# SEMI-STABLE REDUCTION OF FOLIATIONS

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## Introduction

The notion of space be it algebraic, analytic, differentiable, or whatever, presents a manifest lack of functoriality with respect to the ideas. Specifically, unless one is a point set topologist, one invariably wishes to exploit hypothesis such as locally affine algebraic, analytic, $C^\infty$ etc. to arrive to global conclusions. The implicit definition of global, however, implied by the use of the word space insists that it is obtained from the local picture by glueing in the simplest possible way, i.e. along open inclusions. As such, even a relatively straightforward problem such as a fine moduli space for curves cannot be solved in the category of spaces, and requires the 2-category of stacks.

Unquestionably the original French champs is from many points of view preferable to its deliberate mistranslation, since it conveys the ubiquity and applicability of the construction to the globalisation of any reasonable local geometric structure. Indeed by a theorem of Keel and Mori, [K-M], together with a very minor amount of tweaking, I.1–I.3, any proper flat algebraic stack is basically, at least locally, a finite group acting on an affine, or more correctly the classifying stack of an abelian scheme over the same. As such, algebraic stacks represent a harmonious co-existence between finitely generated rings and finite groups, which although rather pleasant, excludes not only most champs, but, unquestionably, those which are the most interesting and which exhibit the most novel phenomenon. Indeed even the most proximate a priori non-étale yet analytic perturbation of [K-M]'s hypothesis, i.e. the dynamics of a rational map on $\mathbb{P}^1$ in general violates one of their principal conclusions, i.e. a postiori not every point (except for some rather special examples) admits an étale-neighbourhood. Our interest, however, will concern a different kind of perturbation in which the étale-condition is guaranteed, but the separation is rather more problematic, i.e. foliations, and more precisely foliations by curves.

Now if we wish to consider a space $X$ together with a foliation by curves $\mathcal{F}$, which certainly may be singular, as a stack, then there is a highly non-trivial problem of definition around sing $(\mathcal{F})$. Worse still, even if we're preparated to sweep this under the carpet and define leaves in the traditional manner only on $X\backslash\text{sing}(\mathcal{F})$, we have two possibilities for defining an equivalence of points on $X$ according as to whether we use the holonomy groupoid $\mathcal{F}_{\text{hol}} \rightrightarrows X$, VI.2, or the homotopy groupoid $\mathcal{F}_{\text{hom}} \rightrightarrows X$, VI.3, and although the unicity of analytic continuation guarantees the separatedness of $\mathcal{F}_{\text{hol}}$ (albeit certainly not that of $[X/\mathcal{F}_{\text{hol}}]$) it appears to be $\mathcal{F}_{\text{hom}}$ that has the better properties (e.g. closure under deformation limits) and this need not be separated. A lack of separation of $\mathcal{F}_{\text{hom}}$, however, can only occur, [Br1], if a sequence of bounded discs in $X\backslash\text{sing}(\mathcal{F})$ converges in the same not to a disc, but a disc with bubbles, and whence the relative canonical class $K_{\mathcal{F}}$ of $X \to [X/\mathcal{F}]$ is not nef., and conversely by [Bo-M] if we don't have a foliation in conics and $K_{\mathcal{F}}$ is not nef., then $\mathcal{F}_{\text{hom}}$ fails to be separated.

Our immediate object of study is, therefore, preliminary in nature and regards what might otherwise be termed relative minimal model theory of $X$ over $[X/\mathcal{F}]$. Clearly to even get this underway we have to insist that the birational geometry of our foliation is well defined, i.e. in a foliated sense it has canonical, or even just log-canonical singularities, I.6-I.7, and unlike spaces we don't for the moment have a Hironaka type theorem to guarantee this except in dimension 2, albeit that the situation in dimension 3 looks very promising, [Ca]. Nevertheless, such singularities are, of themselves highly generic, and certainly the study of smooth varieties wasn't exactly ignored pre-Hironaka, so we'll unhesitantly confine ourselves to what may be termed singularities of Mori category. As such for a families of non-rational algebraic curves, a relative minimal model theorem is synomous with a semi-stable reduction, and, of course, we have a much stronger stable reduction theorem available to us in the form of the existence of the moduli-stacks $\mathcal{M}_{g,n}$ of stable $n$-pointed curves of genus $g$. The general foliated situation, however, does not admit such a simple analysis, since amongst other things, there are likely to be very few $\mathcal{F}$-invariant subvarieties on $X$, and whence the problem is genuinely higher dimensional. Nevertheless, it possesses a couple of salient features inherited from the hypothesis of relative dimension 1 that make it tractable,

(a) The singularities of the Gorenstien covering stack, I.5, or locally, the index one cover in more traditional language, exactly reflect those of the moduli, I.8.

(b) Not only do we have the cone theorem in best possible generality II.4, but the holonomy of a $K_{\mathcal{F}}$-negative curve allows us to assert that a formal neighbourhood of the same is determined by its normal cone II.5-II.8.

Armed with this, and as implied by (a), working throughout in the category of Deligne-Mumford stacks (with characteristic zero implied by (b)) one proceeds to a detailed examination of the locus of extremal rays in §III and to the construction of flips and their termination in §IV. As such the main lemmas here are the minimal theorem for non-conic foliations, IV.7.5, and the reduction by contractions, flips, and if one wishes to preserve projectivity yet another operation, correction, of a foliation in conics to a $\mathbb{P}^1$-bundle over a stack IV.8.5, with both of these holding in the slightly more general form IV.10.2.

With these algebraic preliminaries out of the way we turn, §V, to relating $K_{\mathcal{F}}$ to the curvature not just of leaves, but of invariant curves, and achieve this with essentially optimal isoperimetric inequalities of either a stack flavour V.5.10 or scheme like V.6.1. From there we tidy up with a description of the almost uniqueness of the minimal model VI.1, and the relative uniformisation theorem in the general type case VI.4. More delicate applications of the minimal model lemma together with its essential uniqueness, and the said isoperimetric inequalities will be given elsewhere.

I am imbedded to Abramovich, Bogomolov, Brunella, Cano and Corti for many valuable suggestions and comments, but, as ever, nothing would ever get done without Cécile.

# I. Preliminaries on Foliated Stacks

## I.1. Moduli Problems

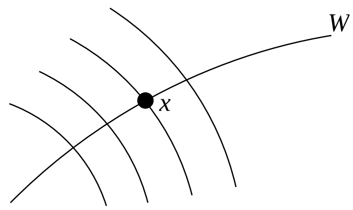
The general moduli problem for groupoids in algebraic spaces is basically solved by [K-M]. Nevertheless there are a couple of outstanding issues which are worth clearing up by way of a closer look at op. cit. To this end let us take up verbatim the problem together with the notations from the same, i.e.

$$R \underset{s}{\overset{t}{\rightrightarrows}} X$$

is a groupoid in, say, finitely presented algebraic spaces over some base, or just finite type should the base be locally Noetherian, such that the source, $s$, and sink, $t$, are flat. Manifestly we require the map $j = s \times t$ to be separated, to have any hope, however, of forming a GC quotient $X/R$ we must also suppose that the stabiliser $j : S := j^{-1}(\Delta_x) \to X$ is proper. Let us, therefore, suppose this and see if following [K-M] we can solve the GC quotient problem. The question is, of course, local on $X$, and the key point of op. cit. is that one can replace $X$ by an appropriate transversal (slice in [K-M]) to the orbits. Specifically, the slightly technical, though cunning, definition of transversal is,

**I.1.1 Definition.** (cf. [K-M] 3.3, step 2) For $x \in X$ a closed geometric point call $W \ni x$ a C-M transversal if around $x$ it is defined by $\dim_x \overline{\mathrm{orb}(x)} := \overline{t(s^{-1}(x))}$ equations and its intersection with $\overline{\mathrm{orb}(x)}$ is zero dimensional at $x$.

The picture is, of course, clear, albeit illuminating, i.e.




as is the existence of such transversals at generic points of $\mathrm{orb}(x)$, i.e. we can freely localise $X$, and so eventually just lift parameters in the local ring of $\overline{\mathrm{orb}(x)}$ at $x$ to $X$. What is less clear, however, is,

**I.1.2 Fact.** ([K-M], 3.3(i)) *Suppose simply that $j = s \times t$ is separated, then for any given geometric point $x \in X$ there is an isomorphic geometric point $w \in X$ together with a C-M transversal $g : W \ni w \hookrightarrow X$ such that,*

$$p : W \times_{(g,t)} R \underset{s}{\longrightarrow} X$$

*is flat. In particular,* [K-M] 3.1, *the groupoid,*

$$R|_W \underset{s}{\overset{t}{\rightrightarrows}} W$$

*defines around $w$, the same G-C quotient problem as the groupoid $R \rightrightarrows X$ around $x$.*

**Proof.** This is precisely the proof of [K-M] 3.3, however, there is a little confusion there about $j$ being quasi-finite, so let's check that $j$ separated suffices. In the first place, being Cohen-Macauly is an open condition, so we may choose a geometric point $w \in \mathrm{orb}(x)$, such that, $\mathrm{orb}(x)$ is C-M at $w$. Now consider, $t : s^{-1}(w) \to X$. This is a principle homogeneous space under $\mathrm{Hom}(w, w)$, so in particular its flat over $\mathrm{orb}(x)$, and $s^{-1}(w)$ is C-M along $t^{-1}(w) \cap s^{-1}(w) = \mathrm{Hom}(w, w)$, which is precisely the condition that we need to guarantee the asserted flatness of $p$. $\square$

Now once we have this, we can throw in the condition of proper stabiliser, and see what happens. Shrinking as necessary we may as well suppose that $\mathrm{orb}(w)$ contains exactly one geometric point, and that any other orbit, $\mathrm{orb}(w')$, $w' \in W$, is finite. Consequently, $j : R \to W \times W$ is quasi-proper with a (quasi) Stein type factorisation $R \underset{j_0}{\longrightarrow} R' \underset{j'}{\longrightarrow} W \times W$, where $j_0$ is actually proper, and $j'$ quasi-finite. Better still, since the orbits are finite and without loss of generality $W$ is affine both $s$ and $t$ are also quasi-proper, with quasi-Stein factorisations $s = s' \circ s_0$, $t = t' \circ t_0$ through the same $R'$, and all of this is independent of flat base change. As such, the GC quotient problem for $R \rightrightarrows W$, will follow from the main theorem of [K-M] provided that $R' \rightrightarrows W$ is a flat groupoid. This is however a well known theorem about abelian schemes, i.e. $s' : R \to R'$ is a family of equidimensional abelian varieties with a section so by [Mu1] 6.1.4, it is an abelian scheme, so in particular $s_0$ is flat whence $s'$ is flat. This is in fact best possible provided the desired object of a moduli problem is a separated space, so let's summarise by way of,

**I.1.3 Complement to [K-M].** *Let $R \rightrightarrows X$ be a flat groupoid with proper stabiliser, then $X/R$ has a G-C quotient as a possibly non-separated algebraic space. The said quotient is separated if $R \to X \times X$ is proper.*

## I.2. Stacks versus Orbifolds

Let us immediately apply our previous moduli considerations to separated algebraic stacks for the *fppf* topology. Indeed let $\mathcal{X}$ be such a stack, and $V \to \mathcal{X}$ a *fppf* atlas then we can form the flat groupoid,

$$R := V \times_{\mathcal{X}} V \underset{s}{\overset{t}{\rightrightarrows}} V$$

where the maps are simply the necessarily flat projections. We therefore have,

**I.2.1 Variant.** *Let $\mathscr{X}$ be a $fppf$ algebraic stack, and suppose to be on the safe side the base is locally Noetherian, then the following are equivalent,*

(I) *$\mathscr{X}$ is separated.*

(II) *There is a proper moduli map $\pi : \mathscr{X} \to X$, with $X$ a separated algebraic space.*

(III) *For some (whence all) $fppf$ atlases $V \to \mathscr{X}$, $s \times t : R \to V \times V$ is proper.*

**Proof.** (II) $\Rightarrow$ (I) is immediate, (I) $\Rightarrow$ (III) is basically [L-MB] 7.8, along with the general results of § 10 of the same. This leaves (III) $\Rightarrow$ (II), and we already have a separated moduli space $X := R/V$ by I.1.3, so it remains to check that $\pi$ is proper, which follows by op. cit. 7.3 and the universal submersivity of $\pi$. □

Notice also that in the course of constructing $X$, we constructed an intermediate $fpqf$ stack $\mathscr{X} \underset{\rho}{\longrightarrow} \mathscr{X}' \underset{\sigma}{\longrightarrow} X$, where now $\sigma$ isn't just proper but finite. Observe further that if $S \underset{j}{\longrightarrow} V$ is the stabiliser group scheme of our atlas $V \to \mathscr{X}$, then we've already establised in the sequel to I.1.2 that the geometrically reduced connected component $S^0$ of the identity is an abelian scheme over $V$. We have moreover a $fppf$ groupoid,

$$R^0 = \{\alpha \times g \times \beta\, ;\ \beta \circ g = g \circ \alpha\} \subset S^0 \times_s R_t \times S^0 \rightrightarrows S^0$$

together with a flat map of groupoids $[R^0 \rightrightarrows S^0] \to [R \rightrightarrows V]$, while the properness of $R \to V \times V$ implies the same for $R^0 \to S^0 \times S^0$. Consequently we may apply I.1.2, to conclude the solution of the GC quotient problem for $[R^0 \rightrightarrows S^0]$ in the form of an abelian "scheme" (inverted commas seem preferable to writing relative abelian algebraic space) $\alpha : A \to X$. Equally we have a map, $S^0 \to A \times_X V$, which is an isomorphism on geometric points, and whence by rigidity an isomorphism stricta dictum. In consequence, and with the aid of [L-MB] 10.7, we may deduce,

**I.2.2 Yet another Variant.** *Things as in* I.2.1, *then yet another equivalent condition is,*

(IV) *There is a proper map $\rho : \mathscr{X} \to \mathscr{X}'$ to a separated algebraic $fpqf$ stack, suth that $\rho$ is a bundle of classifying spaces for $A$ in the flat topology (or indeed the étale topology should it exist) of $\mathscr{X}'$, i.e. for $W \to \mathscr{X}'$ a small open, $\mathscr{X} \times_{\mathscr{X}'} W = B_{\text{ét}}\, A \times_X W$.*

Plainly this more or less reduces the study of separated $fppf$ stacks to that of separated $fpqf$ stacks, so let's concentrate on the latter, although we'll still have a few further comments on how to reduce the general case even more. Regardless we now wish to show, following [V], that a separated $fpqf$ stack is an orbifold (in fact provided that one allows non-separated algebraic spaces, separation isn't important). The question is of a local nature, be it by I.2.1(II) or equivalently [K-M] § 4, i.e. we wish to show that a $fpF$ (where $F$ here, and elsewhere means finite) groupoid $R \rightrightarrows V$, with $V$ the spectrum of a strictly Henselian local ring, is equivalent to a classifying stack $[W/G]$ for $G$ some group scheme, and $W \to V$ finite flat. In the first place $R$ is a finite disjoint union, say, $\coprod_g V_g$, of spectra of local Henselian rings. Moreover base change by a finite flat strictly Henselian $\mathcal{O}_V$ algebra neither changes the GC quotient problem, nor does it increase the number of connected components of $R$, so without loss of generality we may suppose that each $s : V_g \to V$ has a section $f_g$. Now let $G$ be an index set for the set of connected components, with distinguished element 1 corresponding to the connected component of the identity, and observe:

**I.2.3 Facts.** (a) *The groupoid structure on $R \rightrightarrows V$, induces a group structure on $G$, according to $V_{gt} \times_s V_h \to V_{hg}$.*

(b) *Each $V_g$ is isomorphic to $V_1$, by way of,*

$$V_1 \to V_g : \alpha \mapsto f_g(t(\alpha)) \circ \alpha \qquad V_g \to V_1 : \beta \mapsto f_{g-1}(t\beta) \circ \beta\,.$$

Indeed (a) is clear, and while it's not a priori wholly transparent that the maps in (b) are actually inverse, it's never the less the case since over the closed point it's true, so it follows everywhere by Nakayama's lemma and the flatness of $R$ over $V$.

As such in characteristic zero, we're actually done, since quite generally $V_1 \rightrightarrows V$ is a groupoid, and without loss of generality the orbit of the closed point is itself, so restricting to the closed point we find an affine group scheme with 1 point, from which, [K-M] 6.5, the classifying stack $[V_1 \rightrightarrows V]$ is an honest space if the characteristic is neither positive nor mixed. In general, however, things are a bit more complicated. In the first place, one notes that if $\tilde{V} \subset V_1$ is the first order thickening of the identity then $\tilde{V} \rightrightarrows V$ is still a flat groupoid, and by virtue of working to $1^{\text{st}}$-order, there is an isomorphism $\tilde{V}_s \times_s \tilde{V} \xrightarrow{\sim} \tilde{V}_t \times_s \tilde{V}$ which gives $\tilde{V} \underset{s}{\longrightarrow} V$ the structure of a group scheme over $V$ acting on $V$. One then descends this to the moduli, and goes through the rigmorale of [SGA3] VII to find a flat group scheme on the same which acts on $V$. Nevertheless such a refinement isn't of immediate necessity, so we'll omit its pursuit, and make,

**I.2.4 Hypothesis.** The implicit base of this discussion has characteristic zero, or more generally the characteristic of every closed geometric point is prime to the order of the stabiliser groups of $\mathscr{X}'$.

In which case we've established,

**I.2.5 Fact.** *Suppose* I.2.4, *and let* $\mathscr{X}'$ *be a* $fpqf$ *stack, then in fact it's a Deligne-Mumford stack, and,* cf. [V] 2.8, *it's even an orbifold, i.e. there exists an étale cover* $\coprod_\alpha U_\alpha \to X$ *of the moduli, such that* $\mathscr{X}' \times_X U_\alpha = [V_\alpha / G_\alpha]$, *where* $\coprod_\alpha V_\alpha \to \mathscr{X}'$ *is an atlas, and* $G_\alpha$ *a finite group acting on* $V_\alpha$.

Basically I.2.5, is the stack version of [K-M] 6.5, i.e. algebraic $fpqf$ spaces are algebraic spaces. In any case, the usual definition of orbifold pre-supposes that $G_\alpha$ is a subgroup of automorphisms of $V_\alpha$ (it also supposes $V_\alpha$ smooth, which clearly we won't) so let's try and get to grips with this. As a result, let us further make,

**I.2.6 Hypothesis.** $\mathscr{X}'$, equivalently $X$, has only one generic point.

What this amounts to at the level of the $V_\alpha$'s is that provided $U_\alpha$ is irreducible then each one has finitely many components $V_{\alpha i}$, all of which are isomorphically permuted by $G_\alpha$. As a result if $H$ is the automorphism group of the generic geometric point, then we have an exact sequence,

$$0 \to G'_\alpha \to G_\alpha \twoheadrightarrow G''_\alpha \hookrightarrow \mathrm{Aut}(V_\alpha)$$

whith $H$ isomorphic to $G'_\alpha$. More generally, even if $U_\alpha$ isn't irreducible, then on restricting to an appropriate open subset, one sees that $H$ is still the kernel of $G_\alpha \to \mathrm{Aut}(V_\alpha)$, and we obtain a stack $\mathscr{X}'/H$ with moduli $X$, which is locally the orbifold $[V_\alpha / G''_\alpha]$. Notice in addition,

**I.2.7 Fact.** *Suppose* I.2.4 *and* I.2.6, *then for any test scheme* $T$ *we have a fibre square,*

$$\begin{tikzcd}
\mathscr{X}' \arrow[d] & {[T/\mathbb{H}]} \arrow[l] \arrow[d] \\
\mathscr{X}'/H & T \arrow[l]
\end{tikzcd}$$

*where* $\mathbb{H}/T$ *is a group scheme over* $T$, *isomorphic to* $H$ *at geometric points.*

**Proof.** Locally on $T$ this is clear. In general, the patching data for the $G'_\alpha$'s, when restricted to $T$ gives exactly the patching data of a group scheme $\mathbb{H}/T$, which is manifestly $H$ over the geometric points. □

A useful corollary of which is,

**I.2.8 Corollary.** *Let $\mathcal{C}/\mathbb{C}$ be a smooth one dimensional Deligne-Mumford stack, with $H$ its generic stabiliser, then if $\mathcal{C}/H$ is not a so called bad orbifold there is an étale map $C \to \mathcal{C}$, for some honest curve $C$, and even in the bad case there is a map, albeit ramified, $\mathbb{P}^1 \to \mathcal{C}$.*

This also seems an appropriate, for want of a better, place to make,

**I.2.9 Warning.** Given a Deligne-Mumford stack $\mathcal{X}$ with moduli $X$, and $Y \hookrightarrow X$ a closed integral subspace the closed integral sub-stack associated to $Y$, $\mathcal{Y}$, say, in $\mathcal{X}$ is given locally, in terms of our $\pi_\alpha : V_\alpha \to U_\alpha$ by taking $\pi_\alpha^{-1}(Y)_{\mathrm{red}}$, i.e. the analogue of irreducible subvariety, at least if $U_\alpha$ is irreducible, is a reduced variety whose irreducible components are permuted by a group action.

Finally let's put all of this together for a separated $fppf$ stack $\mathcal{X}$ under the hypothesis I.2.4 and I.2.6. Specifically we can find an étale cover $\coprod_\alpha U_\alpha \to X$ of the moduli, together with an étale atlas $\coprod_\alpha V_\alpha \to \mathcal{X}$ such that for some abelian scheme $A/X$ we have finite groups $G_\alpha \in \mathrm{Ext}(H_\alpha, H)$, with $H_\alpha$ a subgroup of $\mathrm{Aut}(V_\alpha)$ such that, $\mathcal{X} \times_X U_\alpha \xrightarrow{\sim} [V_\alpha / G_\alpha \times A_{U_\alpha}]$. Denoting by $\mathcal{X}/A$ our old $\mathcal{X}'$, and $\mathcal{X}/A/H$ our old $\mathcal{X}'/H$, it's definitely not true that $\mathcal{X}/A$ can be realised as a product of $\mathcal{X}/A/H$ with the classifying stack $BH$, nor is it true that we can relate $\mathcal{X}$ to $\mathcal{X}/A$ in this sort of way. Nevertheless the obstruction to the latter is quite small, i.e. $\mathrm{Ext}^1_{\mathcal{X}/A}(\mathcal{I}, \pi^* A)$, where $\mathcal{I}$ is the inertia group stack of $\mathcal{X}/A$, and is as near zero as makes no difference, i.e.

**I.2.10 Fact.** *Let $T \to \mathcal{X}/A/H$ be a test scheme, then for $\mathbb{H}$ as in* I.2.7, *and some étale cover $T' \to T$ we have a fibre square,*

$$\begin{tikzcd} \mathcal{X} \arrow[d] & BA \times_X [T'/\mathbb{H}] \arrow[l] \arrow[d] \\ \mathcal{X}/A/H & T' \arrow[l] \end{tikzcd}$$

**Proof.** Indeed the obstruction to splitting $\mathcal{X}_T$ lies in, $H^1_{\text{ét}}(T, \mathrm{Hom}(\mathbb{H}, A))$ so this is clear. $\square$

Regardless the important point is that there obviously isn't much to an arbitrary separated $fppf$ stack beyond what one finds in the Deligne-Mumford set up, and so we'll make,

**I.2.11 Convention.** From now on, unless stated to the contrary, stack will mean separated Deligne-Mumford stack of finite type over some base (ultimately $\mathbb{C}$, but for the moment let's permit otherwise).

## I.3. Topologies

Let's now give a letter, say, $S$, to our implicit base, supposed locally Noetherian to be on the safe side, and consider the category $\mathcal{C}$ of projective schemes over $S$. On $\mathcal{C}$ we have 4-topologies of interest, i.e. ét $F$, ét, $fpF$, $fpqf$, with respect to which we may form either spaces (i.e. quotients by an equivalence relation for the given topology) or stacks. As such consider, the following diagram of 2-categories, and categories, where the maps are the obvious ones, i.e.

$$
\begin{tikzcd}
\textit{fpF}\text{ stacks} \arrow[rr] \arrow[dr, "\sim", "\text{if I.2.4 holds}"'] \arrow[dd] & & \textit{fpqf}\text{-stacks} \arrow[dr, "\sim", "\text{if I.2.4 holds}"'] \arrow[dd] & \\
& \text{ét } F \text{ stacks} \arrow[rr, crossing over] & & \text{ét stacks (= Deligne-Mumford)} \arrow[dd] \\
\textit{fpF}\text{ spaces} \arrow[rr] \arrow[dr, "\sim"] \arrow[dd, "\wr"] & & \textit{fpqf}\text{ spaces} \arrow[dr, "\sim"] \arrow[dd, "\wr"] & \\
& \text{ét } F \text{ spaces} \arrow[from=uu, crossing over] \arrow[rr, crossing over] \arrow[dl, "\sim"] & & \text{ét spaces} \arrow[dl, "\sim"] \\
\begin{array}{c}\text{Schemes of}\\ \text{finite type}/S\end{array} \arrow[rr] & & \begin{array}{c}\text{Algebraic spaces of}\\ \text{finite type}/S\end{array} &
\end{tikzcd}
\tag{I.3.1}
$$

Now the basic point is that the vast majority of this diagram is redundant. The isomorphisms in the bottom triangles are [SGA3] V.7, and [K-M] 4.2 and 6.5 respectively. In fact we could even replace $F$ (= finite) by proper, and $qf$ (= quasi-finite) by proper, and quasi-proper, all of which is just as well since it's about as easy to describe a local ring in the flat topology as it is to define the set of all sets. We've also seen that under what amounts to a near enough best possible hypothesis, I.2.4, that the vertical arrows in the upper square are isomorphisms. It's also true that a Deligne-Mumford stack is proper over its moduli, so the only two things that really make a difference are,

(a) The bottom arrow from schemes to algebraic spaces.

(b) The difference between the quotient $V/G$ of an affine by a group (or more generally group scheme), and the classifying stack $[V/G]$.

Indeed this distinction is wholly sharp, since we assert,

**I.3.2 Claim.** *The left hand face of the diagram is the fibre over schemes of the right hand face, i.e. a Deligne-Mumford stack has projective moduli iff it is an ét$F$ stack.*

**Proof.** The if direction is again [SGA3] V.7, at least if the moduli is separated, but even if it's not, it's not separated for the wholly scheme like reason of having too many points, and whence coincides with the algebraic space quotient of [K-M]. Conversely let, $\pi : \mathcal{X} \to X$ be the supposed scheme like moduli of a Deligne-Mumford stack. The question is local on $X$, so we can suppose that $X$ is the spectrum of a local ring, and let $q : U \to X$ be its strict Henselisation. By [V] 2.8 (and/or I.2.5), there is a strictly Henselian local ring $V$ and a finite group $G$, such that, $\mathcal{X} \times_X U \xrightarrow{\sim} [V/G]$. As ever denote by $p_i$, $i = 1$ or 2, the projections from $U \times_X U$ to $U$, then since $\mathcal{X}$ is defined over $X$, we have a descent datum of classifying stacks,

$$\phi : p_1^*[V/G] = [p_1^* V/G] \xrightarrow{\sim} p_2^*[V/G] = [p_2^* V/G]\,.$$

Now $V$ is strictly Henselian, and without loss of generality $G$ is the stabiliser of the closed point, so in fact this data amounts to two different descent data, i.e.

$$\varphi : p_1^* V \xrightarrow{\sim} p_2^* V\,, \qquad \Phi : G \xrightarrow{\sim} G$$

such that $\varphi(v^g) = \varphi(v)^{\Phi(g)}$, and of course compatibility with the $p_{ij}$'s from $U \times_X U \times_X U$ to $U \times_X U$. Consequently $V$ descends to a local affine $Y$, finite over $X$, and $G$ to a finite flat group scheme $\mathbb{G}/X$ acting on $Y$. In particular, over geometric points $\mathbb{G}$ is $G$, and $\mathscr{X} = [Y/\mathbb{G}]$. $\square$

Notice that the argument of the proof applies to other contexts, for example,

**I.3.3 Sub-Fact.** ("*GAGA* for stacks") *Let $\mathscr{X}_{\mathrm{an}}$ be an analytic Deligne-Mumford stack whose moduli (which necessarily exists by an easy variant of* [K-M]*) is a compact Moishezon manifold $X_{\mathrm{an}}$, then $\mathscr{X}_{\mathrm{an}}$ is algebraic, i.e. the analytic stack associated to a Deligne-Mumford stack.*

**Proof.** The compactness of $X_{\mathrm{an}}$ implies, and in fact this is all we need it for, that the ramification of $\mathscr{X}_{\mathrm{an}} \to X_{\mathrm{an}}$ is algebraic. As such the question is again local, so we can proceed as above on replacing $X$ by a strictly Henselian ring, and $U$ by its analytification, or indeed, completion. $\square$

Taking all of this into account, we therefore have several different topologies for a stack $\pi : \mathscr{X} \to X$, together with the obvious functors,

$$\begin{array}{ccccc}
\mathscr{X}_{\mathrm{fl}} & \longrightarrow & \mathscr{X}_{\mathrm{\acute{e}t}} & \longrightarrow & \mathscr{X}_{\mathrm{Zar}} \\
{\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\
X_{\mathrm{fl}} & \longrightarrow & X_{\mathrm{\acute{e}t}} & \longrightarrow & X_{\mathrm{Zar}}
\end{array}$$

where we avoid discussing the right most of these if $X$ is not a scheme. The important thing (as Abramovich pointed out to me) is,

**I.3.4 Further Fact.** (cf. [A-V]) *Regardless of whether we're is the flat, étale, or Zariski topology, or for that matter analytic if things are over $\mathbb{C}$, $\pi_*$ is acyclic on coherent sheaves on supposing* I.2.4.

**Proof.** As per op. cit., for flat, étale or analytic, this just amounts to the vanishing of $H^i(G, M)$, for $M$ a $G$-module, with $\frac{1}{|G|} \in \mathrm{Aut}(M)$, which is exactly what I.2.4 guarantees. In the Zariski case, we're looking at $[Y/\mathbb{G}]$ in the notations of the proof of I.3.3, so for a suitable étale cover $\tilde{X} \to X$, and $M$ a coherent $\mathbb{G}$-module on $Y$, we have a spectral sequence,

$$\check{H}^i(\tilde{X}/X, H^j(G, M \otimes_{\mathcal{O}_X} \mathcal{O}_{\tilde{X}})) \Rightarrow H^{i+j}(\mathbb{G}, M)$$

so again we're done, since all the rings in question are local. $\square$

Consequently for $\mathcal{M}$ a coherent sheaf on a stack, the symbol $H^i(\mathscr{X}, \mathcal{M})$ is not just wholly unambiguous, but equal to the equally unambiguous symbol, $H^i(X, \pi_* \mathcal{M})$. In particular if $X$ is projective, then $\mathscr{X}$ is cohomologically projective, i.e.

**I.3.5 Sub-Fact.** *Let $H$ be ample on $X$ and suppose* I.2.4, *then for $\mathcal{M}$ on $\mathscr{X}$ coherent, there is an integer $n_0 = n_0(\mathcal{M})$, such that for $n \geq n_0$, $H^i(\mathscr{X}, \pi^* H^{\otimes n} \otimes \mathcal{M}) = 0$, $i \geq 1$.*

We should of course also check,

**I.3.6 Hilbert 90.** Let $\mathscr{X}$ be an algebraic stack, then,

$$H^1_{\mathrm{fl}}(\mathscr{X}, \mathrm{GL}_n) = H^1_{\mathrm{\acute{e}t}}(\mathscr{X}, \mathrm{GL}_n) = H^1_{\mathrm{Zar}}(\mathscr{X}, \mathrm{GL}_n)$$

where the last group is only understood to have sense if the moduli is a scheme.

**Proof.** This is just the usual Hilbert 90, together with a $G$-action that respects descent. $\square$

At which point it's pretty clear that a stack with projective moduli is every bit as good as projective variety, and so it seems appropriate to make,

**I.3.7 Parenthesis.** Indeed the only difference between the stack and its moduli is that the former has too many tangent vectors (whence I.3.5 does not imply, as it couldn't, that a stack with a cohomologically ample bundle is embeddable in projective space), as such a separated stack is not separated in the classical sense and is more akin to a non-separated algebraic space with the right number of points, but the wrong number of tangents (cf. [K-1]). Unlike the non-separated space, however, which locally thinks itself wholly separated, the stack knows that it has too many tangent vectors, and if the moduli is projective it is in many many ways a better object than a non-projective scheme (e.g. I.3.5, Kodaira vanishing holds too, etc.). The conclusion then, following Gromov, [G], would be to consider stacks (at least separated ones) as a natural soft category for the ideas of algebraic geometry, i.e. just as integrability isn't that important for complex structure, neither is separation, in its classical manifestation, for things which are polynomial.

## I.4. Resolution and Reduction

At this point we confine ourselves to characteristic zero, and of course finite type. In the first place recall the definition of algorithmic resolution of singularities according to [B-M], i.e.

**I.4.1 Definition.** Given a reduced scheme $V$ (or even algebraic space, or even formal scheme or formal algebraic space) there is a sequence of modifications,

$$V = V_0 \leftarrow V_1 \leftarrow \cdots \leftarrow V_n = \tilde{V}$$

such that $V_{i+1} \to V_i$ is a blow up in a smooth centre $Z(V_i)$, $\tilde{V}$ is non-singular, and the process is invariant with respect to smooth maps, i.e. if $\rho : W \to V$ is smooth then the algorithmic resolution,

$$W = W_0 \leftarrow W_1 \leftarrow \cdots \leftarrow W_n = \tilde{W}$$

is the cartesian product of that of $V$ by $\rho$, or if one prefers, $Z(W_i) = \rho^{-1} Z(V_i)$.

The applicability of this to stack resolution is clear. Indeed even in the case of Artin stacks which we momentarily will permit. Specifically let, $\coprod_\alpha X_\alpha = X_0 \to \mathcal{X}$ be a smooth presentation of an integral Artin stack, by integral schemes and consider the subschemes $Z_{\alpha 0}$ of $X_\alpha$ corresponding to the smooth centres of the $1^{\text{st}}$-stage of algorithmic resolution. Necessarily we have a fibre square of smooth maps,

$$\begin{tikzcd}
X_\alpha \arrow[d, "p_\alpha"'] & X_\alpha \times_{\mathcal{X}} X_\beta \arrow[l, "p_\beta"'] \arrow[d, "p_\alpha"] \\
\mathcal{X} & X_\beta \arrow[l, "p_\beta"']
\end{tikzcd}$$

Although perhaps not integral, $X_\alpha \times_{\mathcal{X}} X_\beta$ is reduced, with a well defined algorithmic centre $Z_{\alpha\beta}$, and of course $p_\beta^{-1}(Z_\alpha) = p_\alpha^{-1}(Z_\beta)$. Consequently the $Z_{\alpha 0}$ patch to a smooth sub-stack $\mathcal{Z}_0$ of $\mathcal{X}$ and almost by definition we have a diagram,

$$\begin{tikzcd}
 & \coprod_\alpha \mathrm{Bl}_{Z_{\alpha_0}}(X_\alpha) \arrow[dl] \arrow[dd, "\tilde{p}_\alpha"] \\
\coprod_\alpha X_\alpha \arrow[dd, "p_\alpha"'] & \\
 & \mathrm{Bl}_{\mathcal{Z}_0}(\mathcal{X}) \arrow[dl] \\
\mathcal{X} &
\end{tikzcd}$$

with $\tilde{p}_\alpha$ a smooth presentation, and whence,

**I.4.2 Fact.** (characteristic 0) *Any integral Artin stack $\mathcal{X}$ admits a resolution $\rho : \widetilde{\mathcal{X}} \to \mathcal{X}$ by a sequence of modifications in smooth sub-stacks. Better still the process is even algorithmic with respect to smooth maps of stacks $\mathcal{Y} \to \mathcal{X}$, and enjoys an embedded variant for closed substacks $\mathcal{Z}$ of $\mathcal{X}$.*

Equally however, one is frequently presented with a smooth stack $\mathscr{X}$, whose moduli space $X$ is singular. Here algorithmic resolution doesn't a priori help, since it leaves $\mathscr{X}$ unmodified, and it would certainly be desirable to have a sequence of modifications which didn't just smooth $\mathscr{X}$ but also its moduli. This is however, and again Abramovich pointed it out, impossible by [O], and so let's turn to that which is directly relevant, i.e. semi-stable reduction. Specifically let $\rho : \mathscr{X} \to \mathcal{C}$ be a proper family of 1-dimensional stacks over a normal 1-dimensional stack with $\mathscr{X}$ smooth. We already know that we can find a map $C \to \mathcal{C}$ from a smooth curve, and since our interest is reduction by base change, we may as well suppose that $\mathcal{C}$ is a honest curve. Consequently the fibres $\mathscr{X}_c$, for $c \in C$, are well defined substacks to which we may apply embedded resolution and so arrive to the situation that $\rho : \mathscr{X} \to C$ is a family of 1-dimensional stacks, such that the singularities of $\mathscr{X}_c$ are no worse than nodes. Notice also that since $\mathscr{X}$ has a moduli space, the components of $\mathscr{X}_c$ are well defined and so as ever we can write,

$$\mathscr{X}_c = \sum_{i=1}^{t} n_i(c)\, \mathscr{X}_i$$

for $n_i \in \mathbb{N}$ the multiplicities. For each singular point $c$, choose $n(c)$ such that each $n_i(c)$ divides it, let $B$ be a map from a curve to $C$ with the orbifold/stack structure defined by the $n(c)$, and $\mathscr{X}_B \to B$ the base change, with $\widetilde{\mathscr{X}}_B$ its algorithmic resolution. For any $U \to \mathscr{X}_B$ étale, the corresponding resolution $\tilde{U}$ of $U$ is, by virtue of being algorithmic, the unique minimal resolution of $U$ and so thanks to the standard argument for stable reduction we have concluded,

**I.4.3 Fact.** (characteristic 0) *Any proper map $\rho : \mathscr{X} \to \mathcal{C}$ from a smooth 2-dimensional stack to a smooth 1-dimensional stack with fibres at worst nodes admits a semi-stable reduction, i.e. there is a map from an honest smooth curve $B \to \mathcal{C}$ such that if $\widetilde{\mathscr{X}}_B$ is the algorithmic resolution of the base change then the fibres are still no worse than nodes, and are even reduced.*

Notice additionally that we can combine all of this with the uniformisation theorem of Corollary I.2.8 to obtain,

**I.4.4 Lemma.** *Let $\rho : \mathscr{X} \to \mathcal{C}$ be as in the previous fact and suppose further that modulo its generic stabiliser the generic point of $\mathscr{X}$ over $\mathcal{C}$ is not a bad orbifold then there is a family $\tilde{\rho} : \mathscr{Y} \to B$ of 1-dimensional semi-stable stacks (i.e. with reduced fibres) over a curve $B$, which is scheme like in codimension 2, together with a commutative diagram,*

$$\begin{tikzcd} \mathscr{X} \arrow[d, "\rho"'] & \mathscr{Y} \arrow[l] \arrow[d, "\tilde{\rho}"] \\ \mathcal{C} & B \arrow[l] \end{tikzcd}$$

*such that the ramification of $\mathscr{Y}/\mathscr{X}$ is supported only in the fibres of $\tilde{\rho}$.*

**Proof.** We may as well apply I.2.8 in order to say that $\mathcal{C}$ is actually a curve $C$ and $\mathscr{X}/C$ is semi-stable. Denote by $K$ the function field of $C$, and apply I.2.8 to $\mathscr{X}_{\overline{K}}$. Consequently for some curve $C' \to C$, and $Y \to C'$ a semi-stable family we have a diagram,

$$\begin{tikzcd} Y \arrow[r, dashed, "\varphi"'] \arrow[d] & \mathscr{X} \arrow[d] \\ C' \arrow[r] & C \end{tikzcd}$$

with the dotted arrow $\varphi$ rational, but étale over the generic fibre. Now form the graph of $\varphi$, which may develop non-scheme like points, and apply semi-stable reduction to obtain a family $\mathcal{Y} \to B$ for an appropriate $B$, satisfying all of the lemma except perhaps that it may have non-scheme like points in codimension 1. However if this were to happen, then in local coordinates, $x$ on the base, and $y$ around a component of the fibre we would have a finite group $G$ of automorhisms of the bi-disc leaving $x$ fixed with generic stabiliser non-zero on $x = 0$, which in turn would force $\mathcal{Y}$ to be non-scheme like generically, so we're done. □

## I.5. Foliations and Gorenstien Stacks

Suppose now that $\mathcal{X}$ is a normal stack, then since it is the quotient of an étale groupoid, albeit in the 2-category of stacks, there is no difficulty in defining its cotangent sheaf $\Omega_{\mathcal{X}}$, and whence the notion of foliated stack, i.e.

**I.5.1 Definition.** A foliation $\mathcal{F}$ on $\mathcal{X}$ is a rank 1 (coherent) quotient of $\Omega_{\mathcal{X}}$.

Arguably this is a less than intelligent point of view, nevertheless to avoid getting into technical issues about 3-categories we'll postpone a more sensible discussion till later, and confine ourselves to the directly relevant question of the well definedness of the canonical bundle, viz,

**I.5.2 Definition.** If the 1st Chern class of $\mathcal{F}$ exists as a line bundle on $\mathcal{X}$, e.g. $\mathcal{X}$ non-singular, then we say that $(\mathcal{X}, \mathcal{F})$ is foliated Gorenstien, and write $K_{\mathcal{F}}$ for the corresponding bundle. In the case that $K_{\mathcal{F}}$ is only a $\mathbb{Q}$-bundle, then we say that $(\mathcal{X}, \mathcal{F})$ is $\mathbb{Q}$-foliated Gorenstien. Note also the notation $T_{\mathcal{F}}$ for the dual bundle, or more generally $\mathbb{Q}$-bundle.

An important, indeed the only necessary, example of a foliated Gorenstien stack is that associated to a $\mathbb{Q}$-foliated Gorenstien normal scheme, or algebraic space, $X$. In order to convey the extra structure of the scheme case let us concentrate upon it. Here we can take an open covering $U_\alpha$ by sufficiently small affines, together with $n_\alpha$ the smallest integer such that $\mathcal{O}_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha}(n_\alpha K_{\mathcal{F}})$ by way of some local section $s_\alpha$, and subsequently take the corresponding root $V_\alpha$, i.e.

**I.5.3 Construction.** (characteristic 0, algebraically closed base) Let $U'_\alpha \subset U_\alpha$ be the open set where $\mathcal{O}_{U_\alpha}(K_{\mathcal{F}})$ is a bundle (necessarily $\mathrm{codim}(U_\alpha \backslash U'_\alpha) \geq 2$, since $X$ is normal). Cover $U'_\alpha$ by small affines $U_{\alpha_i}$, and write $s_\alpha = f_{\alpha_i} \partial_{\alpha_i}^{-n_\alpha}$ where $\partial_{\alpha_i}$ is a local generator of $\mathcal{O}_{U_{\alpha_i}}(T_{\mathcal{F}})$, and consider the sheaf of ideals $\mathcal{I}_\alpha$ in $\mathbb{V}(\mathcal{O}_{U'_\alpha}(T_{\mathcal{F}}))$ defined locally by $\partial_{\alpha_i}^{n_\alpha} - f_{\alpha_i}$, with $V'_\alpha$ the corresponding subschme, necessarily non-singular where $U'_\alpha$ is, so in particular normal. Naturally we put $V_\alpha$ to be the integral closure of $U_\alpha$ in the function field of $V'_\alpha$, and observe that the extension of function fields $k(V_\alpha)/k(U_\alpha)$ is a cyclic Kummer extension with Galois group $G_\alpha = \mathbb{Z}/n_\alpha$.

The final observation results, of course, from the triviality of cohomology with values in any locally constant sheaf in the Zariski topology, and of course we obtain an action of $G_\alpha$ on $V_\alpha$. We wish, now, to consider how these constructions patch, so consider what happens over the overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$. To this end let $d_{\alpha\beta}$ be the g.c.d. of $n_\alpha$ and $n_\beta$, $m_\beta = n_\alpha/d_{\alpha\beta}$, $m_\alpha = n_\beta/d_{\alpha\beta}$, and $n_{\alpha\beta} = n_\alpha n_\beta/d_{\alpha\beta}$ then over $U_{\alpha\beta}$ there is a unit $h_{\alpha\beta}$ such that,

$$s_\alpha^{\otimes m_\alpha} = h_{\alpha\beta}\, s_\beta^{\otimes m_\beta} \in \mathcal{O}_{U_{\alpha\beta}}(n_{\alpha\beta} K_{\mathcal{F}})\,.$$

Over the Gorenstien locus, everything is intrinsic so consider an affine $U_{\alpha\beta i} \subset U'_{\alpha\beta} = U'_\alpha \cap U'_\beta$, where $K_{\mathcal{F}}$ admits a trivialisation with respect to which $s_\alpha$, $s_\beta$ correspond to non-vanishing functions $f_\alpha$ and $f_\beta$ respectively then the coordinate ring of $V_\alpha \times_{U'_{\alpha\beta i}} V_\beta$ has the form,

$$\frac{\mathcal{O}_{U'_{\alpha\beta i}}[S, T]}{(S^{n_\alpha} - f_\alpha,\, T^{n_\beta} - f_\beta)}$$

for some indeterminates $S$ and $T$. We can consider this as some disjoint union of components of,

$$S^{n_\alpha} - f_\alpha = \prod_{\zeta \in \mu_{m_\beta}} (T^{n_\beta} - \zeta f_\beta) = 0 \subset U'_{\alpha\beta i} \times \mathbb{A}^2$$

where of course the above product is just $T^{n_{\alpha\beta}} - h_{\beta\alpha} S^{n_{\alpha\beta}}$ so that if $\tilde{U}_{\alpha\beta}$ is $U_{\alpha\beta}(h_{\alpha\beta}^{1/n_{\alpha\beta}})$ then $V_\alpha \times_{U'_{\alpha\beta}} V_\beta$ is open in $V_\alpha \times_{U'_{\alpha\beta}} \tilde{U}_{\alpha\beta}$ which in turn is open in $V_\alpha \times_{U_{\alpha\beta}} \tilde{U}_{\alpha\beta}$, whence the normalisation $V_{\alpha\beta}$ of $V_\alpha \times_{U_{\alpha\beta}} V_\beta$ is naturally contained in $V_\alpha \times_{U_{\alpha\beta}} \tilde{U}_{\alpha\beta}$ and thus étale over $V_\alpha$, and indeed by symmetry over $V_\beta$.

The data $(V_\alpha, G_\alpha, U_\alpha)$ therefore determines a $Q$-variety, which in turn defines a stack $\mathcal{X}$ by way of the groupoid,

$$\coprod_{\alpha,\beta} V_{\alpha\beta} \rightrightarrows \coprod_{\alpha} V_\alpha$$

which enjoys the particularly simple property of being covered by Zariski open substacks of the form $[V_\alpha/G_\alpha]$. Manifestly its moduli space is $X$, and the induced foliation on $\mathcal{X}$ is Gorenstien, by virtue of $\mathcal{X}$ being $S_2$.

Let us note this more explicitly by way of a definition, and partial converse, viz:

**I.5.4 Definition.** Let $(X, \mathcal{F})$ be a $\mathbb{Q}$-Gorenstien foliated scheme then the above constructed stack $\mathcal{X}$ along with the induced foliation $\mathcal{F}$ will be called the associated foliated Gorenstien stack. Conversely if $(\mathcal{X}, \mathcal{F})$ is a foliated Gorenstien stack then its moduli space $(X, \mathcal{F})$ is a $\mathbb{Q}$-foliated-Gorenstien algebraic space. The constructions are not however inverse to each other as the stack may posses a generic stabiliser.

In any case the extent to which $(\mathcal{X}, \mathcal{F})$ has non-scheme like points is easily determined. Indeed let us introduce,

**I.5.5 Definition.** For $x \in X$, the index of $\mathcal{F}$ at $x$ is the smallest integer $n(x)$ such that $\mathcal{O}_{X,x}(n(x)K_{\mathcal{F}})$ is a bundle.

Then unsurprisingly,

**I.5.6 Fact.** *The index is the same irrespective of whether we consider the local ring $\mathcal{O}_{X,x}$ in the Zariski or étale topology or even in its completion at the maximal ideal, while for $\xi \in \mathcal{X}$ any geometric point over a geometric point $x \in X$,* $\mathrm{Aut}(\xi) = \mathbb{Z}/n(x)\mathbb{Z}$.

**Proof.** To begin with let $V$ be an étale neighbourhood of $x$, lying over the Zariski neighbourhood $U$, with $\mathcal{O}_V(mK_{\mathcal{F}})$ a bundle. Plainly we can suppose $V$ irreducible, $m \le n$, and consider the associated Gorenstien loci $V'$ over $U'$. Denoting as ever $p_{ij} : V \times_U V \times_U V \to V \times_U V$, $p_i : V \times_U V \to V$ the various projections then over $U'$ we have the standard isomorphism, $\varphi : p_1^* \mathcal{O}_{V'}(K_{\mathcal{F}}) \xrightarrow{\sim} p_2^* \mathcal{O}_{V'}(K_{\mathcal{F}})$, which extends to an isomorphism $\varphi^m : p_1^* \mathcal{O}_V(mK_{\mathcal{F}}) \xrightarrow{\sim} p_2^* \mathcal{O}_V(mK_{\mathcal{F}})$ by virtue of the $S_2$ condition. On the other hand $\varphi$ satisfies the cocycle condition, $p_{31}^*\varphi = p_{32}^*\varphi\, p_{21}^*\varphi$ over $V'$, so the same is true of $\varphi^m$ over $V$, whence, $\mathcal{O}_U(mK_{\mathcal{F}})$ is a bundle, and the étale and Zariski indices coincide. Better still if formally we have an isomorphism, $s : \hat{\mathcal{O}} \xrightarrow{\sim} \hat{\mathcal{O}}(mK_{\mathcal{F}})$, $m \in \mathbb{N}$, with $\hat{\mathcal{O}}$ the corresponding complete local ring, then by Artin's implicit function theorem, [A1], together with Nakayama's lemma we have the same on an étale neighbourhood, from which our initial assertion.

For the second assertion, we start off with a Galois covering $V \to U$ with group $G = \mathbb{Z}/n(x)\mathbb{Z}$ defined over some Zariski open neighbourhood $U \ni x$ by way of I.5.3. Now let $y \in V$ be a geometric point over $x$, and denote by $H$ the stabiliser of $y$ in $G$. Consider the corresponding quotient $W = V/H \to U$, then $G/H$ acts freely on the pre-image of $x$, so shrinking $U$ as necessary we may as well say that $W \to U$ is étale. Now cover the Gorenstien locus $U'$ of $U$ by affines $U_i$ with, $\mathcal{O}_{U_i}(T_{\mathcal{F}}) = \mathcal{O}_{U_i}\,\partial_i$, and $f_i\,\partial_i^{-n(x)}$ the corresponding section defining $V$, then the Galois extension $k(W)/k(U)$ contains $f_i^{1/d}$ for $d = \#(G/H)$, which yield a nowhere vanishing section of $\mathcal{O}_{W'}\left(\frac{n(x)}{d} K_{\mathcal{F}}\right)$. On the other hand $W$ is certainly normal, so we even have a nowhere vanishing section of $\mathcal{O}_W\left(\frac{n(x)}{d} K_{\mathcal{F}}\right)$, with $W \to U$ étale, from which $\frac{n(x)}{d} \ge n(x)$, and thus the lemma. $\square$

We may also note in consequence that the index is upper semi-continuous, so the non-scheme like locus is closed. In any case the non-scheme like nature of points on $\mathcal{X}$ also admits, under certain circumstances, an interpretation in terms of holonomy. Indeed consider the rather particular case where the non-scheme locus $\mathcal{B}$ is such that $\mathcal{X}$ and the foliation are smooth around $\mathcal{B}$, and everywhere transverse to each other. For a geometric point $b \in \mathcal{B}$ we may therefore apply the Frobenius theorem to find a pointed analytic disc $f : (\Delta, 0) \to (\mathcal{X}, b)$ invariant by $\mathcal{F}$. Projecting to $X$, we have that locally around $\pi(b)$ both $X$ and $\mathcal{F}$ are smooth away from $B := \pi(\mathcal{B})$, as such if $j : L = \pi \circ f(\Delta^\times) \xrightarrow{\sim} \Delta^\times := \Delta \backslash \{0\} \hookrightarrow X$ is our local leaf we have a locally constant sheaf of algebras $j^{-1}\mathcal{O}_\mathcal{F} := \{z \in \mathcal{O}_{X_{\text{an}}} : \partial(z) = 0\}$ where $-$an denotes analytic functions and $\partial$ is understood to generate the foliation on an appropriate open. Consequently if $X$ has dimension $(n+1)$ we can consider the holonomy representation,

$$\rho : \mathbb{Z} = \pi_1(\Delta^\times) \to \mathrm{Aut}(\mathbb{C}\,[z_1, \ldots, z_m])\,.$$

We assert,

**I.5.7 Claim.** *If $n$ is the index at $\pi(b)$, then* $\mathrm{Im}(\rho) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$.

**Proof.** Provided our original disc $f : \Delta \to \mathcal{X}$ is sufficiently small, then manifestly the holonomy is trivialised on the covering $f(\Delta^\times)$ of $L$, so, $\#\,\mathrm{Im}(\rho) \leq \mathbb{Z}/m\mathbb{Z}$. Now suppose conversely that the holonomy is actually smaller than this. It will suffice to consider its linearisation, i.e. the local system on $\Delta^\times$ whose corresponding bundle is $f^*\pi^* N_{L/X}|_{\Delta^\times}$, which we already know has constant part $dz_1, \ldots, dz_m$ with respect to the appropriate connection, for some functions $z_1, \ldots, z_m$ in an étale neighbourhood of $b$ defining the foliation. To assert that the holonomy is smaller then $\mathrm{Aut}(b)$ amounts to claiming that we can find a non-trivial subgroup $G$ of $\mathrm{Aut}(b)$, and $dw_1, \ldots, dw_m \in \Gamma(\Delta^\times, f^*\pi^* N_{L/X})$ invariant by $G$ yet giving the same constant subsheaf as that defined by the $dz_i$. Consequently since the $dw_i$'s are related to the $dz_j$'s by a constant matrix in $\mathrm{GL}_m(\mathbb{C})$ over $\Delta^\times$, we see that $G$ operates trivially on the $dz_i$'s. On the other hand for any $\bar{d}$, $G$ is a commutative group of automorphisms of the finite dimensional vector space, $\frac{\mathcal{O}_{\mathcal{X},b}}{m(b)^{\bar{d}}}$, so in the completion $\hat{\mathcal{O}}_{\mathcal{X},b}$ we certainly have a linearisable action which on the tangent space at $b$, acts non-trivially in the foliation direction and trivially normal to it. Whence if $V \ni b$ is the corresponding étale neighbourhood then $V/G$ is non-singular so the index is strictly less than itself, which is nonsense. $\square$

## I.6. Intermission on Singularities

The discussion will be local and purely scheme like, so let $X = \mathrm{Spec}\,\mathcal{O}_{X,Y}$ be the germ of a normal scheme of finite type localised at a subvariety $Y$ of codimension $\geq 2$. We not only wish to consider a foliated germ $(X, \mathcal{F})$ on the same, but also a boundary divisor $B$. The foliation will, of course, be supposed $\mathbb{Q}$-Gorenstien as will the possibly empty set of prime components $B_i$ be supposed $\mathbb{Q}$-Cartier with $B_i \supset Y$. Along the boundary there will be a symbol for invariance, i.e. $\varepsilon_i = 0$ or 1, according to whether $B_i$ is invariant by $\mathcal{F}$ or not, together with weights $e_i \in \mathbb{N} \cup \{\infty\}$. The weights, however, will not be in the standard Mori theory sense, cf. [K-2], but rather given by,

**I.6.1 Warning/Definition.** If $B_i$ are the prime components of $B$, and $e_i \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ the weights, then $B = \sum\limits_i \left(1 - \frac{1}{e_i}\right) B_i$.

In any case, we have well defined multiplicity and discrepancy functions,

$$\nu_i, a_\mathcal{F} : \{\text{prime valuations of } \mathbb{C}(X) \text{ centered on } Y\} \to \mathbb{Q}$$

which we think of in the usual way, i.e.

**I.6.2 Comment/Definition/Further Warning.** To such a valuation we can associate a proper bi-rational modification $\rho : \tilde{X} \to X$, indeed even by a sequence of blow ups in smooth centres if we wish, and an exceptional divisor $E$ so that the valuation ring is $\mathcal{O}_{\tilde{X},E}$. An important invariant of this situation is,

$$\varepsilon(E) := \begin{cases} 0 & \text{if } E \text{ is invariant by the induced foliation } (\tilde{X}, \tilde{\mathcal{F}}) \\ 1 & \text{if } E \text{ is not invariant by the induced foliation } (\tilde{X}, \tilde{\mathcal{F}}) \end{cases}$$

In particular the Chern class of a log-foliation, say, $(\tilde{X}, \rho^{-1}(Y), \tilde{\mathcal{F}})$ is not calculated as in the $K_X$ case but in terms of $\varepsilon$. Specifically the log-canonical class of $(X, B, \mathcal{F})$ is $K_{\mathcal{F}} + \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i B_i$ and we define the germ $(X, B, \mathcal{F})$ to be,

(I) Terminal if $a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i \nu_i(E) > 0$

(II) Canonical if $a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i \nu_i(E) \geq 0$

(III) Log-terminal if $a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i \nu_i(E) > -\varepsilon(E)$

(IV) Log-canonical if $a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i \nu_i(E) \geq -\varepsilon(E)$

where the various inequalities are understood to hold for all $E$.

Manifestly boundary components $B_i$ which are invariant by $\mathcal{F}$ in no way effect the definitions, i.e. they may be freely added or removed without changing the nature of the singularity. Whence without loss of generality we will subsequently suppose $\varepsilon_i = 0$, and introduce, $\nu(E) = \sum_i \left(1 - \frac{1}{e_i}\right) \nu_i(E)$.

Now basically the problem with these definitions is that they're absolutely impossible to calculate locally unless everything is Gorenstien, and the boundary components are Cartier. Equally evidently we can reduce to this case by covering constructions, therefore we will now make,

**I.6.3 Hypothesis.** For purposes of classifying singularities of types (I)–(IV) we will suppose for the rest of this section $(X, \mathcal{F})$ Gorenstien and each component $B_i$ Cartier.

It is also a pain in the neck to work with non-smooth things, so consider:

**I.6.4 More Notation.** Fix an ambient smooth embedding $X \hookrightarrow M$ of dimension the embedding dimension of $X$ at $M$. Denote by $\partial$ a local generator for $\mathcal{F}$ around $Y$, which we may, without loss of generality, suppose restricted from $M$ in such a way that the lifted foliation $(M, \mathcal{F})$ leaves $X$ invariant. Now fix a quasi coefficient field $K \subset \mathcal{O}_{M,Y}$ of $\mathbb{C}(Y)$, then we can write (non-canonically),

$$\partial = \partial_K \oplus \delta$$

where $\partial_K \in \mathrm{Der}_K(\mathcal{O}_{M,Y})$, $\delta \in \mathcal{O}_{M,Y} \otimes_K \mathrm{Der}_{\mathbb{C}}(K)$. Nevertheless if $\mathcal{F}$ is singular at $Y$, $\partial_K$ may be considered functorially to $1^{\text{st}}$ order, i.e. modulo $\mathfrak{m}^2_{M,Y}$, be it in,

$$\mathrm{End}\left(\frac{\mathfrak{m}_{X,Y}}{\mathfrak{m}^2_{X,Y}}\right) \quad \text{or} \quad \mathrm{End}\left(\frac{\mathfrak{m}_{M,Y}}{\mathfrak{m}^2_{M,Y}}\right)$$

which are in fact the same thing, since we took $M$ to have the embedding dimension of $X$. Consequently we may talk about $\partial_K$ being nilpotent or not, according to whether the said linearisation is, or not. We can also do all of this at the level of completions in $Y$ (so $K = \mathbb{C}(Y)$), where we'll use the same notations up to adding a $\wedge$. Furthermore we will employ,

**I.6.5 (FCP).** (Formal checking principle) Completion is faithfully flat, so realising valuations by blow ups in smooth centres, we can simply calculate the discrepancy by way of the completion. We can also make converse type statements, provided we're careful not to blow up in non-algebraic centres.

We begin with a lemma, viz:

**I.6.6 Lemma.** *Hypothesis as per* I.6.3, *and suppose further that $B$ has precisely one component, transverse to $\mathcal{F}$, and of multiplicity* 1 *at $Y$ then for any finite weight $(X, B, \mathcal{F})$ is terminal, and canonical if the weight is infinite.*

Before proceeding with the proof, observe:

**I.6.7 Sub-Lemma.** *Everything as above, with $E$ a rank 1-discrete valuation centered on $Y$ then $\varepsilon(E) = 0$.*

**Proof.** Let $\tilde{\partial}$ in $\mathcal{O}_{\tilde{X},E}$ be a local generator of the induced foliation. Passing to completions we can find an element $x \in \mathfrak{m}_{X,Y}$ such that $\tilde{\partial} x = 0$. Consequently if $\mu$ is the valuation of $x$, $\pi$ a uniformising parameter, then for some $\tilde{x}$ of valuation zero, we have:

$$0 = \mu\, \pi^{\mu-1}\, \tilde{x}\, \tilde{\partial}\, \pi + \pi^{\mu}\, \tilde{\partial}\, \tilde{x}$$

so indeed $\tilde{\partial}\, \pi \in \mathfrak{m}_{\tilde{X},E}$ as required.

Observe in passing a little fact implicit in the sub-lemma,

**I.6.8 Fact.** *Suppose there is a $x_1 \in \mathfrak{m}_{M,Y}$ with $\partial x_1 \neq 0$, so without loss of generality 1, then we can find embedding coordinates $x_1, \ldots, x_d \in \mathfrak{m}_{M,Y}$ such that not only do we have, $\mathcal{O}_{\hat{M}} = \mathbb{C}(Y)[[x_1, \ldots, x_d]]$, but $\hat{\partial} = \partial/\partial x_1$.*

**Proof.** Apply the formal Frobenius theorem (or more accurately its proof, cf. II.1), and profit from the fact that $\mathbb{C}(Y)$ is finite over some $\mathbb{C}(y_1, \ldots, y_{\dim(Y)})$.

In any case back at the lemma, we may now proceed to,

**Proof.** (of I.6.6) Let $e_1$ be the weight of the said component, $B_1$, with $x_1$ a local equation. Moreover take $\tilde{\partial}$, $\pi$, $\mathcal{O}_{\tilde{X},E}$ etc. as per the proof of I.6.7/I.6.8, and write, $x_1 = \pi^{\nu_1}\, \tilde{x}_1$, with $\tilde{x}_1 \in \mathcal{O}^*_{\tilde{X},E}$, $\partial = \pi^{-a}\, \tilde{\partial}$, then,

$$\mathcal{O}^*_{X,Y} \ni \partial x_1 = \pi^{\nu_1 - a}\, \tilde{\partial}\, \tilde{x}_1 + \nu\, \pi^{\nu_1 - a - 1}\, \tilde{x}_1\, \tilde{\partial}\, \pi$$

whence $a \geq \nu_1$, so if $e_1$ is the weight, $a - \left(1 - \frac{1}{e_1}\right) \nu_1 \geq \frac{\nu_1}{e_1}$, as required.

Note that in particular,

**I.6.9 Corollary.** *$(X, \mathcal{F})$ is terminal iff $\mathcal{F}$ is smooth at $Y$, and transverse to it.*

**Proof.** In the if direction, by hypothesis we can find a $B$ satisfying I.6.6, so $(X, B, \mathcal{F})$ and a fortiori $(X, \mathcal{F})$ is terminal. Conversely if $\mathcal{F}$ is either singular at $Y$ or leaves it invariant, then blowing up in $Y$ followed by an invariant/algorithmic resolution yields valuations with $a_{\mathcal{F}} = 0$.

In fact we've even proved,

**I.6.10 Sub-Corollary.** *$(X, \mathcal{F})$ is terminal iff it's log-terminal.*

**Proof.** Terminal implies log-terminal is trivial. Conversely everything is Gorenstien, so log-terminal implies canonical. However if $\mathcal{F}$ is either singular at $Y$ or not transverse to it, then, the blow ups employed in I.6.9 yield valuations with $\varepsilon(E) = 0$, and discrepancy 0.

This is of course useful in establishing,

**I.6.11 Principle Fact.** *The following are equivalent,*

(I) *$(X, B, \mathcal{F})$ is terminal.*

(II) *$(X, B, \mathcal{F})$ is log-terminal.*

(III) *Either $B$ consists of a single component of multiplicity 1, transverse to $\mathcal{F}$ and of finite weight, or $B$ is empty, and in both cases $(X, \mathcal{F})$ is terminal.*

**Proof.** (I) $\Rightarrow$ (II) is trivial, so suppose $(X, B, \mathcal{F})$ is log-terminal, then $(X, \mathcal{F})$ is too, so by I.6.10 $(X, \mathcal{F})$ is terminal. Now consider the sequence of modifications

$$X = X_0 \underset{\alpha}{\longleftarrow} X_1 \underset{\beta}{\longleftarrow} X_2 \underset{\gamma}{\longleftarrow} \tilde{X}$$

where $X_1$ is the blow up of $X$ in $Y$, with $E_1$ the exceptional divisor, $X_2$ is the blow up of $X_1$ in the singular locus of the induced foliation, $E_2$ the exceptional divisor, and $\tilde{X}$ an invariant/algorithmic resolution of $X_2$. Observe also by I.6.8, that the induced singular locus in $X_1$ is a section, $S$, say, over $Y$ so irrespective of whether $X_1$ is normal or not $\alpha^* T_{\mathcal{F}}(-E_1)$ is already saturated in $\mathcal{J}_{X_1}$, as such $K_{\mathcal{F}_1} = \alpha^* K_{\mathcal{F}} + E_1$ is an honest canonical class of the induced foliation $\mathcal{F}_1$. Better still since $\gamma$ and $\beta$ are equivariant blow ups,

$$\alpha^* \beta^* \gamma^* K_{\mathcal{F}} + \gamma^* \beta^* E_1 = \gamma^* \beta^* K_{\mathcal{F}_1} \geq K_{\tilde{\mathcal{F}}}$$

so that if $E$ is a valuation on $\tilde{X}$ with centre on $X_1$ contained in $E_1$ and $e$ the multiplicity of $E$ then, $a_{\mathcal{F}}(E) \leq e$. However if $\nu_{i,Y}$ is the multiplicity of $B_i$ at $Y$ then, $\nu_i(E) \geq e\, \nu_i(E)$ so if $\varepsilon(E) = 0$ we must have,

$$0 < a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \nu_i(E) \leq e \left\{1 - \sum_i \left(1 - \frac{1}{e_i}\right) \nu_i\right\}$$

while if $\varepsilon(E) = 1$, $a_{\mathcal{F}}(E) \leq e - 1$, so in either case, profiting from $e_i \geq 2$,

$$\sum_i \nu_{i,Y} < 2$$

which certainly establishes that we have at most one component of multiplicity 1, and manifestly of finite weight.

Now suppose that $B_1$ is not transverse to $\mathcal{F}$, then its proper transform in $X_1$ must contain $S$ with multiplicity $\mu$ say. Consequently if the centre of $E$ on $X_1$ is actually $S$ (and of course such an $E$ exists, and we can even take its centre on $X_2$ to be any component of $E_2$) with $e'$ the multiplicity of $E_2$ along $E$, then, arguing as before,

$$0 < e' \left\{1 - \left(1 - \frac{1}{e_1}\right) \nu_{1,Y} - \left(1 - \frac{1}{e_1}\right) \mu\right\}$$

which is rubbish since $\mu, \nu_{1,Y} \in \mathbb{N}$. As such $(X, B, \mathcal{F})$ necessarily satisfies I.6.6, which in turn imply terminality. $\square$

As to canonical and log-canonical singularities the important cases are,

**I.6.12 Further Fact.** *Suppose $B = \varphi$ then the germ $(X, \mathcal{F})$ is canonical iff $\mathcal{F}$ is smooth at $Y$ or the field $\partial_K$ does not have nilpotent linearisation nor is it of the form $\lambda_i \frac{\partial}{\partial x_i}$, summation convention, for $\lambda_1, \ldots, \lambda_d$ positive integers.*

**Proof.** This is [**?**] I.1.1, at least if $X$ is smooth, but actually the proof goes through verbatim for any saturated Gorenstien germ even irrespective of normality. $\square$

In a sense therefore, log-canonical is cleaner, and in many ways preferable since we have,

**I.6.13 Further Fact bis.** *Again $B = \varphi$ then a singular germ $(X, \mathcal{F})$ is log-canonical iff the linearisation of $\partial_K$ is non-nilpotent.*

**Proof.** For $X$ smooth, one can just blow up in the obvious way using the Euclidean algorithm for the $\lambda_i$, and reduce to I.6.12. This is surprisingly tedious to justify if $X$ is normal, however [**?**] I.1.1(b) actually shows for a valuation $E$,

(i) If the discrepancy is $\leq -2$ then $\partial_K$ is nilpotent.

(ii) If the discrepancy is $-1$, and $\varepsilon(E) = 0$ then $\partial_K$ is nilpotent.

Consequently if $\partial_K$ is not nilpotent, we always have $a_{\mathcal{F}}(E) \geq -\varepsilon(E)$, which a postiori even proves (by uniqueness of the valuation with discrepancy $-1$) that the aforesaid obvious resolution $\rho : (\tilde{X}, \rho^{-1}(Y), \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ is log-terminal. $\square$

Although we'll have no need of it, this seems an appropriate place to list what else can occur, albeit without proof by virtue of the aforesaid utilitarian considerations, i.e.

**I.6.14 Fact.** *Let $(X, B, \mathcal{F})$ be a germ of a foliation singularity, with $D \neq \varphi$, which is log-canonical then in fact it's canonical, and should it not be terminal the possibilities are,*

1) *$(X, \mathcal{F})$ is terminal and the weight is infinite.*

2) *$(X, \mathcal{F})$ is terminal, the weight is $2$, $B$ has multiplicity $1$ at $Y$, and there is a simple tangency between $B$ and $\mathcal{F}$, i.e. if $f$, $\partial$ are a local equation, and a local generator respectively then, $\partial^2 f \neq 0$.*

3) *$(X, \mathcal{F})$ is terminal, the weight is $2$, $B$ is a possibly non-irreducible, i.e. there may be two components, tacnode enjoying a simple tangency with $\mathcal{F}$.*

Notice that despite lack of immediate relevance, these are not just curiosities. Obviously (1) is essential for a sensible theory in the infinite weight case, while the importance of (2) rests on the fact that although a Bertini type theorem, i.e. for $H$ generic ample $(X, H, \mathcal{F})$ has log-canonical singularities if $(X, \mathcal{F})$ does, is impossible for infinite weight, it is perfectly possible for weight 2. Furthermore, it is the case, cf. IV.9, that the entire log-minimal model programme for foliations follows from the weight 2 case, so that ultimately a perfectly general definition would have been given had all the weights been taken to be 2.

## I.7. Smooth Singular Stacks

Intermission over let's return to stacks, i.e. $(\mathcal{X}, \mathcal{F})$ will be a normal foliated integral stack with moduli $(X, \mathcal{F})$, projective, albeit that this isn't necessary for the present discussion, and of course $\pi : \mathcal{X} \to X$ the moduli map. For any geometric point $\xi$ of $\mathcal{X}$ we can perfectly well make,

**I.7.1 Definition.** $\mathcal{F}$ has terminal, canonical, log-terminal, log-canonical singularities at $\xi$ iff for some étale (and in fact any) étale neighbourhood $V \ni \xi$ with the image of $\xi$ a closed subvariety $Y$, the germ $(\operatorname{Spec} \mathcal{O}_{V,Y}, \mathcal{F})$ is terminal, canonical, etc.

Indeed the definitions are independent of étale coverings, so there's no ambiguity in what we've said, and we can talk about $\mathcal{O}_{\mathcal{X},\xi}$ being terminal, canonical etc. Our goal is to relate the singularities of $\mathcal{F}$ or $\mathcal{X}$ to those of $X$. To begin with we assert,

**I.7.2 Claim.** *Let $(X, \mathcal{F})$ be $\mathbb{Q}$-Gorenstien, with $\pi : (\mathcal{X}, \mathcal{F}) \to (X, \mathcal{F})$ its Gorenstien covering stack, then the singularities of $(X, \mathcal{F})$ are terminal, respectively canonical, respectively etc. iff those of $(\mathcal{X}, \mathcal{F})$ are.*

**Proof.** By definition $\mathcal{X}$ and $X$ have the same geometric points, so the question is purely local, whence let $Y \subset X$ be a subvariety with $\mathcal{Y} \subset \mathcal{X}$ a closed substack above it, then of course we have,

$$\pi : (\operatorname{Spec} \mathcal{O}_{V,Z}, \mathcal{F}) \to (\operatorname{Spec} \mathcal{O}_{X,Y}, \mathcal{F})$$

for an appropriate $V \to \mathcal{X}$ étale, with $\pi$ finite and étale in codimension 2. A priori for $\mathcal{E}$ a discrete rank 1 valuation of $\mathcal{O}_{V,Z}$ lying over a valuation $E$ of $\mathcal{O}_{X,Y}$ the almost étale nature of $\pi$ yields the formula,

$$a_{\mathcal{F}}(\mathcal{E}) = a_{\mathcal{F}}(E)e + \varepsilon(E)(e - 1)$$

where $e$ is the order of ramification at $\mathcal{E}$. So one way round this is clear, i.e. whatever we have on $(X, \mathcal{F})$ we certainly obtain on $(\mathcal{X}, \mathcal{F})$. Conversely we have to be careful about $\varepsilon$. By construction $(\mathcal{X}, \mathcal{F})$ is Gorenstien so if the singularity is terminal (equivalently log-terminal) then by I.6.7, $\varepsilon(E) = 0$, so the same holds for $(X, \mathcal{F})$. In a similar vein we can consider the canonical case. Indeed if $R$ is the valuation ring of $\mathcal{E}$ then the precise local uniformisation procedure whereby $\mathcal{E}$ is obtained from $\operatorname{Spec} \mathcal{O}_{V,Z}$ by a way of a sequence of blow ups is rather easy, i.e.

$$\operatorname{Spec} \mathcal{O}_{V,Z} = V_0 \leftarrow V_1 \leftarrow \cdots \leftarrow V_n = \tilde{V}$$

where $V_{i_n}$ is the blow up of $V_i$, in the centre $Z_i$, say, of the valuation on $V_i$. As such we have two possibilities,

(a) All the centres, $Z_0 = Z, \ldots, Z_{n-1}$ are invariant by a local generator $\partial$ of $\mathcal{F}$ around $Z$.

(b) There exists $0 \le p < n-1$, such that $Z_0, \ldots, Z_{p-1}$ are invariant by $\partial$, but $Z_p$ is not.

Now if $\rho_p : V_p \to V_0$ is the projection, then in the latter case we may apply the Frobenius theorem à la I.6.8 to $\rho_p^* \partial$ around $Z_p$ since this doesn't require normality, to conclude that $\varepsilon(\mathcal{E}) = 0$. In the former case, $\rho_n^* \partial$ is still an honest derivation, so the discrepancy is non-positive, but everything is canonical so in fact $\rho_n^* \partial$ generates the induced foliation around $\mathcal{E}$, and direct calculation shows that again $\varepsilon(\mathcal{E}) = 0$. Exactly the same line of reasoning works for (b) in the log-canonical case, so in fact the only thing that could go wrong is that $\mathcal{E}$ is the valuation of discrepancy $-1$, in which case our discrepancy formula (more by luck than design) gives $a_{\mathcal{F}}(E) = -1$, so $(X, \mathcal{F})$ is log canonical too. □

Now let's go a bit further. Specifically let $(\mathcal{X}, \tilde{\mathcal{F}})$ be any smooth foliated stack, and $(X, \mathcal{F})$ its necessarily normal, and supposed projective moduli. Appealing to I.5.3, we know that things must factor, i.e. there is a diagram,

$$\begin{tikzcd} (\mathcal{X}, \tilde{\mathcal{F}}) \arrow[r, "\sigma"] \arrow[dr, "\pi"'] & (\mathcal{X}', \mathcal{F}) \arrow[d, "\rho"] \\ & (X, \mathcal{F}) \end{tikzcd}$$

with $(\mathcal{X}', \mathcal{F})$ the Gorenstien covering stack. Associated to this data we have a ramification locus with prime components $B_i$ in $X$, covered by some integral substacks $\mathcal{D}_i'$, and $\mathcal{D}_i$ of $\mathcal{X}$ and $\mathcal{X}'$ respectively. We're interested in the properties of $K_{\mathcal{F}}$ so quotienting out by the generic stabiliser of $\mathcal{X}$ is harmless, whence suppose indeed that the generic stabiliser is zero, then the stabiliser of a generic geometric point of $\mathcal{D}_i$ is of the form $\mathbb{Z}/e_i$ for $e_i \in \mathbb{N}_{\ge 2}$, and indeed,

$$K_{\tilde{\mathcal{F}}} = \pi^* K_{\mathcal{F}} + \sum_i \varepsilon(B_i) \, \pi^* \left\{ \left(1 - \frac{1}{e_i}\right) B_i \right\}$$

while of course $K_{\mathcal{F}}$ is equally the canonical on $\mathcal{X}'$. In order to keep the notation manageable, let's abusively talk about valuations $\mathcal{E}$ of $\mathcal{X}$ centered on geometric points $\xi$, since as we've observed this is well defined up to étale covers, which themselves don't effect the singularities. With this in mind the formula for the discrepancy of a valuation $\mathcal{E}$ lying over $E$ in $X$ with multiplicity $e$ becomes,

$$\frac{1}{e} a_{\tilde{\mathcal{F}}}(\mathcal{E}) = a_{\mathcal{F}}(E) - \sum_i \varepsilon_i(E) \left(1 - \frac{1}{e_i}\right) \nu_i(E) + \varepsilon(E) \left(1 - \frac{1}{e}\right) E \, .$$

Thus if we do the obvious thing and consider the boundary $B = \sum_i \left(1 - \frac{1}{e_i}\right) B_i$ on $X$, then,

$$(\mathcal{X}, \tilde{\mathcal{F}}) \text{ terminal at } \xi \Rightarrow (X, B, \mathcal{F}) \text{ terminal at } \pi(\xi)$$

since as we've seen, in this case $\varepsilon(E) = \varepsilon(\mathcal{E}) = 0$. For the same reason, canonical at $\xi$ gives canonical at $\pi(\xi)$, and as per the proof of I.7.2, accident rather than design, gives log canonical at $\xi$ implies $(X, B, \mathcal{F})$ log canonical at $\pi(\xi)$. The converse implications are of course trivial.

Consequently the key, and rather simple, proposition that keeps the singularities under control while running the minimal model programme may now be stated, viz:

**I.7.3 Fact.** *The following are equivalent,*

(I) *There is a smooth foliated stack* $(\mathcal{X}, \tilde{\mathcal{F}})$ *with moduli* $(X, \mathcal{F})$ *which at the non-scheme like points is terminal.*

(II) *The Gorenstien covering stack,* $(\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$ *is terminal at its non-scheme like points, and* $\mathcal{X}'$ *is everywhere smooth.*

(III) $(X, \mathcal{F})$ *is a normal* $\mathbb{Q}$*-foliated Gorenstien scheme, which is terminal at every point of* $\mathrm{sing}\,(X)$*.*

**Proof.** (II) $\Rightarrow$ (I) is trivial, (I) $\Rightarrow$ (III) is the above discussion, which leaves (III) $\Rightarrow$ (II), so drop the $'$ in the notation for the Gorenstien cover. In any case $\mathcal{X}$ is certainly smooth, indeed isomorphic to $X$, outside of $\mathrm{sing}\,(X)$, and again, by the above terminal at its non-scheme like points. Now suppose $\mathcal{X}$ wasn't smooth then algorithmic resolution I.4.1 guarantees the existence of valuation $\mathcal{E}$ with centre contained in $\mathrm{sing}\,(\mathcal{X})$ such that (employing the aforesaid minor abuse of notation) every vector field on $\mathcal{X}$ lifts without poles to $\mathcal{O}_{\mathcal{X},\mathcal{E}}$. However this would mean that the centre in question was no better than canonical, which is nonsense. $\square$

Notice also that we can even employ these ideas to classify an appropriate class of smooth foliated stacks on which we will work,

**I.7.4 Further Fact.** *There is a one to one correspondence between the following sets,*

(A) *Smooth foliated stacks* $(\mathcal{X}, \tilde{\mathcal{F}})$ *with log-canonical singularities, terminal at their non-scheme like points, so in particular* $\tilde{\mathcal{F}}$ *transverse to the divisional ramification of* $\mathcal{X}$ *over* $X$*.*

(B) *Foliated* $\mathbb{Q}$*-Gorenstien, normal log-varieties* $(X, B, \mathcal{F})$ *with log-canonical singularities, no boundary component invariant, and terminal singularities at every point of* $\mathrm{sing}(X) \cup |D|$*, where* $|\ |$ *denotes support.*

The precise way to go between (A) and (B) will be explained in the course of the proof. However rather plainly $B$ will be the boundary divisor associated to the ramification as previously introduced.

**Proof.** Indeed the said previous discussion, has already associated such a data (B) to an object of (A). To go from (B) to (A), we first take the Gorenstien covering stack $\pi : (\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$. By [V] 2.5, this is a smooth stack, and to each prime component $B_i$ of the boundary, we have a corresponding integral divisorial stack $\mathcal{D}'_i$, from which a boundary stack $\mathcal{D}'$ with the obvious weights. Better still, $\mathcal{X}' \to X$ is almost étale, so the easy part of I.7.2 (although the less trivial converse is still true) gives,

(i) $\mathcal{X}'$ is terminal at its non-scheme like points

(ii) $(\mathcal{X}', \mathcal{D}', \mathcal{F})$ is terminal at $|\mathcal{D}'|$.

As such we may apply I.6.11, to conclude that $|\mathcal{D}'|$ is smooth, i.e. the $\mathcal{D}'_i$ are smooth and disjoint, and the weights $e_i$ are finite. Now let $\coprod_\alpha U_\alpha \to X$ be a covering by appropriately small affines, such that $\mathcal{X}'$ is covered by classifying stacks of the form $[V_\alpha/G_\alpha]$, for $\pi_\alpha : V_\alpha \to U_\alpha$ an index 1-cover. By virtue of (ii) and I.6.11, any $\mathcal{D}'_i$ pulled back to $V_\alpha$ is a smooth (though perhaps not connected) divisor $B'_{i\alpha}$ invariant by $G_\alpha$ equal to the reduction of the pull-back of $B_i \cap U_\alpha$, with the total divisor $\sum_i B'_{i\alpha}$ smooth, albeit very definitely not connected if the boundary has more than 1 component. Now all the patches are affine, and without loss of generality sufficiently small to ensure that each $\mathcal{O}_{V_\alpha}(-B'_{i\alpha})$ is generated by a single equation $f_{i\alpha}$, of which we may simple extract the $e_i^{\mathrm{th}}$-root for every $i$ without in any way affecting smoothness since all the components are disjoint. Consequently we obtain coverings, $\tilde{V}_\alpha \to V_\alpha$ ramified over $\sum_i B'_{i\alpha}$, and in fact there would be no loss of generality in a priori insisting that for each $\alpha$, there is at most one boundary component meeting $U_\alpha$, say $i(\alpha)$, so that $U_\alpha = \tilde{V}_\alpha/\tilde{G}_\alpha$ where $\tilde{G}_\alpha$ is an extension of the form,

$$0 \to \mathbb{Z}/\mathbb{Z}e_{i(\alpha)} \to \tilde{G}_\alpha \to G_\alpha \to 0\,.$$

Regardless the construction clearly respects the patching data on $\mathcal{X}'$, i.e. we have a commutative diagram with the horizontal arrows étale,

$$\begin{array}{ccccc} \tilde{V}_\alpha & \longleftarrow & (\tilde{V}_\alpha \times_X \tilde{V}_\beta)^\# & \longrightarrow & \tilde{V}_\beta \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ V_\alpha & \longleftarrow & (V_\alpha \times_X V_\beta)^\# & \longrightarrow & V_\beta \end{array}$$

where $\#$ denotes normalisation. As such the desired stack $\mathcal{X}$ is, therefore, the classifying stack of the groupoid,

$$\coprod_{\alpha,\beta} (\tilde{V}_\alpha \times_X \tilde{V}_\beta)^\# \rightrightarrows \coprod_\alpha \tilde{V}_\alpha\,.$$

□

Of course none of this requires algebraicity of the moduli, and is wholly valid for arbitrary foliated (separated) stacks with appropriate singularities. However since the index is the same for the Zariski and étale topologies, we even have,

**I.7.5 Remark.** If the moduli of a foliated stack $(\mathcal{X}, \mathcal{F})$ of type (A) is a scheme then in fact $\mathcal{X}$ is a Zariski stack in the sense of being locally in the Zariski topology an orbifold of the form $[V/G]$, for $G$ not just a group scheme, but a group.

While we may equally note,

**I.7.6 Summary/Conclusion.** Let $(X, B, \mathcal{F})$ be a normal $\mathbb{Q}$-Gorenstien foliated log-variety with finite weights on the non-invariant components of the boundary, enjoying terminal singularities on both the same as well as $\mathrm{sing}(X)$, along with log-canonical singularities elsewhere then,

There is a particularly nice class of smooth stacks $(\mathcal{X}, \mathcal{F})$, as described by $(A)$, whose minimal model theory accurately reflects that of $(X, B, \mathcal{F})$.

## I.8. Remarks on Intersection Theory

What follows are a few essentially obvious remarks on the structure of the Picard group of a stack over an algebraically closed field of characteristic zero, so let's say $\mathbb{C}$, along with the calculation of intersection numbers, amongst which the most important example will be the adjunction formula for invariant sub-stacks in the presence of a foliation. Nevertheless there seems to be a little bit of confusion in the literature as to the simplicity of what's involved, e.g. the emphasis in [Mu2] on Cohen Macauly coverings, while we wish to emphasise bundles as opposed to cycles as found in [V].

Regardless let's begin with the purely local case, i.e. $U$ and $V$ the spectra of strictly Henselian local rings, and $G$ a finite group such that $\pi : [V/G] \to U$ is the moduli map of the classifying stack of some $G$-action on $V$. In this case the Serre-Hochschild spectral sequence yields an exact sequence,

$$0 \to H^1(G, \mathcal{O}_V^\times) \to H^1([V/G], \mathbb{G}_m) \to H^1(V, \mathbb{G}_m)^G\,.$$

The last group is of course 0, so the calculation of the local Picard group reduces to considering the exact sequence,

$$0 \to \mathcal{O}_U^\times/\mathbb{C}^\times \to H^0(G, \mathcal{O}_V^\times/\mathbb{C}^\times) \to \mathrm{Hom}(G, \mathbb{C}^\times) \to H^1(G, \mathcal{O}_V^\times) \to H^1(G, \mathcal{O}_V^\times/\mathbb{C}^\times)\,.$$

On the other hand $\mathcal{O}_V$ is a local ring, so $\mathcal{O}_V^\times/\mathbb{C}^\times$ is torsion free. Better still $\mathcal{O}_U$ is Henselian and an invariant element $\mathcal{O}_V^\times/\mathbb{C}^\times$ lifts to an element of $\mathcal{O}_V^\times$ satisfying a monic polynomial over $\mathcal{O}_U$, so

$$\mathbb{C}^\times\backslash\mathcal{O}_U^\times \to H^0(G, \mathcal{O}_V^\times/\mathbb{C}^\times)$$

is surjective. Equally, and wholly generally, torsion free divisible $G$-modules are acyclic, from which we conclude,

$$\mathrm{Hom}(G, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Pic}([V/G])\,.$$

In particular if $\pi : \mathcal{X} \to X$ is the moduli, projective or not, and $n$ the highest common multiple of the stabilisers of geometric points, then in the étale topology,

$$R^1\pi_*\,\mathbb{G}_m \otimes \mathbb{Z}\left[\frac{1}{n}\right] = 0\,.$$

Continuing to work in the étale topology, the Leray spectral sequence yields a short exact sequence,

$$0 \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^1_{\text{ét}}(\mathcal{X}, \mathbb{G}_m) \to H^0_{\text{ét}}(X, R^1\pi_*\,\mathbb{G}_m)$$

and so we certainly deduce,

$$\pi^* : \mathrm{Pic}(X) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \xrightarrow{\sim} \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right]\,.$$

Let us make explicit the inverse isomorphism. To this end let $L$ be a line bundle on $\mathcal{X}$, and $\coprod_\alpha U_\alpha \to X$ a sufficiently fine étale covering with, $\coprod_\alpha [V_\alpha/G_\alpha] \to \mathcal{X}$ the corresponding covering of $\mathcal{X}$. Refining the cover as necessary we may suppose that we have isomorphisms,

$$s_\alpha : \mathcal{O}_{V_\alpha} \xrightarrow{\sim} L_{V_\alpha}$$

which in turn yield a $G_\alpha$ cocycle according to the rule,

$$G_\alpha \to \mathcal{O}_{V_\alpha}^\times : g \mapsto s_\alpha^g\, s_\alpha^{-1}$$

and without loss of generality we may suppose that this is zero modulo $\mathbb{C}^\times$, and so, $s_\alpha^g = \chi(g)\,s_\alpha$, for $\chi$ a character of $G_\alpha$. Consequently,

$$Ns_\alpha := \prod_g s_\alpha^g : \mathcal{O}_{V_\alpha} \xrightarrow{\sim} L_{V_\alpha}^{\otimes|G_\alpha|}$$

is a $G_\alpha$-invariant isomorphism, and $g_{\alpha\beta} = (Ns_\alpha)^{n/|G_\alpha|}\,(Ns_\beta)^{-n/|G_\beta|} \in \mathcal{O}_{U_{\alpha\beta}}^\times$ yields the desired cocycle on $X$.

In any case if we subsequently suppose that $X$ is a scheme then we can make,

**I.8.1 Provisional Definition.** Denoting by $\pi_*$ the inverse of the above isomorphism, put:

$$c_1 : \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \to \mathrm{CH}^1(X) \otimes \mathbb{Z}\left[\frac{1}{n}\right] : L \mapsto c_1(\pi_* L)\,.$$

The use of the word provisional is, of course, a result of the related notion of Cartier divisor on $\mathcal{X}$, i.e. the group,

$$\mathrm{Div}(\mathcal{X}) := \Gamma(\mathcal{X}, \mathcal{K}_{\mathcal{X}}^\times/\mathcal{O}_{\mathcal{X}}^\times)\,.$$

As such to give a Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ amounts, for a sufficiently fine cover as above, to the giving of rational functions $f_\alpha$ on $V_\alpha$, together with transition functions $g_{\alpha\beta} \in \mathcal{O}_{V_{\alpha\beta}}^*$. Consequently by sending a divisor to its transition functions, we obtain an exact sequence,

$$0 \to k(X)^\times \to \mathrm{Div}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})$$

where the last map is surjective if the moduli is a scheme without generic stabilisers, by way of

$$H^0(\mathcal{X}, \pi^* \mathcal{K}_X \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D})) = H^0(X, \mathcal{K}_X \otimes \pi_* \mathcal{O}_{\mathcal{X}}(D)) \,,$$

at least supposing I.2.4, and even otherwise there's always a map in the right direction. Furthermore we have, following [V], a Chow group $\mathrm{CH}^1(\mathcal{X})$ of integral divisorial substacks modulo rational equivalence together with a cycle class map,

$$c_1 : \mathrm{Div}(\mathcal{X}) \to \mathrm{CH}^1(\mathcal{X})$$

defined in the usual way, i.e. send the local equations of the Cartier divisor $f_\alpha$ to the corresponding integral substacks with multiplicity. As ever this $c_1$ is injective modulo $k(X)^\times$, respectively surjective, according to whether $\mathcal{X}$ is normal, respectively factorial. Also continuing to follow op.cit. we have a map,

$$\pi_* : \mathrm{CH}^1(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \to \mathrm{CH}^1(X) \otimes \mathbb{Z}\left[\frac{1}{n}\right]$$

defined in the usual way, i.e. an integral substack $\mathcal{Y}$ of $\mathcal{X}$ with image $Y$ in $X$, gets sent to $\deg(\mathcal{Y}/Y)Y$, where $\deg(\mathcal{Y}/Y) \in \mathbb{Z}\left[\frac{1}{n}\right]$ is as per op. cit. 1.15. In order, then, to turn our provisional definition I.8.1 into an actual definition we wish to check,

**I.8.2 Fact.** *The following diagram commutes,*

$$\begin{array}{ccc}
\mathrm{Div}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] & \longrightarrow & \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \\
\downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} \\
\mathrm{CH}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] & \xrightarrow[\pi_*]{} & \mathrm{CH}(X) \otimes \mathbb{Z}\left[\frac{1}{n}\right]
\end{array}$$

**Proof.** For a sufficiently fine cover as above let $f_\alpha$ be local equations for the Cartier divisor $\mathcal{D}$. By a minor variation of our previous considerations, we may, without loss of generality suppose $f_\alpha^g = \chi_\alpha(g)\, f_\alpha$, for $g \in G_\alpha$, and $\chi_\alpha$ a character. As such $(Nf_\alpha)^{n/|G_\alpha|}$ are local equations for an étale Cartier divisor on $X$. On the other hand $X$ is a scheme, so modulo refining the cover Hilbert 90 implies that the $(Nf_\alpha)^{n/|G_\alpha|}$ are actually rational functions on $X$, so that the image of the horizontal followed by the vertical is simply $\frac{1}{|G_\alpha|}\operatorname{div}(Nf_\alpha)$, independently of $\alpha$.

The other direction is just an exercise in unravelling the definitions of [V], and indeed is purely local about the generic point of the image $D$ in $X$ of a component of the support of $\mathcal{D}$. In addition by op. cit. 3.7, or more correctly the proof, we can suppose both $\mathcal{X}$ and $X$ are normal and irreducible. As such if $H$ is the generic stabiliser of $\mathcal{X}$, then $\mathcal{X}/H$ is a classifying stack of the form $[\tilde{X}/\mathbb{Z}/n]$ where $\rho : \tilde{X} \to X$ is the extraction of a $n^{\mathrm{th}}$-root of $D$. Consequently if $\mathcal{D}$ has multiplicity $m$ at the said root, then the said definition (op. cit. 3.6) gives,

$$\pi_*\, c_1(\mathcal{D}) = \frac{m}{n(\#\, H)}\, D$$

which is indeed $\frac{1}{|G|}\operatorname{div}(f)$, for $f$ a local equation of $\mathcal{D}$ on the étale neighbourhood $\tilde{X}$, and $G$ the group affording the stack structure. $\square$

Putting all of this together, we can certainly declare an end to the provisional character of I.8.1, and furthermore make,

**I.8.3 Definition.** Let $\mathcal{C}$ be a 1-dimensional stack with $\pi : \mathcal{C} \to C$ the necessarily projective moduli, then we have a map,

$$\deg : \mathrm{Pic}(\mathcal{C}) \to \mathbb{Q} : L \mapsto \deg(c_1(\pi_* L)) \,.$$

So that if $f : \mathcal{C} \to \mathcal{X}$ is a map to a stack. Then we have,

$$\mathrm{Pic}(\mathcal{X}) \xrightarrow[f^*]{} \mathrm{Pic}(\mathcal{C}) \xrightarrow[\deg]{} \mathbb{Q} : L \mapsto L \cdot_f \mathcal{C} := \deg(c_1(\pi_* f^* L)) \,.$$

Notice in particular that if we consider the square of maps,

$$\begin{array}{ccc} \mathscr{X} & \xleftarrow[f]{} & \mathcal{C} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ X & \xleftarrow[f]{} & C \end{array}$$

and view $L$ as a $\mathbb{Q}$-divisor on $X$, then provided every generic point has trivial stabiliser the degree $L ._f C$ coincides with that defined in the usual way (i.e. $L^{\otimes n}$ is Cartier, so $\frac{1}{n}\deg(f^*L^{\otimes n})$ is well defined) by virtue of the identity $f^*\pi^* = \pi^* f^*$ at the level of Pic. Nevertheless there is an assumption involved here, so let us make,

**I.8.4 Warning.** There will arise situations in which $\pi : \mathcal{C} \to C$ does not have a trivial stabiliser at each of its generic points, so that for $D$ a $\mathbb{Q}$-Cartier divisor on $X$ which is Cartier on $\mathscr{X}$ it will not be the case that,

$$D ._f \mathcal{C} = D ._f C\,.$$

On the other hand if $C$ is irreducible, and $G$ the generic stabiliser, then:

$$D ._f \mathcal{C} = \frac{1}{|G|}\, D ._f C$$

so that the important property of positivity/negativity of the intersection number is independent of whether we employ I.8.3 or the above more classical considerations.

As an example to illustrate the correctness of the above degree conventions, we may consider,

**I.8.5 Example.** (Riemann-Hurwitz) Let $\mathcal{C}$ be a smooth irreducible 1-dimensional stack with generic stabiliser $G$ and moduli $\pi : \mathcal{C} \to C$. Furthermore consider $\mathcal{C}/G$ as an orbifold in the usual way, i.e. with signatures $e_i \in \mathbb{N}\backslash\{1\}$ at points $c_i \in C$, then,

$$\deg(\omega_{\mathcal{C}}) = \frac{1}{|G|}\left\{\deg(\omega_C) + \sum_i \left(1 - \frac{1}{e_i}\right)\right\}.$$

In addition if $f : \mathcal{C}' \to \mathcal{C}$ is a map between such stacks, then,

$$\deg(\omega_{\mathcal{C}'}) = (\mathcal{C}' : \mathcal{C})\deg(\omega_{\mathcal{C}}) + \deg(\mathrm{Ram}_f)\,.$$

**Proof.** In fact we have a map, $\pi^*\omega_C \to \omega_{\mathcal{C}}$, so $\omega_{\mathcal{C}}$ is actually a Cartier divisor, and if we consider $c_i$ as points on $\mathcal{C}$ then, at the level of $\mathrm{CH}^1(\mathscr{X})$,

$$c_1(\omega_{\mathcal{C}} \otimes \pi^*\omega_C^\vee) = \sum_i (e_i - 1)\, c_i$$

and of course each $c_i$ has degree $\frac{1}{|G|\,e_i}$, so this establishes the first part. The second is immediate by the very definition of $\mathrm{Ram}_f$, although what should be born in mind is that,

$$(\mathcal{C}' : \mathcal{C}) = \frac{\#\, G'}{\#\, G}\,(C' : C)$$

where $G'$, $C'$ etc. have the obvious meaning. $\square$

In any case let us progress from these essentially obvious considerations, to the equally obvious but rather more pertinent adjunction formula. The set up is, of course, that $(\mathscr{X}, \mathcal{F})$ is a foliated-Gorenstien stack, and

$f : \mathcal{C} \to \mathscr{X}$ a smooth invariant 1-dimensional stack not wholly contained in the singular locus $\mathcal{Z}$. We have a map,

$$\Omega_{\mathscr{X}} \to K_{\mathcal{F}} \cdot \mathcal{I}_{\mathcal{Z}} \to 0$$

and whence an embedding,

$$i : \mathrm{Bl}_{\mathcal{Z}}(\mathscr{X}) \xrightarrow{\sim} \mathrm{Proj}\left(\bigoplus_{n\geq 0} K_{\mathcal{F}}^n \cdot \mathcal{I}_{\mathcal{Z}}^n\right) \hookrightarrow \mathbb{P}(\Omega_{\mathscr{X}})\,.$$

In particular if $\rho$ denotes the projection of either of these stacks to $\mathscr{X}$, with $L$ the tautological bundle on the latter and $\mathcal{E}$ the exceptional divisor on the former, then:

$$i^* L = \rho^* K_{\mathcal{F}}(-\mathcal{E})\,.$$

Furthermore the derivative $f' : \mathcal{C} \to \mathbb{P}(\Omega_{\mathscr{X}})$ factors generically through the blow up, and since everything is irreducible this is true everywhere, so in fact,

$$f^* K_{\mathcal{F}} = (f')^* L + \tilde{f}^* \mathcal{E}$$

where of course $\tilde{f}$ is the lifting of $f$ to the blow up. As such we introduce,

**I.8.6 Definition.** Let things be as above then the Segre class of $f$ at $\mathcal{Z}$ is defined to be,

$$s_{\mathcal{Z}}(f) = \deg(\tilde{f}^* \mathcal{E})\,.$$

This is of course wholly independent of any foliation hypothesis, and is an easily computed invariant of a purely local character, e.g. if $f^{-1}(\mathcal{Z})$ is a single smooth scheme like point $c$ whose image in $\mathscr{X}$ is also scheme like, with $a_1, \ldots, a_m$ local equations for $\mathcal{Z}$ about $f(c)$ then,

$$s_{\mathcal{Z}}(f) = \min_i \mathrm{ord}_c(f^* a_i)\,.$$

In addition $(f')^* L$ is the image of the natural map,

$$f^* \Omega_{\mathscr{X}} \twoheadrightarrow (f')^* L \hookrightarrow \omega_{\mathcal{C}}$$

so by the very definition of ramification, we obtain,

**I.8.7 Fact.** (Adjunction Formula) *Everything as above, including conventions on degrees, then:*

$$K_{\mathcal{F}} \cdot_f \mathcal{C} = \deg(\omega_{\mathcal{C}}) + s_{\mathcal{Z}}(f) - \deg(\mathrm{Ram}_f)$$

Now there is a critical, and wholly specific to characteristic to zero, adjunction inequality. To state it properly observe that every closed geometric point of $\mathcal{C}$ is a Cartier divisor, so that $f^{-1}(\mathcal{Z})_{\mathrm{red}}$ is a well defined Cartier divisor on $\mathcal{C}$, so that with this in mind we have,

**I.8.8 Sub-Fact.** (Adjunction Inequality) *Again things as above then,*

$$K_{\mathcal{F}} \cdot_f \mathcal{C} \geq \deg(\omega_{\mathcal{C}}) + \deg f^{-1}(\mathcal{Z})_{\mathrm{red}}\,.$$

**Proof.** By the Frobenius theorem (cf. II.1 for the proper generality) $f(\mathcal{C})$ is smooth away from $\mathcal{Z}$, so identifying $\tilde{f}^* \mathcal{E}$ with $f^{-1}(\mathcal{Z})$ it will suffice to prove the inequality of Cartier divisors,

$$f^{-1}(\mathcal{Z}) \geq \mathrm{Ram}_f + f^{-1}(\mathcal{Z})_{\mathrm{red}}\,.$$

This is a purely local question about closed points $c \in \mathcal{C}$, with $f(c) \in \mathcal{Z}$, so we may suppose that everything is scheme like. As such let $y$ be a coordinate around $c$, $x_1, \ldots, x_n$, embedding coordinates around $f(c)$, with $a_1, \ldots, a_m$ local equations for $\mathcal{Z}$. By construction, $(a_1, \ldots, a_m) \subset (x_1, \ldots, x_n)$, so that,

$$f^{-1}(\mathcal{Z}) \geq (\min_i \operatorname{ord}_c(f^* x_i))[c] \,.$$

On the other hand,

$$\min_i \operatorname{ord}_c(f^* x_i) = \operatorname{ord}_c(\operatorname{Ram}_f) + 1$$

so we're done. □

As such we have,

**I.8.9 Corollary/Remarks.** *Certainly then if $K_{\mathcal{F}} \cdot_f \mathcal{C} < 0$ then $\deg(\omega_{\mathcal{C}}) < 0$, and $\mathcal{C}$ is a rational stack. Manifestly we could list in terms of various signatures the possibilities for such curves in terms of the non-scheme like points, order of ramification, etc. The cleanest case, however, is when $f^{-1}(\mathcal{Z})$ is scheme like, so that* $\deg f^{-1}(\mathcal{Z})_{\mathrm{red}} \geq 1$, *and the number of possibilities is dramatically reduced.*

## I.9. Weighted Projective Stacks

The normal definition of a weighted projective space, $\mathbb{P}(a_0, \ldots, a_r)$ is the quotient of $\mathbb{P}^r$ by the natural coordinate action of $\mu_{a_0} \times \cdots \times \mu_{a_r}$. This might, perhaps, lead to the belief that weighted projective stack simply means the classifying stack of such a quotient. Such a supposition would, however, be wholly false. The actual objects in question have a couple of different flavours, of which the more important one is more properly to be considered the higher dimensional generalisation of the so called bad orbifold which has moduli $\mathbb{P}^1$ and non-trivial automorphism group at precisely one point. Let us, therefore, proceed from this latter standpoint by way of,

**I.9.1 Definition.** Let $a_1, \ldots, a_r \in \mathbb{N}$ be given, and denote by $\mu_{\underline{a}}$ the product of multiplicative groups $\mu_{a_1} \times \cdots \times \mu_{a_r}$. We first of all form the weighted projective space, $\mathbb{P}(1, a_1, \ldots, a_r)$ by way of the standard coordinate action,

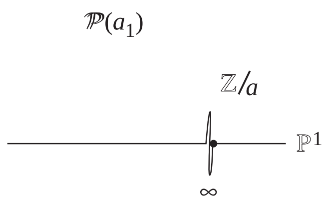
$$\mu_{\underline{a}} \times \mathbb{P}^r \to \mathbb{P}^r : (\zeta_1, \ldots, \zeta_r) \times [X_0, \ldots, X_r] \mapsto [X_0, \zeta_1 X_1, \ldots, \zeta_r X_r] \,.$$

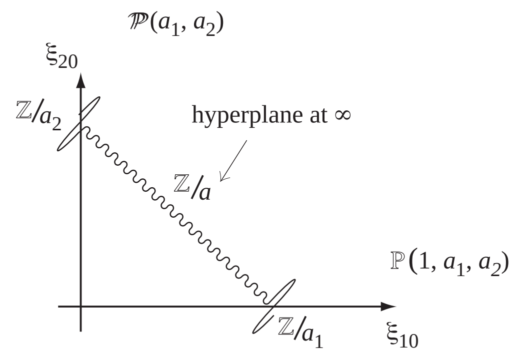
Over $\mathbb{P}(1, a_1, \ldots, a_r)$ there is a unique stack, $\pi : \boldsymbol{\mathcal{P}} \to \mathbb{P}(1, a_1, \ldots, a_r)$ which is smooth, and such that any other smooth stack with the same moduli factors through $\boldsymbol{\mathcal{P}}$, cf. the proof of [V] 2.5. Furthermore on $\boldsymbol{\mathcal{P}}$ there is a smooth connected divisor $\boldsymbol{\mathcal{P}}_\infty$, the hyperplane at $\infty$, corresponding to the coordinate $X_0 = 0$, we extract an $a^{\text{th}}$-root of this divisor, where $a = gcd(a_1, \ldots, a_r)$, and the stack so obtained will be called a weighted projective stack, and denoted $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$. Explicitly in coordinates, we have for each $0 \leq j \leq r$, the $j^{\text{th}}$-coordinate patch isomorphic to $\mathbb{A}^r$ with standard coordinate functions, $\xi_{ij} = (X_i/X_j)^{a_i}$, $0 \leq i \leq r$, $i \neq j$, and $a_0 = 1$, together with a $\mu_{a_j}$-action according to the rule,

$$\mu_{a_j} \times \mathbb{A}^r : \zeta_j \times (\xi_{ij}) \mapsto (\zeta_j^{-a_i} \xi_{ij})$$

which yield Zariski open substacks $[\mathbb{A}^r/\mu_{a_j}]$ of $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$. The patching maps are the obvious ones deduced from these on $\mathbb{P}^r$, or equivalently the normalised fibre product over $\mathbb{P}(1, a_1, \ldots, a_r)$.

Pictures of $\boldsymbol{\mathcal{P}}(a_1)$, and $\boldsymbol{\mathcal{P}}(a_1, a_2)$ should illustrate the structure pretty clearly,




Basically the weighted projective stacks $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$ enjoy just about every property of a standard projective space $\mathbb{P}^r$ with the obvious exception of schemeness. A relevant place to start, which in turn helps in other ways is,

**I.9.2 Fact.** *With coordinates as above, the foliation by curves on $\mathbb{P}^r$ corresponding to the pencil of lines through the origin $[1, 0, \ldots, 0]$ descends to a foliation denoted $\mathcal{R}$ (for radial in foliation parlance, or perhaps rational in an algebraic context) on $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$, and the resulting foliated stack, $(\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r), \mathcal{R})$, will be called a linearly foliated weighted projective stack. The foliation in question has a unique singular point at the necessarily scheme like origin, where in the above coordinate system on the $X_0 \neq 0$ patch it is generated by the vector field,*

$$\partial = \sum_{i=1}^{r} a_i \, \xi_{i0} \, \frac{\partial}{\partial \xi_{i0}} \, .$$

The tangent bundle of this foliation (strictly speaking the pre-foliation unsaturated at the origin if $r = 1$) will be denoted $\mathcal{O}_{\underline{a}}(1)$. Notice further that if we introduce the linear coordinate stacks $\mathcal{L}_j$ (i.e. $X_i = 0$, for $i \neq j$, $1 \leq j \leq r$) then by the adjunction formula,

$$\mathcal{O}_{\underline{a}}(1).\, \mathcal{L}_j = \frac{1}{a_j} \, .$$

Unsurprisingly our previous considerations on intersection theory yield,

**I.9.3 Fact.**

$$\mathrm{Pic}(\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)) = \mathbb{Z} \cdot \mathcal{O}_{\underline{a}}(1) \, .$$

**Proof.** In the first place consider the moduli space $P = \mathbb{P}(1, a_1, \ldots, a_r)$. This has quotient singularities, and is in fact $\mathbb{Z}\left[\frac{1}{m}\right]$ factorial, where $m$ is the lowest common multiple of the $a_i/a$. Better still the complement of the hyperplane at $\infty$, $P_\infty$, say is an $\mathbb{A}^r$, so the usual short exact sequence of Chow groups yields,

$$\mathrm{CH}^0(P_\infty) \twoheadrightarrow \mathrm{CH}^1(P)$$

so that up to $m$ torsion, $\mathrm{Pic}(P)$ is generated by $P_\infty$, and so up to $ma$ torsion $\mathrm{Pic}(\boldsymbol{\mathcal{P}}(a_1, \ldots a_r))$ is generated by $\mathcal{O}_{\underline{a}}(1)$.
To proceed to a conclusion, we avail ourselves of the natural maps (as we will do throughout this section, so we'll even fix notation),

$$\mathbb{P}^r \underset{\sigma}{\longrightarrow} [\mathbb{P}^r/\mu_{\underline{a}}] \underset{\rho}{\longrightarrow} \boldsymbol{\mathcal{P}}(\underline{a}) \, .$$

In the first place we can relate $\mathrm{Pic}\, \boldsymbol{\mathcal{P}}(\underline{a})$ to $\mathrm{Pic}[\mathbb{P}^r/\mu_{\underline{a}}]$ by the Leray spectral sequence, and $\mathrm{Pic}[\mathbb{P}^r/\mu_{\underline{a}}]$ to $\mathrm{Pic}\, \mathbb{P}^r = \mathbb{Z}\, \mathcal{O}_{\mathbb{P}^r}(1)$ (which incidentally is easily seen to be $\sigma^* \rho^* \mathcal{O}_{\underline{a}}(1)$ by virtue of the intersection numbers with coordinate lines) by the Hochschild-Serre spectral sequence. Putting this together amounts to a commutative diagram with exact rows and columns of the form,

$$
\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & \mathrm{Hom}(\mu_{\underline{a}}, \mathbb{C}^\times) \arrow[d] & \\
0 \arrow[r] & \mathrm{Pic}\, \boldsymbol{\mathcal{P}}(a_1, \dots, a_r) \arrow[r] & \mathrm{Pic}([\mathbb{P}^r/\mu_{\underline{a}}]) \arrow[r] \arrow[d] & H^0_{\text{ét}}(\boldsymbol{\mathcal{P}}(a_1, \dots, a_r), R^1_{\rho_*}\mathbb{G}_m) \\
 & & \mathrm{Pic}(\mathbb{P}^r) \arrow[d] & \\
 & & 0 &
\end{tikzcd}
$$

In particular $\mathrm{Pic}\, \boldsymbol{\mathcal{P}}(\underline{a})$ modulo its torsion is certainly generated by $\mathcal{O}_{\underline{a}}(1)$. Equally if $L$ is a torsion bundle on $\boldsymbol{\mathcal{P}}(\underline{a})$, then $\rho^* L$ comes from a character of $\mu_{\underline{a}}$. Now the bundles that come from characters, which we'll slightly abusively denote $\mathrm{Hom}(\mu_{\underline{a}}, \mathbb{C}^\times)$, restrict injectively to the bundles on the distinguished affine $\mathbb{A}^r$ given by $X_0 \neq 0$, indeed we even have,

$$\mathrm{Res}^* : \mathrm{Hom}(\mu_{\underline{a}}, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Pic}[\mathbb{A}^r/\mu_{\underline{a}}]\,.$$

On the other hand, $\mathrm{Res}^* \rho^* = \rho^* \mathrm{Res}^*$, and the corresponding $\mathbb{A}^r$ in $\boldsymbol{\mathcal{P}}(\underline{a})$ is wholly scheme like, so there are no torsion bundles on $\boldsymbol{\mathcal{P}}(\underline{a})$, from which we conclude. $\square$

In a similar vein, albeit rather more specific to characteristic zero, we have:

**I.9.4 Fact.** *$\boldsymbol{\mathcal{P}}(\underline{a})$ is simply connected.*

**Proof.** For $\pi_1^{\text{ét}}$ this is clear, since $\mathbb{A}^r$ is a Zariski open, and $\boldsymbol{\mathcal{P}}(\underline{a})$ is smooth, so any connected étale cover has the same generic point. More generally for a possibly analytic cover $h : \mathcal{X} \to \boldsymbol{\mathcal{P}}(\underline{a})$ with covering group $\Gamma$, we have $\#\,\Gamma$ disjoint copies of our $\mathbb{A}^r$. A priori these might not admit a Zariski closure, but by the simple expedient of lifting to $\mathcal{X}$ the 1-dimensional and indeed simply connected, stacks invariant by $\mathcal{R}$, then taking the stack covered by those through a given lifting of the origin, we see that they do, so again $\mathcal{X}$ would have too many generic points if it were non-trivial. $\square$

Now let's proceed to some cohomological calculations. The best place of course to start is the $H^0$, where we'll state the result by way of,

**I.9.5 Fact.** *We have an isomorphism of graded rings,*

$$\bigoplus_{n \in \mathbb{Z}} H^0(\boldsymbol{\mathcal{P}}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) \xrightarrow{\sim} \mathbb{C}\,[Y_0, \dots, Y_r]$$

*where on the right $Y_i$ corresponds to a section of $\mathcal{O}_{\underline{a}}(a_i)$, so in particular has degree $a_i$.*

**Proof.** Arguing as in I.3.4, we see that $\sigma_*$ is exact, supposing, as ever, I.2.6 so indeed for any $i$ and $n$,

$$H^i(\boldsymbol{\mathcal{P}}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) = H^i([\mathbb{P}^r/\mu_{\underline{a}}], \sigma^* \mathcal{O}_{\underline{a}}(n))$$

and we can calculate the latter using the Hochschild-Serre spectral sequence,

$$H^i(\mu_{\underline{a}}, H^j(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))) \Rightarrow H^{i+j}([\mathbb{P}^r/\mu_{\underline{a}}], \sigma^* \mathcal{O}_{\underline{a}}(n))$$

which also degenerates, so that unsurprisingly,

$$H^i(\boldsymbol{\mathcal{P}}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) = H^i(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))^{\mu_{\underline{a}}}\,.$$

As such in terms of standard coordinates, we obtain our assertion by putting $Y_i = X_i^{a_i}$. □

Manifestly we also have

**I.9.6 Further Fact.**

$$H^i(\boldsymbol{\mathcal{P}}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) = 0\,, \quad 1 < i < r\,, \quad \forall n \in \mathbb{Z}\,.$$

As to the top dimensional cohomology we have, of course, an explicit description in terms of negative monomials, i.e.

**I.9.7 Fact.** *In terms of the monomials $Y_i$ of I.9.5, the top cohomology is, $n \in \mathbb{Z}$,*

$$H^r(\boldsymbol{\mathcal{P}}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) = \bigoplus_{\substack{\sum_{i=0}^{r} a_i s_i = n \\ s_j \in \mathbb{Z}_{<0}}} \mathbb{C} \cdot Y_0^{s_0} \dots Y_r^{s_r}\,.$$

In particular we have the re-assuring fact that the largest $n$ for which cohomology occurs is $-\sum_i a_i$, that the dimension of the said group is 1, and indeed $\omega_{\boldsymbol{\mathcal{P}}(\underline{a})} = \mathcal{O}_{\underline{a}}\left(-\sum_i a_i\right)$. Consequently we have in the usual and wholly explicit fashion,

**I.9.8 Fact.** *The natural pairing of weighted monomials,*

$$H^0(\boldsymbol{\mathcal{P}}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) \times H^r(\boldsymbol{\mathcal{P}}(\underline{a}), \omega_{\boldsymbol{\mathcal{P}}(\underline{a})}(-n)) \to H^r(\boldsymbol{\mathcal{P}}(\underline{a}), \omega_{\boldsymbol{\mathcal{P}}(\underline{a})})$$

*is a perfect pairing.*

Probably the most important example in what follows is the so called bad orbifold $\boldsymbol{\mathcal{P}}(a_1)$, so let's consider:

**I.9.9 Example.** $\boldsymbol{\mathcal{P}}(a_1)$ in detail. In this case we'll write $\mathcal{O}\left(\frac{1}{a}\right)$ for $\mathcal{O}_{\underline{a}}(1)$, $a = a_1$, evidently. So the cohomology calculation yields,

$$h^0\left(\boldsymbol{\mathcal{P}}(\underline{a}), \mathcal{O}\left(\frac{n}{a}\right)\right) = \left[\frac{n}{a}\right] + 1\,,\ n \geq 0\,;\ h^1\left(\boldsymbol{\mathcal{P}}(\underline{a}), \mathcal{O}\left(\frac{n}{a}\right)\right) = -\left[\frac{n}{a}\right] - 1\,,\ n \leq -(a+1)\,.$$

It will be useful to be able to recognise global sections of $\mathcal{O}\left(\frac{n}{a}\right)$ in terms of the standard $\mathbb{A}^1$-coordinate patches $U$, scheme like neighbourhood of 0, $V$ neighbourhood of the non-scheme like point at $\infty$. To this end let $x$ be a standard coordinate on the former, and $\xi$ on the latter, and identify $U \times_{\boldsymbol{\mathcal{P}}(a)} V$ with $\mathbb{G}_m$ having coordinate $t$ such that, $x = t^a$, and $\xi = 1/t$, then in standard notation, the transition function for $\mathcal{O}\left(\frac{n}{a}\right)$ is

$$g_{VU} = t^n\,.$$

Consequently a global section of $\mathcal{O}\left(\frac{n}{a}\right)$ amounts to two functions $f_0$, and $f_\infty$ such that,

$$f_0(t^a) = t^n f_\infty\left(\frac{1}{t}\right)\,.$$

There is also a patching condition on $V \times_{\boldsymbol{\mathcal{P}}(a)} V$, but this is implied by the above. In any case the important thing to recognise is that in terms of $x$, the transition function is $x^{1/a}$. The not unuseful exercise of unravelling definitions out of the way, let's consider a vector bundle $E$ on $\boldsymbol{\mathcal{P}}(\underline{a})$, and proceed by the usual procedure of induction on the rank to prove that it splits. By the exactness of $\pi : \boldsymbol{\mathcal{P}}(\underline{a}) \to \mathbb{P}^1$, $E$ has meromorphic sections so we can find,

$$s \in H^0\left(\boldsymbol{\mathcal{P}}(\underline{a}), E \otimes \mathcal{O}\left(-\frac{n}{a}\right)\right)$$

with $n$ maximal. Everything is smooth, so by induction we have an exact sequence,

$$0 \to \mathcal{O}\left(\frac{n}{a}\right) \to E \to \bigoplus_i \mathcal{O}\left(\frac{m_i}{a}\right) \to 0\,.$$

Now suppose $m_j \geq n+1$, for some $j$, then we would have an exact sequence,

$$H^0\left(\boldsymbol{\mathcal{P}}(a), E\left(-\frac{(n+1)}{a}\right)\right) \to \bigoplus_i H^0\left(\boldsymbol{\mathcal{P}}(a), \mathcal{O}\left(\frac{m_i - a - 1}{a}\right)\right) \to H^1\left(\boldsymbol{\mathcal{P}}(a), \mathcal{O}\left(-\frac{1}{a}\right)\right)$$

however the latter group is zero, so this would contradict the maximality of $n$, and we deduce that $E$ splits by virtue of I.9.7. Of course just as in the $\mathbb{P}^1$-case, there's nothing unique about the splitting, but rather the unique thing is the $H-N$ filtration, or what amounts to the same thing in this particular case, the weights,

$$\frac{n}{a} = \frac{m_1}{a} \geq \frac{m_2}{a} \geq \cdots \geq \frac{m_k}{a}$$

or equivalent if $d_i/a$, $i \leq 1$ is a repetition free list of the same and $\iota_i$ the number of occurences, the filtration,

$$E = E^\iota \supset E^{\iota-1} = \bigoplus_{i \leq \iota-1} \mathcal{O}\left(\frac{d_i}{a}\right)^{\oplus \iota_i} \supset \cdots \supset E^1 = \mathcal{O}\left(\frac{d_1}{a}\right)^{\oplus \iota_1} \supset E^0 = 0\,.$$

In any case the example over, this pretty much completes what we need to know about weighted projective stacks, beyond:

**I.9.10 Fact.** *All of this goes through verbatim (excepting* I.9.4*) in the relative situation of a bundle, $\nu : \boldsymbol{\mathcal{P}} \to S$ of weighted projective stacks over a scheme, or algebraic space, or for that matter stack, provided $S$ itself is over $\mathbb{C}$ (or a field of characteristic prime to the weights) i.e. $R^1\nu_*\mathbb{G}_m$ is generated by some $\mathcal{O}_{\underline{a}}(1)$, $\bigoplus_n R^0\nu_*\mathcal{O}_{\underline{a}}(n)$ is a coherent sheaf of weighted graded rings, etc.*

**I.9.11 Proof/Remarks/Warning.** Granted the calculation of Pic used CH at some point, so strictly speaking its only verbatim over a $\mathbb{Q}$-factorial base. Nevertheless, what's at stake is only the calculation mod torsion so [K-3] II.4.1.9 could be used instead.

Note however,

There may fail to exist a global covering by a $\mathbb{P}^r$-bundle. Our assertion is, on the other hand, of a purely local character where such a covering will certainly exist, so this is irrelevant to the claim.

Now let's turn to the general case. We'll retake the notation of Definition I.9.1, up to some superficial changes to encourage consistency with further chapters and thus emphasise the different roles that the two definitions enjoy in the minimal model theory, namely:

**I.9.12 Definition.** Let $b_1, \ldots, b_t \in \mathbb{N}$ be given, put $\mu_{\underline{b}} = \mu_{b_1} \times \cdots \times \mu_{b_t}$, and consider the standard coordinate action on $\mathbb{P}^{t-1}$, i.e.

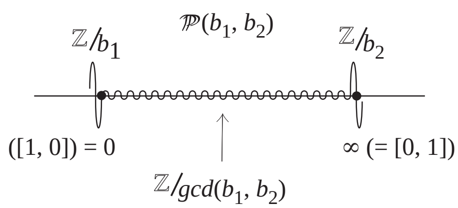
$$\mu_{\underline{b}} \times \mathbb{P}^{t-1} \to \mathbb{P}^{t-1} : (\zeta_1, \ldots, \zeta_t) \times [X_1, \ldots, X_t] \mapsto [\zeta_1 X_1, \ldots, \zeta_t X_t]$$

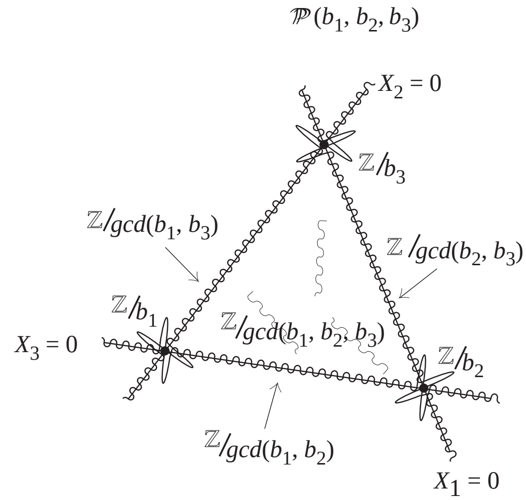
then the generalised weighted projective stack $\boldsymbol{\mathcal{P}}^{t-1}(b_1, \ldots, b_t)$ has as moduli the weighted projective space $\mathbb{P}(b_1, \ldots, b_t)$, while the system of affine coordinate patches $\mathbb{A}^{t-1}$, $1 \leq j \leq t$, with standard coordinates $\xi_{ij} = (X_i/X_j)^{a_i}$, $i \neq j$, together with $\mu_{b_j}$-action,

$$\mu_{b_j} \times \mathbb{A}^{t-1} : \zeta_j \times \xi_{ij} \mapsto \zeta_j^{-b_i} \xi_{ij}$$

yield Zariski open substacks $[\mathbb{A}^{t-1}/\mu_{b_j}]$, together with the not wholly obvious patching maps, i.e. strictly speaking to turn this data into a stack requires some choices (this was also true in op. cit., but the effect was negligible) amongst which is a splitting of, $\mu_b \to \mu_{\underline{b}}$, where $b = gcd(b_1, \ldots, b_t)$, and we'll denote the corresponding character of $\mu_{\underline{b}}$ by $\chi$.

Again let's draw the pictures in dimensions 1 and 2, i.e.




So in particular a generalised weighted projective stack $\mathcal{P}^{t-1}(b_1, \ldots, b_t)$ is permitted to be everywhere non-scheme like with stabiliser group $\mathbb{Z}/b$, $b = gcd(b_1, \ldots, b_t)$. We list the salient facts, whose proofs are just a minor variation of the previous ones, viz:

**I.9.13 Facts.** (Characteristic 0, or I.2.6) *Let $\mathcal{P}^{t-1}(b_1, \ldots b_t)$ be a generalised weighted projective stack, with $\chi$ the character implicit in the patching data of the definition. Then,*

(i) *There is a map, $\sigma : [\mathbb{P}^{t-1}/\mu_{\underline{b}}] \to \mathcal{P}^{t-1}(b_1, \ldots, b_t)$ and a bundle $\mathcal{O}_{\underline{b}}(1)$ on the latter, such that the free $\oplus$ torsion decomposition of the Picard group is,*

$$\mathrm{Pic}(\mathcal{P}^{t-1}(b_1, \ldots, b_t)) = \mathbb{Z}\mathcal{O}_{\underline{b}}(1) \oplus \mathbb{Z}\chi$$

*and indeed $\sigma^*\mathcal{O}_{\underline{b}}(1) = \mathcal{O}(b)$, where $\mathcal{O}(1)$ is the standard tautological bundle on $\mathbb{P}^{t-1}$ descended to the classifying stack.*

(ii) *The fundamental group of $\mathcal{P}^{t-1}(b_1, \ldots, b_t)$ is $\mathbb{Z}/b$.*

(iii) *If $Y_i = X_i^{b_i}$, and $S = \mathbb{C}[Y_1, \ldots, Y_t]$ the graded algebra assigning to $Y_i$ weight $b_i$, then for $n \in \mathbb{Z}$,*

$$\bigoplus_{m\in\mathbb{Z}} H^0(\mathcal{P}^{t-1}(b_1, \ldots, b_t), \mathcal{O}_{\underline{b}}(m) \otimes \chi^n) = \begin{cases} S, & n \equiv 0(b) \\ 0, & \textit{otherwise} \end{cases}$$

(iv) *For any $0 < q < t-1$, and $m, n \in \mathbb{Z}$,*

$$H^q(\mathcal{P}^{t-1}(b_1, \ldots, b_t), \mathcal{O}_{\underline{b}}(m) \otimes \chi^n) = 0\,.$$

(v) *If $\omega$ is the canonical bundle, then $\omega = \mathcal{O}_{\underline{b}}(-b_1/b \cdots - b_t/b)$, so in particular, $\sigma^*\omega = \mathcal{O}(-b_1 \cdots - b_t)$.*

(vi) *The cohomology is given by the negative monomials, i.e. for $m, n \in \mathbb{Z}$*

$$H^{t-1}(\mathcal{P}^{t-1}(b_1, \ldots, b_t), \mathcal{O}_{\underline{b}}(m) \otimes \chi^n) = \begin{cases} \bigoplus_{s_i<0} \mathbb{C}\, Y_1^{s_1} - Y_m^{s_m}, & \sum b_i s_i = mb < 0 \\ & n \equiv 0(b) \\ 0, & \textit{otherwise} \end{cases}$$

(vii) *In particular, we have an explicit duality pairing, $m, n \in \mathbb{Z}$,*

$$H^0(\boldsymbol{\mathcal{P}}^{t-1}(b_1, \ldots, b_t), \mathcal{O}_{\underline{b}}(m) \otimes \chi^n) \times H^{t-1}(\boldsymbol{\mathcal{P}}^{t-1}(b_1, \ldots, b_t), \mathcal{O}_{\underline{b}}(-m) \otimes \omega \otimes \chi^{-n})$$
$$\to H^{t-1}(\boldsymbol{\mathcal{P}}^{t-1}(b_1, \ldots, b_t), \omega)\,.$$

# II. $K_{\mathcal{F}}$ negative curves

## II.1. Foliations as birational groupoids

In reality the point of view of a foliation as an integrable quotient of the cotangent sheaf is completely wrong. Rather a foliation should be considered as an infinitesimal equivalence relation outside of its singularities, and the equivalence of this definition to that involving linear 1st order data as a non-trivial theorem (not withstanding the triviality of the proof) specific to characteristic zero. In any case let us begin by reviewing the equivalence, whence let $X$ be a normal affine variety over $\mathbb{C}$ and $\mathcal{F}$ a smooth foliation on $X$. Notice that $X$ may be singular, so $\mathcal{F}$ smooth means that for some (and indeed any) embedding of $X$ in a smooth variety $M$ the composition,

$$T_{\mathcal{F}} \to \mathcal{J}_X \to T_M \otimes \mathcal{O}_X$$

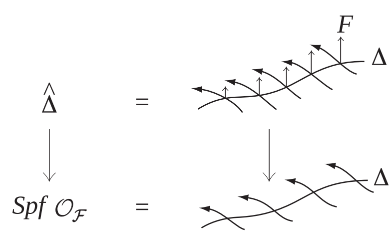
is an injection of bundles. Now consider the diagonal $\Delta$ in $X \times X$, with $p_i$ the projections, and $p_2^* T_{\mathcal{F}}$ the foliation obtained by pull-back from the 2nd direction. Dualising commutes with flat pull-back so this is notationally unambiguous, so that shrinking $X$ as necessary we can find a local generator $\partial$ of $T_{\mathcal{F}}$ and $f \in I_\Delta$ such that $p_2^*\partial(f)$ is non-zero on $X$. We put $\delta = (p_2^*\partial(f))^{-1}\partial$, and for any function $g$ on $X \times X$ define,

$$\tilde{g} := \sum_{n=0}^{\infty} (-1)^n \frac{f^n \delta^n(g)}{n!} \in \hat{\mathcal{O}}_\Delta := \varprojlim_n \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^n$$

then $\delta\tilde{g} = 0$, and better still if $\hat{\Delta}$ is the completion of $X \times X$ in $\Delta$ then the inclusion of rings,

$$\mathcal{O}_{\mathcal{F}} := \{h \in \hat{\mathcal{O}}_\Delta : \partial h = 0\} \subset \hat{\mathcal{O}}_\Delta$$

corresponds to a relatively smooth fibration of formal schemes,




such that the pull-back of the image of $\Delta$ in Spf $\mathcal{O}_{\mathcal{F}}$ is the corresponding infinitesimal equivalence relation, i.e. the formal sub-scheme of $\hat{\Delta}$ defined by the ideal generated by $\mathcal{O}_{\mathcal{F}} \cap \mathcal{I}_\Delta$ or equivalently the maximal sub-ideal of $\mathcal{I}_\Delta$ invariant by $\mathcal{F}$. Rather more picturesquelly what we have done is add a small germ in the $p_2^*T_{\mathcal{F}}$ direction for each point in the diagonal.

To extend this to stacks, even separated ones, is a little delicate since unless the stack is in fact an algebraic space the diagonal will fail to be an embedding. To remedy this it suffices to observe that we've actually been working in,

$$\mathfrak{P}_X := \operatorname{Spf} \mathcal{P}_X\,, \quad \mathcal{P}_X = \varprojlim_n \mathcal{P}_X^{(n)}$$

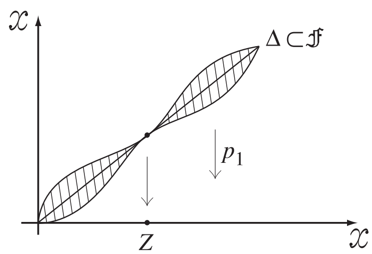
where $\mathcal{P}_X^{(n)}$ is Grothendieck's sheaf of $n$-jets consider as a nilpotent $\mathcal{O}_X$-algebra by way of the 1st-projection. Being by supposition of Deligne-Mumford type a stack $\mathcal{X}$ is defined by étale equivalence relations so there are well defined sheaves of nilpotent $\mathcal{O}_{\mathcal{X}}$-algebras, $\mathcal{P}_{\mathcal{X}}^{(n)}$ of $n$-jets, and of course idem, modulo replacing nilpotent by topologically so, for the inverse limit $\mathcal{P}_{\mathcal{X}}$. Equally the formation of the formal spectrum is a local construction, while both the projectors and the diagonal embedding patch, so we obtain an object which we summarise by way of,

**II.1.1 Definition.** The jet groupoid of a stack $\mathcal{X}$ is the formal stack $\mathfrak{P}_{\mathcal{X}} = \operatorname{Spf} \mathcal{P}_{\mathcal{X}} \rightrightarrows \mathcal{X}$ with source map $p_1$, sink $p_2$, and identify the diagonal.

Notice in particular that the diagonal is actually embedded in the jet groupoid, so its worth emphasising what's happening. Specifically for a geometric diagonal point $x \times x$ in $\mathcal{X} \times \mathcal{X}$, its automorphism group is simply $\mathrm{Aut}(x) \times \mathrm{Aut}(x)$. Inside this group we have a copy of $\mathrm{Aut}(x)$ sitting diagonally. Now any attempt to define diagonal type subgroups of automorphisms for off diagonal points, and whence define an actual étale "neighbourhood" in which $\mathcal{X}$ embeds in some sort of diagonal way, is doomed to failure. At the infinitesimal level this can, however, be achieved.

Turning then to stacks foliated by curves, or indeed even foliated full stop, the corresponding foliations on étale neighbourhoods of the stack are again by supposition invariant by the corresponding étale groupoid so that we may once again apply the expedient of summary by way of definition, i.e.

**II.1.2 Definition.** Let $(\mathcal{X}, \mathcal{F})$ be a foliated stack, $\mathcal{Z}$ its singular locus, and $\mathcal{U} = \mathcal{X} \backslash \mathcal{Z}$ the smooth locus then the infinitesimal equivalence relation $\mathfrak{f} \rightrightarrows \mathcal{U}$ defined according to the correspondence which associates to $\mathcal{F}$ a formal subscheme of the jet groupoid will be denoted the smooth infinitesimal groupoid of $\mathcal{F}$. This construction may, however, fail catastrophically over $\mathcal{Z}$, i.e. consider:




then over $\mathcal{Z}$ we may have an essential singularity, so that the smallest closed formal sub-stack of $\mathfrak{P}_{\mathcal{X}}$ containing $\mathfrak{f}$ is $\mathfrak{P}_{\mathcal{X}}$ itself.

To remedy this latest difficulty we conisder the possibility of birational groupoids, i.e. such that the identity map is simply birational. With this extra flexibility we can complete across the singularities. Specifically let $\tilde{\mathfrak{P}}_{\mathcal{X}}$ be the blow up of $\mathfrak{P}_{\mathcal{X}}$ in the diagonal embedding $\Delta(\mathcal{Z})$ of $\mathcal{Z}$ understood with any implied nilpotent structure on the singular locus. Now let $U \to \mathcal{X}$ be an étale neighbourhood of a geometric point $z \in \mathcal{Z}$ with $U \hookrightarrow M$ an embedding into a smooth. Consider coordinates $x_1, \ldots, x_n$ on $M$ restricting to functions on $U$, then for $\mathcal{F}|_U$ Gorenstien, and shrinking $U$ as necessary we may suppose that the foliation is defined by a vector field $\partial$ on $U$, which we write using the summation convention as,

$$\partial = a_i \frac{\partial}{\partial x_i}$$

so that $\mathcal{I}_{\mathcal{Z}}|_{\mathcal{U}} = (a_i)$. Now introduce $x_i, y_i$ as coordinates on $U \times U$ obtained from our initial coordinates by way of $1^{\text{st}}$ and $2^{\text{nd}}$ pull-back respectively, and put $z_i = x_i - y_i$, then in $z_i, x_i$ coordinates,

$$p_2^* \partial = p_2^* a_i \frac{\partial}{\partial y_i} = -p_2^* a_i \frac{\partial}{\partial z_i} .$$

Consequently if $\pi : \tilde{\mathfrak{P}}_{\mathcal{X}} \to \mathfrak{P}_{\mathcal{X}}$ is the projection, restricted to $U$ on the $p_1^* a_i \neq 0$ patch,

$$\partial \left( \frac{z_i}{p_1^* a_i} \right) = \frac{-p_2^* a_i}{p_1^* a_i} = 1 + \frac{(p_1^* a_i - p_2^* a_i)}{p_1^* a_i} .$$

On the other hand the ideal of the diagonal embedding of $\mathcal{Z} \times_{\mathcal{X}} U$ has ideal $(p_1^* a_i, z_i)$ so on the proper transform $\tilde{\Delta}$ of $\Delta$ in $\tilde{\mathfrak{P}}_{\mathcal{X}}$ not only can we locate each point in some $p_1^* a_i \neq 0$ patch for an appropriate $i$, but indeed the function $z_i / p_1^* a_i \in I_{\tilde{\Delta}}$ enjoys a non-zero derivation with respect to $\pi^* \partial$. Better still we have blown up in a centre invariant by $p_2^* \mathcal{F}$ so the induced foliation $\widetilde{p_2^* \mathcal{F}}$ on $\tilde{\mathfrak{P}}_{\mathcal{X}}$ is both smooth in a neighbourhood of

$\tilde{\Delta}$ and everywhere transverse to it. Whence we can just repeat our minor variant of the classical Frobenius theorem to obtain,

**II.1.3 Fact/Definition/Summary.** Let $(\mathscr{X}, \mathcal{F})$ be a foliated Gorenstien stack, then there is a formal substack $\tilde{\mathfrak{F}}$ of $\tilde{\mathfrak{P}}_{\mathscr{X}}$ together with projection maps, $p_i \circ \pi$, $i = 1$ or $2$ defining a birational groupoid, i.e.

$$\tilde{\mathfrak{F}} \rightrightarrows \mathscr{X}$$

where the identity and composition are rational maps. In addition the projection $p_1 \circ \pi$ factors as,

$$\tilde{\mathfrak{F}} \to \tilde{\Delta} \to \mathscr{X}$$

with the former map relatively smooth of dimension 1. We call this structure the infinitesimal birational groupoid.

Notice in particular,

**II.1.4 Fact.** *There is an isomorphism,* $N_{\tilde{\Delta}/\tilde{\mathfrak{F}}} \xrightarrow{\sim} \mathcal{O}_{\tilde{\Delta}}(p_2^* T_{\mathcal{F}})$.

## II.2. Chow's Lemma

We'll confine ourselves to that which is strictly necessary for applications. Our interest centres on smooth formal stacks $\mathfrak{F}$ whose underlying stack $\mathcal{C}$ is smooth of dimension 1. From our utilitarian point of view we'll confine ourselves to the case where $\dim \mathfrak{F} = 2$. Irrespectively there is a well defined normal bundle $N_{\mathcal{C}/\mathfrak{F}}$, and we make,

**II.2.1 Definition.** $\mathfrak{F}$ is a concave formal neighbourhood of $\mathcal{C}$ if $\deg(N_{\mathcal{C}/\mathfrak{F}}) > 0$.

Unsurprisingly the classical Chow lemma continues to hold, i.e.

**II.2.2 Lemma.** (Chow, Grauert et al.) *Let $L$ be a line bundle on $\mathfrak{F}$ then there is a quadratic polynomial $P_L$, depending on $L$, such that for all $n \in \mathbb{N}$,*

$$h^0(\mathfrak{F}, L^{\otimes n}) \leq P_L(n)\,.$$

**Proof.** Let $\mathfrak{F}_m$ be the $m^{\text{th}}$-thickening of $\mathcal{C}$ then we have an exact sequence,

$$0 \to \mathrm{Sym}^m N^\vee_{\mathcal{C}/\mathfrak{F}} \to \mathcal{O}_{\mathfrak{F}_{m+1}} \to \mathcal{O}_{\mathfrak{F}_m} \to 0\,.$$

On the other hand if $h^0(\mathcal{C}, L^n \otimes \mathrm{Sym}^m N^\vee_{\mathcal{C}/\mathfrak{F}}) \neq 0$, then,

$$m \deg(N_{\mathcal{C}/\mathfrak{F}}) \leq n \deg_{\mathcal{C}}(L)\,.$$

Consequenlty for any $n \in \mathbb{N}$,

$$H^0(\mathcal{O}_{\mathfrak{F}_{m+1}} \otimes L^n) \hookrightarrow H^0(\mathcal{O}_{\mathfrak{F}_m} \otimes L^n)$$

is injective, provided $m > \dfrac{n \deg_{\mathcal{C}}(L)}{\deg(N_{\mathcal{C}/\mathfrak{F}})}$, so that if $M$ is the aforesaid lower bound then,

$$h^0(\mathfrak{F}, L^{\otimes n}) = \varprojlim_m h^0(\mathfrak{F}_m, L^{\otimes n}) \leq \sum_{k=0}^{M} h^0(\mathcal{C}, L^n \otimes N^{-k}_{\mathcal{C}/\mathfrak{F}})\,.$$

Moreover we can find a map, $\rho : C \to \mathcal{C}$ from an honest curve, while for any bundle $E$, $h^0(\mathcal{C}, E) \leq h^0(C, \rho^* E)$, so we conclude by Riemann-Roch. $\square$

## II.3. Bend & Break

We are now in a position to extend the results of [Bo-M], so to this end let $(\mathcal{X}, \mathcal{F})$ be a foliated Gorenstien stack with projective moduli space $\pi : \mathcal{X} \to X$, and $H$ an ample bundle on the latter. As ever the basic object of study is $K_{\mathcal{F}}$ negative curves on $\mathcal{X}$, i.e. maps $f : \mathcal{C} \to \mathcal{X}$ from smooth 1-dimensional stacks such that $K_{\mathcal{F}} \cdot_f \mathcal{C} < 0$. We impose further the condition that $f$ does not factor through the singular locus $\mathcal{Z} = \mathrm{sing}(\mathcal{F})$. Consequently if we consider the infinitesimal birational groupoid as fibered over $\tilde{\mathcal{X}} = \mathrm{Bl}_{\mathcal{Z}}(\mathcal{X})$ by way of the $1^{\mathrm{st}}$ projection, then $f$ admits a lifting $\tilde{f} : \mathcal{C} \to \tilde{\mathcal{X}}$ and we may form the fibre square,

$$\begin{tikzcd}
\tilde{\mathfrak{F}} \arrow[d, "p"'] & \tilde{\mathfrak{F}}_{\mathcal{C}} \arrow[l] \arrow[d, "p"] \\
\tilde{\mathcal{X}} & \tilde{\mathcal{C}} \arrow[l, "\tilde{f}"]
\end{tikzcd}$$

In addition the identity map of the groupoid gives a section $s$ of $p$ of the left, so a fortiori of the right, vertical arrow, which is everywhere well defined since we're working with $\tilde{\mathcal{X}}$ rather than $\mathcal{X}$. Consequently $\tilde{\mathfrak{F}}_{\mathcal{C}}$ is a concave neighbourhood of $s(\mathcal{C})$. In addition if $C$ is the modulus of $\mathcal{C}$ then we have maps,

$$\tilde{\mathfrak{F}}_{\mathcal{C}} \hookrightarrow \mathcal{C} \times \tilde{\mathfrak{P}}_{\mathcal{X}} \to \mathcal{C} \times \mathfrak{P}_{\mathcal{X}} \to C \times \mathfrak{P}_X \hookrightarrow C \times X \times X$$

where $\tilde{\mathfrak{P}}_{\mathcal{X}}$ is as the prequel to II.1.3, and we claim,

**II.3.1 Fact.** *The Zariski closure of the image of $\tilde{\mathfrak{F}}_{\mathcal{C}}$ in $C \times X \times X$ is integral of dimension* 2.

**Proof.** Indeed let $Y$ be the Zariski closure, which is integral since $\tilde{\mathfrak{F}}_{\mathcal{C}}$ is integral. Moreover if $L$ is an ample line bundle on $C \times X \times X$ then by definition,

$$H^0(Y, L) \to H^0(\tilde{\mathfrak{F}}_{\mathcal{C}}, L)$$

is injective by the definition of $Y$, so we're done by the Chow lemma. $\square$

Now certainly we have that the projection of $Y$ to $X$ by the second factor of $X \times X$ is a $\mathcal{F}$-invariant subvariety, but equally its projection to $C$ is a $1^{\mathrm{st}}$ integral, so $(Y, \mathcal{F})$ is a pencil of curves, albeit perhaps a rather singular one.

Unfortunately since the diagonal of $\mathcal{X}$ will only be an embedding if $\mathcal{X}$ is a scheme to proceed from here to a well defined family of invariant stacks parametrised by $\mathcal{C}$ which is infinitesimally $\tilde{\mathfrak{F}}_{\mathcal{C}}$, or something close to it, requires a little care. Now what we can certainly do is to consider the $\mathcal{F}$-invariant 2-dimensional integral substack $\mathcal{Y}_0$ of $\mathcal{X}$ obtained by projecting $Y$ to $X$ then pulling back to $\mathcal{X}$. The induced foliation $(\mathcal{Y}_0, \mathcal{F})$ with tangent bundle $\mathcal{O}_{\mathcal{Y}_0}(T_{\mathcal{F}})$ may only be a pre-foliation, but this is actually better rather than worse. In any case $\mathcal{F}$ is by hypothesis generically smooth around $f(\mathcal{C})$, so by algorithmic resolution (or more precisely the invariance of the resolving centres under vector fields) we know that the said image is not contained in the singular locus of $\mathcal{Y}_0$. Consequently if $\rho : \tilde{\mathcal{Y}} \to \mathcal{Y}_0$ is the algorithmic resolution and $\tilde{\mathcal{F}}$ the induced saturated foliation then $f : \mathcal{C} \to \mathcal{Y}_0$, lifts to $\tilde{f} : \mathcal{C} \to \tilde{\mathcal{Y}}$, say, and $K_{\tilde{\mathcal{F}}} \cdot_{\tilde{f}} \mathcal{C} \leq K_{\mathcal{F}} \cdot_f \mathcal{C}$. Additionally $(\tilde{\mathcal{Y}}, \tilde{\mathcal{F}})$ admits a meromorphic $1^{\mathrm{st}}$ integral $\varphi : \tilde{\mathcal{Y}} \dashrightarrow \mathbb{P}^1$, and we can blow up in the crossings of the image of $\tilde{f}$ with the singular locus of $\tilde{\mathcal{F}}$ to ensure that $\tilde{\mathcal{F}}$ is everywhere smooth in a neighbourhood of $\tilde{f}(\mathcal{C})$, and whence $\varphi$ is defined on all of the same. Again such blow ups are in invariant centres, so the canonical degree is certainly still bounded above by $K_{\mathcal{F}} \cdot_f \mathcal{C}$, whence we may as well suppose that we have these additional hypothesis. However at this point the indeterminacy points of $\varphi$ are all of dimension 0, from which some more blowing up all of which takes place outside a neighbourhood of $\tilde{f}(\mathcal{C})$ leads to $\varphi$ being defined everywhere. Abusing notation slightly let's continue to denote this modification by $\tilde{\mathcal{Y}}$ and consider the fibre square,

$$\begin{tikzcd}
\tilde{\mathcal{Y}} \arrow[d, "\varphi"'] & \tilde{\mathcal{Y}}_{\mathcal{C}} \arrow[l] \arrow[d] & \mathcal{C} \arrow[l, "s=\tilde{f}\times\mathrm{id}"'] \\
\mathbb{P}^1 & \mathcal{C} \arrow[l, "\varphi\circ f"'] &
\end{tikzcd}$$

It may, of course, happen that $\tilde{\mathcal{Y}}_{\mathcal{C}}$ is not irreducible, but it is generically smooth about the image of $s$, since $\varphi$ itself is generically smooth. Consequently there is a unique irreducible component $\mathcal{Y}$ containing the image of $s$. Better still $\mathcal{Y}$ is actually smooth, albeit that it's evidently only ramified over $\tilde{\mathcal{Y}}$ in the fibre direction, and this would be wholly sufficient. Regardless to see that it's smooth we can work with small étale neighbourhoods $U$ and $V$ of $\tilde{\mathcal{Y}}$ and $\mathcal{C}$ in the analytic topology. Around $\tilde{f}(\mathcal{C})$ the foliation is smooth with transverse coordinate $z$, say, and parallel coordinate $x$. Without loss of generality we can write $\varphi = z^m$, so that for $c$ a coordinate on $\mathcal{C}$, $\tilde{\mathcal{Y}}_{\mathcal{C}}$ is, around $s$, covered by étale neighbourhoods of the form,

$$\frac{\mathbb{C}[x,z,c]}{z^m - z(c)^m}$$

where of course, $f(c) = (x(c), z(c))$. Thus around the ramification of $\varphi$, an étale neighbourhood of $\tilde{\mathcal{Y}}_{\mathcal{C}}$ has the form,

$$\frac{\mathbb{C}[x,z,c]}{z - z(c)}$$

which is certainly smooth. Moreover the ramification of $\mathcal{Y} \to \tilde{\mathcal{Y}}$ is precisely along the fibres, so in a further abuse of notation if $(\mathcal{Y}, \mathcal{F})$ is the induced foliation then $K_{\mathcal{F}}$ is just the pull-back of $K_{\tilde{\mathcal{F}}}$. Whence to summarise,

**II.3.2 Fact.** *There is a proper family $p : \mathcal{Y} \to \mathcal{C}$ of stacks, together with a section $s$ such that the induced foliation $\mathcal{F}$ on $\mathcal{Y}$ is the fibration $p$. Better still,*

(a) $K_{\mathcal{F}} \cdot_s \mathcal{C} \leq K_{\mathcal{F}} \cdot_f \mathcal{C}$.

(b) $(\mathcal{Y}, \mathcal{F}) \to (\mathcal{X}, \mathcal{F})$ *is everywhere defined, and respects the various foliations.*

Armed with our family $\mathcal{Y}$ we've more or less constructed invariant rational curves through every point of $\mathcal{C}$. Indeed we may apply, provided modulo its generic stabiliser the generic fibre of $\mathcal{Y}$ is not a bad orbifold, our combination of semi-stable reduction and uniformisation, I.4.4, to find an honest curve $B$, a semi-stable family $\mathcal{S} \to B$ scheme like in codimension 2 along with a commutative diagram,

$$\begin{tikzcd}
\mathcal{Y} \arrow[d] & \mathcal{S} \arrow[l, "\rho"'] \arrow[d] \\
\mathcal{C} & B \arrow[l]
\end{tikzcd}$$

such that $\mathcal{S} \to \mathcal{Y}$ is ramified only on fibres, from which $\omega_{\mathcal{S}/B} = \rho^* K_{\mathcal{F}}$. In addition our section $s : \mathcal{C} \to \tilde{\mathfrak{F}}_{\mathcal{C}} \to \mathcal{Y}$, can be lifted, at least after a possibly larger base extension, to $\tilde{s} : B \to \mathcal{S}$, and

$\omega_{\mathcal{S}/B} \cdot_{\tilde{s}} B = (B : \mathcal{C}) K_{\mathcal{F}}.\mathcal{C} < 0$. Consequently to prove that the generic fibre of $\mathcal{S} \to B$ is a rational curve it suffices by [Bo-M] (or for that matter a classical theorem of Arakelov, [Sz]) to show that $\mathcal{S}$ is scheme like around $\tilde{s}(B)$. Suppose, then, this fails at some point $b$, with $x$ a coordinate coming from the base, and $y = 0$ defining the section on some étale neighbourhood $V$. Blowing up, if necessary, we can suppose $x, y$ are a coordinate system. For the situation to be non-scheme like we would have to be able to find a finite group $G$ acting on $V$ leaving $x$ invariant as well as the ideal $(y)$. However in $\hat{\mathcal{O}}_{V,b}$ this can be linearised for a possibly different choice of $y$ (essentially since $H^1(H, L) = 0$ for any cyclic group $H$, and coherent $\mathbb{Q}[H]$ module $L$) as,

$$(x, y) \mapsto (x, y)^g = (x, \chi(g)y)$$

for $\chi : G \to \mathbb{C}^\times$ some character. This would imply, however, non-scheme like behaviour around all of the section, and so we have almost arrived at,

**II.3.3 Proposition.** *Let $(\mathcal{X}, \mathcal{F})$ be a foliated Gorenstien stack with projective moduli $X$, and $f : \mathcal{C} \to \mathcal{X}$ a map from a 1-dimensional smooth stack not factoring through the singular locus of $\mathcal{F}$, with $K_{\mathcal{F}} \cdot_f \mathcal{C} < 0$ then for every geometric point $c \in \mathcal{C}$ there is a map $g_c : \mathcal{L}_c \to \mathcal{X}$ with $\mathcal{L}_c$ a stack of positive Euler characteristic such that for $M$ any* nef. *$\mathbb{R}$-divisor on $\mathcal{X}$,*

$$M \cdot_{g_c} \mathcal{L}_c \leq 2 \, \frac{M \cdot_f \mathcal{C}}{-K_{\mathcal{F}}.\mathcal{C}}$$

*and $g_c : \mathcal{L}_c \to \mathcal{X}$ is invariant by $\mathcal{F}$.*

**Proof.** According to our previous set up, if the generic fibre of $\mathcal{Y}$ modulo the generic stabiliser is not a bad orbifold, then for each $b \in \tilde{s}(B)$ we may find a rational curve $g_b : \mathbb{P}^1 \to \mathcal{S}$ factoring through the fibres of $\mathcal{S} \to B$ such that,

$$M \cdot_{g_b} \mathbb{P}^1 \leq 2 \, \frac{\rho^* M \cdot_{\tilde{s}} B}{-\omega_{\mathcal{S}/B} \cdot_{\tilde{s}} B} = 2 \, \frac{M \cdot_f \mathcal{C}}{-K_{\mathcal{F}} \cdot_f \mathcal{C}} .$$

Otherwise generically $\mathcal{Y}/G$, with $G$ its generic stabiliser, is a bad orbifold so certainly of positive Euler characteristic. Consequently for each geometric point $c \in \mathcal{C}$, the component of the fibre of $\mathcal{Y}$ over $\mathcal{C}$ meeting $s$ gives a map from a stack $g_c : \mathcal{L}_c \to \mathcal{X}$ which is invariant by the foliation. The degree bound on the other hand is, therefore, just that for the fibres of $\mathcal{Y} \to \mathcal{C}$, which is equally that for the fibres of the moduli $Y \to C$, say, which in turn is just the index theorem for a 2-dimensional normal algebraic space. $\square$

**II.3.4 Remark.** The same proof works, in more complete generality, if $f(\mathcal{C})$ meets the pre-image of the open subset where the moduli are algebraic.

## II.4. The Cone of Curves

We may now apply the basic estimate II.3.3 to the cone of curves of a $\mathbb{Q}$-foliated Gorenstien variety $(X, \mathcal{F})$ over $\mathbb{C}$. Indeed more precisely we have,

**II.4.1 Fact.** *Let $(X, \mathcal{F})$ be a $\mathbb{Q}$-foliated Gorenstien variety with log-canonical singularities then there are countably many rational curves $L_i$ invariant by $\mathcal{F}$ with, $0 < -K_{\mathcal{F}}.\, L_i \leq 2$ and,*

$$\overline{\mathrm{NE}}\,(X)_{\mathbb{R}} = \overline{\mathrm{NE}}\,(X)_{K_{\mathcal{F}} \geq 0} + \sum_i \mathbb{R}_+ \, L_i$$

*where $\overline{\mathrm{NE}}\,(X)_{K_{\mathcal{F}} \geq 0}$ is the sub-cone of the closed cone of curves on which $K_{\mathcal{F}}$ is non-negative. Better still the $\mathbb{R}_+ \, L_i$ are locally discrete, and if $R \subset \overline{\mathrm{NE}}\,(X)_{\mathbb{R}}$ is an extremal ray in the half space $\mathrm{NE}_{K_{\mathcal{F}} < 0}$ then it is of the form $\mathbb{R}_+ \, L_i$.*

**Proof.** This is a wholly formal consequence, i.e. modulo the existence of Hilbert schemes linear algebra and cf. [K-3] III.1.2, of the following variant of II.3.3, viz:

**II.4.2 Sub-Fact.** *Let $(X, \mathcal{F})$ be as above, $f : C \to X$ a curve with $K_{\mathcal{F}\cdot_f} C < 0$ then for all $c \in C$ there is a rational curve $L_c \ni f(c)$ invariant by $\mathcal{F}$ such that for all* nef. *$\mathbb{R}$-divisors $M$ on $X$,*

$$M.\, L_c \leq 2\, \frac{(M.\, C)}{-K_{\mathcal{F}}.\, C}\,.$$

**Sub-Proof.** Let $\pi : \mathscr{X} \to X$ be the corresponding Gorenstien stack, then provided $f(c)$ is not contained in the image of the singular locus of $(\mathscr{X}, \mathcal{F})$ then we simply apply II.3.3, and take $L_c$ to be the moduli of the corresponding parabolic stack $\mathcal{L}_c$. Otherwise consider the linearisation map, i.e. the composition of,

$$D : \mathcal{I}_{\mathcal{Z}}/\mathcal{I}_{\mathcal{Z}}^2 \xrightarrow[d]{} \Omega_{\mathscr{X}} \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_{\mathcal{Z}} \longrightarrow K_{\mathcal{F}} \otimes \mathcal{I}_{\mathcal{Z}}/\mathcal{I}_{\mathcal{Z}}^2$$

where $\mathcal{Z} \subset \mathscr{X}$ is the singular locus of $\mathcal{F}$. The linearisation is $\mathcal{O}_{\mathcal{Z}}$ linear, and since the singularities are canonical for any $z \in \mathcal{Z}$ some symmetric function of $D$ defines a global section of some power $K_{\mathcal{F}}^{\otimes n}$ of $K_{\mathcal{F}}$ over $\mathcal{Z}$ which is non-vanishing at $z$, so for a curve contained in $\pi(\mathcal{Z})$, $K_{\mathcal{F}}.\, C \geq 0$, cf. [Bo-M]. □

While in many ways highly satisfactory one should bear in mind the following facts about the cone theorem, viz:

**II.4.3 Caveats.** (i) It is a relatively simple consequence of the bend & break estimate II.3.3. Moreover the existence of a stack $\mathcal{L}_0$ with positive Euler characteristic mapping to the rational curve $L_c$ is substantially stronger than the assertion of rationality, and in due course is what rather more refined estimates will be based on.

(ii) The need for canonical singularities is slightly artificial since they're only used to prove the non-negativity of $K_{\mathcal{F}}$ restricted to the singular sub-stack. As such the cone theorem could have been stated with this weaker hypothesis, which, incidentally, is implied by log-canonical in dimension 1.

Finally let us examine the possibilities for $K_{\mathcal{F}}$ negative invariant parabolic stacks under the additional hypothesis that the foliation is smooth and terminal at the non-scheme like points of $\mathscr{X}$. Whence let $f : \mathcal{L} \to \mathscr{X}$ be the normalisation of such a stack, then since it can't factor through the non-scheme like locus of $\mathscr{X}$ it's actually an orbifold. Better still adjunction, I.8.7, gives

$$K_{\mathcal{F}\cdot_f}\, \mathcal{L} = -\chi(\mathcal{L}) + s_{\mathcal{Z}}(\mathcal{L}) - R_f$$

and all points of $\mathcal{Z}$ are scheme like, so: $s_{\mathcal{Z}}(\mathcal{L}) - R_f \geq \#\, f^{-1}(\mathcal{Z})$. In addition if $l_i$ are the non-scheme like geometric points and $d_i$ the order of their stabilisers then,

$$-\chi = -2 + \sum_i \left(1 - \frac{1}{d_i}\right)\,.$$

So the only possibility is that there is at most one non-scheme like point $l \in \mathcal{L}$ with stabiliser $\mathbb{Z}/d$, and exactly one singular point $z \in f^{-1}(\mathcal{Z})$. Consequently, unless $f^{-1}(\mathcal{Z}) = \varphi$, and whence $\mathcal{F}$ is a pencil of rational curves, then we have,

**II.4.4 Fact/Definition.** Hypothesis as per the above discussion, then a $K_{\mathcal{F}}$-negative $\mathcal{F}$-invariant curve, $f : \mathcal{L} \to \mathscr{X}$ is $-1/d\,\mathbb{F}$, i.e. the automorphism group at the unique (if it exists) non-scheme like point $l$ of $\mathcal{L}$, as well as the automorhism group of $f(l)$ is $\mathbb{Z}/d$. In particular $\mathcal{L}$ is the orbifold $\mathbb{P}^1$ with signature $d$ at the said point, and $f$ is an embedding everywhere except possibly at the scheme like point $f^{-1}(\mathcal{Z})$ where it may have a cusp. In any case,

$$K_{\mathcal{F}}.\, \mathcal{L} = -1/d\,,$$

and indeed $\mathbb{Z}/d$ is the holonomy, as per I.5.7, if $\mathscr{X}^f$ is the Gorenstien covering stack of its moduli.

## II.5. Linear Holonomy of Smooth $-\frac{1}{d}\,\mathbb{F}$ Curves

To extract more subtle information on the structure of $-\frac{1}{d}\,\mathbb{F}$ curves, we consider the consequences for the singularities of $(\mathcal{X},\mathcal{F})$ under the hypothesis of smoothness and terminality of $\mathcal{F}$ at the non-scheme like points of $\mathcal{X}$. We begin with the linear holonomy for smooth $-\frac{1}{d}\,\mathbb{F}$ curves. Independently of this observe quite generally that if $\mathcal{C}$ is a smooth embedded substack of $\mathcal{X}$ then descent yields a well defined normal bundle $N_{\mathcal{C}/\mathcal{X}}$ to which we can specialise $\mathcal{F}$. Indeed if $V \to \mathcal{X}$ is étale, and $U$ the pull-back of $\mathcal{C}$ then the pull-back of the associated cone is,

$$\operatorname{Spec} S := \bigoplus_{n=0}^{\infty} \frac{I^n_{U,V}}{I^{n+1}_{U,V}}\,.$$

On the other hand the foliation leaves $I_{U,V}$ invariant, so a local generator $\partial$ of $T_{\mathcal{F}}$ passes to a graded derivation of $S$ by way of applying it to any lifting of an element in the $n^{\text{th}}$-graded piece, and then reducing modulo $I^{n+1}_{U,V}$. This process may not immediately lead to a foliation, but only a pre-foliation, i.e. the specialisation may not be saturated. Let us therefore distinguish the naively specialised foliation by the subscript $p$, i.e. $\mathcal{F}^p$, should it only be a pre-foliation, and reserve the letter $\mathcal{F}$ for saturated foliations. On the other hand the only way to occasion a pre-foliation would be at the singularities of $Z$, while quite generally if $y$ is a coordinate along $U$, and $x_i$ normal coordinates then the specialisation of $\partial$ takes the form,

$$\partial : y \mapsto b(y) = \partial y (\operatorname{mod} I_{U,V})\,, \qquad x_i \mapsto a_{ij}(y)\, x_j = \partial x_i (\operatorname{mod} I_{U,V})$$

where we employ the summation convention. Consequently $\mathcal{F}^p \neq \mathcal{F}$ at a singular point $z$ iff the linearisation $\nabla_z$ of $\partial$ in $\operatorname{End}(N_{U,V}\otimes\mathbb{C}(z))$ is zero. Now consider the particular case of a smooth $-\frac{1}{d}\,\mathbb{F}$ curve $\mathcal{L}$ then by I.5.6 and I.6.11 the linear holonomy is $\mathbb{Z}/e$, for some $e \mid d$, while in an affine, i.e. $\xrightarrow{\sim} \mathbb{A}^1$, neighbourhood of the singular point considered as the origin we can normalise our specialised derivation so that $b(y) = y$. As a first step let us observe,

**II.5.1 Fact.** *The specialised foliation is in fact saturated.*

**Proof.** Indeed suppose otherwise, then on the foliated stack $(N_{\mathcal{L}/\mathcal{X}},\mathcal{F})$ the zero section has flat normal bundle. In particular $e = 1$, so $\mathcal{L} \xrightarrow{\sim} \mathbb{P}(d)$, and there are analytic coordinate functions $x_i$ normal to $\mathcal{L}$ in the neighbourhood of our given affine, $V$, such that $\partial x_i = 0$, so in fact around the singularity the foliation we can take a scheme like open analytic neighbourhood $\Delta$ with coordinate functions $x_i$, $y$ normal and parallel to $\mathcal{L}$ such that,

$$\partial x_i = 0 (\operatorname{mod} I^2_{\mathcal{L}})\,.$$

On the other hand the triviality of the holonomy equates to the existence of functions $\xi_i$ on an analytic étale neighbourhood $U$ of $\mathcal{L}\backslash\{0\}$ such that,

$$\partial \xi_i \equiv 0\,.$$

So without loss of generality we can say that $\xi_i|_V \equiv x_i|_U (\operatorname{mod} I^2_{\mathcal{L}})$. Now proceed inductively supposing that we've found coordinates $x_i$ such that this holds over $U$, $\operatorname{mod} I^n_{\mathcal{L}}$ for $n \geq 2$, and write,

$$\xi_i = x_i + b_{iJ}\, x^J$$

for $b_{iJ}$ analytic functions on $U \times_{\mathcal{L}} V$, $x^J$ monomials of degree $n$, and employ summation convention. On the other hand $\partial x_i|_U$ is zero $\operatorname{mod} I^n_{\mathcal{L}}$, so the same holds over $\Delta$, whence,

$$\partial x_i = c_{iJ}\, x^J$$

where now the $c_{iJ}$ are holomorphic on all of $\mathcal{L}\cap\Delta$. Combining we obtain, for $y = t^d$,

$$b'_{iJ}(t) = -d\, c_{iJ}(t^d)\, t^{-1}\,, \quad \forall\, i, J$$

thus the $c_{iJ}(0)=0$ for all $i, J$, so the $b_{iJ}$ are in fact holomorphic on $\mathcal{L}\cap\Delta$. As a result on the completion $\hat{\Delta}$ of $\Delta$ in $\mathcal{L}$, the foliation is given by a formal fibration $\xi_1\times\cdots\times\xi_n = x_1\times\cdots\times x_n$ over a formal affine space, whence is smooth, which is absurd. $\Box$

As such we can cease to worry about the difference between foliations and pre-foliations. Furthermore the divisor $g=0$ in $N_{\mathcal{L}/\mathcal{X}}$ is invariant by $\mathcal{F}$, and $t\mapsto t^d = y$ defines a covering $\nu:\mathbb{P}^1\to\mathcal{L}$, ramified only in $y=0$, so we even have an induced foliation on $\nu^* N_{\mathcal{L}/\mathcal{X}}$ whose canonical bundle $K_{\mathcal{F}}$ is not only the pull-back of the same on $\mathcal{X}$ but, $\mathcal{O}_{\mathbb{P}^1}(-1)$. Additionally there is an action of $\mathbb{Z}/d$ on this foliation which we identify with an action of $\mu_d$ in the usual way by choosing a $d$-th root of unity $\zeta$, acting on $t$ by, $t\mapsto\zeta t$. Now over the coordinates $\mathbb{A}^1$ containing $\infty$, whose image we identify with the non-scheme like point if it exists, we have a basis $\xi_i$, in the analytic topology, of sections of $\nu^* N^\vee_{\mathcal{L}/\mathcal{X}}$ satisfying $\partial_\infty(\xi_i)=0$, for $\partial_\infty$ a generator of the specialised foliation over the said patch. Such sections are permuted by the Galois group, so we may as well say that they've been chosen so that the action is linear, i.e.

$$(\zeta,\,\xi_i)\mapsto\zeta^{-\frac{b_i d}{e}}\xi_i\,,\quad b_i\in\mathbb{Z}$$

where necessarily $\gcd(b_1,\ldots,b_n,\,e)=1$, for $n$ the codimension of $\mathcal{L}$. Over our original patch containing the origin, however, we can express the $\xi_i$ as functions of the $x_i$'s, i.e.

$$\xi_i = f_{ij}\,x_j$$

where the $f_{ij}$ are analytic functions on $\mathbb{G}_m=\mathbb{P}^1\backslash\{0,\,\infty\}$, necessarily meromorphic at $\infty$, and we employ the summation convention. The matrix $F=[f_{ij}]$ must, however, satisfy the differential equation,

$$F^{-1}\dot{F} = -\frac{d}{t}\,A$$

where $A=[a_{ij}(y)]$, while by [D], the limit of the holonomy about a circle of radius $r$ around the origin as $r\to 0$ is $\exp(-A(0))$, so in the first place, without loss of generality, $A(0)$ is diagonal with eigenvalues $\alpha_i/e$, $\alpha_i\in\mathbb{Z}$. Better still an explicit solution of this equation is furnished by,

$$\exp(-d\log t\,A(0)+B)$$

where $tB$ is a primitive of $-\frac{d}{t}\,(A-A(0))$, so certainly holomorphic. Furthermore any other solution is a constant, i.e. $\mathrm{GL}(n,\,\mathbb{C})$, multiple of this one so if $-\lambda$ is the smallest eigenvalue then $F$ satisfies an estimate of the form,

$$\|F\|\ll\frac{1}{|t|^\lambda}$$

in a neighbourhood of the origin, so the $\xi_i$ are actually meromorphic sections of $\nu^* N^\vee_{\mathcal{L}/\mathcal{X}}$. Naturally we introduce their valuation, i.e.

$$v_i = v(\xi_i) := \min_j v_{ij} := v(f_{ij})\in\mathbb{Z}$$

where $v$ is the order of vanishing/polarity at the origin. Observe that $t^{-v_i}\xi_i$ is a regular section of $\nu^* N^\vee_{\mathcal{L}/\mathcal{X}}$ over our $\mathbb{A}^1\ni 0$ which is non-zero in $N^\vee_{\mathcal{L}/\mathcal{X}}\otimes\mathbb{C}(z)$, and satisfies, $\partial(t^{-v_i}\xi_i) = -v_i/d\,(t^{-v_i}\xi_i)$. Indeed we can certainly order things to insist that,

$$v_1\le v_2\le\cdots\le v_d$$

while we may also note that the Taylor expansion of each $f_{ij}$ is of the form,

$$f_{ij}(t) = \sum_{n+\frac{b_i d}{e}\equiv 0(d)} c_{ijn}\,t^n$$

whence $v_i\equiv -b_i(d)$, and $t^{-v_i}\xi_i$ descends. With this in mind, we assert,

**II.5.2 Claim.** *For a possibly different $\mu_d$ linear basis $\xi_i$ of solutions of $\partial_\infty = 0$, the sections $t^{-v_i}\xi_i$ of $\nu^* N_{\mathcal{L}/\mathcal{X}}$ form a basis of nowhere vanishing sections of $\nu^* N_{\mathcal{L}/\mathcal{X}}$ at the origin, and whence, indeed, everywhere over the $\mathbb{A}^1 \ni 0$.*

**Proof.** The indeed part is clear. We proceed inductively to the effect that we've found a $G$-linear basis $\xi_i$ which respects the ordering of the valuation such that $t^{-v_1}\xi_1, \dots, t^{-v_k}\xi_k$, $1 \le k < n$ are linearly independent at the origin. The case $k = 1$ is clear. Furthermore having the inductive hypothesis in hand we can suppose without loss of generality that $x_i = t^{-v_i}\xi_i$ for $1 \le i \le k$. Now, $t^{-v_j}\xi_j(0)$ is an eigenfactor of the linearisation of $\partial \in \mathrm{End}\,(N_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(0))$ with eigenvalue $v_j/d$, so we're certainly done unless $v_k = v_{k+1}$. Writing things out in terms of our basis $x_i$, we have for any $i > k$,

$$\xi_i = \sum_{v_j < v_k} f_{ij}\, x_j + \sum_{\substack{v_j = v_k \\ j \le k}} f_{ij}\, x_j + \sum_{j > k} f_{ij}\, x_j$$

we again see that we're done unless $v_{ij} > v_k$ for $j > k$, and every $i$ such that $v_i = v_k$. So suppose this happens then for suitable constant functions $c_{ij}$ we can replace $\xi_i$ for $v_i = v_k$ by,

$$\tilde{\xi}_i := \xi_i + \sum_{\substack{v_j = v_k \\ j \le k}} c_{ij}\, \xi_j$$

so that $v(\tilde{\xi}_i) > v_k$ if $v_i = v_k$. Indeed we need only check that $v_{ij} > v_k$ if $v_i = v_k$ and $v_j < v_k$, but this is clear since $t^{-v_i}\xi_i(0)$ is an eigenvector with eigenvalue $v_k/d$, while for $v_j < v_k$, $x_j(0)$ is an eigenvector of smaller eigenvalue. Better still this is a $G$-linear operation, so re-ordering the new basis, $\xi_i$, $i \le k$, $\tilde{\xi}_i$, $v_i = v_k$, $i > k$, $\xi_i$, $v_i > v_k$ according to increasing valuation we're done. $\square$

Now certainly $-v_i/d$ are the eigenvalues $\alpha_i/e$ of the linearisation of $\partial$ in $\mathrm{End}\,(\nu^* N_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(0))$, i.e. $A(0)$, but we've also constructed an isomorphism,

$$\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(v_i) \xrightarrow[\oplus_i\, t^{-v_i}\xi_i]{} \nu^* N^{\vee}_{\mathcal{L}/\mathcal{X}}$$

so in fact $N_{\mathcal{L}/\mathcal{X}} \xrightarrow{\sim} \underset{i}{\oplus}\, \mathcal{O}_{\mathcal{L}}(a_i/d)$, where $a_i = \alpha_i\, \frac{d}{e}$. One should however bear in mind that we have much more, i.e. a meromorphic $1^{\text{st}}$ integral $\xi_1 \times \cdots \times \xi_n$ of the foliation, where $\xi_i$ is thought of as a meromorphic function on the cone, $\rho : \nu^* N_{\mathcal{L}/\mathcal{X}} \to \mathbb{P}^1$, or perhaps better, $t^{-v_i}\xi_i$ is thought of as a global section of $\rho^* \mathcal{O}_{\mathcal{L}}(a_i/d)$, defining an invariant divisor of $N_{\mathcal{L}/\mathcal{X}}$ isomorphic to,

$$\bigoplus_{j \neq i} \rho^* \mathcal{O}_{\mathcal{L}}\left(\frac{a_j}{d}\right).$$

In particular the canonical or Harder-Narismhan filtration of $N_{\mathcal{L}/\mathcal{X}}$ corresponds to an increasing filtration of invariant sub-cones,

$$[0] = N_0 \subsetneqq N_1 \subsetneqq N_2 \subsetneqq \cdots \subsetneqq N_k = N_{\mathcal{L}/\mathcal{X}}$$

such that the normal bundle of $N_i$ in $N_{i+1}$ restricted to the zero section is a trivial bundle twisted by some $\mathcal{O}_{\mathcal{L}}(a_i/e)$ where $a_i$ decreases as one proceeds up the chain.

## II.6. Formal Holonomy

We wish to extend the previous discussion of linear holonomy of smooth $-\frac{1}{d}\,\mathbb{F}$ stacks to the rather more delicate case of formal holonomy. We retake verbatim the notation of the previous section with $(\mathcal{X}, \mathcal{F})$ a non-singular Gorenstien foliated stack such that $\mathcal{F}$ is terminal at the non-scheme like points, and $f : \mathcal{L} \to \mathcal{X}$ a smooth $-\frac{1}{d}\,\mathbb{F}$ curve. Consequently by hypothesis there is a formal analytic neighbourhood $V$ of the $\mathbb{A}^1$

over $\infty$ together with a $\mu_d$ action on the same such that the formal classifying stack $\nu := [V/\mu_d]$ is a formal analytic neighbourhood of $[\mathbb{A}^1/\mu_d]$ over which the foliation trivialises, i.e. on $V$ we have analytic coordinate functions $\xi_i$, $s$ normal and parallel to our $\mathbb{A}^1$ respectively such that over $V$ the foliation is just the formal fibration $\xi_1 \times \cdots \times \xi_n : V \to \hat{\Delta}^n$, where the latter space is a $n$-polydisc completed in the origin. Plainly we can take $s$ to be meromorphic on a formal neighbourhood of $\mathcal{L}$, while the algebra $\mathbb{C}\,[[\xi_1, \ldots, \xi_n]]$ comes equipped with a $\mu_d$ action which, modulo the maximal ideal, is nothing other than that of the linear holonomy. In addition, the said algebra is an inverse limit of finite dimensional vector spaces so the action may be supposed linear and the $\xi_i$ compatible with the basis of II.5.2.

Now suppose, to begin with, that we can choose $\partial$ on a formal neighbourhood $U$ of the $\mathbb{A}^1$ patch around 0 together with a coordinate function $y$ on $U$ restricting to the same on $\mathbb{A}^1$ such that $\partial y \equiv y(\mathcal{I}_{\mathcal{L}}^m)$. Observe, additionally, that on any formal space an element congruent to 1 modulo an ideal of definition has a logarithm. Consequently if $\mathcal{X}_m$, $U_m$, $V_m$, etc. denote the reduction of our various spaces modulo $\mathcal{I}_{\mathcal{L}}^m$, then on extracting a $d^{\text{th}}$-root $t$ of $y$, on which $\mu_d$ acts by multiplication by $\zeta$, we obtain a covering $\mu : \tilde{U}_m \to U_m$ to which $\partial$ lifts and of which $U_m \times_{\mathcal{X}} V_m$ may be identified with an open formal analytic subspace. With this in mind we assert,

**II.6.1 Claim.** *For a possibly different basis $\xi_i$ of the algebra $\mathbb{C}\,[[\xi_1, \ldots, \xi_n]]$, but which is nevertheless compatible with any previous choice of the same modulo $\mathcal{I}_{\mathcal{L}}^{m'}$ for $m' < m$, and which in any case is compatible with the above considerations, the functions $t^{a_i}\xi_i$ define global sections of the $\mathcal{O}_{\mathcal{X}_m}$ module $\mathcal{I}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}^{m+1}(+a_i/d)$, where $\mathcal{O}_{\mathcal{X}_m}\left(\frac{1}{d}\right)$ is the bundle with transition function $t$ on $U_m \times_{\mathcal{X}} V_m$.*

**Proof.** We proceed inductively on $m$, the case $m = 1$ being the already established linear result. Our first concern is what happens over $\tilde{U}_m$. By the induction hypothesis for $m-1$, $m \geq 2$, we can find coordinate functions $x_i$ normal to $\mathcal{L}$ whose reduction modulo $\mathcal{I}_{\mathcal{L}}^2$ are a basis of the normal bundle over $U$ and such that,

$$t^{a_i}\xi_i - x_i \,|_{U_m \times_{\mathcal{X}} V_m} \in \Gamma(U_m \times_{\mathcal{X}} V_m, \mathcal{I}_{\mathcal{L}}^m) = 0\,.$$

Consequently, $\partial x_i - a_i\, x_i$ restricted to $U_m \times_{\mathcal{X}} V_m$ is zero, so in fact it's zero on $\tilde{U}_m$. Whence if we consider the $x_i$ as elements of the $\mathcal{O}_{U_m}$ module $\mathcal{I}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}^{m+1}\,|_{U_m}$, then,

$$\partial x_i = \frac{a_i}{d}\, x_i + a_{iJ}(t)\, x^J$$

where $x^J$ is the monomial $x_1^{j_1} \ldots x_n^{j_n}$, $j_1 + \cdots + j_n = m$, the summation convention is employed, and $a_{iJ}(t)$ is a holomorphic function of $t$. On the other hand,

$$t^{a_i}\xi_i = x_i + b_{iJ}(t)\, x^J$$

with the same conventions, but where, now, $b_{iJ}$ is only holomorphic over our $\mathbb{A}^1$ minus the origin, which for convenience we'll denote $\mathbb{G}_m$. Combining we obtain,

$$t\,\dot{b}_{iJ} + b_{iJ}(a_J - a_i) = -a_{iJ}\, d \in \mathcal{O}_{\mathbb{G}_m}$$

where $a_J = \sum_i j_i\, a_i$, and no summation is implied. Again we can explicitly integrate this, by way of,

$$\frac{d}{dt}\,(t^{(a_J - a_i)}\, b_{iJ}) = -d\, a_{iJ}\, t^{a_J - a_i - 1}\,.$$

As such the $b_{iJ}$ are certainly meromorphic, and no $a_{iJ}\, t^{a_J - a_i - 1}$ has a residue, from which we deduce,

$$b_{iJ} = h_{iJ}(t) + \frac{\lambda_{iJ}}{t^{a_i - a_J}}$$

where $h_{iJ}$ is holomorphic, and $\lambda_{iJ}$ is a constant. As a result if we replace $\xi_i$ by,

$$\tilde{\xi}_i := \xi_i - \lambda_{iJ}\xi^J$$

where we sum over $J$ with $a_i > a_J$, and observe that $a_J \equiv 0(d)$ if $\lambda_{iJ} \neq 0$, we see that the $\tilde{\xi}_i$ form a $\mu_d$-linear basis of $\mathcal{I}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}^{n+1}\mid_{V_m}$ compatible with our previous choices such that $t^{a_i}\tilde{\xi}_i$ are holomorphic on $\tilde{U}_m$. Better still on replacing the $\xi_i$ by $\tilde{\xi}_i$, the $t^{a_i}\xi_i = x_i$ are invariant by $\mu_d$, so if $\mathcal{O}_{\mathcal{X}_m}\left(\frac{1}{d}\right)$ is the bundle on the $m^{\text{th}}$-thickening of $\mathcal{L}$ with transition function $t$ on the overlap $U_m \times_{\mathcal{X}} V_m$ then we have global sections as claimed. □

Before proceeding further let's clear up what's analytic, and what's algebraic. In the first place provided $t$ is algebraic, the bundle $\mathcal{O}_{\mathcal{X}_m}\left(\frac{1}{d}\right)$ is algebraic, i.e. given by a transition function in $\Gamma(\mathcal{O}^*_{U_m \times_{\mathcal{X}} V_m})$, albeit that the said function may not lead in the limit to a meromorphic function on the completion of $\mathcal{X}$ in $\mathcal{L}$. Now given the algebraicity of $t$, and using GAGA for the case $m = 1$, the proof shows that the $t^{a_i}\xi_i$ are algebraic on $U_m$ while a similar, and rather easier, induction at $\infty$ shows that the $\xi_i$ are algebraic on $V_m$. In any case let us proceed to find an appropriate $t$ by re-considering the situation mod $\mathcal{I}_{\mathcal{L}}^{m+1}$, $m \geq 1$. Over $U$ we have, in the notation/spirit of the proof,

$$\partial y = y + c_J\, x^J (\mathcal{I}_{\mathcal{L}}^{m+1})\,, \quad \partial x_i = \frac{a_i}{d}\, x_i + c_{iK}\, x^K (\mathcal{I}_{\mathcal{L}}^{m+2})$$

where the summation convention is back in force, with respect to multi-indices $J$ and $K$ of degrees $m$, $m+1$ respectively, and all the $c_*$'s are regular algebraic functions of $y$. Over $U_{m+2} \times_{\mathcal{X}} V_{m+2}$, $y$ has a $d^{\text{th}}$-root, as ever denoted $t$, with standard $\mu_d$ action, so without loss of generality,

$$\xi_i \mid_{U_{m+2}\times_{\mathcal{X}} V_{m+2}} = t^{-a_i}\, x_i + b_{iK}\, x^K \mid_{U_{m+2}\times_{\mathcal{X}} V_{m+2}} (\operatorname{mod} \mathcal{I}_{\mathcal{L}}^{m+1})$$

where of course $b_{iK}$ may only be analytic functions of $t$ defined on our $\mathbb{G}_m$, and summation over the multi-index $K$ of degree $m+1$ is implied. Combining these, yields for each multi-index $J$ of weight $m$, and any $i$,

$$a_i\, c_J\, t^{a_J-(d+1)} = \sum_{x^K = x^J x_i} d\, c_{iK}\, t^{(a_J-1)} + \frac{d}{dt}\left(t^{a_K}\, b_{iK}\right).$$

Consequently, given we know the existence of a non-zero eigenvalue by II.5.1, if $a_J = d \geq 1$, then $c_J(0) = 0$. Consider on the other hand the obstruction to finding a coordinate $\tilde{y}$ over $U$ restricting to the same on $\mathcal{L}$ such that,

$$\partial \tilde{y} = (1+\lambda)\, \tilde{y}\ (\mathcal{I}_{\mathcal{L}}^{m+1})\,, \quad \lambda = \lambda_J\, x^J \in \mathcal{I}_{\mathcal{L}}^m\,, \quad \lambda_J \in \mathcal{O}_{U\cap\mathcal{L}}\,.$$

If we look for such a $\tilde{y}$ in the form, $y + \Lambda_J\, x^J$, with $\Lambda_J$ constants, then we require to solve,

$$\left(\frac{a_J}{d} - 1\right)\Lambda_J - \lambda_J\, y = -c_J$$

for all $J$. However if $a_J \neq d$, then $\Lambda_J = -c_J(0)\left(\frac{a_J}{d} - 1\right)^{-1}$, and $\lambda_J$ whatever, will do, while if $a_J = d$, then we can freely take $\Lambda_J = 0$, and $\lambda_J = c_J\, y^{-1}$. Whence we inductively obtain a compatible system of coordinates $y_m$, and generators $\partial_m$ of the foliation over $U_m$ such that $\partial_m\, y_m = y_m$. In consequence passing to the limit we obtain that the canonical/Harder-Narismhan filtration of $N_{\mathcal{L}/\mathcal{X}}$ extends to the entire formal neighbourhood, i.e.

**II.6.2 Proposition/Summary.** *Let $\hat{\mathcal{X}}$ be the formal stack obtained by completing $\mathcal{X}$ in $\mathcal{L}$ then there is a bundle $\mathcal{O}_{\hat{\mathcal{X}}}\left(\frac{1}{d}\right)$ extending $\mathcal{O}_{\mathcal{L}}\left(\frac{1}{d}\right)$ and indeed $\mathcal{O}_{\hat{\mathcal{X}}}(D) = \mathcal{O}_{\hat{\mathcal{X}}}(1)$ for $D$ a smooth formal invariant divisor transverse to $\mathcal{L}$ passing through the unique point $z$ of $\mathcal{L} \cap \operatorname{sing}(\mathcal{F})$, together with a filtration of formal invariant sub-stacks,*

$$\mathcal{L} = \mathcal{X}_0 \subsetneqq \mathcal{X}_1 \subsetneqq \cdots \subsetneqq \mathcal{X}_k = \hat{\mathcal{X}}$$

*such that if $\frac{\alpha_1}{e} > \cdots > \frac{\alpha_k}{e}$ are the distinct eigenvalues of $\partial$ considered linearised in $\operatorname{End}(N_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(z))$ with $n_1, \ldots, n_k$ the dimensions of the corresponding eigenspaces then $\mathcal{X}_i$ is defined by $\mathcal{F}$-invariant $\mathcal{O}_{\hat{\mathcal{X}}}$-global sections $\gamma_j$ of $\mathcal{O}_{\hat{\mathcal{X}}}(\frac{\alpha_j}{e})$, $j > i$, and $n_j$-sections for each $j$. In particular,*

$$N_{\mathcal{L}/\mathcal{X}_i} \xrightarrow{\sim} \bigoplus_{j\leq i} \mathcal{O}_{\mathcal{L}}\left(\frac{\alpha_j}{e}\right)$$

*and for $\mathcal{X}$ almost étale over its moduli, the automorphism group of the non-scheme like point of $\mathcal{L}$ is generated by $\prod_j \exp(2\pi\sqrt{-1}\,\alpha_j/d) \subset (\mathbb{C}^\times)^k$.*

## II.7. Jordan Decomposition

We briefly interupt our discussion of $K_{\mathcal{F}}$-negative invariant stacks to recall some salient facts on Jordan decomposition which will be relevant both to our study of cusps, and the local uniqueness of the Harder-Narismhan filtration. The situation is entirely local and scheme-like, i.e. $\mathcal{O}$ is the ring of formal power series $\mathbb{C}\,[[x_1,\ldots,x_n]]$, $\mathfrak{m}$ its maximal ideal, and $\partial$ a $\mathbb{C}$-derivation of $\mathcal{O}$ with a singularity at the origin. Recall that since $\mathcal{O}$ is an inverse limit of finite dimensional vector spaces $\partial$ admits a Jordan decomposition, i.e. $\partial = \partial_S + \partial_N$, where the semi-simple part $\partial_S$ acts as a semi-simple matrix on each $\mathcal{O}/\mathfrak{m}^n$, $n \in \mathbb{N}$, $\partial_N$ is nilpotent, and of course $[\partial_S, \partial_N] = 0$. In particular if $\partial_S = \lambda_i\, x_i\, \frac{\partial}{\partial x_i}$, summation convention, then a conventional choice of basis for the nilpotent fields commuting with $\partial_S$ is,

**II.7.1 Recollection.** (cf. [Ma]) Notations as above then $\partial_N = \sum\limits_{i=1}^{n} \sum\limits_{Q_i} a_{Q_i}\, x^{Q_i}\, x_i\, \frac{\partial}{\partial x_i}$, $a_{Q_i} \in \mathbb{C}$, where either,

(i) $Q_i = (q_1,\ldots,q_n)$, $q_j \in \mathbb{N}\cup\{0\}$, $x^{Q_i} = x_1^{q_1}\ldots x_n^{q_n}$, $\Lambda\cdot Q_i = 0$, or

(ii) $Q_i = (q_1,\ldots,q_n)$, $q_i = -1$, $q_j \in \mathbb{N}\cup\{0\}$, $j \neq i$, $x^{Q_i} = x_1^{q_1}\ldots x_n^{q_n}$, $\Lambda\cdot Q_i = 0$.

Now the Jordan decomposition of a vector field is certainly unique, and whence the property of semi-simplicity of a vector field is wholly unambiguous. For a foliation however the situation is rather more delicate since there is a question of rescaling by units. Whence suppose our field $\partial$ is semi-simple, and consider a field $\tilde{\partial} = u\partial$, where $u \equiv 1(\mathfrak{m})$ to avoid stupidity. Furthermore let's say, without loss of generality, that $\partial = \lambda_i\, x_i\, \frac{\partial}{\partial x_i}$ then we assert,

**II.7.2 Claim.** *There is a change of coordinates of the form, $\xi_i = u_i\, x_i$, $u_i \equiv 1(\mathfrak{m})$, and $\varepsilon \equiv 0(\mathfrak{m})$ with $\partial\varepsilon = 0$ such that the Jordan decomposition of $\tilde{\partial}$ is,*

$$\tilde{\partial} = \lambda_i\, \xi_i\, \frac{\partial}{\partial \xi_i} + \varepsilon\, \lambda_i\, \xi_i\, \frac{\partial}{\partial \xi_i}$$

*i.e. $\tilde{\partial}$ may not be semi-simple, but the extent to which it is not is very particular.*

**Proof.** Consider the following inductive proposition for $k \in \mathbb{N}$,
there are coordinates $x_{ik} = u_{ik}\, x_i$, $u_{ik} \equiv 1(\mathfrak{m})$, $\tilde{\partial} = u_k\, \partial_k$, $\partial_k = \lambda_i\, x_{ik}\, \frac{\partial}{\partial x_{ik}}$, $u_k \equiv 1(\mathfrak{m})$ such that $u_k^{-1} = 1+\varepsilon_k+\delta_k$, where $\varepsilon_k$, $\delta_k$ are defined by way of the Jordan decomposition of $\mathfrak{m}$ as $\operatorname{Ker}\partial_k \oplus \operatorname{Im}\partial_k$, and $\delta_k \in \mathfrak{m}^k$. The case $k=1$ is simply our given data. Otherwise consider trying to improve the situation by putting, $x_{ik+1} = v_{ik}\, x_{ik}$, $v_{ik} \equiv 1(\mathfrak{m})$ to be chosen. If such a change were to actually render the situation semi-simple then we would have to solve,

$$\partial_k \log v_{ik} = \lambda_i \left(\frac{1}{u_k} - 1\right) = \lambda_i(\varepsilon_k + \delta_k)$$

which plainly may not be possible if $\lambda_i \neq 0$, and $\varepsilon_k \neq 0$. However we can solve $\partial_k \log v_{ik} = \lambda_i\, \delta_k$, so that in particular, $v_{ik} \equiv 1(\mathfrak{m}^k)$, while in the new coordinates,

$$\tilde{\partial} = \frac{1+\delta_k}{1+\varepsilon_k+\delta_k}\, \lambda_i\, x_{ik+1}\, \frac{\partial}{\partial x_{ik+1}}$$

which is indeed what we're looking for, since putting $u_{k+1} = (1+\delta_k)\, u_k$ then,

$$u_{k+1}^{-1} = 1 + \varepsilon_k(1+\delta_k)^{-1} = 1 + \varepsilon_k + \sum_{n=1}^{\infty} (-1)^n\, \varepsilon_k\, \delta_k^n$$

so that $\delta_{k+1} \in \mathfrak{m}^{k+1}$.

Certainly therefore the $\delta_k \to 0$, but the proof also shows that for each $i$ the infinite product, $\prod_k v_{ik}$ converges to some $u_i$, so putting $\xi_i = u_i\, x_i$ we're certainly done on observing that $\partial \varepsilon = 0$ obliges,

$$\left[\lambda_i\, \xi_i\, \frac{\partial}{\partial \xi_i}\, ,\ \varepsilon\, \lambda_i\, \xi_i\, \frac{\partial}{\partial \xi_i}\right] = 0\,.$$

□

The consequence of the fact that not only can Jordan decomposition of a rescaling of semi-simple only fail in a very controlled way, but also that Jordan decompositions of rescalings are related in a very simple way suggests that we introduce,

**II.7.3 Definition.** A germ of a foliation $(\hat{\mathbb{A}}^n,\, \mathcal{F})$ on a formal affine space with a not necessarily isolated singularity at the origin is said to be semi-simple, if $T_{\mathcal{F}} = \mathcal{O}_{\hat{\mathbb{A}}^n} \partial$ for some semi-simple vector field $\partial$.

As an important example/application consider the situation of blowing up in the origin, i.e. $\rho : (X, \tilde{\mathcal{F}}) \to (\hat{\mathbb{A}}^n,\, \mathcal{F})$ is the said modification with induced foliation and $X$ is the completion in the exceptional divisor of the blow up of $\operatorname{Spec} \mathcal{O}$. Denoting by, $\partial = \partial_S + \partial_N$ a Jordan decomposition of any generator $T_{\mathcal{F}}$ we have,

**II.7.4 Further Fact.** *Suppose $\partial_S \neq 0$ and $(X, \tilde{\mathcal{F}})$ is not everywhere smooth (which in any case could only happen if in suitable coordinates $\partial = x_i\, \frac{\partial}{\partial x_i}$) then the following are equivalent,*

1. *$(\hat{\mathbb{A}}^n,\, \mathcal{F})$ is semi-simple.*
2. *$(X,\, \tilde{\mathcal{F}})$ is semi-simple at all of its singular points.*
3. *$(X,\, \tilde{\mathcal{F}})$ is semi-simple at one of its singular points, and $(\hat{\mathbb{A}}^n,\, \mathcal{F})$ is semi-simple modulo $\mathfrak{m}^2$.*
4. *$(X,\, \tilde{\mathcal{F}})$ is semi-simple at one of its singular points, which is itself a singular point of $\rho^*\, \partial_S$, and $\rho^*\, \partial_S$ is semi-simple at the said point.*

**Proof.** Since $(X,\, \tilde{\mathcal{F}})$ is not everywhere smooth the induced foliation is given everywhere by $\rho^*\, \partial$ (cf. I.6.2) so trivially (1) ⇒ (2) ⇒ (3) ⇒ (4). Consider therefore (4) ⇒ (1). Write $\partial_S = \lambda_i\, x_i\, \frac{\partial}{\partial x_i}$, and say, without loss of generality, the singularity finds itself on the $x_1 \neq 0$ patch, at a point $p$. At the completion of the local ring $\mathcal{O}_{X,p}$ in $\mathfrak{m}(p)$, by hypothesis $\rho^*\, \partial_S + \rho^*\, \partial_N$ is a Jordan decomposition of $\rho^*\, \partial$. However, there is some generator $\tilde{\partial}$ at $p$ which is semi-simple, so an application of II.7.2 yields $\varepsilon \in \hat{\mathcal{O}}_{X,p}$ such that $\rho^*\, \partial(\varepsilon) = 0$, and,

$$\varepsilon\, \rho^*\, \partial_S = \rho^*\, \partial_N\,.$$

Consider, therefore the following cases,

(a) $\lambda_1 \neq 0$, with $\partial_N(x_1) = f$, then, $\lambda_1\, x_1\, \varepsilon = \rho^* f$, from which,

$$0 = \partial(f/x_1) = \frac{\partial f}{x_1} - f/x_1^2(\lambda_1\, x_1 + f)$$

so $x_1 \mid f$, and $\varepsilon$ is actually a function on $\hat{\mathbb{A}}^n$, from which we conclude, or

(b) $\lambda_1 = 0$, and without loss of generality $\lambda_2 \neq 0$, with $\partial_N(x_2) = f$, from which $\varepsilon\, \lambda_2\, x_2 = f$, so as before $x_2 \mid f$, and $\varepsilon$ comes from $\hat{\mathbb{A}}^n$. □

A further question which we may reasonably address here is the uniqueness, or lack thereof, of the Jordan decomposition. Even without rescaling the particular choice of coordinates in which we may write a semi-simple field as $\lambda_i\, x_i\, \partial/\partial x_i$ may be catastrophically non-unique. Plainly the worst possible case is when all the $\lambda_i$ are rational, or equivalently up to a harmless rescaling integers. Even this is of course not unique but it's not too bad since of course any rational point in some $\mathbb{P}^N(\mathbb{Q})$ is up to multiplication by $\pm 1$ uniquely represented by a tuple of relatively prime integers, consequently let's establish some notation,

**II.7.5 Notation.** Let $\partial$ be a semi-simple derivation of $\mathcal{O}$ with integer eigenvalues $a_1, \dots, a_r, -b_1, \dots, -b_t$, $a_i, b_j \in \mathbb{N}$, $s$ zeroes, $r \geq 1$, although possibly $t = 0$, i.e. no negatives, and $(a_1, \dots, a_r, b_1, \dots, b_t) = 1$, then we will suppose these ordered by decreasing size, i.e.

$$a_1 \geq a_2 \geq \cdots \geq a_r \geq 0 \geq -b_1 \geq \cdots \geq -b_t$$

and by $\alpha_1, \dots, \alpha_k$, $k \leq r$, $\beta_1, \dots, \beta_l$, $l \leq t$ a complete repetition free list of the same, so that,

$$a_1 = \alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$$

$$0 > -b_1 = -\beta_1 > -\beta_2 > \cdots > -\beta_l\,.$$

Now for a given choice of basis of a semi-simple derivation $\partial$ with the said eigenvalues i.e. a particular way of writing it as $a_i\, y_i\, \frac{\partial}{\partial y_i} - b_j\, x_j\, \frac{\partial}{\partial x_j}$, we can introduce,

**II.7.6 Definition.** The Harder-Narismhan pair of $(\hat{\mathbb{A}}^n, \mathcal{F})$ with respect to the data $(\partial, y_i, x_j)$ is the invariant formal sub-schemes, $X_+$, $X_-$ whose ideals are generated by the non-positive, respectively non-negative, eigenvectors of $\partial$. If instead we take strictly negative, respectively strictly positive, eigenvectors then the resulting subschemes, denoted $X_+^{\geq 0}$, $X_-^{\leq 0}$, will be called the non-negative Harder-Narismhan pair.

Manifestly, apart from abreviating Harder-Narismhan to H-N, the important thing is that the H-N pairs are well defined up to $\pm 1$, i.e.

**II.7.7 Fact.** *Fix a choice of semi-simple $\partial$ with integer eigenvalues normalised as per II.7.5, then the following are equivalent,*

(1) *$\{X_+, X_-\}$, respectively $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, is the H-N, resp. non-negative, pair with respect to $\partial$ in the basis $\{x_i, y_j\}$.*

(2) *$\{X_+, X_-\}$, resp. $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, is the H-N, resp. non-negative, pair with respect to $\partial$ in any semi-simple basis.*

(3) *$\{X_+, X_-\}$, resp. $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, is the H-N, resp. non-negative, pair of any semi-simple $\tilde{\partial} = u\partial$ in any semi-simple basis for the same, where $u \equiv 1(\mathfrak{m})$.*

**Proof.** (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are all trivial, so consider (1) $\Rightarrow$ (3). By II.7.2, we know that we can find units $u_i$, $v_j \equiv 1(\mathfrak{m})$ such that if $\eta_i = u_i\, y_i$, $\eta_j = v_j\, x_j$ then $\tilde{\partial} = a_i\, \eta_i\, \frac{\partial}{\partial \eta_i} - b_j\, \xi_j\, \frac{\partial}{\partial \xi_j}$. As such $\{X_+, X_-\}$, resp. $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, is the H-N, resp. strict, pair of $\tilde{\partial}$ in the basis $\{\xi_i, \eta_j\}$. Now suppose $\tilde{\partial} = a_i\, f_i\, \frac{\partial}{\partial f_i} - b_j\, g_j\, \frac{\partial}{\partial g_j}$ in some other basis $f_i$, $g_j$. At the mod $\mathfrak{m}^2$ level this is just a question of the uniqueness of diagonalisation/the commutator of a diagonal matrix, so without loss of generality let's say $f_i \equiv \xi_i$, and $g_j \equiv \eta_j(\mathfrak{m}^2)$. For higher order terms, consider the Taylor expansion,

$$f_i = \xi_i + \sum_{\#J + \#K \geq 2} c_{iJKL}\, \xi^J\, \eta^K\, \zeta^L\,,$$

where, as ever, $\xi^J$ etc. is the monomial $\xi_1^{j_1} \dots \xi_r^{j_r}$, and $\zeta_1, \dots, \zeta_s$ are the null vectors. Now $\tilde{\partial}\, f_i = a_i\, f_i$ so $\tilde{\partial}\, \xi^J = a_i$, and whence,

$$c_{iJKL} \neq 0 \Rightarrow \sum_\alpha a_\alpha\, j_\alpha - \sum_b b_\beta\, k_\beta = a_i\,.$$

Consequently if $f_i \notin (\xi_1, \ldots, \xi_r)$, then we have a manifest absurdity, and so conclude by symmetry. $\square$

The dependence on $\pm 1$ is, however, unavoidable, except in the sense of,

**II.7.8 Summary/Definition.** Let $(\hat{\mathbb{A}}^n, \mathcal{F})$ be a germ of a singular semi-simple foliation such that the eigenvalues of a linearisation in $\mathfrak{m}/\mathfrak{m}^2$ are in $\mathbb{P}^{n-1}(\mathbb{Q})$ then there are two canonical pairs (as opposed to two pairs of canonical) of invariant subschemes in the form of the H-N pair, $\{X_+, X_-\}$, and the non-negative H-N pair $\{X_+^{\geq 0}, X_-^{\leq 0}\}$. The former intersecting in the origin, the latter in the whole singular locus. If no-confusion is likely, the suffices may be dropped.

As to why this is in any way of interest, the point is,

**II.7.9 Reason for the above discussion.** Let $(\mathcal{X}, \tilde{\mathcal{F}})$ be a smooth foliated stack with terminal singularities at the non-scheme like points, and $f : \mathcal{L} \to \mathcal{X}$ any smooth $-1/d\,\mathbb{F}$ curve, containing the wholly scheme like singular point $z$, then after completion at $z$, $\mathcal{L}$ lies in precisely one of $\mathcal{X}_+$ or $\mathcal{X}_-$, respectively, $\mathcal{X}_+^{\geq 0}$, $\mathcal{X}_-^{\leq 0}$. An important further task will be to extend this to cusps.

It's also the case, and for more or less the same reason, that $X_+$, say, can be canonically filtered as,

$$0 = X_+^0 \subsetneqq X_+^1 \subsetneqq \cdots \subsetneqq X_k^+ \subsetneqq X_{k+1}^+ = X^+$$

where $X_+^i$ has for ideal eigenvectors of eigenvalue $> \alpha_i$, and similarly for $X_-$, $X_+^{\geq 0}$, $X_-^{\leq 0}$. This is, however, not so important, nor is it true that the filtration could be extended to all of $\hat{\mathbb{A}}^n$.

## II.8. Cusps

We consider the consequences of the previous discussion for cuspidal $-\frac{1}{d}\,\mathbb{F}$ curves, so as ever $(\mathcal{X}, \mathcal{F})$ is a non-singular foliated Gorenstien stack, with $\mathcal{F}$ terminal at the non-scheme like points. As such a cusp can only occur at a scheme like point. Consequently to begin with let us simply consider the situation of a map $f : \mathbb{A}^1 \to X$ to a smooth scheme having a cusp at the origin. The important thing is the embedding dimension, $k$, say, so actually we simply want to consider a formal $\hat{\mathbb{A}}^1$ mapping to a formal $\hat{\mathbb{A}}^k$, by way of a cusp of embedding dimension $k$. Denoting by $v$ the order function at the origin of the $\hat{\mathbb{A}}^1$ we can associate to such a data a prefered system of coordinates $y_1, \ldots, y_k$ with valuations $v_1, \ldots, v_k$ such that,

$$v_1 < v_2 < \cdots < v_k\,, \quad \text{and } v_{i+1} \notin \mathbb{Z}_{\geq 0}\, v_1 + \cdots + \mathbb{Z}_{\geq 0}\, v_k\,.$$

Indeed the only non-trivial point is that the given topology on $\mathcal{O}_{\hat{\mathbb{A}}^k}$ is the same as the valuation topology, but this is a theorem of Chevalley. Now suppose, further, that the situation is invariant by a formal vector field $\delta$ then by algorithmic resolution $f^*\delta$ is again a vector field, so for $\eta$ a coordinate function on the $\hat{\mathbb{A}}^1$ we can normalise $\delta$, up to scalar multiplication, in such a way that,

$$v(f^*\delta\,\eta - \eta) \geq 2$$

provided that the Segre class of $\delta$ with respect to $f$ at the origin is 1. A fact that we will, naturally enough, be able to suppose. As such the basis $y_i$ enjoys the property that, $v(\delta y_i) = v_i$, so if $a_{ij} \in gl(k, \mathbb{C})$ is the matrix corresponding to the linearisation of $\delta$ at the origin in this basis then the property that, $v_{i+1} \notin \mathbb{Z}_{\geq 0}\, v_1 + \cdots + \mathbb{Z}_{\geq 0}\, v_k$ imply that $a_{ij}$ is upper semi-triangular with diagonal $v_1, \ldots, v_k$, so without loss of generality we may actually suppose that it's diagonal. Better still the said condition that $v_{i+1}$ is not a non-negative sum of the $v_i$ is exactly what's required to guarantee a linearisation of $\delta$ at the origin, so we may as well say that $\delta y_i = v_i\, y_i$ in $\mathcal{O}_{\hat{\mathbb{A}}^k}$. Consequently after a linear diagonal change of coordinates we can even assert that the cusp takes the form,

$$\eta \mapsto (\eta^{v_1}, \ldots, \eta^{v_k})\,.$$

As such the Euclidean algorithm for $v_i$ resolves the cusp by a sequence of blow ups such that the order of vanishing of the proper transform of $f$ at the unique point where it crosses the exceptional divisor is the gcd of $v_1, \ldots, v_k$, so in fact the $v_1, \ldots, v_k$ are relatively prime.

Now let's return to the specifics of $-1/d\,\mathbb{F}$ curves which are cusps of embedding dimension $k$. Locally about the singular point we can find coordinate functions $x_1, \ldots, x_\ell$ vanishing on the cusp, and embedding coordinates $y_1, \ldots, y_k$. As such the $\mathcal{F}$-invariant ideal of the cusp is of the form $I = (x_i,\, g_i(y))$, while a local generator $\partial$ of the foliation takes the form,

$$\partial = a_i\,\frac{\partial}{\partial x_i} + (b_j + c_j)\,\frac{\partial}{\partial y_j}$$

where $a_i \in I$, $b_j \in (x_1, \ldots, x_\ell)$, $c_j$ is a pure function of the $y$'s, and we employ the summation convention. Consequently the projection of the cusp onto the $y$-plane is invariant by the field $\delta = c_j\,\frac{\partial}{\partial y_j}$, which of course satisfies our Segre class 1-assumption. Consequently in the completion $\hat{\mathcal{O}}_{\mathcal{X},z}$ of $\mathcal{O}_{\mathcal{X},z}$ in $\mathfrak{m}_{\mathcal{X}}(z)$ we can suppose that the $y_i$ and their valuations $v_i$ have the form of the previous discussion, and of course we normalise our generator $\partial$ so that $v(\partial\eta - \eta) \geq 2$ for $\eta$ a coordinate on the normalisation of the cusp.

Unsurprisingly our immediate concern is to prove that $\mathcal{F}$ is semi-simple at $z$. However if we are to do this by way of II.7.4, then we'll require a certain minimum amount of information about the comportment mod $\mathfrak{m}_{\mathcal{X}}(z)^2$. To this end consider the exact sequence,

$$0 \to N_f \to f^*\,\Omega_{\mathcal{X}} \to \omega_{\mathcal{L}}(-R_f) \to 0$$

where $R_f$ is the ramification, and $N_f$ by definition the kernel of the map on the right. If we denote by $\mathcal{L}^*$, $\mathcal{L} - 0$, where $f(0) = z$, then we know that specialising the foliation corresponds to a sort of connection,

$$\nabla : N_f \otimes \mathcal{L}^* \to N_f \otimes \mathcal{L}^*$$

we say sort of since usually a connection satisfies, $\nabla(gn) = dg \cdot n + g\nabla n$, for $g$ a function and $n$ a section, whereas locally ours satisfies $\nabla(gn) = f^*\partial(g)\cdot n + g\nabla n$, where $f^*\partial$ is the necessarily regular (by virtue of algorithmic resolution) derivation of $\mathcal{L}$ obtained by restricting a generator of $\mathcal{F}$. In any case what's required is that $\nabla$ should extend over the puncture. Now by virtue of considering the situation on a resolution $\rho : \tilde{X} \to X$ obtained by blowing up in points according to the Euclidean algorithm it's certainly the case that the situation is at worst meromorphic, i.e. there is a map,

$$\bar{\nabla} : N_f \to N_f \otimes \mathcal{O}_{\mathcal{L}}(n0)$$

for $n \in \mathbb{N}$, satisfying, $\bar{\nabla}(gn) = f^*\partial(g)\,n + g\bar{\nabla}n$, with $g$, $n$, $f^*\partial$ etc. as before. Consequently to check that we can actually take $n = 0$ we can simply work in the complete local rings $\hat{\mathcal{O}}_{\mathcal{X},z}$ and $\hat{\mathcal{O}}_{\mathcal{L},0}$. We proceed by brute force, viz:

**II.8.1 Step 1.** Just as in the case where the embedding dimension is the dimension we can for possibly different $y_i$'s but which nevertheless have the same valuations achieve a linearisation of $\partial$ in $\hat{\mathcal{O}}_{\mathcal{X},z}$ such that, $c_j = v_j\,y_j$.

**II.8.2 Step 2.** Again as per the case $l = 0$, we may as well therefore say that the cusp has the form,

$$\eta \mapsto (\eta^{v_1}, \ldots, \eta^{v_k}, 0, \ldots, 0)$$

so that generators of $N_f \otimes \hat{\mathcal{O}}_{\mathcal{L},0}$ take the form, $dx_1, \ldots, dx_\ell$, $dy_i - \frac{v_i}{v_1}\,\eta^{v_i - v_1}\,dy_1$, $2 \leq i \leq k$, and $f^*\partial(\eta) = \eta$.

**II.8.3 Step 3.** $f^*\,I/I^2$ certainly maps to $N_f$, so just calculate. The case of the $x_i$'s is automatic since, as we've noted, $da_i \in N_f$, whereas,

$$\begin{aligned}\nabla\left(dy_i - \frac{v_i}{v_1}\,\eta^{v_i-v_1}dy_1\right) &= f^*(db_i + v_i\,dy_i) - \frac{v_i}{v_1}\,(v_i - v_1)\,\eta^{v_i-v_1}dy_1 - \frac{v_i}{v_1}\,\eta^{v_i-v_1}f^*(db_1 + v_1dy_1)\\ &= v_i\left(dy_i - \frac{v_i}{v_1}\,\eta^{v_i-v_1}dy_1\right) \quad \bmod f^*I/I^2\end{aligned}$$

with this out of the way we can therefore use our knowledge of the linear holonomy to conclude that the induced linearisation $\bar{\nabla}_0 \in \mathrm{End}\,(N_f \otimes \mathbb{C}(0))$ is semi-simple, and whence for possibly different coordinates $x_i$, $y_i$ but nevertheless satisfying the same hypothesis on embedding and valuation, the Jordan decomposition of $\partial$ in $\hat{\mathcal{O}}_{\mathcal{X},z}$ takes the form $\partial = \partial_S + \partial_N$, where

$$\partial_S = v_i\, y_i\, \frac{\partial}{\partial y_i} + a_j\, x_j\, \frac{\partial}{\partial x_j}\,, \quad \partial_N = \varepsilon\, x_1\, \frac{\partial}{\partial y_1} + \delta\,, \quad \delta \in \mathfrak{m}^2\, \mathrm{Der}(\hat{\mathcal{O}}_{\mathcal{X},z})$$

and $\varepsilon = 0$ or 1, so in the latter case $a_1 = v_1$. As such we're already pretty much done by II.7.4, unless indeed $\varepsilon = 1$. In any case consider a blow up $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$, formal or otherwise, in $z$ then the proper transform $\tilde{f}$ finds itself at the origin $\tilde{z}$, say, in the standard coordinate system, of the $y_1 \neq 0$ patch. In particular, by direct calculation, $\rho^*\partial_S + \rho^*\partial_N$ is still the Jordan decomposition of $\rho^*\partial$, so by a descending induction on the number of blow ups required to resolve the cusp, we obtain,

**II.8.4 Fact.** *If $f : \mathcal{L} \to \mathcal{X}$ is a $-1/d\mathbb{F}$ cusp then there are formal coordinates $x_i$, $y_j$ about the singular point $z$, with the latter embedding coordinates of valuation $v_j \notin \mathbb{Z}_{\geq 0}\, v_1 + \cdots + \mathbb{Z}_{\geq 0}\, v_{j-1}$, $1 \leq j \leq k$, $\gcd(v_1, \ldots, v_k) = 1$, along with integers $a_i\, e$ enjoying $\gcd(e, a_1, \ldots, a_\ell) = 1$, such that a suitable formal generator $\partial$ of $T_{\mathcal{F}}$ takes the form,*

$$\partial = \frac{a_i}{e}\, x_i\, \frac{\partial}{\partial x_i} + v_j\, y_i\, \frac{\partial}{\partial y_j}\,.$$

In particular, if in the notation of II.7.6, $X_+$, $X_-$ are the H-N pair of formal subschemes of the completion of $\mathcal{X}$ in $z$, then the formal germ of the cusp at $z$ is in either $X_+$ or $X_-$, while if $\mathcal{X}$ is the Gorenstien cover of its moduli then $e = d$.

This is already sufficient to avoid much further study of cusps, but for completeness let's do a little better. As such let $\hat{X}$ be a smooth formal affine neighbourhood of a $-1/d\,\mathbb{F}$-cusp minus the non-scheme like point. The Euclidean algorithm determines a sequence of formal blow ups,

$$\hat{X}_n \to \hat{X}_{n-1} \to \cdots \to \hat{X}_1 \to \hat{X}_0 = \hat{X}$$

where the final $\hat{X}_n$ is the completion, without loss of generality, of our previous $\tilde{X}$ in the proper transform of the cusp. Additionally on each $\hat{X}$ we have a distinguished singular point $z_i$ be it of the corresponding foliation or the cusp itself provided in the latter case $i < n$. Bearing in mind our previous normalisation of a generator $\partial$ for the foliation at $z_i$, and noting that we've already proved that the eigenvalues, at the origin, not amongst the $v_i$ are of the form $a_j/e$, $a_j \in \mathbb{Z}$, with $\gcd(a_1, \ldots, a_\ell, e) = 1$, we wish to prove,

**II.8.5 Curiosity.** The formal holonomy is of the form $\mathbb{Z}/e$ iff we can find coordinates $y_1, \ldots, y_k, x_1, \ldots, x_\ell$ with the $y_i$'s embedding coordinates $x_i$ vanishing on the cusp and a generator $\partial$ for the foliation on $\hat{X}$ such that,

$$\partial = v_i\, y_i\, \frac{\partial}{\partial y_i} + \frac{a_j}{e}\, x_j\, \frac{\partial}{\partial x_j}$$

with $\gcd(a_1, \ldots, a_\ell, e) = 1$, and as ever summation covention.

**Proof.** The if direction is trivial, the affine cusp has no Picard group, so neither does $\hat{X}$, so generators $\partial_i$ over $\hat{X}_i$ exist for each $i$, and we proceed by descending induction to choose them in the required form.

To begin with simplify the notations to consider a formal blow up $\rho : \tilde{X} \to \hat{X}$ in $z \in \hat{X}$, with $\tilde{\partial}$ on $\tilde{X}$ of the required form with respect to coordinates $\tilde{y}_i$, $\tilde{x}_i$. We know that in $\hat{\mathcal{O}}_{\hat{X},z}$ i.e. $\mathcal{O}_{\hat{X},z}$ completed in $\mathfrak{m}(z)$, we can find an eigenfunction $y_1$ of a formal generator $\hat{\partial}$, with eigenfunction $v_1$, such $\rho^* y_1$ has the same property in $\hat{\mathcal{O}}_{\tilde{X},\tilde{z}}$, where, obviously, $\tilde{z}$ is the lifting of $z$. Equally, up to permutations of the $y_i$, and noting that the valuation of $\rho^* y_1$ is still $v_1$, we may as well say that $\tilde{y}_1$ has the same property, including that of valuation $v_1$.

However, by general non sense about eigenvalues we can write,

$$\rho^* y_1 = \tilde{y}_1 + \sum_{I,J} c_{IJ}\, \tilde{y}^I\, \tilde{x}^J$$

for $\tilde{y}^I = \tilde{y}_1^{i_1} \dots \tilde{y}_k^{i_k}$, etc., $c_{IJ}$ constants, $\# I \geq 2$, and $\tilde{y}^I\, \tilde{x}^J$ of eigenvalue $v_1$. However for $\# J$ fixed there are only finitely many possibilities for $I$, so the sum is convergent in all of $\tilde{X}$, i.e. $\rho^* y_1$ extends to a function on $\tilde{X}$ which we may as well say is $\tilde{y}_1$. Armed with this identification the same convergence argument shows that without loss of generality, $\tilde{y}_i = y_i/y_1$, $\tilde{x}_j = x_j/y_1$, $i \geq 2$, all $j$, for some formal coordinates in $\hat{\mathcal{O}}_{\hat{X},z}$. On the other hand $\mathcal{O}_{\hat{X},z} \to \mathcal{O}_{\tilde{X},z}$ is finite, and completion in $\mathfrak{m}(z)$ is faithfully flat, so in fact the said $y_i$, $x_j$ are actually coordinate functions on $\hat{X}$. $\square$

Profiting from the Euclidean algorithm to solve $c_1\, v_1 + \cdots + c_k\, v_k = 1$, for some integers $c_i$, we could make a more strict analogue of II.6.2. Indeed in the above notation, $\phi = y_1^{c_1} \dots y_k^{c_k}$ is a rational function on the affine cusp of degree 1, which we can use to form an explicit patch with an étale affine neighbourhood of the non-scheme like point at infinity according to the relation $\phi^d = s^{-1}$, for $s$ a standard coordinate on the latter. Equally there is a bundle on the completion in $\mathcal{L}$, say, $\mathcal{O}(\frac{1}{d})$, with transition function $\phi^{1/d}$, which we can use to encode the poles and zeroes of the trivialised holonomy at infinity. Nevertheless a precise statement is irrelevant to what follows, so we'll omit it.

# III. Extremal Subvarieties

## III.1. Generalities

Unless specified otherwise, throughout this chapter $(\mathcal{X}, \mathcal{F})$ will be a foliated non-singular stack, with $\mathcal{F}$ smooth and terminal at the non-scheme like, together with log-canonical foliation singularities elsewhere. We switch our attention from $K_{\mathcal{F}}$ negative curves, to $K_{\mathcal{F}}$ negative extremal rays $R$. The moduli $X$ is of course supposed projective so if $H_R$ is a nef. Cartier divisor supporting the ray, i.e. $H_R.\, \alpha = 0$, and $\alpha$ in the closed cone of curves iff $\alpha \in R$, then for sufficiently large $m \in \mathbb{N}$, $A_R := m\, H_R - K_{\mathcal{F}}$ is ample. In any case following Kollàr, Mori, et al., cf. [K-3] III.1, we introduce our main object of study, by way of,

**III.1.1 Definition.** The locus of $R$, $\mathrm{Loc}\,(R)$ is the set of closed points $x \in X(\mathbb{C})$ such that there is a curve $x \in C \subset X$ with $[C] \in R \subset NS_1(X)$.

Observe that a priori $\mathrm{Loc}\,(R)$ is not a subvariety of $X$. Indeed for $m \in \mathbb{N}$, we can filter $\mathrm{Loc}\,(R)$ by sub-schemes $\mathrm{Loc}_m\,(R)$ on demanding that $x \in \mathrm{Loc}_m\,(R)$ if we can take the curve $C$ of the definition to have $A_R.\, C \leq m$. That $\mathrm{Loc}_m\,(R)$ is a sub-scheme is immediate from the existence of the Hilbert scheme. To remedy this let us consider,

**III.1.2 Further Definition.** A $R$-pre-extremal subvariety is an irreducible subvariety $Y \subset \mathrm{Loc}\,(R)$ maximal amongst the set of irreducible varieties contained in the locus.

Trivially, the dimension in chains of proper inclusions of irreducible varieties must increase so $R$-pre-extremal subvarieties exist, any $x \in \mathrm{Loc}\,(R)$ is contained in one, and $\mathrm{Loc}\,(R)$ is the a priori countable union of all of them. Now if $Y$ is $R$-pre-extremal, and $y \in Y$ then there is a $C_y$ with $[C_y] \subset R$ containing $y$. However applying II.4.2, we know there is an invariant rational curve $L_y \ni y$ such that,

$$H_R.\, L_y \leq 2\, \frac{H_R.\, C_y}{-K_{\mathcal{F}}.\, C_y} = 0\,.$$

So in fact $L_y \in R$, and $A_R.\, L_y \leq 2$. Additionally $L_y$ cannot be contained in $\mathrm{sing}\,(\mathcal{F})$ since it has $K_{\mathcal{F}}$-negative degree, so we can make a $\mathcal{F}$-invariant subvariety $W$ by adding to generic points of $Y$ an appropriate $L_y$. On the other hand $Y$ is by hypothesis $R$-pre-extremal, so $W = Y$, i.e. $Y$ is $\mathcal{F}$ invariant, with the induced foliated variety $(Y, \mathcal{F})$ being a pencil of rational curves of $A_R$ degree at most 2. Hilbert schemes, however, exist, and being invariant is a closed condition so in fact there are at most finitely many $R$-pre-extremal subvarieties for a given $R$. Better still the Hilbert scheme yields for any $R$-pre-extremal subvariety $Y$ a flat family, $L \to T$, for some irreducible sub-scheme $T$ of the Hilbert scheme such that the projection of $L$ to $X$ factors as a generically finite map over $Y$. Surprisingly the slightly awkward case arises when $X$ is itself a $R$-pre-extremal subvariety, i.e. $(X, \mathcal{F})$ is a pencil of rational curves. As a result we introduce,

**III.1.3 Definition/More Terminology.** A $R$-extremal subvariety $Y$ is a subvariety of a $R$-pre-extremal subvariety $Y'$ which is maximal amongst the subvarieties of $Y'$ which are covered by invariant curves passing through at least one point of the image in $X$ of the singular locus of $(\mathcal{X}, \mathcal{F})$.

So indeed unless $(X, \mathcal{F})$ is a pencil of rational curves then extremal and pre-extremal coincide, while in the awkward case an extremal variety will be specified by taking the invariant curves passing through an appropriate component of the singular locus. Now pulling everything back by the moduli map, $\pi : \mathcal{X} \to X$, define $R$-extremal stack in the obvious way, idem for the locus, denoted $\mathcal{L}\mathrm{oc}\,(R)$, and observe,

**III.1.4 Fact.** *The locus $\mathcal{L}\mathrm{oc}\,(R)$ of an extremal ray, is a finite union of $R$-pre-extremal stacks. Denote by $\mathcal{L}\mathrm{oc}'\,(R)$ the subvariety which is the union of $R$-extremal stacks, then any $\mathcal{Y} \subset \mathcal{L}\mathrm{oc}'\,(R)$ making up this union is covered by $-1/d\,\mathbb{F}$ curves, where $d$ may vary from curve to curve. There is however a family $\mathcal{L} \to T$ of stacks, possibly non-flat at the non-scheme like points, such that, $(\mathcal{L}, \mathcal{L}/T) \to (\mathcal{Y}, \mathcal{F})$ is a generically finite map of foliated stacks, where of course $(\mathcal{L}, \mathcal{L}/T)$ is the foliation corresponding to the fibration $\mathcal{L} \to T$.*

## III.2. Digression on Deformation

As remarked in the introduction the Hilbert scheme of a stack is not known to exist in general, and whence we're a priori missing some natural way to go from formal positivity to actualy being able to move in algebraic families. Fortunately however our stacks are all orbifolds over schemes so following Kollàr, [K-1], we may easily remedy this. Specifically suppose $\pi : \mathcal{X} \to X$ is an integral stack with projective moduli, and generically finite stabiliser, the for $U = \coprod_{\alpha} U_\alpha \to X$ a sufficiently fine étale cover of $X$ by integral affines then we know I.2.5 that $\mathcal{X} \times_X U_\alpha = [V_\alpha / G_\alpha]$, for $G_\alpha$ a finite group of automorphisms of $V_\alpha$. Put $V = \coprod_{\alpha} V_\alpha$, and let $V^{\text{ét}}$ be the dense open subset where $V \to X$ is étale, then we may consider the étale equivalence relation,

$$R = V \cup (V^{\text{ét}} \times_X V^{\text{ét}} \backslash \Delta) \quad \underset{\mathrm{id} \cup p_2}{\overset{\mathrm{id} \cup p_1}{\rightrightarrows}} \quad V$$

where $p_i$ are the projections. By [A2], the quotient exists as an algebraic space, and is what Kollàr calls a bug-eyed cover of $X$, since unless $V \to X$ were actually étale, equivalently $\mathcal{X} = X$, then it is non-separated by virtue of having too many tangent directions. In any case we denote the said quotient $\pi^b : X^b \to X$.

Now if $Y$ is an irreducible proper, so in particular separated, algebraic space then $\mathrm{Hom}\,(Y,\, X^b)$ exists as an algebraic space, and satisfies the usual kind of deformation estimates. Furthermore inside $X^b$ we have a rather strict analogue of the non-scheme like points of $\mathcal{X}$, namely the non-separated points, which we'll denote $N^b$ and $\mathcal{N}$ respectively. Equally although we may know little about it we nevertheless have a functor $\mathrm{Hom}\,(Y,\, \mathcal{X})$ of algebraic spaces to sets given by,

$$\mathrm{Hom}\,(Y,\, \mathcal{X})(T) = \{T \text{ morphisms } Y \times T \to \mathcal{X} \times T\}\,.$$

Inside both these functors we can consider the sub-functors $\mathrm{Hom}'(Y,\, X^b)$, and $\mathrm{Hom}'(Y,\, \mathcal{X})$ of maps which don't factor through the non-separated and non-scheme like points respectively. Unsurprisingly we have,

**III.2.1 Fact.** *Notations as above then,* $\mathrm{Hom}'(Y,\, X^b) = \mathrm{Hom}'(Y,\, \mathcal{X})$*, not just as functors but as an actual natural identification between elements.*

**Proof.** To give an element $f_T^b$, or $\mathrm{f}_T$ is simply to give a $T$-morphism $f_T : Y \times T \to X \times T$ which can be lifted be it to the bug eyed cover or the stack, and which of course doesn't factor through the closed sub-scheme, $\pi(N^b) = \pi(\mathcal{N})$. Lifting is however a local discussion, so if $Y_\alpha = f_T^{-1}(U_\alpha \times T)$ then we require to lift, $f_\alpha : Y_\alpha \to U_\alpha \times T$ to $V_\alpha^b \times T$, respectively the classifying stack $[V_\alpha / G_\alpha] \times T$. On the other hand both of these admit $V_\alpha \times T$ as an étale cover; so in either case this amounts to the fibre product,

$$\begin{array}{ccc} V_\alpha \times T & \xleftarrow[\tilde{f}_\alpha]{} & \tilde{Y}_\alpha \\ \downarrow & & \downarrow \\ U_\alpha \times T & \xleftarrow[f_\alpha]{} & Y_\alpha \end{array}$$

having étale right hand arrow. Any lifting, if it exists, is under the given hypothesis unique, so we're done. □

Of course, this is very much a case of better still, since the co-tangent bundles of $X^b$ and $\mathcal{X}$, respectively, are defined by the identical subterfuge of the respective étale coverings by $V$, so that if $f^b$, $f$ are liftings of the same map $f : Y \to X$, as ever not factoring through $\pi(N^b) = \pi(\mathcal{N})$, then $(f^b)^*\, \Omega^1_{X^b} = f^*\, \Omega^1_{\mathcal{X}}$. Putting these observations together with op. cit. 2.9, we therefore obtain,

**III.2.2 Better Fact.** *Suppose $\mathcal{X}$ is a smooth stack, and $f : Y \to \mathcal{X}$ a map not factoring through the non-scheme like points then there is an algebraic space $T$, possibly non-separated, of dimension at least,*

$$h^0(f^*\, T_{\mathcal{X}}) - h^1(f^*\, T_{\mathcal{X}})$$

*together with a deformation $F : Y \times T \to \mathcal{X} \times T$ of $f$ such that the corresponding map, $T \to \mathrm{Hom}\,(Y, X)$ is injective.*

The better fact does, of course, suggest that $T$ comes equipped with a natural separated stack structure. This could probably be proved by going through the details of [A4]. In cases where it may be of relevance, however, we'll easily be able to prove this, and more besides, wholly directly.

## III.3. Finding Weighted Projective Spaces

As ever let $(\mathcal{X}, \mathcal{F})$ be a smooth foliated stack with log canonical foliation singularities, terminal at the non-scheme like points. Now let $f : \mathcal{L} \to \mathcal{X}$ be a smooth $-\frac{1}{d}\,\mathbb{F}$ curve with eigenvalues $\frac{a_1}{d} \geq \frac{a_2}{d} \geq \cdots \geq \frac{a_n}{d}$ at the unique point $z$ where $f$ meets the singular locus where of course eigenvalue refers to the eigenvalues of the habitual normalisation of a linearisation $\nabla_z$ of a local generator $\partial$ of $T_{\mathcal{F}}$ about $z$ in $\mathrm{End}\,(N_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(z))$. If $a_1 \leq 0$, then we simply have nothing to say for the moment. Otherwise there is a formal invariant sub-stack $\hat{\mathcal{Y}}$ of the completion of $\mathcal{L}$ in $\mathcal{X}$ such that $N_{\mathcal{L}/\hat{\mathcal{Y}}}$ is ample. Consequently we may apply the Chow lemma to conclude that the minimal integral sub-stack $\mathcal{Y}$ of $\mathcal{X}$ containing $\hat{\mathcal{Y}}$ has the same dimension as $\hat{\mathcal{Y}}$, so in fact $\hat{\mathcal{Y}}$ is the completion of $\mathcal{Y}$ in $\mathcal{L}$. By construction $\mathcal{Y}$ is smooth in a neighbourhood of $\mathcal{L}$, so we can apply the deformation estimates to the cover $\mu : \mathbb{P}^1 \to \mathcal{L}$, $y \mapsto y^d$, ramified in the origin $z$ with of course $y^d$ a coordinate on the same. Now a priori the deformation guaranteed by III.2.2 is not complete, nor are deformations obtained by pulling back from the moduli deformations stricta dictum since we may lose flatness. However $\mathcal{Y}$ is invariant so we can apply a minor variant of the cone theorem to conclude,

**III.3.1 Initial Fact.** *If $\mathcal{L}$ corresponds to an extremal ray $R$ in Néron-Severi, with supporting function $H_R$, and ample bundle $A_R = mH_R - K_{\mathcal{F}}$, then for all $y \in \mathcal{Y}$, there is a $-1/d(y)\,\mathbb{F}$ stack $\mathcal{L}_y \ni y$ in $\mathcal{Y}$ parallel to $R$ in Néron-Severi.*

**Proof.** Indeed if $\mathcal{C} \subset \mathcal{X}$ is any integral 1-dimensional sub-stack lying over a curve $C$ in $X$ then $\pi_*[\mathcal{C}]$ is a multiple of $C$ in Néron-Severi. On the other hand our initial deformation in $\mathcal{X}$ can be completed to a deformation of $\pi \circ f \circ \mu$ in $X$, and of course if $\sum_i a_i\, C_i$ is some effective 1-cycle numerically equivalent to $\pi_*[\mathcal{L}]$ then every $C_i$ is so equivalent by virtue of the extremality. $\square$

Now the singular point is scheme like, so if $L \hookrightarrow X \times \mathbb{P}^1 \times T$ is a flat family in $X \times \mathbb{P}^1$ completing our initial deformation, with $p$ to $X$ the projection then the completion in $p^{-1}(z)$ is a formal subscheme mapping properly by $p$ to the completion of $X$ (or equally $\mathcal{X}$) in $z$. Profiting from the closed nature of invariance we can insist that $p(L_t)$ is $\mathcal{F}$-invariant for every $t$, and so by a minor variant of our initial fact we may deduce for $\mathcal{L}$ parallel to some extremal $R$ in Néron-Severi (which, incidentally, we'll suppose without further comment).

**III.3.2 Further Fact.** *For each eigendirection $\frac{\partial}{\partial x_i}$ of $\nabla_z$ in $\mathrm{End}\,(N_{\mathcal{L}/\mathcal{Y}} \otimes \mathbb{C}(z))$ there is a smooth $-1/d_i\,\mathbb{F}$ invariant stack $f_i : \mathcal{L} \to \mathcal{X}$ through $z$ and parallel to $R$ in Néron-Severi.*

Additionally points in $\mathbb{P}^t(\mathbb{Q})$, $t \in \mathbb{N}$, are, up to $\pm 1$, uniquely represented by $t+1$ tuples of integers with $\gcd = 1$, so if we change to a more homogeneous notation, viz:

**III.3.3 New Notation.** Linearise a local generator $\partial$ of $T_{\mathcal{F}}$ in the completion of $\hat{\mathcal{O}}_{\mathcal{X},z}$ of $\mathcal{O}_{\mathcal{X},z}$ in $\mathfrak{m}_{\mathcal{X}}(z)$ by way of, $\partial = a_1\, y_1\, \frac{\partial}{\partial y_1} + \cdots + a_r\, \frac{\partial}{\partial y_r} - b_i\, x_i\, \frac{\partial}{\partial x_i}$, $a_i \in \mathbb{N}$, $b_i \in \mathbb{N} \cup \{0\}$, $(a_1, \ldots, a_r, b_1, \ldots, b_s) = 1$, with $x_i$ local equations for $\mathcal{Y}$, and the summation convention in the obvious way, with $s$ the codimension of $\mathcal{Y}$.

Then for each eigendirection $\frac{\partial}{\partial y_i}$ we have a smooth invariant $-1/d_i\,\mathbb{F}$ stack $f_i : \mathcal{L}_i \to \mathcal{X}$ parallel to $R$ in Néron-Severi, with $a_i \mid d_i$. Observe further that if $a$ is the gcd of $a_1, \ldots, a_r$, then the generic invariant stack in $\mathcal{Y}$ is a $-1/da\,\mathbb{F}$ curve, so in particular,

**III.3.4 Intermediary Fact.** *The fundamental group of a leaf of the induced smooth foliation $(\mathcal{Y}\backslash z, \mathcal{F})$ is generically $\mathbb{Z}/da$ for some $d \in \mathbb{N}$.*

Note, additionally, that $\mathcal{Y}$ is a smooth stack. Indeed if it were not then there would be a smooth sub-stack $\mathcal{B}$ in the singular locus invariant by the induced foliation. On the other hand the said foliation is smooth on $\mathcal{Y}\backslash z$, so $\mathcal{B}$ meets any punctured neighbourhood of $z$. However by construction $\mathcal{Y}$ is smooth at $z$, and smoothness is open, so this is nonsense. Continuing in this vein let $Y$ be the moduli of $\mathcal{Y}$, and $Y_0$ the image of $\mathcal{Y}$ in $\mathcal{X}$ then, $Y \to Y_0$ is 1 to 1 on closed points, $\mathcal{Y}$ is a closed sub-stack of a separated stack, so $Y$ is separated whence it's actually a scheme, and a projective one at that, while by [V] 2.8 the smoothness of $\mathcal{Y}$ implies that $Y$ is geometrically uni-branch with at worst quotient singularities.

Now let's put this together to calculate the Picard group of $\mathcal{Y}$. Quite generally we have that $\mathrm{Pic}\,(\mathcal{Y})_{\mathbb{Q}} = \mathrm{Pic}\,(Y)_{\mathbb{Q}}$, so let's start with $\mathrm{Pic}\,(Y)$. Since $Y$ has only quotient singularities they are in particular rational so if $\rho : \tilde{Y} \to Y$ is a resolution of singularities then $H^1(Y, \mathcal{O}_Y) = H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$. However $\tilde{Y}$ is covered by rational curves through a point, so $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$ and whence we have a commutative diagram,

$$\begin{array}{ccc} \mathrm{Pic}\,(Y)_{\mathbb{Q}} & \xrightarrow[\rho^*]{} & \mathrm{Pic}\,(\tilde{Y})_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ NS^1(Y)_{\mathbb{Q}} & \xrightarrow[\rho^*]{} & NS^1(\tilde{Y})_{\mathbb{Q}} \end{array}$$

where the horizontal arrows are quite generally injective, and we've established the injectivity of the right vertical, whence (rather obviously, but a good reference wasn't clear), $\mathrm{Pic}\,(Y)_{\mathbb{Q}} \xrightarrow{\sim} NS^1(Y)_{\mathbb{Q}}$. However $NS_1(Y)_{\mathbb{Q}}$ is known, e.g. [K-3] II.4.21, so: $\mathrm{Pic}\,(\mathcal{Y})_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}$, and the remaining issue is the possibility of torsion. So suppose there was torsion, then we would obtain an irreducible étale cover $\tau : \mathcal{Y}' \to \mathcal{Y}$, with fibre over $z$, $z_1, \ldots, z_d$, say. On the other hand $\mathcal{Y}$ is covered by simply connected stacks, so every invariant stack $\mathcal{M} \ni z$, is covered by $d$-copies of the same, with exactly one through each $z_i$, which is obvious nonsense since the family of invariant curves through any $z_i$ would define an irreducible sub-stack of $\mathcal{Y}'$ not equal to it, i.e. $\mathrm{Pic}\,(\mathcal{Y}) \xrightarrow{\sim} \mathbb{Z}$. It remains to establish the exact relation of the divisors $\mathcal{D}_i$ in $\mathcal{Y}$ which are infinitesimally $y_i = 0$ in our coordinate system at $z$, where the existence of the $\mathcal{D}_i$ as well-defined divisorial sub-stacks of $\mathcal{Y}$ is just another application of the Chow lemma in the spirit of III.3.1. The said infinitesimal calculation already yields, however, for $\mathcal{L}_j$ the invariant stack tangent to $\frac{\partial}{\partial y_j}$, $\mathcal{D}_i \cdot \mathcal{L}_j = \frac{a_i}{a_j}$, and we know that $\mathcal{O}_{\mathcal{L}_j}(K_{\mathcal{F}})$ generates $\mathrm{Pic}\,(\mathcal{L}_j)$ for any $j$, by I.9.3. Consequently $K_{\mathcal{F}}$ generates $\mathrm{Pic}(\mathcal{Y})$, and $d_i = a_i d$, for some $d \in \mathbb{N}$ independent of $i$, with $\mathcal{D}_i = -d_i\, K_{\mathcal{F}}$.

With this behind us we can now proceed to extract roots of the sections, $\gamma_i \in \mathcal{O}_{\mathcal{Y}}(-d_i\, K_{\mathcal{F}})$ defining the $\mathcal{D}_i$ by the usual procedure. We do this systematically, observe that no $y_j|_{\mathcal{L}_j}$ has a root of any sort around $z$, so the $d_1$-th root cover of $\gamma_1$ in $\mathbb{V}(d_1\, K_{\mathcal{F}})$ is smooth and irreducible, and since $\mathcal{D}_1$ is invariant by $\mathcal{F}$, the induced foliation on the cover has canonical bundle isomorphic to the pull-back of $K_{\mathcal{F}}$. On the other hand the reduced pull-back of $\mathcal{D}_1$ is isomorphic to $\mathcal{D}_1$, so the $\mathcal{L}_j$, $j \geq 2$ are unchanged, while equally the $\gamma_i$ are still sections of $K_{\mathcal{F}}^{-d_i}$ without $d_i$-th roots for $i \geq 2$, so continuing in this way we obtain a cover $\rho : \tilde{\mathcal{Y}} \to \mathcal{Y}$ ramified only in the $\mathcal{D}_i$ such that the induced foliation $(\tilde{\mathcal{Y}}, \tilde{\mathcal{F}})$ has canonical bundle $\rho^* K_{\mathcal{F}}$. Notice that $\tilde{\mathcal{Y}}$ is separated, whence so is its moduli space $\tilde{Y}$ which in turn is finite over $Y$, so $\tilde{Y}$ is projective. Now if $z \in \mathcal{M} \subset \mathcal{X}$ is a generic invariant stack through the origin, then for any $j$, $\mathcal{D}_j.\,\mathcal{M} = +\,d_j/d$, which is just the multiplicity of $\mathcal{Y}_j$ on the cusp, so apart from $z$, $\mathcal{M}$ doesn't meet the ramification locus, whence if $\tilde{\mathcal{M}} \to \mathcal{M}$ is a lifting then over $\mathcal{M} - z$, $g$ is unramified, thus $(\tilde{\mathcal{M}} : \mathcal{M})$ divides $da$. Notice, however, if $H_j$ is the reduced pull-back of $\mathcal{D}_j$ then the intersection of $H_j$ with $\tilde{\mathcal{M}}$ is supported only in the unique point $\tilde{z}$ of $z$ around which everything is scheme like, so:

$$\mathbb{N} \ni H_j.\,\tilde{\mathcal{M}} = \frac{1}{d_j}\,\rho^*\,\mathcal{D}_j.\,\tilde{\mathcal{M}} = \frac{(\tilde{\mathcal{M}} : \mathcal{M})}{da}$$

whence $H_j.\,\tilde{\mathcal{M}} = 1$, for all $j$, and $(\tilde{\mathcal{M}} : \mathcal{M}) = da$, so in fact $\tilde{\mathcal{M}} \xrightarrow{\sim} \mathbb{P}^1$. Repeating this calculation for the generic invariant stack in $\mathcal{D}_j$, then the $\mathcal{D}_j \cap \mathcal{D}_i$, right down to the $\mathcal{L}_k$, we conclude that $\tilde{\mathcal{Y}}$ is smoothly foliated by $\mathbb{P}^1$'s. In particular, away from $\tilde{z}$, $(\tilde{\mathcal{Y}}, \tilde{\mathcal{F}})$ is locally a product of the form $\Delta \times [\Delta^{r-1}/G]$ of a disc with a smooth classifying stack, with $\tilde{\mathcal{F}}$ the projection to the classifying stack, from which the non-scheme like locus of $\tilde{\mathcal{Y}}$ is invariant by $\tilde{\mathcal{F}}$, so the image under $\rho$ in $\mathcal{Y}$ of the non-scheme like locus gives a non-scheme like sub-stack of $\mathcal{X}$ invariant by the foliation, contrary to the hypothesis of terminal singularities everywhere along the non-scheme like locus. Consequently $\tilde{\mathcal{Y}} = \tilde{Y}$ is projective, with leaf space isomorphic to $P(T_{\tilde{\mathcal{Y}}} \otimes \mathbb{C}(\tilde{z})) \xrightarrow{\sim} \mathbb{P}^{r-1}$. Better still if $y_i = \eta_i^{d_i}$, for $\eta_1, \ldots, \eta_r$ coordinates on $\tilde{Y}$ around $\tilde{z}$ then $d\rho^* \partial$ is precisely $\eta_1 \frac{\partial}{\partial \eta_1} + \cdots + \eta_r \frac{\partial}{\partial \eta_r}$ in the completion of $\mathcal{O}_{\tilde{Y}, \tilde{z}}$ in $\mathfrak{m}(\tilde{z})$, so blowing up in $\tilde{z}$ resolves the singularity, and the holonomy of every leaf is trivial, so in fact the said blow up is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{r-1}$ with a section, the exceptional divisor, whence, cf. [K-3] V.3, $(\tilde{\mathcal{Y}}, \tilde{\mathcal{F}})$ is actually $(\boldsymbol{\mathcal{P}}^r, \mathcal{R})$ i.e. $\mathbb{P}^r$ with the foliation the pencil of lines through some point. To conclude to the final result observe that the choice of linearising coordinates for a field with positive eigenvalues is unique up to the centraliser of the associated diagonal matrix, so $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{P}^r$ and the centraliser of $\mathrm{diag}\{d_1, \ldots, d_r\}$, on $\mathcal{Y}$. In either case $\eta_1, \ldots, \eta_r$ are not just local, but actual standard affine coordinates on some standard affine patch $\mathbb{A}^r$ on $\mathbb{P}^r$. Additionally our knowledge of the completion at $\tilde{z}$ even tells us that the relative Galois group $G = \mu_{a_1} \times \cdots \times \mu_{a_r}$ acts on this patch by, $(\eta_1, \ldots, \eta_r) \times (\zeta_1, \ldots, \zeta_r) \mapsto (\eta_1 \zeta_1, \ldots, \eta_r \zeta_r)$. As such the moduli is certainly a weighted projective space with weights $\left(1, \frac{a_1}{a}, \ldots, \frac{a_r}{a}\right)$ in standard coordinates. Now let $(\boldsymbol{\mathcal{P}}\left(\frac{a_1}{a}, \ldots, \frac{a_r}{a}\right), \mathcal{R})$ be the Gorenstien covering stack of the induced foliation by lines then since $\mathcal{Y}$ is already Gorenstien the map $\mathcal{Y} \to Y$ must factor through the covering stack. Equally our standard coordinate patch $\mathrm{mod}\, G$ gives a smooth affine neighbourhood of $Y$, and by our habitual intersection theory considerations we know the pull-back to $\mathbb{P}^r$ from $\mathcal{Y}$ of the non-scheme like locus is a plane, which in these coordinates is necessarily the hyperplane at infinity. As such $\mathcal{Y} \to \boldsymbol{\mathcal{P}}\left(\frac{a_1}{a}, \ldots, \frac{a_r}{a}\right)$ is a map of smooth stacks one to one on closed points and with ramification only over the plane at infinity, necessarily corresponding to an automorphism group $\mathbb{Z}/da$ by [V] 2.8, generically over the same, whence, in fact,

**III.3.5 Proposition/Summary.** *Let $f : \mathcal{L} \to \mathcal{X}$ be a smooth $-?\,\mathbb{F}$ extremal ray, with $a_1, \ldots, a_r$ the positive eigenvalues of a linearisation in $\mathrm{End}\,(T_{\mathcal{X}} \otimes \mathbb{C}(z))$ of a generator $\partial$ of the foliation around the unique point where $\mathcal{L}$ meets the singular locus, normalised so that the eigenvalue tangent to $f$ is positive, and the totality of positive and negative values is relatively prime, then there is in fact a smooth invariant $(\boldsymbol{\mathcal{P}}(a_1 d, \ldots, a_r d), \mathcal{R})$, some $d \in \mathbb{N}$, centered on $z$ whose lines are parallel to $\mathcal{L}$ in Néron-Severi, and of course $N_{\mathcal{L}/\boldsymbol{\mathcal{P}}_l(da_1, \ldots, da_r)}$ is the ample part of $N_{\mathcal{L}/\mathcal{X}}$.*

## III.4. Ignoring Cusps

So far we haven't discussed what may happen if our extremal ray $R$ is represented by an invariant stack $f : \mathcal{L} \to \mathcal{X}$ which has a cusp at the unique singular point $z$ where $f$ meets $\mathrm{sing}\,(\mathcal{F})$. Thus, to that end, let $f$ indeed be such, and indeed let's say that it's $-1/d\,\mathbb{F}$, so $\mathcal{L}$ is the bad orbifold with signature $d$ at $\infty$. As ever a resolution of the cusp is dictated by the Euclidean algorithm for $v_1, \ldots, v_k$ where we normalise a local generator $\partial$ to $T_{\mathcal{F}}$ in the completion of $\hat{\mathcal{O}}_{\mathcal{X},z}$ of $\mathcal{O}_{\mathcal{X},z}$ in $\mathfrak{m}_{\mathcal{X}(z)}$ as per II.8, and indeed precisely retake the notations therein. In particular if $\tilde{f} : \mathcal{L} \to \tilde{\mathcal{X}}$ is the resolution obtained via blowing up in points according to the Euclidean algorithm, then there is a part of the Harder-Narismhan filtration (or perhaps more correctly a sub-quotient) of $N_{\mathcal{L}/\tilde{\mathcal{X}}}$ of the form $\bigoplus_{i=1}^{k} \mathcal{O}_{\mathcal{L}}\left(\frac{\tilde{v}_i}{e}\right)$ where $\tilde{v}_1 = 1$, and $\tilde{v}_2, \ldots, \tilde{v}_k > 1$ with $e \mid d$ the order of the holonomy at infinity. Now it may happen that $\tilde{f}$ is not extremal, nevertheless the ubiquitus Chow lemma combined with the full force of II.6.2, tell us that we can find a closed invariant sub-stack $\tilde{\mathcal{Y}}$ of $\tilde{\mathcal{X}}$ containing $\mathcal{L}$ with $N_{\mathcal{L}/\tilde{\mathcal{Y}}} \xrightarrow{\sim} \bigoplus_{i=1}^{k} \mathcal{O}_{\mathcal{L}}\left(\frac{\tilde{v}_i}{e}\right)$. Necessarily the standard covering map, $\nu^* \tilde{f} : \mathbb{P}^1 \to \tilde{\mathcal{Y}}$ moves as it should thanks to III.2.2, and by way of the moduli of $\tilde{\mathcal{X}}$ we can complete this in the weak sense of being able to cover $\tilde{\mathcal{Y}}$ by invariant stacks. If we project everything down to $\mathcal{X}$, with $\mathcal{Y}$ the image of $\tilde{\mathcal{Y}}$, the situation is much better. By construction the completion of $\mathcal{Y}$ at $z$ contains a formal affine of the same dimension so

$\mathcal{Y}$ is smooth everywhere around $\mathcal{L}$. Better still our deformation of $\nu^*\tilde{f} : \mathbb{P} \to \tilde{\mathcal{Y}} \to \mathcal{Y}$, completed by way of the moduli of $\mathcal{X}$, assures that, at not just every point of $\mathcal{Y}$, but at every formal invariant curve in $\hat{\mathcal{O}}_{\mathcal{Y},z}$ there is a $-1/d\,\mathbb{F}$ curve, evidently for varying $d$, parallel to $R$ in Néron-Severi. Thus as per the smooth case, algorithmic resolution implies that $\mathcal{Y}$ is smooth, while for each eigen direction $\frac{\partial}{\partial y_i}$ in $T_y \otimes \mathbb{C}(z)$, there is even a smooth $-1/d_i\,\mathbb{F}$ curve, $f_i : \mathcal{L}_i \to \mathcal{X}$, parallel to $R$ in Néron-Severi. Bearing in mind, however, that in the notation of II.8.4, $\gcd(ev_1, \dots, ev_k, a_1, \dots, a_l) = 1$, the completion $\hat{\mathcal{Y}}$ of $\mathcal{Y}$ at $z$ must be contained in the unique formal invariant subvariety determined by the positive eigen-directions. On the other hand we already know that this is a weighted projective space by III.3.5, so in fact $(\mathcal{Y}, \mathcal{F})$ with the induced foliation is $(\boldsymbol{\mathcal{P}}(dv_1, \dots, dv_k), \mathcal{R})$. So a $d$-th multiple of $f$, can certainly be moved into a $dv_i$-th multiple of $f_i$, or perhaps more correctly $\nu^* f : \mathbb{P}^1 \to \mathcal{X}$ to $\nu_i^* f_i : \mathbb{P}^1 \to \mathcal{X}$, for $\nu_* : \mathbb{P}^1 \to \mathcal{L}_*$ the standard covering appear in the same family. In any case, we can certainly move a cusp to a smooth, and so ignore the former.

## III.5. Structure of Extremal Stacks

To begin with let $f : \mathcal{L} \to \mathcal{X}$ be a smooth $-\frac{1}{d_1}\mathbb{F}$ stack parallel to an extremal ray $R$ in Néron-Severi. As ever we normalise a local generator $\partial$ of the foliation in the complete local ring $\hat{\mathcal{O}}_{\mathcal{X},z}$, for $z = f^{-1}(\operatorname{sing}\mathcal{F})$, according to II.3.3 with $d_1 = a_1 d$. Furthermore we may take the standard covering map, $\mu : \mathbb{P}^1 \to \mathcal{L} : y \mapsto y^{d_1}$, and apply a minor variant of the deformation estimates of III.2.2, to $\mu^* f : \mathbb{P}^1 \to \mathcal{Y}_{\geq 0}$, where $\mathcal{Y}_{\geq 0}$ is the formal sub-stack of the completion of $\mathcal{X}$ in $\mathcal{L}$ whose normal bundle is,

$$\bigoplus_{i=2}^{r} \mathcal{O}_{\mathcal{L}}\left(\frac{a_i}{a_1}\right) \oplus \mathcal{O}_{\mathcal{L}}^{\oplus s}$$

with $s$ the dimension of the kernel of $\partial$ reduced mod $\mathfrak{m}_{\mathcal{X}}(z)^2$. Consequently deforming $\mu^* f$ in $\mathcal{Y}_{\geq 0}$ is unobstructed and gives a formal sub-algebraic space supported at $\mu^* f$ of $\operatorname{Hom}(\mathbb{P}^1, X^b)$, for $X^b$ the associated bug eyed cover, with the same tangent space, so it is in fact all of the deformation space completed at $\mu^* f$. Profiting from the habitual subterfuge of completing the deformation in the moduli $X$, then passing to the associated closed integral sub-stack, we obtain an integral invariant sub-stack $\mathcal{Y}$ of $\mathcal{X}$ whose completion at $\mathcal{L}$ is $\mathcal{Y}_{\geq 0}$, and through every point of which there is a $-1/e\,\mathbb{F}$ stack, for varying $e$, parallel to $R$ in Néron-Severi. Now let $Z$ be the necessarily scheme like intersection of $\mathcal{Y}$ with the singular locus of $\mathcal{F}$, and suppose that $Z$ isn't connected, say a disjoint union of components $Z'$, $Z''$, then we may consider the sub-stacks $\mathcal{Y}'$, $\mathcal{Y}''$ covered by $K_{\mathcal{F}}$-negative extremal 1-dimensional stacks through $Z'$ and $Z''$ respectively, consequently if $y \in \mathcal{Y}' \cap \mathcal{Y}''$ it is a singular point of some extremal 1-dimensional invariants stacks $\mathcal{L}'$, $\mathcal{L}''$, so in $Z' \cap Z''$, which is nonsense, so in fact $Z$ is connected. Better still at $z$ we know that $Z$ is irreducible and smooth of $\dim = s$, so there is some irreducible component $Z_0$ of $\operatorname{sing}(\mathcal{F})$ of dimension $s$ contained wholly in $Z$. However for any $\zeta \in Z$, there is a $-1/e(\zeta)\,\mathbb{F}$ stack $\mathcal{L}_\zeta \ni \zeta$, and indeed contained in $\mathcal{Y}$, so $\operatorname{sing}(\mathcal{F})$ is smooth at $\zeta$ by II.8.4, and whence $\zeta \mapsto \dim_\zeta \operatorname{sing}(\mathcal{F})$ is not just upper semi-continuous but continuous, i.e. constant $= s$, since $Z$ is connected. Consequently $Z_0 = Z$, and is smooth irreducible of dimension $s$.

Now consider the ideal $I_Z$ of $Z$ in $\mathcal{X}$, along with the ideal $I_{Z,\mathcal{Y}}$ of $Z$ in $\mathcal{Y}$ then our linearisation procedure for vector fields at singular points globalises to $\mathcal{O}_Z$-linear maps, fitting into a diagram,

$$\begin{array}{ccc} I_Z/I_Z^2 & \xrightarrow[D_Z]{} & I_Z/I_Z^2 \otimes K_{\mathcal{F}} \\ \downarrow & & \downarrow \\ I_{Z,\mathcal{Y}}/I_{Z,\mathcal{Y}}^2 & \xrightarrow[D_{Z,\mathcal{Y}}]{} & I_{Z,\mathcal{Y}}/I_{Z,\mathcal{Y}}^2 \otimes K_{\mathcal{F}} \end{array}$$

and defined initially, in a neighbourhood of $Z$, as the composition of the maps,

$$I_Z \xrightarrow[d]{} \Omega_X \longrightarrow K_{\mathcal{F}}.I_Z\,.$$

If we double dualise the lower map we get a map of reflexive sheaves,

$$D^{\vee}_{Z,\mathcal{Y}} : (I_{Z,\mathcal{Y}}/I^2_{Z,\mathcal{Y}})^{\vee\vee} \to (I_{Z,\mathcal{Y}}/I^2_{Z,\mathcal{Y}})^{\vee\vee} \otimes K_{\mathcal{F}}$$

so that the trace gives a section of $\mathcal{O}_Z(K_{\mathcal{F}})$ in codimension 2, and $Z$ is smooth so in fact a section everywhere, which is certainly non-zero at $z$. On the other hand at every point $\zeta$ of $Z$, $D_Z$ is an isomorphism, since it's residue is nothing other than the identity on the non-zero eigenspaces up to a multiplicative constant. Consequently, $\mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_Z(\{\operatorname{codim} Z\} K_{\mathcal{F}})$, and $\mathcal{O}_Z(K_{\mathcal{F}})$ has a non-zero section, so $K_{\mathcal{F}} \mid_Z$ is in fact $\mathcal{O}_Z$. As a result the eigenvalues of $D_Z$ are well defined constant functions up to a choice of generator of $\mathcal{O}_Z(K_{\mathcal{F}})$, which we choose in such a way to have compatibility with our formal linearisation at $z$, i.e. the eigenvalues of $D_Z$ are everywhere $a_1, \ldots, a_r, -b_1, \ldots, -b_t$, with $a_i, b_j \in \mathbb{N}$, and $\gcd(a_1, \ldots, a_r, b_1, \ldots, b_t) = 1$. In any case, for every $\zeta \in Z$, there are well defined positive and negative eigenspaces, $T_{+(\zeta)}, T_{-(\zeta)}$ of $T_{\mathcal{X}} \otimes \mathbb{C}(\zeta)$, and every $K_{\mathcal{F}}$-negative 1-dimensional invariant stack has tangent space at $\zeta$ contained in precisely one of these. To fully profit from this we will have to extend from the normal bundle to a formal neighbourhood of $Z$, which probably shows that being lazy about convergence wasn't perhaps an optimal use of time. The discussion is local over affine neighbourhoods $U$ of $Z$ over which the normal bundle and $K_{\mathcal{F}}$ trivialise, and which we consider centered on a point $\zeta$ of $Z$. To momentarily simplify the notations let $\lambda_i$ denote the necessarily non-zero eigenvalues of the normal bundle, and consider the following inductive proposition,

Let $\hat{\mathcal{O}}_U$ be the completion of $\mathcal{O}_U$ in $\mathfrak{m}_{\mathcal{X}}(\zeta)$, and for $k \in \mathbb{N}$, we have coordinates $x_i$ normal to $Z$ (evidently giving a basis for $N^{\vee}_{Z/x}$) and a generator $\partial$ of $\mathcal{F}$ over $U$ such that,

(1) $\partial x_i = \lambda_i \, x_i \pmod{I_Z^k}$

(2) There is a semi-simple generator $\hat{\partial}$ of $T_{\mathcal{F}} \otimes \hat{\mathcal{O}}_{U,\zeta}$ of the form $\lambda_i \, \xi_i \frac{\partial}{\partial \xi_i}$, for $\xi_i \in \hat{\mathcal{O}}_{U,\zeta}$ and $\xi_i = x_i \pmod{I_Z^k}$.

The case $k = 1$ trivially follows from the previous discussion, so consider going from $k$ to $k+1$, which evidently we wish to be compatible with restriction so that things converge. In any case, in terms of our usual notations about monomials and summation conventions we have, mod $I_Z^{k+1}$,

$$\partial x_i = \lambda_i \, x_i + a_{iJ} \, x^J \,,\ a_{iJ} \in \mathcal{O}_U \,,\ \xi_i = x_i + b_{iJ} \, x^J \,,\ b_{iJ} \in \hat{\mathcal{O}}_{U,\zeta} \,.$$

Furthermore, $\hat{\partial} = u\partial$, $u \in \hat{\mathcal{O}}_{U,\zeta}$, and, $u = 1 + u_{iK} \, x^K$, $u_{iK} \in \hat{\mathcal{O}}_{U,\zeta}$, with $\# J = k$, $\# K = k-1$. Now if we just put these equations together then we obtain,

$$a_{iJ} = (\lambda_i - \lambda_J) \, b_{iJ} - u_{iK} \, \lambda_i \,, \qquad \text{if } x^K \, x_i = x^J$$
$$a_{iJ} = (\lambda_i - \lambda_J) \, b_{iJ} \,, \qquad \text{otherwise}$$

without any summations. The second case is rather good since if $\lambda_i \neq \lambda_J := j_p \, \lambda_p$ we conclude that the $b_{iJ}$ are algebraic, so if without loss of generality we replace $x_i$, by,

$$x_i \mapsto x_i + \sum_{\substack{\lambda_i \neq \lambda_J \\ x_i \nmid x^J}} b_{iJ} \, x^J$$

then in fact we conclude that $a_{iJ} = 0$ if $x_i \nmid x^J$. As for the 1$^{\text{st}}$-case we do what we can. Specifically, again without loss of generality we can replace $x_i$ by,

$$x_i \mapsto x_i + \sum_{\lambda_i \neq \lambda_J} \frac{a_{iJ}}{\lambda_i - \lambda_J} \, x^J$$

so that $a_{iJ} = 0$ if $\lambda_i \neq \lambda_J$, while if $\lambda_i = \lambda_J$ we conclude that $u_{iK}$ is algebraic. Thus if we replace $\partial$ by,

$$\partial \mapsto \left( 1 + \sum_{\lambda_K = 0} u_{iK} \, x^K \right) \partial$$

then $u_{iK} = 0$ if $\lambda_K = 0$, so in fact we can suppose $a_{iJ} = 0$ for all $J$. Consequently, $\hat{\partial}$ has the form,

$$\left(1 + \sum_{\lambda_K \neq 0} u_{iK}\, x^K\right) \partial\,.$$

However if we replace $\hat{\partial}$ by,

$$\tilde{\partial} = \left(1 + \sum_{\lambda_K \neq 0} \tilde{u}_{iK}\, \xi^K\right)^{-1} \hat{\partial}$$

for $\tilde{u}_{iK}$ appropriate functions of coordinates $z$ in $\hat{\mathcal{O}}_{Z,\zeta}$ which restrict from coordinates in $\hat{\mathcal{O}}_{U,\zeta}$ annihilated by $\partial$, and of course $\tilde{u}_{iK} = u_{iK} \pmod{I_Z}$, then by II.7.2 $\tilde{\partial}$ is still semi-simple, with respect to a possibly different basis $\tilde{\xi}_i$ of the form $v_i\, \xi_i$, $v_i \equiv 1\, (I_Z^{k-1})$. To complete the induction, therefore, it suffices to observe, on supposing without loss of generality that $\xi_i = \tilde{\xi}_i$, that,

$$\xi_i \mapsto \xi_i - \sum_{\substack{\lambda_K = 0 \\ x^J = x_i\, x^K}} \tilde{b}_{iJ}(z)\, \xi^J$$

for $\tilde{b}_{iJ}$ satisfying much the same prescriptions as the $\tilde{u}_{iK}$ is still a trivialising basis for $\hat{\partial}$.

Consequently over an appropriately small affine $U$ containing $\zeta$, and bearing in mind that for any $\zeta' \in Z$ we know we can find appropriate coordinates in $\hat{\mathcal{O}}_{\mathcal{X},\zeta'}$ annihilated by $\partial$, we obtain formal subschemes $U_+$, $U_-$ of the completion $\hat{U}$ of $U$ in $Z$, whose subsequent completion at any $\zeta' \in Z \cap U$ is the non-negative Harder-Narismhan pair of II.7.8. Since we've already eliminated the possibility of plus going into minus by way of our tangent space calculation, we therefore obtain that these patch to formal subschemes, $X_+$, $X_-$ of the completion of $\mathcal{X}$ in $Z$, which completed at any point is the non-negative H-N pair, and of course we normalize so that $\forall\, \zeta \in Z$, $T_+(\zeta) = T_{X_+} \otimes \mathbb{C}(\zeta)$, $T_-(\zeta) = T_{X_-} \otimes \mathbb{C}(\zeta)$.

With this out of the way we can quickly proceed to a conclusion. To begin with complete $\mathcal{Y}$ in $Z$, call it $\hat{\mathcal{Y}}$, say, then this is an integral formal scheme supported on $Z$ whose completion at $z$ is, by construction, $X_+$, which itself is integral, so $\hat{\mathcal{Y}} = X_+$. In particular by the usual propagation of singularities along leaves/algorithmic resolution argument $\mathcal{Y}$ is not just smooth around $Z$, but smooth everywhere. Next consider projecting $\mathcal{Y}$ to $Z$, by sending an invariant 1-dimensional stack to its unique singular point, which we'll call $p$. At the completion level for any $\zeta \in Z$, $\hat{\mathcal{Y}}_\zeta$ is nothing other than the plus of the positive H-N pair. On the other hand $\mathcal{Y}_\zeta$ is covered by $-\mathbb{F}$ stacks, so $\hat{\mathcal{Y}}_\zeta$ is nothing other than the completion at $\zeta$ of a weighted projective stack $\mathcal{P}(da_1, \ldots, da_r)$ obtained by first deforming from cuspidal $-\mathbb{F}$ stacks to smooth ones, then applying the recipe of III.3.5. Equally $\mathcal{Y}_\zeta$ is a closed stack, so it certainly contains the $\mathcal{P}(da_1, \ldots, da_r)$, and since it cannot contain any other $-\mathbb{F}$-stacks except those in the weighted projective stack at worst its reduction is the said stack. $\mathcal{Y}_\zeta$ is, however, regularly embedded, since $Z$ is smooth, and equal to the $\mathcal{P}(da_1, \ldots, da_r)$ in a neighbourhood of $\zeta$, so in fact they are equal everywhere. Consequently $p : \mathcal{Y} \to Z$ is a relatively smooth fibration as is the non-scheme locus, whence $d$ is independent of $\zeta$, giving a fibration in weighted projective stacks, and as we've seen our particular weighted projective stacks are simply connected so the fibration is actually a bundle in the analytic topology, and in actual fact, as we'll establish later, in the Zariski. In any case to summarise we've established,

**III.5.1 Large Fact.** *Given a $-1/d_1\, \mathbb{F}$ curve $f : \mathcal{L} \to \mathcal{X}$ parallel to an extremal ray $R$ in Néron-Severi, with $z$ the unique singular point, then after multiplication by a suitable constant, a linearisation in* $\mathrm{End}(\Omega_{\mathcal{X}} \otimes \mathbb{C}(z))$ *of a generator $\partial$ of the foliation is a diagonal matrix* $\mathrm{diag}\{a_1, \ldots, a_r, 0, -b_1, \ldots, -b_t\}$*, $a_i$, $b_j \in \mathbb{N}$ without common divisor and $s$ zeroes. Better still, normalising so that the tangent space to $f(\mathcal{L})$ lies in the positive eigenspace, there is an $R$-extremal stack $\mathcal{Y}$ containing $f$ such that,*

(a) $\mathcal{Y}$ *contains a unique, smooth $s$-dimensional component $Z$ of the singular locus of $\mathcal{F}$.*

(b) $\mathcal{Y}$ *retracts onto $Z$ by $p$, with $p : \mathcal{Y} \to Z$ a bundle in the analytic topology of weighted projective stacks of the form $\boldsymbol{\mathcal{P}}(da_1, \dots, da_r)$, for some $d \in \mathbb{N}$, with $d_1 = da_1$ if $\mathcal{L}$ is tangent to the $a_1$ eigenvector.*

(c) *This holds good not just at the level of spaces, but for the induced foliated structure too, i.e. $(\mathcal{Y}, \mathcal{F}|_y) \xrightarrow{\sim} (\boldsymbol{\mathcal{P}}(da_1, \dots, da_r), \mathcal{R})$.*

(d) *Every extremal stack is of this form.*

It only remains to check the various assertions on extremality, and since $\mathcal{Y}$ is certainly contained in an extremal subvariety $\mathcal{Y}'$ we may as well just check (d) for arbitrary $\mathcal{Y}'$. Now the argument that the intersection of $\mathcal{Y}$ with $\mathrm{sing}(\mathcal{F})$ was a smooth component of the singular locus works for $\mathcal{Y}'$ verbatim, so call the said component $Z'$ (necessarily $Z$ if $\mathcal{Y}' \supset \mathcal{Y}$). Additionally for some (and indeed any) $z' \in Z'$, there is a $-\mathbb{F}$-stack $f' : \mathcal{L} \to \mathcal{Y}'$ containing $z'$. Whence by III.4 there is a smooth $-\mathbb{F}$-stack $g : \mathcal{L} \to \mathcal{X}$, perhaps not a priori in $\mathcal{Y}'$, containing $Z'$, so the previous discussion about the H-N pair all goes through to establish that there are formal sub-schemes $X'_+$, $X'_-$ of the completion of $\mathcal{X}$ in $Z'$ which contain the germ of any $-\mathbb{F}$-stack (extremal or otherwise) passing through $Z'$. As such the completion of $\mathcal{Y}'$ in $Z'$, say $\hat{\mathcal{Y}}'$, is contained in either $X'_+$ or $X'_-$, so we're already done if $\mathcal{Y} \subset \mathcal{Y}'$, since everything is irreducible. Otherwise, say without loss of generality $\hat{\mathcal{Y}}'$ is $X'_+$, so we can even take our smooth $-\mathbb{F}$-stack $g$ to have germ at $z'$ factoring through $X'_+$. At this point however, we know that $X'_+$ is the completion of an appropriate $\boldsymbol{\mathcal{P}}(a'_1, \dots, a'_r)$ bundle over $Z'$ in a section, so indeed $\mathcal{Y}'$ is the said bundle. Note in particular, therefore, that the number of extremal subvarieties (independent of what rays are involved) is no more than twice the number of smooth components of the singular locus, and they're either all disjoint, or if they do intersect they do so at exactly everywhere in a mutual smooth component of $\mathrm{sing}(\mathcal{F})$, locally about which, of course, their completions would be the H-N pair. As it happens we'll subsequently see that this latter possibility only occurs for foliations which are pencils of rational curves, although for the moment we'll content ourselves with observing,

**III.5.2 Corollary to the above discussion.** *The number of extremal rays in the half space, $\mathrm{NE}_{K_{\mathcal{F}}<0}$ is finite.*

Finally let us introduce,

**III.5.3 More Notation.** An extremal substack which at one point $z$ (and whence any) of its intersection with the singular locus $\mathcal{F}$ affords an appropriately normalised linearisation of a generator $\partial$ in $\mathrm{End}(\Omega_{\mathcal{X}} \otimes \mathbb{C}(z))$ as $\mathrm{diag}(a_1, \dots, a_r, 0, -b_1, \dots, -b_t)$ will be denoted $\mathcal{Y}(da_i, s, b_j)$ where $s$ is the dimension of the kernel, and of course $a_i, b_j \in \mathbb{N}$ satisfy $\gcd(a_1, \dots, a_r, b_1, \dots, b_t) = 1$, with,

$$a_1 \geq a_2 \geq \dots \geq a_r \geq 0 \geq -b_1 \geq \dots \geq -b_t\,, \quad d \in \mathbb{N}\,.$$

## III.6. More on the H-N filtration

Our first task will be to establish that an extremal stack is actually a bundle of weighted projective stacks in the Zariski topology. To this end let $\mathcal{Y}(da_i, s, b_j)$ be the said stack, and $Z$ its intersection with the singular locus $\mathcal{F}$, then wholly unsurprisingly we have,

**III.6.1 Minor Fact.** *Let $\pi : \mathcal{Y}(da_i, s, b_j) \to Y(da_i, s, b_j)$ be the moduli, then $Y(da_i, s, b_j)$ is projective.*

**Proof.** If $Y'$ in $X$, the moduli of $\mathcal{X}$, is the image of $\mathcal{Y}(a_i, s, b_j)$ then $Y(a_i, s, b_j) \to Y'$ is quasi-finite. Moreover $\mathcal{Y}(a_i, s, b_j)$ is separated, whence so is $Y(a_i, s, b_j)$ and a fortiori $Y(a_i, s, b_j) \to Y'$, whence by [A1] 3.3, $Y(a_i, s, b_j)$ is a scheme. On the other hand $Y(a_i, s, b_j)$ is proper, so $Y(a_i, s, b_j) \to Y'$ is finite, and $Y(a_i, s, b_j)$ is projective.

Now let $\mathbb{P}\left(1, \frac{a_i}{a}\right)$, $a = \gcd(a_1, \dots, a_r)$ be the weighted projective space $\mathbb{P}\left(1, \frac{a_1}{a}, \dots, \frac{a_r}{a}\right)$ as opposed to a weighted projective stack, and consider the functor on $Z$ schemes,

$$\mathrm{Hom}\left(Y(da_i,\, s,\, b_j),\, \mathbb{P}\left(1, \frac{a_i}{a}\right)\right)(T) = \mathrm{Hom}\left(Y(da_i,\, s,\, b_j) \times_Z T,\, \mathbb{P}\left(1, \frac{a_i}{a}\right) \times T\right)$$

then $\mathrm{Hom}\left(Y(da_i,\, s,\, b_j),\, \mathbb{P}\left(1, \frac{a_i}{a}\right)\right)$ is locally of finite type, and for $z \in Z$, with $\hat{Z}$ the completion at $z$, we have by virtue of $\mathcal{Y}(a_i,\, s,\, b_j)$ being an analytic bundle and/or our reflections on deformations, sections of the said Hom scheme over $\hat{Z}$. Consequently we may apply the implicit function theorem of [A1], to find an étale neighbourhood $U$ of $Z$ such that we have maps,

$$Y(a_i,\, s,\, b_j) \times_Z U \to \mathbb{P}(1,\, a_i/d) \times U$$

which are isomorphisms on an appropriately large thickening of $Z$ at $z$, so by shrinking $U$ as necessary, we conclude that $Y(da_i,\, s,\, b_j)$ is a bundle of weighted projective spaces in the étale topology.

The next thing to consider is another minor variant of the Harder-Narismhan filtration, this time of $\mathcal{Y}(da_i,\, s,\, b_j)$ completed in $Z$. Specifically we have a filtration of formal schemes supported on $Z$,

$$Z_1 = \hat{Y}_0 \subset \hat{Y}_1 \subset \cdots \subset \hat{Y}_k = \mathcal{Y}(\widehat{a_i,\, s,\, b_j})$$

according to a complete repetition free list of eigenvalues $\alpha_1, \dots, \alpha_k$, which when completed at any $z \in Z$ is for some (and indeed any) semi-simple generator of the foliation is in the $i^{\text{th}}$ place cut out by eigenfunctions of eigenvalue $> \alpha_i$. The existence of such of filtration at a point has already been noted in II.7.9, and the extension over $Z$ is a small perturbation of the discussion in the previous section. In any case the germ of a $-\mathbb{F}$ stack at its singular point factoring through any given $\hat{Y}_i$ is a closed condition on the moduli of the moduli of such stacks, while we know that at any $z$ the fibre of $\hat{Y}_i$ over $z$ is covered by the germs of the same, so in fact the $\hat{Y}_i$ are actually completions of closed sub-stacks $\mathcal{Y}_i$ of $\mathcal{Y}$ in $Z$. Argueing precisely as before, i.e. $\mathcal{Y}_i$ is formally smooth around $Z$, so its smooth etc. we conclude that each $\mathcal{Y}_i$ is a bundle of weighted projective stacks of the form $\boldsymbol{\mathcal{P}}(da_1, \dots, da_{r_i})$ for $a_1, \dots, a_{r_i}$ the eigenvalues at least $\alpha_i$, whose moduli $Y_i$ is a bundle of weighted projective spaces in the étale topology. At this point we can simply proceed by induction. Indeed let $S_i$ be the sheaf of $\mathcal{O}_Z$-algebras,

$$\bigoplus_e (p_i)_* \, K_{\mathcal{F}}^{-e}$$

where $p_i : \mathcal{Y}_i \to Z$ is the projection. Now the moduli $Y_i$ of each $\mathcal{Y}_i$ certainly maps to $\mathrm{Proj}\, S_i$, and the map is everywhere an étale isomorphism, so it's a Zariski isomorphism, and it remains to find appropriate weighted generators in the Zariski topology, which we do by induction. The case of $i = 1$ is essentially that of projective bundles, so there's nothing to do (except to note that the smallest $d_1 \in \mathbb{N}$, such that $p_* K_{\mathcal{F}}^{-d_1}$ has rank 2 gives us the normalising factor $d$, by way of, $d_1 = d\alpha_1$). Furthermore étale maps are faithfully flat, so we have an exact sequence,

$$0 \to J_i \to S_{i+1} \to S_i \to 0$$

where the last map is a map of graded algebras, all of whose graded pieces are, by a standard descent arguement, locally free $\mathcal{O}_Z$-modules. Consider therefore the $d\alpha_{i+1}$-th graded piece, then $J_i(d\alpha_{i+1})$ is locally free on generators $X_{i+1,p}$, $1 \le p \le k_{i+1}$, say, where $k_{i+1}$ is the dimension of the eigenspace for $\alpha_{i+1}$. Moreover the sequence of graded pieces can be locally split in the Zariski topology, so inductively we obtain a collection $X_{qp}$, $q \le i+1$, $1 \le p \le k_q$ of local Zariski sections of degree $d\alpha_1$ which freely generate $S_{i+1}$. As such we're established that each $Y_i$, and in particular, $Y(da_i,\, s,\, b_j)$ is a bundle of weighted projective spaces in the Zariski topology. To conclude from this that $\mathcal{Y}(da_i,\, s,\, b_j)$ is a bundle, we can first put the minimal smooth stack structure $\mathcal{Y}'$, say, with moduli $Y(da_i, s, b_j)$ – which is in fact the Gorenstien cover of the induced radial foliation – and argue as above to identify the ramification locus of $\mathcal{Y}(da_i,\, s,\, b_j) \to \mathcal{Y}'$ with a Zariski bundle of weighted projective stacks, and so deduce,

**III.6.2 Concluding Fact.** *We can add to* III.5.1 *and* III.6.1 *the precision, the map $p : (\mathcal{Y}, \mathcal{F}) \to Z$ presents an extremal substack as a Zariski bundle of $(\boldsymbol{\mathcal{P}}(da_1, \dots, da_r), \mathcal{R})$'s.*

# IV. Flip, Flap, Flop

## IV.1. Contractions

The basis of the flip theorem will actually be nothing other than a minor variant of Castelnuovo's contraction theorem. To motivate this consider,

**IV.1.1 Data.** *$\pi : \mathcal{X} \to X$, a smooth stack with projective moduli, $\mathcal{D}$ an integral Cartier divisor on $\mathcal{X}$ and $D$ the corresponding $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$.*

Now it may not be true that $\mathcal{D}$ is the fibre product $\pi^{-1}(D)$, however if $eD$, $e \in \mathbb{N}$, is actually Cartier, then we have,

**IV.1.2 Fact.** *If $\mathcal{D}$ is not contained in the ramification locus $\pi^{-1}\,\mathcal{O}_X(-eD) = \mathcal{O}_{\mathcal{X}}(-e\mathcal{D})$, i.e. with natural scheme structures, $e\mathcal{D} = \pi^{-1}(eD)$.*

**Proof.** This is just the usual vanishing of local cohomology in the presence of the $S_2$-condition, so it would even work with $\mathcal{X}$ normal.

Observe moreover that since $\pi_*$ is exact, we have an exact sequence,

$$0 \to \pi_*\mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \to \mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{D}} \to 0\,.$$

Thus $\mathcal{O}_D = \pi_*\mathcal{O}_{\mathcal{D}}$, and whence for any line bundle $B$ on $X$,

**IV.1.3 Further Small Fact.** $H^0(\mathcal{D}, \pi^*B) = H^0(D,\, B)$.

As such following [K-3] III.1.8.1, we can combine all of this to form a contraction criteria, viz:

**IV.1.4 Criteria.** Let $A$ be a very ample bundle on $X$, and suppose for $k \in \mathbb{N}$, divisible by $e$,

(a) $H^1(X,\, A) = 0$

(b) $H^1(\mathcal{D},\, \mathcal{O}_{\mathcal{D}} \otimes \pi^*A(j\mathcal{D})) = 0$, $1 \le j \le k-1$

(c) $\mathcal{O}_{\mathcal{D}} \otimes \pi^*A(k\mathcal{D})$, or equivalently by, $\mathcal{O}_D \otimes A\left(\frac{k}{e}\, D\right)$, is generated by global sections, with cont : $D \to D_0$ the corresponding map.

Then in fact $A(kD)$ is generated by global sections, so that if Cont : $X \to X_0$ is the corresponding map, then,

(a)' Cont : $X\backslash D \xrightarrow{\sim} X_0\backslash D_0$

(b)' $\mathrm{Cont}|_D = \mathrm{cont} : D \to D_0$

(c)' $X_0$ is projective.

**Proof.** Just use the exactness of $\pi_*$. □

To apply the criteria we'll need appropriate cohomological vanishing. The following minor variant of Serre vanishing, cf. [E], will suffice, viz:

**IV.1.5 Vanishing Fact.** *Let $\mathcal{F}$ be a coherent sheaf on a scheme $W$ with, $A$ an ample bundle, and $H_1, \ldots, H_n$ semi-ample bundles then for all but finitely many integers $p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$,*

$$H^i(W,\, \mathcal{F} \otimes A^p \otimes H_1^{p_1} \otimes \cdots \otimes H_n^{p_n}) = 0\,, \quad i \ge 1\,.$$

**Proof.** We proceed exactly as per Serre vanishing, i.e. on replacing $A$ and the $H_i$ by suitable tensor powers $d$, $d_i$, respectively, not to mention applying the desired outcome to $\mathcal{F} \otimes A^q \otimes H_1^{q_1} \otimes \cdots \otimes H_n^{q_n}$, $1 \leq q < d$, $0 \leq q_i < d_i$, we can assume without loss of generality that $A$ is very ample, and the $H_i$ are generated by global sections. As such we can reduce to supposing that $W$ is a product of projective spaces provided we prove a slightly different fact, i.e.

**IV.1.6 Similar Fact.** *Let $P = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a product of projective spaces, $L_i$ the pull-back of the tautological bundle on the $i^{\text{th}}$-factor, and $\mathcal{F}$ a coherent sheaf on $P$ then for all but finitely many $p_1, \ldots, p_k \in \mathbb{N}$,*

$$H^i(P, \mathcal{F} \otimes L_1^{p_1} \otimes \cdots \otimes L_k^{p_k}) = 0\,, \qquad i \geq 1$$

which of course we just do by the usual decreasing induction on $i$, which itself is trivially true for $i \geq n_1 + \cdots + n_k$. Otherwise we have an exact sequence,

$$0 \to \mathcal{E} \to (L^{-m})^{\oplus a} \to \mathcal{F} \to 0$$

where $L = L_1 \otimes \cdots \otimes L_k$, $m \in \mathbb{N}$ is sufficiently large, and $a \in \mathbb{N}$. On the other hand, $H^i(P, L_1^{p_1} \otimes \cdots \otimes L_k^{p_k}) = 0$, for say, $i \geq 1$, and $p_j \geq 0$, $1 \leq j \leq k$, so we're done.

Observe in particular that we have,

**IV.1.7 Corollary.** *Let $A$, $L_1, \ldots, L_n$ be bundles on a stack $\mathcal{W}$ such that a suitable multiple of $A$ is ample on the moduli $\pi : \mathcal{W} \to W$ and such that multiples of the $L_i$ are generated by global sections, then for all but finitely many $p \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$,*

$$H^i(\mathcal{W}, A^p \otimes L_1^{p_1} \otimes \cdots \otimes L_n^{p_n}) = 0\,, \qquad i \geq 1\,.$$

**Proof.** Let $d$, $d_1, \ldots, d_n$ be the said multiples, then necessarily $A^{\otimes d}$, $L_i^{\otimes d_i}$, $1 \leq i \leq n$, are bundles on $W$. Consequently putting $p = nd + q$, $p_i = n_i d_i + q_i$, $0 \leq q < d$, $0 \leq q_i < d$, we can just use the exactness of $\pi_*$ and apply IV.1.5, with $\mathcal{F} = \pi_*(A^q \otimes L_1^{q_1} \otimes \cdots \otimes L_n^{q_n})$, $H_i = L_i^{\otimes d_i}$, and of course $A = A^{\otimes d}$ for want of having been more notationally careful. $\square$

Now let's apply all of this to an extremal sub-stack $\mathcal{Y}(da_1, \ldots, da_r, s, b)$ which is a divisor, and which for convenience we'll denote $\mathcal{D}$. As ever $(\mathcal{X}, \mathcal{F})$ is our ambient smooth stack with terminal singularities at the non-scheme like points, and $\pi : \mathcal{X} \to X$ its moduli. To keep things clear $D$ will be the image on $\mathcal{D}$ in $X$, which may be distinct from its moduli $|\mathcal{D}|$. In any case, associated to the implicit extremal ray $R$, there is a supporting Cartier divisor $H_R$, and without loss of generality we may suppose that $A = H_R - K_{\mathcal{F}}$ is an ample $\mathbb{Q}$-divisor on $X$. As such if $i : Z \hookrightarrow X$ is the embedding of the intersection of $\mathcal{D}$ with the singular locus of $\mathcal{F}$, then we already know that $i^* K_{\mathcal{F}}$ is trivial, so in fact $i^* H_R$ is ample. Moreover for some bundle $B$ on $Z$ we have by I.9.10, that $\mathcal{O}_{\mathcal{D}}(\mathcal{D}) = p^* B \otimes K_{\mathcal{F}}^b$, whence up to replacing $H_R$ by a sufficiently large multiple, we can suppose, without loss of generality, that $i_Z^* H_R$ and $i_Z^* H_R + B$ are very ample. We wish to apply the criteria IV.1.4 to a sufficiently large and divisible multiple of $A + qH$, where $q \in \mathbb{N}$ is itself a sufficiently large integer to be chosen. Whence in reverse order we observe,

(c) There is a $m_0 \in \mathbb{N}$ such that for all $q$, $\mathcal{O}_{\mathcal{D}}\left(m\,A + mq\,H_R + \frac{m}{b}\mathcal{D}\right)$ is generated by global sections provided $m \geq m_0$ and sufficiently divisible, i.e. the bundle in question is a bundle not just on $\mathcal{D}$ but on $|\mathcal{D}|$.

Indeed if $m = bm'$, then the bundle is just, $m'(H + B + (q + b - 1)H)$, which is a sum of very ample bundles, so this is obvious, as is the fact that, $\mathrm{cont} : \mathcal{D} \to Z$ is nothing other than the projection $p$.

(b) There is a $q_0 \in \mathbb{N}$ such that for all $m$ divisible by $b$,

$$H^1(\mathcal{D}, \mathcal{O}_{\mathcal{D}}(m\,A + mq\,p^* H_R + j\mathcal{D})) = 0\,, \quad q \geq q_0\,, \quad 1 \leq j \leq \frac{m}{b} - 1\,.$$

Here we can simply rewrite the bundle in question as,

$$(m - jb)\, A + (mq + j(b-1))\, p^* H_R + j\, p^*(H_R + B)$$

so that IV.1.7, guarantees the existence of such a $q_0$.

(a) Take $q \geq q_0$ as in (b), and $m \geq m_0$ sufficiently large and divisible to guarantee that $m(A + q\, H_R)$ is very ample and without cohomology.

Consequently we've certainly constructed a contraction $\rho : X \to X_0$, where to be on the safe side we take the Stein factorisation of the Cont of in order to guarantee that $X_0$ is unique. The criteria also gives for free,

**IV.1.8 Fact.** *$X_0$ is $\mathbb{Q}$-factorial.*

**Proof.** Indeed a minor variant of the above argument works with an essentially arbitrary $A$, or to be more precise if $F$ is any ample bundle on $X$ then we can construct $X_0$ by way of the global sections of sufficiently large and divisible multiples of $F + \alpha D + qH$, where $\alpha \in \mathbb{Q}$ is determined by $F$, and any $q$ bigger than some $q_F$ depending on $F$ will do. In particular, $\rho_* F + q\rho_* H$ is a $\mathbb{Q}$-divisor on $X_0$, for all $q \geq q_F$. However if $E$ is any divisor on $X_0$, then its proper transform is the difference $F - F'$ of a pair of amples, so that taking an appropriately large $q$, e.g. $\max\{q_F,\, q_{F'}\}$, we see that $E$ is $\mathbb{Q}$-Cartier as required. □

So while this is perfectly sufficient for applications, let's restore the stack structure. By I.7.4 we know that the stack structure of $\mathscr{X}$ is uniquely defined by the log-structure $(X,\, B,\, \mathcal{F})$ associated to the divisorial ramification of $\mathscr{X}$ over $X$. Certainly we have a contraction $(X_0,\, B_0,\, \mathcal{F}_0)$ of this data, and we require to prove,

**IV.1.9 Claim.** *$(X_0,\, B_0,\, \mathcal{F}_0)$ has log-canonical singularities, which are in fact terminal at $|B_0| \cup \operatorname{sing}(X_0)$.*

**Proof.** Putting, $K_{\tilde{\mathcal{F}}} = K_{\mathcal{F}} + \sum_i \left(1 - \frac{1}{e_i}\right) B_i$, and $K_{\tilde{\mathcal{F}}_0} = K_{\mathcal{F}_0} + \sum_i \left(1 - \frac{1}{e_i}\right) B_{0i}$, and denoting the exceptional divisor now by $E$, then by construction,

$$K_{\tilde{\mathcal{F}}} = \rho^* K_{\tilde{\mathcal{F}}_0} + \frac{1}{db}\, E\,.$$

Consequently if $F$ is an exceptional rank 1 discrete valuation of the function field, distinct from $E$ with things modeled on $X_0$, then,

$$a_{\mathcal{F}_0, D_0}(F) = a_{\mathcal{F}, D}(F) + \frac{\mu(F)}{eb}$$

where $\mu$ is the multiplicity of $E$ along $F$. As such $(X_0,\, D_0,\, \mathcal{F}_0)$ certainly has log-canonical singularities. Better still if the centre of $F$ in $X_0$ is contained in the locus $Z_0$ to which $E$ is contracted, then its center on $X$ is contained in $E$, and $\mu(F) > 0$. Whence, at every point of $Z_0$, $(X_0,\, D_0,\, \mathcal{F}_0)$ is log-terminal, and thus in fact terminal by I.6.11, while terminality along $|D_0|$ is obvious from the same at $|D|$. On the other hand $\operatorname{sing}(X_0) \subset Z_0 \cup \rho(\operatorname{sing} X)$, and terminality along $\operatorname{sing}(X)$ is a priori given by our hypothesis on $(\mathscr{X},\, \mathcal{F})$ thanks to I.7.3. □

With this out of the way we can simply appeal to I.7.4, to restore the stack structure, i.e. we have a foliated smooth stack $(\mathscr{X}_0, \tilde{\mathcal{F}}_0)$ determined by the log-data $(X_0,\, D_0,\, F_0)$ which fits into a diagram,

$$\begin{tikzcd}
(\mathscr{X},\, \tilde{\mathcal{F}}) \arrow[r, dashed, "\rho"'] \arrow[d, "\pi"'] & (\mathscr{X}_0,\, \tilde{\mathcal{F}}_0) \arrow[d, "\pi"] \\
(X,\, D,\, F) \arrow[r, "\rho"'] & (X_0,\, D_0,\, F_0)
\end{tikzcd}$$

Note however, that the contraction map on stacks cannot in general be everywhere defined, whether at the stack level $\mathcal{Y}(da_1, \ldots, da_r, s, b)$ could be contracted to a singular stack is a mute point of little relevance which will be ignored. It is, therefore, perhaps useful to summarise by way of,

**IV.1.10 Summary/Remarks/Terminology.** Given a divisorial extremal substack of a foliated smooth stack with terminal singularities at the non-scheme like points and log-canonical singularities elsewhere there is a flop, $\rho : (\mathcal{X}, \tilde{\mathcal{F}}) \dashrightarrow (\mathcal{X}_0, \tilde{\mathcal{F}}_0)$ to a foliated smooth stack enjoying the same prescriptions on the singularities. Here the definition of flop is not the usual one, but rather the data of the above diagram, where we think of the rightmost $\pi$ as being almost log-étale, $\rho \circ \pi$ as a "bad" contraction, and note that $\rho$ is well defined on objects but not on arrows.

## IV.2. A Formal Approach

Although rich in information, and adequate for applications to foliations which are not fibrations in rational curves, the contraction theorem of IV.1 uses global information on extremal rays to prove something which is in fact of a purely local character. Not surprisingly this is not wholly desirable, since the requirement of verifying global hypothesis before concluding a local statement is just plain stupid. As such we wish to consider the following contraction problem,

**IV.2.1 Problem.** *Let $\mathfrak{X}$ be a smooth formal stack, with underlying stack $\mathcal{X}_0$, such that $\mathcal{X}_0 \hookrightarrow \mathfrak{X}$ is a divisor with $\mathcal{X}_0$ a bundle of weighted projective stacks $\mathbf{P}(a_1, \ldots, a_r)$ over a smooth scheme (or indeed separated algebraic space) $Z$, then can we contract $\mathcal{X}_0$ if $\mathcal{O}_{\mathcal{X}_0}(-\mathcal{X}_0)$ is ample relative to $Z$.*

The precise meaning of contraction is as follows. The formal stack $\mathfrak{X}$ will have some moduli, $\pi : \mathfrak{X} \to \Xi$, where $\Xi$ is a separated formal algebraic space (notice, as we've said stacks, formal or otherwise, are always separated so the existence of $\Xi$ follows from a minor variant of [K-M], or alternatively in applications everything is actually a completion) with underlying space $\Xi_0$, so that in the spirit of IV.1.10, what we'll actually do is construct a formal contraction of $\Xi_0$ (in the sense of [A3]) with a view to a leisurely restoration of the stack structure.

We begin by discussing the case where $Z$ is affine, and denote by $\mathcal{O}(1)$ the tautological bundle on $\mathcal{X}_0$, so that by hypothesis $\mathcal{O}_{\mathcal{X}_0}(\mathcal{X}_0) = \mathcal{O}(-b)$ for some $b \in \mathbb{N}$. In so much as it exists the contraction is basically the Stein factorisation of $\mathfrak{X}$ over its functions so we simply compute this. We could of course work at the $\Xi$ level, however this involves working with symmetric powers of ideals, rather than powers, and would in any case just be a complicated way of obscuring the correct structure. In any case the appropriate diagram, with exact rows and columns, is as follows,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathcal{X}_0}(-n\mathcal{X}_0) \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathcal{X}_n}(-\mathcal{X}_0) \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{X}_{n-1}}(-\mathcal{X}_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathcal{X}_0}(-n\mathcal{X}_0) \arrow[r] & \mathcal{O}_{\mathcal{X}_{n+1}} \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{X}_n} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_{\mathcal{X}_0} \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathcal{X}_0} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Now we know by I.9.7, that $\mathcal{O}_{\mathcal{X}_0}(-n\mathcal{X}_0)$, $n \geq 0$, is acyclic, so the same is true of $\mathcal{O}_{\mathcal{X}_m}(-n\mathcal{X}_0)$, for any $m, n \geq 0$ by an obvious induction. Consequently on taking global sections everything is still exact, i.e. we have a diagram, with the notation established by the same,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_{bn} \arrow[r] \arrow[d, equal] & I_{n+1} \arrow[r] \arrow[d] & I_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_{bn} \arrow[r] & \Gamma_{n+1} \arrow[r] \arrow[d] & \Gamma_n \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_Z \arrow[r, equal] \arrow[d] & \mathcal{O}_Z \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where $S = \bigoplus_k S_k$ is the weighted graded algebra of I.9.5, and we confuse (in the tradition of the subject) functions on $Z$ with the structure sheaf. The important thing however is that Mittag-Leffler is valid for the directed systems defined by $I$ and $\Gamma$ so in fact we have an exact sequence,

$$0 \longrightarrow I = \varprojlim_n I_n \longrightarrow \Gamma = \varprojlim_n \Gamma_n \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

with associated graded algebra, $\operatorname{gr}\Gamma = \bigoplus_{n=0}^{\infty} S_{bn}$, where of course we regard $\Gamma$ as an admissible ring for the topology defined by,

$$0 \longrightarrow I^{(n)} \longrightarrow \Gamma \longrightarrow \Gamma_n \longrightarrow 0\,.$$

There is equally no difficulty in making similar diagrams with $\mathcal{O}_{\mathcal{X}_m}$ on the bottom row in place of $\mathcal{O}_{\mathcal{X}_0}$, and thus identifying $I^{(n)}$ with $\varprojlim_{p\geq n} H^0(\mathcal{O}_{\mathcal{X}_p}(-n\,\mathcal{X}_0))$. Better still $\operatorname{gr}\Gamma$ is a finitely generated $\mathcal{O}_Z$ algebra, so $\Gamma$ is noetherian (indeed formally of essentially finite type) and we could of course identify the $I^{(n)}$ topology with the $I^n$ topology. However formal schemes are really built from their nilpotent pieces, so it's unsurprisingly more appropriate to stick with the $I^{(n)}$, and identify the potential contraction with $\mathfrak{Y} := \operatorname{Spf}\Gamma$. In addition we have a free $\mathcal{O}_Z$-basis of $I^{(n)}/I$ which we may identify with monomials of the form $X_0^{t_0}\,X_1^{t_1}\ldots X_r^{t_r}$ such that $t_0 + a_1t_1 + \cdots + a_rt_r = bn$. Indeed choosing a quasi-coefficient ring for $\mathcal{O}_Z$, or for that matter just shrinking $Z$ since we're working in the étale topology, we can identify $\Gamma$ with the subring of $\mathcal{O}_Z[[X_0, \ldots, X_r]]$ generated by monomials of this form. Regardless, the important thing is that we define ideals $J^{(n)}$ of $\Gamma$ as the ideals generated by the said free basis of $I^{(n)}/I$, and form the graded algebra $T := \bigoplus_n J^{(n)}$, with linear topology defined by the ideals $I^{(n)}T$. With this in mind, we assert,

**IV.2.2 Claim.** *$\Xi$ is isomorphic to the weighted formal blow up* $\operatorname{Prof} T$.

Before proceeding to a proof, let's explain what Prof is since for some reason it's one of those notions that didn't make it into EGA.

**IV.2.3 Intermission** (on formal Proj). Quite generally let $X = \operatorname{Spf} A$ be an affine formal scheme, and $T$ a graded $A$-module, with an admissible linear topology $T_\lambda$, $\lambda \in \Lambda$, of graded ideals such that $j : A \to T$ is continuous. We can subsequently form the schemes, $\operatorname{Proj}(T/T_\lambda) \to \operatorname{Spec}(A/j^{-1}T_\lambda)$, and take the direct limit in the category of formal schemes (or better formal formal schemes) to obtain a formal scheme $\operatorname{Prof} T$ over $X$. As such there are actually several different notions of formal/weighted formal blowing up according to the topology employed. The particular choice in IV.2.2 corresponds to completing a usual weighted blow up in the pre-image of the support of the irrelevant ideal. Consequently since we're local and everything is algebraisable (i.e. $\mathrm{Spf}\,\Gamma$ is the completion of $\mathrm{Spec}\,\Gamma$ in $I$) the claim automatically implies that $\Xi \to \mathrm{Spf}\,\Gamma$ is a formal modification in the sense of Artin.

To prove the claim observe that we have a map, $\pi : \mathfrak{X} \to \mathrm{Proj} \bigoplus_n H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(-n\,\mathcal{X}_0))$ , which for the sake of precision could be identified with Prof in the discrete topology. Equally we have such a map for $\mathrm{Prof}\, T$, and even an embedding,

$$\mathrm{Prof}\, T \hookrightarrow \mathfrak{Y} \times \mathrm{Proj} \bigoplus_n H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(-n\,\mathcal{X}_0))$$

and of course the projection $\rho : \mathfrak{X} \to \mathfrak{Y}$. Manifestly $\rho \times \pi$ factors through $\mathrm{Prof}\, T$, and whence by the universal property of $\Xi$ we obtain a map,

$$\rho \times \pi : \Xi \to \mathrm{Prof}\, T$$

and it remains to check that this is indeed an isomorphism. To this end an alternative description of $\mathrm{Prof}\, T$ will be useful. In the fist place consider the cone over $\mathbb{P}^r_Z$ embedded in $|\mathcal{O}_{\mathbb{P}^r_Z}(b)|$ by the full linear system, and let $\mathfrak{Y}^\#$ be the completion of the said cone in the singular section, so in fact $\mathfrak{Y}^\#$ is the formal quotient $\hat{\mathbb{A}}^{r+1}_Z/\mu_b$, where $\mu_b$ acts diagonally on a standard formal affine coordinate system. Better still we have a $\mu_{a_1} \times \cdots \times \mu_{a_r}$ action on the final $r$-coordinates in the standard way, so that if $\lambda : \tilde{\mathfrak{Y}} \to \mathfrak{Y}^\#$ is the formal blow up of $\mathfrak{Y}^\#$ in the singular section then the $\mu_{a_1} \times \cdots \times \mu_{a_r}$ action-extends, Prof commutes with group action, and we conclude to a commutative diagram,

$$\begin{tikzcd}
\mathrm{Prof}\, T = \tilde{\mathfrak{Y}}/\mu_{a_1} \times \cdots \times \mu_{a_r} \arrow[d, "\lambda"] & \tilde{\mathfrak{Y}} \arrow[l] \arrow[d, "\lambda"] \\
\mathfrak{Y} = \mathfrak{Y}^\#/\mu_{a_1} \times \cdots \times \mu_{a_r} & \mathfrak{Y}^\# \arrow[l]
\end{tikzcd}$$

We can also do a similar thing on the $\Xi$ side. Indeed any non-negative multiple of $\mathcal{O}_{\mathcal{X}_0}(1)$ is acyclic by I.9.6/7, so there is no difficulty in lifting to $\mathfrak{X}$ the sections defining coordinate hyperplanes à la III.5. Having made such liftings as per op. cit. we may extract their roots to obtain a formal covering stack $\tilde{\mathfrak{X}} \to \mathfrak{X}$ whose underlying stack is now the classifying stack $[\mathbb{P}^r/\mu_{a_1} \times \cdots \times \mu_{a_r}]$, with exactly the same moduli $\Xi$. Equally the various $\mu_{a_1} \times \cdots \times \mu_{a_r}$ actions are compatible, and $\mathbb{P}^r \to [\mathbb{P}^r/\mu_{a_1} \times \cdots \times \mu_{a_r}]$ is étale, which in turn yields a formal covering stack with underlying stack $\mathbb{P}^r$, so that we're reduced to the case of $\mathcal{X}_0 \xrightarrow{\sim} \mathbb{P}^r$. However in this case $\mathrm{Prof}\, T = \tilde{\mathfrak{Y}}$ is smooth as is $\Xi$, while if $E$ is the exceptional divisor on $\tilde{\mathfrak{Y}}$ then $(\rho \times \pi)^* E = \Xi_0$ by construction, so $\Xi \to \tilde{\mathfrak{Y}}$ is étale, and $\Xi$ is connected, so we're done.

Fortunately contraction is unique not just up to isomorphism, but up to the identity, so on covering our original formal stack $\mathfrak{X}$ of IV.2.1 by sufficiently small affines we conclude to a positive solution of our original problem. The problem solved we can appeal to Artin's convergence theorem to obtain,

**IV.2.4 Fact/Summary** (according to the proper generality in which the above works, and not just the special subcase that was discussed). Let $X$ be an algebraic (or complex analytic) space with quotient singularities, and $D \subset X$ a divisor which is a bundle of weighted projective spaces over some smooth $Z$, which becomes a bundle of weighted projective stacks, or indeed generalised weighted projective stacks, on the natural smooth almost étale covering $\mathcal{X} \to X$ of [V] 2.5, then there is a weighted blow down $\rho : X \to X_0$ of $D$ to a unique algebraic (or complex analytic) space.

Better still our discussion of foliation singularities I.7 in no way employs projectivity, and so we obtain,

**IV.2.5 Corollary.** *Let $(\mathcal{X}, \tilde{\mathcal{F}})$ be a foliated smooth stack with log-canonical singularities which are terminal at the non-scheme like points and $\pi : (\mathcal{X}, \tilde{\mathcal{F}}) \to (X, B, \mathcal{F})$ it's not necessarily projective logarithmic moduli then if $\mathcal{Y}(a_i d, s, b)$ is an invariant divisor of the type encountered in* III.3.3, *there is in the sense of* IV.1.10 *a flop of $\mathcal{Y}(a_i d, s, b)$ to a smooth stack $(\mathcal{X}_0, \tilde{\mathcal{F}}_0)$.*

A postiori it will emerge that such a $\mathcal{Y}(a_i d,\, s,\, b)$ is always extremal, provided that $(\mathcal{X},\, \mathcal{F})$ is not a pencil of rational curves, but we are now in the reasonably advantageous position that we can construct our minimal model as an algebraic space from purely local considerations, which themselves will subsequently be sufficient to imply that projectivity was never lost in the first place.

## IV.3. The H-N Filtration again

We will require knowledge of the normal bundle to an extremal sub-stack. Consequently, as ever, $(\mathcal{X},\, \mathcal{F})$ is a foliated smooth stack with log-canonical singularities, projective moduli $\pi : (\mathcal{X},\, \mathcal{F}) \to (X,\, \mathcal{F})$, and $\mathcal{Y}$ an extremal sub-stack of the form $\mathcal{Y}(da_1, \ldots, da_r,\, s,\, b_1, \ldots,\, b_t)$. Given the said data we may specialise to the projective normal cone, $\mathcal{P} := \mathbb{P}(N^\vee_{\mathcal{Y}/\mathcal{X}}) = P(N_{\mathcal{Y}/\mathcal{X}})$, to obtain another foliated smooth stack $(\mathcal{P},\, \tilde{\mathcal{F}})$ again with log-canonical singularities.

Our primary interest is the local variation of $N_{\mathcal{Y}/\mathcal{X}}$ over the base/singular locus $Z$, so to begin with, and essentially without loss of generality, we'll restrict attention to the case $s = 0$. Naturally there are two tautological bundles of relevance, namely that on the weighted projective stack $\mathcal{Y}$, which is in fact $K^\vee_{\mathcal{F}}$, and the relative tautological bundle of $\rho : \mathcal{P} \to \mathcal{Y}$ which we'll denote $H$, while $P$, $Y$ etc. will be the corresponding moduli. To begin with we seek to bound the effective cone of divisors on $\mathcal{P}$ so suppose that $H + x\rho^* K^\vee_{\mathcal{F}}$, $x \in \mathbb{R}$, is nef. Now suppose that $\mathcal{L} \to \mathcal{Y}$ is a smooth invariant substack, then of course,

$$\mathcal{P} \times_{\mathcal{Y}} \mathcal{L} \xrightarrow{\sim} P\left( \bigoplus_{j=1}^{t} \mathcal{O}_{\mathcal{L}}(b_j\, K_{\mathcal{F}}) \right)$$

where as ever we order things according to $b_1 \leq b_2 \leq \ldots \leq b_t$. In particular if $b$ is the smallest of the $b_j$, and is supposed to occur $q$ times amongst the $b_j$, then we have an embedding,

$$\mathcal{P} \times_{\mathcal{Y}} \mathcal{L} \hookleftarrow \mathcal{L} \times \mathbb{P}^{q-1} : i\, .$$

Moreover this occurs in such a way, that if $H_0$ is the standard normalisation of the tautological bundle on $\mathbb{P}^{q-1}$ pulled back to the said product then, $i^* H = H_0 + b\,\rho^* K^\vee_{\mathcal{F}}$, so in fact $x \geq -b_1$. Now observe that either $K_{\tilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$ or $q = t$, $r = 1$ and all the $b_j$ are equal. In the latter case we already know the answer, so ignoring it for simplicity, we can certainly appeal to the cone theorem to conclude the existence of a $K_{\tilde{\mathcal{F}}}$-negative extremal ray $R$, or better a smooth invariant parabolic stack $\mathcal{R}$ representing the same. Indeed taking $\mathcal{R}$ to be what might be termed a coordinate line in terms of the explicit coordinate system guaranteed by our holonomy considerations of we see that $\rho(\mathcal{R})$ is without loss of generality equal to $\mathcal{L}$, so, in fact, $\mathcal{R}$ is extremal in $\mathcal{P} \times_{\mathcal{Y}} \mathcal{L}$. Consequently for $p$ an appropriate geometric point in our $\mathbb{P}^{q-1}$, $\mathcal{R} = \mathcal{L} \times p$, and $x = -b_1$ whenever $H + x\,\rho^* K^\vee_{\mathcal{F}}$ is both extremal and effective.

In addition if for our standard linearisation of a generator of the foliation about the singularity $z$, we denote by $V$ the eigenspace of $b = b_1$ contained in $N_{\mathcal{Y}/\mathcal{X}} \otimes \mathbb{C}(z)$ then canonically we may identify the aforesaid $\mathbb{P}^{q-1}$ with $P(V)$, which in turn is the singular locus of $\tilde{\mathcal{F}}$. As such our considerations on extremal subvarieties imply that the extremal rays sweep out a $\boldsymbol{\mathcal{P}}(da_1, \ldots, da_r)$ bundle $Q$ parametrised by the said $\mathbb{P}^{q-1}$. Conversely we also have at our disposal the projection of $Q$ to $\mathcal{Y}$, so in fact $Q \xrightarrow{\sim} \mathcal{Y} \times \mathbb{P}^{q-1}$, and embeds in $\mathcal{P}$ as a sub-projective bundle. Consequently applying $\rho_*$ gives a surjection,

$$N^\vee_{\mathcal{Y}/\mathcal{X}} = \rho_*\, \mathcal{O}_{\mathcal{P}}(H) \twoheadrightarrow \rho_*\, \mathcal{O}_Q(H) = \mathcal{O}_{\mathcal{L}}(-bK_{\mathcal{F}})^{\oplus q}\, .$$

To deduce from here the structure of the H-N filtration of $N_{\mathcal{Y}/\mathcal{X}}$ it suffices to observe,

(a) Let $N$ be the sub-bundle of $N^\vee_{\mathcal{Y}/\mathcal{X}}$ corresponding to the kernel of the above map, and $p$ any geometric point of our $\mathbb{P}^{q-1}$, then we have a natural isomorphism,

$$N^\vee_{Q/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{Y} \times p} \xrightarrow{\sim} N\, .$$

(b) The hypothesis $s = 0$ plays no essential role. Indeed for arbitrary $s$, $Q$ is globally defined as an extremal subvariety of $\mathcal{P}$, and its local structure over the base of $\mathcal{Y} \to Z$ requires only that the base is a sufficiently small affine.

As such appealing inductively to the above considerations by way of (a) we conclude,

**IV.3.1 Fact.** *There is a canonical filtration of $N_{\mathcal{Y}/\mathcal{X}}$ by invariant sub-bundles for the induced foliation,*

$$\mathcal{O} = N_0 \subsetneqq N_1 \subsetneqq N_2 \subsetneqq \cdots \subsetneqq N_k = N_{\mathcal{Y}/\mathcal{X}}$$

*such that if $\beta_1 < \cdots < \beta_k$ is a complete repetition free list of the $b_1, \ldots, b_t$, and $q_j$, $1 \le j \le k$ the corresponding multiplicities, then locally over $Z$,*

$$N_j/N_{j-1} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}}(+\beta_j K_{\mathcal{F}})^{\oplus q_j}\,.$$

Unsurprisingly we continue to refer to this as the H-N filtration, and observe that we even have a splitting (at least locally over $Z$), albeit non-canonically, of the form,

$$N_{\mathcal{Y}/\mathcal{X}} \xrightarrow{\sim} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathcal{Y}}(b_j K_{\mathcal{F}})\,.$$

## IV.4. Split Neighbourhoods & Weighted Blow Ups

Let us consider in more detail the rather special situation presented to us by the H-N filtration of the normal bundle of an extremal subvariety $\mathcal{Y}$ of a foliated smooth stack $(\mathcal{X}, \mathcal{F})$ with of course log-canonical singularities, terminal at the non-scheme like points. In the particular case where $\mathcal{Y}$ has dimension 1, we already know that the corresponding formal neighbourhood $\hat{\mathcal{X}}$ of $\mathcal{Y}$ is actually *split*, i.e. there are invariant formal divisorial, substacks $\mathfrak{D}_1, \ldots, \mathfrak{D}_t$, such that $\mathcal{O}_{\mathcal{Y}}(\mathfrak{D}_j)$, or $N_{\mathcal{Y}/\mathfrak{D}_j}$, define the various terms in the H-N filtration. Such a splitting would seem to be specific to the additional structure provided by the foliation, but if $\mathcal{Y}$ is a $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$ for $r \ge 2$ with $N_{\mathcal{Y}/\mathcal{X}} \xrightarrow{\sim} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathcal{Y}}(-b_j)$, $b_1 \le \cdots \le b_t$ positive integers then such a splitting continues to hold without supposing the existence of any such foliation, albeit that the presence of a foliation is precisely what guarantees such simplicity of the H-N filtration. Regardless suppose indeed we have a H-N decomposition for the normal bundle of a bundle $\mathcal{X}_0$ of weighted projective stacks $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$ over a smooth scheme (or algebraic space) $Z$, or at least locally over $Z$, inside a formal neighbourhood $\mathfrak{X}$ of the same. Certainly, cf. IV.5, we have, on supposing $Z$ a sufficiently small affine, that $\mathrm{Pic}(\mathfrak{X}) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{X}_0)$, and the liftability to $\mathfrak{X}$ of the divisors,

$$\mathcal{D}_j := \bigoplus_{\substack{k=1\\k\neq j}}^{t} \mathcal{O}_{\mathcal{Y}}(-b_k) \subset N_{\mathcal{X}_0/\mathfrak{X}}$$

for each $1 \le j \le t$, follows immediately from the acyclicity of $\mathcal{O}_{\mathcal{Y}}(n)$, for each $n \in \mathbb{Z}$. Let us therefore summarise this discussion by way of,

**IV.4.1 Summary/Definition.** A smooth formal neighbourhood $\mathfrak{X}$ of a bundle of weighted projective stacks $\mathcal{X}_0$ will be called split if there are smooth formal divisorial stacks $\mathfrak{D}_1, \ldots, \mathfrak{D}_t$, $t = \operatorname{codim} \mathcal{X}_0$, such that,

$$\mathcal{X}_0 = \mathfrak{D}_1 \cap \cdots \cap \mathfrak{D}_t$$

where the implied stack structure is the stack/scheme theoretic one. Under the hypothesis discussed above (which will certainly be sufficient for applications) we are indeed in the situation of a split neighbourhood.

Observe that once we have a split neighbourhood, with $\mathcal{O}_{\mathcal{X}_0}(\mathfrak{D}_j) = \mathcal{O}_{\mathcal{X}_0}(-b_j)$, at least relative to $Z$, we can form a weighted blow up with weights $b = (b_1, \dots, b_t)$. Indeed for each $n \in \mathbb{N} \cup \{0\}$, and $x_j$ a local equation for $\mathfrak{D}_j$ define sheaves of ideals by,

$$\mathcal{I}^{(n)} = (x_1^{s_1} \cdots x_t^{s_t} \mid s_1 b_1 + \cdots + s_t b_t \geq n)$$

and consider the graded algebra,

$$T = \bigoplus_n \mathcal{I}^{(n)}$$

with linear topology $\mathcal{I}^{(m)} T$, $m \in \mathbb{N}$, then we have the weighted formal blow up defined by,

$$\tilde{\mathfrak{X}} := \operatorname{Prof} T\,.$$

Now a priori the weighted blow up would seem to depend on two things,

(a) The particular splitting of $N_{\mathcal{X}_0/\mathfrak{X}}$, which unlike the H-N filtration is far from unique.

(b) The particular liftings of the divisors $\mathcal{D}_j$ from the normal bundle to the full neighbourhood.

In reality however it depends on neither. To begin with consider (a), and suppose $y_1, \dots, y_t$ are the local equations of divisors on $\mathfrak{X}$ corresponding to a different splitting. Necessarily these are related to the equations $x_1, \dots, x_t$ by a matrix of functions $a_{ij}$ which is upper semi-triangular, i.e. $a_{ij} = 0$ if $j < i$, yet constant on the diagonal. In consequence not just $\tilde{\mathfrak{X}}$, but the actual graded algebra $T$ is indepndent of (a). In a similar vein, if we fix the splitting and $y_1, \dots, y_t$ correspond to a different lifting then for coefficient functions $c_l \in \mathcal{O}_Z$, $l = (l_1, \dots, l_t)$ (i.e. shrink $Z$ in the étale topology),

$$y_i = x_i + \sum_{l_1 b_1 + \cdots + l_t b_t \geq b_i} c_l\, x_1^{l_1} \dots x_t^{l_t}\,.$$

So again even the algebra $T$ is unchanged, and of course, a combination of this discussion removes any dependence should both (a) & (b) occur simultaneously. Consequently, in conclusion, we have:

**IV.4.2 Fact/Summary.** A split formal neighbourhood $\mathfrak{X}$ of a bundle of weighted projective stacks $\mathcal{X}_0$, admits a well defined weighted blow up $\sigma : \tilde{\mathfrak{X}} \to \mathfrak{X}$ with weights $b = (b_1, \dots, b_t)$.

A more amenable description of the situation is, of course, provided in terms of an appropriate $\mu_{\underline{b}} := \mu_{b_1} \times \cdots \times \mu_{b_t}$ action. To begin with we work locally, so that $\mathfrak{U} \to \mathfrak{X}$ is a formal étale neighbourhood, and $x_1, \dots, x_t$ local equations for the $\mathfrak{D}_j$. We can extract a $b_j$-th root, $\xi_j$, say, of each $x_j$, so as to realise $\mathfrak{U}$ as the coarse quotient $\mathfrak{V}/\mu_{\underline{b}}$ of some formal affine. The underlying formal scheme of $\mathfrak{V}$ is still the same as that of $\mathfrak{U}$, i.e. $U_0 = \mathfrak{U} \cap \mathcal{X}_0$, and indeed is the set theoretic pre-image of the same. Consequently if $\sigma : \tilde{\mathfrak{V}} \to \mathfrak{V}$ is the usual blow up of $\mathfrak{V}$ in $U_0$ then we have a not necessarily Cartesian diagram,

$$\begin{array}{ccc} \mathfrak{V} & \xleftarrow[\sigma]{} & \tilde{\mathfrak{V}} \\ \downarrow & & \downarrow \\ \mathfrak{U} = \mathfrak{V}/\mu_{\underline{b}} & \xleftarrow[\sigma]{} & \tilde{\mathfrak{X}} \times_{\mathfrak{X}} \mathfrak{U} = \tilde{\mathfrak{V}}/\mu_{\underline{b}} \end{array}$$

This description admits various different globalisations, although the precise one in question is rather dependent on whether $Z$ is affine or not. As such let us not singular out anyone in particular, and content ourselves with the conclusion,

**IV.4.3 Fact.** $\tilde{\mathfrak{X}}$ *has at worst quotient singularities; as such following* [V] 2.8, *we can, if we wish, associate to its moduli the minimal smooth stack structure associated to the same.*

Now let us turn to the issue of algebraicisability of the weighted blow up. Specifically suppose that in fact $\mathfrak{X}$ is the completion of some stack $\mathcal{X}$ in $\mathcal{X}_0 = \mathcal{Y}$. A priori as we've defined the algebra $T$ affording $\mathfrak{X}$ it may not be an extension of an appropriate $\mathcal{O}_{\mathcal{X}_0}$-algebra. On the other hand the very argument employed to verify the dependence (or more precisely the lack thereof) of $\tilde{\mathfrak{X}}$ on the liftings of the $\mathcal{D}_j$ actually shows that if $x_j$ is a local equation for $\mathfrak{D}_j$, and $y_j$ any function such that,

$$y_j \equiv x_j(\mathcal{I}_{\mathcal{X}_0}^{b_j})$$

then in fact the weighted blow ups defined by way of the $y_j$ and the $x_j$, or more precisely the ideals,

$$(y_1^{s_1} \dots y_t^{s_t} \mid s_1 b_1 + \cdots + s_t b_t \geq n) \quad \text{and} \quad (x_1^{s_1} \dots x_t^{s_t} \mid s_1 b_1 + \cdots + s_t b_t \geq n)$$

coincide. Better still if $\pi : \mathcal{X} \to X$ is the moduli of $X$, and $\pi : \tilde{\mathcal{X}} \to \tilde{X}$ that of $\tilde{\mathcal{X}}$, then by the invariance of the moduli under flat base change,

$$\tilde{X} = \operatorname{Proj}(\pi_* T)$$

while $\pi_* T$ is itself a graded $\mathcal{O}_X$-algebra generated by its pieces in degree at most $n$, for some sufficiently large $n$ determined by the $b_j$'s. As such we even have,

**IV.4.4 Fact.** *If $\mathfrak{X}$ is the completion of a stack $\mathcal{X}$ in some appropriate sub-stack $\mathcal{Y}$, then $\sigma : \tilde{\mathfrak{X}} \to \mathfrak{X}$ is the completion of a modification $\sigma : \tilde{\mathcal{X}} \to \mathcal{X}$ in the exceptional divisor. Moreover if $\mathcal{X}$ has projective moduli $X$, then $\tilde{\mathcal{X}}$ also enjoys a projective moduli space, say $\tilde{X}$.*

Finally let us note the ultimate motivation for this discussion. For simplicity let us suppose that $\mathcal{X}_0 = \mathbb{P}^r$, and write $\mathfrak{X} = \mathbf{X}'$ to distinguish the fact that our formal neighbourhood is now an honest formal scheme or algebraic space. Extracting the $b_j$-th root of the divisor $\mathfrak{D}_j$ yields a covering $\mathbf{X} \to \mathbf{X}'$, together with the accompanying $\mu_{b_j}$ action in such a way that $\mathbf{X}' = \mathbf{X}/\mu_{\underline{b}}$ and as we've said the weighted blow up $\tilde{\mathbf{X}}'$ is simply the quotient of the standard blow up of $\tilde{\mathbf{X}}$ in $\mathbb{P}^r$ modulo the induced $\mu_{\underline{b}}$ action. On the other hand the normal bundle of our $\mathbb{P}^r$, $X_0$, in $\mathbf{X}$ is $\mathcal{O}_{\mathbb{P}^r}(-1) \otimes \mathbb{C}^t$, so that the underlying sub-space of $\tilde{\mathbf{X}}$, say, is isomorphic to $\mathbb{P}^r \times \mathbb{P}^{t-1}$, and of course,

$$\mathcal{O}_{\tilde{\mathbf{X}}_0}(-\tilde{\mathbf{X}}_0) = \mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathcal{O}_{\mathbb{P}^{t-1}}(1)\,.$$

As such, having formed the weighted blow up in the "$b$-direction" we're manifestly in the position where we'll be able to make a weighted blow down in the "$a$-direction".

## IV.5. Formal Flipping

We could certainly deduce a flip theorem from the H-N filtration, the contraction theorem, the cone theorem, and the Euclidean algorithm. This would, however, as the previous mouthful suggests involve a certain, and as it happens unwaranted, degree of combinatorial complexity, not to mention an obfuscation of the correct structure. Consequently we'll proceed in the spirit of IV.2, beginning with,

**IV.5.1 Problem.** *Let $\mathfrak{X}$ be a smooth formal stack, with underlying stack $\mathcal{X}_0$ such that $\mathcal{X}_0 \hookrightarrow \mathfrak{X}$ is regularly embedded of codimension $t$. Suppose further,*

(a) *$\mathcal{X}_0$ is a bundle of weighted projective stacks $\boldsymbol{\mathcal{P}}(a_1, \dots, a_r)$ over a smooth scheme (or separated algebraic/complex analytic space) $Z$.*

(b) *Locally over $Z$, $N_{\mathcal{X}_0/\mathfrak{X}} \xrightarrow{\sim} \bigoplus_{j=1}^{t} \mathcal{O}(-b_j)$, where $b_1 \leq \cdots \leq b_t$ are positive integers, and $\mathcal{O}(1)$ the tautological bundle on $\boldsymbol{\mathcal{P}}_l(a_1, \dots, a_r)$.*

*Then can we flip $\mathcal{X}_0$?*

Again let us explain precisely what is intended. In the first place observe that by virtue of the exponential sequence,

$$0 \longrightarrow S^n I/I^2 \longrightarrow \mathcal{O}^*_{\mathscr{X}_{n+1}} \longrightarrow \mathcal{O}^*_{\mathscr{X}_n} \longrightarrow 0$$

and the relative acyclicity of symmetric powers of the co-normal bundle of $\mathscr{X}_0$ in $\mathfrak{X}$, that for $Z$ affine,

$$\mathrm{Pic}(\mathfrak{X}) \xrightarrow{\sim} \mathrm{Pic}(\mathscr{X}_0)\,.$$

As such we necessarily suppose $Z$ a sufficiently small affine, and denote equally by $\mathcal{O}_{\mathfrak{X}}(1)$ the lifting of the tautological bundle of $\mathcal{O}_{\mathscr{X}_0}(1)$. Unsurprisingly flip is to be understood with respect to this bundle, so to this end let's actually write our initial data as $\mathfrak{X}^-$, $\mathscr{X}_0^-$, etc. with $\Xi^-$, $\Xi_0^-$ etc., the formal moduli. For the sake of conformity with standard procedure we wish to first contract $\mathscr{X}_0$ to $Z$, i.e. find a formal modification,

$$\rho_- : \Xi^- \to \Xi$$

which restricted to $\mathscr{X}_0$ is nothing other than the projection map to $Z$.

The most convenient way to proceed is by changing the stack structure on $\mathfrak{X}^-$. Indeed as per the prequel to IV.2.4 we can extract roots of the coordinate hyperplanes not just on $\mathscr{X}_0$, but actually on $\mathfrak{X}^-$ to obtain a new formal stack, which rather abusively we will continue to denote by $\mathfrak{X}^-$ whose moduli continues to be $\Xi^-$ but whose underlying stack $\mathscr{X}_0$ is actually the classifying stack $[\mathbb{P}^r_Z/\mu_{\underline{a}}]$, where of course $\mu_{\underline{a}} = \mu_{a_1} \times \cdots \times \mu_{a_r}$. Now in addition to IV.5.1, let us add to the discussion,

**IV.5.2 Further Hypothesis.** Suppose the neighbourhood $\mathfrak{X}$ of $\mathscr{X}_0$ of is actually split.

Consequently in the obvious extension of the definition of IV.4.1, the neighbourhood $\mathfrak{X}^-$ of the classifying stack $\mathscr{X}_0 = [\mathbb{P}^r_Z/\mu_a]$ is also split. Unsurprisingly we form a weighted blow up of $\mathfrak{X}^-$ by extracting $b_j$-th roots of the splitting divisors $\mathfrak{D}_j$ while additionally specifying a smooth stack structure $\tilde{\mathfrak{X}}$ in such a way that we have ramification of order $b_j$ around the proper transforms of the $\mathfrak{D}_j$. The moduli of $\tilde{\mathfrak{X}}$ is itself, nothing other than the moduli of the weighted blow up of the original neighbourhood $\mathfrak{X}$, which in turn we denote $\tilde{\Xi}$. The advantage of this picture is of course that, $\mathbb{P}^r_Z \to \mathscr{X}_0$ is étale, so we may suppose that $\mathfrak{X}^-$ is the classifying stack, $[\mathbf{X}_1^-/\mu_a]$ for some honest formal algebraic space $\mathbf{X}_1^-$. Better still the weighted blow up $\tilde{\mathbf{X}}_1^-$ of $\mathbf{X}_1^-$ is also the moduli of a classifying stack, this time of the form $[\tilde{\mathbf{X}}/\mu_b]$ where $\tilde{\mathbf{X}}$ is the usual blow up of some formal neighbourhood $\mathbf{X}^-$ of $\mathbb{P}^r_Z$ with the property that $\mathbf{X}_1^- = \mathbf{X}^-/\mu_b$. All the stacks in question are of course smooth, and we arrive to the rather desirable position where the weighted blow up $\sigma : \tilde{\Xi} \to \Xi$ is the moduli of,

$$\begin{array}{ccc} [\tilde{\mathbf{X}}/\mu_{\underline{a}} \times \mu_{\underline{b}}] & \xrightarrow[\pi]{} & \tilde{\Xi} \\ \big\downarrow{\scriptstyle \sigma_-} & & \big\downarrow{\scriptstyle \sigma_-} \\ [\mathbf{X}^-/\mu_{\underline{a}} \times \mu_{\underline{b}}] & \xrightarrow[\pi]{} & \Xi \end{array}$$

for $\mathbf{X}^-$ a smooth split formal neighbourhood of $\mathbb{P}^r_Z$, with $\tilde{\mathbf{X}}$ the blowing up in the same, and the $\mu_{\underline{a}} \times \mu_{\underline{b}}$ acting in the natural way on the projective coordinates $X_0, \ldots, X_r$, and splitting divisors $\mathfrak{D}_1, \ldots, \mathfrak{D}_t$.

It almost goes without saying that a minor variant of the prequel to IV.2.4 as alluded to above establishes that the formal functions on $\tilde{\mathbf{X}}$ are simply those over the cone of the veronese embedding of $\mathbb{P}^r_Z \times_Z \mathbb{P}^{t-1}_Z$ considered as a formal subvariety of the formal affine space $\hat{\mathbb{A}}^{(r+1)t}_Z$, and as a result in the natural topology determined by the underlying space,

$$\tilde{\mathbf{X}} \longrightarrow \mathrm{Spf}\,\Gamma(\mathcal{O}_{\tilde{\mathbf{X}}})$$

is not just a formal modification, but the blow down of the said underlying space to $Z$. Appealing, therefore, as ever to the exactness of $\pi_*$ we conclude that,

$$\tilde{\Xi} \longrightarrow \mathrm{Spf}\,\Gamma(\mathcal{O}_{\tilde{\Xi}})(= \Gamma(\mathcal{O}_{\Xi})) := \Xi_0$$

is a formal modification, and in the obvious sense of the word, a bi-weighted blow down. Better still the argument works equally well for $\mathcal{P}(a_1, \ldots, a_r)$ as the classifying stack $[\mathbb{P}^r/\mu_a]$, and is local for the étale topology with respect to which the underlying stack of $[\tilde{\mathbf{X}}/\mu_{\underline{a}}]$ is $[\mathbb{P}^r_Z/\mu_{\underline{a}}] \times_Z [\mathbb{P}^{t-1}_Z/\mu_{\underline{b}}]$, so that in fact we have a weighted blow down,

$$\tilde{\Xi} \longrightarrow \Xi^+$$

which at the stack level contracts $[\mathbb{P}^r_Z/\mu_{\underline{a}}] \times_Z [\mathbb{P}^{t-1}_Z/\mu_{\underline{b}}]$ to $[\mathbb{P}^{t-1}_Z/\mu_{\underline{b}}]$. We may summarise the situation by way of,

$$\begin{array}{ccccc} & & \tilde{\Xi} & & \\ \text{weighted blow up} = \sigma_- & \swarrow & & \searrow & \sigma_+ = \text{weighted blow down} \\ & \Xi^- & - - - - \rightarrow & \Xi^+ & \\ \rho_- & \searrow & & \swarrow & \rho_+ \\ & & \Xi_0 & & \end{array}$$

and observe,

**IV.5.3 Fact/Definition/Summary/Warning.** The above diagram, or better still simply the upper triangle will be called a flap. Note that in terms of the $\mathbb{Q}$-Cartier divisor $\mathcal{O}(1)$ on $\Xi$ we have not verified the irrelevant technical definition of formal flip, i.e.

$$\Xi_- = \operatorname{Prof} \bigoplus_{n=0}^{\infty} \mathcal{O}_{\Xi_0}(n(\rho_-)_* \mathcal{O}(-1)), \quad \Xi_+ = \operatorname{Prof} \bigoplus_{n=0}^{\infty} \mathcal{O}_{\Xi_0}(n(\rho_+)_* \mathcal{O}(1))$$

and that the given algebras are finitely generated, nor have we even verified that $\rho_-$, and $\rho_+$ are formal modifications in the technical sense of the word, albeit that we have done this for the composites $\rho_- \circ \sigma_- = \rho_+ \circ \sigma_+$. Nevertheless the divisors $\mathcal{O}(-1)$, and $\mathcal{O}(+1)$ are relatively ample at the level of the maps of the underlying spaces of $\rho_-$, and $\rho_+$, respectively, and whence we will not hesitate to equally employ the word flip to describe the lower triangle.

Turning to questions of convergence we may apply Artin's convergence theorem together with I.7 not to forget the uniqueness of contraction, to conclude,

**IV.5.4 Corollary.** *Let $\mathcal{X}^-$ be a smooth stack and $\mathcal{Y}$ a closed sub-stack which is a $\mathcal{P}(a_1, \ldots, a_r)$ bundle over a smooth scheme or algebraic space $Z$, such that locally over $Z$, the completion $\hat{\mathcal{X}}^-$ of $\mathcal{X}^-$ in $\mathcal{Y}$ is a split neighbourhood, then, employing as ever latin letters $X^-$, $Y$ etc. for the moduli,*

(a) *The contraction, $\rho_- : X^- \to X_0$ of $Y$ to $Z$ exists as an algebraic space.*

(b) *There is an algebraic space, $\rho_+ : X^+ \to X_0$, which modulo the caveat of IV.5.3 is locally the formal flip of the unique generator of* $\operatorname{Pic}(\hat{X}^-)$.

(c) *All of this fits into a flip/flap diagram of algebraic spaces,*

$$\begin{array}{ccccc} & & \tilde{X} & & \\ \text{weighted blow up} = \sigma_- & \swarrow & & \searrow & \sigma_+ = \text{weighted blow down} \\ & X^- & - - - - \rightarrow & X^+ & \\ \rho_- & \searrow & & \swarrow & \rho_+ \\ & & X_0 & & \end{array}$$

As already noted if $X^-$ is projective then so is $\tilde{X}$, although we have no such a priori conlusions regarding $X^+$, and $X_0$. In addition due to our aforesaid lacadaisical study of $\hat{X}^-$, we even use the a priori existence of $\tilde{X}$ to conclude to the existence of $X_0$.

## IV.6. Back at Foliations

We now wish to put our previous considerations into action in the case of foliated stacks. Consequently, as ever $(\mathcal{X}_-, \tilde{\mathcal{F}}_-)$ will be a smooth foliated stack with log-canonical singularities terminal at the non-scheme like points and projective moduli, $\pi : (\mathcal{X}_-, \tilde{\mathcal{F}}_-) \to (X_-, B_-, \mathcal{F}_-)$. Up to the laxism implicit in our definition of the same, IV.5.3, we wish to construct the flip of an extremal subvariety $\mathcal{Y}$, as a stack $(\mathcal{X}_+, \tilde{\mathcal{F}}_+)$ enjoying the same list of properties enunciated for $(\mathcal{X}_-, \tilde{\mathcal{F}}_-)$ with the exception of projectivity of the moduli. Unsurprisingly the Gorenstien covering stack, $\pi : (\mathcal{X}'_-, \mathcal{B}'_-, \mathcal{F}_-) \to (X_-, B_-, \mathcal{F}_-)$ will play an important intermediary role.

To begin with we form the weighted blow up $\sigma_- : \tilde{X} \to X_-$, which we may certainly do since our split neighbourhood condition is guaranteed by II.6.2. In addition, as remarked in the prequel to IV.5.3, $\tilde{X}$ has at worst quotient singularities so there is a minimal smooth stack $\pi : \tilde{\mathcal{X}}' \to \tilde{X}$, étale in codimension 2. Note, however, that if $\mathcal{Y}$ is a $\mathcal{Y}(d\underline{a}, s, \underline{b})$, for general $\underline{b}$, $\tilde{\mathcal{X}}'$ will be bigger than the Gorenstien covering stack of $\tilde{X}$ in the induced foliation $\mathcal{F}$. Nevertheless we have a map of stacks $\sigma_- : \tilde{\mathcal{X}}' \to \mathcal{X}'_-$, and by the various lack of dependences, cf. sequel to IV.4.1, of the weighted blowing on anything other than the H-N filtration, not to mention the identity of the same over a coordinate line in any $(\mathbf{\mathcal{P}}(\underline{a}), \mathcal{R})$ with that on the $\mathbf{\mathcal{P}}(\underline{a})$, and the fact that the former is defined by way of invariant divisors we conclude that,

$$K_{\mathcal{F}} = (\sigma_-)^* K_{\mathcal{F}_-} .$$

In addition if $x_1, \ldots, x_t$ are any local functions in the étale site of $\mathcal{X}'_-$ employed in the formulation of the weighted blow up, with $y$ a local equation for $\mathcal{B}'_-$ then the $x_1, \ldots, x_t, y$ are simple normal crossing divisors. As such,

(a) The proper transform $\tilde{\mathcal{B}}'$ of $\mathcal{B}'_-$ is in fact its pull-back. Idem for any sufficiently large multiple of $B_-$ which is Cartier, in terms of the proper transform $\tilde{B}$.

(b) There is no difficulty in forming a covering stack $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\tilde{\mathcal{X}}', \mathcal{F})$ ramified over the smooth divisor $\tilde{\mathcal{B}}'$ with appropriate weight implicit in the definition of $B_-$.

Necessarily we have a map, $\sigma_- : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}_-, \tilde{\mathcal{F}}_-)$, and, even $K_{\tilde{\mathcal{F}}} = (\sigma_-)^* K_{\tilde{\mathcal{F}}_-}$, from which we conclude that $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ also has log-canonical singularities. It may certainly happen, however, that $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ is not terminal at the non-scheme like points, so to go further requires a little care. To begin with observe that if $E$, $\mathcal{E}'$ are the exceptional divisors with respect to $\sigma_-$ for $\tilde{X}$, and $\tilde{\mathcal{X}}'$ respectively then of course $\mathcal{E}' = \pi^* E$, while for $\mathcal{L} \subset \mathcal{E}'$ a $K_{\mathcal{F}}$-negative smooth stack,

$$\mathcal{O}_{\mathcal{L}}(b\, K_{\mathcal{F}}) = \mathcal{O}_{\mathcal{L}}(\mathcal{E}')$$

where $b$ is the gcd of $b_1, \ldots, b_t$. Even better since $\mathcal{B}'_-$ is transverse to $\mathcal{E}'$, then in fact for $\mathcal{E}$ the exceptional divisor on $\tilde{\mathcal{X}}$, $\mathcal{E} = \pi^* E$. As such if $K_{\tilde{\mathcal{F}}_+}$ is the canoncial class of the foliated logarithmic $\mathbb{Q}$-factorial algebraic space $(X_+, B_+, \mathcal{F}_+)$ whose existence is guaranteed by IV.5.4, then as per, IV.1.9,

$$K_{\tilde{\mathcal{F}}} = (\sigma_+)^* K_{\tilde{\mathcal{F}}_+} + \frac{1}{db} E .$$

Now we can argue exactly as per op. cit. The important being that any new singularities on $X_+$ are terminal, and whence by I.7.3,

**IV.6.1 Fact/Summary.** There is a smooth foliated stack $(\mathcal{X}_+, \tilde{\mathcal{F}}_+)$ with log-canonical singularities, terminal at the non-scheme like points, albeit that its moduli $\pi : (\mathcal{X}_+, \tilde{\mathcal{F}}_+) \to (X_+, \mathcal{F}_+)$ may only be an algebraic space. Nevertheless it fits into a flip/flap diagram

$$\begin{array}{ccc} & \tilde{X} & \\ \swarrow & & \searrow \\ X^+ & \dashrightarrow & X^- \\ \searrow & & \swarrow \\ & X_0 & \end{array}$$

whose completion around $\mathcal{Y}$ is the formal flip/flap of IV.5.4. In particular, from the conformal surgery point of view, the bundle of weighted projective stacks $\mathcal{P}(da_1, \dots, da_r)$ over $Z$ is flipped to a bundle of generalised projective stacks $\mathcal{P}_g(db_1, \dots, db_t)$ over the same.

## IV.7. Projectivity and Termination

Plainly the minimal model theorem for foliations is very close to being established. Indeed the only outstanding issues are the projectivity, not to mention termination, of flipping. The said issues are in fact rather closely linked, so let us begin by retaking the notations of the previous section, as well as introducing the key invariant $Z = Z(\mathcal{Y})$, i.e. the unique smooth connected component of the singular locus of the foliation contained in the extremal sub-stack $\mathcal{Y}$ which we have flipped. Observe that over the weighted blow up, $\sigma_- : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}_-, \tilde{\mathcal{F}}_-)$, the induced singular locus $\tilde{\mathcal{Z}}$, which may very well be non-scheme like, is the fibre of the exceptional divisor $\mathcal{E}$ over $\mathcal{Z}$, and so indeed forms a $\mathcal{P}_g(b_1, \dots, b_t)$ bundle over $Z$ transverse to the direction of the weighted blow down $\sigma_+$. As such, at least at the level of the moduli, $\mathcal{Y}$ is flipped to the image $Z_+$ of the moduli of $\tilde{\mathcal{Z}}$, and as we've said, $\sigma_+ : \tilde{\mathcal{Z}} \to Z_+$ is certainly one to one on closed points. In any case what's important is that the logarithmic moduli $(X_+, B_+, \mathcal{F}_+)$ is log-terminal at every scheme point of $Z_+$, so that for $\mathcal{Z}_+ \subset \mathcal{X}_+$, and $\mathcal{Z}'_+ \subset \mathcal{X}'_+$ the corresponding closed sub-stacks whether of the flip or the Gorenstien covering stack of $(X_+, \mathcal{F}_+)$ the induced foliations, are:

**IV.7.1 Fact.**

(a) *Smooth at every geometric point of $\mathcal{Z}_+$, and $\mathcal{Z}'_+$ respectively.*

(b) *The ramification, respectively branching, divisor $\mathcal{B}_+$, respectively $\mathcal{B}'_+$, are smooth and everywhere transverse to the induced foliation at every geometric point of $\mathcal{Z}_+$, respectively $\mathcal{Z}'_+$.*

Now let's consider the consequences of this observation for the projectivity of the moduli of $\mathcal{X}_+$. In the first place let us denote by $R$ the extremal ray in Néron-Severi with respect to which $\mathcal{Y}$ is extremal. On forming the weighted blow up $\sigma_- : \tilde{\mathcal{X}} \to \mathcal{X}_-$, we have over each $K_{\tilde{\mathcal{F}}}$-negative curve in $\mathcal{Y}$, a $\mathcal{P}_g(b_1, \dots, b_t)$ of $K_{\tilde{\mathcal{F}}}$-negative curves in $\tilde{\mathcal{X}}$, all of which are parallel to a not necessarily extremal ray $\tilde{R}$ in the Néron-Severi group of $\tilde{\mathcal{X}}$. Indeed if $L$ in Néron-Severi of $\tilde{\mathcal{X}}$ is the class of the image of an exceptional line, then,

$$\tilde{R} = \sigma_-^* \, R + L \, .$$

On the other hand $\tilde{\mathcal{X}}$ has projective moduli, so the cone theorem is valid over it, and whence there are precisely two a priori obstructions to the extremality of $\tilde{R}$, namely :

**IV.7.2 Obstructions.** The ray $\tilde{\mathcal{R}}$ is not extremal in $\tilde{\mathcal{X}}$ iff one of the following occurs,

(i) There is an extremal subvariety $\mathcal{Y}_-$ in $\mathcal{X}_-$ of the form $\mathcal{Y}_-(d_-b_1, \dots, d_-b_t, s, a_1, \dots, a_r)$ corresponding to the negative part of the H-N filtration around $Z$, which is itself covered by $K_{\tilde{\mathcal{F}}}$-negative extremal stacks parallel to $R$.

(ii) There is some, necessarily finite, set of disjoint extremal subvarieties $\mathcal{Y}_1, \dots, \mathcal{Y}_k$, with say $\mathcal{Y} = \mathcal{Y}_1$, and $k \geq 2$, all of codimension at least 2, yet all swept out by $K_{\tilde{\mathcal{F}}}$-negative extremal stacks parallel to $\mathcal{R}$.

So suppose (i) were to occur, and let $\mathcal{L}_-$ be a $K_{\tilde{\mathcal{F}}}$-negative invariant 1-dimensional stack in $\mathcal{Y}_-$, then its proper transform $\mathcal{L}_+$ in $\mathcal{X}_+$ does not by meet the singular locus of $\tilde{\mathcal{F}}_+$. On the other hand $\mathcal{L}_+$ is certainly parabolic, and indeed $K_{\tilde{\mathcal{F}}_+}$ negative, so that IV.7.1 implies that $\mathcal{F}$ is a foliation in parabolic stacks. This case is rather particular, and meritorious of individual discussion, as such we note,

**IV.7.3 Hypothesis.** Suppose in fact that $(\mathcal{X}, \tilde{\mathcal{F}}_-)$ (so in particular $(X, \mathcal{F}_-)$) is not a foliation in parabolic stacks, which in any case is a necessary hypothesis for constructing a minimal model in the sense of a nef. canonical class.

Of course it could happen that we have some combination of (i) and (ii), i.e. at some $\mathcal{Y}_p \neq \mathcal{Y}_1$, both parts of the H-N filtration correspond to extremal sub-stacks. However, rather obviously, this is equally excluded by the said hypothesis, and so the only obstruction to the projectivity of $\mathcal{X}_+$, is that there is more than one extremal subvariety swept out by $-\mathbb{F}$-stacks parallel to $R$, albeit, as we've noted, the said sub-stacks are necessarily disjoint. Now whether this could actually happen or not is open to question. Nevertheless, it is easily dealt with. Specifically we form weighted blow up in each $\mathcal{Y}_p$, $1 \leq p \leq k$, in turn to get a smooth stack $\tilde{\mathcal{X}}(R)$ with projective moduli $\tilde{X}(R)$, naturally denoted $\sigma_-$, and exceptional divisors $\mathcal{E}_1, \dots, \mathcal{E}_k$. Furthermore in an obvious extension of our previous notations we have classes $\tilde{R}_1, \dots, \tilde{R}_k$, and $L_1, \dots, L_k$ in Néron-Severi of $\tilde{\mathcal{X}}(R)$ corresponding to $K_{\tilde{\mathcal{F}}(R)}$-negative stacks, and exceptional lines in the said divisors, with of course,

$$\tilde{R}_p = \sigma_-^* \, R + L_p \,, \qquad 1 \leq p \leq k \,.$$

As such each $\tilde{R}_p$ is extremal in $\tilde{\mathcal{X}}(R)$, and by IV.1.4 etc. we can make a weighted blow down of each in turn without loosing projectivity of the moduli. Unsurprisingly, therefore, this gives the flip $\mathcal{X}_+(R)$ of $R$ as the graph of the various flips of the individual $\mathcal{Y}_p$'s. Better still if $H_R$ is a supporting function for $R$, then the $\tilde{R}_p$'s and $L_p$'s generate the extremal face defined by $\sigma_-^* \, H_R$ in Néron-Severi of $\tilde{\mathcal{X}}(R)$ so a minor variant of our considerations in IV.1, shows that the simultaneous blow down of the $\mathcal{E}_1, \dots, \mathcal{E}_k$ to the implied smooth components $Z_1, \dots, Z_k$ of the singular locus of $\mathcal{F}_-$ (which is of course the blow down of the $\mathcal{Y}_1, \dots, \mathcal{Y}_k$ to the same) yields a projective variety $X_0(R)$, so we even have;

**IV.7.4 Fact.** *The previous technical caveat about flips has been removed, i.e. in terms of the underlying moduli,*

$$\begin{array}{ccccc} (X_-,\, B_-,\, \mathcal{F}_-) & & - - - \rightarrow & & (X_+(R),\, B_+(R),\, \mathcal{F}_+(R)) \\ & \rho_- \searrow & & \swarrow \rho_+ & \\ & & (X_0,\, B_0(R),\, F_0(R)) & & \end{array}$$

*is not just a flap, but a flip of $R$ in the proper technical sense of* [K-2], 2.26.

Better still, flipping manifestly terminates for the simple reason that the number of connected components of the singular locus decreases by at least 1 with the flip of any extremal ray, and so we conclude,

**IV.7.5 Lemma.** *Let $(\mathcal{X}, \tilde{\mathcal{F}})$ be a foliated smooth stack with projective moduli $(X, \mathcal{F})$ and ramification divisor $B$. Suppose further that $(\mathcal{X}, \tilde{\mathcal{F}})$ has log-canonical (respectively canonical) singularities which are terminal at the non-scheme like points, and that the said foliation is not a foliation in parabolic stacks then there is a sequence of flips and flops in the sense of IV.1.10 and IV.5/7.4 (or alternatively just flaps),*

$$(\mathcal{X}, \tilde{\mathcal{F}}) = (\mathcal{X}_0, \tilde{\mathcal{F}}_0) \dashrightarrow (\mathcal{X}_1, \tilde{\mathcal{F}}_1) \dashrightarrow \cdots \dashrightarrow (\mathcal{X}_n, \tilde{\mathcal{F}}_n) = (\mathcal{X}_{\min}, \tilde{\mathcal{F}}_{\min})$$

*such that each $(\mathcal{X}_i, \tilde{\mathcal{F}}_i)$ enjoys all the properties enunciated above of $(\mathcal{X}, \tilde{\mathcal{F}})$, and in addition $K_{\tilde{\mathcal{F}}_{\min}}$ is nef.* $\square$

## IV.8. Rational Foliations

We now wish to consider more precisely the case of smooth foliated stacks $(\mathcal{X}, \tilde{\mathcal{F}})$ enjoying the property that through every geometric point there is a rational 1-dimensional stack, or alternatively a rational curve through every point of the moduli $\pi : (\mathcal{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$. We'll begin with the case that $\pi$ is actually the Gorenstien covering stack, so indeed $\tilde{\mathcal{F}} = \mathcal{F}$, and continue to suppose that the singularities are terminal at the non-scheme like points. On the other hand we'll now ask that the singularities are canonical rather than just log-canonical, since,

**IV.8.1 Claim.** *A foliation with canonical singularities $(X, \mathcal{F})$ on a normal variety $X$ is a rational foliation iff through a generic closed point $x$ of $X$ there is a smooth invariant rational curve $L_x \ni x$ contained in the smooth locus of $X$ and $\mathcal{F}$, so in particular with $K_{\mathcal{F}}.L_x = -2$.*

**Proof.** The if direction is clear by the existence of the Hilbert scheme, and the closedness of the invariance condition. Conversely let $(X, \mathcal{F})$ be a rational foliation, and denote by $W$ and $Z$ the singular loci of $X$ and $\mathcal{F}$ respectively. Consider in the first instance the possibility that every invariant curve meets $Z$, so in fact the generic invariant curve meets some irreducible subvariety $Z_0$ in its generic point. Now call $X$, $X_0$, and blow up in $Z_0$, then either there is a subvariety $Z_1$ of the induced singular locus, where the generic rational invariant curve meets the generic point of the same, or there is not. In any case calling the blow up $X_1$, and proceeding inductively we eventually reach the situation that on some $X_n$, the exceptional divisor $E$ over the generic point of $Z_{n-1}$ meets the proper transform of a generic rational invariant curve generically, and whence cannot be invariant. However by the very definition, this is impossible if the singularities are canonical, so indeed the generic invariant curve is contained in the smooth locus of $\mathcal{F}$. Now suppose further that the generic invariant curve meets an irreducible subvariety $W_0$ of $\mathrm{sing}\,(X)$ in its generic point. By virtue of our previous discussion we can suppose $W_0 \not\subset \mathrm{sing}\,(\mathcal{F})$, so at the generic point of $W_0$, the foliation is a relatively smooth fibration. Consequently $W_0$ must be of codimension 1, which is nonsense given $X$ normal. As such we obtain that the generic invariant rational curve is contained in $X\backslash\{\mathrm{sing}\, X \cup \mathrm{sing}\, \mathcal{F}\}$, from which we conclude. $\square$

Notice, indeed, that the assumption of canonical is essentially optimal since,

**IV.8.2 Clarification.** There exist rational foliations with log-canonical singularities and $K_{\mathcal{F}}$ nef. For example on $\mathbb{P}^2$, the vector field given in standard affine coordinates by,

$$\partial = px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y}$$

$p, q \in \mathbb{N}$, relatively prime, affords such an example.

In any case what we wish to do is to analyse more carefully the particular obstruction to the projectivity of flipping which may be present for rational foliations, viz:

**IV.8.3 Obstruction.** For some smooth, and necessarily scheme like, component $Z$ of $\mathrm{sing}\,(\mathcal{F})$ there are closed invariant weighted projective extremal sub-stacks $\mathcal{Y}_+ = \mathcal{Y}(\underline{a}, s, \underline{b})$, and $\mathcal{Y}_- = \mathcal{Y}(\underline{b}, s, \underline{a})$ corresponding to the H-N pair at $Z$, and swept out by $K_{\mathcal{F}}$-negative extremal stacks parallel to the same extremal ray $R$.

The maintenance of projectivity in this situation must inevitably involve a new operation in the form of an anti-contraction. As such there is a reasonable argument for just working in the category of algebraic spaces (certainly bend & break holds trivially for rational foliations, while the more delicate analysis of III.5 is largely unaffected by destroying projectivity at points disjoint from the locus under consideration) with a view to restoring projectivity by way of a suitable modification of the moduli of an appropriate model of $X/\mathcal{F}$. Nevertheless an explicit analysis is relatively straightforward, so let's finish what we've started.

Regardless, the obvious basic construction when presented with IV.8.3, is to first form the weighted blow up, $\sigma : \tilde{\mathcal{X}} \to \mathcal{X}$ in $\mathcal{Y}_+$, yielding an exceptional divisor $\tilde{\mathcal{Y}}_+$, followed by the weighted blow up, $\tau : \tilde{\tilde{\mathcal{X}}} \to \tilde{\mathcal{X}}$ in the proper transform $\tilde{\mathcal{Y}}_-$ of $\mathcal{Y}_-$, then weighted blowing down either, but not both, of the exceptional divisors $\tilde{\mathcal{Y}}_+$ or $\tilde{\mathcal{Y}}_-$ on $\tilde{\mathcal{X}}$ in the opposite direction. The minor difficulty here is that this will yield non-scheme like points which are no longer terminal, and whence we'll have to make a more precise analysis of the stack structure, as opposed to our previous more leisurely approach.

In order to get underway, we'll first consider the Gorenstien structure $(\mathcal{X}_+, \mathcal{F}_+)$ on the weighted blow down of $\tilde{\mathcal{Y}}_+$ in the opposite direction. Evidently we're interested in what happens around the new non-scheme like locus $\mathcal{Z}_+$ in the notation of IV.7. This is however susumed into a more general analysis of the structure around an invariant rational stack $\mathcal{L}$ wholly contained in the smooth locus of $\mathcal{F}_+$. We know, of course, that the non-scheme like points of $\mathcal{X}_+$ are transverse to $\mathcal{F}_+$, so $\mathcal{L}$ is generically space like with moduli $\mathbb{P}^1$. Better still by the definition of the Gorenstien covering stack, $\mathcal{L}$ cannot be a bad orbifold, so we can find an étale cover $h : \mathbb{P}^1 \to \mathcal{L} \hookrightarrow \mathcal{X}_+$, which indeed is nothing other than the holonomy cover of $\mathcal{L}$ minus its non-space like points. As a result we can certainly find, in the analytic topology, an étale neighbourhood $\mathcal{U}$ of $\mathcal{L}$ such that $\mathcal{U} \xrightarrow{\sim} \mathbb{P}^1 \times \Delta$, for $\Delta$ some smooth polydisc. Furthermore if $X_\alpha \to \mathcal{X}_+$ is an affine étale neighbourhood contained in the smooth locus we can form a quotient $X_\alpha/\mathcal{F}_+$ as the ring of invariant functions, which by virtue of flat descent come equipped with patching maps $\coprod_{\alpha\beta} X_\alpha \times_{x_+} X_\beta/\mathcal{F}_+ \rightrightarrows \coprod_\alpha X_\alpha/\mathcal{F}_+$. Consequently over the smooth locus $\mathcal{X}_+^{\mathrm{sm}}$ of $\mathcal{F}_+$, we obtain a well defined (possibly non-separated) quotient $\mathcal{X}_+^{\mathrm{sm}}/\mathcal{F}_+$. On the other hand around $\mathcal{L}$, $q : \mathcal{X}_+^{\mathrm{sm}} \to \mathcal{X}_+^{\mathrm{sm}}/\mathcal{F}_+$ is analytically a $\mathbb{P}^1$-bundle so by Artin's approximation theorem, it is in fact a $\mathbb{P}^1$-bundle for the étale topology of $\mathcal{X}_+^{\mathrm{sm}}/\mathcal{F}_+$, as such given that $\mathcal{X}_+$ is separated, then at least around the image $\ell$ of $\mathcal{L}$, $\mathcal{X}_+^{\mathrm{sm}}/\mathcal{F}_+$ is separated too.

Now it may be that in terms of the minimal smooth stack structure $\tilde{\mathcal{X}}$ on the weighted blow up of $\mathcal{X}$ that we do not have a map between $\tilde{\mathcal{X}}$ and $\mathcal{X}_+$. If, however, we augment the structure on $\tilde{\mathcal{X}}$ to $\tilde{\mathcal{X}}_b$, by extracting a $b^{\mathrm{th}}$-root of the exceptional divisor, $b = \gcd(b_1, \dots, b_t)$ then in fact we obtain a weighted blow down of stacks, $\sigma_+ : \tilde{\mathcal{X}}_b \to \mathcal{X}_+$. In addition using the coordinates associated to a Jordan decomposition of an appropriate local generator $\partial = a_i\, y_i\, \frac{\partial}{\partial y_i} - b_j\, x_j\, \frac{\partial}{\partial x_j}$ of the foliation around the singularity of $\mathcal{Y}_+$ we find that formally about $\mathcal{Z}_+$ on say the $x_1 \neq 0$ piece, the stack structure of $\mathcal{X}_+$ is simply the formal classifying stack of,

$$\mu_{b_1} \times \hat{\mathbb{A}}^{r+t} \rightrightarrows \hat{\mathbb{A}}^{r+t} : \zeta \times \eta_1 \times \cdots \times \eta_r \times \xi_1 \times \cdots \times \xi_t \mapsto \zeta^{a_1}\eta_1 \times \cdots \times \zeta^{a_r}\eta_r \times \zeta\xi_1 \times \zeta^{b_2}\xi_2 \times \cdots \times \zeta^{b_t}\xi_t\,.$$

As such, $\tilde{\mathcal{X}}_b \to \mathcal{X}_+$ is the weighted blowing up in $\eta_1 = \cdots = \eta_r = \xi_1 = 0$, and $\tilde{\mathcal{X}}_b \to \tilde{\mathcal{X}}_b$ (where $\tilde{\mathcal{X}}_b$ is $\tilde{\mathcal{X}}$ with a $b^{\mathrm{th}}$-root of $\tilde{\mathcal{Y}}_+$ extracted) the weighted blow up in the proper transform of $q^{-1}(q\mathcal{Z}_+)$, i.e. $\eta_1 = \cdots = \eta_r = 0$ with weights $a_1, \dots, a_r$. Now we know that we can make a weighted blow down of the moduli of $\tilde{\mathcal{X}}_b$ in $\tilde{\mathcal{Y}}_+$, and equip it with a minimal smooth stack structure, say, $\tilde{\mathcal{X}}_+$, all of which necessarily fits into a diagram (just use the above coordinate system),

$$\begin{array}{ccc} \tilde{\mathcal{X}}_b & \longrightarrow & \tilde{\mathcal{X}}_+ \\ {\scriptstyle \text{weighted blow up in the proper transform of } q^{-1}(q\mathcal{Z}_+)}\downarrow & & \downarrow{\scriptstyle \text{weighted blow up in } q^{-1}(q\mathcal{Z}_+)} \\ \tilde{\mathcal{X}}_b & \longrightarrow & \mathcal{X}_+ \end{array}$$

weighted blow up of $\mathcal{Z}_+$ .

Better still, at least formally around $q^{-1}(q\mathcal{Z}_+)$ the algebra describing $\tilde{\mathcal{X}}_+ \to \mathcal{X}_+$ is invariant by $\mathcal{F}_+$, so descends to an algebra on $\mathcal{X}_+^{\mathrm{sm}}/\mathcal{F}_+$, and of course we can form a quotient $\tilde{\mathcal{X}}_+^{\mathrm{sm}}/\mathcal{F}_+$, so that we even have a

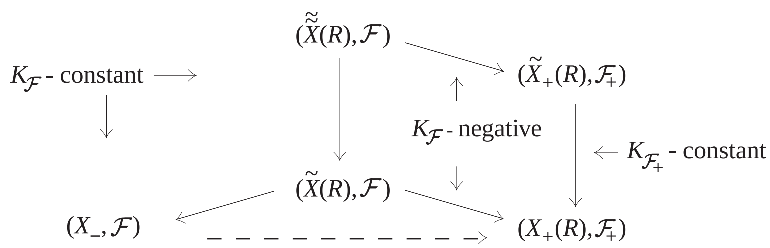
Cartesian square,

$$\begin{array}{ccc} \tilde{\mathcal{X}}_+^{\mathrm{sm}} & \longrightarrow & \tilde{\mathcal{X}}_+^{\mathrm{sm}}/\mathcal{F}_+ \\ \downarrow & \square & \downarrow \quad = \text{weighted blow up in } q(\mathcal{Z}_+)\,. \\ \mathcal{X}_+^{\mathrm{sm}} & \longrightarrow & \mathcal{X}_+^{\mathrm{sm}}/\mathcal{F}_+ \end{array}$$

Putting all of this together, we eventually reach some sort of conclusion, viz:

**IV.8.4 Intermediary Fact.** $\tilde{\mathcal{X}}_+^{\mathrm{sm}} \to \tilde{\mathcal{X}}_+^{\mathrm{sm}}/\mathcal{F}_+$ *is around neighbourhoods of rational stacks wholly contained in the smooth locus a* $\mathbb{P}^1$*-bundle, i.e. despite the presence of non-space like points on* $\tilde{\mathcal{X}}_+$ *where* $\mathcal{F}_+$ *does not have terminal singularities the foliation is nevertheless smooth there, and the leaves are not bad orbifolds.*

The good thing about $\tilde{\mathcal{X}}_+$ is of course that its moduli $\tilde{X}_+$ is actually a scheme, and $\tilde{X}_+$ is itself, or more precisely something close to it, projective. Indeed rather analogously to the flipping procedure, given an extremal ray $R$ in a smooth foliated stack $(\mathcal{X}_-, \mathcal{F})$ which is the Gorenstien cover of its moduli $(X_-, \mathcal{F})$ for which two extremal subvarieties have non-empty intersection, we take the totality of extremal sub-stacks $\mathcal{Y}_+^1, \mathcal{Y}_-^1, \dots, \mathcal{Y}_+^p, \mathcal{Y}_-^p, \mathcal{Y}_1, \dots, \mathcal{Y}_q$ swept out by rays parallel to $R$, where of course $\mathcal{Y}_+^i, \mathcal{Y}_-^i$ are an intersecting pair, while the $\mathcal{Y}_k$'s are disjoint from everything except themselves (actually no such $\mathcal{Y}_k$ exist, given some intersecting pair, but that's irrelevant). Subsequently we make weighted blow ups in $\mathcal{Y}_+^1, \dots, \mathcal{Y}_+^p, \mathcal{Y}_1, \dots, \mathcal{Y}_q$, take appropriate roots of the exceptional divisors $\tilde{\mathcal{Y}}_+^1, \dots, \tilde{\mathcal{Y}}_+^p, \tilde{\mathcal{Y}}_1, \dots, \tilde{\mathcal{Y}}_q$, and call this $\tilde{\mathcal{X}}(R)$. Making weighted blow downs in the opposite direction gives $\mathcal{X}_+(R)$, although naturally we make subsequent weighted blow ups in the proper transforms $\tilde{\mathcal{Y}}_-^1, \dots, \tilde{\mathcal{Y}}_-^p$ of $\mathcal{Y}_-^1, \dots, \mathcal{Y}_-^p$ on $\tilde{\mathcal{X}}(R)$ to give some $\tilde{\tilde{\mathcal{X}}}(R)$ before weighted blowing down in the opposite direction of the proper transforms $\tilde{\tilde{\mathcal{Y}}}_+^1, \dots, \tilde{\tilde{\mathcal{Y}}}_+^p$ of the $\tilde{\mathcal{Y}}_+^1, \dots, \tilde{\mathcal{Y}}_+^p$, to obtain a foliated smooth stack $\tilde{\mathcal{X}}_+(R)$ which by IV.1.4, and sequel has projective moduli. At the moduli level this all fits into a diagram of spaces,




all of which are projective except $X_+(R)$, which is necessarily not so should this situation occur.

As it happens this situation actually yields an everywhere smooth foliated stack $(\tilde{\mathcal{X}}_+(R), \mathcal{F}_+)$, so we'll call it a *correction*. Nevertheless rather than actually prove this, observe that although $\tilde{\mathcal{X}}_+(R)$ has non-scheme like points which are not terminal, $\mathcal{F}_+$ is wholly smooth around these so they're irrelevant to the analysis of extremal sub-stacks in III.5. In addition the number of components of the singular locus of the foliation continues to decrease, so a sequence of flops, flips, and corrections will certainly terminate. Furthermore if in rather more generality we begin with an arbitrary foliated smooth stack $(\mathcal{X}, \tilde{\mathcal{F}})$ with canonical singularities, terminal at the non-scheme like points, and of course projective moduli $(X, \mathcal{F})$ then a $K_{\tilde{\mathcal{F}}}$-extremal ray is $K_{\mathcal{F}}$-extremal for the Gorenstien covering stack $(\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$, and whence the existence of corrections in the Gorenstien case establishes them in general, equivalently in the presence of a boundary divisor $B_-$, i.e.

$$
\begin{tikzcd}
K_{\mathcal{F}}\text{- constant} \longrightarrow \arrow[d, phantom, "\downarrow"] & (\tilde{\tilde{X}}(R), \tilde{\tilde{B}}, \mathcal{F}) \arrow[d] \arrow[r] & (\tilde{X}_+(R), B_+, \mathcal{F}_+) \arrow[dd, "\leftarrow K_{\mathcal{F}_+}\text{- constant}"] \\
 & (\tilde{X}(R), \tilde{B}, \mathcal{F}) \arrow[dl] \arrow[dr] \arrow[r, phantom, "\uparrow\ K_{\mathcal{F}}\text{- negative}\ \downarrow"] & \\
(X_-, B_-, \mathcal{F}) \arrow[rr, dashed] & & (\tilde{X}_+(R), B_+, \mathcal{F}_+)
\end{tikzcd}
$$

Indeed we may even observe that $B_+$ is transverse to $\mathcal{F}_+$, so there is no difficulty in restoring the full-stack structure including the branching over $B_+$ at the $\tilde{X}_+(R)$ level. With all of this out of the way let us conclude by way,

**IV.8.5 Fact/Summary.** Let $\pi : (\mathcal{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ be a smooth foliated stack with projective moduli, and log-canonical singularities terminal at the non-scheme like points such that a sequence of flops and flips of proper extremal sub-stacks does not produce a foliated stack with nef. canonical class. In such circumstances let,

$$(\mathcal{X}, \tilde{\mathcal{F}}) = (\mathcal{X}_0, \mathcal{F}_0) \dashrightarrow (\mathcal{X}_1, \mathcal{F}_1) \dashrightarrow \cdots \dashrightarrow (\mathcal{X}_n, \tilde{\mathcal{F}}_n) = (\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$$

be a maximal sequences of flops and flips of proper extremal sub-stacks, then, either,

(a) The singularities of $(\mathcal{X}, \tilde{\mathcal{F}})$ are not canonical, and $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ is a bundle (in the Zariski topology) of linearly foliated projective stacks $(\boldsymbol{\mathcal{P}}(da_1, \ldots, da_r), \mathcal{R})$ over a smooth projective variety $Z$.

(b) The singularities of $(\mathcal{X}, \tilde{\mathcal{F}})$ are canonical, then there is a separated stack $\tilde{\mathcal{X}}/\tilde{\mathcal{F}}$ with possibly non-projective moduli, such that $\tilde{\mathcal{X}} \to \tilde{\mathcal{X}}/\tilde{\mathcal{F}}$ is a $\mathbb{P}^1(p, q)$ bundle for some relatively prime $p, q \in \mathbb{N}$ in the étale topology of $\tilde{\mathcal{X}}/\tilde{\mathcal{F}}$, and $\tilde{\mathcal{F}}$ is the corresponding fibration.

Should case (b) occur, we can make not just a maximal sequence of flips and flops, but flips, flops, and at the very last stage if necessary a correction to conclude,

(b)$'$ At the price of allowing non-scheme like points on a correction $\tilde{\mathcal{X}}_+$ of $\tilde{\mathcal{X}}$ which are not terminal, we have (b) together with the assertion that both $\tilde{\mathcal{X}}_+$ and $\tilde{\mathcal{X}}_+/\tilde{\mathcal{F}}$ have projective moduli.

**Proof.** Pretty much everything has been established, and it's really just a matter of clearing up loose ends. We begin with,

**IV.8.6 Sub-Claim.** *Suppose that we are in the situation where the obstruction of IV.8.3 to the projectivity of flipping presents itself with respect to an extremal ray $R$, then* $\mathrm{Loc}\,(R) = X$.

**Sub-Proof.** In our usual notations with $X = X_-$, consider the possibly non-projective flip $X_+$ of $\mathcal{Y}_+$ in terms of the flap,

$$
\begin{tikzcd}
 & \tilde{X} \arrow[dl, "\sigma_-"'] \arrow[dr, "\sigma_+"] & \\
X_- & & X_+
\end{tikzcd}
$$

and denote by $C_+$, $C_-$ generic invariant curves in $Y_+$ and $Y_-$ respectively. Furthermore denoting by $\tilde{C}_+$, $\tilde{C}_-$ invariant curves in $\tilde{X}$ lying over $C_+$ and $C_-$ respectively, then we know that $\sigma(\tilde{C}_-)$ is numerically equivalent to a generic invariant rational curve $M$ not just wholly contained in the smooth locus of both $\mathcal{F}_+$ and $X_+$, but missing any codimension 2-subset that we may wish to specify, e.g. $Z_+$. As such $M$ is actually in the smooth locus of both $X_-$ and $\mathcal{F}_-$, while for some positive rationals $p$, $q$,

$$\sigma_+^* M \sim p\,\tilde{C}_+ + q\,\tilde{C}_- \in \mathrm{NS}_1(\tilde{X})_{\mathbb{Q}}$$

from which we conclude that $M$ is parallel to $R$ in Néron-Severi.

Keeping the same notations, let us continue in this vein by way of,

**IV.8.7 Further Claim.** Suppose that we're in the conditions of the sub-claim, but now with a smooth stack $\pi : (\mathcal{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ with the said moduli and log-canonical singularities terminal at the non-scheme like points, then:

Every component $Z$ of the singular locus $\tilde{\mathcal{F}}$ is smooth, and there are proper extremal sub-stacks of codim $\geq 2$, $\mathcal{Y}_+, \mathcal{Y}_- \supset Z$ corresponding to the H-N pair swept out by invariant stacks parallel to $R$. In particular the singularities are canonical, not just log-canonical.

**Proof.** Invariance by $\mathcal{F}$ is a closed condition on Hilb $(X)$, so we can find a subvariety $T$ of Hilb $(X)$ containing $M$, and smooth there of dimension: $\dim X - 1$. In particular if,

$$\begin{array}{ccc} C & \xrightarrow[q]{} & X \\ {\scriptstyle p}\big\downarrow & & \\ T & & \end{array}$$

is the universal family over $T$, then $q$ is surjective. Now let $t \in T$, and $C_t = \Sigma\, a_i\, C_i$ the cycle theoretic fibre, then in $\mathrm{NS}_1(X)$,

$$R = M = q_* \,C_t = \Sigma\, a_i (q_*\, C_i)$$

so every component of $C_t$ is a $K_{\tilde{\mathcal{F}}}$-negative extremal invariant curve parallel to $R$.

Now let $Z$ be a component of sing $(\tilde{\mathcal{F}})$, then for $z \in Z$, there is a $t \in T$ such that, $C_t \ni z$, so in fact $z \in \mathcal{L}$ for some smooth $K_{\tilde{\mathcal{F}}}$-negative extremal rational stack, which in turn defines an extremal subvariety $\mathcal{Y}_+(Z)$ à la III.5. Better still the existence of $M$ forces $\mathcal{Y}_+(Z)$ to be proper of codim $\geq 2$, and of course around $Z$ we can find formal coordinates $y_i$, $x_i$ defining $Z$, $1 \leq i \leq r$, $1 \leq j \leq t$, $a_i$, $b_j \in \mathbb{N}$ so that the foliation is of the form,

$$a_i\, y_i \,\frac{\partial}{\partial y_i} - b_j\, x_j\, \frac{\partial}{\partial x_j}$$

$t \geq 2$, so $Z$ is of course smooth, and the corresponding singularity of $\tilde{\mathcal{F}}$ canonical. Performing the naive flip of $\mathcal{Y}_+(Z)$, we obtain the algebraic space $X_+$ which still has a proper Hilbert-scheme so $\mathcal{Y}_-(Z)$ exists too, and we're done. $\square$

To put all of this together and establish the main claim, observe that if $(\mathcal{X}, \tilde{\mathcal{F}})$ has log-canonical singularities which are not canonical, then by I.6.12 there is an irreducible component $Z$ of sing $(\tilde{\mathcal{F}})$ around the generic point of which we can find formal coordinates $y_1, \ldots, y_r$ defining $Z$, positive integers $a_i$, and a derivation $\partial$ defining the foliation, such that:

$$\partial = a_i\, y_i\, \frac{\partial}{\partial y_i} + f \cdot \delta$$

where $f$ vanishes on $Z$, and $\delta$ is a derivation along $Z$. Manifstly such a singularity is not destroyed by flipping, and flopping proper sub-stacks, so in this situation a maximal (if not unique) chain of flips, and flops of proper sub-stacks is well defined, and terminates in some smooth foliated stack $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ with projective moduli, and log-canonical singularities, terminal at the non-scheme like points. Nevertheless $K_{\tilde{\mathcal{F}}}$ is not nef., nor as the above discussion shows could an extremal ray be represented by an invariant stack in the smooth locus of $\tilde{\mathcal{F}}$. As such the only remaining possibility, by way of the obvious variant of III.5 is precisely (a), with $a_i$ the above integers, provided gcd $(a_1, \ldots, a_r) = 1$, and $d$ the branching order of the hyperplane bundle at infinity.

As to the properly canonical case, we carry out flips and flops of extremal subvarieties until presented with the obstruction IV.8.3 or otherwise. This necessarily gives us in the former case $(\mathcal{X}_{n-1}, \mathcal{F}_{n-1})$, which we can naively flip to $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$, albeit that the moduli may not be projective, although in the latter case where the obstruction doesn't occur $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ will be projective without proper extremal subvarieties so that any invariant representative of a $K_{\tilde{\mathcal{F}}}$-negative extremal ray is smooth. In either case if $(\mathcal{X}', \mathcal{F}) \to (\tilde{X}, \mathcal{F})$ is the Gorenstien cover of the moduli, then $\mathcal{X}'/\mathcal{F}$ is a well defined separated stack, and $\mathcal{X}' \to \mathcal{X}'/\mathcal{F}$ a $\mathbb{P}^1$-bundle, so the only possibility for a branching divisor is that we have two components $B_1$, $B_2$ with weights $m$ and $n$ (where we permit the possibility that these are 1 to avoid notational complexity) such that the corresponding closed stacks $\mathcal{B}_1'$, $\mathcal{B}_2'$ in $\mathcal{X}'$ are smooth everywhere transverse to $\mathcal{F}$. As such taking $p$, $q$ to be $m$, $n$ divided by their gcd, we have (b), as well as (b)$'$ by virtue of the preamble to the claim, with the exception of the projectivity of the moduli $\tilde{\mathcal{X}}_+/\tilde{\mathcal{F}}$, or for that matter $\tilde{\mathcal{X}}/\tilde{\mathcal{F}}$ were the obstruction not to occur. In either case, say the former, the moduli map,

$$\tilde{X}_+ \to \tilde{X}_+/\mathcal{F}$$

is a well defined family of algebraic cycles in the sense of [K-3] I.3.11, and so we have a finite map of $\tilde{X}_+/\mathcal{F}$ to the Chow-scheme of $\tilde{X}_+$. On the other hand the latter is projective so indeed, $\tilde{\mathcal{X}}_+/\tilde{\mathcal{F}}$ has projective moduli. $\square$

## IV.9. Logarithmic Remarks

If we translate back to varieties the minimalist type of stacks that we have discussed, viz: $(\mathcal{X}, \tilde{\mathcal{F}})$ smooth foliated with log-canonical singularities, terminal at the non-scheme like points, then in terms of the moduli $(X, B, \mathcal{F})$ and its branching divisor we know by I.7.4 that this translates as $X$ normal, and log-canonical singularities terminal at $\mathrm{sing}\,(X) \cup |B|$. Translating once more, but this time to the Gorenstien covering stack, $(\mathcal{X}', \mathcal{B}, \mathcal{F})$ we have log-canonical singularities, terminal at the non-scheme like points, and $\mathcal{B}$ a smooth (albeit not necessarily connected) divisor everywhere transverse to $\mathcal{F}$ with finite weights. Now it's pretty clear that we can easily extend the programme to allowing infinite weights on whatever components of $\mathcal{B}$ we may choose by the simple expedient of introducing sufficiently large weights to be determined. There is, however, a rather easier strategy which quite generally produces better models, viz: introduce the 2-cover, $(\mathcal{X}_2, \mathcal{F}_2) \to (\mathcal{X}', \mathcal{F})$ by extracting a square root of every component of $\mathcal{B}$, and run the minimal model programme for $(\mathcal{X}_2, \mathcal{F}_2)$. This results in a smooth foliated stack $(\mathcal{X}_2^{\#}, \mathcal{F}_2^{\#})$ with all of the enunciated properties, and in particular a smooth divisor $\mathcal{B}^!$ of the Gorenstien covering stack of the moduli, $\mathcal{X}^!$, everywhere transverse to $\mathcal{F}^!$. In particular if we restore the full-stack structure $(\mathcal{X}^{\#}, \tilde{\mathcal{F}}^{\#})$, say, associated with an arbitrary system of finite weights $b_i$ then in terms of $\mathbb{Q}$ divisors on $\mathcal{X}^!$,

$$K_{\tilde{\mathcal{F}}^{\#}} = K_{\mathcal{F}_2^{\#}} + \sum_i \left(\frac{1}{2} - \frac{1}{b_i}\right) \mathcal{B}_i^! \,.$$

If however $\mathcal{L}$ is an invariant sub-stack representing a $K_{\tilde{\mathcal{F}}^{\#}}$-negative extremal ray, then $\mathcal{B}_i^! \cdot \mathcal{L} \geq 0$ by transversality, so in fact $K_{\mathcal{F}_2^{\#}} \cdot \mathcal{L} < 0$, and indeed finiteness of the weights is irrrelevant, from which we deduce,

**IV.9.1 One more Fact.** Suppose more generally that we consider log-varieties $(X, B, \mathcal{F})$ with log-canonical singularities, terminal at $\mathrm{sing}\,(X)$, with $B$ (on the Gorenstien cover) smooth and everywhere transverse to $\mathcal{F}$, not to mention possibly infinite weights. Then a maximal sequence of flips and flops for the 2-cover (in the above sense) yields a model $(X^{\#}, B^{\#}, \mathcal{F}^{\#})$ with either,

(a) $K_{\tilde{\mathcal{F}}^{\#}}$ nef., and all the said properties.

(b) A rational foliation, necessarily as per IV.8.5, except in the case of the existence of a genuinely infinite weight which considered as an appropriate affine stack is an $\mathbb{A}^1$-bundle, with, at least after a correction, projective moduli.

## IV.10. Arbitrary Remarks

One might reasonably ask whether the previous body of theory is sufficient for the construction of minimal models of foliated varieties irrespective of the initial model. In this generality the question is slightly ambiguous, since "Mori theory" is a priori non-sensical without (log)-canonical singularities. We can, however, consider the question starting from a foliated stack $(\mathcal{X}, \mathcal{F})$ with projective moduli, $\pi : \mathcal{X} \to X$, and, of course log-canonical singularities. Manifestly, by a minor variant of I.5 we can, without loss of generality, suppose that $(\mathcal{X}, \mathcal{F})$ is foliated Gorenstien. Furthermore by algorithmic resolution we can blow up in $\mathcal{F}$-invariant centres to obtain a smooth modification $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ such that the canonical bundle of the induced foliation is still $\rho^* K_{\mathcal{F}}$. Indeed if the singularities are canonical this is clear, while if they're only log-canonical I.6.12 tells us that sing $(\mathcal{F})$ is regularly embedded where this holds, so this case is okay too. As such provided we're prepared to blow up, we may further suppose that $\mathcal{X}$ is smooth. Now associated to such a $\mathcal{X}$, we have not only the Gorenstien covering stack $(\mathcal{X}_0^*, \mathcal{F}_0)$, of the moduli, but the smallest stack $\mathcal{X}_1$ which is smooth over the moduli. In order to get from $\mathcal{X}$ to $\mathcal{X}_1$, we write the former locally as a classifying stack $[V/G]$, quotient out by the generic stabiliser, then systematically kill pseudo-reflections, cf. [V]. Necessarily a pseudo-reflection fixes a divisor, which itself is either invariant or not. Consequently we obtain a non-invariant divisor $\mathcal{B}$ on $\mathcal{X}_1$, with components $\mathcal{B}_i$ and weights $b_i$ such that,

$$K_{\mathcal{F}} = K_{\mathcal{F}_0} + (1 - 1/b_i)\,\mathcal{B}_i$$

where, for simplicity of notation, we omit the implied pull-backs, and of course, employ the summation convention. Now we can appeal to the pre-quel to I.7.3 to conclude that all of the triples $(\mathcal{X}_1, \mathcal{B}, \mathcal{F}_0)$, $(\mathcal{X}_0, (\pi_{01})_* \mathcal{B}, \mathcal{F}_0)$, $(X, B, \mathcal{F}_0)$ have log-canonical singularities, where $\pi_{01} : \mathcal{X}_1 \to \mathcal{X}_0$, and latin letters denote moduli. Now the role of $\mathcal{B}$ here is pretty trivial except possibly if we're in I.6.14(2), but by hypothesis in this case we can blow up in the tangency locus between $\mathcal{F}_0$ and $\mathcal{B}$, followed by a blow up in the intersection of the proper transform of $\mathcal{B}$ with the exceptional divisor to obtain a model which is terminal at points of the proper transform of $\mathcal{B}$ and at worst canonical at any other points we may have introduced. This operation will raise the canonical class by $1/2$ of the proper transform of the $1^{\text{st}}$ exceptional divisor, but if we're really fussy about not changing $K_{\mathcal{F}}$ then we can blow this down, so, regardless, we may suppose that $\mathcal{F}_0$ is terminal at $\mathcal{B}$, with the latter everywhere transfer. As such what is critical is the difference between $\mathcal{X}_1$ and $\mathcal{X}_0$. To this end let us momentarily introduce a smooth resolution $\rho : \tilde{\mathcal{X}}_0 \to \mathcal{X}_0$ of the Gorenstien covering stack, then on replacing $\mathcal{X}_1$ by the smallest smooth stack covering the moduli of $\tilde{\mathcal{X}}_0$ we can suppose that every point of $\mathcal{X}_1$ admits a neighbourhood of the form $[V/G]$, with $V$ smooth, and $G$-abelian without generic fixed point or pseudo-reflections. Better still if $\zeta$ is a non-scheme like geometric point of $\mathcal{X}_1$, then there is a character $\chi : G \to \mathbb{C}^\times$, without kernel, and a generator $\partial$ of the foliation, such that $\partial^\sigma = \chi(\sigma)\partial$, for all $\sigma \in G$. Consequently we can even suppose that $\mathcal{X}_0 = \mathcal{X}_1$, and we distinguish,

**IV.10.1 Cases.**

(a) $\mathcal{F}_0$ is terminal at $\zeta$. Certainly this can happen, but obviously we're pretty happy if it does since this has been a habitual hypothesis throughout the preceeding ruminations. Notice additionally that since our temporarily introduced $\tilde{\mathcal{X}}_0$ was obtained by invariant modification and $\mathcal{F}_0$ was supposed terminal at $\mathcal{B}$, then à la I.7.3, $\tilde{\mathcal{X}}_0$ and $\mathcal{X}_0$ coincided around $\mathcal{B}$, and everything around $\mathcal{B}$ is still terminal.

(b) $\mathcal{F}_0$ is smooth at $\zeta$, but $\zeta$ is invariant. Actually this can't happen, since $\chi$ is a faithful representation of the local stabiliser. In particular there are no invariant, $\mathcal{F}_0$ smooth, non-scheme like points.

(c) $\mathcal{F}_0$ is singular at $\zeta$. This can happen, but it's difficult. For example, this is impossible not just in dimension 2, but codimension 2. As such the first examples are in codimension 3, and indeed since the local stabiliser preserves Jordan decomposition, these are easily seen to be,

  (1) $G = \mathbb{Z}/3$, $\partial = x\frac{\partial}{\partial x} + \zeta y \frac{\partial}{\partial y} + \zeta^2 z \frac{\partial}{\partial z}$, $\zeta^3 = 1$, and $\zeta$ the origin, with the action being permutation of the eigenfuctions.

(2) $G = \mathbb{Z}/2$, and for some functions $a, b, c$ of two variables, vanishing at the origin,

$$\partial = x\{1 + a(z, xy)\}\frac{\partial}{\partial x} - y\{1 + b(z, xy)\}\frac{\partial}{\partial y} + c(z^2, xy)\frac{\partial}{\partial z}$$

where as ever $\zeta$ is the origin, $G$ acts by permuting $x$ and $y$, but sending $z$ to $-z$, and, of course, $a(z, xy) = b(-z, xy)$.

Now, the notable difference between the first and the second example is that the first on blowing up separates the fixed points of $G$ from sing $(\mathcal{F}_0)$, whereas in the second case this isn't an option. Specifically, much as in [O], even though we have a perfectly smooth Gorenstien covering stack, no matter how much we blow up, we can never achieve that the induced Gorenstien covering stack is everywhere scheme like. In a different guise this example was brought to my attention by Felipe Cano, [Ca]. Plainly it is necessarily a "final form" in any resolution game, but, rather more immediately such an example doesn't fit our standard hypothesis of "terminal at the non-scheme points", so let's look at what a $K_{\mathcal{F}}$-negative invariant stack must look like if indeed we have non-scheme like points at the singularities, so back at $(\mathcal{X}, \mathcal{F})$, albeit without loss of generality the Gorenstien covering stack is smooth, and $\mathcal{X} \to \mathcal{X}_0$ is unramified around sing $(\mathcal{F})$. As such let $f : \mathcal{L} \to \mathcal{X}$ be the normalisation of a $K_{\mathcal{F}}$-negative 1-dimensional stack, which by IV.10.1(b) is generically scheme like, so that the adjunction formula of I.8.7 takes the form,

$$K_{\mathcal{F}}.{}_{\mathcal{F}}\,\mathcal{L} = -2 + \sum_i \left(1 - \frac{1}{n_i}\right) + \sum_{\ell \in f^{-1}(\operatorname{sing}\mathcal{F})} \left\{\frac{\operatorname{ord}_l(\operatorname{sing}\mathcal{F}) - \operatorname{ord}_l(\operatorname{Ram}_f)}{\#\operatorname{stab}_{\mathcal{L}}(\ell)} + \left(1 - \frac{1}{\#\operatorname{stab}_{\mathcal{L}}(\ell)}\right)\right\}$$

where the $n_i$ are the orders of the stabilisers of any non-scheme like points not in sing $(\mathcal{F})$, and we profit from the fact that the moduli of $\mathcal{L}$ must be rational. Consequently negativity obliges us to have at most one point in $f^{-1}(\operatorname{sing}\mathcal{F})$, counted without multiplicity, and unless there are actually no singular points (and whence we have a foliation in conics, which we've dealt with in IV.8) at most one non-scheme like points. Let us view, therefore, $\mathcal{L}$ over $\mathbb{P}^1$ as a stack with stabilisers $\mathbb{Z}/m$, and $\mathbb{Z}/n$ over 0 and $\infty$, with the former the point of $f^{-1}(\operatorname{sing}\mathcal{F})$. Manifestly, we may without loss of generality suppose $m > 1$, although we allow the possibility that $n = 1$ to avoid complicating the notation. Now the easy case, occurs when $\operatorname{ord}_l(\operatorname{sing}\mathcal{F}) - \operatorname{ord}_l(\operatorname{Ram}_f)$ is 1 (with the order function understood on a scheme like étale neighbourhood) so that by II.6.2 and II.8.4 or a very minor variant of the same, the eigenvalues (or better their ratios) of any generator $\partial$ of the foliation around $f(0)$ are rational. On the other hand $\mathcal{X}_0$ is equal to $\mathcal{X}$ around $f(0)$, and is the Gorenstien covering stack, so there is a faithful character $\chi : \mathbb{Z}/m \to \mathbb{C}^\times$, and an appropriate $\partial$, such that $\partial^\sigma = \chi(\sigma)\,\partial$, for $\sigma \in \mathbb{Z}/m$. Whence, working even just mod $\mathfrak{m}^2_{\mathcal{X},f(0)}$, we observe that $\mathbb{Z}/m$ permutes the eigenvalues, and the ratios within a permutation are the values of $\chi$. On the other hand roots of unity are inevitably irrational unless $m = 2$, and there is at least one non-zero eigenvalue, so in fact one for every value of $\chi$, i.e. at worst $m = 2$. However, even in this case the tangent space of $\mathcal{L}$ at $f(0)$ is both $\mathbb{Z}/2$ invariant since $\mathcal{L}$ is, yet not so since the sign of the associated eigenvector changes, which is complete nonsense, so under these hypothesis there are no $K_{\mathcal{F}}$-negative 1-dimensional stacks through sing $(\mathcal{F})$.

The remaining alternative is a bit more delicate, so we'll begin with the easy case that $f$ is an isomorphism, and introduce a local coordinate $t$ on an étale neighbourhood of 0 together with a generator $\partial$ of the foliation, such that, $\partial t = t^{p+1}\,u(t) \bmod I_{\mathcal{L}}$, where $p \in \mathbb{N}$, and $u$ is a unit, such that, without loss of generality $u(0) = 1$. Now, certainly the holonomy around 0 is no worse than $\mathbb{Z}/n$, and it's possibly less if $\mathcal{X} \to \mathcal{X}_0$ is ramified at $\infty$. Furthermore, we may suppose that around 0, $\mathbb{Z}/m$ acts on $t$ by a possibly different character $\psi$, i.e. $t^\sigma = \psi(\sigma)\,t$, $\psi : \mathbb{Z}/m \to \mathbb{C}^\times$, $\sigma \in \mathbb{Z}/m$, which is necessarily faithful, and whence: $\chi(\sigma)\,\psi(\sigma)^p = 1$, and $u = u(t^m)$. Equally there is at least one non-zero eigenvalue/vector in the normal direction, which is necessarily permuted by Im $(\chi)$, so independently of any Gorenstien covering hypothesis, the cardinality of the holonomy is at least $m/p$, and at most $n$, so $K_{\mathcal{F}}.{}_{\mathcal{F}}\,\mathcal{L} = p/m - 1/n \geq 0$. Thus although equality can occur, even in dimension 2, we cannot obtain a $K_{\mathcal{F}}$ negative curve in this way. Better still even if the image of $\mathcal{L}$ had a cusp around $f(0)$, then for our ubiquitous local generator $\partial$, $f^*\partial$ is a vector field on $\mathcal{L}$, such that for any uniformising parameter $t$ at 0, $\operatorname{ord}_0(f^*\partial t) > 1$, and this condition persits under blowing up in the singularity of the cusp, so we reduce to the smooth case, and, whence, a contradiction. Consequently although IV.10.1(c)2 shows that we cannot guarantee our condition of "terminal at the non-scheme like points" everywhere, even after arbitrary modification, we can guarantee it around $K_{\mathcal{F}}$ negative invariant 1-dimensional stacks, so by II.4.1 etc. a fortiori around extremal rays, from which we arrive to,

**IV.10.2 Corollary.** *Let $(\mathcal{X}, \mathcal{F})$ be a $\mathbb{Q}$-Gorenstien foliated normal stack with projective moduli and (log)-canonical singularities (equivalently $(X, B, \mathcal{F})$ is a log-triple such that $K_{\mathcal{F}} + \left(1 - \frac{1}{b_i}\right) B_i$ is $\mathbb{Q}$-Cartier, $X$ is normal projective, and the foliation singularities are log-canonical) then there is a modification $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ with $K_{\tilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$ together with a sequence of flips and flops in the sense of* IV.1.10 *and* IV.5/7.4,

$$(\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) = (\mathcal{X}_0, \mathcal{F}_0) \dashrightarrow (\mathcal{X}_1, \mathcal{F}_1) \dashrightarrow \ldots \dashrightarrow (\mathcal{X}_n, \mathcal{F}_n) = (\mathcal{X}_{\min}, \mathcal{F}_{\min})$$

*such that $(\mathcal{X}_n, \mathcal{F}_n)$ is either one of the rational foliations discussed in* IV.8.5, *or $K_{\mathcal{F}_{\min}}$ is nef.*

Indeed we've done everything, expect the rational case, and out with this case we could even offer the precision that the Gorenstien cover of the moduli of each $(\mathcal{X}_i, \mathcal{F}_i)$ is smooth. The rational case needs no real comment since the very difficult implicit in IV.10.1(c)2 cannot occur.

# V. Value Distribution

## V.1. Remarks on Intersection Theory

Our ultimate set up will be inverse to that of the uniformisation theorem, albeit that the key objects will remain $\mathbb{P}^1$, $\mathbb{C}$, and the unit disc $\Delta$. Unsurprisingly, however, the theory over the unit disc encompasses its equivalents over $\mathbb{P}^1$ and $\mathbb{C}$, so we'll concentrate on the latter, while confining ourselves to parenthetical remarks on its parabolic companions.

In the first place, provided we fix a radius $r \in [0,1)$ we have a well defined notion of degree on $\Delta$ defined by way of Nevanlinna theory. Indeed if $\mathrm{Div}(\bar{\Delta})$ is the group of metricised Cartier divisors on $\Delta$ then we have,

**V.1.1 Definition.** Let $\bar{D}$ be a metricised Cartier divisor on $\Delta$, with $1\!\!1_D \in \Gamma(\mathcal{O}_\Delta(D))$ the tautological section, then,

$$\not\!\!\oint_{\Delta(r)} c_1(\bar{D}) := \sum_{0<|z|<r} \mathrm{ord}_z(D) \log\left|\frac{r}{z}\right| + \mathrm{ord}_0(D)\log r - \int_{\partial\Delta(r)} \log\|1\!\!1_D\| \cdot \frac{d\theta}{2\pi} + \lim_{|z|\to 0} \log\frac{\|1\!\!1_D\|}{|z|^{\mathrm{ord}_0(D)}}\,.$$

To those unfamiliar with Nevanlinna theory, this a priori may just look like sad rubbish. In reality, however, it is a delicate instrument which for effective Cartier divisors pulled back by maps to compact varieties will yield a positive intersection number under the condition that the image of the origin under the map is not "too close" to the divisor. The efficacity of the definition, however, results from an equivalent formulation via integration by parts/Jensen's formula, *i.e.*

**V.1.2 Fact.** *Identifying* $\mathrm{Div}(\bar{\Delta})$*, with the group of metricised line bundles* $\mathrm{Pic}(\bar{\Delta})$ *we have,*

$$\not\!\!\oint_{\Delta(r)} c_1(\bar{D}) = \int_0^r \frac{dt}{t}\int_{\Delta(t)} c_1(\bar{D})$$

*where the integrals on the right are understood with respect to standard Lebesgue measure on* $[0,r]$ *and the disc* $\Delta(t)$ *of radius* $t$ *respectively. In particular the Nevanlinna type integral* $\not\!\!\oint$ *extends to smooth* $(1,1)$ *forms by way of,*

$$\not\!\!\oint_{\Delta(r)} : A^{1,1}(\Delta) \to \mathbb{C} : \tau \mapsto \int_0^r \frac{dt}{t}\int_{\Delta(t)} \tau$$

*and more generally to* $(1,1)$ *forms of appropriate regularity, say* $A^{1,1}_?(\Delta)$*, to guarantee the finiteness of the right hand side.*

The appropriate level of generality, however, involves finite coverings $p : \mathcal{C} \to \Delta$ of the unit disc by smooth analytic stacks. To this end observe,

**V.1.3 Fact/Definition.** *There is a well defined map,*

$$p_* : A^{1,1}(\mathcal{C}) \to A^{1,1}_?(\Delta)\,.$$

**Proof/Justification.** Indeed let, $q : \coprod_\alpha C_\alpha \to \mathcal{C}$ be an analytic atlas for $\mathcal{C}$, and $\tau$ a smooth $(1,1)$ form on $\mathcal{C}$. In addition as per I.2.5, we can suppose that $\mathcal{C}$ has moduli $C$, with $[C_\alpha/G_\alpha] = \mathcal{C} \times_C U_\alpha$, for $\coprod_\alpha U_\alpha \to C$ a covering of $C$, and $G_\alpha$ finite groups acting on $C_\alpha$. Denoting as ever by $\pi$ the moduli map, we have in the first place,

$$\pi_* : A^{1,1}(C_\alpha) \to A^{1,1}_?(U_\alpha) : \sigma \mapsto \frac{1}{|G_\alpha|}\,(\pi\,|_{C_\alpha})_*(\sigma)$$

where now ? simply indicates some appropriate regularity class. Manifestly this extends to the whole of $\mathcal{C}$ according to,

$$\pi_* : A^{1,1}(\mathcal{C}) \to A^{1,1}_?(C) : \tau \to \pi_*(\tau\,|_{C_\alpha})$$

where the right hand side is understood to be the measurable function calculated at $c \in C$ by choosing a $U_\alpha \ni c$, and applying the previous local definition. As such, having got this far, we can just apply the standard push forward from $C \to \Delta$ in order to conclude, while, observing that this agrees with our previous degree conventions in I.8.3.

Therefore, and essentially without need of comment, we make

**V.1.4 Definition.** Let things be as above then we define,

$$\fint_{\mathcal{C}(r)} : A^{1,1}(\mathcal{C}) \to \mathbb{C} : \tau \mapsto \fint_{\Delta(r)} p_* \tau \,.$$

Once more, and despite its triviality, what's important here is a Riemann-Hurwitz type formula. To begin with we need to understand what is meant by "canonical degree of a disc", which in turn is nothing other than the Nevanlinna degree of the Poincaré metric, *i.e.*

**V.1.5 Definition.** As ever for $r \in [0,1)$ put,

$$\mathrm{discr}_\Delta(r) := \fint_{\Delta(r)} dd^c \log(1-r^2)^{-1} = \log(1-r^2)^{-1} \,.$$

Returning to finite covering stacks $p : \mathcal{C} \to \Delta$, with generic stabiliser $G$, observe that the Riemann-Hurwitz formula of I.8.5 naturally suggests that we make,

**V.1.6 Definition.** Let things be as above, with $C \to \Delta$ the moduli of $p$ then,

$$\mathrm{discr}_{\mathcal{C}}(r) := \frac{(C:\Delta)}{|G|} \log(1-r^2)^{-1} + \sum_{0<|p(c)|<r} \mathrm{ord}_c(\mathrm{Ram}_{\mathcal{C}/\Delta}) \log\left|\frac{r}{p(c)}\right| + \sum_{c \in p^{-1}(0)} \mathrm{ord}_c(\mathrm{Ram}_{\mathcal{C}/\Delta}) \log r$$

where the geometric closed points $c$ of $\mathcal{C}$ are identified with those of $C$, but the order function at $c$ is the order function on an étale neighbourhood divided by the order of the stabiliser of $c$.

While there are, perhaps somewhat surprisingly in light of the uniformisation theorem, situations that require this level of generality (*cf.* ??), most of the cases to be immediately discussed will involve nothing more complicated than a stack whose moduli is the disc itself together with a generically trivial stabiliser. As such let us note,

**V.1.7 Small Fact.** *Suppose $p : \mathcal{C} \to \Delta$ is in fact the moduli map, the stabiliser is generically trivially, and the non-scheme like points $c_i$ of $\mathcal{C}$ have automorphism group $\mathbb{Z}/e_i\mathbb{Z}$ with no $c_i$ lying over the origin, then,*

$$\mathrm{discr}_{\mathcal{C}}(r) = \log(1-r^2)^{-1} + \sum_{0<|p(c_i)|<r} \left(1 - \frac{1}{e_i}\right) \log\left|\frac{r}{p(c_i)}\right| \,.$$

## V.2. Positivity

As already suggested post definition V.1.1, the crucial property of these Nevanlinna type definitions is that they lead to positivity of intersection between effective Cartier divisors and discs under relatively mild hypothesis. Indeed consider what will be the principle case of interest according to the following arrows, which in turn fixes our notation, *i.e.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"'] \arrow[d, "\pi"'] & \mathcal{X} \arrow[d, "\pi"] \\ C \arrow[r, "f"'] \arrow[d, "p"'] & X \\ \Delta & \end{tikzcd}$$

where as ever $\pi$ is the moduli, and the stack $\mathcal{X}$ has a projective (or even just compact) moduli space $X$, and we equally denote by $p$ what would be $p \circ \pi$ in the diagram. Note in addition that we will abusively employ,

**V.2.1 Notation.** If $\alpha$ is a function on $\mathcal{C}$, then,

$$\alpha(0) := (p_* \, \alpha)(0)$$

where $p_*$ on functions is defined analogously to the same on forms as per V.1.3.

With this in mind, Jensen's formula yields,

**V.2.2 Small (but important) Fact.** *Let $\bar{\mathcal{D}}$ be an effective metricised Cartier divisor on $\mathcal{X}$, with $\mathbb{I}_{\mathcal{D}} \in H^0(\mathcal{O}_{\mathcal{X}}(\mathcal{D}))$ the tautological section, then,*

$$\displaystyle\not\!\!\oint_{\Delta(r)} f^* \, c_1(\bar{\mathcal{D}}) \geq \log \| f^* \, \mathbb{I}_{\mathcal{D}} \| \, (0) \, .$$

One should, however, bear in mind,

**V.2.3 Warning.** There is absolutely no a priori reason why $\log \| f^* \, \mathbb{I}_{\mathcal{D}} \| \, (0)$ need not be ridiculously small in comparison to the degree function. As such it will be a hypothesis appropriate to the situation which guarantees this.

Warnings apart, the positivity implicit in V.2.2, is best combined with some further observations resulting from the equivalent formulation of the degree function given by V.1.2. Specifically,

**V.2.4 Definition/Discussion.** Consider the situation of countably many maps $f_n : \mathcal{C}_n \to \mathcal{X}$, from countably many stacks $p_n : \mathcal{C}_n \to \Delta$ and suppose that for some $r_0 \in (0,1)$,

$$\frac{1}{(\mathcal{C}_n : \Delta)} \displaystyle\not\!\!\oint_{\Delta(r_0)} f_n^* \, c_1(\bar{H})$$

tends to infinity, where $\bar{H}$ is a metricised ample divisor on $X$. Independently of these hypothesis observe that for every $n$ and $r$ we can (suppose $\mathcal{X}$ smooth for simplicity) define a current,

$$T_n(r) : A^{1,1}(\mathcal{X}) \to \mathbb{C} : \tau \mapsto \frac{1}{(\mathcal{C}_n : \Delta)} \left( \displaystyle\not\!\!\oint_{\Delta(r)} f_n^* \, c_1(\bar{H}) \right)^{-1} \displaystyle\not\!\!\oint_{\Delta(r)} f_n^* \, \tau \, .$$

Manifestly the said current is positive, and by Jensen's formula more or less harmonic. Under the initial hypothesis of our discussion, however, we have,

**V.2.5 Fact.** *Suppose additionally that $f_n(0)$ (in the sense of* V.2.1*, so more precisely $f_n(p^{-1}(0))$) is bounded away from the branch locus of $\mathcal{X} \to X$ independently of $n$ (or more generally an appropriate growth condition in $n$) then for $r \geq r_0$ outside a set of finite hyperbolic measure (*i.e. *$(1-r^2)^{-1} \, dr$) in $(0,1)$ any convergent weak limit in $n$, of the $T_n(r)$ defines a closed positive current.*

**Proof/Sub-Discussion.** This is essentially just ??. There is, nevertheless, a small difference arising from the branching of $\mathcal{X} \to X$. As such what we have to check is that if $U_\alpha \to \mathcal{X}$ is an étale neighbourhood, and $\omega_\alpha$ a non-negative $(1,1)$ form on it with compact support then for a suitable constant (depending on $\omega_\alpha$),

$$\displaystyle\not\!\!\oint_{\Delta(r)} f_n^* \, \omega_\alpha \ll \displaystyle\not\!\!\oint_{\Delta(r)} f_n^* \, c_1(\bar{H}) \, .$$

To this end, consider a smooth modification $\rho : \tilde{X} \to X$, such that $\rho^{-1}$ of the branch locus of $\pi$ is a smooth normal crossing divisor, and denote by $\tilde{\mathcal{X}}$ the normalisation of $\tilde{X}$ in $\mathcal{X}$. Let us further introduce the divisors $E_i$ contracted by $\rho$, and note that for an appropriately large constant $\alpha$,

$$\omega := \alpha \, c_1(\tilde{H}) - \sum_i dd^c \log \log^2 \| \mathbb{I}_{E_i} \|$$

is a complete metric on $\tilde{X} \backslash \underset{i}{\cup} E_i$, where, say, $\tilde{H}$ is an ample divisor on $\tilde{X}$ in a Fubini-Study type metric and $\|\mathbb{1}_{E_i}\|$ is the norm of the tautological section of $\mathcal{O}_{\tilde{X}}(E_i)$ in some norm. As such if $\tilde{f}_n$ denotes a lifting of $f_n$ $\tilde{\mathcal{X}}$, then indeed, independently of $n$,

$$\fint_{\Delta(r)} f_n^* \omega_\alpha \ll \fint_{\Delta(r)} \tilde{f}_n^* \omega \, .$$

So that it only remains to verify,

$$\fint_{\Delta(r)} \tilde{f}_n^* \omega \ll \fint_{\Delta(r)} f_n^* c_1(\bar{H}) \, .$$

This, however, follows immediately from the fact that the class in Pic of $\tilde{H}$ is that of $H$ minus some exceptional $\mathbb{Q}$-divisor together with our hypothesis on $f_n(0)$ and the ubiquitous Jensen formula. □

Returning then to the main theme we obtain as weak limits of the $T_n(r)$, closed positive currents $T(r)$ for $r$ outside of some set of finite hyperbolic measure. In principle, of course, not to mention reality, we're likely to have some dependence on some subset(s) of $\mathbb{N}$ over which we subsequence. In practice, however, this is never a problem (*cf.* ??), so we'll tend to use the notation $T(r)$, or even just $T$, as if it were wholly unambiguous. Indeed, such is the lack of ambiguity, that we even have a sort of functoriality, *i.e.*

**V.2.6 Claim.** *Let $\sigma_m : \mathcal{X}_m \to \mathcal{X}$ be a countable sequence of proper maps, $f_{nm}$ liftings of $f_n$, then we can define closed positive currents $T_m$ on $\mathcal{X}_m$, such that for some constants $\lambda_m$,*

$$(\sigma_m)_* T_m = \lambda_m T \, .$$

It may of course happen that the $\lambda_m$ are zero, but this is usually what we want to know, so that to all intents and purposes, we may without loss of generality deploy the $\lambda_m$ in the normalisation of the $T_m$ in order to assert,

$$(\sigma_m)_* T_m = T \, .$$

We can also arrange all of the various subsequencing so that,

**V.2.7 Fact.** *Again let $\sigma_m : \mathcal{X}_m \to \mathcal{X}$ be a countable sequence of proper maps, and $\mathcal{Y} \subset \mathcal{X}$ a proper closed substack such that $f_n(0)$ is bounded away from $\mathcal{Y}$ independently of $n$. Suppose in addition that everything is as before (*i.e. V.2.5 *and* V.2.6 *hold) then for any effective Cartier divisor $\mathcal{D}$ on some $\mathcal{X}_m$ whose support is contained in $\sigma_m^{-1}(\mathcal{Y})$,*

$$\mathcal{D}.\, T_m \geq 0 \, .$$

Evidently the current $T$ serves as a "homological limit" of the maps $f_n$, which we can think of as a homology class of the data $\{f_n \,|_{\Delta_r} : n \in \mathbb{N}\}$, with $r \geq r_0$ appropriately chosen. In order to get a little feeling for the definitions and their properties we may consider the analogues for $\mathbb{P}^1$, and $\mathbb{C}$, *i.e.*

**V.2.8 Parenthetical Remark 1.** Suppose our interest is some set of closed 1-dimensional irreducible substacks $\mathcal{C}_n$ of $\mathcal{X}$ or better maps, $f_n : \mathcal{C}_n \to \mathcal{X}$, with $\mathcal{C}_n$ smooth. Manifestly there is a priori no issue about boundaries, so that we already have closed positive currents,

$$T_n : A^{1,1}(\mathcal{X}) \to \mathbb{C} : \tau \mapsto (H \cdot_{f_n} \mathcal{C}_n)^{-1} \int_{\mathcal{C}_n} f_n^* \tau$$

or perhaps with a further normalising factor $(\mathcal{C}_n : \mathbb{P}^1)$ on choosing maps to $\mathbb{P}^1$, although this would only really be appropriate for rational points over function fields (and even in the general disc case $(\mathcal{C}_n : \Delta)$ is always bounded in applications). In any case there is certainly some weak limit $T$ of some subsequence, enjoying the same sort of "functoriality" that we find in V.2.6. Better still the condition of V.2.7, that $f_n(0)$ be bounded away from $\mathcal{Y}$, may be replaced by $\mathrm{Im}(f_n) \not\subset \mathcal{Y}$, under which condition, and the hypothesis of V.2.5, we obtain once more,

$$\mathcal{D}.\, T_m \geq 0\,.$$

Furthermore, if in addition the curves $\mathcal{C}_n$ were Zariski dense, and the $\sigma_m$ birational, then we would even have,

$$\mathcal{D}.\, T_m \geq 0$$

for every $m$, and every effective Cartier divisor $\mathcal{D}$ on $\mathcal{X}_m$. The main thing here, however, to reflect on is that in this case we have an object of study which is wholly well defined, even in positive characteristic, yet the nature of the limit is rather analytic, which in turn poses the question of whether is there a better way of taking a limit of algebraic cycles. Certainly we could work at the level of Néron-Severi, but surprisingly this has a tendancy to loose information in a way that is not entirely desirable.

**V.2.9 Parenthetical Remark 2.** Consider, this time, a proper covering stack $p : \mathcal{C} \to \mathbb{C}$, together with a map $f : \mathcal{C} \to \mathcal{X}$. Obviously, provided things are defined on the boundary all of the previous discussion continues to make sense, where appropriate, for $r = 1$, and in this way we may introduce the currents $T_f(r)$ defined as,

$$T_f(r) : A^{1,1}(\mathcal{X}) \to \mathbb{C} : \tau \mapsto \left( ⨏_{\mathcal{C}(r)} f^*\, c_1(\bar{H}) \right)^{-1} ⨏_{\mathcal{C}(r)} f^*\, \tau\,,$$

where for $r \in \mathbb{R}$ now, $\mathcal{C}(r)$ is regarded as a covering of the unit dist by way of,

$$\begin{array}{ccc} \mathcal{C}(r) & \longrightarrow & \mathcal{C} \\ {\scriptstyle p}\big\downarrow & \square & \big\downarrow{\scriptstyle p} \\ \Delta & \longrightarrow & \mathbb{C} \\ z & \mapsto & rz \end{array}$$

As per our earlier remark, the closure of an appropriate limit of the $T_f(r_n)$, $r_n \to \infty$ outside a set of finite Lebesgue measure is much easier (and indeed more intuitive) than its disc analogue, while again, as for complete curves, the hypothesis on $f_n(0)$ used to guarantee V.2.5, can be replaced by $f(\mathcal{C}) \not\subset \mathcal{Y}$, while again Zariski deneseness of $\mathrm{Im}(f)$ guarantees the positivity of intersection with effective Cartier divisors.

A final remark is that although metrics are necessary in the definition of how we intersect divisors and discs, even without the various closure properties that we've discussed, the role of exactly which metric is employed is almost invariably without importance, and, as such, will be frequently dropped from the notation.

## V.3. Comparison of Conformal Structures

The minimal correct generality here is to consider smooth stacks $\mathcal{X}$ together with a simple normal crossing boundary $\mathcal{B}$. The natural conformal structure of the pair $(\mathcal{X}, \mathcal{B})$ amounts to a complete metric on $\mathcal{X}\backslash\mathcal{B}$, which in turn defines a metric on $T_{\mathcal{X}}(-\log \mathcal{B})$ with rather mild singularities, *i.e.* if on some étale neighbourhood $\mathcal{B}$ is defined by $x_1 \ldots x_n = 0$ and $t = \lambda_i\, x_i\, \frac{\partial}{\partial x_i} + \mu_j\, \frac{\partial}{\partial y_j}$ with $y_j$ the other coordinates, then,

$$\|t\|^2 \sim \sum_i |\lambda_i|^2 \log^{-2} |x_i|^2 + \sum_j |\mu_j|^2\,.$$

In any case it's certainly sufficiently regular to permit us to metricise the tautological bundle $L$ of $\mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{B}))$ in such a way that it makes sense to intersect the said metricised bundle, $L^c$, say, with maps from coverings of discs, provided that the latter don't factor through $\mathcal{B}$. Ironically, despite our previous protestations, this is a case where there is some importance in the metric (essentially because its singular rather than smooth) so we'll denote a smooth metric on the same by $L$ without further need for notational precision. Supposing all of this, and introducing a sequence of maps $f_n : \mathcal{C}_n \to \mathscr{X}$ from covering stacks $p_n : \mathcal{C}_n \to \Delta$ of the disc satisfying the hypothesis that $f_n(0)$ is bounded away from the branching locus of $\pi : \mathscr{X} \to X$ over the moduli, we introduce an appropriate singular $(1,1)$ form $\omega$ on $\mathscr{X}$ which defines the metric on $T_{\mathscr{X}}(-\log \mathcal{B})$ and write,

$$f_n^* \,\omega = \frac{\|f_n'\|^2_{\log}}{|p_n'|^2}\, p_n^* \left(\frac{dz\, d\bar{z}}{-2\pi\sqrt{-1}}\right) = \frac{\|f_n'\|^2}{|p_n'|^2}\, p_n^*\, d\mu\,.$$

Now simply apply Jensen's formula to the logarithmic derivatives $f_n' : \mathcal{C}_n \to \mathbb{P}(\Omega_{\mathscr{X}}(\log \mathcal{B}))$ to obtain,

**V.3.1 Claim.** *Suppose for simplicity $f_n$, $p_n$ unramified over the origin, and $f_n(0) \not\subset \mathcal{B}$, then*

$$\begin{aligned}
\log \frac{\|f_n'\|^2}{|p_n'|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^*\, c_1(L^c) \quad &= \quad \int_{\partial \mathcal{C}_n(r)} \log \frac{\|f_n'\|^2_{\log}}{|p_n'|^2}\, p_n^* \left(\frac{d\theta}{2\pi}\right) + \sum_{0<|p_n(c)|<r} \mathrm{ord}_c(\mathrm{Ram}_{p_n}) \log \left|\frac{r}{p_n(c)}\right| \\
&+ \quad \sum_{0<|p_n(c)|<r} \min\left\{\mathrm{ord}_c(f_n^*\,\mathcal{B}), \frac{1}{|\mathrm{Aut}(c)|}\right\} \log \left|\frac{r}{p_n(c)}\right| \\
&- \quad \sum_{\substack{0<|p_n(c)|<r \\ f_n(c)\notin \mathcal{B}}} \mathrm{ord}_c(\mathrm{Ram}_{f_n}) \log \left|\frac{r}{p_n(c)}\right|
\end{aligned}$$

*where as ever* $\mathrm{ord}_c$ *for c a closed geometric point is computed as the order function on an étale neighbourhood divided by* $|\mathrm{Aut}(c)|$.

To avoid long winded writing of complicated expressions let's give every term occuring in the sum a name. The term counting the $p_n$ ramification is as per V.1.6, and equal to what we add to the Poincaré term to compute the discriminant so lets call it $\mathrm{discr}^{\#}_{\mathcal{C}_n}(r)$. The next term counts the intersection with $\mathcal{B}$ without multiplicity, so lets call it the radical, and write, $\mathrm{rad}_{f_n,\mathcal{B}}(r)$, while of course the last term counts what's left of the $f_n$ ramification so we'll write $\mathrm{Ram}_{f_n}(r)$. To get a feeling for the term under the integral sign, we'll introduce the length in the complete metric, *i.e.*

$$\ell_{n,\log}(r) := \frac{1}{(\mathcal{C}_n : \Delta)} \int_{\partial \mathcal{C}_n(r)} \frac{\|f_n'\|^2}{|p_n'|^2}\, p_n^* \left(\frac{d\theta}{2\pi}\right)$$

where our conventions on relative degrees are as per I.8. As such the Jensen type identity affords by way of the concavity of the logarithm,

**V.3.2 Sub-Claim.** *Everything as above,*

$$\log \frac{\|f_n'\|^2_{\log}}{|p_n'|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^*\, c_1(L^c) \leq (\mathcal{C}_n : \Delta) \log \ell_{n,\log}(r) + \mathrm{discr}^{\#}_{\mathcal{C}_n/\Delta}(r) + \mathrm{rad}_{f_n,\mathcal{B}}(r) - \mathrm{Ram}_{f_n}(r)\,.$$

Notice, however, that notational improvements apart, V.3.2 is strictly less interesting than V.3.1, since we've now got a much less delicate measure of the "ramification of the boundary", whose exact measurement is critical to the difference between isoperimetric inequalities and curvature. Nevertheless what it is suggestive of is the basic observation of Nevanlinna theory, *i.e.* the lemma of the logarithmic derivative/tautological inequality,

**V.3.3 Fact.** *If we put* $S_n(r) = \fint_{\mathcal{C}_n(r)} f_n^*\, c_1(L^c)$ *then,*

$$\ell_{n,\log}(r) = \frac{1}{r(\mathcal{C}_n : \Delta)}\, (r\, S_n'(r))'\,.$$

As such basic measure theory easily allows estimation, essentially independent of $n$, under hypothesis such as V.2.7 with $\mathcal{B} \subset \mathcal{Y}$, of the form,

$$\ell_{n,\log}(r) \ll (1-r^2)^{-1}$$

outside a set of finite hyperbolic measure. Rather than engage in precise statements we'll simply cross reference to ??, and note that essentially the sub-claim now reads,

$$\log \frac{\|f'_n\|^2_{\log}}{|p'_n|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^* c_1(L^c) \leq \mathrm{discr}_{\mathcal{C}_n/\Delta}(r) + \mathrm{rad}_{f_n,\mathcal{B}}(r) - \mathrm{Ram}_{f_n}(r)\,.$$

Unfortunately, there is a non-trivial difference between using the complete metric and the smooth metric. As it happens this is only of importance for computing rather delicate phenomenon about discs arbitrarily close to $\mathcal{B}$, but since it does occur let us observe,

**V.3.4 Fact.** *Denoting by* $\|\ \|_{\log_c}$, $\|\ \|_{\log_{sm}}$ *the complete and smooth metrics on* $T_{\mathcal{X}}(-\log \mathcal{B})$ *respectively, we have,*

$$\begin{aligned} \log \frac{\|f'_n\|^2_{\log_{sm}}}{|p'_n|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^* c_1(L) \quad &\leq \quad \log \frac{\|f'_n\|^2_{\log_c}}{|p'_n|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^* c_1(L^c) \\ &+ \quad \log O\left(1, \fint_{\mathcal{C}_n(r)} f_n^* c_1(H), |\log f_n^* \|\mathbb{I}_{\mathcal{B}}\|\,(0)|\right). \end{aligned}$$

So indeed, up to essentially, $\log\log^+ \|f_n^*\,\mathbb{I}_{\mathcal{B}}\|\,(0)$, there is no difference between working with $L^c$ as opposed to $L$. Nevertheless examples do exist where this seemingly negligible term makes all the difference in the world, albeit that they require the $f_n$ to be arbitrarily close to $\mathcal{B}$.

Anyway, be that as it may, the thing to retain from this discussion is,

**V.3.5 Summary.** Given some bunch of maps $f_n : \mathcal{C}_n \to \mathcal{X}$ of unbounded degree we associate a limiting current $T$. Equally we may do the same for their logarithmic derivatives $f'_n : \mathcal{C}_n \to \mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{B}))$ and provided then $f_n(0)$ aren't too close to $\mathcal{B}$ we can do this so that the associated current $T'$ satisfies a bound of the form,

$$L.\,T' \leq \mathrm{discr} + \mathrm{rad} - \mathrm{Ram}$$

where disc, rad, ram are as per their counterparts in V.3.2 after dividing out by the degree with respect to an ample $H$. The right hand side may be thus be infinite, but that's inevitably what we're trying to prove, so we may as well suppose that its finite and $T'$ pushes forward to $T$.

## V.4 Invariant curves and singularities

As a prelude to extending the previous, and essentially tautological, generalities on comparison of conformal structures to something rather more refined in the folitation context, we will require some knowledge of how invariant curves not wholly contained in the singular locus may meet the latter. To begin with our situation is completely local, *i.e.* if $\hat{\Delta}^n$ is formal affine space of dimension $n$, then we consider a derivation $\partial$ of $\mathcal{O}_{\hat{\Delta}^n}$ with a not necessarily isolated singularity at the origin, along with an invariant map, $f : \hat{\Delta}(= \hat{\Delta}^1) \to \hat{\Delta}^n$ not factoring through the singular locus. We suppose that the singularity of $\partial$ is not just log-canonical, but rather canonical, and introduce the Jordan decomposition $\partial_S + \partial_N$ of $\partial$ into its semi-simple and nilpotent part with respect to some appropriate coordinates $x_1, \ldots, x_n$ which afford the standard representation of I.7.1. Observe, in particular, that the naturality of the Jordan decomposition implies that $f$ is not just invariant by $\partial$, but by both $\partial_S$ and $\partial_N$. Invariance under a semi-simple field is, of course, rather easy to analyse, *i.e.*

**V.4.1 Fact.** *Let $\partial_S = \lambda_i\, x_i\, \frac{\partial}{\partial x_i}$ (summation convention, here, and throughout) be semi-simple with $f : \hat{\Delta} \to \hat{\Delta}^n$ $\partial_S$ invariant, and $f^* x_1 \neq 0$, then after a re-ordering of coordinates, there is a coordinate $z$ on $\hat{\Delta}$ such that,*

$$f(z) = (u_1\, z^{e_1}, \dots, u_n\, z^{e_n})\,,\ e_i \in \mathbb{N}$$

*where $u_i \in \mathbb{C}$ is constant, and should it be non-zero then not only is $\lambda_i \neq 0$ but, $\lambda_i/\lambda_1 = e_i/e_1$.*

Now let's keep the hypothesis and notation of the above fact, modulo re-ordering to obtain $\lambda_1 \leq \ldots \leq \lambda_m$, $f^* x_i \neq 0$, $1 \leq i \leq m$, $f^* x_i = 0$, $i > m$, for some $1 \leq m \leq n$, and proceed to an analysis which takes account of $\partial_N$. Necessarily $\partial_N$ is supported on monomials of the form,

$$x_i\, x^Q\,,\ Q \in \mathbb{Z}^n_{\geq 0}\,,\ \text{or } q_i = -1\,,\ q_j \geq 0\,,\ j \neq i\,,\ \text{and in any case } \Lambda \cdot Q = 0\,.$$

For such monomials, denote by $|Q|$ the integers $k$ between 1 and $n$ such that $x_k$ occurs in $x_i\, x^Q$ to a positive power. Manifestly if $|Q| \cap \{m, \dots, n\} \neq \varphi$, then $f^*(x_i\, x^Q) = 0$. Furthermore for $1 \leq i \leq n$, and $Q \in \mathbb{Z}^n_{\geq 0}$, $\Lambda \cdot Q = 0$ is an impossibility without $|Q| \cap \{m, \dots, n\} \neq \varphi$, so indeed $f^*(x_i\, x^Q) = 0$ whenever $Q \in \mathbb{Z}^n_{\geq 0}$. As to the remaining cases, one notes that if $x_1, \dots, x_k$ are the non-zero under $f^*$ functions with eigenvalue $\lambda_1$ then any $x_j\, x^Q$, $1 \leq j \leq k$, whose pull-back by $f$ is non-zero, is of degree 1. Now consider a Jordan-block associated with $\lambda_1$, let's say,

$$\delta = (\lambda_1\, y_1 + y_2)\, \frac{\partial}{\partial y_1} + \cdots + (\lambda_1\, y_{r-1} + y_r)\, \frac{\partial}{\partial y_{r-1}} + \lambda_1\, y_r\, \frac{\partial}{\partial y_r}$$

for some appropriate $\{y_1, \dots, y_r\} \subset \{x_1, \dots, x_n\}$, and suppse $1 \leq t \leq r$ is the largest integer such that $f^* y_t \neq 0$. Now for some meromorphic function $\varphi(z)$ we have, $y_t'(z) = \lambda_1\, \varphi\, f^*\, y_t$, so in fact $\varphi(z) = \frac{e_1}{\lambda_1 z}$. As a result if $t > 1$, then, $\varphi\, f^*\, y_t = 0$, which is nonsense, and whence, $f^*(x_j\, x^Q) \equiv 0$ for any $1 \leq j \leq k$. Moreover if we turn to the other eigenvalues, then we've established that $f_* \left(\frac{\partial}{\partial z}\right) = \frac{e_1}{\lambda_1 z}\, f^* \partial$, so that direct calculation yields $f^* \partial_N = 0$. Let's summarise by way of,

**V.4.2 Fact.** *Notations as per* I.7.1*, but put $\lambda_1 = 1$. Furthermore let $H(\Lambda)$ be the big height in the standard $\ell_1$-metric of the point $\frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_M}{\lambda_1} \in \mathbb{A}^{M-1}(\mathbb{Q})$ (including $M = 1$) where $M \geq m$, and $\lambda_i/\lambda_1$ are all the eigenvalues in $\mathbb{Q}_{>0}$, then,*

(i) *For any $i > M$, $f^* x_i = 0$.*

(ii) *$f^* \partial_N \in f^* T_{\hat{\Delta}^n}$ is identically zero.*

(iii) *The order of vanishing at the origin of $f^*(x_1 \dots x_M)$ is at least $H(\Lambda)$.*

So to give an indication of where this is going, basically the point is that we're finding formal subvarieties which have a priori nothing to do with $f$, but in fact must vanish on it, or in a worst case scenario (viz: (iii)) vanish to high order unless $\Lambda$ is some rather small rational point. We should, however, also consider the possibility that there is no such $x_1$, but this is precisely its own alternative, *i.e.*

**V.4.3 Fact.** *Things as before, but without the hypothesis on $f^* x_1 \neq 0$, nor the same after re-ordering, then in fact,*

(iv) *$f^* \partial_S \in f^* T_{\hat{\Delta}^n}$ is identically zero.*

Equally (iv) also holds if $f^* x_1 \neq 0$ but $\lambda_1 = 0$.

In order to subsequently profit from these observations we will require to understand how Jordan decomposition varies from point to point. Ultimately our discussion will be local for the étale topology in various formal neighbourhoods of the foliation singularities and so our set up is as follows: $\mathcal{O}$ will be a regular ring, complete in the adelic topology of an ideal $I$ defined by coordinate functions $x_1, \dots, x_m$ which in turn cut out a smooth connected subscheme $Y$ of $\operatorname{Spec} \mathcal{O}$, and we'll even suppose that $\mathcal{O} = \mathcal{O}_Y[[x_1, \dots, x_m]]$ with $Y$

admitting an étale map to an affine space, so that there are functions $y_1, \dots, y_n$ on $Y$ such that $dx_i$, $dy_j$ freely generate $\Omega_{\mathcal{O}}$. There will of course be a derivation $\partial$ of $\mathcal{O}$ with canonical singularities whose singular locus contains $Y$ in a not necessarily isolated way. Furthermore we will suppose that the linearisation $D$ of $\partial$ modulo $I^2$, which necessarily factors as,

$$D : \Omega_{\mathcal{O}} \otimes \mathcal{O}_Y \to I/I^2 \hookrightarrow \Omega_{\mathcal{O}} \otimes \mathcal{O}_Y$$

is already in Jordan normal form, so that indeed any nilpotent part is of the form, $x_i \frac{\partial}{\partial x_p}$ or $x_i \frac{\partial}{\partial y_q}$ for suitable $i, p, q$, and the eigenvalues $\lambda_1, \dots, \lambda_m$ are well defined functions in $\mathcal{O}_Y$, having, without loss of generality, $dx_i$ as the corresponding eigenfunctions under the semi-simple part. Unfortunately it will be necessary to distinguish several case, beginning with,

**V.4.4 Case 1.** We single out a distinguished coordinate $x_1$, with everywhere non-zero eigenvalue $\lambda_1$ which is supposed an invariant hypersurface for $\partial$. Consequently we renormalise by a constant in order to suppose $\lambda_1 = 1$, and suppose that not all the remaining eigenvalues are in $\mathbb{C}$. As such let's say, $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, and $\lambda_{k+1}, \dots, \lambda_m \in \mathcal{O}_Y$ are non-constant functions for some $1 \leq k < m$.

In this situation it will not be possible to produce a Jordan decomposition in $\mathcal{O}$ stricta dictum. However, what we will aim for is,

**V.4.5 Definition/Warning.** In this context by an $I$-adic Jordan decomposition (or should it be necessary to be more precise, a Jordan decomposition secundum quid) of $\partial$ we will mean a decomposition of the form $\partial' + \partial''$ where for some functions $\xi_i, \eta_j$ in an appropriate formal localisation $\mathcal{O}_{\{S\}}$, to be determined, in the directions normal and parallel to $Y$ respectively,

$$\partial' = \lambda_i \, \xi_i \, \frac{\partial}{\partial \xi_i} \, , \; \partial'' = a_{iQ} \, \xi_i \, \xi^Q \, \frac{\partial}{\partial \xi_i} + a_{jP} \, \xi^P \, \frac{\partial}{\partial \eta_j}$$

with $a_{iQ}, a_{jP} \in \mathcal{O}_{Y,S}$, $P \in \mathbb{Z}^m_{\geq 0}$, conventions on $Q$ as before, and in any case $\Lambda \cdot Q = \Lambda \cdot P = 0$ in $\mathcal{O}_Y$. Although similar in appearance to Jordan decomposition $\partial', \partial''$ may not commute except in the directions normal to $Y$.

Bearing in mind that things are already Jordan modulo $I^2$, let's consider in more detail what's involved in this process of Jordanisation secundum quid. Manifestly we proceed inductively and suppose that we've found appropriate coordinates $x_i, y_j$ modulo $I^p$, and look for a solution modulo $I^{p+1}$ by way of a perturbation,

$$\tilde{x}_i = x_i + c_{iQ} \, x_i \, x^Q \, , \; \tilde{y}_j = y_j + c_{jP} \, x^P$$

with $c_{iQ}, c_{jP} \in \mathcal{O}_Z$, and the $x_i \, x^Q$ (possibly $q_i = -1$), $x^P$ the obvious monomials in $x_1, \dots, x_m$ of degree $p$. Explicit computation shows that we require to solve linear equations of the form,

$$Lc = b$$

for $b \in I^p/I^{p+1} \otimes T_{\mathcal{O}}$, $c_*$ as above, and $L \in \mathrm{End}(I^p/I^{p+1} \otimes T_{\mathcal{O}})$ some linear operator. Better still the mod $I^2$ decomposition shows that if $L_S + L_N$ is the Jordan decomposition of $L$ into semi-simple and nilpotent parts, then $L_S$ has eigenvectors, $x_i \, x^Q \otimes \frac{\partial}{\partial x_r}$, $x^P \otimes \frac{\partial}{\partial y_t}$ for suitable $i, Q, r, P, t$, with eigenvectors $\Lambda \cdot Q$ and $\Lambda \cdot P$ respectively. As such the said equations are solvable after localising by $S_p$, where $S_p$ is the multiplicative set generated by the $\Lambda \cdot Q$ and $\Lambda \cdot P$ which are not identically zero in $\mathcal{O}_Y$, and which correspond to the appropriate monomials $x_i \, x^Q, x^P$.

Now the unfortunate thing is that a solution of such equations is not unique, albeit that at worst in the normal direction, say, any two solutions would differ by a power series in the $x_i \, x^Q$, $\Lambda \cdot Q = 0$. As such if we start with any two sets of normal coordinates, $\tilde{x}_i$, $x_i$, say, which agree modulo $I^2$, and proceed with the above Jordanisation algorithm to Jordan coordinates $\tilde{\xi}_i, \xi_i$ then,

$$\tilde{\xi}_i = \xi_i + \sum_Q a_{iQ} \, \xi_i \, \xi_Q \, , \; \Lambda \cdot Q = 0$$

with the $a_{iQ} \in \mathcal{O}_Y$.

The real problem, however that we must address is that if $S$ is the smallest (and, by the way, necessarily topologically non-nilpotent) multiplicative set containing all the $S_p$, then all of this is only defined in the formal localisation $\mathcal{O}_{\{S\}}$. As such it could perfectly well happen that the smallest ideal of $\mathcal{O}$ containing a Jordan coordinate $\xi_i \in \mathcal{O}_{\{S\}}$ is $\mathcal{O}$ itself. On the other hand for each such $i$, and $p \in \mathbb{N}$, there is a well defined Zariski closure, $J_{ip}$, say, in $\mathcal{O}$ of $\xi_j + I^p$ considered as an ideal of the usual localisation $\mathcal{O}_{S_p}$. In order to get to grips with what $J_{ip}$ looks like around a closed point 0 of $Y$, let us begin by further supposing that $\mathcal{O}$ is $\mathfrak{m} = \mathfrak{m}(0)$-adically complete. As such we introduce Jordan coordinates stricta dictum $x_i, y_j$, and let $\xi_i^p, \eta_j^p$ be their $I$-adic counterparts obtained by following the above procedure up to order $p$, then we further assert,

**V.4.6 Claim.** *Notations as above, then for any $i, j$, there are $s_i, s_j \in S_p$ and functions $f_i, f_j$ of the form $\sum a_{iQ}\, x_i\, x^Q$, $\Lambda \cdot Q(0) = 0$, respectively $\sum a_{jP}\, x^P$, $\Lambda \cdot P(0) = 0$, $a_{iQ}\, a_{jQ} \in \mathcal{O}_Z$, such that,*

$$\xi_i^p = x_i + f_i/s_i\,,\ \eta_j^p = y_j + f_j/s_j\,.$$

**Proof.** We, of course, proceed inductively since everything is true for $p = 1$ by virtue of our set up. Now write, for example,

$$\partial\, \xi_i^p = \partial\, x_i + \frac{\partial f_i}{s_i} - f_i\, \frac{\partial s_i}{s_i^2}$$

and consider the $I$-adic expansion after localising by $S_p$. Certainly $\partial x_i$ is $\lambda_i\, x_i$ plus a series of the form $\sum a_{iQ}\, x_i\, x^Q$, but the same is also true of $\partial f_i$, as one sees by explicit computation of expressions of the form,

$$x_j\, x^P\, \frac{\partial}{\partial x_j}\, (a_{iQ}\, x_i\, x^Q)\,,\ \text{and}\ x^R\, \frac{\partial}{\partial y}\, (a_{iQ}\, x_i\, x^Q)$$

where as ever $\Lambda \cdot P(0) = \Lambda \cdot R(0)$, and we permit the possibility of $p_j = -1$. As a result the element $b$ of $I^p/I^{p+1} \otimes T_{\mathcal{O}}$ for which we require to solve $Lc = b$ is supported on elements of the form $x_i\, x^Q \otimes \frac{\partial}{\partial x_r}$, respectively $x^P \otimes \frac{\partial}{\partial y_t}$, from which we conclude. $\square$

With this out of the way, we can quickly move to a conclusion by way of,

**V.4.7 Fact.** *Let things be as in Case 1, with no further hypothesis on $\mathcal{O}$ beyong $I$-adic completeness. Furthermore for $2 \leq i \leq m$, and $p \in \mathbb{N}$ let $J_{ip}$ be the ideal of the Zariski closure of $\xi_i + I^p$ where $\xi_1, \ldots, \xi_m$ are $I$-adic Jordan coordinates secundum quid. Observe also that there is a perfectly well defined $J_{1p} = \xi_1 + I^p$ independently for any closure considerations by virtue of the aforesaid hypothesis. Finally, and bearing in mind that $\lambda_1 = 1$, for $y \in Y$ a closed point put $M_y \subset \{1, \ldots, m\}$ to be the set of indices such that $\lambda_i \in \mathbb{Q}_{>0}$, and denote by $H(\Lambda(y))$ the big height computed in the standard $\ell_\infty$-metric of $(\lambda_i)_{i \in M_y} \in \mathbb{A}^{M_y}(\mathbb{Q})$ for $M_y \geq 1$, and $\infty$ otherwise, then for $f : \hat{\Delta} \to \operatorname{Spf} \hat{\mathcal{O}}_{Y,y}$ (completion in $\mathfrak{m}(y)$) an invariant curve,*

$$\operatorname{ord}_0(f^{-1}\, J_{1p} \cap \ldots \cap J_{mp}) \geq \min\{p, H(\Lambda(y))\}\,.$$

**Proof.** Since the Zariski closure could only decrease under $\mathfrak{m}(y)$-adic completion, we may, more or less suppose that $\mathcal{O}$ is $\mathfrak{m}$-adically complete. We say more or less since we have to keep in mind the fact that the $\xi_i$ are a priori given. However starting from an $\mathfrak{m}$-adic Jordan basis stricta dictum, $x_i, y_j$, we can Jordanise secundum quid to $\tilde{\xi}_i, \tilde{\eta}_j$, so that for $S_p, S$ etc. as before we have the relations,

$$\tilde{\xi}_i = x_i + \sum a_{iQ}\, x_i\, x^Q,\ \Lambda \cdot Q(y) = 0\,,\ \xi_i = \tilde{\xi}_i + \sum b_{iP}\, \tilde{\xi}_i\, \tilde{\xi}^P,\ \Lambda \cdot P = 0 \in \mathcal{O}_Y$$

where $a_{iQ} \in \mathcal{O}_{Y,S}$, the $b_{iP} \in \mathcal{O}_Y$, and the monomials $x_i\, x^Q, \tilde{\xi}_i\, \tilde{\xi}^P$ which appear are, without loss of generality, supposed to have degree at least 2. Consequently for some $c_{iQ} \in \mathcal{O}_{Y,S}$ we obtain,

$$\xi_i = x_i + \sum c_{iQ}\, x_i\, x^Q\,,\ \Lambda \cdot Q(y) = 0\,.$$

Consequently if $g \in J_{ip}$, $i \geq 2$, then modulo $I_Y^p$ it is of the form, $d_i\, x_i + \sum d_{iQ}\, x_i\, x^Q$, $\Lambda \cdot Q(y) = 0$, where now $d_i, d_{iQ} \in \mathcal{O}_Y$. Consequently for $e_i$ as per V.4.1, we obtain,

$$\mathrm{ord}_0(f^* g) \geq \min\{e_i, p\}$$

and the result follows. □

Fortunately the remaining cases are far more straightforward. We begin with,

**V.4.8 Case 2.** Everything as per case 1, but with all the eigenvalues constant.

From the arguments of the previous case, we see that there is no difficulty in finding $I$-adic Jordan coordinates $\xi_i, \eta_j \in \mathcal{O}$ stricta dictum, *i.e.* we have over the whole of $\mathcal{O}$ a decomposition $\partial = \partial_S + \partial_N$, where,

$$\partial_S = \lambda_i\, \xi_i\, \frac{\partial}{\partial \xi_i}\,,\ \partial_N = a_{iQ}\, \xi_i\, \xi^Q\, \frac{\partial}{\partial \xi_i} + a_{jP}\, \xi^P\, \frac{\partial}{\partial \eta_j}$$

with as ever $P \in \mathbb{Z}^m_{\geq 0}$, much the same for $Q$ excepting the possibility $q_i = -1$, and of course $\Lambda \cdot Q = \Lambda \cdot P = 0$, with $[\partial_S, \partial_N] = 0$. In particular for $y \in Y$ a closed point, on replacing the $\eta_j$ by $\tilde{\eta}_j = \eta_j - \eta_j(y)$, we get an $\mathfrak{m}(y)$-adic Jordan decomposition of $\partial$. In addition as before we have ideals $J_{ip} = \xi_i + I^P$, although, fortunately these are defined over the whole of $\mathcal{O}$, and we distinguish sub-cases/sub-facts by way of,

**V.4.9 Fact.** *Everything as above, but suppose that not all the eigenvalues are in $\mathbb{Q}_{>0}$, then for $f : \hat{\Delta} \to \mathrm{Spf}\, \hat{\mathcal{O}}_{Y,y}$ a formal invariant curve through a closed point $y \in Y$,*

$$\mathrm{ord}_y(f^{-1}\, J_{1p} \cap \ldots \cap J_{mp}) \geq p\, \mathrm{ord}_y(f^{-1}\, I)\,.$$

There remains, of course, the sub-case that all of the eigenvalues are in $\mathbb{Q}_{>0}$. This could only happen, however, if $Y$ were a generically isolated component of $\mathrm{sing}\,(\partial)$. Better still by I.6.12, $\partial_N \neq 0$ (otherwise the singularity would not be canonical) consequently if $J_p^{\mathrm{nil}}$ is the ideal generated by the coefficients of $\partial_N(\xi_i)$, $1 \leq i \leq m$, and $I_Z^p$ then $J_p^{\mathrm{nil}}$ is a proper ideal of $\mathcal{O}$ for $p \gg 0$, and by V.4.2, we have,

**V.4.10 Fact bis.** *Everything as per V.4.2(ii), but with all the eigenvalues in $\mathbb{Q}_{>0}$, then,*

$$\mathrm{ord}_y(f^{-1} J_p^{\mathrm{nil}}) \geq p\, \mathrm{ord}_y(f^{-1} I)\,.$$

The remaining case is, of course,

**V.4.11 Case 3.** We continue to single out a distinguished coordinate $x_1$ corresponding to an invariant hypersurface for $\partial$, but suppose that it is nilpotent, *i.e.* $\frac{\partial x_1}{x_1} \in I$. The remaining non-zero eigenfunctions $\lambda_2, \ldots, \lambda_k$, say, may or may not be constant.

As such we may need to face the problem of only having an $I$-adic Jordan decomposition secundum quid, not to mention the formal localisation issue. Nevertheless we're interested in a much simpler formal sub-scheme, $W$, say, cut out by $(\xi_2, \ldots, \xi_k)$ in $\mathcal{O}_{\{S\}}$ for $\xi_i$ Jordan coordinates. As ever let $J_W^p$ be the Zariski closure of $(\xi_2, \ldots, \xi_k) + I^P$, then it follows immediately by what we've seen in the discussion of case 1, that $J_W^p \subset J_{W(y)}^p$ where $y \in Y$ is a closed point and $W(y)$ is the sub-scheme of $\hat{\mathcal{O}}_{Y,y}$ ($\mathfrak{m}(y)$-adic completion) defined by the coefficients of the semi-simple part of the $\mathfrak{m}(y)$-adic jordan decomposition stricta dictum. Consequently on combining with V.4.2, we arrive to,

**V.4.12 Fact.** *Things as per case 3, with $f : \hat{\Delta} \to \mathrm{Spf}\, \hat{\mathcal{O}}_{Y,y}$ a formal invariant curve through a closed point $y \in Y$ such that $f^* x_1 \neq 0$, then,*

$$\mathrm{ord}_y(f^{-1} J_W^p) \geq p\, \mathrm{ord}_y(f^{-1} I)\,.$$

Before abandonning this topic, let us note,

**V.4.13 Reason for the above discussion.** Ultimately we'll wish to estimate the "size" of intersection of invariant curves with the singular locus, and what will lead to non-trivial estimates is the above series of facts which more or less say that such a curve must factor through a formal subscheme of dimension strictly smaller than the ambient dimension.

## V.5 Refined Tautology

Our set up is as follows: $(\mathcal{X}, \mathcal{F}, \mathcal{B})$ is a foliated smooth stack with simple normal crossing non-invariant boundary $\mathcal{B}$, whose singularities are supposed canonical rather than just log-canonical. We will also suppose the existence of an invariant simple normal crossing divisor $\mathcal{E}$ which contains every component of sing $(\mathcal{F})$. Notice that this latter hypothesis is very much without loss of generality, *i.e.* given $(\mathcal{X}, \mathcal{F}, \mathcal{B})$ satisfying the initial hypothesis then by I.6, $\mathcal{B}$ is disjoint from sing $(\mathcal{F})$. Furthermore by algorithmic resolution we can, by way of a sequence of blow ups in smooth invariant centres, find a modification $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \mapsto (\mathcal{X}, \mathcal{F})$ such that $\rho^{-1}(\operatorname{sing}\mathcal{F})$ is an invariant simple normal crossing divisor, with $K_{\tilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$, and of course $\rho^{-1}(\operatorname{sing}\mathcal{F}) \supset \operatorname{sing}(\tilde{\mathcal{F}})$. As a result $\mathcal{B} + \mathcal{E}$ is a simple normal crossing divisor, and we have a short exact sequence of the form,

$$0 \to \mathcal{N} \to \Omega_{\mathcal{X}}(\log \mathcal{B} + \mathcal{E}) \to K_{\mathcal{F}}(\mathcal{B}) \cdot I_{\mathcal{Z}} \to O$$

where $\mathcal{Z}$ is some closed invariant sub-stack contained in sing$(\mathcal{F})$, and $\mathcal{N}$ a reflexive sheaf. Consequently if $f : \mathcal{C} \to \mathcal{X}$ is an invariant map from a proper covering stack $p : \mathcal{C} \to \Delta$, not factoring through $\mathcal{B}$ or $\mathcal{E}$ then we may apply V.3.2 to conclude,

**V.5.1 Fact.** *Suppose everything as above, then:*

$$\begin{aligned} \log \frac{\|f'\|_{\log}}{|p'|}(0) + \fint_{\mathcal{C}(r)} c_1(K_{\mathcal{F}} + \mathcal{B}) \quad &\leq \quad (\mathcal{C} : \Delta) \log \ell_{\log}(r) + \operatorname{discr}^{\#}_{\mathcal{C}/\Delta}(r) + \operatorname{rad}_{f, \mathcal{B}+\mathcal{E}}(r) \\ &+ \quad s_{f,\mathcal{Z}}(r) + 0\left(1, \fint_{\mathcal{C}(r)} c_1(H),\, \log\log^+ f^* \|\mathbb{1}_{\mathcal{B}+\mathcal{E}}\|(0)\right) \end{aligned}$$

*where we compute everything with respect to smooth metrics, and the new term,* i.e. *the Segre class of $f$ around $\mathcal{Z}$ is the Nevanlinna counterpart of* I.8.6*, or more precisely let $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow up in $\mathcal{Z}$, $\mathcal{E}_{\mathcal{Z}}$ the exceptional divisor, and $\tilde{f}$ the lifting of $f$, then $s_{f,\mathcal{Z}}(r) = \fint \tilde{f}^* c_1(\mathcal{E}_{\mathcal{Z}})$.*

Now the things that we want to try and get rid of in the right hand side are $\operatorname{rad}_{f,\mathcal{E}}(r)$, and $s_{f,\mathcal{Z}}(r)$. There is of course an immediate improvement we can make, viz:

**V.5.2 Further Fact.** *Hypothesis as before then,*

$$\operatorname{rad}_{f,\mathcal{E}}(r) = \operatorname{rad}_{f,\operatorname{sing}(\mathcal{F})}(r)$$

*where the radical of $f$ around* sing$(\mathcal{F})$ *is computed in the obvious way by way of blowing up* sing$(\mathcal{F})$*.*

**Proof.** $\mathcal{E}$ is invariant by $\mathcal{F}$.

Naturally we're going to try and make use of the previous sections, which leads us to pose:

**V.5.3 Sub-Problem** (of a purely scheme like nature). *Let $X$ be a projective variety, $H$ ample on $X$, $Y$ a subvariety of $X$ and $\mathfrak{V}$ a formal subscheme of the formally formal (*cf. [M4]*) generic point of the completion of $X$ in $Y$. As such for each $p \in \mathbb{N}$, there is a Zariski neighbourhood $U_p$ of $Y$, and subscheme $V_p$ of $\mathcal{O}_{U_p/I_Y^p}$, or equivalently an ideal $I_V$ of $\varprojlim_p \frac{\mathcal{O}_{X,Y}}{I_Y^p}$. In any case for $s,t \in \mathbb{N}$, $t \geq s$, we have ideals $I_V^s + I_Y^t$ with well defined Zariski closure, say, $I_{V,Y}^{(s,t)}$ in $\mathcal{O}_X$, and we wish to estimate, for $r \in \mathbb{N}$,*

$$h^0(X, H^r \cdot I_{V,Y}^{(s,t)}) \,.$$

For the purposes of this calculation, and indeed elsewhere, we'll use the symbol $\ll$ to denote less than or equal up to a linear sum in irrelevant constants. It will also be convenient, in the beginning, to fix $r, s$ and let $t$ be $s+n$, $n \in \mathbb{N}$. In any case for $k \in \mathbb{N} \cup \{0\}$, we certainly have a short exact sequence,

$$0 \to I_{V,Y}^{(s,s+k+1)} \to I_{V,Y}^{(s,s+k)} \to Q_{V,Y}^{(s,k)} \to 0$$

where by definition $Q^{(s,k)}_{V,Y}$ is nothing other than the quotient, with a more precise investigation of the same being only momentarily postponed. This apart, the standard exact sequence for $H^0$ certainly gives,

$$h^0(X, H^r \cdot I^{(s,s+n)}_{V,Y}) \geq h^0(X, H^r \cdot I^s_Y) - \sum_{k=0}^{n-1} h^0(X, H^r \otimes Q^{(s,k)}_{V,Y})$$

while a similar argument profiting from the short exact sequence,

$$0 \to I^{k+1}_Y \to I^k_Y \to I^t_Y / I^{k+1}_Y \to 0$$

eventually yields,

$$h^0(X, H^r I^{(s,s+n)}_{V,Y}) \geq h^0(X, H^r) - \sum_{k=0}^{s-1} h^0\left(X, H^r \cdot \frac{I^{k-1}_Y}{I^k_Y}\right) - \sum_{k=0}^{n-1} h^0(X, H^r \otimes Q^{(s,k)}_{V,Y})\,.$$

In order to proceed to a conclusion, we cut $Y$ by $\dim Y$ very general hyperplanes under the further hypothesis that $H$ is very ample. This gives exact sequences of the form,

$$0 \to H^i \otimes Q^{(s,k)}_V \to H^{i+1} \otimes Q^{(s,k)}_{V,Y} \to H^{i+1} \otimes Q^{(s,k)}_{V\cap H, Y\cap H} \to 0$$

$$0 \to H^i \otimes I^k_Y / I^{k+1}_Y \to H^{i+1} \otimes I^k_Y / I^{k+1}_Y \to H^{i+1} \otimes I^k_{Y\cap H} / I^{k+1}_{Y\cap H} \to 0$$

etc., so that ultimately it suffices to understand the case where $Y$ is zero dimensional, or more precisely a finite number of reduced points. Certainly, however, we have estimates of the form,

$$\begin{aligned} h^0(X, H^i \otimes Q^{(s,k)}_{V\cap H_1 \cap \ldots \cap H_{\dim Y}, Y\cap H_1\cap\ldots\cap H_{\dim Y}}) &= h^0(X, Q^{(s,k)}_{V\cap H_1 \cap \ldots \cap H_{\dim Y}, Y\cap H_1\cap\ldots\cap H_{\dim Y}}) \\ &\ll s^{\dim X - \dim V}\,(s+k)^{\dim V - \dim Y - 1} \end{aligned}$$

$$h^0(X, H^i \otimes \frac{I^k_{Y\cap H_1\cap\ldots\cap H_{\dim Y}, H_1\cap\ldots\cap H_{\dim Y}}}{I^{k+1}_{Y\cap H_1\cap\ldots\cap H_{\dim Y}, H_1\cap\ldots\cap H_{\dim Y}}}) = h^0(X, \frac{I^k_{Y\cap H_1\cap\ldots\cap H_{\dim Y}, H_1\cap\ldots\cap H_{\dim Y}}}{I^{k+1}_{Y\cap H_1\cap\ldots\cap H_{\dim Y}, H_1\cap\ldots\cap H_{\dim Y}}}) \ll k^{\dim X - \dim Y - 1}.$$

Notice in addition that we've been rather course with much of our estimation. In particular we don't actually need to bound above, $h^0(X, H^i \otimes Q^{(s,k)}_{V,Y})$, or for that matter the equivalent object after cutting by a hyperplane, but only things such as,

$$\dim\left\{\mathrm{Im}(H^0(X, H^i \otimes I^{(s,k)}_{V,Y}) \to H^0(X, H^i \otimes Q^{(s,k)}_{V,Y}))\right\}$$

or more generally, $\dim\left\{\mathrm{Im}(H^0(X, H^i \otimes I^{(s,k)}_{V\cap\Lambda, Y\cap\Lambda}) \to H^0(X, H^i \otimes Q^{(s,k)}_{V\cap\Lambda, Y\cap\Lambda}))\right\}$ for $\Lambda$ an intersection of very general hyperplanes. As such when applying the standard exact sequence and truncating, we can always pretend that if $\dim Y\cap\Lambda > 0$, $h^0(X, Q^{(s,k)}_{V\cap\Lambda, Y\cap\Lambda})$ " $= 0$", where inverted comma means: in so far as it's important to our calculation. Consequently, we eventually obtain,

$$h^0(X, H^r \cdot I^{(s,t)}_{V,Y}) - h^0(X, H^r) \gg -r^{\dim Y}(s^{\dim X - \dim Y} + s^{\dim X - \dim V} t^{\dim V - \dim Y})\,.$$

Now let us apply these observations to the computation of the radical. Specifically,

**V.5.4 Fact.** *Let $\varepsilon > 0$, then there is a proper closed substack $\mathcal{Z}_\varepsilon$ of $\mathcal{X}$ such that for $f : \mathcal{C} \to \mathcal{X}$ an invariant map from a covering $p : \mathcal{C} \to \Delta$ of the disc which doesn't factor through $\mathcal{E}$,*

$$\mathrm{rad}_{f,\mathcal{E}}(r) \leq \varepsilon \fint_{\mathcal{C}(r)} f^* c_1(H) + \mathcal{O}_\varepsilon(\log \mathrm{dist}\,(f(0), \mathcal{Z}_\varepsilon))$$

*where as ever $H$ is an ample bundle, on the, supposed projective, moduli $X$.*

**Proof.** Fix some rational $\delta > 0$ to be chosen a posteriori. In addition stratify $\mathrm{sing}\,(\mathcal{F})$ by finitely many substacks $\mathcal{Y}_i$ and projections $\psi_i : \mathcal{Y}_i \to \mathbb{P}^{n_i}$ in such a way that every closed geometric point $y \in \mathrm{sing}\,(\mathcal{F})$ is contained in the open subset of some $\mathcal{Y}_i$ where,

(a) $\mathcal{Y}_i$ is smooth, and indeed $\psi_i$ is étale.

(b) The roots of the characteristic polynomial of the linearisation of $\mathcal{F}$ around $\mathcal{Y}_i$ (*i.e.* the map $\Omega_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{Y}_i} \to \Omega_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{Y}_i}(K_{\mathcal{F}})$, defined pre V.4.4) define an étale covering.

(c) The Jordan decomposition of the map of (b) on the said étale neighbourhood is constant, or more correctly the dimension of the eigenspaces of the semi-simple part and the nilpotent part are constant.

Furthermore introduce a subindex set, $I'$, say, of the set of such $\mathcal{Y}_i$'s, according to the rule, $i \in I'$ iff the eigenvalues (equivalently the characteristic polynomial) are non-constant, considered as elements of the appropriate projective space. We also let $N$ denote a large number to be chosen, and consider the substacks of some appropriate étale neighbourhood of $\mathcal{Y}_i$ defined by,

$$\lambda_2 = e_2, \ldots, \lambda_M = e_m$$

provided there is some component $\mathcal{E}_1$ of $\mathcal{E}$ containing $\mathcal{Y}_i$, such that for $x_1$ a local equation of the same, and $\partial$ a local generator of $\mathcal{F}$, $\frac{\partial x_1}{x_1}$ is non-zero, so that $\lambda_2, \ldots, \lambda_M$ are the non-constant eigenvalues normalised in such a way that $\partial x_1 = x_1$ and $e_2, \ldots, e_M \in \mathbb{A}^M(\mathbb{Q}_{>0})$ is a positive rational point of big height at most $N/\delta$. This introduces around $(N/\delta)^{M+1}$ additional stacks $\mathcal{Y}_{ij}$ which we further stratify to ensure (a), (b), (c) on each of them. We may also assume, and for convenience we shall, that we have such a stratification $\mathcal{Y}_i$, $i \in I'$, of $\operatorname{sing}(\mathcal{F})\backslash\mathcal{Z}$, $I' \subset I$ as above, and $\mathcal{Y}_k$, $k \in K$, of $\mathcal{Z}$ in turn, with $\mathcal{Z}$ as defined prior to V.5.1.

We now introduce appropriate formal subschemes around the generic points of the various $\mathcal{Y}_i$'s, $\mathcal{Y}_{ij}$'s, $\mathcal{Y}_k$'s etc. as dictated by the analysis of V.4. In particular,

**V.5.5 Case 1.** For $i \in I'$, take $\mathcal{V}_{i\alpha}$ to be the formal substack of the completion in $\mathcal{Y}_i$ at its generic point defined by a Jordan coordinate, in the sense of V.4.5, $\xi_\alpha$ in the normal direction to $\mathcal{Y}_i$.

**V.5.6 Case 2.** For $i \in I\backslash I'$, or $i \in I'$ and $\mathcal{Y}_{ij}$, $j \in J_i$, say, arising from the further stratification, again take $\mathcal{V}_{i\alpha}$ respectively $U_{ij\alpha}$ to be the formal substack defined by a Jordan coordinate unless all the eigenvalues are generically positive rational, where we take an appropriate component of $\mathcal{E}$ as the means to obtain a distinguished eigenfunction $x_1$.

**V.5.7 Case 2 bis.** If in fact all the eigenvalues are rational, with things as per Case 2, take $\mathcal{V}_i$, respectively $\mathcal{V}_{ij}$, to be cut out by the coefficients of the nilpotent part of the Jordanisation.

**V.5.8 Case 3.** $k \in K$, so in fact $\mathcal{Y}_k \subset \mathcal{Z}$, then take $\mathcal{W}_k$ to be the formal substack of the completion in $\mathcal{Y}_k$ defined generically by the vanishing of the coefficients of the semi-simple part of the Jordanisation.

With these various choices, and extending our notations from the scheme like discussion of V.4, in the obvious way, we consider sections of,

$$H^0\left(\mathcal{X}, \pi^* H^p \otimes \bigcap_{\substack{i\alpha \\ i\in I'}} I^{(q,r)}_{\mathcal{U}_{i\alpha},\mathcal{Y}_i} \bigcap_{\substack{i\alpha \\ i\notin I'}} I^{(1,s)}_{\mathcal{V}_{i\alpha},\mathcal{Y}_i} \bigcap_{\substack{ij\alpha \\ i\in I'}} I^{(1,t)}_{\mathcal{V}_{ij\alpha},\mathcal{Y}_{ij}} \bigcap_{\substack{k \\ k\in K}} I^{(1,u)}_{\mathcal{W}_k,\mathcal{Y}_k}\right)$$

whose dimension we'll denote $h(p,r,s,t,u)$. As such our previous dimension count, or more precisely codimension count, yields,

$$\begin{aligned} h(p,r,s,t,u) - h^0(X, H^p) \quad &\gg \quad -\sum_{i\in I'} p^{\dim\mathcal{Y}_i}(q^{\dim\mathcal{X}-\dim\mathcal{Y}_i} + r^{\dim\mathcal{X}-\dim\mathcal{Y}_i-1}) \\ &\quad -\sum_{i\in I\backslash I'} p^{\dim\mathcal{Y}_i}\, s^{\dim\mathcal{X}-\dim\mathcal{Y}_i-1} - \sum_{k\in K} p^{\dim\mathcal{Y}_k}\, u^{\dim\mathcal{W}_k-\dim\mathcal{Y}_k} \\ &\quad -\sum_{i\in I'}\sum_{j\in J_i} p^{\dim\mathcal{Y}_{ij}}\, t^{\dim\mathcal{X}-\dim\mathcal{Y}_{ij}-1}. \end{aligned}$$

We take, in consequence, $q = \delta p$, for $p$ sufficiently divisible, so that necessarily we take $r = M_p$, for $M$ about, $\delta^{-\frac{1}{\dim \mathcal{X} - \dim \mathcal{Y}_{i-1}}}$. Equally we can safely take $s, u$ to be of the order $p^{1+\eta}$, for $\eta > 0$, depending only on $\dim \mathcal{X}$, and we can even do the same for $t$ provided $N, \delta$ are fixed, say of orders about $1/\varepsilon$ and $\varepsilon^{\dim \mathcal{X}}$ respectively.

In particular with these kind of choices, $h(p, r, s, t, u) > 0$ (in fact even of order $p^{\dim \mathcal{X}}$) while for $\gamma$ a global section of the corresponding $H^0$, and $c \in \mathcal{C}$ a closed point,

$$\operatorname{ord}_c(f^*\gamma) \geq p/\varepsilon \min\{\operatorname{ord}_c(f^{-1}\operatorname{sing}\mathcal{F}), 1\}$$

and whence the assertion by way of the ubiquitous Jensen formula. $\square$

Notice also that in the particular example of case 3, the above discussion yields much more. Indeed for $(\mathcal{X}_i, \mathcal{F}_i)$ a foliated stack, with $\mathcal{E}_i, \mathcal{B}_i$ as before, let us call $\mathcal{Z}_i$ of an exact sequence such as that prior to V.5.1, the locus where things are not log-flat. Now blow up in $\mathcal{Z}_i$, and call the resulting stack $\tilde{\mathcal{X}}_i$ with $\mathcal{X}_{i+1}$ a resolution of the ideal $\mathcal{I}_{Z_i}$ by a sequence of blow ups in smooth invariant centres, which necessarily exists by algorithmic resolution, while $\mathcal{E}_{i+1}$ will be the reduced pull-back of $\mathcal{E}_i$, and $\mathcal{B}_{i+1}$ the proper transform of $\mathcal{B}_i$. Necessarily if we take $\mathcal{X} = \mathcal{X}_0$, then we obtain a tower of stacks of the form

$$\begin{tikzcd}
 & & & {} \arrow[dl, "\rho_{23}"'] \\
 & & (\mathcal{X}_2, \mathcal{F}_2) \arrow[dl, "\rho_{12}"'] \arrow[d, "\tau_{21}"] & {} \arrow[l, "\sigma_2"] \\
 & (\mathcal{X}_1, \mathcal{F}_1) \arrow[dl, "\rho_{01}"'] \arrow[d, "\tau_{10}"] & (\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{F}}_1) \arrow[l, "\sigma_1"] & \\
(\mathcal{X}, \mathcal{F}) = (\mathcal{X}_0, \mathcal{F}_0) & (\widetilde{\mathcal{X}}_0, \widetilde{\mathcal{F}}_0) \arrow[l, "\sigma_0"] & &
\end{tikzcd}$$

Now the remaining term in that we have not so far got into the shape that we desire (*i.e.* small and/or intrinsic to $\mathcal{C}$) is the $s_{f,\mathcal{Z}}(r)$ term, and even here the nature of case 3 is such that we already know,

**V.5.9 Intermediary Fact.** *Let $\varepsilon > 0$ be given, then there is a proper closed substack $\mathcal{W}_\varepsilon \supset \mathcal{Z}$ such that,*

$$s_{f,\mathcal{Z}}(r) \leq \int_{\partial \mathcal{C}(r)} \log \frac{\|f^* \mathbb{1}_{\mathcal{Z}}\|}{\|f^* \mathbb{1}_{\mathcal{Z}}\|(0)}\, p^* \left(\frac{d\theta}{2\pi}\right) + \varepsilon \fint_{\mathcal{C}(r)} f^* c_1(H) + \mathcal{O}_\varepsilon(\log \|f^* \mathbb{1}_{\mathcal{W}_\varepsilon}\|(0))$$

*where for $\mathcal{Y} \subset \mathcal{X}$ a closed substack, $\mathbb{1}_{\mathcal{Y}}$ is the tautological section of the exceptional divisor on the blow up in $\mathcal{Y}$, and we'll use the standard notation $m_{f,\mathcal{Z}}(r)$ for the 1st integral on the right,* i.e. *the so called proximity function.*

**Proof.** Indeed this is an immediate and minor variation of V.5.4, where, evidently in the notations of the proof of the same, we look at sections of,

$$H^0\left(\mathcal{X}, \pi^* H^p \bigcap_{k \in K} \mathcal{I}^{(1,t)}_{\mathcal{W}_k, \mathcal{Y}_k}\right)$$

which is certainly non-zero for $t$ of order $p^{1+\eta}$, $\eta > 0$ depending only on $\dim \mathcal{X}$.

In order to remove the last offending term, *i.e.* the proximity around $\mathcal{Z}$, we could, of course, repeat the arguments of V.4 at "infinity". On the other hand this involves just a little bit more analysis than we're at home to, so instead let us observe that for $\mathcal{X}' \to \mathcal{X}$ a blow up in a smooth centre contained in $\mathcal{Z}$, and $\mathcal{Z}'$ the new locus where things are not log-flat the proper transforms $\mathcal{W}'_k$, $k \in K$, or more correctly their union, contain at every closed point $z' \in \mathcal{Z}'$ the formal subscheme of $\mathcal{X}'$ completed in $z'$ cut out by the

Jordan coordinates corresponding to non-zero eigenfunctions of the semi-simple part of a local generator of the foliation. As such given a section of,

$$s \in H^0\left(\mathcal{X}, \pi^* H^p \bigcap_{k \in K} \mathcal{I}^{(1,t)}_{\mathcal{W}_k, \mathcal{Y}_k}\right)$$

with as ever $t$ around $p^{1+\eta}$, and $p \gg 0$ to be chosen, there is some modification $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$, with $\tilde{\mathcal{X}}$ one of the $\mathcal{X}_N$'s prior to V.5.9 (with $N$ probably about $t$, but this is unimportant) such that,

$$\mathrm{ord}_{\tilde{\mathcal{Z}}}(\rho^* s) \gg t$$

with as ever $\tilde{\mathcal{Z}}$ the new non-log flat locus. Consequently if we let $\tilde{\mathcal{C}}_{\log}(r)$ be the length of some lifting $\tilde{f}$ of $r$, then finally we arrive to our goal,

**V.5.10 Main Fact.** *Let everything be as above (in particular, and critically, $(\mathcal{X}, \mathcal{B}, \mathcal{F})$ has not just log-canonical, but canonical singularities) then for $\varepsilon > 0$ there is a closed proper substack $\mathcal{Z}_\varepsilon$ of $\mathcal{X}$ and an invariant modification $\rho : \tilde{\mathcal{X}}_\varepsilon \to \mathcal{X}$ (with in fact $\tilde{\mathcal{X}}_\varepsilon$ some $\mathcal{X}_N$ as above) such that for $f : \mathcal{C} \to \mathcal{X}$ any invariant map from a covering $p : \mathcal{C} \to \Delta$ of the disc, the following isoperimetric type inequality holds,*

$$\begin{aligned}\log\left\|\frac{\tilde{f}'}{p'_{\log}}\right\|(0) + \fint_{\mathcal{C}(r)} f^* c_1(K_{\mathcal{F}} + \mathcal{B}) \quad &\leq \quad (\mathcal{C} : \Delta)\log \tilde{\mathcal{C}}_{\log}(r) + \mathrm{discr}^{\#}_{\mathcal{C}/\Delta}(r) + \mathrm{rad}_{f,\mathcal{B}}(r) \\ &+ \quad \varepsilon \fint_{\mathcal{C}(r)} f^* c_1(H) + O_\varepsilon(1, \log \mathrm{dist}(f(0), \mathcal{Z}_\varepsilon)) \,.\end{aligned}$$

## V.6 Schematic Interpretation

Let us close this section by translating the main fact V.5.10 into schematic language under the habitual hypothesis that $(\mathcal{X}, \mathcal{B}, \tilde{\mathcal{F}})$ is terminal at the non-scheme like points. Naturally we note the moduli as $(X, \mathcal{B}, \mathcal{F})$, and, of course, we have our old friend the Gorenstien cover $\pi : (\mathcal{X}', \mathcal{B}', \mathcal{F}') \to (X, \mathcal{B}, \mathcal{F})$. We are now a priori interested in invariant maps $f : C \to X$, where $p : C \to \Delta$ is an everywhere space like proper covering of the disc. We have of course,

$$K_{\tilde{\mathcal{F}}} + \mathcal{B} = \pi^*(K_{\mathcal{F}} + \mathcal{B}') = \pi^*(K_{\mathcal{F}} + B)$$

where the latter is interpreted as the pull-back of a $\mathbb{Q}$-Cartier divisor in the usual way. Furthermore since $f$ is invariant, and everything terminal at the non-scheme like points, the fibre products $\mathcal{C}' = C \times_X \mathcal{X}'$, $\mathcal{C} = C \times_X \mathcal{X}$ are irreducible with moduli $C$ so replacing the same by their normalisation, albeit without changing notation, we see that for $c$ a geometric point be it of $C$, $\mathcal{C}'$ or $\mathcal{C}$ we have inequalities such as,

$$\mathrm{ord}_{\mathcal{C},c'}(f^*\mathcal{B}') \leq \frac{1}{n_B(f(c))}\,\mathrm{ord}_{C,c}(f^*B)$$

where the former is understood as the order function calculated in $\mathcal{C}'$, *i.e.* divide by the stabiliser, the latter is the standard order function extended to $\mathbb{Q}$-Cartier divisors, and $n_B(x)$ is the smallest integer such that $n_B \cdot B$ is Cartier at $x$. Consequently if we define (supposing for convenience $f(0) \notin B$),

$$\mathrm{rad}_{f,B}(r) := \sum_{0<|p(r)|<r} \min\left\{\frac{1}{n_B(f(c))}, \mathrm{ord}_c(f^*B)\right\}\log\left|\frac{r}{z}\right|$$

then in fact, $\mathrm{rad}_{f,B}(r) \geq \mathrm{rad}_{f,\mathcal{B}'}(r)$. It may happen, however, that $\mathrm{discr}^{\#}_{\mathcal{C}'/\Delta}(r)$ is bigger than $\mathrm{discr}^{\#}_{C/\Delta}(r)$, not to mention a similar problem at infinity. However if $\mathcal{Y}$ in $\mathcal{X}'$ is the non-scheme like locus, and $\mathcal{W}$

the formal substack of the completion of $\mathcal{X}'$ in $\mathcal{Y}$ obtained by adding in the formal curves in the foliation direction then we may easily estimate this difference by way of sections of

$$H^0(\mathcal{X}', \pi^* H^p \cdot (\mathcal{I}_{\mathcal{W}} + I^t_{\mathcal{Y}}))$$

with as ever $t$ around $p^{1+\eta}$, $\eta > 0$. From which we arrive to,

**V.6.1 Variant.** *Things as above, and let $\varepsilon > 0$, then there is a proper closed subvariety $Z_\varepsilon$ of $X$, and a modification $\rho : \tilde{X}_\varepsilon \to X$ (manifestly the moduli of that in* V.5.10*) such that,*

$$\begin{aligned} \log \left\| \frac{f'}{p'} \right\|_{\log} (0) + \fint_{C(r)} f^* c_1(K_{\mathcal{F}} + B) \quad &\leq \quad \log \mathcal{C}_{\log}(r) + \mathrm{discr}^{\#}_{C/\Delta}(r) + \mathrm{rad}_{f,B}(r) \\ &\quad + \varepsilon \fint_{C(r)} f^* c_1(H) + O_\varepsilon(1, \log \mathrm{dist}(f(0)\, Z_\varepsilon)) \,. \end{aligned}$$

# VI. Bubbling

## VI.1. Uniqueness

Historically, there is an alternative use of the word minimal in bi-rational geometry to the effect of something like, $X$ is minimal provided every map $\tilde{X} \dashrightarrow X$ in its birational equivalence class is everywhere defined, *cf.* [W]. Evidently such a variety really would be minimal, and indeed the only candidate for such a model is, in current terminology, a canonical model. For foliations by curves, however, something quite close to the extension property continues to hold even if the model is no better than minimal. To begin with we'll need a couple of clarifying remarks, and even a little terminology so as to speed things up, *e.g.*

**VI.1.1 Terminology.** Call the singularities of a foliated log-variety $(X, B, \mathcal{F})$ with finite weights of minimalist (as opposed to minimal) Mori category if they are canonical $\mathbb{Q}$-foliated Gorenstien, no boundary component is invariant, terminal at every point of $\operatorname{sing}(X) \cup |B|$, and of course $X$ is normal.

Of course by virtue of I.7.4, this just amounts to the smoothness of the Gorenstien covering stack $\pi : (\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$, as well as that of the branched covering $\beta : (\mathcal{X}, \tilde{\mathcal{F}}) \to (\mathcal{X}', \mathcal{F})$ defined by $B$, in conjunction with the habitual properties of terminal at the non-scheme like points and both $(\mathcal{X}, \tilde{\mathcal{F}})$, $(\mathcal{X}', \mathcal{F})$ enjoying canonical singularities. The reasons for introducing the new terminology are two fold. In the first place, we'd like to emphasise that we have an acceptable class of singularities in which we can run the minimal model programme, and whence, although we certainly could, there's no real practical benefit to be achieved from working in greater generality. More importantly, however, there are technical issues involved in defining rational maps between stacks that we'd rather avoid, as such we make,

**VI.1.2 Definition.** A rational map $(X, B, \mathcal{F}) \dashrightarrow (Y, D, \mathcal{G})$ between foliated log-varieties of minimalist Mori category is a rational map, $\psi : X \dashrightarrow Y$ such that,

(a) The image of the natural composition,

$$\psi^* N_{Y/\mathcal{G}} \to \psi^* \Omega_Y \dashrightarrow \Omega_X \to K_{\mathcal{F}}$$

is zero, where, of course, $N_{Y/\mathcal{G}} \hookrightarrow \Omega_Y$ is the co-normal sheaf of $\mathcal{G}$.

(b) For every prime Weil divisor $F$ on $X$ such that $\psi_* F \subset |D|$ either, $F$ is invariant or its support is contained in that of $B$. In the latter case we further suppose that $\psi_* F$ is in fact a Weil divisor on $Y$, and the weight $b$ of $B$ on $F$ is related to the same, $d$ of $D$ on $\psi_* F$, by $d \mid b\nu$ where $\nu$ is the multiplicity of $\psi^* \psi_* F$ around $F$.

A modification $\rho : (X^{\#}, B^{\#}, \mathcal{F}^{\#}) \to (X, B, \mathcal{F})$ is of course an honest map with a rational inverse, which we a priori allow to have more singularities. Somewhat less trivially it is invariant if the log-canonical bundles coincide. With this out of the way we can now state,

**VI.1.3 Lemma.** *A projective log-canonical variety $(Y, D, \mathcal{G})$ of minimalist Mori category is a minimal model iff for every dominant rational map $\psi : (X, B, \mathcal{F}) \dashrightarrow (Y, D, \mathcal{G})$ from a minimalist Mori category triple there is an invariant resolution,* i.e. *a diagram,*

$$\begin{tikzcd}
(X^{\#}, B^{\#}, \mathcal{F}^{\#}) \arrow[d, "\rho"] \arrow[dr, "\tilde{\psi}"] & \\
(X, B, \mathcal{F}) \arrow[r, dashed, "\psi"'] & (Y, D, \mathcal{G})
\end{tikzcd}$$

*with $\rho$ a proper invariant modification.*

Notice that in the course of running the minimal model programme we've already proved that the condition is necessary. Specifically if $(Y, D, \mathcal{G})$ weren't minimal, then we could take $\psi$ to be an anti-contraction or an anti-flip to obtain a contradiction. In the opposite direction we, somewhat inevitably, re-introduce our friends the Gorenstien covering stack $\pi : (\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$, and the branched covering $\beta : (\mathcal{X}, \tilde{\mathcal{F}}) \to (\mathcal{X}', \mathcal{F})$. Thanks to algorithmic resolution we can resolve the composite $\psi \circ \pi \circ \beta$ by a sequence, $\rho_{ij} : (\mathcal{X}_j, \tilde{\mathcal{F}}_j) \to (\mathcal{X}_i, \tilde{\mathcal{F}}_i)$ of blow ups in smooth centres $\mathcal{Z}_i$ say defining $\rho_{i+1,i}$ in such a way that the restriction of our rational map to $(\mathcal{X}_i, \tilde{\mathcal{F}}_i)$ is undefined at $\mathcal{Z}_i$. Now if every $\mathcal{Z}_i$ were invariant we'd be done, so let $\mathcal{Z}_n$ denote the last centre in our chain, if it exists, which is generically transverse to the induced foliation $\tilde{\mathcal{F}}_n$. By direct calculation we observe that the exceptional divisor $\mathcal{E}_{n+1}$ over $\mathcal{Z}_n$ is generically a bundle of weighted projective stacks. Better still $\mathcal{E}_{n+1}$ is invariant, and the induced foliation on the same is in fact a bundle of linear foliations on the said weighted projective stacks. Furthermore, by hypothesis, all subsequent blow ups are invariant so if $\mathcal{E}$ is the proper transform of $\mathcal{E}_{n+1}$ in our ultimate modification, $(\mathcal{X}_N, \tilde{\mathcal{F}}_N)$, which we'll denote $(\mathcal{X}^{\#}, \tilde{\mathcal{F}}^{\#})$, with $E$, $(X^{\#}, B^{\#}, \mathcal{F}^{\#})$ etc. the moduli, all of these salient features continue to be preserved, so, for example, $E$ is covered by invariant rational curves in such a way that if $C_t$ denotes a generic such, then $K_{\tilde{\mathcal{F}}^{\#}} \cdot C_t < 0$. Observe in addition that even though $(X^{\#}, B^{\#}, \mathcal{F}^{\#})$ may not be in our minimalist Mori category, every divisor contracted by $\rho$ is invariant (by an obvious variation of the above discussion) so unambiguously $B^{\#}$ is the proper transform of $B$ with the same weights $b_i$ associated to the components $B_i^{\#}$ and, of course, $K_{\tilde{\mathcal{F}}^{\#}} = K_{\mathcal{F}^{\#}} + \sum_i (1 - 1/b_i)\, B_i$.

Manifestly, our next task is to relate $K_{\tilde{\mathcal{F}}^{\#}}$ to $K_{\tilde{\mathcal{G}}} := K_{\mathcal{G}} + \sum_i (1 - 1/d_i)\, D_i$, where $d_i$ are the weights of our boundary on $Y$. To this end, we assert,

**VI.1.4 Claim.** *For some effective divisor $F$ on $X^{\#}$, $K_{\tilde{\mathcal{F}}^{\#}} = \tilde{\psi}^* K_{\tilde{\mathcal{G}}} + F$.*

**Proof.** To begin with, note that VI.1.2(a) implies that we have a map,

$$\tilde{\psi}^*(K_{\mathcal{G}} \cdot \mathcal{I}_{\mathrm{sing}(\tilde{\mathcal{G}})}) \to K_{\mathcal{F}^{\#}} \cdot \mathcal{I}_{\mathrm{sing}(\mathcal{F}^{\#})}$$

or more correctly the same but for some sheaves $\mathcal{K}_{\mathcal{G}}$, $\mathcal{K}_{\mathcal{F}^{\#}}$ whose double duals are the respective canonical classes. Regardless, the map is certainly non-trivial since $\psi$ is dominant, so at worst we find effective $\mathbb{Q}$-Cartier divisors $F_+, F_-$ such that, $K_{\tilde{\mathcal{F}}^{\#}} = \tilde{\psi}^* K_{\tilde{\mathcal{G}}} + F_+ - F_-$. Now quite generally, *cf.* [K-3], every irreducible divisor $F$ on $X^{\#}$ either dominates $Y$ or defines a discrete prime rank 1-valuation $v$, say, which can be resolved in the standard way when $Y$ is smooth, *i.e.* $V$ has a centre $W_0$ on $Y$, blow up $Y$ in $W_0$, call the new centre $W_1$ on $\mathrm{Bl}_{W_0}(Y) := Y_1$, and continue until we obtain a divisor, say $F_0$, all the while taking only an interest in the part of $Y$ around the smooth locus of $W_i$, whence we obtain some normal modification $\tilde{Y} \to Y$ together with a map of germs,

$$\tilde{\psi}^* : \mathcal{O}_{\tilde{Y}, F_0} \to \mathcal{O}_{X^{\#}, F}\,.$$

Evidently, in this situation, the canonical class around $F$ is the canonical class around $F_0$ plus some effective divisor, and since the singularities of $(Y, D, \mathcal{F})$ are supposed canonical, we obtain,

**VI.1.5 Sub-Claim.** *Notations as above, if $F_-$ exists and $F$ is an irreduclible component of its support then $F$ defines a rank* 1 *discrete valuation with centre contained in* $\mathrm{sing}\,(Y)$.

Indeed, outside of $\mathrm{sing}\,(\mathcal{G}) \cup \mathrm{sing}\,(Y)$ we have an honest map between canonical classes, so certainly $K_{\mathcal{F}^{\#}} = \tilde{\psi}^* K_{\mathcal{G}} + F'_+ - F'_-$ where at worst $F'_-$ defines a valuation with centre in $\mathrm{sing}\,(Y)$. If we subsequently throw in the boundary then the only further candidate for a divisor supported in $F_-$ must be invariant on $X^{\#}$, and contract to a centre of codimension at least 2 on $Y$, since by hypothesis $D$ is transverse. In such a scenario, however, the corresponding divisor $F_0$ would be invariant by I.6.6, so this cannot pose a problem either. The outstanding possibility that some $F$ in the support of $F_-$ could map to $\mathrm{sing}\,(Y)$ equally follows for the same reason on noting that in the resolution used to obtain $F_0$ smoothness of the ambient space around the centre is irrelevant provided that we're prepared to blow up in Weil divisors, and of course the valuation ring $\mathcal{O}_{\tilde{Y}, F_0}$ must have invariant maximal ideal by I.6.7 and VI.1.1. □

Applying this to the particular situation where we further suppose $K_{\tilde{\mathcal{G}}}$ nef., we necessarily obtain for $C_t$ a generic invariant curve in what we previously denoted $E$, $0 > \tilde{\psi}^* K_{\tilde{\mathcal{G}}} \cdot C_t + F \cdot C_t \geq F \cdot C_t$, so indeed $E$ is in the support of $F$. Better still $E$ cannot dominate $Y$, since otherwise the branched covering stack $(\mathcal{Y}, \tilde{\mathcal{G}})$ would be a foliation in rational stacks with nef. canonical bundle which is impossible by IV.8.1. As such $E$ is certainly in some sense contracted by $\tilde{\psi}$, and what must be proved is that this doesn't happen in a direction opposite to that of $\rho$. Fortunately the direction in question is that of the foliation, so that unsurprisingly, we have,

**VI.1.6 Further Claim.** *The generic invariant rational curve, $C_t$, in $E$, maps to a point under $\tilde{\psi}$.*

**Proof.** Suppose the opposite, then $\tilde{\psi}(E)$ is invariant, so in particular it's generic point is not in $\operatorname{sing}(Y)$. Better still from the point of view of the resolution process of the corresponding discrete valuation as found in the proof of VI.1.4, this is in fact a sequence of blow ups in invariant centres. Consequently we obtain an invariant modification $\tilde{Y} \to Y$ with a map of germs, $\tilde{\psi}^* : \mathcal{O}_{\tilde{Y}, E_0} \to \mathcal{O}_{X^\#, E}$ together with a map of canonical classes $\tilde{\psi}^* K_{\mathcal{G}} \to K_{\mathcal{F}^\#}$, which are, equally the log-canonical classes around the germ since everything is invariant. Furthermore, by hypothesis, this map vanishes on $E$, which leads to the absurdity that the $C_t$ map to points. $\square$

Now at this point we're done, since the rigidity lemma, [K-3], applies to show that when we were resolving $\psi$ there was never any need to modify on account of what was actually defined in a neighbourhood of the last transverse centre in which we blew up, which is, of course, contrary to hypothesis. Consequently let's amuse ourselves by reviewing some special known cases of the lemma,

**VI.1.7 Example 1.** Let $Y$ be a curve of positive genus, then every rational map $X \dashrightarrow Y$ from a smooth variety is, in fact, a map.

**Proof.** We consider $Y$ as a foliation by curves whoses only leaf is itself, so the condition of a relative foliated mapping $(X, \mathcal{F}) \dashrightarrow Y$ is empty beyond the existence of the map itself. As such just choose an appropriately large number of foliations $\mathcal{F}_i$ on $X$, so that at least one of them is smooth and generically transverse to any points where $X \dashrightarrow Y$ is supposed not to exist. $\square$

**VI.1.8 Example 2.** The moduli stack $\mathcal{M}_g$ of curves of genus $g$ is proper.

**Proof.** By definition this amounts to checking that if $X, Y \to \operatorname{Spec} R$ are families of stable curves of genus $g \geq 1$ over a DVR $R$ with quotient field $K$ which are generically isomorphic, then indeed they are isomorphic. Plainly we view the said families as foliations $\mathcal{F}$, $\mathcal{G}$ with relative canonical bundles $K_{\mathcal{F}}$, $K_{\mathcal{G}}$ respectively, and apply the lemma to obtain a diagram,

$$\begin{tikzcd} & (W, \mathcal{H}) \arrow[dl, "\theta"'] \arrow[dr, "\psi"] & \\ (X, \mathcal{F}) \arrow[rr, dashed, leftrightarrow] & & (Y, \mathcal{G}) \end{tikzcd}$$

with everything being Gorenstien, and $K_{\mathcal{H}} = \theta^* K_{\mathcal{F}} = \psi^* K_{\mathcal{G}}$. Now take a rational curve $C$ in the special fibre blown down by $\theta$, then $K_{\mathcal{G}} \cdot \psi_* C = 0$. On the other hand, the very definition of stable amounts to $K_{\mathcal{G}} \cdot E > 0$ for every rational curve in the singular fibre, so $C$ must be blown down by $\psi$ too. $\square$

**VI.1.9 Remark.** The "uniqueness lemma" VI.1.3 is rather close in spirit to the no bubbling lemma of [Br1] (*cf.* lemma 1 of op. cit. as corrected in [Br2]). Nevertheless they are very different. Specifically VI.1.3 is not valid in a non-algebraic setting, even for germs around DVR's of the function field which are non-prime. The no-bubbling lemma is, however, local and analytic, with its validity arising from more restrictive hypothesis.

## VI.2 The Holonomy Groupoid

We wish to associate to a foliated stack $(\mathcal{X}, \mathcal{F})$ a representable groupoid in analytic stacks. As such the word analytic stack must be understood for the analytic étale topology, so that in particular étale maps are allowed to be of infinite degree. To begin with, we consider the case of Gorenstien covering stacks, and as ever we have projective moduli $\pi : (\mathcal{X}, \mathcal{F}) \to (X, \mathcal{F})$, with of course $\mathcal{X}$ smooth, and needless to say $\mathcal{F}$ terminal at the non-scheme like points. As a result, the singular locus $Z$ is scheme like, and we'll only attempt to define things over the smooth locus $\mathcal{U} = X \backslash Z$, whose moduli will be denoted $U$.

To begin with we need to find a convergent version of the smooth infinitesimal groupoid $\mathfrak{F} \rightrightarrows \mathcal{U}$ of II.1.2. Technically it's more convenient to do this at the level of the moduli. Away from $\operatorname{sing}(X)$ there's no problem making a convergent neighbourhood, since the Frobenius theorem holds convergently. Equally, by hypothesis, around a complex point $x$ of $\operatorname{sing}(X)$ we can find an étale polydisc $\Delta$ of $x$ considered as a complex point of $x$, together with a smooth fibration $\Delta \to \Delta'$, compatible with the local monodromy, and whose fibres are the local leaves. Taking the moduli of the same we get a fibration, in possibly singular, analytic spaces $D \to D'$, and $D \times_{D'} D$ glues to our previous construction around the smooth points of both $X$ and $\mathcal{F}$. Thus while not having done much, let us note this under,

**VI.2.1 Fact.** *There is an analytic space $F$ around the diagonal of $U$ in $U \times U$, such that $F \rightrightarrows U$ extends convergently the moduli of the smooth infinitesimal groupoid $\mathfrak{F} \rightrightarrows \mathcal{U}$.*

We next wish to make $F$ as big as possible, so we let $\mathbf{F}_{\text{big}} \to U \times U$ be a domain of holomorphy for $F \to U \times U$. Such a space may in fact be rather too big, but it will usefully provide a vechile over which we'll be able to identify an appropriate open set, which in turn admits the right kind of stack structure. To this end we adopt the standard convention,

**VI.2.2 Definition.** A leaf $\mathcal{L}$ of $(\mathcal{X}, \mathcal{F})$ is a maximal connected 1-dimensional invariant analytic substack of $\mathcal{U}$.

This definition certainly skates over several issues about singular points, but we'll leave remarking on these till later. Regardless, if we further suppose that $(X, \mathcal{F})$ is not a foliation in conics, which in any case we've dealt with in IV.8.5, then $\mathcal{L}$ is not a so called bad orbifold. Better still for $x \in \mathcal{L}$ non-scheme like we've already seen, I.5.7, that $\operatorname{Aut}(x)$ is naturally the local holonomy group. Consequently the holonomy covering $\tilde{L} \to \mathcal{L}$ is scheme like, and we denote by $\Gamma_{\mathcal{L}}$ the corresponding covering group. In particular, and essentially by definition, there is a representation $\rho_{\mathcal{L}} : \Gamma_{\mathcal{L}} \to \operatorname{Aut}(\Delta^n)$ of germs of convergent automorphisms fixing the origin of a complex polydisc of dimension $n := \dim \mathcal{X} - 1$, such that in the obvious notation, the foliation around $\mathcal{L}$ is the map of classifying stacks,

$$[\tilde{L} \times \Delta^n / \Gamma_{\mathcal{L}}] \to [\Delta^n / \Gamma_{\mathcal{L}}]$$

where the former action is the diagonal one. With this notation we assert,

**VI.2.3 Claim.** *There is a natural open map,*

$$[\tilde{L} \times \Delta^n / \Gamma_{\mathcal{L}}] \times_{[\Delta^n / \Gamma_{\mathcal{L}}]} [\tilde{L} \times \Delta^n / \Gamma_{\mathcal{L}}] \to \mathbf{F}_{\text{big}} \,.$$

**Proof.** Evidently, $[\Delta^n / \Gamma_{\mathcal{L}}]$ may have no moduli, although the classifying stack $[\tilde{L} \times \Delta^n / \Gamma_{\mathcal{L}}]$ does. Indeed it is a small open (even around the points of $\operatorname{sing}(X)$) neighbourhood $\Lambda$ of the moduli $L$ of $\mathcal{L}$ (which certainly does exist since the stabilisers are finite). Furthermore depending on whether we wish to emphasise the 1$^{\text{st}}$ or 2$^{\text{nd}}$ factors, say the first $p_1$ which is the usual for a leaf's source, we may identify the fibre product with more obvious things such as,

$$[\tilde{L} \times \Delta^n / \Gamma_{\mathcal{L}}] \times \tilde{L}$$

with maps $p_1, p_2$ to $U$, where on the first factor the classifying stack goes to its moduli $\Lambda$, and $p_2$ is the inclusion of a leaf into $\mathcal{U}$ followed by projection to its moduli. If furthermore we choose a base complex geometric point $x \in \mathcal{L}$, and identify it with some $\tilde{x}$ on $\tilde{L}$, then locally around $\tilde{x} \times \tilde{x}$ the moduli of our fibre product is what we previously called $D \times_{D'} D$, with of course, compatible projections, which itself is an open subset of the moduli of $[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times \tilde{L}$. However, by definition, $\mathbf{F}_{\mathrm{big}}$ is a domain of holomorphy, so the assertion follows. $\square$

Unsurprisingly then, we replace $\mathbf{F}_{\mathrm{big}}$ by the open set obtained by taking the union over all leaves of the images of the maps appearing in VI.2.3, and call this $\mathbf{F}$. Better still the $[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}]$'s are open substacks of $\mathcal{X}$ and we have some patching data between them by way of the fibre product,

$$\begin{array}{ccc} [\tilde{M} \times \Delta^n/\Gamma_{\mathcal{M}}] \times_{\mathcal{X}} [\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] & \longrightarrow & [\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \\ \big\downarrow & & \big\downarrow \\ [\tilde{M} \times \Delta^n/\Gamma_{\mathcal{M}}] & \longrightarrow & \mathcal{X} \end{array}$$

where $\mathcal{M}$ is another leaf, the notation is the obvious one, and all the maps in question are both open and representable. Equally, and essentially by hypothesis, the said patching is compatible with the foliation, so we obtain an analytic stack, which rather abusively we'll denote $\mathcal{F}_{\mathrm{hol}}$, with moduli $\mathbf{F}$ and containing as open substacks, the various fibre products,

$$[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times_{[\Delta^n/\Gamma_{\mathcal{L}}]} [\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}]\,.$$

In particular, not only does $\mathcal{F}_{\mathrm{hol}}$ map to $U \times U$, but it maps to $\mathcal{U} \times \mathcal{U}$, and we obtain a groupoid $\mathcal{F}_{\mathrm{hol}} \rightrightarrows \mathcal{U}$ in analytic stacks. Now the crucial technical point that avoids all of this degenerating into a discussion about how to define a 3-category is,

**VI.2.4 Fact.** *The groupoid $\mathcal{F}_{\mathrm{hol}} \rightrightarrows \mathcal{U}$ constructed above is representable.*

**Proof.** The question is certainly local, and to simplify matters, the various fibre products $[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times_{[\Delta^n/\Gamma_{\mathcal{L}}]}$ $[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}]$ are even open, so this just amounts to using various identities like,

$$[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times_{[\Delta^n/\Gamma_{\mathcal{L}}]} \tilde{L} \times \Delta^n = [\tilde{L}_1 \times \tilde{L}_2 \times \Delta^n \times \Gamma_{\mathcal{L}}/\Gamma_{\mathcal{L}}]$$

where the ultimate action is $(x_1 \times x_2 \times z \times \gamma)^g = x_1 \times x_2^g \times z^g \times \gamma g^{-1}$, so the classifying stack in question is the wholly space like $\tilde{L}_1 \times \tilde{L}_2 \times \Delta^n$. $\square$

The upshot of which is, of course,

**VI.2.5 Fact/Definition.** Suppose our foliation is not in rational curves, then there is a well defined classifying stack $[\mathcal{U}/\mathcal{F}_{\mathrm{hol}}]$ in the 2-category of smooth stacks in analytic spaces. Specifically it's the classifying stack deduced by étale "base change" $\coprod_{\alpha} U_\alpha \to \mathcal{U}$ for any scheme like covering $U_\alpha$ of the source of the smooth holonomy groupoid in analytic stacks $\mathcal{F}_{\mathrm{hol}} \rightrightarrows \mathcal{U}$. More precisely the inverted commas are to be understood in the sense of [K-M] 2.6.

Having completed the Gorenstien case, we may now, slightly more generally, consider a branched covering $\sigma : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ of the same, with say weighted branching divisor $B$ in $X$, not containing any singularities of $\mathcal{F}$, or better the pull-back $\mathcal{B}$ of the said divisor to $\mathcal{X}$. Under these hypothesis the pull-backs $s^*\mathcal{B}$, $t^*\mathcal{B}$ by the source and sink of $\mathcal{F}_{\mathrm{hol}} \rightrightarrows \mathcal{U}$ fit together to a simple normal crossing divisor $s^*\mathcal{B} + t^*\mathcal{B}$, and indeed there is only ever at most 2-components through any geometric point. Consequently we can take an appropriate, *i.e.* dictated by the weights of $\mathcal{B}$, branched cover $\tilde{\mathcal{F}}_{\mathrm{hol}} \rightrightarrows \tilde{\mathcal{U}}$, where $\tilde{\mathcal{U}}$ is the smooth locus in $\tilde{\mathcal{X}}$, which we may equally note by way of,

**VI.2.6 Fact/Definition bis.** For $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ as above, we continue to have a representable groupoid $\tilde{\mathcal{F}}_{\mathrm{hol}} \rightrightarrows \tilde{\mathcal{U}}$ in analytic stacks, and as such a well defined classifying stack $[\tilde{\mathcal{U}}/\tilde{\mathcal{F}}_{\mathrm{hol}}]$ in the 2-category of smooth stacks in analytic spaces.

This exercise in making sure that we haven't accidentally ran into an insurmountable problem in set theory out of the way, let's finish off in making,

**VI.2.7 Remarks/Caveats.** 1) The entire content of the above simply amounts to verifying that the naive equivalence relation which identifies points belonging to the same leaf has sense. Manifestly the holonomy groupoid is the smallest object which captures this notion, *i.e.* any smooth stack in analytic spaces whose complex points are identified should they belong to the same leaf must factor through $[\mathcal{U}/\mathcal{F}_{\mathrm{hol}}]$, or more generally $[\tilde{\mathcal{U}}, \tilde{\mathcal{F}}_{\mathrm{hol}}]$.

2) In the smooth category, the holonomy groupoid may fail to be separated, *cf.* [C]. Fortunately (although a posteriori, *cf.* VI.3, this could have been avoided for somewhat deeper reasons) this is prevented by the unicity of analytic continuation.

3) There is something of an Ockham's razor in this definition, since if it were applied to the foliation associated to the fibration $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ of stable $n$-pointed curves of genus $g$ over those with $(n-1)$ points, we won't get back $\mathcal{M}_{g,n-1}$ but rather some non-separated covering of it. The way to avoid this is, obviously, to profit from the whole of the domain of holonomy $\mathbf{F}_{\mathrm{big}}$. This hasn't been done since our immediate interest is the behaviour of the Poincaré metric along the leaves, which, even at the level of $\mathcal{M}_{g,n}$, is necessarily defined for leaf understood in the sense of VI.2.2.

## VI.3 The Homotopy Groupoid

What we'd like to do now is to pass from the holonomy groupoid to a rather more simple object in which each leaf is replaced by its universal cover. The possibility that the universal cover is compact for even one leaf is completely dealt with by way of IV.8.5. Consequently we'll suppose that $(\mathcal{X}, \mathcal{F})$ is a minimal model of a foliated stack with log-canonical singularities terminal at the non-scheme like points, with of course $\mathcal{X}$ smooth, $(X, \mathcal{F}_0)$ its moduli, and $\pi : (\mathcal{X}_0, \mathcal{F}_0) \to (X, \mathcal{F}_0)$ the Gorenstien cover of the same. To begin with let us consider the arrows of the holonomy groupoid of VI.2.5 fibring over $\mathcal{X}$ by way of the source $s$, or more correctly its composite with the obvious inclusion. Quite generally when presented with such a smooth fibring $s : \mathcal{F}_{\mathrm{hol}} \to \mathcal{X}$, there is no difficulty in constructing an analytic stack whose fibres over $\mathcal{X}$ are the universal covers of the fibres of $s$, what, however, may be less than pleasant is that the stack so constructed is non-separated. The essential of what may go wrong is that for some geometric point $x \in \mathcal{X}(\mathbb{C})$, one finds loops in $s^{-1}(x)$ not homotopic to the identity, but which displaced to nearby fibres bound a disc. As such to eliminate this phenomenon on $\mathcal{F}_{\mathrm{hol}}$ it suffices to show that this fibrewise construction produces a separated stack when applied to $s^{-1}(\mathcal{J})$ for $\mathcal{J} \hookrightarrow \mathcal{X}$ a small embedded everywhere transverse smooth curve. In the particular case of $\mathcal{X}$ a surface this is the content of [Br1] lemma 1, and this particular case is very nearly the general case. Indeed op. cit. (or more correctly, the correction of the same in [Br2]) provides, by way of connecting the non-trivial loop to nearby trivial ones via a domain of holomorphy, a meromorphic map $t$ (in fact it's the restriction of the sink) from a fibring $s : \mathcal{S} \to \mathcal{J}$ in discs over $\mathcal{J}$, together with a section $\sigma$, such that $t$ is undefined at $\sigma(0)$, for $0 \in \mathcal{J}(\mathbb{C})$, yet defined everywhere else. Now such data also occurs in other situations for example discs converging to discs with bubbles, *cf.* [M2], but what is special here is that each fibre of $\mathcal{S}$ maps by $t$ to the smooth locus of $\mathcal{F}$ in $\mathcal{X}$. Consequently if we resolve $t : \mathcal{S} \dashrightarrow \mathcal{X}$ by way of,

$$\begin{tikzcd} \widetilde{\mathcal{S}} \arrow[d, "\rho"'] \arrow[dr, "\tau"] & \\ \mathcal{S} \arrow[r, dashed, "t"'] & \mathcal{X} \end{tikzcd}$$

with $\tilde{\mathcal{S}}$ normal and $\rho$ relatively minimal amongst maps affording a resolution, then for $\mathcal{G}$ on $\tilde{\mathcal{S}}$ the foliation induced by the fibring $s$, $\tau^* K_{\mathcal{F}}$ and $K_{\mathcal{G}}$ can only disagree on curves contracted by $\rho$. So let $\mathcal{E}$ be such a curve, then certainly $\tau(\mathcal{E})$ is $\mathcal{F}$-invariant, and if it's not contained in sing$(\mathcal{F})$, $\tau^* K_{\mathcal{F}}$ and $K_{\mathcal{G}}$ agree around $\mathcal{E}$. Otherwise $\mathcal{E}$ maps to sing $(\mathcal{F})$ under $\tau$. In this case although $\tau(S)$ may not be normal around $\mathcal{E}$, it can be made normal by invariant blow ups without changing $K_{\mathcal{F}}$ nor for that matter the fact that $\tau$ is a proper map, since after all $\tilde{\mathcal{S}}$ is already normal, and bi-rational to its image. Consequently around such a curve, should it exist, $K_{\mathcal{G}}$ is strictly smaller than $K_{\mathcal{F}}$, so if $\mathcal{E}_i$ are the curves contracted by $\rho$ then there are $a_i \in \mathbb{Q}_{\geq 0}$ such that $K_{\mathcal{G}} = \tau^* K_{\mathcal{F}} \left( -\sum_i a_i \, \mathcal{E}_i \right)$. The singularities of the fibring $s : \mathcal{S} \to \mathcal{J}$ are, however, terminal so for some $b_i \in \mathbb{Q}_{>0}$ we have $K_{\mathcal{G}} = \sum_i b_i \, \mathcal{E}_i$, and so conclude that $\sum_i (b_i + a_i) \, \mathcal{E}_i = \tau^* K_{\mathcal{F}}$ is nef., which is of course nonsense unless $t$ is already defined everywhere. As a result we can replace the fibres of the sink by their universal covers without destroying separatedness, then do the same for the source, and so obtain,

**VI.3.1 Fact/Definition.** Let $(\mathcal{X}, \mathcal{F})$ be as above then there is a representable (by a minor variant of VI.2.5) smooth groupoid, the homotopy groupoid, $\mathcal{F}_{\text{hom}} \rightrightarrows \mathcal{U} = \mathcal{X} \backslash \text{sing}(\mathcal{F})$ in separated analytic stacks whose fibres be it under the source or sink are isomorphic either to a disc or a complex line. In particular there is a well defined classifying stack, $[\mathcal{U}/\mathcal{F}_{\text{hom}}]$ in the 2-category of smooth analytic stacks.

We will primarily use the homotopy groupoid to keep book on the leaves. Nevertheless, there's good reason to believe that it is in fact the fundamental object of study. Unlike the holonomy groupoid it has good deformation invariance properties, as the following example shows,

**VI.3.2 Example.** ("Non-Commutative Algebraic Tori") Consider the Kronecker type foliation on $\mathbb{G}_m^2$ with standard coordinates $x, y$ given by $x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$, $\lambda \in \mathbb{G}_m$. For $\lambda \notin \mathbb{Q}$ all the leaves are isomorphic to $\mathbb{C}$ so the homotopy and holonomy groupoids coincide, and have classifying stack isomorphic to $[\mathbb{G}_m/q^{\mathbb{Z}}]$ where $q = \exp(2\pi i \lambda)$, which is even an honest scheme, in fact an elliptic curve, for $\lambda \notin \mathbb{R}$. Alternatively for $\lambda \in \mathbb{Q}$, the leaves are $\mathbb{G}_m$'s and the classifying stack of the holonomy groupoid is $\mathbb{G}_m$ itself, which does not deform to an elliptic curve. In this respect the homotopy groupoid is better since its classifying stack is $[\mathbb{G}_m/\mathbb{Z}]$ with $\mathbb{Z}$ acting trivially, and this is indeed an honest deformation limit of $[\mathbb{G}_m/q^{\mathbb{Z}}]$ for $q \to 1$. The example becomes more amusing still if we compactify to $\mathbb{P}^1 \times \mathbb{P}^1$, where the induced foliation has canonical singularities for $\lambda \notin \mathbb{Q}$, and log-canonical singularities otherwise. In the former case the homotopy and holonomy groupoids continue to coincide with classifying stack $[\mathbb{P}^1/q^{\mathbb{Z}}]$, while in the latter case if $|\lambda| = p/q$ for $(p, q)$ relatively prime, the classifying stack of the holonomy groupoid is the generalised weighted projective stack $\boldsymbol{\mathcal{P}}^1(p, q)$, which again fails to be a deformation limit of the irrational case, unlike its homotopy groupoid which is $[\mathbb{P}^1/\mathbb{Z}]$ for $\mathbb{Z}$ acting by way of,

$$\mathbb{Z} \to \text{Aut}(\mathbb{P}^1) : [S, T]^n \mapsto [\zeta^{pn} S, \zeta^{qn} T]$$

in appropriate standard coordinates with $\zeta$ a $(pq)^{\text{th}}$-root of unity. The classifying stacks of the homotopy groupoids in the compactified case are, therefore, essentially "non-commutative projective curves" in the sense of [AZ], while the $\mathbb{G}_m$-case is simply a variant on the "non-commutative real torus" of [C]. As to why, however, one would wish to render a perfectly good étale classifying stack non-commutative is another matter. After all for $H^*$ any reasonable cohomology theory, $H^*([\mathbb{P}^1/\mathbb{Z}])$ can be computed from the Hoschild-Serre spectral sequence, while it's equally geometrically false, albeit algebraically true, that the algebra $\mathbb{C}\,[x, y]/xy = qyx$ is a deformation of the homogeneous coordinate ring of $\mathbb{P}^1$. The correct geometric statement is that the trivial action deforms. Indeed since $\mathbb{P}^1$ doesn't deform, the deformation space of $[\mathbb{P}^1/\mathbb{Z}]$ under the trivial action, is $\text{Aut}(\mathbb{P}^1)$.

## VI.4 Uniform Uniformisation

We can tie our considerations of the homotopy groupoid together with the value distribution theory of §V rather nicely in the case that our foliated variety has general type. Indeed if $(\mathcal{X}, \mathcal{F})$ is of minimalist Mori category, then the uniqueness lemma VI.1.3 guarantees that its numerical Kodaira dimension is well defined, i.e.

**VI.4.1 Definition.** If $(\mathcal{X}, \mathcal{F})$ is a foliated stack whose branched moduli $(X, B, \mathcal{F})$ is minimalist Mori category and is not a foliation in conics, then its numerical Kodaira dimension $\nu(\mathcal{F})$ is given by,

$$\max, 0 \leq i \leq \dim \mathcal{X}, \text{ such that } K_{\mathcal{F}}^i \neq 0$$

where, here, $K^i_{\mathcal{F}}$ is viewed as a numerical equivalence class.

and, if this is maximal, i.e. $(\mathcal{X}, \mathcal{F})$ is of general type, equal to the actual Kodaira dimension. As such V.5.10 guarantees that a sequence of invariant discs, $f_n : \Delta \to \mathcal{X}$, or even, by V.6.1, $f_n : \Delta \to X$ with origin, i.e. $f_n(0)$, bounded away from some algebraic set – determined by the base locus of $|K_{\mathcal{F}}|$ and its tensor powers – converges to a disc with bubbles. If, however, we assume that the discs are actually leaves in $\mathcal{X}$ then provided we bound the $f_n(0)$ away from a possibly larger algebraic subset to ensure that it contains the singularities of $\mathcal{F}$, then provided $(\mathcal{X}, \mathcal{F})$ is minimal there can be no bubbling, i.e.

**VI.4.2 Fact.** *Let $(\mathcal{X}, \mathcal{F})$ be a minimal model of a foliated variety of general type and minimalist Mori category then there is a, necessarily invariant, proper algebraic substack $\mathcal{Y}$ such that the space of discs mapping to leaves of $\mathcal{F}$ is compact modulo $\mathcal{Y}$, cf.* [Br2], [M1].

Alternatively we can profit from the psh. variation of the leafwise Poincaré metric, [Br2], to assert,

**VI.4.3 Complement.** Let things be as VI.4.2, with $\bar{K}_{\mathcal{F}}$ the metricisation of $K_{\mathcal{F}}$ by way of the leafwise Poincaré metric then $c_1(\bar{K}_{\mathcal{F}}) \geq 0$ as a current, and outside of $\mathcal{Y}$ this metric is continuous with leafwise curvature $-1$.

All of this has, of course, been achieved with little, or no, regard for the transverse dynamic, and this is exactly what will be exploited elsewhere in order to extend VI.4.2. For the moment, however, let us note that we have by V.6.1,

**VI.4.4 Fact.** *Let $(\mathcal{X}, \mathcal{F})$ be a minimal model of a foliated variety, and suppose that for every proper algebraic substack $\mathcal{Y}$ the space of invariant discs with bubbles mapping to the branched moduli $(X, B, \mathcal{F})$ (i.e. the disc cannot lie in $B$, and its order of ramification at any component must be divisible by the weight) is not compact modulo $\mathcal{Y}$ then there is a transverse invariant measure $d\mu$ such that,*

*(a)* $D.\, d\mu \geq 0$, *for every effective Cartier divisor* $D$.

*(b)* $K_{\mathcal{F}}.\, d\mu = 0$.

Consequently we will in the next installment proceed to a more delicate study of the "leafwise" positivity of $K_{\mathcal{F}}$, especially in the case of numerical Kodaira dimension = dimension $\mathcal{X} - 1$, by viewing $d\mu$ as a very special representative for $K_{\mathcal{F}}^{\dim \mathcal{X} - 1}$ and bringing into play the transverse dynamic so as to relate the boundedness of invariant discs to the purely algebraic condition $K_X \cdot K_{\mathcal{F}}^{\dim \mathcal{X} - 1} > 0$.

## References


[A1] Artin, M. The implicit function theorem in algebraic geometry, *Algebraic Geometry Internat. Colloq.*, Tata Inst. Fund. Res., Bombay (1968), 13-34, Oxford Univ. Press, London.

[A2] Artin, M. Algebrisation of formal moduli I, Global Analysis (papers in honour of K. Kodaira) 21-71 (1969).

[A3] Artin, M. Algebrisation of formal moduli II, Existence of modifications, *Ann. of Math.* **91** (1970), 88-135.

[A4] Artin, M. Versal deformations and algebraic stacks, *Invent. Math.* **27** (1974), 165-189.

[AZ] Artin, M.; Zhang J.J. Non-commutative projective schemes, *Adv. Math.* **109** (1994), 228-287.

[A-V] Abramovich, D.; Vistoli, A. Compactifying the space of stable maps, *preprint*.

[B-M] Bierstone, E.; Milman, P. Canonical desingularisation in characterictic zero by blowing up the maximum strata of a local invariant, *Invent. Math.* **128** (1997), 207-302.

[Bo-M] Bogomolov, F.; McQuillan, M. Rational curves on foliated varieties, *IHES preprint* (2000).

[Br1] Brunella, M. Subharmonic variation of the leafwise Poincaré metric, *Invent. Math.* **152** (2003), 119-148.

[Br2] Brunella, M. Plurisubharmonic variation of the leafwise Poincaré metric, *Internat. J. Math.* **14** (2003), 139-151.

[C] Connes, A. Non-commutative geometry, Academic Press, San Diego 1994.

[Ca] Cano, F. Discussions over dinner, Latin American Congress of Mathematics, 2004.

[D] Deligne, P. Equations différentielles à points singuliers réguliers, *Lecture Notes in Math.* **163** (1970).

[E] Edixhoven, B et al, Diophantine approximation and abelian varieties, *Lecture Notes in Math.* **1566** (1993).

[G] Gromov, M. Spaces and questions, *Geom. Funct. Anal.* **13** (2003), 73-146.

[K-1] Kollár J. Cone theorems and bug-eyed covers, *J. Algebraic Geom.* **1** (1992), 293-323.

[K-2] Kollár J. et al, Flips and abundance for algebraic threefolds, *Astérisque* **211** (1992).

[K-3] Kollár J. Rational curves on algebraic varieties, Springer-Verlag, Berlin (1996).

[K-M] Keel, S.; Mori, S. Quotients by groupoids, *Ann. of Math.* **145** (1997), 193-213.

[L-MB] Laumon, G.; Moret-bailly, L. Champs algébriques, Springer-Verlag, Berlin (2000).

[M1] McQuillan, M. Non-commutative Mori theory, *IHES preprint* (2001).

[M2] McQuillan, M. Bloch hyperbolicity, *IHES Preprint* (2002).

[M3] McQuillan, M. Integrating $\partial\bar{\partial}$, *Proc. ICM*, Vol. 1, 547-554, Higher Ed. Press, Peking, 2002.

[M4] McQuillan, M. Formal formal schemes, *Contemp. Math.* **B14** (2002), 187-198.

[Ma] Martinet, J. Normalisations de champs de vecteurs holomorphes (d'après Brjuna), *Sém. Bourbaki*, Exp. 564 (1980).

[Mu1] MUMFORD, D. Geometric invariant theory, Springer-Verlag, Berlin-New York (1965).

[Mu2] MUMFORD, D. Towards an enumerative geometry of the moduli space of curves, *Progr. Math.* **36** (1983), 271-328.

[O] ODA, T. Convex bodies and algebraic geometry. An introduction to the theory of toric varieties, Springer-Verlag, Berlin (1988), pg. 80.

[SGA3] GROTHENDIECK, A. ET AL, Séminaire de Géométrie Algébrique.

[Sz] SZPIRO, L. Séminaire sur les pinceaux des courbes de genre au moins deux, *Astérisque* **86**.

[V] VISTOLI, A. Intersection theory on algebraic stacks and on their moduli spaces, *Invent. Math.* **97** (1989), 613-670.

[W] WEIL, A. Foundations of algebraic geometry, AMS, Providence, R.I., 1962.