeting $\widetilde{S}$ (including the non-central ones) is the intersection of a component of the exceptional divisor with $\widetilde{S}$, which is still simple normal crossing in $\widetilde{S}$.

(c) The induced foliation in $\widetilde{S}$ has canonical, and even for convenience reduced, i.e. no eigenvalue in $\mathbb{Q}_+$, singularities.

Furthermore if X were an étale neighbourhood of a foliated Deligne-Mumford stack $\mathcal{X} \to [\mathcal{X}/\mathcal{F}]$ with canonical singularities, then all of this can be done globally, i.e. the sequence π is as above but globally, and our particular instance is the restriction to X of the same. This straightforward, if lengthy discussion, may be brought to a close by way of observing:

V.1.9 Corollary Let things be as per V.1.1, and for π of the form V.1.8, and indeed even augmented by further blow ups in the induced singular loci contained in central components, we can étale or analytically locally (according to the context) achieve after completion in the central components of our bi-rational modification a formal generator of the induced foliation of the form,

$$z\frac{\partial}{\partial z} + x^p y^q \left\{ a(x,y)x\frac{\partial}{\partial x} + b(x,y)y^{\varepsilon}\frac{\partial}{\partial y} \right\}$$

where $x = 0$ is always the local equation of an exceptional divisor, $p \in \mathbb{N}$, and $q \in \mathbb{Z}_{\geqslant 0}$. If $q > 0$ then $y = 0$ is also to be understood as an exceptional divisor, and $\varepsilon = 1$, or 0 according as this is the case; which may or may not define a central component depending on whether we're at a beast or not, and the field δ which multiplies $x^p y^q$ is saturated with canonical singularities everywhere (following standard usage smooth is allowed). Notice the completion is in the ideal (z, xy^*), with $* = 1$ if $z = y = 0$ is central and 0 otherwise, so the entire singular locus may be covered by finitely many opens where this holds, should it be compact, or always in the étale "type fini" setting, if that were our context.

Proof. By V.1.6 the centre manifold is already a well defined formal sub-scheme of the singular locus encountered in X, so a fortiori after the above completion. As such, let V be a formal affine in the completion in the ideal I defined in V.1.9, then shrinking V as necessary we may suppose that we have a local equation $z = 0$ for $\widetilde{S}$ and a formal generator $\widehat{\partial}$ such that $\widehat{\partial}(z) = z$.

We distinguish the case y non-central, so that I may be written (z, x) and the asserted form in x and y is certainly true modulo z. Convergence in the topology of completion in $z = 0$ is stronger than that in $\mathcal{O}_V$ (strictly speaking $\Gamma(\mathcal{O}_V)$ but this is cumbersome), so we proceed modulo z^k, starting at $k = 1$, we may suppose the assertion holds modulo z^k, with a,b the functions encountered modulo z. If, say, $q = 0$, we attempt a substitution $x \mapsto X(1+\xi z^k)$, $y \mapsto Y+\eta z^k$, ξ, η functions of x, y. Then, modulo z^{k+1} we find,

$$\begin{aligned} z^{-k}\Big(\frac{\partial X}{X} - X^p a(X,Y)\Big) &= z^{-k}\Big(\frac{\partial x}{x} - x^p a(x,y)\Big) + x^p\{p\xi a + x\xi a_x + \eta a_y\} \\ &\quad - x^p\delta(\xi) - k\xi \\ z^{-k}\Big(\partial Y - X^p b(X,Y)\Big) &= z^{-k}\Big(\partial y - x^p b(x,y)\Big) + x^p\{p\xi b + x\xi b_x + \eta b_y\} \\ &\quad - x^p\delta(\eta) - k\eta \end{aligned}$$

So there is a purely linear equation to solve in $\mathcal{O}_V\big|_{(z=0)}$ of the form

$$k\begin{bmatrix}\xi\\ \eta\end{bmatrix} + N\begin{bmatrix}\xi\\ \eta\end{bmatrix} = z^{-k}\begin{bmatrix}x^{-1}\partial x - x^p a(x,y)\\ \partial y - x^p b(x,y)\end{bmatrix}$$

and N is topologically nilpotent since x is and $\delta(x) \in (x)$. The other cases are similar, i.e. y non-central but $\varepsilon > 0$, as above but use the substitution $Y = y(1+z^k\eta)$, which again is the substitution to use in the central case, where one appeals to the nilpotency of xy. ▫

V.2 Analytic Preparations

From V.1.9 the reader may safely infer, or jump to, the form of the PDE that we must solve to find the centre manifold. At first glance, one might think that everything has been done, and this is true when, in the notations of V.1.9, $q > 0$. Otherwise there is a lacuna because the polynomiality of the operator in the fields as hypothesised in III.3.1(c), and IV.1.1 is not satisfied a priori because we are missing an invariant hypersurface,i.e. the "$y = 0$" plane, which inter alia, is really necessary to afford sufficient space to apply the implicit function theorem in these cases.

The first step to remedying the said lacuna is to find what corresponds to the "$y = 0$" plane intersected with the formal centre manifold. We have two cases to consider, viz:

V.2.1 Set Up Suppose we're in the situation V.1.8 then we require to find an invariant surface "$y = 0$", so, a priori the curve "$y = z = 0$" in the cases where the Jordan form in the complete local ring for a formal generator of the foliation may be written as,

(a) Linear Case: $z\dfrac{\partial}{\partial z} + \dfrac{x^p}{1+\nu x^p}\Big(x\dfrac{\partial}{\partial x} + \lambda y \dfrac{\partial}{\partial y}\Big)$, $p \in \mathbb{N}$, $\nu, \lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) < 0$.

(b) Linear Case bis: As above but $\lambda \in \mathbb{Q}_{<0}$, so the Jordan form is more complicated, but it's precise form of no actual importance, which would not be the case if $\lambda \in \mathbb{Q}_{>0}$.

(c) Nodal Case: $z\dfrac{\partial}{\partial z} + x^p\Big(R(x)y\dfrac{\partial}{\partial y} + \dfrac{x^{r+1}}{1+\nu x^{r+p}}\dfrac{\partial}{\partial x}\Big)$, notation as per IV.2.2.

Now observe that in all of these cases the following preparations may be achieved modulo the maximal ideal $\mathfrak{m}$ at the point to any order k without prejudicing the convergence, viz.:

$$\partial z = z \ \ (\mathfrak{m}^k), \quad \partial y = \lambda x^p u(x,y) y \ \ (\mathfrak{m}^k), \quad \frac{\partial x}{x} = x^{p+r} v(x,y) \ \ (\mathfrak{m}^k)$$

where $r = 0$ in the linear cases, and $\lambda = R(0)$ in the nodal case, with $u, v \in 1+\mathfrak{m}$. Blowing up in a point we improve to a $q > 0$ situation except around the proper transform of the formal curve in question. We may, however, now multiply by a unit so that, $x^{-1}\partial x = x^{p+r}$. Changing the notation accordingly with $\mathfrak{m}$ the maximal ideal in the blow up, we may now suppose:

$$\partial x = x^{p+r+1}, \quad \partial z = u(x,y)z \ \ (\mathfrak{m}^k), \quad \partial y = \lambda x^p v(x,y) y \ \ (\mathfrak{m}^k)$$

and, again, $u, v \in 1+\mathfrak{m}$, but λ possibly different, i.e. $\lambda_{\mathrm{old}} - 1$ in the linear cases. Blowing this up, again in the point, we may further suppose that $u, v \in 1 + (x)$ and the congruences hold modulo x^k (actually x^{k-1} so we change notation appropriately). Now we look for our curve in the form $x \mapsto (x, \eta(x), \zeta(x))$ which amounts to solving the ODE,

$$\begin{aligned} x^{p+r+1}\eta'(x) &= \lambda x^p v(x,\eta)\eta + x^k a(x,\eta,\zeta) \\ x^{p+r+1}\zeta'(x) &= u(x,\eta)\zeta + x^k b(x,\eta,\zeta) \end{aligned}$$

where k is as large as we like, and a, b are convergent functions. In particular we can divide the first equation by x^p, and we find a functional derivative,

$$\begin{bmatrix}\eta \\ \zeta\end{bmatrix} \longmapsto \begin{bmatrix}\lambda - x^{r+1}\frac{\partial}{\partial x} & 0 \\ 0 & 1 - x^{r+p+1}\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}\eta \\ \zeta\end{bmatrix} + x^k A \begin{bmatrix}\eta \\ \zeta\end{bmatrix}$$

where A is a matrix of functions in x, and we change k as we please. Now let's consider the case $A = 0$, and look for a right inverse of the form,

$$\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

We have 2 cases to consider. The first, $r = 0$, is rather easy. The construction of K_2 is as per II.4 with total aperture $3\pi/p$, and branching on the right as described in II.4.2. We can't do better than this, so there's no point in fussing over the analyticity of K_1. Consequently we immediately pass to the surface of the logarithm, say $x = e^{\xi}$, $\mathrm{Re}(\xi) \ll 0$, and ξ is a strip of width 3π, so that:

$$(K_1 g)(\xi) = -e^{\lambda\xi} \int_{-\infty}^{\xi} e^{-\lambda t} g(t) dt$$

is a bounded right inverse since $\mathrm{Re}(\lambda) < 0$. Now shrinking the disc in x as appropriate and applying the usual power series for bounded perturbations with a bounded right inverse, yields a bounded right inverse for the functional derivative.

The case $r > 0$ is much more fastidious. We have to respect the conditions of the implicit function theorem, actually here it's just the contraction mapping principle, and the combinatorics are sufficiently fastidious that we won't even bother trying to get the best possible, i.e.

V.2.2 Digression on Branching For any value of the argument of $\zeta = 1/x$ there is an open sector about the same such that we can construct K_1, K_2 as bounded integral operators by integrating from the point where $\mathrm{Re}(\lambda\zeta^r)$, respectively $\mathrm{Re}(\zeta^{p+r})$ has a minimum, thus possibly, but by no means necessarily, $-\infty$. To avoid any inconvenience when restricting the radius of x as the inner boundary for the domain of ζ, rather than a piece of circle take a suitable straight line $\mathrm{Re}(\rho\zeta) =$ constant such that $d\,\mathrm{Re}(\rho\zeta) d\,\mathrm{Re}(\lambda\zeta^r)$, respectively $d\,\mathrm{Re}(\rho\zeta) d\,\mathrm{Re}(\zeta^{p+r})$ are uniformly bounded below by $c_1|\zeta|^{r+1}$, $c_2|\zeta|^{p+r+1}$ for suitable constants c_1, c_2 and all ζ in the sector.

Let us therefore summarise our conclusions,

V.2.3 Fact For x restricted to a sector S of width $3\pi/p$ in the linear cases and otherwise as per V.2.2 in the nodal case, with branching in the former case as per II.4.2, there is a smooth invariant curve $x \mapsto (x, \eta(x), \zeta(x))$, $x \in S$, as tangent to the normal form given by $y = z = 0$ as we please, i.e. look for a solution $\eta = x^e \widetilde{\eta}$, $\zeta = x^e \widetilde{\zeta}$ for any e but with $k \gg e$ so as not to change any of the above.

Effecting the change of coordinates $y \mapsto y - \eta(x)$, $z \mapsto z - \zeta(x)$ we therefore still have for any k we wish,

$$\partial z = uz + x^k b, \quad \partial y = \lambda x^p v y + x^k a$$

for a, b now functions on $S \times \Delta^2$. Now however (y, z) is an invariant ideal, so it must contain a, b. Let us therefore write,

$$\partial z = \widetilde{u} z + x^k \beta y, \quad \partial y = \lambda x^p \widetilde{v} y + x^k \alpha z$$

for $\widetilde{u} = u(x, y) + x^k c(x, z)$, $\widetilde{v} = v(x, y) + x^k d(x, y)$, and some functions α, β of all the variables. Now let's look for an invariant hypersurface $\widetilde{y} = 0$ in the form $y = \widetilde{y} + z f(x, z)$. This amounts to solving the PDE:

$$f\Big\{\widetilde{u}(x, zf, z) + x^k \beta(x, zf, z) f\Big\} + \partial f = \lambda x^p \widetilde{v}(x, zf) f + x^k \alpha(x, zf, z)$$

where ∂ is the operator,

$$f \longmapsto \big(\widetilde{u}(x,zf,z) + x^k\beta(x,zf,z)f\big)zf_z + x^{p+r+1}f_x$$

so that the relevant functional derivative of the appropriate operator $f \mapsto P(f)$ sending $0 \mapsto 0$ is,

$$\begin{aligned} P'(0) \ : \ f \longmapsto & \Big\{\widetilde{u}(x,0,z)z\frac{\partial f}{\partial z} + x^{p+r+1}\frac{\partial f}{\partial x}\Big\} + f\Big\{\widetilde{u}(x,0,z) - \lambda x^p \widetilde{v}(x,0)\Big\} \\ & - x^k z f \alpha_y(x,0,z) \end{aligned}$$

and we have to achieve $P(f) = x^k\alpha(x,0,z)$. Nevertheless we have the inconvenience that the node in the above functional derivative is not in normal form, albeit, $\widetilde{u}(x,0,z) = u(x,0) + x^k c(x,z)$, so the situation is not too bad. Dividing through $P(f)$ by $\widetilde{u}(x,0,0)$, and making an appropriate Schlict mapping in x, we already achieve the preparatory steps encountered in the proof of IV.2.3. Here the situation is in a sense easier, and we only have to conjugate the variable z, which we attempt to do in the same way. When $r = 0$, the situation is exactly as per op.cit., and the argument in x drops from $3\pi/p$ to $2\pi/p$, since the appropriate infinity is, again, the negative of that encountered in the construction of K_2 prior to V.2.2. In the situation $r > 0$, we've made no attempt to do anything except keep a small open sector about our direction of interest. If in the notation pre IV.2.3, in the variable $\zeta = x^{-(p+r)}$, this sector contains $+\infty$, nothing changes. Otherwise the only way this doesn't happen is if our sector is wholly in $\mathrm{Re}(\zeta) < 0$, in which case we take a domain of the form shown in figure V.2.3, for s a first integral. Such a leaf is bounded, but we're interested

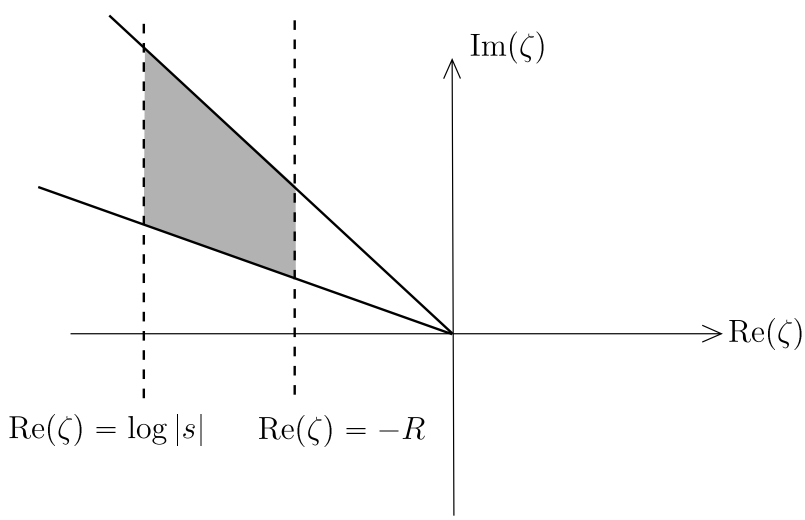

Figure V.2.3

in the minimum value of $\mathrm{Re}(-\zeta)$ to find our base point, so anything on the line $\mathrm{Re}(-\zeta) = -R$ will do, or better move the line a little off vertical so that the minimum is unique. Essentially this is similar to something like IV.2.4(a), since the problem with bounded leaves is when the base point is on the mobile (function of s) as opposed to the fixed boundary.

Consequently we obtain new coordinates (x, z), still on $S \times \Delta$, but now S has aperture $2\pi/p$ if $r = 0$, such that the functional derivative has the form,

$$P'(0)f = \left(x^{p+r+1}\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial z}\right) + f + \varepsilon(x,z)f$$

where as ever $\varepsilon(x, z)$ goes to zero as $x \to 0$, which, by coincidence, up to ε is exactly the same operator that we had to deal with in constructing the conjugation. Again this operator has a bounded right inverse independent of the radius in x, so the habitual power series gives a bounded right inverse to $P'(0)$ even after adjoining small discs about the base point if necessary when $r > 0$, and whence we deduce:

V.2.4 Quick Note on Branching For $r > 0$, since we're limiting our ambition to an open sector around every direction nothing changes from V.2.3. For $r = 0$, the conformal mapping $x \mapsto x^{-(p+r)}$ must be branched within $\pm\pi/2$ of -1, and we have a sector of width $2\pi/p$.

V.2.5 Fact In the linear and nodal cases envisaged by V.2.1, every value of the argument of x is contained in an open sector S of some small radius (the prescription being given by V.2.4) such that on $S_x \times \Delta^2_{(y,z)}$ we can find an invariant hypersurface which is as tangent as we please (i.e. as per V.2.3, do $y = \tilde{y} + x^e z f$, for $k \gg e \gg 0$, to get whatever one wants modulo powers of (x) in the initial convergent coordinates) to the invariant formal hypersurface denoted by the same letter in the formal normal forms V.2.1(a)-(c).

V.3 Solving the PDE

Plainly most the the work has been done in §II-IV and it remains only to check that the equation of the centre manifold is of a type encountered therein. There are some subtleties as suggested in V.2, so we put ourselves in the situation of V.1.8, and apply V.1.9 according to cases of ascending difficulty.

To begin with, suppose we are in the situation of only one central component and apply V.1.9 by way of multiplication by a unit to obtain a convergent generator ∂ of the foliation on a polydisc or even étale open such that for (convergent) coordinates x, y, z:

$$\partial z = z \;\; (I^k), \quad \frac{\partial x}{x} = x^p a(x,y) \;\; (I^k), \quad \frac{\partial y}{y^\varepsilon} = x^p b(x,y) \;\; (I^k)$$

where $k \in \mathbb{N}$ is some large integer to be chosen, $I = (z, x)$, and a, b are possibly truncated versions of their manifestations in V.1.9 since the latter were functions after completion in x. Now blow up in the central component $z = x = 0$, then the congruences for $x^{-1}\partial x$, $y^{-\varepsilon}\partial y$ become congruences modulo $(x)^k$. Whence, replacing V.1.8 by a blow up in the central component (which in no way changes $\widetilde{S}$), multiplying through by the unit $u = \big(1 - \partial x/x\big)^{-1}$, and changing k to $k-1$, a to ua etc., we may suppose,

$$\partial z = z + x^k \gamma, \quad \frac{\partial x}{x} = x^p a + x^k \alpha, \quad \frac{\partial y}{y^\varepsilon} = x^p b + x^k \beta$$

for a, b functions of x and y, and α, β, γ functions of x, y, z. Now let's look for the centre manifold in the form of a graph $\tilde{z} = z - f(x,y)$, which yields the PDE,

$$P(f) = -x^k \gamma(x,y,0)$$

where $f \mapsto P(f)$ is the operator:

$$\begin{aligned} f + x^k\big(\gamma(x,y,f) - \gamma(x,y,0)\big) - x^p\big(a(x,y) + x^{k-p}\alpha(x,y,f)\big)x f_x \\ - x^p\big(b(x,y) + x^{k-p}\beta(x,y,f)\big)y^\varepsilon f_y \end{aligned}$$

This has functional derivative,

$$\begin{aligned} P'(0) \ : \ f \longmapsto \big(1 + x^k \gamma_z(x,y,0)\big) f - x^p\big(a(x,y) + x^{k-p}\alpha(x,y,0)\big)x f_x \\ - x^p\big(b(x,y) + x^{k-p}\beta(x,y,0)\big)y^\varepsilon f_y \end{aligned}$$

So take $-D$ to be the vector field that corresponds to the part of the functional derivative of order -1, then, modulo the order zero terms (i.e. functions of f),

$$P(f) = -Df - x^k\Big[\alpha(x,y,f) - \alpha(x,y,0)\Big]x f_x - x^k\Big[\beta(x,y,f) - \beta(x,y,0)\Big]y^\varepsilon f_y$$

In all of the set ups encountered in §II-IV there is announced a basis of derivations in which the operator should be polynomial. Here it's actually linear, but regardless, this basis always contains D and the module $x^n T(-\log x, -\varepsilon \log y)$, for some fixed n (basically p, but maybe not in III.1 and IV.3), and T the module of derivations in x and y, provided we're not in the situation of III.3.1(c) or IV.2.1, where things are more delicate, and the discussion is postponed. Whence taking $k \gg 0$, we have in all other cases where $q = 0$, no problem in satisfying the condition enunciated in the various set ups, plainly at which point we can no longer work étale locally and we restrict to the analytic topology to satisfy the equation on neighbourhoods governed by the type of the singularity of D, which are everywhere canonical around $x = 0$, and even reduced, where appropriate to eliminate one case. Better still if we look for a solution $\tilde{z} = z - x^e f(x,y)$, for any $0 \ll e \ll k$, the functional derivative's -1 part is unchanged on dividing through by x^e, and the formula for the -1 part of $P(f)$ is as above except that α, β are evaluated at $x^e f$. So, we can achieve any approximation to the formal central manifold in the completion of the central components that we please.

We now pass to the case where $z = y = 0$ is a central component, so necessarily $q > 0$. Our preparation by way of V.1.9 is wholly analogous, the displayed formulae at the first step being exactly the same up to fixing $\varepsilon = 1$, replacing x^p by $x^p y^q$, and taking I to be the ideal (z, xy). One resolves this ideal in 2-steps, viz: first blow up one central component then the other, the order being irrelevant, and $\widetilde{S}$ remains unchanged. Continuing, as before, to denote an approximate local equation for it by $z = 0$, multiplying through by the unit $\big(1-\partial x/x-\partial y/y\big)^{-1}$, and so forth, we may thus assume that we have a prepared form:

$$\partial z = z + (xy)^k\gamma, \quad \frac{\partial x}{x} = x^p y^q a + (xy)^k \alpha \quad \frac{\partial y}{y} = x^p y^q b + (xy)^k \beta$$

for a, b functions of x and y, and α, β, γ functions of x, y, z. Whence, if as before $-D$ is the -1 part of the functional derivative, the difference between this and the appropriate operator $P(f)$ in order -1 is given by,

$$P(f) = -Df - (xy)^k\Big[\alpha(x,y,f) - \alpha(x,y,0)\Big]xf_x - (xy)^k\Big[\beta(x,y,f) - \beta(x,y,0)\Big]yf_y$$

The basis of polynomial terms occurring in §II-IV where this discussion applies all contain the module of derivations $(x^p y^q)^N T(-\log x, -\log y)$, for some N, in fact $N = 1$ except for the bad case around the negative real axis in IV.4. Consequently we can do all of these cases and by an appropriately large choice of $k \gg e \gg 0$, even find the centre manifold in the form $\widetilde{z} = z - (xy)^e f(x,y)$, for any e.

It therefore remains to apply V.2.5 in order to deal with the problem cases of type III.3.1(c) or IV.2.1. Here $q = \varepsilon = 0$, and the difficulty is stable whether under blowing up in points or central components. Furthermore, completing in a maximal ideal then blowing up, as opposed to blowing up and completing in the exceptional divisor is the same operation where we have a problem, so we can make our preparation either according to V.2, or as we've done initially here, i.e. both the current preparations modulo $(x)^k$ and those of V.2 are valid simultaneously. We then make a coordinate change implied by V.2.3 and V.2.5 of the form $y \mapsto y + x^k f(x,z)$, $z \mapsto z + x^k g(x)$, for f, g functions on $S \times \Delta$, or S' respectively, with S, S' as per V.2.5 and V.2.3. Consequently on $S \times \Delta^2$ we arrive to,

$$\partial z = z + x^k\gamma, \quad \frac{\partial x}{x} = x^p a + x^k \alpha, \quad \frac{\partial y}{y} = x^p b + x^k \beta$$

For $a, b, \alpha, \beta, \gamma$ as before except that the domain of x is now restricted to S. This yields the polynomiality in the operator required by III.3.1(c) or IV.2.1 for exactly the same reason as already encountered above, but at the price of restricting the domain of x. Worse still the operator in question is rather sensitive to coordinate changes, and this operation really changes the -1 part, D, of the functional derivative, albeit only modulo x^k, nevertheless its domain of definition need not be a bi-disc in (x, y), but is only a priori $S_x \times \Delta_y$, so that III.3.3 and IV.2.5 cannot be applied as stated.

Fortunately the right inverses constructed in these cases are all bounded, and rather robust, so §III.3 and §IV.2 can still be applied provided we can achieve the conjugations implied by III.3.1(b) and IV.1.3.

The latter we have already seen in solving the equation that afforded V.2.5. Indeed in IV.2, we construct a bounded right inverse, whence the same for a small bounded perturbation, so we only need to conjugate the foliation (not the field as per IV.1) to normal form, so the computation is exactly the same as V.2, even if the particular node is different the principle is the unchanged, i.e. nodes may be conjugated to normal form in any open sector provided one starts from a position where the form along the strong branch is a sufficiently high order of approximation to the normal form. Here, of course, one can start from an arbitrarily good approximation, and whether in IV.2 or V.2 we never attempted to obtain more than an open sector in x, while the domain of y remains a disc whether in V.2 or in the conjugation currently envisaged, so IV.2 can be applied mutatis mutandis to solve the requisite PDE and again with an arbitrary order of tangency to the formal solution modulo powers of (x). If one wishes to calculate the optimal aperture where this can be achieved one should be cautious because of the problem already encountered in V.2.2, which is twice compounded (i.e. notations as per op.cit., we subsequently have to construct K_1', K_2' with branching in the opposite direction to K_1, K_2 found therein) via V.2.4 and the above, and then one still has to apply IV.2.

This leaves us to prove an analogue of III.3.1(b), but under weaker hypothesis. Again the situation is one where a bounded right inverse was achieved, so we only need to bring the foliation into normal form, whence the following will do,

V.3.1 Lemma Let ∂ be a vector field on a sector S (in practice of width $2\pi/p$) times a disc Δ with coordinates $x \in S$, $y \in \Delta$ such that,

$$\partial = x\frac{\partial}{\partial x} + y\big(\lambda + xg(x,y)\big)\frac{\partial}{\partial y}$$

with $\mathrm{Re}(\lambda) < 0$ and g holomorphic and bounded on $S \times \Delta$, then there exists a Schlict mapping $(x, Y)\colon S \times \Delta \to S \times \Delta$ (modulo appropriate shrinking of radii and loss of epsilon in aperture) conjugating the field to:

$$x\frac{\partial}{\partial x} + \lambda Y\frac{\partial}{\partial Y}$$

Proof. We first prepare the situation modulo y, i.e. for $Y = e^{f(x)}y$ attempt to solve $Y^{-1}\partial Y = \lambda\,(\mathrm{mod} y)$. This amounts to the ODE,

$$f'(x) = -g(x,0)$$

which can even be solved with $\sup_{x\in S}|f|/|x| \leqslant \|g\|$, so this is very comfortably Schlict. Whence without loss of generality, $y\big|g$, so, in a minor confusion of notation we replace $g(x,y)$ by $yg(x,y)$. Now consider a normal form,

$$D = x\frac{\partial}{\partial x} + \lambda Y\frac{\partial}{\partial Y}$$

and look for a Schlict mapping, $(x,Y) \mapsto \big(x, y(x,Y)\big)$ to our given field in the form, $y = Ye^{Yf(x,Y)}$. Whence we have to solve the PDE,

$$\lambda f + Df = e^{Yf}xg(x, Ye^{Yf})$$

which can even be done by the contraction mapping principle. Alternatively written as $P(f) = xg(x, Y)$ for a suitable operator P, the functional derivative is,

$$P'(0) \ : \ f \longmapsto \lambda f + Df + fxY\Big(g(x,Y) + g_y(x,Y)\Big)$$

and $P(f) - P'(0)f$ is an order 0 operator, i.e. one can take ϕ a constant in I.3.2(b). Irrespectively, in the first instance we need a right inverse for,

$$\mathcal{L}f = \lambda f + Df$$

and to avoid confusion put $\lambda = -\nu$, where $\mathrm{Re}(\nu) > 0$. We pass to logarithmic coordinates $e^{-\xi/\nu} = x$, $e^{\eta} = y$, $\sigma = \eta - \xi$, and divide through by λ, to obtain

$$\lambda^{-1}\mathcal{L}f = f + \frac{\partial f}{\partial \xi}$$

in σ, ξ coordinates. Now the domain of η is a left half plane $\mathrm{Re}(\eta) < -R$, while that of ξ, T say, is a right half strip rotated through ν. Consequently for an appropriate choice of strip/determination of $\log x$ or even $|x|$ sufficiently small, the domain of σ and η coincide. Viewing our original $S \times \Delta$ as a fibring over the domain of σ, the fibres are, therefore,

$$L_\sigma = \big(\sigma + T\big) \cap \mathrm{Dom}(\eta)$$

leading to a domain in ξ space, over which we must integrate, of the form shown in figure V.3.1. Unsurprisingly therefore, notation as per the diagram,

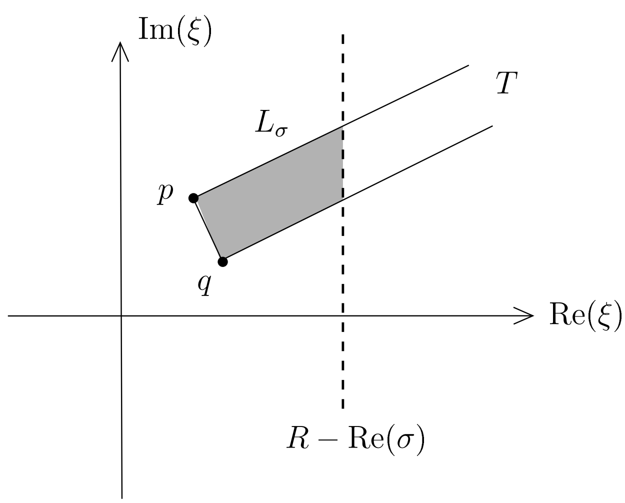

Figure V.3.1

$$\left(K_0 g\right)(\xi) = e^{-\xi}\int_p^{\xi} g(\xi,\sigma)e^{\xi}d\xi$$

works fine, and $P'(0)K_0 = 1 + \varepsilon$, ε a small function divisible by x and Y. So shrinking the radius in x alone is wholly adequate to provide a right inverse to $P'(0)$. This solves the PDE by the implicit function theorem, and the resulting mapping, as per IV.1.3, is comfortably Schlict on admitting a small loss of radius and aperture in x. $\square$

Evidently this has a certain nuisance, which we observe by way of,

V.3.2 Warning If, notations of the proof of V.3.1, one looks for $f = x^n F$, some $n \in \mathbb{N}$, then the leading part of the functional derivative will change to,

$$\big(n + \lambda\big)\mathbb{1} + D$$

Consequently if we take n too big, we fall into a fairly inconvenient situation where we'd have to take base points on the line $-R - \mathrm{Re}(\sigma) = \mathrm{Re}(\xi)$, which tends to result in some messy restriction in $\log y$, or worse σ. Plainly, however, we're okay provided that $n < -\mathrm{Re}(\lambda)$. It's also true that from a valuation theory perspective the problem is, supposed non-algebraic for simplicity of exposition, the discrete valuation v obtained by taking the order of vanishing at the origin on the formal curve $y = z = 0$. In particular blowing up here, $\lambda \mapsto \lambda - 1$, and one can follow v, and blow up ad nauseum to have $\mathrm{Re}(\lambda)$ as negative as one wishes. In such a sense one can achieve arbitrary good approximations to the formal central manifold around the central components, but only on a model that depends on the approximation which is desired.

This said we can therefore summarise our conclusions by way of,

V.3.3 Fact Suppose the central components are compact, or start from a germ à la V.1.1. Then following a bi-rational modification of the form V.1.8, possibly augmented by further blowing up in points to accommodate the problems associated with singularities of the type described in V.2.1, the trace (i.e. the central components) of the formal central manifold can be covered by finitely many open sets, isomorphic to either discs or a neighbourhood of a plane node. In the former case take y to be a coordinate on a disc, in the latter x, y plane coordinates and $xy = 0$ the node, and similarly $xy = 0$ the singular locus even if we're in a neighbourhood where central and non-central components meet. In any case there is always a further coordinate x whose zero locus is an exceptional divisor intersecting the central sub-scheme in a central component. This yields a bi-disc $\Delta^2 \ni (x, y)$. Denote by I the ideal of the reduced scheme structure of the singular locus, and by V.1.9, let z be a third coordinate which is an approximation to some fixed order e of the central sub-scheme $\hat{z} = 0$ in the formal I-adic completion. Then, modulo the precision V.3.2, there is a finite covering $\{U_k\}$ of either $\Delta^2 \backslash V(I)$, or possibly only $\Delta^2 \backslash (xy = 0)$, in the situation of V.2.1 with $y = 0$ understood as the curve encountered in V.2.3 (so really, $S \times \Delta \backslash (xy = 0)$) and finitely many sectors S in x) - the exact form of the covering being determined by the formal type of the induced foliation in the formal central manifold, and being detailed on a case by case base in §II-IV - such that on each U_k there is a bounded holomorphic function $\zeta_k(x, y)$ such that the zero locus of $z - \zeta_k(x, y)$ is invariant and agrees with $\hat{z} = 0$ to some prescribed order determined by the approximation z, and indeed goes to infinity as the quality of the approximation z goes to infinity. Plainly, modulo the above caveat that occurs for singularities of the form V.2.1, an open neighbourhood of the singular locus punctured in the said locus admits a finite covering where such invariant hypersurfaces may be found. In this sense the formal central sub-scheme may be said to "converge" in dimension 3.

VI Normal Forms

VI.1 Scalar vs. Vector

So far we've largely concentrated on obtaining right inverses to scalar rather than vector equations. The general relation between these is a whole can of worms that it's preferable to avoid, so let's confine ourselves to some useful facts as we will employ them, albeit we can certainly permit a wholly general,

VI.1.1 Set Up Let $U \subseteq \mathbb{C}^n$ be a domain , with $U(\underline{d})$ as encountered in I.3.2(a), and $P : E \to E$ a uniformly $C^{1,\alpha}$, $\alpha > 0$, operator on an analytic vector bundle in the sense of I.3.2(b). Suppose further that there is a vector field D on U, and a linear endomorphism $A \in \Gamma\big(U, \mathrm{Hom}(E, E)\big)$ such that,

$$P'(0) = D\mathbb{1} + A$$

and that $E\big|_U$ is trivial, so that A may be identified with a $r \times r$ matrix, for r the rank of E.

Plainly in such a situation we have good possibilities to reduce to scalar equations. The precise values of the scalars may be important so let us formalise this by way of,

VI.1.2 Definition Let $\chi \in \mathbb{C}$ (in practice non-zero), and D as per VI.1.1, then the spectral problem at χ is said to admit a polynomial solution if, as per I.3.2(c), there are a family of right inverses $K(\underline{d})$ for domains $U(\underline{d})$ between $U(\underline{\delta})$ and U to $\chi + D$ such that, for $U(\underline{\delta}) \subset U(\underline{d}) \subset U(\underline{e}) \subset U$, there are constants C_χ, $n \geqslant 0$, independent of $\underline{d}$ and $\underline{e}$ for which,

$$\big\|Kh\big\|_{U(\underline{d})} \leqslant C_\chi \big\|h\big\|_{U(\underline{e})} \prod_{i=1}^{n} (d_i - e_i)^{-n}$$

In the particular case that $n = 0$, consistent with the previous usage, we say that K is bounded.

The bounded case is particularly simple, and was already encountered in the proof of IV.1.3. We formalise it as follows:

VI.1.3 Triviality Let $U(\underline{d})$ be as per I.3.3 for P a polynomial operator in the implied standard fields $\frac{\partial}{\partial z_i}$ in the sense of I.3.4. Suppose further that A is semi-simple, modulo a small perturbation ε, and that the solution to the spectral problem for each eigenvalue χ is bounded. Then, provided ε is sufficiently small ($\|\varepsilon\| \max_\chi C_\chi$, in the notation of V.1.2) the Cauchy problem for $P'(0)$ has a non-ludicrous solution in the sense of I.3.2(c), in fact even of polynomial type as encountered in I.3.6.

Proof. By hypothesis, A is diagonal with eigenvalues $\chi_1, \ldots, \chi_r$, and for each eigenspace there is a right inverse K_i to the operator $\chi_i \mathbb{1} + D$ which is bounded. Whence the diagonal operator K_0, consisting of K_i on each eigenspace affords,

$$P'(0)K_0 = \mathbb{1} + \varepsilon K_0$$

so provided ε is sufficiently small, the usual power series may be applied. ▫

The situation is potentially much more subtle for unbounded solutions of the spectral problem but we've already hinted at how to do it in I.4.5, which we now make explicit, viz:

VI.1.4 Claim Let things be initially be as per VI.1.3 but now suppose that A is semi-simple, modulo a small perturbation ε. Suppose further that the spectral problems not just for the χ_i, but also the differences $\chi_i - \chi_j$, including $i = j$, admit a polynomial solution, and, in the sense of I.3.6, ε decreases at a sufficiently rapid polynomial rate in comparison with that for the totality of the $\chi_i - \chi_j$ spectral problems. Then potentially for a different U which has been shrunk accordingly, and on which the χ_i spectral problems are still polynomial, the Cauchy problem for $P'(0)$ has a non-ludicrous solution in the sense of I.3.2(c), and even of polynomial type in the sense of I.3.6.

Proof. Let χ be the diagonal matrix with entries χ_i. A conjugation of the operator $D\mathbb{1} + \chi$ by an invertible matrix of functions Q yields the operator,

$$D\mathbb{1} + \big(Q^{-1}\big(DQ\big) + Q^{-1}\chi Q\big)$$

so for $Q = \exp(q)$, $q \in \mathrm{gl}_r(\mathcal{O}_U)$, we want to solve,

$$\exp(-q)D\exp(q) + \exp(-q)\chi\exp(q) - \chi = \varepsilon$$

which has a functional derivative,

$$D + \big[q, \chi\big]$$

and, of course, $q \mapsto [q, \chi]$ is a semi-simple endomorphism of gl_r with eigenvalues $\chi_i - \chi_j$. By hypothesis, therefore, the implied operator in q satisfies the conditions of the implicit function theorem, and we have a solution as soon as ε is sufficiently small. Of course, this may be difficult to guarantee if we're not in the situation envisaged in I.3.6, or similar, at which point we may have to shrink the domain U to guarantee a solution. At this point, however we have found a matrix of functions, such that the implied change of basis puts $P'(0)$ in the form,

$$D\mathbb{1} + \chi$$

so, again, by hypothesis, we conclude. ▫

While this is a perfect sufficient lemma for a situation such as I.4.5, the condition that the spectral problem in zero, i.e. $\chi_i = \chi_j$, admits a solution is not practical, so we make,

VI.1.5 Remark Let us put ourselves in the situation of VI.1.4 but suppose now that A is semi-simple with distinct eigenvalues χ_i, and only the spectral problems for $\chi_i - \chi_j$, $i \neq j$, admit a polynomial solution. Under these hypothesis, with the same hypothesis on ε, we can, on an appropriately smaller domain, effect a change of basis by a matrix of functions such that $P'(0)$ has the form,

$$D + A$$

where A is semi-simple, but the eigenvalues are now functions $\widetilde{\chi}_i$ which up to a small perturbation are the χ_i. In practice we're only concerned about situations similar to IV.3, where the implied difficulty is already overcome by way of the normal form IV.1.3, or, to a much lesser extent, II.2, since, where we used power series there, which was by no means essential, the normal form was trivial to obtain.

VI.2 Quasi-formal preparation

We blow up sufficiently as per V.1.8 and consider the analytic space X (or stack, albeit this can be largely eschewed) around the singular components, which we complete in the central components, in the sense implied by V.1.9, to form a formal analytic space (or stack) $\widehat{X}_{\mathrm{an}}$. As such the formal affines have rings of functions of the form,

$$\varprojlim_n \frac{\mathcal{O}_X}{(z, x^p y^q)^n}$$

notations as per V.1.9 with $\mathcal{O}_X$ holomorphic functions. This permits some simplification over the general formulae of op.cit. at the price of a moderately long list and a couple of caveats, viz:

VI.2.1 List Possibly after blowing up of $\widehat{X}_{\mathrm{an}}$, every geometric point has at worst an analytic neighbourhood (i.e. often étale is possible if our context were algebraic) with coordinates x, y, z, conventions as per V.1.9 on which we find a generator of the induced foliation which enjoys one of the following normal forms,

(a) $z\dfrac{\partial}{\partial z} + x^p \dfrac{\partial}{\partial y}, \quad p \in \mathbb{N}.$

(b) $z\dfrac{\partial}{\partial z} + \dfrac{x^{p+1}}{1+\nu(y)x^p}\dfrac{\partial}{\partial x}, \quad p \in \mathbb{N},\ \nu \in \mathbb{C}\{y\}.$

(c) $z\dfrac{\partial}{\partial z} + \dfrac{x^{p+1}y^q}{1+\nu(y)x^p}\dfrac{\partial}{\partial x}, \quad p, q \in \mathbb{N},\ \nu \in \mathbb{C}\{y\}.$

(d) $z\dfrac{\partial}{\partial z} + x^p y \dfrac{\partial}{\partial y}, \quad p \in \mathbb{N}.$

(e) $z\dfrac{\partial}{\partial z} + \dfrac{x^p y^q}{1+\nu x^p y^q}\Big(x\dfrac{\partial}{\partial x} + \lambda y \dfrac{\partial}{\partial y}\Big), \quad p \in \mathbb{N},\ q \in \mathbb{Z}_{\geqslant 0},\ \nu \in \mathbb{C},\ \lambda \notin \mathbb{Q}.$

(f) $z\dfrac{\partial}{\partial z} + x^p\Big(R(x)y\dfrac{\partial}{\partial y} + \dfrac{x^{r+1}}{1+\lambda x^{p+r}}\dfrac{\partial}{\partial x}\Big), \quad p, r \in \mathbb{N},\ \deg R \leqslant r,\ \lambda \in \mathbb{C}.$

(g) $z\dfrac{\partial}{\partial z} + \dfrac{x^p y^q}{1+y^q\big(R(x)+\lambda x^{p+r}\big)}\Big(y\dfrac{\partial}{\partial y} + \dfrac{x^r}{1+\nu x^r}\big(qx\dfrac{\partial}{\partial x} - py\dfrac{\partial}{\partial y}\big)\Big),$

where $p,q,r \in \mathbb{N}$, $\deg R \leqslant r-1$ and $\lambda,\nu \in \mathbb{C}$. The above normal form (g) cannot, in general, be obtained in a neighbourhood of $\widehat{X}_{\mathrm{an}}$, but only in the weaker completion at a point, or, equivalently after blowing up, in the divisor $x=0$. Nevertheless the remaining cases, even though some reduce to (g), can be obtained in neighbourhoods of $\widehat{X}_{\mathrm{an}}$, and are all for a negative rational eigenvalue $-k/l$, with k,l relatively prime integers, viz:

$$\text{(h)}\ \ z\frac{\partial}{\partial z}+\frac{x^iy^j(x^ky^l)^n}{1+x^iy^j\big(R(x^ky^l)+\lambda(x^ky^l)^{n+r}\big)}\Big\{\big(lx\frac{\partial}{\partial x}-ky\frac{\partial}{\partial y}\big)+\frac{(x^ky^l)^r}{1+\nu(x^ky^l)^r}\big(jx\frac{\partial}{\partial x}-iy\frac{\partial}{\partial y}\big)\Big\},$$

where $x^py^q=(x^iy^j)(x^ky^l)^n$ with at least one of $i,j\in\mathbb{Z}_{\geqslant 0}$, less than k, respectively l, $(i,j)\neq 0$, $n\in\mathbb{Z}_{\geqslant 0}$, $r\in\mathbb{N}$, $\lambda,\nu\in\mathbb{C}$, with convention on (p,q) as per V.1.9, and otherwise R of degree at most $r-1$.

$$\text{(i)}\ \ z\frac{\partial}{\partial z}+(x^ky^l)^n\Big\{R(x^ky^l)\big(lx\frac{\partial}{\partial x}-ky\frac{\partial}{\partial y}\big)+\frac{(x^ky^l)^r}{1+\nu(x^ky^l)^{n+r}}x\frac{\partial}{\partial x}\big)\Big\},$$

for $n,r\in\mathbb{N}$, $\deg R\leqslant r$, $R(0)\neq 0$, $\nu\in\mathbb{C}$.

$$\text{(j)}\ \ z\frac{\partial}{\partial z}+\frac{x^iy^j(x^ky^l)^n}{1+x^iy^j\ \nu(x^ky^l)}\Big(lx\frac{\partial}{\partial x}-ky\frac{\partial}{\partial y}\Big),$$

for ν a formal function, and the multiplier of $(x^ky^l)^n$ understood to be 1 if $(i,j)=0$. Otherwise, prescriptions on i,j,n are exactly as per (h).

VI.2.1 (bis) Proofs of cases (a)-(g) This is occasionally a bit more tricky than one might think, since by V.1.9 one is only reduced to finding a normal form for a plane field in an object which itself is only a formal analytic space with trace the central components, i.e. it is not the completion of an analytic space, so certain formulae that one might think of applying are not a priori justifiable.

In the first instance consider (a). By hypothesis we are in a neighbourhood of a point, 0 say, in the trace together with a formal field on the formal centre manifold of the form,

$$x^p\partial=x^p\Big\{u(x,y)\frac{\partial}{\partial y}+xa(x,y)\frac{\partial}{\partial x}\Big\}$$

where u is a unit, so without loss of generality $u(0)=1$, and the formal function $x=0$ defines the trace. In particular by the definition of $\widehat{X}_{\mathrm{an}}$, the restriction of ∂ to the trace is convergent. Consequently restricting the neighbourhood of 0 and changing coordinates appropriately we may suppose that $u\equiv 1(\operatorname{mod} x)$, for some convergent function y on the trace. Now consider a change of variables $\xi=e^fx$, for which $\partial\xi=0$. This amounts to solving,

$$\partial f=-a(x,y)$$

To do this choose a mapping from functions on the trace $\mathcal{O}_\Delta$, say, to $\mathcal{O}_{\widehat{X}_{\mathrm{an}}}$, so we can write $\mathcal{O}_{\widehat{X}_{\mathrm{an}}} = \mathcal{O}_\Delta[[x]]$. Whence we can expand in Taylor series and integrate term by term to obtain a continuous right inverse K_0, say, to $\frac{\partial}{\partial y}$, and whence if $u(x,y) = 1 + xb(x,y)$ then,

$$\partial K_0 = \mathbb{1}(1 + xb(x,y))$$

which is a topologically nilpotent perturbation of the identity so ∂ has a right inverse, and we can suppose $\partial x = 0$. Finally, therefore, we look for $\eta = e^g y$ such that $\partial\eta = x^p$, which amounts to the equation,

$$e^g\Big(1 + y\frac{\partial g}{\partial y}\Big) = u^{-1}$$

Now we can apply I.4.1. The functional derivative is,

$$P'(0) \ : \ g \longmapsto g + y\frac{\partial g}{\partial y}$$

which has continuous right inverse by power series in x, followed by term by term power series expansion in y, which, inter alia, respects the filtration by the ideals (x^n), $n \in \mathbb{N}$, so $PK - \mathbb{1}$ is nilpotent, and the fact that we've already achieved our solution $\bmod x$, guarantees the conditions of I.4.1.

The case (b) is much simpler. Here by hypothesis we start with a plane field of the form,

$$x^p\partial = x^{p+1}\Big\{u(x,y)\frac{\partial}{\partial x} + a(x,y)\frac{\partial}{\partial y}\Big\}$$

with u a unit. Here if $\widetilde{\partial}$ denotes the division by u, the formula,

$$\widetilde{y} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n\widetilde{\partial}}{n!}y$$

is convergent in $\mathcal{O}_{\widehat{X}_{\mathrm{an}}}$, even étale locally if that were our context, so without loss of generality $\partial y = 0$. Consequently functions of y are now like constants, and proceeding inductively we obtain the asserted form, and again even étale locally if that where the context, i.e. $\nu(y)$ in the strict Henselisation of the trace.

Now the argument of (a), respectively (b), applies mutatis mutandis to establish (c) after completion in $y = 0$, respectively $x = 0$. Of itself, however, this does not establish (c), since the topology here is convergence modulo powers of xy. This latter assertion is true, but since we're prepared to blow up ad nauseum, we can eschew the tedium of actually checking it by observing that, after blowing up, completion in x and y, separately, becomes completion in the singular locus. This subterfuge introduces new cases of type (a), but we know how to do that, together with one case of type (c), and another singular one that we'll come to directly. It also introduces a multiplier in the field in x and y, but in a straightforward way that is easily eliminated.

The same blowing up considerations, in combination with the ubiquitous case (a), imply that it suffices to verify cases with $q = 0$ in the completion at a point, and those with $pq \neq 0$ in, without loss of generality, the completion in $x = 0$ should some symmetry be present. Consequently (d) is easy, and (e) is reduced to the Jordan form V.2.1 for $q = 0$.

Applying the same considerations to (e) for $q \neq 0$, we may suppose that we have our normal form to any order in x and try to conjugate to the desired one in a way that converges modulo powers of y. Arguing as per op.cit. with $\partial = \xi\partial/\partial\xi + \lambda y\partial/\partial y$, then after a linear change of coordinates the functional derivative becomes,

$$\begin{bmatrix} 0 & 0 \\ 0 & -(p+q\lambda) \end{bmatrix} + \mathbb{1}\partial$$

Our preparations assure that this is un-obstructed mod y, after which we can expand in powers y^n of y, and the situation remains un-obstructed for each n since $\lambda \notin \mathbb{Q}$, so that we obtain the desired solution by I.4.1.

This brings us to nodes, (f) having already been done in IV.1.2. To do (g), which is only true mod completion in $x = 0$, we know that after blowing up it is equivalent to completion in a point, so we'll avail ourselves of such a simplification. Whence, by hypothesis, we have a formal plane field,

$$D = x^p y^q \partial$$

with ∂ the generator of a node, $pq \neq 0$, and $x = 0$, respectively $y = 0$, the strong, respectively weak, branch. Looking at ∂ mod y, after scaling in x and y we may suppose,

$$\partial = u(x)y\frac{\partial}{\partial y} + v(x)x^{r+1}\frac{\partial}{\partial x}$$

For u, v formal units with $u(0) = 1$, $v(0) = q$, so in an obvious abuse of notation say, $\partial = \partial_0$ mod y, for

$$\partial_0 \doteqdot u(x)y\frac{\partial}{\partial y} + x^r v(x)\Big(qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}\Big) \doteqdot u(x)D_1 + x^r v(x)D_2$$

where now $u(0) = v(0) = 1$. In particular, we can write $\partial = \partial_0 + \partial_1$ for $\partial_1 \in yT_{\widehat{X}}(-\log x, -\log y)$, for which a convenient basis is, evidently, yD_1, yD_2. Now such fields have formally convergent exponentials, so consider attempting to conjugate $x^p y^q \partial_0$ to D by such, i.e. attempt to solve,

$$\exp(E)x^p y^q \partial_0 \exp(-E) - x^p y^q \partial_0 = x^p y^q \partial_1$$

where we view the left hand side as a formal operator $0 \mapsto 0$, with functional derivative,

$$E \longmapsto [E, x^p y^q \partial_0]$$

which in terms of our basis D_1, D_2 amounts to,

$$aD_1 \longmapsto x^p y^q \big(qa\partial_0 - \partial_0(a)D_1\big), \qquad bD_2 \longmapsto -x^p y^q \partial_0(b)D_2 + bx^p y^q [D_2, \partial_0]$$

and a, b formal functions in the ideal (y), which up to a topologically nilpotent operator amounts to,

$$aD_1 \longmapsto x^p y^q \big(qa - y\frac{\partial}{\partial y}\big) D_1, \qquad bD_2 \longmapsto -x^p y^{q+1} \frac{\partial b}{\partial y} D_2$$

which, expanding in powers of y, is invertible except on y^q mod y^{q+1}. Now observe that we can make an a priori coordinate change $x \mapsto e^{f(x)}x$, $y \mapsto e^{g(x)}y$ such that $u = 1$ and $v = (1 + \nu x^r)^{-1}$, for some $\nu \in \mathbb{C}$. Furthermore, we may successfully perform a conjugation of ∂ into ∂_0 modulo y^q.

For $a = y^q \alpha(x)$ and $b = y^q \beta(x)$ our functional derivative modulo y^{q+1} simplifies to,

$$y^q \alpha D_1 \longmapsto x^p y^q \left\{ \frac{qx^r}{1+\nu x^r}(p\alpha - x\alpha_x) y^q D_1 + \frac{qx^r \alpha}{1+\nu x^r} y^q D_2 \right\}$$

$$y^q \beta D_2 \longmapsto x^p y^q \left\{ -q\beta + \frac{qx^r}{1+\nu x^r}\Big(p\beta - x\beta_x - \frac{r\beta}{1+\nu x^r}\Big) \right\} y^q D_2$$

so we can get everything except terms of the form $R(x)y^q D_1$, $\deg R \leqslant r-1$, and $x^{p+r}y^q D_1$. Consequently we organise things by way of surjectivity in D_2, so that the missing terms take the form of a multiplier of ∂_0, while as already observed modulo y^n, for $n > q$ there are no further obstructions, whence the normal form (g).

The remaining cases may be reduced to this one. Indeed, say the eigenvalue of the implied linear part is $-k/l$, for k, l relatively prime positive integers. Then the implied divisor may be written as $x^i y^j (x^k y^l)^n$ for $i, j, n \in \mathbb{Z}_{\geqslant 0}$ with $i < k$ and/or $j < l$. Furthermore whether in the completion in a point or around a branch, one can achieve an expression for the formal plane field in the form,

$$x^i y^j (x^k y^l)^n \left\{ l\Big(1 + a(x^i y^j, x^k y^l)\Big) x\frac{\partial}{\partial x} - k\Big(1 + b(x^i y^j, x^k y^l)\Big) y\frac{\partial}{\partial y} \right\}$$

for some formal functions a, b vanishing at the origin. Now there are various cases to consider, viz:

VI.2.1 (bis) Case (h) The couple $(i, j) \neq 0$ and the descended plane foliation in $X = x^k y^l$, $Y = x^i y^j$ is singular, then we have the normal form,

$$\frac{x^i y^j (x^k y^l)^n}{1 + x^i y^j \big(R(x^k y^l) + \lambda (x^k y^l)^{n+r}\big)} \Big\{ \big(lx\frac{\partial}{\partial x} - ky\frac{\partial}{\partial y}\big) + \frac{(x^k y^l)^r}{1 + \nu (x^k y^l)^r} \big(jx\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y}\big) \Big\}$$

for some $r \in \mathbb{N}$, $\lambda, \nu \in \mathbb{C}$ and R a polynomial of degree at most $r - 1$.

VI.2.1 (bis) Case (j) As above but the descended plane foliation is smooth, then we have the normal form,

$$\frac{x^i y^j (x^k y^l)^n}{1 + x^i y^j \ \nu(x^k y^l)} \Big(lx \frac{\partial}{\partial x} - ky \frac{\partial}{\partial y} \Big)$$

for some formal function ν. In particular, the difference between (h) and (j) is according to whether the foliation has a formal first integral or not.

VI.2.1 (bis) Case (i) The couple (i,j) is zero but $n \neq 0$. In this case we descend to a plane field in $X = x^k y^l$, $Y = y$ and if the implied foliation in X and Y is singular then we have a normal form,

$$(x^k y^l)^n \Big\{ R(x^k y^l) \big(lx \frac{\partial}{\partial x} - ky \frac{\partial}{\partial y} \big) + \frac{(x^k y^l)^r}{1 + \nu (x^k y^l)^{n+r}} x \frac{\partial}{\partial x} \big) \Big\}$$

for $r \in \mathbb{N}$, R a polynomial of degree at most $r \in \mathbb{N}$, $R(0) \neq 0$ and $\nu \in \mathbb{C}$.

VI.2.1 (bis) Case (j′) As above but the descended foliation is smooth and we obtain the normal form,

$$(x^k y^l)^n \Big(lx \frac{\partial}{\partial x} - ky \frac{\partial}{\partial y} \Big)$$

Finally, it remains to make the same distinctions in the case $n = 0$, i.e.

VI.2.1 (bis) Case (h′) The couple (i,j) is non-zero but $n = 0$, so, in particular by way of cases already considered only that of $q = j = 0$. As such we descend to a plane field in $X = x^k y^l$, $Y = x$ and obtain, whenever the descended foliation is singular, a normal form,

$$\frac{x^p}{1 + x^p R(x^k y^l)} \Big\{ \big(lx \frac{\partial}{\partial x} - ky \frac{\partial}{\partial y} \big) + \frac{(x^k y^l)^r}{1 + \nu (x^k y^l)^{n+r}} y \frac{\partial}{\partial y} \Big\}$$

for some $\nu \in \mathbb{C}$, and R a polynomial of degree at most $r \in \mathbb{N}$.

VI.2.1 (bis) Case (j″) As above but for the descended foliation being smooth, so that we have a normal form,

$$\frac{x^p}{1 + \nu(x^k y^l) x^p} \Big(lx \frac{\partial}{\partial x} - ky \frac{\partial}{\partial y} \Big)$$

for some formal function ν, and, plainly, VI.2.1(bis) (j), (j′), (j″) may be distinguished from VI.2.1(bis) (h), (h′), (i) by the fact that they admit a formal first integral.

It is in addition rather convenient to have on hand further conjugations of these normal forms that bring us closer to one of the formulae employed in §III-IV, to wit,

VI.2.2 Fact (e) The normal form of VI.2.1(e) under the change of coordinates, or better spiralling change of coordinates,

$$X = \frac{x}{\big(1 - \nu x^p y^q \log(x^p y^q)\big)^{1/(p+\lambda q)}}\ , \quad Y = \frac{y}{\big(1 - \nu x^p y^q \log(x^p y^q)\big)^{\lambda/(p+\lambda q)}}$$

becomes the pure monomial form,

$$z\frac{\partial}{\partial z} + X^p Y^q \Big(X\frac{\partial}{\partial X} + \lambda Y \frac{\partial}{\partial Y}\Big)$$

A suitable conjugation of VI.2.1(f) has already been achieved in IV.1.3, while the case of VI.2.1(g) is more subtle, viz:

VI.2.2 Fact (g) For x in a sector up to $2\pi/r$ (actually somewhat better as we'll see in the course of the proof) and y (or better $\log y$) in a spiralling domain as per figure II.2.2, there is a conjugation $x \mapsto X(x,y)$, $y \mapsto Y(x,y)$ under which the normal form VI.2.1(g) becomes the monomial form;

$$z\frac{\partial}{\partial z} + X^p Y^q \left(Y\frac{\partial}{\partial Y} + X^r\Big(qX\frac{\partial}{\partial X} - pY\frac{\partial}{\partial Y}\Big)\right)$$

Proof. Suppose in the first instance $q > 1$ and $\nu = 0$. Then for any $n \in \mathbb{N}$, $n < q$, we can attempt to find the conjugation from the given normal form in x and y to that in X and Y by way of,

$$x \longmapsto X e^{fY^n}, \qquad y \longmapsto Y e^{gY^n}$$

This amounts to solving a coupled system of PDE's with a functional derivative of the form,

$$\begin{bmatrix} f \\ g \end{bmatrix} \longmapsto A \begin{bmatrix} f \\ g \end{bmatrix} + D \begin{bmatrix} f \\ g \end{bmatrix}$$

where D is the field $Y\partial/\partial Y + X^r\Big(qX\partial/\partial X - pY\partial/\partial Y\Big)$, and A is a matrix of the form

$$\begin{bmatrix} n & 0 \\ -p & n-q \end{bmatrix} + \varepsilon$$

where as ever ε is a matrix in small functions. Whence up to ε, diagonal with eigenvalues $f \mapsto nf$, $(pf + qg) \mapsto (n-q)(pf+qg)$. In an attempt to preserve the notation of §IV, we can pass to $\xi = (X^p Y^q)^{-1}$, $s = \xi^{-1}\exp\big(X^{-r}/r\big)$, so that for a certain domain fibred as $s : L \to B$, and embedded by $(s, z(X))$ in $B \times \mathbb{C}$, for $z(X) = X^{-r}/r$,

$$D = -q\frac{\partial}{\partial z}$$

where, to re-iterate, the above appearance of z is a regrettable notational confusion, i.e. its a function of X alone and has nothing to do with the ambient 3-fold. In any case for $\tilde{z} = z(X) - (p/r)\log z(X)$, the fibres of L for σ the logarithm of s are described by

$$\tilde{z}(X) \in \sigma - q \cdot \mathrm{Domain}(\log Y)$$

The difference between $\tilde{z}$ and z is, as per §IV.4, of little importance. What is important, however, is that in the z-plane we must integrate both from right to left (eigenvalue of f), and left to right (eigenvalue of $pf + qg$). The former poses no problem and is essentially, exactly the situation encountered in IV.1.3, the latter presents the difficulty in the full domain of the logarithm of a purely imaginary boundary on the left. Consequently, we restrict the domain of $\log Y$ as per figure II.2.2, which results in a domain in $\tilde{z}(X)$ space as shown in figure VI.2.2. The constant R is of course to be chosen, and the lines meeting $\mathrm{Im}(\tilde{z}) =$

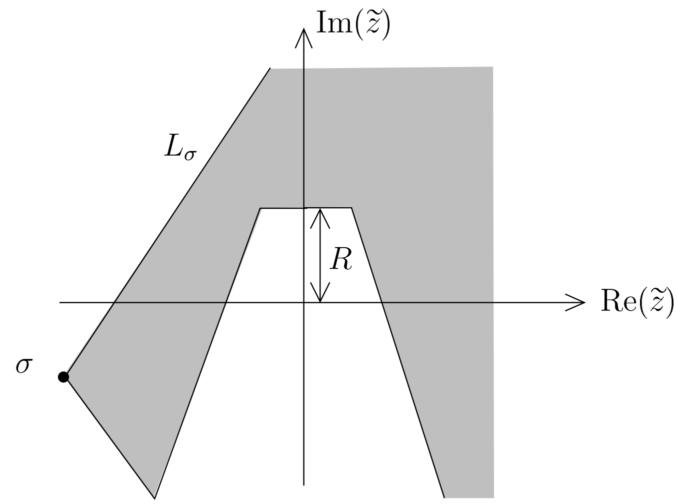

Figure VI.2.2

R simply reflect the sector up to $2\pi/r$ that we're aiming for, so anything not purely imaginary will do, and evidently a similar diagram for $\mathrm{Im}(\tilde{z}) = -R$ to cover everything. At which point there's no problem constructing a bounded right inverse for the functional derivative modulo the usual caveats, viz: add a Euclidean disc about the $(pf + qg)$ base point σ in every fibre according to the conditions of the implicit function theorem and use the usual power series to get the case of $\varepsilon \neq 0$.

Consequently we have achieved our conjugation for $\log y$ in a spiralling domain and x^r in a sector up to 2π. In particular the restrictions on q and ν are seen to be groundless. The latter as per IV.1.3, and the former since Y^n really means, $\exp(n \log Y)$ for $n < q$, which has perfect sense even for $q = 1$ provided we drop the condition that n is an integer, albeit it must be taken positive to ensure that the solution of the PDE leads to a Schlict mapping. □

The case (h) is very much a combination of (e) and (g), i.e.

VI.2.2 Fact (h) Conventions on x, y as per V.1.9, then for one of x or y belonging to a spiralling domain and the other in a disc or a half plane à la figure II.2.2, if $pq \neq 0$, or x in such a domain and y in a disc if $q = 0$, the normal form

(h) may be conjugated via $x \mapsto X(x,y)$, $y \mapsto Y(x,y)$ to the monomial form

$$z\frac{\partial}{\partial z} + X^p Y^q \left(lX\frac{\partial}{\partial X} - kY\frac{\partial}{\partial Y} \right)$$

Proof. Despite its complication we can always assume that the normal form is at least as good as,

$$z\frac{\partial}{\partial z} + x^p y^q \left\{ lx\big(1 + x^k y^l a(x,y)\big)\frac{\partial}{\partial x} - ky\big(1 + x^k y^l b(x,y)\big)\frac{\partial}{\partial y} \right\}$$

Now for $M = X^u Y^v$ any monomial with $u \leqslant k$, $v \leqslant l$, we can attempt to look for the conjugation in the form,

$$x \longmapsto Xe^{fM}, \qquad y \longmapsto Ye^{gM}$$

This leads to a partial differential operator in f, g with functional derivative,

$$\begin{bmatrix} f \\ g \end{bmatrix} \longmapsto \left\{ \begin{bmatrix} \chi - lp & -ql \\ kp & \chi + kp \end{bmatrix} + \varepsilon \right\} \begin{bmatrix} f \\ g \end{bmatrix} + D \begin{bmatrix} f \\ g \end{bmatrix}$$

for D the field $lX\partial/\partial X - kY\partial/\partial Y$, and $\chi = (DM)/M$. For $\varepsilon = 0$, the matrix in question has eigenvalues $\chi, \chi - (lp - kq)$, which in the case of (h) are distinct. Consequently we must invert the scalar operators,

$$\mu + D$$

for μ the eigenvalues. Now observe if $q = 0$, then just as in V.3.1, we can take $\chi < 0$, e.g. $M = Y$ in op.cit., so both eigenvalues are negative. Otherwise according to the sign of $lp - kq$, one chooses M so that both eigenvalues have the same sign.

A priori the logarithms, $e^\xi = X$, $e^\eta = Y$, lie in left half planes, and for $\sigma = k\xi + l\eta$, again in a left half plane, we have a fibring of the domain U of interest, by way of σ over B with fibres in, say, ξ,

$$\mathrm{Domain}(\xi) \cap \left\{ \frac{\sigma}{k} - \frac{l}{k}\mathrm{Domain}(\eta) \right\}$$

Now if $q = 0$, one is always integrating from right to left, so leaving η in a half plane, and taking ξ is a spiralling domain works - the resulting integral operator being periodic in η. For $q \neq 0$, this still works if both eigenvalues are negative, and again things descend to a disc. Otherwise if both eigenvalues are positive then one must take σ as the base point, and integrate from left to right in ξ space, whence loosing, potentially periodicity in η. As ever the usual conditions of adding a Euclidean disc around base point in the fibres should be applied, while one treats the perturbation by the habitual power series. ▫

The previous proof only used that $lp - kq \neq 0$, so it immediately applies to VI.2.1(j) for the couple $(i, j) \neq 0$, and otherwise VI.2.1(j) is already monomial. Consequently, there only remains to treat,

VI.2.2 Fact (i) Provided $k/l \neq -1$ (a condition that can always be achieved by blowing-up, or taking a square root in x or y) for x belonging to a spiralling domain and y in a disc, there is a conjugation $x \mapsto X(x,y)$, $y \mapsto Y(x,y)$ of the normal form VI.2.1(i) to the monomial form,

$$z\frac{\partial}{\partial z} + \left(X^k Y^l\right)^n \left(lX\frac{\partial}{\partial X} - kY\frac{\partial}{\partial Y}\right)$$

Proof. We keep the initial notations of the proof of VI.2.2(h), but for M, N monomials in X, Y to be chosen, and look for the conjugation in the form

$$x \longmapsto Xe^{Mf} \qquad y \longmapsto Ye^{(Ng-kMf)/l}$$

Consequently we have to solve the coupled system of PDE's,

$$\left(\frac{DM}{M}\right) f + Df = lM^{-1}\Big\{\left(e^{nNg}-1\right) + X^k Y^l e^{(n+1)Ng} a\Big\}$$
$$\left(\frac{DN}{N}\right) g + Dg = -lkN^{-1}X^k Y^l (a-b)e^{nNg}$$

As such if we choose $M \neq N$, with $M\big|N$ and $N\big|X^kY^l$, but not equal to it, the functional derivative is a bounded perturbation of the left hand side of the above. In particular, if, without any serious loss of generality $l > 2$, then $M = Y$ and $N = Y^2$ is a good choice, resulting for the same reason as VI.2.2(h), in a solution on the type of domain described. ▫

VI.3 Integrable Cases

It is convenient to import the distinctions of §II and §III-IV into a division of cases for the existence of normal forms, particularly for the ubiquitous V.2.1(a) which is rather different. To this end consider,

VI.3.1 (a) Set up We have a holomorphic foliation in $2+n+m$ variables, $z, y, x_1, \ldots, x_n, t_1, \ldots, t_m$ with normal form, up to a formal unit:

$$z\frac{\partial}{\partial z} + x_1^{p_1}\cdots x_n^{p_n}\frac{\partial}{\partial y}$$

We prepare as follows: let ∂ be an actual convergent generator, and, of course, blow up in central components so that all congruences modulo $(z, x_1^{p_1}\cdots x_n^{p_n})$ or powers thereof, become congruences modulo $x_1^{p_1}\cdots x_n^{p_n}$, around the formal centre manifold. Whence solving for the centre manifold à la V.3, and dividing through by a unit congruent to 1 modulo as large a power of $x_1^{p_1}\cdots x_n^{p_n}$ as we please, we have

VI.3.2 (a) Reduction We may suppose that on a domain $S_1 \times \cdots \times S_n \times \Delta^{2+m}$, i.e. sectors S_i in x_i of width θ_i with $p_1\theta_1 + \cdots + p_n\theta_n < \pi$ that our foliation has a generator of the form

$$\partial = z\frac{\partial}{\partial z} + x_1^{p_1}\cdots x_n^{p_n}\Big\{a\frac{\partial}{\partial y} + b_i x_i\frac{\partial}{\partial x_i} + c_j\frac{\partial}{\partial t_j}\Big\}$$

where $a \equiv 1$, $b_i, c_j \equiv 0$ modulo z and any power of $x_1^{p_1} \cdots x_n^{p_n}$ we please, and the summation convention is employed.

Proof. Divisibility by the monomial $x_1^{p_1} \cdots x_n^{p_n}$ of the relevant terms is a consequence of blowing up as already encountered in V.3 et sequel. Consequently it remains only to simplify the field modulo z, which one does in the usual way, i.e. if $\widetilde{D}$ is a field with $\widetilde{D}y = 1$ then,

$$f \longmapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} y^n \widetilde{D}^n f$$

defines an invariant function, and this converges by I.1.6 on $S_1 \times \cdots \times S_n \times \Delta^{m+1}$, since the x_i are invariant. One can then obtain an appropriate change of variable in y by way of a path integral. ▫

Now consider conjugating to the normal form in $\zeta, \eta, \xi_i, \tau_j$ variables by way of a conjugation,

$$z \mapsto \zeta, \quad y \mapsto \eta + \zeta f, \;\; x_i \mapsto \xi_i e^{\zeta g_i}, \;\; t_j \mapsto \tau_j + \zeta h_j$$

This amounts to solving the system of PDE's,

$$f + \check{\partial} f = \xi^{\underline{p}} \Big(\frac{e^{\zeta p_k g_k}}{\xi} - 1 \Big) + \xi^{\underline{p}} e^{\zeta p_k g_k} \alpha, \; g_i + \check{\partial} g_i = \xi^{\underline{p}} e^{\zeta p_k g_k} \beta_i, \; h_j + \check{\partial} h_j = \xi^{\underline{p}} e^{\zeta p_k g_k} \gamma_j$$

where $\check{\partial}$ is the normal form, $\xi^{\underline{p}} = \xi_1^{p_1} \cdots \xi_n^{p_n}$, $a = 1 + z\alpha$, $b_i = z\beta_i$, $c_j = z\gamma_j$. Up to a bounded perturbation, this system has a functional derivative

$$\mathbb{1} + \check{\partial}$$

understood in $\mathcal{O}^{m+n+1}$. One might be tempted to invert this by way of an expansion in power series in ζ, but, unfortunately, the trick of putting a Euclidean disc in the appropriate variable (here $\xi_1^{-p_1} \cdots \xi_n^{-p_n} \eta$) around the base point to guarantee a sheaf like inverse as per I.3.2(a) doesn't work, so it's better to just accommodate the extra variable in the fibration. More precisely if $Y = \xi_1^{-p_1} \cdots \xi_n^{-p_n} \eta$, then over and above the fibring functions encountered in II.1 we have a further function, $\zeta \exp(-Y)$, so, say, $\sigma = Z - Y$ for $\exp(Z) = \zeta$ and Z in a left half plane. Consequently we have a fibration $s : L \to B$, with fibre over (σ, ξ, τ) exactly,

$$L_{(\xi,\tau)} \cap \big(-\sigma + H \big)$$

where $L_{(\xi,\tau)}$ is the fibre encountered in II.1, and H the domain of Z. Manifestly this leads to no change in the key features enunciated in II.1.2-3, and we may safely add a Euclidean disc in Y around the base point in each fibre to guarantee the conditions I.3.2 of the implicit function theorem, taking into account, as ever, the bounded perturbation by way of the usual power series. Consequently, we obtain,

VI.3.3 (a) Fact Let things be as per VI.3.1(a) then for a domain as per VI.3.2(a) there is a generator ∂ of the foliation conjugate to the normal form,

$$z\frac{\partial}{\partial z} + x_1^{p_1}\cdots x_n^{p_n}\frac{\partial}{\partial y}$$

The remaining cases are very similar. Due to its close proximity to the above, we'll turn our immediate attention to,

VI.3.1 (b) Set Up Everything as per VI.3.1(a), but suppose that the normal form is,

$$z\frac{\partial}{\partial z} + x_1^{p_1}\cdots x_n^{p_n}y\frac{\partial}{\partial y}$$

where now, implicitly, $y = 0$ is the equation of an invariant, even algebraic hypersurface, albeit just as V.1.8 the component defined by $z = y = 0$ would not be central.

Now we can certainly achieve a similar reduction as previously, viz:

VI.3.2 (b) Reduction Denote by X the monomial $x_1^{p_1}\cdots x_n^{p_n}$, and S a domain of the form $|X| \leqslant r$, $\mathrm{Im}(1/X) > 0$ if $\mathrm{Re}(X) > 0$. Then on $S \times \Delta^{2+m}$ we can find a generator of the foliation of the form,

$$\partial = z\frac{\partial}{\partial z} + x_1^{p_1}\cdots x_n^{p_n}\{ay\frac{\partial}{\partial y} + b_i x_i\frac{\partial}{\partial x_i} + c_j\frac{\partial}{\partial t_j}\}$$

for a, b_i, c_j as per VI.3.2(a). Similarly for $S = S_1 \times \cdots \times S_n$ a product of sectors resulting in a small sector about the positive real axis in $1/X$ and $\Omega_S(R)$ the spiralling sector in the logarithm of y, as per II.2, we can obtain the same but on $\Omega_S(R) \times S \times \Delta^{1+m}$.

Proof. Modulo notation (i.e. $\xi = X^{-1}$) and changing from negative to positive real axis to reflect the normalisation implicit in the normal form, the first assertion follows from II.2.2 with the same proof as VI.3.2. The second assertion is more subtle. Observe that the co-normal sheaf I/I^2, for I the ideal of the singular locus is a free $\mathcal{O}/I$-module of rank two. Furthermore for appropriate (convergent) coordinates and $J = (\partial z, \partial y)$, $J \otimes \mathcal{O}/I$ is also a free rank two $\mathcal{O}/I$-module - the assertion may be checked after completion, since it is independent of the same - so by inspection in the completion, and Nakayama's Lemma, $J = I$. Consequently one can achieve a priori, the preparation $\partial z = z$ mod I^2, in fact even mod I^n, but never fully in the completion, cf. post II.2.2. In particular after finding the centre manifold, and taking $\exp(Y) = y$, Y in a left half plane, we may suppose that on the centre manifold our field has the form,

$$X\Big\{a\frac{\partial}{\partial Y} + b_i x_i\frac{\partial}{\partial x_i} + c_j\frac{\partial}{\partial t_j}\Big\}$$

with a a unit, and b_i, c_j vanishing at $-\infty$, in fact belonging to the pull-back under the exponential of the ideal (y). At this point the formula of VI.3.2(a) rendering the directions in x_i and t_j fully invariant now converges, albeit at the price of a decrease in the angle of the cone, cf. figure II.2.2, defining the domain $\Omega_S(R)$. ▫

Plainly we attempt the same king of conjugation as before, adopting the same variables, but with $y = e^{\zeta f}\eta$. As such we obtain the system of PDE's,

$$f + \check{\partial} f = \xi^{\underline{p}}\Big(\frac{e^{\zeta p_k g_k}}{\xi} - 1\Big) + \xi^{\underline{p}} e^{\zeta p_k g_k}\alpha,\ g_i + \check{\partial} g_i = \xi^{\underline{p}} e^{\zeta p_k g_k}\beta_i,\ h_j + \check{\partial} h_j = \xi^{\underline{p}} e^{\zeta p_k g_k}\gamma_j$$

where still, $a = 1 + z\alpha$, and modulo evaluating $\alpha, \beta_i, \gamma_j$ at $(e^{\zeta f}\eta, e^{\zeta g_i}x_i, \tau_j + \zeta h_j)$ rather than $(\eta + \zeta f, e^{\zeta g_i}x_i, \tau_j + \zeta h_j)$ the same system of PDE's. A priori VI.1.5 does not apply. This may be remedied as follows, for $\mathrm{Im}(1/X)$ bounded away from zero there is no difficulty solving modulo arbitrarily large powers of ζ. Consequently to obtain the distinct eigenvalue condition of op.cit. one should solve to a suitable large power of ζ, i.e. $n + m + 1$, change basis to f, h_j, $g_2, \dots, g_n$, $p_1 g_1 + \cdots p_n g_n = G$, and weight the conjugation appropriately,

$$y \mapsto e^{\zeta f}\eta, \quad \tau_j \mapsto \tau_j + \zeta^{1+j}h_j, \quad x_i \mapsto e^{g_i \zeta^{m+1+i}}\xi_i, \quad X \mapsto e^{\zeta^{m+n+1}G}\xi_1^{p_1}\cdots\xi_n^{p_n}$$

In the particular case of immediate interest in dimension 3, this amounts to $x^{-1}\partial x$ being 0 mod z^2, and taking $x \mapsto e^{\zeta^2 g}\xi$. After such a change one can then construct a fully analytic inverse to $P'(0)$ by expanding as power series in $y^a z^b$ by way of the exact the same estimates of §II.2 provided $\mathrm{Im}(1/X)$ is bounded away from zero. Otherwise we must divide by cases. In the first place we have sectors $\xi_i \in S_i$, such that $\xi_1^{-p_1}\cdots\xi_n^{-p_n}$ is close to a positive real. As per case (a) take Z to be a logarithm of ζ, and $Y = \xi_1^{-p_1}\cdots\xi_n^{-p_n}\log\eta$, $\sigma = Z - Y$, so that we have a fibration over (σ, ξ, τ) with fibre,

$$L_{(\xi,\tau)} \cap \big(-\sigma + H\big)$$

the fibre $L_{(\xi,\tau)}$ being exactly as in figure II.2.2, bearing in mind the change of normalisation, i.e. X close to positive real, so to find the centre manifold we integrate from right to left, whereas to obtain the conjugation we must go from left to right. This latter operation may be done in a bounded way with $-\infty$ as base point so VI.1.3 applies. Similarly for ξ_i in sectors S_i such that X is close to negative real, the domain of Y is the negative of that in figure II.2.2 intersected with a translation of a left half plane, so, again, we can integrate from the apex of the cone and VI.1.3 applies. Consequently we achieve,

VI.3.3 (b) Fact Let S be one of the connected components of the set defined by $\mathrm{Im}\big(x_1^{-p_1}\cdots x_n^{-p_n}\big) > \varepsilon$, $|x_i|$ sufficiently small with formal normal form as per VI.3.1(b) then on a domain of the form $S \times \Delta^{2+m}$, i.e. full analyticity in the other variables, we can achieve a conjugation so that the foliation admits a generator of the form,

$$z\frac{\partial}{\partial z} + x_1^{p_1}\cdots x_n^{p_n} y \frac{\partial}{\partial y}$$

Otherwise for $S_i \ni x_i$ sectors of appropriately small radii, such that $x_1^{-p_1}\cdots x_n^{-p_n}$ is close to positive real (sectorially), respectively (sectorially) negative real, and

y belonging to the domain $\Omega_S(R)$ we can achieve an identical conjugation but on the domain $S \times \Omega_S(R) \times \Delta^{1+m}$.

Obviously, as per II.2 and the discussion of the beast, this is the best possible and, in a sense it's already implicit in V.3.1 that it is possible. Notice, however:

VI.3.4 Remark It's tempting to suggest that the restriction to x_i in a sector S_i such that $x_1^{p_1} \cdots x_n^{p_n}$ is close to positive or negative real could be replaced by a single condition involving only $x_1^{p_1} \cdots x_n^{p_n}$. This may be true, but it wouldn't follow from the implicit function theorem, which requires bounds on the derivations $x_i\partial/\partial x_i$ which arise from the aforesaid restriction $x_i \in S_i$.

Again for want of a better place for it, let us treat the generalisation of classical nodes, viz:

VI.3.1 (c) Set Up We have a holomorphic foliation in $2+n+m$ variables $z, x, y_1, \ldots, y_n, t_1, \ldots, t_m$ with normal form, up to a formal unit,

$$z\frac{\partial}{\partial z} + \frac{x^{p+1}}{1+\nu(y,t)x^p} y_1^{q_1} \cdots y_n^{q_n} \frac{\partial}{\partial x}$$

in the topology of completion in $(z, x^{p+1}y_1^{q_1} \cdots y_n^{q_n})$, $p \in \mathbb{N}$, $q_i \in \mathbb{Z}_{\geqslant 0}$, $\nu(y,t)$ convergent, and for $q_i \neq 0$, the hypersurface $y_i = 0$ is supposed to define an invariant hypersurface which arises from blowing up, and defines a central component $z = y_i = 0$ inside the formal centre manifold.

As ever we perform appropriate simplification, possibly after further blowing up, modulo z, viz:

VI.3.2 (c) Reduction Let S be a domain in x, y such that x^{-p} is within 3π of the argument of $y_1^{q_1} \cdots y_n^{q_n}$ (so -1 if all $q_i = 0$) then on $S \times \Delta^{1+m}$ we may suppose that we have a generator of our foliation of the form,

$$z\frac{\partial}{\partial z} + \frac{x^p}{1+\nu(y,t)x^p} y_1^{q_1} \cdots y_n^{q_n} \Big\{ax\frac{\partial}{\partial x} + b_i y_i \frac{\partial}{\partial y_i} + c_j \frac{\partial}{\partial t_j}\Big\}$$

and a, b_i, c_j as per VI.3.2(a).

Proof. This is very much the same as the reduction VI.3.2, with the minor caveat that one should make a sufficiently good formal approximation to avoid any loss of domain in the x, y variables. ▫

Proceeding, therefore, as before we attempt to conjugate to the normal form in $\zeta, \xi, \eta_i, \tau_j$ variables by way of,

$$z \mapsto \zeta, \quad x \mapsto e^{\zeta f}\xi, \;\; y_i \mapsto e^{\zeta g_i}\eta_i, \;\; t_j \mapsto \tau_j + \zeta h_j$$

which in turn results in the system of PDE's,

$$f + \partial f = \frac{\xi^p \eta^q \zeta^{-1}}{1+\nu\xi^p}\Big\{\frac{e^{pf\zeta + q_k g_k \zeta}}{1+\nu e^{pf\zeta}\xi^p}(1+\nu\xi^p) - 1 + \alpha\zeta\Big\},$$

$$g_i + \partial g_i = \frac{\xi^p \eta^q e^{pf\zeta + q_k g_k \zeta}}{1+\nu\xi^p e^{pf\zeta}}\,\beta_i, \qquad h_j + \partial h_j = \frac{\xi^p \eta^q e^{pf\zeta + q_k g_k \zeta}}{1+\nu\xi^p e^{pf\zeta}}\,\gamma_j$$

for $\underline{\eta}^{\underline{q}} = \eta_1^{q_1} \cdots \eta_n^{q_n}$. Consequently up to a bounded perturbation, the functional derivative is,

$$\mathbb{1} + \xi^{p+1} \frac{\partial}{\partial \xi}$$

for $\mathbb{1}$ the identity on the trivial bundle of rank $m + n + 1$. As such the only real issue here, viz: loss of domain from ξ^{-p} within 3π of argument of $+\underline{\eta}^{\underline{q}}$ to 2π of the argument of $-\underline{\eta}^{\underline{q}}$ has already been encountered in IV.1.3. Unlike cases (a) and (b) we have actually conserved the normalisations of II.4. Consequently if X is the monomial $\xi^{-p}\underline{\eta}^{-\underline{q}}$ (having conjugated to $\nu = 0$), with Z the logarithm of z belonging to a left half plane, we have an invariant function $\sigma = Z + X$. Whence we have a fibring in σ, η, τ coordinates with fibres,

$$L_{(\eta,\tau)} \cap \big(\sigma - H\big)$$

with the leaf $L_{(\eta,\tau)}$ as per II.4.1(b). In terms of the variable X, we now, as per IV.1.3, have to integrate from $+\infty$, as opposed to $-\infty$ for finding the centre manifold. This is, however, done in a bounded way so VI.1.3 applies, and we obtain,

VI.3.3 (c) Fact Let $S, S_1, \ldots, S_n$ be sectors in x and y_i (supposing $q_i \neq 0$, otherwise this may be omitted) such that x^{-p} is within 2π of the argument of $y_1^{q_1} \cdots y_n^{q_n}$ with normal form as per VI.3.3(a). Then for sufficiently small radii in x and/or y_i we can achieve a conjugation on the domain $S \times S_1 \times \cdots \times S_n \times \Delta^{1+m}$ such that on the same, foliation admits a generator of the form,

$$z\frac{\partial}{\partial z} + \frac{x^p}{1 + \nu(y,t)x^p} y_1^{q_1} \cdots y_n^{q_n} x \frac{\partial}{\partial x}$$

VI.4 Singular Cases

The singular cases in the list VI.2.1 permit a certain unity of treatment, albeit, conventions as per V.1.9, modulo a need for care about the difference between $q = 0$ versus $q > 0$. Nevertheless let us establish some notation by way of,

VI.4.1 Set Up As ever we're studying a holomorphic foliation around the possibly purely formal centre manifold, and we'll suppose that we're at a point where the induced foliation is singular so that we have a normal form in $\widehat{X}_{\mathrm{an}}$, notation as per VI.2 with caveat for VI.2.1(g),

$$z\frac{\partial}{\partial z} + x^p y^q \Big\{ A(x,y) x \frac{\partial}{\partial x} + B(x,y) y \frac{\partial}{\partial y} \Big\}$$

Plainly we wish to achieve,

VI.4.2 Proposed Reduction For a suitable domain U in (x, y) to be specified and a disc Δ in z we have on $U \times \Delta$ a generator of the foliation of the form,

$$z\frac{\partial}{\partial z} + x^p y^q \Big\{ a(x,y,z) x \frac{\partial}{\partial x} + b(x,y,z) y \frac{\partial}{\partial y} \Big\}$$

where $a = A + z\alpha$, $b = B + z\beta$, α, β holomorphic on $U \times \Delta$.

Again a certain amount of caution will have to be exhibited here, particularly in light of V.2.3/5. Nevertheless, for a normal form in ζ, ξ, η coordinates we will therefore look for a conjugation in the form,

$$z \longmapsto \zeta, \qquad x \longmapsto e^{\zeta f}\xi, \qquad y \longmapsto e^{\zeta g}\eta$$

Consequently we will have to solve the coupled system,

$$\begin{aligned}
&f + \partial f = \\
&\xi^p\eta^q\zeta^{-1}\Big(e^{p\zeta f + q\zeta g}A(e^{\zeta f}\xi, e^{\zeta g}\eta) - A(\xi,\eta)\Big) + \xi^p\eta^q\alpha(e^{\zeta f}\xi, e^{\zeta g}\eta, \zeta)e^{p\zeta f + q\zeta g} \\
&g + \partial g = \\
&\xi^p\eta^q\zeta^{-1}\Big(e^{p\zeta f + q\zeta g}B(e^{\zeta f}\xi, e^{\zeta g}\eta) - B(\xi,\eta)\Big) + \xi^p\eta^q\beta(e^{\zeta f}\xi, e^{\zeta g}\eta, \zeta)e^{p\zeta f + q\zeta g}
\end{aligned}$$

In most cases we'll be able to employ VI.1.3, but for nodes with $q = 0$ we'll have to look to VI.1.5 and make further preparation - in fact in the x-variable modulo z^2.

Now let us implement this plan according to the various degrees of difficulty that are implicit therein, so starting from VI.2.1(e) with $q > 0$. As such we have a domain U, according to the various cases considered in §III, in x and y on which we have found the centre manifold, and this can be done up to as large a power of $x^p y^q$ as we please. In particular restricting to the centre manifold we may suppose that we have a plane field in the domain U of the form,

$$\delta \doteqdot \frac{x^p y^q}{1 + \nu x^p y^q} D \doteqdot \frac{x^p y^q}{1 + \nu x^p y^q}\Big\{x\big(1 + (x^p y^q)^k \alpha\big)\frac{\partial}{\partial x} + \lambda y\big(1 + (x^p y^q)^k \beta\big)\frac{\partial}{\partial y}\Big\}$$

with $k \in \mathbb{N}$ as large as we please and α, β bounded functions on U. We aim to conjugate this to a normal form,

$$\frac{\xi^p\eta^q}{1 + \nu\xi^p\eta^q}\check{D} \doteqdot \frac{\xi^p\eta^q}{1 + \nu\xi^p\eta^q}\Big\{\xi\frac{\partial}{\partial \xi} + \lambda\eta\frac{\partial}{\partial \eta}\Big\}$$

over U, by way of a transformation,

$$x \longmapsto e^{f(\xi^p\eta^q)^n}\xi, \qquad y \longmapsto e^{g(\xi^p\eta^q)^n}\eta$$

for some sufficiently large n to be chosen. This amounts to solving a coupled system of PDE's which up to a bounded perturbation has functional derivative,

$$\begin{bmatrix} n(p + q\lambda) - p & -q \\ -p\lambda & n(p + q\lambda) - \lambda q \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} + \check{D}\begin{bmatrix} f \\ g \end{bmatrix}$$

where the matrix in question enjoys eigenvalues $n(p + q\lambda), (n-1)(p + q\lambda)$. In order to try to maintain notation compatible with §III, let $e^X = x$, $e^Y = y$ define logarithms of X and Y, $\sigma = \lambda X - Y$ (or possibly $X - Y/\lambda$) a first integral and

$\tau = -\big(pX + qY\big)$. As such by VI.1.3 on dividing through by $p + q\lambda$, we require to solve in (σ, τ) coordinates scalar equations of the form,

$$\Big(n - \frac{\partial}{\partial \tau}\Big)(F) = G$$

for $n \in \mathbb{N}$. Unsurprisingly in the situation of III.2, τ lies in a strip like domain unbounded in $\mathrm{Re}(\tau) \to \infty$, which is exactly what one requires for an infinite base point. In case of §III.4-5, the fibres in σ are bounded, but the pre-images of the base points envisaged are also those where $\mathrm{Re}(\tau)$ is maximum, and since the variation of the imaginary part of τ is bounded one can simply take paths which have constant imaginary part followed by an appropriate displacement in the purely imaginary direction. Consequently in these cases we can obtain the proposed reduction VI.4.2.

There remains, therefore, to achieve such a reduction for $q = 0$. Plainly this requires in the first instance finding an invariant hypersurface $y = 0$. For $\mathrm{Re}(\lambda) < 0$ this has already been done in V.2.5. For $\mathrm{Re}(\lambda) > 0$, we first look for the corresponding invariant curve inside the centre manifold, starting, as ever, from a sufficiently good approximation modulo powers of (x). Plainly we look for the said curve in the form of a graph $\widetilde{y} = y - f(x)$, with $\widetilde{y} = 0$ invariant. This results in an ODE with functional derivative, up to a small bounded perturbation,

$$f \longmapsto \lambda f - x\frac{\partial f}{\partial x}$$

which is, a priori, a little problematic, but for any suitably large e we can replace f by $x^e f$, so the derivative becomes:

$$f \longmapsto (\lambda - e)f - x\frac{\partial f}{\partial x}$$

Consequently for $\mathrm{Re}(\lambda - e) < 0$, we have the right inverse by way of integrating from $-\infty$ in the strip like region which is the domain of the logarithm. At which point we can argue exactly as in V.2.5, to find the hypersurface $\widetilde{y} = 0$ albeit with a loss of domain from a sector of width $3\pi/p$ to $2\pi/p$. In any case, all of this can be achieved modulo arbitrary powers of the ideal (x), so we have an identical preliminary reduction as in the case of $q > 0$, and the entire argument goes through as before to obtain VI.4.2 with the only caveat being that of the reduction of the domain as per V.2.4 in the case corresponding to III.1.1.

The reduction achieved we can, modulo having ran out of letters, move relatively swiftly to a conclusion. To actually solve the equation we employ the conjugation VI.2.2(e), or more correctly followed by a homothety in X, Y (notations as per op.cit.) so as to multiply our plane field by $(p + q\lambda)^{-1}$. As such the main protagonists of §III were the variables $\Xi = (X^p Y^q)^{-1}$ and some invariant function s according to which we fibred the domain U, via s, over B, with U embedding in $\mathbb{C}^2$ by way of $\Xi \times s$, and our linearised equation was of the form:

$$\big(\mathbb{1} + \varepsilon\big) + \frac{\partial}{\partial \Xi}$$

for ε a small function, and the left to right integration of §III for finding the centre manifold having been preserved. Consequently under the conjugation implied by VI.2.2(e) in ζ, Ξ, s coordinates the field $\check{\partial}$ becomes

$$\zeta\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\Xi}$$

So now we fibre the domain $\widetilde{U} = U \times \Delta$ over $\widetilde{B}$, where the latter is the domain of s and $t \doteqdot \xi\exp(\Xi)$. As such, by VI.1.3 we must solve the scalar equation,

$$\Big(\mathbb{1} - \frac{\partial}{\partial\Xi}\Big)(F) = G$$

in a bounded way, where the fibre of $\widetilde{U}$ over (s,t) is,

$$U_s \cap \big(\log t - H\big)$$

where the left half plane H is a domain of the logarithm Z of ζ. As far as the good cases III.1/2 are concerned this poses no additional problem to those encountered in V.2.3, i.e. the loss of domain from 3π in Ξ to 2π, which is consistent with finding the hypersurface $y = 0$ in case $q = 0$, and involves no further loss of domain.

As far as the case of bounded leaves occurring in III.3-5 is concerned, the situation is similar, but obviously more unpleasant. Specifically the fibres of $U \to B$ are necessarily branched on the right in Ξ, in order to find the centre manifold, and, unlike the good cases III.1-2 there is no way to change this. We can, however, take a branch which is very close to purely imaginary. Thus, although we now have to integrate from right to left in the leaves, rather than left to right, we still can do everything that we were able to do before except for a small sector around an imaginary axis. There being two of these, we still succeed in covering everything, and so obtain,

VI.4.3 Fact Suppose for irrational λ, the foliation admits in a neighbourhood of $\widetilde{X}_{\mathrm{an}}$ a formal generator of the form,

$$z\frac{\partial}{\partial z} + \frac{x^p y^q}{1+\nu x^p y^q}\Big(x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}\Big)$$

Then for U a domain, according to the cases as documented in §III.1-5, in (x,y) with the above small loss in cases §III.3-5, and that of 3π to 2π in §III.1-2, on a small disc Δ in z, there is a conjugation in $U \times \Delta$ of the generator of the foliation to the normal form. In particular, when we have to take further arguments in a variable such as $x^{-\lambda}y$ or similar, as per §III.3-5, these may be supposed arguments of an invariant function.

Now let us turn our attention to nodes as encountered in VI.2.1(f). The necessary work to be done is that to achieve VI.4.2. For the case of bounded sectors as encountered in IV.2, everything has already been done in §V.2-3. For unbounded sectors we proceed much as in the good cases implicit in VI.4.3.

Whence in the first place we restrict to the central subvariety and seek the weak branch of the node therein by way of a graph $y = \tilde{y} + f(x)$, $\tilde{y} = 0$ the desired branch. At the risk of a certain notational confusion, this amounts to solving in a domain U in (x, y) where the central manifold exists (so basically a disc in y by a sector modulo the precision on critical points in IV.2) an ODE in x with a functional derivative up to a bounded perturbation of the form,

$$f \longmapsto f + \frac{\partial f}{\partial z}$$

where z is the conformal variable encountered in §IV.2-IV.3. Since we're in the hypothesis of IV.3, $\mathrm{Re}(z) \geqslant 0$, there's only really an issue for z purely imaginary. There are cases IV.3.6(a),(b),(d) where one has a full sector in the imaginary direction so one can take as base point, $\mathrm{Re}(z) \to -\infty$. Otherwise one is in IV.3.6(c), which should, and further to will be thus supposed, be treated as the other bounded cases in IV.2. Notice also that when we are around a ray with $\mathrm{Re}(z) > 0$ in a domain that does not admit an intersection with cases IV.3.6(a),(b),(d) that one has to take a finite base point in the z-plane, and the possibilities for the weak branch are highly non-unique. Consequently it is best to construct the right inverse to $\mathbb{1} + \partial/\partial z$ on domains which are maximal for the existence of the ambient centre manifold in 3-space. As ever the solutions in §IV.3 (excepting IV.3.6(c)) patch whenever they have a common intersection where $\mathrm{Re}(x^{p+r}) < 0$. The exact position, however, of the critical points with $\mathrm{Re}(c^{p+r}) > 0$ is an absolute obstruction, so this is far from being as simple as a 2π versus 3π discussion.

Evidently we now wish to argue as in V.2.5, which will inevitably involve some loss of domain, but the situation is not too bad since;

VI.4.4 Claim In all of the cases covered by IV.3 (excepting IV.3.6(c)), the centre manifold in ambient 3-space may be supposed to exist on a domain V in x and discs in y, z, so $V \times \Delta^2$ such that V contains open sectors V_+, V_- on which the rays enjoy $\mathrm{Re}(x^{-(p+r)}) \to +\infty$, respectively $\mathrm{Re}(x^{-(p+r)}) \to -\infty$.

Proof. The assertion about $-\infty$ is actually the content of the various cases considered in IV.3. The right inverses to the functional derivative can be patched when they have a component of $\mathrm{Re}(x^{-(p+r)}) < 0$ in common, so one gets the equivalent statement for $\mathrm{Re}(x^{-(p+r)})$ going to $+\infty$ by going through the exact same set of cases replacing $-\infty$ by $+\infty$. ▫

One also has a more precise statement involving critical points, but for the moment this is unimportant since V.2.5 involves a bounded right inverse, and it's only relevant that we can integrate from $+\infty$. As such, as per VI.4.3 we've already changed the direction of integration from right to left in this preparation alone.

The full preparation to normal form on the centre manifold is exactly as per IV.1.3 - understanding x as the conformal variable z in the sense of IV.2-IV.3. In the particular case of immediate discussion around IV.3 one even has $+\infty$ as a base point, so there is absolutely no change. Finally to apply VI.1.5 we must achieve a preparation in $x^{-1}\partial x$ modulo z^2, where here $z=0$ is once again the centre manifold in 3-space. Attempting the preparation by way of $x \mapsto e^{fz}x$, this amounts to solving the linear equation,

$$f\left\{1-\frac{(p+r)x^{p+r}}{(1+\lambda x^{p+r})^2}\right\}+\check{\partial} f=-x^p\beta$$

where $\check{\partial}$ is the normal form restricted to the centre manifold, and $x^{-1}\partial x = x^{p+r}(1+\lambda x^{p+r})^{-1}+x^p z\beta$. Now one should be a little cautious since the critical lines appearing in IV.3 are somewhat intrinsic to the field in question but rather susceptible to change on multiplication by a unit - equal to 1 modulo x^{p+r+1} would be okay, but this is not our situation. Consequently one should observe that if K is a right inverse to $\big(1+\check{\partial}\big)$ then,

$$f=-\frac{x^{p+r}}{(1+\lambda x^{p+r})}K\left\{\frac{(1+\lambda x^{p+r})}{x^{p+r}}\ x^p\beta\right\}$$

solves the equation. For this to have sense β must be a priori divisible by x^r, but this can certainly be achieved by starting from a sufficiently good approximation.

Here and elsewhere the construction of the right inverse to $\big(1+\check{\partial}\big)$ largely just involves changing plus signs to minus signs in IV.2/IV.3, and being attentive to the loss of domain. One should, however be cautious, in cases IV.3.6(c) and (d). The latter isn't too bad since it just involves a loss of analyticity at it becomes of the former type when the signs change. The former is, however, really quite bad. More precisely, it doesn't seem to be possible to make analytic solutions, i.e. arguments in x only, in an open neighbourhood of the imaginary axis. Unlike bounded domains around negative real arguments the conformal variable z of IV.2, there is not a symmetry between inverting $\mathbb{1}+\partial$ and $\mathbb{1}-\partial$, and the base point for the former implied by using the cone construction of IV.2 for the latter is not holomorphic. This can be remedied by constraining the invariant variable σ of IV.2 to any strip bounded by the ray of IV.3.6(c) in the plane $\mathrm{Re}(z)<0$, and any other parallel to it. A priori the implied base point for an integral operator to invert $\mathbb{1}+\partial$ is still not holomorphic, but since σ is in a strip, this is within finite Euclidean distance in the z-plane of a holomorphic function $h(\sigma)$. Again notation as per op.cit., this does not imply finite Euclidean distance in the ξ-plane, so one should change the leaf L_σ by the simple expedient of adjusting the polygon to have a vertex at $h(\sigma)$ with edges through the same the straight lines between σ and $h(\sigma)$ and say a parallel to the ray that was the other edge through the nearby non holomorphic vertex. This leads to a slightly less tidy domain in σ,ξ coordinates (and of course extended by an open disc in the ξ-plane around the base point to apply the implicit function theorem), but à la III.3.2 and III.5.2 all points are still covered. Otherwise the loss of domain in governed by the usual branching considerations of 2π versus 3π in the analytic sectors together with the position of the critical points in the sense of IV.3.1. With this in mind we can therefore not only safely apply VI.1.5 but argue exactly as prior to VI.4.3 to obtain,

VI.4.5 Fact Suppose in $\widehat{X}_{\mathrm{an}}$ the foliation admits on a neighbourhood a formal generator of the form

$$z\frac{\partial}{\partial z} + x^p\Big(R(x)y\frac{\partial}{\partial y} + \frac{x^{r+1}}{1+\lambda x^{p+r}}\frac{\partial}{\partial x}\Big)$$

Then for U a domain in x, y as per IV.2/IV.3, with the above caveats in cases IV.3.6(c)-(d) on $U \times \Delta$ there is a conjugation of a convergent generator to the same. In particular the extra restriction occurring in IV.3.6(c) in the invariant variable σ explained above is in-fact with respect to an invariant variable for the foliation on the ambient 3-fold.

This brings us to VI.2.1(g), i.e. a node with $q > 0$, so at least we don't have to worry about finding the invariant hypersurface $y = 0$. Furthermore even though VI.2.1(g) cannot be achieved in $\widehat{X}_{\mathrm{an}}$, but only in the weaker topology of completion in $x = 0$ or a point, V.1.9 is sufficient to imply that after blowing up we may suppose $x^{-1}\partial x$ and $y^{-1}\partial y$ are divisible by $x^p y^q$, for the ambient foliated 3-fold. By IV.4, VI.1.3 will prove to be applicable provided we can achieve the reduction VI.4.2. Essentially, this is done as per the proof of VI.2.1(g) modulo appropriate changes. In the first place, one simply cannot argue as per op.cit. because exponentials may not converge. Nevertheless if $\check{D} = \xi^p\eta^q\check{\partial}$ is the normal form in ξ, η coordinates one can proceed to seek a conjugation modulo y^k, $k < q$, modulo y^q, and eventually modulo y^{q+1} for the field restricted to the centre manifold. Consequently it is certainly necessary to begin from a situation which is prepared modulo a large power of (x). In any case, proceeding by way of successive powers of y, one finds a system of ODE's in x with functional derivatives up to a bounded perturbation of the form,

$$\begin{bmatrix} q-k & 0 \\ 0 & -k \end{bmatrix} - qx^{r+1}\frac{\partial}{\partial x}$$

where the derivative can actually be taken as $\partial/\partial z$ in, obvious risk of notational confusion, $z(x)$ as per IV.2/IV.3. For $k < q$ there is an obvious risk of competing signs. To minimise the implied loss of domain, observe that since the centre manifold converges in $\widehat{X}_{\mathrm{an}}$, so modulo powers of $x^p y^q$ after blowing up, we can carry out this step a priori before finding the centre manifold on domains of the form shown in figure VI.4.5, and negatives thereof for some suitable large R, $z(x)$ as IV.2/IV.3, and all lines bounded strictly away from imaginary. Furthermore provided our preparation modulo powers of (x) is sufficiently good, we can also similarly carry out the preparation modulo y^q.

The monomial form VI.2.2(g), which as per VI.4.5 is what we employ to actually solve the equation, casts a little more light on IV.4. In the persistent confusion of notation between the conjugated coordinates and the conformal mappings of IV.2-IV.4 we have, in the notation of the latter, $\zeta(X) = X^{-1}$, $z(X) = X^{-r}/r$, and after a homothety $\partial\xi = -1$, $\xi = X^{-p}Y^{-q}$. Whence in the notation of op.cit., $s = \xi^{-1}\exp(z(X))$, which is rather convenient. Irrespectively we proceed exactly as for VI.4.3. According to the domains encountered in IV.4.2 together with the fact that we must change from integrating from the

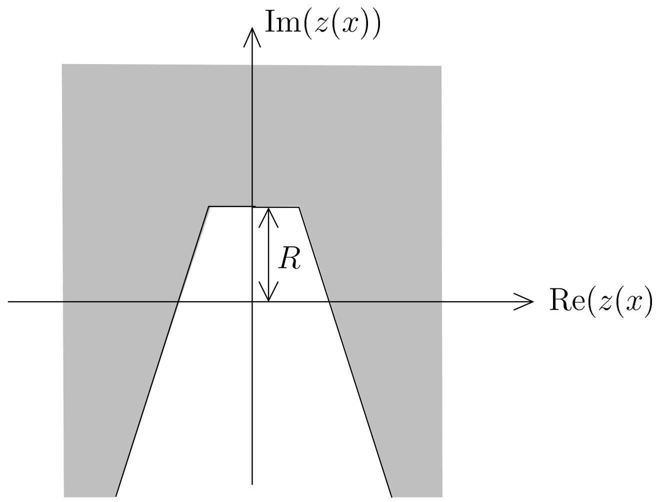

Figure VI.4.5

left right orientation of IV.4.2 (which again has been conserved) for finding the centre manifold, to going from right to left. As such we suffer the usual loss from 3π to 2π in $\big(x^p y^q\big)^{-1}$ in the good regions, i.e. domain of $z(x)$ the larger connected component of VI.4.5, or its negative, on exclusion of a strip of width $2R$ about the negative real axis, while the problematic region around the negative real axis (in the conformal variable $z(X)$) is no worse than before albeit it's a good idea to change the rightmost boundary to one that is not purely imaginary so that we can shrink it and apply the standard perturbation argument in a uniform way. Ultimately, therefore, we obtain:

VI.4.6 Fact Suppose whether in the completion of $\widehat{X}_{\mathrm{an}}$ in a point, or better the divisor $x = 0$, we have the normal form,

$$z\frac{\partial}{\partial z} + \frac{x^p y^q}{1 + y^q\big(R(x) + \lambda x^{p+r}\big)}\Big(y\frac{\partial}{\partial y} + \frac{x^r}{1+\nu x^r}\big(qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}\big)\Big)$$

$r \in \mathbb{N}$, R polynomial of degree $r-1$, $\lambda, \nu \in \mathbb{C}$, then for a domain U in x, y, as per IV.4 with the above further prescriptions, on a domain $U \times \Delta$ we have a conjugation of a convergent generator to the said normal form. In particular when we're obliged in IV.4 to restrict the argument of the variable s^q for the conformal variable $z(X)$ close to negative real, the variable s defining such restricted domain may be supposed to be an invariant function for the ambient foliation in 3-space.

In light of VI.2.2(h), the case VI.2.1(h) follows with exactly the same proofs as VI.4.3 since, as already occurs whether in the proof of VI.2.2(h) or VI.4.3, the only important thing is that $p + q\lambda \neq 0$, which is what distinguishes VI.2.1(h) from VI.2.1(i) or (j), which we therefore make explicit note of by way of,

VI.4.7 Fact Exactly as VI.4.3 but for a rational eigenvalue λ provided $p+q\lambda \neq 0$, which corresponds to the normal form VI.2.1(h) or (j) for the couple $(i,j) \neq 0$, understood by way of any finite truncation of the series ν.

This leaves us with the highly resonant cases VI.2.1 (i)/(j) to do. The strategy is basically as per VI.2.2(i), in order to obtain the reduction VI.4.2. Let us spell this out in order to see that there is a difference between the normal form and simply conjugating to a monomial form. More precisely let us write the normal form of the plane field as,

$$\check{\partial} = \big(X^k Y^l\big)^n \check{D} \doteqdot \big(X^k Y^l\big)^n \Big\{ a(X,Y)\big(lX\frac{\partial}{\partial X} - kY\frac{\partial}{\partial Y}\big) + b(X,Y)X\frac{\partial}{\partial X}\Big\}$$

On the other hand, restricting our field ∂ to the centre manifold we have for (x,y) in some domain U as prescribed in §II.3 an expression for ∂ of the form,

$$\big(x^k y^l\big)^n \left\{ \big(a(x,y) + (x^k y^l)^\rho \alpha\big)\Big(lx\frac{\partial}{\partial x} - ky\frac{\partial}{\partial y}\Big) + \big(b(x,y) + (x^k y^l)^\rho \beta\big)x\frac{\partial}{\partial x}\Big)\right\}$$

for ρ some integer as large as we please, and α, β bounded functions on U. Now let us argue as in VI.2.2(i) but with some change between x and y, i.e. for monomials M, N in X, Y to be chosen, let us look for a conjugation in the form,

$$y \longmapsto Ye^{kMf}, \qquad x \longmapsto Xe^{Ng-lMf}$$

So that we require to solve the system of PDE's,

$$\check{D}(Mf) = -e^{nNg}\widetilde{a}(x,y) + a(X,Y), \qquad \check{D}(Nf) = e^{nNg}\widetilde{b}(x,y) - b(X,Y)$$

where $\widetilde{a}(x,y) = a(x,y) + (x^k y^l)\alpha$, and similarly for $\widetilde{b}$. Now in the particular cases at hand, a and b are functions of $X^k Y^l$ alone whence,

$$\widetilde{a}(x,y) = a(e^{nNg}X^k Y^l) + \big(X^k Y^l\big)^\rho e^{\rho n N g}\alpha(x,y)$$

and similarly for $\widetilde{b}$. Consequently the right hand sides of the above equations are both divisible by the monomial N, provided N divides $\big(X^k Y^l\big)^\rho$, and of course we suppose that $M\big|N$ but is not equal to it. Furthermore the function b is always divisible by $X^k Y^l$, so our functional derivative has the shape,

$$\begin{bmatrix} f \\ g \end{bmatrix} \longmapsto \left\{ \begin{bmatrix} M^{-1}\check{D}M & 0 \\ 0 & N^{-1}\check{D}N \end{bmatrix} + \varepsilon \right\} \begin{bmatrix} f \\ g \end{bmatrix} + \check{D}\begin{bmatrix} f \\ g \end{bmatrix}$$

Consequently we have up to the usual un-troublesome bounded perturbation a reduction by way of VI.1.3 to solving scalar equations,

$$\big(\mu + \check{D}\big)(F) = G$$

say even for the same μ, e.g. $M = \big(X^k Y^l\big)^{\rho-2} Y$, $N = \big(X^k Y^l\big)^{\rho-1} Y$. To solve the equation we employ the conjugation VI.2.2(i) if our context is VI.2.1(i), and otherwise do nothing. Irrespectively we have to proceed according to the proof of VI.2.2(i) and allow spiralling in the variable X. As per V.2.1 we can eschew worrying about base points remaining in the domain, since there is only one derivative in the system of PDE's, and apply the usual power series to take care of the bounded perturbation ε. This achieves the preparation VI.4.2 to an arbitrary order of approximation with respect to powers of the ideal $(x^k y^l)$, with a loss of domain exactly as per VI.2.2(i).

Unfortunately this loss of domain has a price, since in the notations of §II.3 the unbounded domain U_s now consists of the same left hand boundary, but a right boundary as per figure II.2.2. Consequently when we come to employ the strategy of pre VI.4.3 to achieve the actual conjugation on the 3-fold to the normal form we can suffer further loss of domain. Indeed just as the other integrable cases in §VI.3, the orientation is reversed from the notation of §II, i.e. one integrates from right to left to achieve the centre manifold, and, in fact, with the above choices of M and N also for the preparation VI.4.2. Consequently in the equation for the normal form one is integrating from left to right. This implies a problem of holomorphicity of base points unless we also spiral in the y-variable, with both spiralling in x and y occurring in cones in the logarithm of the same which must be adapted to s belonging to a strip domain. We have two cases to consider. In the first place, notation as per II.3, we take s in a sector S of aperture up to π/r so that s^r is bounded away from purely imaginary. This places us in exactly the situation of II.3.1(b), with large spiralling in x and y if S is of small aperture around the real axis, and small spiralling if S is close to full. In either case, we have a holomorphic section as a base point, and everything is as per op. cit. Otherwise S is again a sector of aperture up to π/r in s but this time bounded away from negative real. For large spiralling the

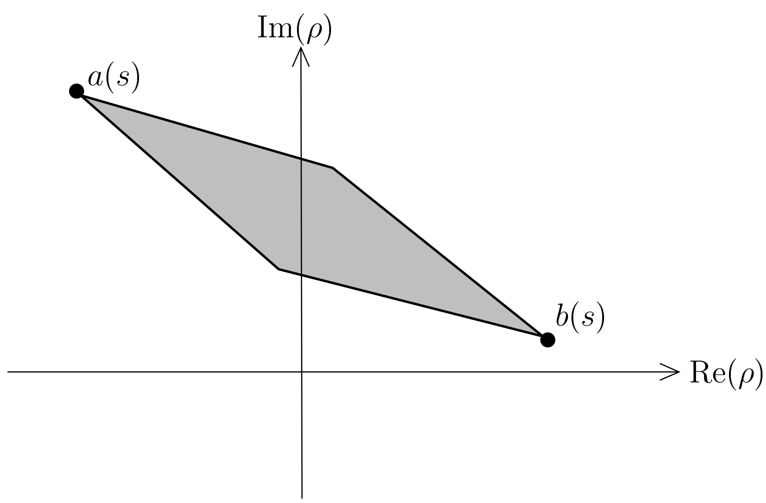

Figure VI.4.7

minimum of $\mathrm{Re}(\rho)$ will, according to the imaginary part of s^r, be at either $a(s)$

or $b(s)$, for all $s \in S$. In neither case, however, is $a(s)$ or $b(s)$ holomorphic. If we re-scale by s^r, i.e. view II.3.1(b) in the ξ plane, there are holomorphic sections $\alpha(s)$, respectively $\beta(s)$, close by. More precisely $\alpha(s)$, respectively $\beta(s)$, is at a finite (as a function of the spiralling) Euclidean distance from $a(s)$, respectively $b(s)$, in the ξ plane. Unfortunately this distance gets magnified by s^{-r} in the variable that counts, i.e. ρ, so we need to make some changes to use either α or β as a base point. This has some similarity with the extreme possibilities about imaginary critical points encountered in VI.4.5. Irrespectively, we simply modify the domain U' of II.3 fibre by fibre to obtain a sub-domain V in which $\mathrm{Re}(\rho)$ has a minimum at α, respectively β, as required, which in turn is joined by straight lines to the holomorphic sections employed in the previous case. By construction, at the price of some loss in radius in x and/or y, and in either variable a small decrease in the spiralling, i.e. the aperture around the real axis in the domain of the logarithm, V contains, therefore, a domain which is much the same as U'. Fortunately, we don't need more of a polynomiality condition on the relevant differential operator than that in III.2.1, so we can apply the implicit function theorem directly in V having constructed a bounded right inverse in the usual way, i.e. put a disc of fixed Euclidean distance in each fibre around our base point, and integrate from it to get a bounded right inverse with power series to deal with any perturbations exactly as per III.2. Whence, to conclude:

VI.4.8 Fact Suppose that in the completion of $\widehat{X}_{\mathrm{an}}$ we have either of the normal forms,

$$z\frac{\partial}{\partial z} + (x^k y^l)^n \Big\{ R(x^k y^l)\big(lx\frac{\partial}{\partial x} - ky\frac{\partial}{\partial y}\big) + \frac{(x^k y^l)^r}{1+\nu(x^k y^l)^{n+r}} x\frac{\partial}{\partial x}\big)\Big\}$$

for $n, r \in \mathbb{N}$, $\deg R \leqslant r$, $R(0) \neq 0$, $\nu \in \mathbb{C}$, or,

$$z\frac{\partial}{\partial z} + \frac{(x^k y^l)^n}{1+\ \nu(x^k y^l)}\Big(lx\frac{\partial}{\partial x} - ky\frac{\partial}{\partial y}\Big)$$

with everything as above, except ν which is now a formal function of a single variable. Then for $s = x^k y^l$ in a sector S of aperture π/n bounded away from one of real or purely imaginary we have a subdomain U of the product of hyperplanes defined by $\log x$, respectively $\log y$ such that U maps to S, with spiralling neighbourhoods of x and y as large as we like for S bounded away from real, or, exactly as per II.3, for spiralling adapted to S for S bounded away from purely imaginary, such that for z varying in a disc Δ we can find a conjugation of a convergent generator ∂ of the foliation on $\Delta \times U$ to the appropriate normal form, and this can be done with an arbitrary large degree of polynomial approximation modulo $x^k y^l$.

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