

# Integrating $\partial\bar{\partial}$

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## Abstract

We consider the algebro-geometric consequences of integration by parts.

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## 1. Jensen's formula

Recall that for a suitably regular function  $\varphi$  on the unit disc  $\Delta$  we can apply integration by parts/Stoke's formula twice to obtain for  $r < 1$ ,

$$\int_0^r \frac{dt}{t} \int_{\Delta(t)} dd^c \varphi = \int_{\partial\Delta(r)} \varphi - \varphi(0) \quad (1.1)$$

where  $d^c = \frac{1}{4\pi i} (\partial - \bar{\partial})$  so actually we're integrating  $\frac{1}{2\pi i} \partial\bar{\partial}$ . In the presence of singularities things continue to work. For example suppose  $f : \Delta \rightarrow X$  is a holomorphic map of complex spaces and  $\bar{D}$  a metricised effective Cartier divisor on  $X$ , with  $f(0) \notin D$ , and  $\varphi = -\log f^* \|\mathbb{I}_D\|$ , where  $\mathbb{I}_D \in \mathcal{O}_X(D)$  is the tautological section, then we obtain,

$$\begin{aligned} \int_0^r \frac{dt}{t} \int_{\Delta(t)} f^* c_1(\bar{D}) &= - \int_{\partial\Delta(r)} \log \|f^* \mathbb{I}_D\| + \log \|f^* \mathbb{I}_D\|(0) \\ &+ \sum_{0 < |z| < r} \text{ord}_z(f^* D) \log \left| \frac{r}{z} \right|. \end{aligned} \quad (1.2)$$

Obviously it's not difficult to write down similar formulae for not necessarily effective Cartier divisors, meromorphic functions, drop the condition that  $f(0) \notin D$  provided  $f(\Delta) \not\subset D$ , extend to ramified covers  $p : Y \rightarrow \Delta$ , etc., but in all cases what is clear is,

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**Facts 1.3.** (a) If  $X$  is compact then  $\int_0^r \frac{dt}{t} \int_{\Delta(t)} f^* c_1(\overline{D})$  is very close to being positive if  $f(\Delta) \not\subset D$ , e.g. in the particular hypothesis preceding (1.2),

$$\int_0^r \frac{dt}{t} \int_{\Delta(t)} f^* c_1(\overline{D}) \geq \log \|f^* \mathbb{1}_D\|(0) + O_{\overline{D}}(1)$$

(b) There is no such principle for the usual area function  $\int_{\Delta(r)} f^* c_1(\overline{D})$  except in extremely special cases such as  $D$  ample.

Equally the essence of the study of curves on higher dimensional varieties lies in understanding their intersection with divisors, and, of course, the principle that a curve not lying in a divisor intersects it positively is paramount to the discussion. Consequently the right notion of intersection number for non-compact curves is the so-called characteristic function defined by either side of the identity (1.2). On the other hand intersection number and integration are interchangeable in algebraic geometry, and whence we will write,

**Notation 1.4.** Let  $\omega$  be a  $(1,1)$  form on  $\Delta$  then for  $r < 1$ ,

$$\int_{\Delta(r)} \omega := \int_0^r \frac{dt}{t} \int_{\Delta(t)} \omega.$$

The more traditional characteristic function notation is reserved for the current associated to a map, i.e.

**Definition 1.5.** Let  $f : \Delta \rightarrow X$  be a map of complex spaces then for  $r < 1$ , we define,

$$T_f(r) : A^{1,1}(X) \rightarrow \mathbb{C} : \omega \mapsto \int_{\Delta(r)} f^* \omega.$$

Evidently in many cases one works with forms which are not quite smooth, so there are variations on the definition. In any case in order to motivate our intersection formalism let us pause to consider,

## 2. Convergence

The basic theorem in the study of subvarieties of a projective variety is Grothendieck's existence and properness of the Hilbert scheme, or if one prefers a sequence of subvarieties of bounded degree has a convergent subsequence. Of course families of smooth curves do not in general limit on smooth curves but rather semi-stable ones, and as such we must necessarily understand convergence of discs in the sense of Gromov [G], i.e.

**Definition 2.1.** A disc with bubbles  $\Delta^b$  is a connected 1-dimensional complex space with singularities at worst nodes exactly one of whose components is a disc  $\Delta$  and such that every connected component of  $\Delta^b \setminus \{\Delta \setminus \text{sing}(\Delta^b)\}$  is a tree of smooth rational curves.

For  $z \in \text{sing}(\Delta^b) \cap \Delta$  and  $R_z$  the corresponding tree of rational curves, and provided  $0 \notin \text{sing}(\Delta^b)$  we can extend our integral (1.4) to this more general situation by way of,

$$\int_{\Delta^b(r)} \omega := \int_{\Delta(r)} \omega + \sum_{z \in \Delta(r) \cap \text{sing}(\Delta^b)} \log \frac{r}{|z|} \int_{R_z} \omega$$

while if  $f : \Delta^b \rightarrow X$  is a map then we have a graph,

$$\Gamma_f := (\text{id} \times f)(\Delta) \cup \bigcup_{z \in \Delta \cap \text{sing}(\Delta^b)} z \times f(R_z) \subset \Delta \times X.$$

An appropriate formulation of Gromov's compactness theorem is then,

**Fact 2.2.** *Let  $\overline{\text{Hom}}(\Delta, X)$  be the space of maps from discs with bubbles into a projective variety  $X$  topologised by way of the Hausdorff metric on the graphs then for  $C : (0, 1) \rightarrow \mathbb{R}_+$  any function and  $K \subset \text{Aut}(\Delta)$  compact the set,*

$$\left\{ f \in \overline{\text{Hom}}(\Delta, X) : \exists \alpha \in K, \int_{\Delta(r)} \alpha^* f^* c_1(\overline{H}) \leq C(r) \right\}$$

where  $\overline{H}$  is a metricised ample divisor, is compact.

Under more general hypothesis on  $X$ , 2.2 continues to hold, but in the special projective case one has an essentially trivial proof thanks to the ubiquitous Jensen formula, cf. [M] V.3.1. Equally although the appearance of automorphisms looks like an unwarranted complication they are necessitated by,

**Remarks 2.3.** (a) The possibility of bubbling at the origin.

(b) The defect of positivity for the intersection number as per (1.3)(a).

Observe moreover that the introduction of  $\overline{\text{Hom}}(\Delta, X)$  and its precise relation to  $\text{Hom}(\Delta, X)$  are both necessary, and easy respectively, i.e.

**Fact 2.4.** ([M] V.3.5) *Let  $T \supset \text{Hom}(\Delta, X)$  be such that the bounded subsets in the sense of 2.2 are relatively compact then  $T \supset \overline{\text{Hom}}(\Delta, X)$ . Moreover assuming that  $X$  is not absurdly singular then  $\text{Hom}(\Delta, X) = \overline{\text{Hom}}(\Delta, X)$  iff  $X$  contains no rational curves.*

One can equally generalise this to a log, or quasi-projective situation by introducing a divisor  $D$ , whose components  $D_i$  should be  $\mathbb{Q}$ -Cartier at which point the appropriate variation thanks to a lemma of Mark Green, [Gr], is,

**Fact 2.5.**  $\overline{\text{Hom}}(\Delta, X \setminus D) \subset \text{Hom}(\Delta, X)$  iff  $X \setminus D$  and  $D_i \setminus \bigcup_{j \neq i} D_j$  do not contain any affine lines.

In particular  $\text{Hom}(\Delta, X \setminus D)$  is relatively compact in  $\text{Hom}(\Delta, X)$  if and only if  $\overline{\text{Hom}}(\Delta, X \setminus D)$  is compact and the boundary is *mildly hyperbolic* in the sense that  $D_i \setminus \bigcup_{j \neq i} D_j$  does not contain affine lines. The latter question is purely algebraic

and closely related to the log minimal model programme. In the case of foliations by curves an even more delicate result holds since as Brunella has observed, [B], the equivalence of  $\overline{\text{Hom}}$  with  $\text{Hom}$  for invariant maps into the orbifold smooth part of a foliated variety is itself equivalent to the said foliated variety being a minimal model.

### 3. The Bloch principle

Bloch's famous dictum, "Nihil est in infinito quod non fuerit prius in finito", might thus be translated as,

**Question 3.1.** Suppose for a projective variety  $X$ , or more generally a log variety  $(X, D)$  there is a Zariski subset  $Z$  of  $X \setminus D$  through which every non-trivial map  $f : \mathbb{C} \rightarrow X \setminus D$  must factor then do we have *hyperbolicity modulo  $Z$* , i.e. is it the case that a sequence  $f_n$  in  $\text{Hom}(\Delta, X)$  not affording a convergent subsequence in  $\overline{\text{Hom}}$  must be arbitrarily close (in the compact open sense) to  $Z \cup D$ .

In the particular case that 2.5 is satisfied we can replace  $Z \cup D$  by  $\overline{Z}$  and ask for *complete hyperbolicity modulo  $Z$* , but outside of surfaces (2.5) seems difficult to guarantee. Regardless in his thesis Brody, [Br], provided an affirmative answer for both  $Z$  and  $D$  empty by way of his reparameterisation lemma which was subsequently extended by Green to the case of  $Z$  empty and every  $D_i \setminus \bigcup_{j \neq i} D_j$  not containing holomorphic lines.

Bearing in mind the singular variant of Green's lemma implicit in 2.5, which for example makes it applicable to stable families of curves, it would appear that the unique known case not covered by the methods of Brody and Green was a theorem of Bloch himself, [Bl], i.e.  $\mathbb{P}^2 \setminus \{4 \text{ planes in general position}\}$ , and its subsequent extension by Cartan to  $\mathbb{P}^n$ , [C]. However, even here, a moment's inspection shows that 2.5 holds, so one knows a priori that there can be no bubbling, and whence complete hyperbolicity in the sense of 3.1 trivially implies so-called normal convergence modulo the diagonal hyperplanes, and the correct structure is obscured.

Now an extension of the reparameterisation lemma to cover 3.1 would be by far the most preferable way forward, since the non-existence of holomorphic lines is an essentially useless qualitative statement without the quantitative information provided by the convergence of discs. Nevertheless we can vaguely approximate a reparameterisation lemma thanks as ever to Jensen's formula. Specifically consider as given,

**Data 3.2.**

- (a) A  $\mathbb{Q}$ -Cartier divisor  $\partial$  on a log-variety  $(X, D)$ .
- (b) A sequence  $f_n \in \text{Hom}(\Delta, X \setminus D)$  which neither affords a convergent subsequence nor is arbitrarily close to  $\partial \cup D$ .

In light of (b) we can choose convergent automorphisms  $\alpha_n \in \text{Aut}(\Delta)$ , such that  $\alpha_n^* f_n(0)$  is bounded away from  $\partial \cup D$ , and given, modulo subsequencing, the convergence of the  $\alpha_n$  we may as well suppose this. Moreover for each  $0 < r < 1$

we can normalise the current  $T_{f_n}(r)$  of 1.5 by its degree with respect to an ample divisor  $H$ , which we'll denote by  $T_{f_n}^H(r)$  and take a weak limit for a suitable subset  $\mathcal{N}$  of  $\mathbb{N}$  to obtain a current  $T_{\mathcal{N}}^H(r)$ . In addition 3.2(b) also tells us that for some fixed  $0 < s < 1$ , the degrees of the  $f_n$  at  $s$  go to infinity, and whence by (1.1) and (1.2)

**Pre-Fact 3.3.** *For  $r \geq s$ ,  $T_{\mathcal{N}}^H(r)$  is a positive harmonic current such that,  $T_{\mathcal{N}}^H(r) \cdot F \geq 0$  for all effective divisors  $F$  supported in  $\partial \cup D$ .*

What is somewhat less trivial, but once more the key is Jensen's formula, is,

**Fact 3.3(bis).** ([M] V.2.4) *Subsequencing in  $\mathcal{N}$  as necessary, then for  $r \geq s$  outside of a set of finite hyperbolic measure (i.e.  $(1 - r^2)^{-1} dr$ )  $T_{\mathcal{N}}^H(r)$  is closed.*

Obviously there are various choices involved but whenever we're dealing in the context of countably many projective varieties they can all be rendered functorial, up to a constant, with respect to push forward. The constant itself only causes a problem should it be zero which is usually what one wants to prove anyway, and as such the notation  $T(r)$  is relatively unambiguous, and represents in a vague sense a parabolic limit of the sequence  $f_n$ .

## 4. Applications

Applications of course require some knowledge of intersection numbers, and quite generally even for a compact curve  $f : C \rightarrow X$  there is very little that one can say in general beyond,

**Observation 4.1.** *Let  $f' : C \rightarrow P(T_X)$  be the derivative ( $\mathbb{P}(\Omega_X)$  in the notation of EGA) with  $L$  the tautological bundle then,*

$$L_{\cdot, f'} C = (2g - 2) - \text{Ram}_f.$$

This is of course the Riemann-Hurwitz formula if  $\dim X = 1$ , and there's an equally trivially log-variant where on the right hand side we have to throw in the number of points in the intersection with the boundary  $D$  counted without multiplicity the special case of  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  being Mason's "a, b, c" theorem for polynomials. The correct generality for best possible applications is to work with log-smooth Deligne-Mumford stacks (or alternatively just orbifolds since the inertia tends to be irrelevant), however for simplicity let's stick with log-smooth varieties and metricise  $T_X(-\log D)$  by way of a complete metric  $\|\cdot\|_{\log}$  on  $(X, D)$ , which in turn leads to a mildly singular metricisation  $\bar{L}$  of the tautological bundle. Supposing for simplicity that  $f(0) \notin D$  with  $f$  unramified at the origin then Jensen's formula yields,

**Observation 4.2(bis).** *Notations as above,*

$$\begin{aligned} \int_{\Delta(r)} f^* c_1(\overline{L}) &= -\log \left\| f_* \left( \frac{\partial}{\partial z} \right) \right\|_{\log} (0) + \int_{\partial \Delta(r)} \log \left\| f_* \left( \frac{\partial}{\partial z} \right) \right\|_{\log} \\ &+ \sum_{0 < |z| < r} \min\{1, \text{ord}_z(f^* D)\} \log \frac{r}{|z|} \\ &- \sum_{\substack{0 < |z| < r \\ f(z) \notin D}} \text{ord}_z(R_f) \log \frac{r}{|z|}. \end{aligned}$$

Combining the concavity of the logarithm and once more Jensen's formula, but this time for  $dd^c \log \log^2 \|\mathbb{I}_D\|$  for any norm on the boundary divisor  $D$ , immediately yields in the notations of 3.3,

**Fact 4.3.** *Let  $T'(r)$  be the current associated to the logarithmic derivative of a sequence  $f_n \in \text{Hom}(\Delta, X \setminus D)$  with  $f_n(0)$  not arbitrarily close to  $D$  and which does not afford a convergent subsequence then outwith a set of finite hyperbolic measure,*

$$L. T'(r) \leq 0.$$

The so-called tautological inequality 4.3 is well adapted for applications to convergence of discs (note incidentally that it's implicit to the formulation that a smooth metric on the bundle  $T_X(-\log D)$  is being employed). Nevertheless for more delicate questions such as quantifying degenerate/non-convergent behaviour. etc. there is a wealth of information in (4.2) that is lost in the coarser corollary. Indeed even using the concavity of the logarithm distorts a very delicate term measuring the 'ramification at  $\infty$ ', i.e. the distortion of the boundary from its length in the Poincaré metric, which is closely related to the difficulty of extracting an isoperimetric inequality from a knowledge of hyperbolicity in the sense of 3.1. While from the still deeper curvature point of view, 4.2 is simply a doubly integrated tautological Schwarz lemma, since by definition metricising  $T_X(-\log D)$  by way of a metric  $\omega$  of curvature  $\leq -K$  is equivalent to a lower bound of the left hand side of the form,

$$K \int_{\Delta(r)} \omega$$

for all infinitely small, and whence all in the large, possible discs. While on the subject of curvature and isoperimetric inequalities, a variant specific to dimension 1 replaces the current  $\delta_D$  implicitly hidden in (4.2) by the current associated to the boundary of a simply connected region, i.e.

**Variant 4.4.** *Suppose  $\dim X = 1$ , let  $U_i \subset X$  be simply connected,  $h_i : \Delta \xrightarrow{\sim} U_i$  isomorphisms and put*

$$\delta_{\Gamma_i} : A^0(X) \rightarrow \mathbb{C} : \varphi \mapsto \int_{\partial \Delta} h_i^* \varphi.$$

Then specific to dimension 1,  $\delta_{\Gamma_i}$  is closed and may be written,  $c_1(H) + dd^c\gamma_i$  for  $H$  an ample divisor of degree 1. Now apply Jensen's formula to recover an integrated form of Ahlfors's isoperimetric inequality, and the Five Island's Theorem.

Returning to varieties and divisors it's still possible to employ (4.2) to get integrated isoperimetric inequalities for more general situations that preserve some 1-dimensional flavour, i.e. discs which are invariant by foliations by curves, with *canonical foliation singularities* (with the obvious definition of that notion which is functorial with respect to the ideas) and which do not pass through the singularities. The latter hypothesis which is reasonable for the study of the leafwise variation of the Poincaré metric is however somewhat restrictive for other applications, and is probably unnecessary as suggested by the essentially optimal inequality of [M] V.4.4 for foliations on surfaces which employs (4.2) to a very large number of monoidal transformations in the foliation singularities. Regardless here is a genuinely 2-dimensional theorem,

**Theorem 4.5.** ([M] V.5) *Let  $(X, D)$  be a smooth logarithmic surface with  $\Omega_X(\log D)$  big (e.g. log-general type, and  $s_2(\Omega_X(\log D)) > 0$ ) then there is a proper Zariski subset  $Z$  of  $X \setminus D$  such that  $X \setminus D$  is complete hyperbolic (in the sense of 3.1 et sequel) modulo  $Z$ .*

Indeed one can even optimally quantify (cf. op. cit.) the degeneration of the Kobayashi metric (which is evidently continuous and non-zero off  $Z$ ) around  $Z$ . Amusingly the theorem only covers  $\mathbb{P}^2 \setminus \{5 \text{ planes in general position}\}$ , although it's a good exercise in the techniques (cf. op. cit. V.4) to prove Bloch's theorem too, at which point a rather small sequence of blow ups replaces all of the original estimation. In any case (4.5) should only be seen as a stepping stone which in order of ascending difficulty leaves open the following questions, viz,

**Concluding Remarks 4.6.** For concreteness take a smooth algebraic surface  $X$  of general type with  $c_1^2 > c_2$  (otherwise the following should be understood in terms of higher jets, but not for anything more general than a surface) then,

- (a) Do we have an isoperimetric inequality with appropriate degeneration along the subset  $Z$  of (4.5).
- (b) Is the Kobayashi metric negatively curved.
- (c) For each  $x \notin Z$  and  $t$  a tangent direction at  $x$ , is there a unique up to the usual action of  $\mathrm{SL}_2(\mathbb{R})$  pointed disc with maximal tangent in the direction  $t$ , and if so does it continue to be so along its image, i.e. is there a continuous (off  $Z$ ) connection whose geodesics are the discs defining the Kobayashi metric.

## 5. Thanks

In closing it is a pleasure to thank M. Brunella for introducing me to the unit disc and bubbling, together with M. Gromov for continuing this education, but above all Cécile without whom the reader could never have got this far.

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