

Two Ext groups and a residue

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In memoria di Marco Brunella

Abstract

The present constitutes the key lemma which was hitherto missing in the author's proof of, inter alia, the Green-Griffiths conjecture for surfaces with enough 2-jets, *e.g.* $13c_1^2 > 9c_2$. As to why this is so is not the immediate concern of this article since it is a lemma within a whole. By way of what is essentially an appendix to the current paper, a guide to the whole is provided at [M6].

Introduction

Prooemium

A foliation by curves \mathcal{F} on a complex space X , or, indeed champ de Deligne Mumford analytique, is most usefully viewed as a fibration $X \rightarrow [X/\mathcal{F}]$ over the champs classifiant. Supposing X smooth for ease of exposition, then away from the singularities of \mathcal{F} this has its obvious sense, cf. §I.1, otherwise, while one can make definitions at the singularities, the sense of this arrow is ambiguous, and it's role is largely for the memonic purpose of emphasising the dynamical nature of the study. What is unambiguous, however is that there is an exact sequence of coherent sheaves on X ,

$$0 \longrightarrow \Omega_{X/\mathcal{F}} \longrightarrow \Omega_X \longrightarrow K_{\mathcal{F}}\mathcal{I}_Y \longrightarrow 0$$

with $K_{\mathcal{F}}$ a line bundle, which one may think of as the canonical bundle along the leaves so strictly: $K_{X/[X/\mathcal{F}]}$, whence the abbreviation, and \mathcal{I}_Y an ideal supported in co-dimension 2, *i.e.* the singularities. This leads to many invariants, known as Baum-Bott residues, [BB], which express this ambiguity to first order, *e.g.* an invariant measure, $d\mu$ is an unambiguous notion away from the singularities which one usually supposes (I.1.1) to extend across the singularities as a closed positive current, which is not supposing very much, whence there is a residue,

$$d\mu \in \text{Ext}^{n-1}(\mathcal{I}_Y, K_{X/\mathcal{F}}) \xrightarrow[\text{RES}]{} \text{Ext}^n(\mathcal{O}_Y, K_{X/\mathcal{F}})$$

where $K_{X/\mathcal{F}} = K_X - K_{\mathcal{F}}$ may be thought of as the canonical bundle of $[X/\mathcal{F}]$, albeit even this statement is ambiguous at the singularities, and is subject to

other residues, cf. I.1.5. Some limited insight may be had when the foliation is, say, a fibration $X \rightarrow [X/\mathcal{F}]$ of a surface over an algebraic curve, so everything extends un-ambiguously over the singularities, and we find singular fibres,

$$X_b = \sum_i n_i C_i$$

with multiplicities n_i along the reduced components C_i . According to the definition of an invariant measure, every combination,

$$\sum_i \nu_i C_i, \quad \nu_i \in \mathbb{R}_{\geq 0}$$

with at least one $\nu_i \neq 0$ is to be considered such. However RES vanishes on such a combination iff it is parallel to X_b iff the measure descends to the quotient. Similarly straightforward remarks apply in the “universal case” $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$. Already, however, for a slight perturbation, cf. III.4.1 (c), of such algebraic examples, even locally $[X/\mathcal{F}]$ is quite complicated, *i.e.* the action of a diffeomorphism which is no better than conjugate to a rotation up to some finite order, and RES sees only a very small part of the dynamics. Nevertheless, the part which it does see is exactly what is pertinent to algebraic geometry.

To understand this relation, we require a brief parenthesis, [M4] §III, on closed positive currents T of dimension (p, p) (so acting on $2p$ forms) defined in a neighbourhood U of some proper sub-variety V . An example of which would be integration over a holomorphic p -cycle, and in this case one has the Fulton/Macpherson specialisation, [Fu], of the cycle to a p -cycle on the normal cone $C_{V/U}$, and indeed without any hypothesis of the properness of V . In general, however, properness appears to be a necessary assumption to define a specialisation of T to $C_{V/U}$, which again is closed positive (non-negative would be more accurate) of dimension (p, p) . This in turn defines, albeit only unambiguously on closed forms, a closed positive $(p-1, p-1)$ current $s_{V,T}$ on the projectivised cone which yields exactly the Segre class of op. cit. when T is an analytic p -cycle. This is perhaps clearer in the immediate case of relevance: V is a divisor and T of type $(1, 1)$, then T splits into closed positive currents,

$$\mathbb{I}_{U \setminus V} T + \mathbb{I}_V T$$

for \mathbb{I}_\bullet the characteristic function. The Segre class of T around V is the same as that of $\mathbb{I}_{U \setminus V} T$, it is a measure on V , and its total mass is given by,

$$\int_V s_{V,T} = V \cdot \mathbb{I}_{U \setminus V} T$$

with the latter product just being that between H_c^2 and $H^{2(n-1)}$. Plainly for T of dimension $(1, 1)$ the general case can always be reduced to this case by blowing up. In particular the Segre class $s_{V,T}$ is, functorially with respect to the ideas, the winding number of $\mathbb{I}_{U \setminus V} T$ around V .

In contrast to RES the Segre class of an invariant measure, or indeed of any closed positive current, around a sub-variety, which here will be the singular locus Y of the foliation is highly amenable to estimation by algebraic techniques such as counting sections of linear systems vanishing to high order on Y . Whence the following relating the two is central to the global study of foliations,

Lemma *Let $U \rightarrow [U/\mathcal{F}]$ be a foliated 3-fold with (foliated) canonical singularities in which the singular locus Y is proper (so, in practice a tubular neighbourhood of the same) and $d\mu$ an invariant measure then,*

$$s_{Y,d\mu} = 0 \iff \text{RES}(d\mu) = 0$$

Here, by way of clarification, let us note that by hypothesis/definition $\mathbb{I}_Y d\mu$ is 0 otherwise neither side has sense. The notion of canonical singularities is functorial with respect to the ideas, and the existence of such resolutions has been proved in [MP1]. Plainly the content of the lemma is to reduce the study of a dynamical invariant, *i.e.* something about $[X/\mathcal{F}]$ to a much simpler algebraic one. Before over-viewing its proof, let us note an illustrative corollary,

Corollary[M4] *Let S be an algebraic surface of general type with enough 2-jets, *e.g.* $c_1^2(S) > 9/13c_2(S)$, then the Green-Griffiths conjecture holds.*

In fact, way more is true for such surfaces, *e.g.* an optimal version of Gromov's isoperimetric inequality, boundedness of curves of genus g in moduli, a Bloch type dichotomy for convergence of discs in the Gromov-Hausdorff sense, *etc., etc.* One can also apply it to Mordell for surfaces over characteristic zero function fields too, but apart from the 2-jet condition on the generic point, one needs related conditions on the fibres of bad reduction. This is, however, purely illustrative, it is really a lemma about foliated 3-folds of which it almost implies a complete description in the spirit of [M1]. Indeed, what remains to be done to achieve this is the construction of $K_{\mathcal{F}}$ -flops, which in one of life's ironies is more difficult than flips which were done in all dimensions in [M3], and to improve the lemma so that it implies something when the numerical kodaira dimension, $\nu(\mathcal{F})$, of $K_{\mathcal{F}}$ is 1. More precisely not only for $\nu(\mathcal{F}) \neq 1$ are the hypothesis of the lemma adequate, but, quite generally the lemma addresses the most difficult part of the computation of RES, to apply it however to the case of $\nu(\mathcal{F}) = 1$ one needs a more delicate hypothesis than vanishing Segre class close to the beast of §III.5. This is a particular type of foliation singularity with very wild behaviour even formally which the vanishing Segre class hypothesis renders benign, but which will certainly show its teeth as soon as one seeks a more quantitative statement. As a result, even with all of the machinery at our disposal we still cannot quite claim non-trivial analogues of the above corollary for 3-folds.

All of which, apart from the side stepping of a subtle dynamical issue at the beast, is irrelevant except as motivation since the proof of the corollaries is elsewhere, *i.e.* [M4]. Let us, therefore, turn to the question of the proof. The lemma is stable under blowing up in centres inside the singular locus, so we can, and will, blow up as much as we can to improve the situation. Any sequence of such operations has the following effect: we get a modification $\rho : \tilde{X} \rightarrow X$, with a total exceptional divisor E invariant by the induced foliation, again denoted

\mathcal{F} with no change in the canonical, so an exact sequence,

$$0 \longrightarrow \Omega_{\tilde{X}/\mathcal{F}}(\log E) \longrightarrow \Omega_X(\log E) \longrightarrow K_{\mathcal{F}}I_L \longrightarrow 0$$

The resulting ideal I_L is supported in what [M4] describes, correctly, as points where the foliation is not log-flat. Log-flatness is a stable condition under blowing up, and the lemma is trivial when it holds. This leads to a finite list, [M4] I.3, [MP2] V.1.8, VI.2.1 of final forms for the singularities where the lemma has to be proved. There is only one isolated case which can be done in a couple of lines in the same way that the surface version of the lemma was proved in [M2], which we'll here by refer to as the "baby lemma". This is, however, relevant and the proof is recalled in §I.3. Otherwise there is a formal (2 dimensional) centre manifold, Z , in the completion of the singular locus, which itself has dimension 1. If this were to converge, then there is a rather good trick in [M5] for reducing the lemma to a measure supported in the centre manifold. Of course, even for saddles on surfaces the centre manifold does not converge, but there are large sectorial domains where it does, and smaller, albeit only relatively, sectors, where the holonomy on transversals has basins of attraction. The relevant meaning here of large is large enough to define the centre manifold uniquely. As such the motivation for [MP2], was to prove similar behaviour in dimension 3, and deduce the lemma from the trick of [M5]. Although the results of [MP2] are optimal, this particular assertion is hopelessly false, and the basic conclusion of [MP2] is that even locally the general singularity that we must worry about is insanely complicated in a way that makes the theory of 2-dimensional saddles look like a game for age 5 and under. An exception to this is when locally many leaves adhere in the exceptional divisor, at which point one can make statements very similar to those found in dimension 2, and, basically, §II, invariant measures don't exist, which does make the computation rather easy. The generic situation, however, is that locally there are no obstructions to existence, and, worse, where the leaves are asymptotically as if they factored through the centre manifold no local obstruction to the measure being absolutely arbitrary. This, and the fact that the injectivity of distributions implies that locally RES is meaningless anyway is explained in more detail in §I.2. We therefore have the following problems,

- (a) A computation that only has meaning globally.
- (b) A dynamical structure, $[X/\mathcal{F}]$ which is so complicated that even locally it's rare that anything pertinent can be said.
- (c) A centre manifold that probably doesn't converge, but it might, and, regardless, two completely unrelated kinds of behaviour for leaves "outside" it- generally escaping from our neighbourhood, and "inside" it- going nowhere.

The final point is perhaps the most important to understand, and it contains the following "toy lemma": suppose the centre manifold did converge, and the measure was supported in it, how do we prove the lemma ? Well, the singular locus has two kinds of components, those left invariant by the induced foliation in the centre manifold, and those transverse to it. The latter are easy, because on blowing up they can be supposed everywhere transverse, and the zero Segre

class hypothesis implies the measure doesn't exist, albeit conversely, II.1.6, as Kontsevich pointed out to me, the lemma at such components is best possible, *i.e.* it has no quantitative variant. Otherwise the toy lemma does not follow from the baby lemma, since the residue here has poles that are much worse. It almost follows from a much better lemma, [B], of Marco Brunella in dimension 2, to the effect that the completion of the holonomy around each invariant curve must take values in S^1 . This, together with standard existence results about the convergence of branches through foliation singularities on surfaces is, in fact, enough if the dual intersection graph formed by these invariant components has no cycles. Otherwise, one can get a super attracting phenomenon around such cycles which a priori buggers up the calculation without excluding invariant measures, I.4.2. This is due to what might be called logarithmic holonomy, which is introduced in §I.4, but on combining this with Brunella's theorem, one would find that were there to be super attraction around cycles in the graph then the measure would be radially symmetric on transversals, and this permits the computation to be done.

All of which looks like we're moving forward, but remember this was a toy problem, and the only step in the above reasoning that is a priori meaningful in our situation is the first one about the transverse components, §II.1. At the same time, the above might be our situation, so we must be able to do the toy problem with what we have at our disposal. One thing we have at our disposal is the formal holonomy in the formal centre manifold, and, in the context of the toy problem, one realises that this can be used to re-prove Brunella's theorem. Similarly, one may prove approximate radial symmetry, *i.e.* for transversal discs of radius ϵ , radial symmetry up to an error $O(\epsilon^n)$, in the presence of (formally) super attracting cycles, and being careful not to use facts about convergence of branches through foliated surface singularities that are false in dimension 3 yields a solution to the toy problem which doesn't use anything we don't have except the toy hypothesis themselves.

This definitely constitutes progress, and merits a re-examination of the baby lemma. The only case of the baby lemma that is non-trivial is the saddle node, and we now have 3 approaches,

- (1) The original. A quick 2 line trick using nothing but Stokes' theorem, and the first few terms in the formal normal form of the singularity, but, unfortunately, utterly reliant on the singularity being isolated, or, more accurately the complete absence of curvature. In particular it does not imply the toy lemma.
- (2) A theorem by Brunella that the measure itself must be integration over a convergent curve through the singularity, of which there are at most 2. In particular, when there is zero Segre class there is no measure outside the exceptional divisor. This renders the baby lemma trivial, but requires an entire series of theorems about conjugation to normal form on sufficiently large sectorial domains, which are not only much more difficult in dimension 3, but generically, cannot meet the sufficiently large condition. A variant proves the toy lemma.
- (3) A *via media*. Just take anything which agrees with the normal form to sufficiently high order. The measure is approximately invariant by its holonomy, and this affords a proof of Brunella's theorem using only the formal structure

of the singularity. This generalise to a proof of the toy lemma.

To which it should be added that whether it be the invariance of the measure by the holonomy on transversals employed in (2), or the approximate invariance in (3), either is most usefully seen by way of Stokes, and so, post factum, (1) is really just a special case of (3) in which just enough and no more of the Taylor expansion was considered.

At this point, however, all we've done is to understand the obvious, *viz*: without any need for analytic conjugation to normal form, the latter alone will give whatever bound we want on transversals whenever a naive inspection of the normal form suggests there should be transversals with no measure. This might reasonably be described as *approximate holonomy*. However, under the conditions where it would be applicable, not by coincidence, one can, [MP2], obtain actual analytic conjugation to normal form on large sectors, and actually conclude from the holonomy that the measure is zero in a way that is more convenient than approximate holonomy: an infinite sum of zeroes is zero, but an infinite sum of small still needs to be estimated. Whence, there is still a difficulty to overcome, and this is most easily grasped by considering the particular, and improbable, sub-case, §IV.1, where the singular locus has exactly one component and the induced foliation in the centre manifold is everywhere smooth. Let us call this the “warm up lemma”. In such circumstances, on completion in the singular locus (not just at some point, which would be useless) one finds on a small open formal coordinates x, y, z , such that the foliation is given by,

$$z \frac{\partial}{\partial z} + x^p \frac{\partial}{\partial y}$$

for some $p \in \mathbb{N}$. The exceptional divisor is $x = 0$, and the centre manifold is $z = 0$. The curves $x = 0, y = \text{const. } z \neq 0$ converge, but the holonomy around them isn't very interesting, whereas the good candidate for a curve which governs the dynamics is the singular locus itself, which, as it happens, would be an elliptic curve. On the other hand, holonomy around a curve in the singular locus is meaningless. Certainly there is the formal holonomy in the formal centre manifold, but a priori this has no relation whatsoever with the structure of $[X/\mathcal{F}]$, *i.e.* the dynamics. A local analogy, is the pleasing example II.2.1, where the centre manifold exists on large sectors, associated to which, cf. the proof of II.2.3, there is an étale groupoid $R \rightrightarrows T$ isomorphic to an action of $\mathbb{Z}(1)$ on \mathbb{C} whose completion is the formal holonomy, nevertheless this association is simply by way of an abstract gluing operation, and it is not the case that $[T/R]$ maps to $[X/\mathcal{F}]$. In the presence of an invariant measure, however, II.2.3, this abstract structure becomes relevant, and it is how measure in the centre manifold is eliminated. For what pertains to the warm up lemma, there is no comparable analytic conjugation to normal form. This can only be done under the condition that the argument of x does not turn through more than $\pi/p - \delta$. Were it possible to do $\pi/p + \delta$, the centre manifold would be unambiguously defined in such sectors, and on the transversals $y = \text{const.}$ one would find the mass bound ϵ^p for balls of radius ϵ off the centre manifold, whence, off the centre manifold, it would be possible to justify calculating RES

locally, and otherwise one would aim to conclude the warm up lemma by a global analogue of II.2.3 to bring the formal holonomy in the formal centre manifold into the game. We know, however, by [MP2] that this cannot be justified, but like the local example II.2.3, the measure itself brings the formal holonomy in the centre manifold into play and leads to the *almost holonomy* estimate, III.1.2. Unlike a conjugation to normal form on large sectors, it need not (and does not in the above example) provide local information on the mass of transversals. It does, however, provide mass bounds on transversals to the singular locus as a function of the formal holonomy in the centre manifold. In the particular case of the warm up lemma, §IV.1, it's quite easy to see that these bounds are exactly what one needs, and if one is prepared to assume the useful fact the the singular locus is an elliptic curve in the warm up lemma, or the corollary that the formal holonomy in the formal centre manifold is commutative, it's particularly easy to see how it solves all of the above problems (a)-(c). Thus, there is a good case for working through the warm up lemma before attempting anything else.

Once one has the almost holonomy estimate, it's reasonably clear that everything is going to work. Nevertheless the calculation is long, and is organised as follows:

(α) Holonomy/Approximate holonomy at certain singularities. More precisely, apart from the warm up lemma, there will be singularities in the induced foliation in the formal centre manifold, which, in turn lead to complicated singularities in the ambient 3-fold. In particular, there are some where the leaves can be locally very large and adhere in the singular locus. Until we can exclude that there is no measure here this is an obstruction to making use of the almost holonomy. The obstruction can be dealt with by looking either at the holonomy or the approximate holonomy on transversals to curves through the singularities. As we've said [MP2] applies here, and this has been done, §II & III.3.2, by holonomy, but only to avoid repetition of calculations that one finds in op. cit.. In the case, III.3.1.(a), however of a generic node in the centre manifold the vanishing of the measure in the region defined in III.3.2 is only a parenthesis, and is not actually used. This is because such nodes have many formal invariants, and certain combinations of these yield a region of nil measure which appears to be a bit too small to be useful. As such, we use a variant of the original proof of the "baby lemma" to get appropriate local estimates which more properly should be ascribed to approximate holonomy.

(β) Almost Holonomy. Off the singularities, component by component this works exactly like the warm up lemma, and is the basic technique for addressing the non-local nature of the calculation. It also provides the local strategies at singularities, §III, where holonomy/approximate holonomy does not apply, or, II.3.3 & III.3.2, provides only partial information, and, how to glue the local strategies to the global one. Nevertheless, as we've attempted to explain, almost holonomy is not approximate holonomy, but rather an estimate that manages to address simultaneously the distinct dynamical features of problem (c). In particular this estimate has little room for error, and has nowhere near the robustness of approximate holonomy. One might think post §III.2 that it'll be plain sailing nevertheless, but when the almost holonomy estimate is provided

by holonomy tangent to a rational rotation there is need for caution.

(γ) The topology of the intersection graph. The plan, cf. §I.4, was to do this by “logarithmic almost holonomy”, but I couldn’t get the right shape of estimate. Instead a more algebro-geometric strategy has been adopted, which leads to a wholly different difficulty arising from ludicrously improbable combinations of singularities in the centre manifold with a rational eigenvalue.

On a philosophical level, the combination of approximate and almost holonomy addresses what the author has always considered to be the most difficult part of his programme to prove everything you want to know about algebraic surfaces, but were afraid to ask *i.e.* the computation of RES, and one might even say, the “right” proof of the baby lemma. It only uses the natural tools at one’s disposal: the measure regularity (the lemma is rubbish for invariant distributions) and the formal structure of the singularities, which, given that an invariant measure (rather than the restriction of a 2 extension) on completion in the singularities is meaningless is reason to be cheerful. Whence, it’s a bit disappointing that the calculation is so long. The nature of (α) is clear, and it should be possible to produce quite a clean theory based on approximate holonomy in all dimensions. The problem posed by (γ) is real, and it would have been preferable to have sub-ordinated this to (β). Otherwise, the use of the almost holonomy is rather uniform, and could arguably have been presented in a less ad-hoc way. Some singularities, however, have a much more complicated local structure than others, so much so that they have to be looked at closely, and only yield to a combination of approximate and almost holonomy, so, there is an inductive structure here that remains to be organised.

By way of thanks, I am indebted to Adam Epstein for an illuminating tutorial on the dynamics of diffeomorphisms tangent to the identity, to Kontsevich for avoiding a wild goose chase after a more quantitative lemma, and Sibony for a precision on the (non-) uniqueness of the Segre class as a current, albeit in this article it’s highly unique, *i.e.* 0. It should, however, be plain from the introduction that this article owes much to Marco Brunella. In our last mathematical conversation, I posed to him the problem which is the intersection of the toy lemma with the warm up lemma. I was already fairly sure that even such a simple case had to be done by way of his structure theorem for invariant measures on surfaces, but I still hoped that I was overlooking some simple point, which was necessarily much the same point as to why [MP2], which had just been completed, did not imply the lemma. Consequently, I gave him no hints, and after 15 minutes or so, he replied, well the holonomy is S^1 , “ma questo non ti piace”. I had learned holonomy from him, and his memory was of a suspicious amateur. Such days, however, were in the past, and I replied that I agreed that the S^1 holonomy argument was the right one, but the real problem was way more difficult, and I had no idea how a S^1 holonomy argument could be organised, or even what it might mean. My worst fears thus confirmed, yet loathing to accept that I’d spent years thinking in totally the wrong terms, I pleaded for an answer different to what I had just been given, and asked “Sei sicuro che non c’è altro modo?”. He replied, “Sono sicuro, sono sicuro”. Requiescat in pace.

De radice -1

Let \mathbb{C} be the complex plane, with $o_{\mathbb{C}}$ its orientation sheaf, and M , \mathcal{A}^2 , and $\mathcal{A}^{1,1}$ the sheaves of measures, 2-forms, and tensors (not differential forms) of type $(1, 1)$, all with real values, then there are canonical inclusions,

$$i : \mathcal{A}^{1,1} \hookrightarrow M \hookrightarrow \mathcal{A}^2 \otimes o_{\mathbb{C}} : j_{\mathbb{C}}$$

the subscript on j being used to indicate that it is in fact, a natural transformation, to which there is, furthermore, a unique canonical isomorphism, such that the following diagram commutes,

$$\begin{array}{ccc} \mathcal{A}^{1,1} & \xrightarrow{\sim} & \mathcal{A}^2 \otimes o_{\mathbb{C}} \\ \downarrow i & & \downarrow j_{\mathbb{C}} \\ M & \xlongequal{\quad} & M \end{array}$$

Consequently for $\mathbb{R}(1)$ the imaginary numbers there is an isomorphism,

$$a : \mathcal{A}^2 \otimes o_{\mathbb{C}} \xrightarrow{\sim} \mathcal{A}^2(1) : dz \otimes d\bar{z} \mapsto -2\pi dz d\bar{z}$$

which does not depend on the choice of the square root of -1 , albeit that this requirement, plainly, does not determine a uniquely. Irrespectively, the choice of a , so, again independently of a choice of the square root of -1 , defines a map,

$$\int : \Gamma_c(\mathbb{C}, \mathcal{A}^2(1)) \longrightarrow \mathbb{R}$$

Similarly for ∂U the border of a bounded domain $U \subset \mathbb{C}$, there is the Stokes isomorphism, $\sigma_U : o_{\mathbb{C}} \rightarrow o_{\partial U}$, which although conventional, has the pleasing feature of relating two canonically defined quantities in a convenient way, *viz*:

$$\int_U j_{\mathbb{C}}(d(\omega)) = \int_{\partial U} j_{\partial U}(\sigma_U \omega), \quad \omega \in \Gamma(\bar{U}, \mathcal{A} \otimes o_{\mathbb{C}})$$

It therefore follows, still independently of any choice of the root of -1 , that there is a map,

$$\oint_{\partial U} : \Gamma(\partial U, \mathcal{A}^1(1)) \longrightarrow \mathbb{R}$$

depending on the Stokes isomorphism, and the isomorphism a . The agreeable nature of the former having been observed, the latter is chosen so that for U the unit disc Δ ,

$$\oint_{\partial \Delta} \frac{dz}{z} = 1$$

which together with functoriality under conformal mapping serves to define a uniquely. In particular, by way of notation, this is how the contour integral symbol will be understood. In addition, since a is a choice, albeit an elegant

one, when integrating an actual measure we shall eschew it, and work with tensors rather than differential forms.

Related to this we have the exponential mapping,

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$$

which affords a canonical and base point free isomorphism,

$$\pi_1(\mathbb{C}^\times) \longrightarrow \mathbb{Z}(1) \quad (= \mathbb{Z}2\pi\sqrt{-1})$$

using which we will freely identify oriented loops γ around a punctured with their image in $\mathbb{Z}(1)$ which will equally be denoted γ . Being oriented, γ defines a distribution via integration, which in the case that γ borders the unit disc Δ is related to the previous discussion by way of,

$$\oint_{\partial\Delta} = \frac{1}{\gamma} \int_\gamma$$

and, again, no choice of a root of -1 is necessary.

Unfortunately there will be choices that may be “forced”. One way this can arise is when we have a loop γ , identified to its image in $\mathbb{Z}(1)$ and a number $\lambda \in \mathbb{R}(1)$, typically an eigenvalue, so the ratio $\lambda/\gamma \in \mathbb{R}$, and its notationally more convenient to express formulae when the sign of this number is, say, positive rather than negative. In such circumstances we will typically make the choice of a square root of -1 in the form of an imaginary part function Im . Another example which is worse still, and it will occur, is that we find ourselves in a domain which contains one root rather than the other, and again ease of notation forces a choice of Im , and if both of these occur at once, then we should be carefull.

I. Residues

I.1 Measures and co-homology

Let $X \rightarrow [X/\mathcal{F}]$ be a foliation by curves of a smooth complex space, or, indeed smooth champ de Deligne-Mumford analytique, X of dimension n , then by definition we have a short exact sequence,

$$0 \longrightarrow \Omega_{X/\mathcal{F}}^1 \longrightarrow \Omega_X^1 \longrightarrow K_{\mathcal{F}}\mathcal{I}_Y \longrightarrow 0$$

where $K_{\mathcal{F}}$ is a line bundle, $\Omega_{X/\mathcal{F}}$ is reflexive, and the support of the singular locus Y has co-dimension at least 2. Away from the singularities an invariant measure $d\mu$ may be unambiguously identified with a closed positive $(n-1, n-1)$ current lying in $K_{X/\mathcal{F}} \otimes \bar{K}_{X/\mathcal{F}}$ for $K_{X/\mathcal{F}}$ the double dual of the top power of $\Omega_{X/\mathcal{F}}$. More usefully, there is a smooth groupoid,

$$F \begin{array}{c} \xrightarrow{t} \\ \xrightarrow{s} \end{array} X \setminus Y$$

which may be sliced along a not necessarily connected transversal T to yield an étale groupoid,

$$R \begin{array}{c} \xrightarrow{t} \\ \xrightarrow{s} \end{array} T$$

while in addition we have a fibre square,

$$\begin{array}{ccc} X \setminus Y & \xleftarrow{\rho} & L = \coprod_t L_t \\ \pi \downarrow & & \downarrow \pi \\ [X \setminus Y / F] & \xleftarrow{\rho} & T \end{array}$$

where π is the projection to the champ classifiant of the foliation, ρ is étale, and L_t is the leaf through t . Whence,

- $\rho^*d\mu$ is a closed $(n-1, n-1)$ current lying in $\pi^*K_T \otimes \pi^*\bar{K}_T \xrightarrow{\sim} K_{X/\mathcal{F}} \otimes \bar{K}_{X/\mathcal{F}}$, and thus descends to a measure $d\mu(t)$ on T , *i.e.* closure is not just equivalent to a local descent datum, but also a global one since $\rho^*d\mu$ is itself global, and by definition $\rho^*d\mu = \pi^*d\mu(t)$.
- By stokes there is even a descent datum for $R \rightrightarrows T$. Indeed if $f \in R$ is an arrow with source $\tau = t(f)$, and sink $\sigma = s(f)$ then we may find a small neighbourhood $V \ni f$ such that s, t are homeomorphisms about f , then connect points of $t(V)$ to $s(V)$ by way of real paths inside leaves to form a real co-dimension 1 manifold \tilde{V} in L so that for any function g on $t(V)$,

$$0 = \int_{\tilde{V}} d(\pi^*g)d\mu = \int_{s(V)} s_*t^*gd\mu - \int_{t(V)} gd\mu = \int_T (s_*h - t_*h)d\mu$$

for t^*g identified with the function h on $V \subset R$. The final integral will be referred to as a *co-equaliser*. It is well defined whenever both s_*h and t_*h are absolutely integrable, *e.g.* bounded Borel functions.

- Equivalently, but less informatively, $R \rightrightarrows T$ is étale, so $K_{X/\mathcal{F}}$, indeed any power of $\Omega_{X/\mathcal{F}}$, is a bundle on $[X \setminus Y/F] = [R/T]$. The class $d\mu$ is global and locally closed, so it descends to,

$$d\mu \in \Gamma([X \setminus Y/F], M \otimes K_{X/\mathcal{F}} \otimes \bar{K}_{X/\mathcal{F}})$$

where M indicates that the coefficients have measure regularity on the given (whence any) étale transversal $T \rightarrow [X \setminus Y/F]$.

Around the singularities there is no similar discussion. Indeed the groupoid $R \rightrightarrows T$ may only very rarely be completed across the singularities (basically probability zero with respect to Lebesgue measure for any moduli space of foliations) in a way that s, t remain discrete. There are ways to do it by the addition of $B_{\mathbb{G}_m}$'s, but this results in points with negative dimension, which is demonstrably meaningless from the point of view of measure theory, whence we'll take the usual ad-hoc approach, *viz*:

I.1.1 Definition By an invariant measure $d\mu$ for the foliation is to be understood a closed positive $(n-1, n-1)$ current with $\mathbb{I}_Y d\mu = 0$ (where here, and elsewhere, \mathbb{I}_* will be the characteristic function of a set), which lies in $K_{X/\mathcal{F}} \otimes \bar{K}_{X/\mathcal{F}}$ away from the singularities. In particular the only hypothesis at the singularities is that it extends across the same with finite mass.

There is a co-homological shadow of this ambiguity by way of,

$$A_c^{0,1} \otimes K_{\mathcal{F}} \mathcal{I}_Y \longrightarrow A_c^{0,0} \otimes K_{\mathcal{F}} \mathcal{I}_Y \otimes \bar{K}_{\mathcal{F}} \bar{\mathcal{I}}_Y \xrightarrow{d\mu} \mathbb{C}$$

whence $d\mu$ defines a class in $\text{Hom}(A_c^{0,1} \otimes K_{\mathcal{F}} \mathcal{I}_Y, \mathbb{C})$, which by way of the Verdier isomorphism is a closed class in $\text{Hom}_X(\mathcal{I}_Y, \mathcal{D}_X^{0,n-1} \otimes K_{X/\mathcal{F}})$, for A, D , respectively \mathcal{A}, \mathcal{D} , smooth functions, and distributions, respectively their sheafification. The complex,

$$\mathcal{D}_X^{0,*} : \mathcal{D}_X^{0,0} \xrightarrow{\bar{\partial}} \mathcal{D}_X^{0,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{D}_X^{0,n}$$

is an injective resolution of the structure sheaf, so we obtain:

$$d\mu \in \text{Ext}_X^{n-1}(\mathcal{I}_Y, K_{X/\mathcal{F}})$$

As such the obstruction to $d\mu$ being a co-homology class is purely local, *i.e.*

$$\text{H}^{n-1}(X, K_{X/\mathcal{F}}) \longrightarrow \text{Ext}_X^{n-1}(\mathcal{I}_Y, K_{X/\mathcal{F}}) \xrightarrow{\text{RES}} \text{Ext}_X^n(\mathcal{O}_Y, K_{X/\mathcal{F}})$$

For Y compact the final group is in duality with $\text{H}^0(K_{\mathcal{F}} \otimes \mathcal{O}_Y)$, and RES may be made explicit as follows: take U any neighbourhood of Y - in practice small and tubular- with ω_α local liftings to $K_{\mathcal{F}}$ of some τ in $\text{H}^0(K_{\mathcal{F}} \otimes \mathcal{O}_Y)$, and

ρ_α compactly supported bump functions forming a partition of unity on some smaller neighbourhood $V \subset\subset U$ of Y , then,

$$\tilde{\tau} = \sum_{\alpha} \omega_{\alpha} \bar{\partial} \rho_{\alpha} \in A_c^{0,1} \otimes K_{\mathcal{F}}(U)$$

may be identified with an integral (1, 1 form, and:

$$\text{RES}(d\mu)(\tau) = \int_U \tilde{\tau} d\mu = \lim_{\epsilon \rightarrow 0} \int_{\partial Y_{\epsilon}} \omega d\mu, \quad \omega = \sum_{\alpha} \omega_{\alpha}$$

where ∂Y_{ϵ} is any continuous boundary around Y at some distance ϵ , tending to zero in any sequence whatsoever since $\tilde{\tau}$ is integrable. All of which is equally true for any Ext class with measure regularity. Otherwise, identifying the connecting homomorphisms,

$$\text{Ext}_X^{n-1}(\mathcal{I}_Y, K_{X/\mathcal{F}}) \xrightarrow[\text{RES}]{\delta} \text{Ext}_X^n(\mathcal{O}_Y, K_{X/\mathcal{F}}), \quad \text{H}^0(K_{\mathcal{F}} \otimes \mathcal{O}_Y) \xrightarrow{\delta^{\vee}} \text{H}_c^1(U, K_{\mathcal{F}} \mathcal{I}_Y)$$

with $\bar{\partial}$, *i.e.* $\delta^{\vee} = \bar{\partial}\omega$, as above, and $\delta\phi = \bar{\partial}\theta$ for ϕ an Ext class, $\theta \in D^{0,n-1} \otimes K_{X/\mathcal{F}}(U)$, one has,

$$\phi(\bar{\partial}\omega) = \theta(\bar{\partial}\omega) = (\bar{\partial}\theta)(\omega) = (\bar{\partial}\theta)(\tau)$$

which is rather less than being able to write the obstruction as an unambiguous limit of an honest integral, even if it remains a “residue”, *i.e.* concentrated around the singularities. Consequently although our interest is confined to invariant measures we do quite generally have a pairing,

I.1.2 Definition For U any neighbourhood of a supposed compact connected singular locus Y ,

$$\text{RES}_Y : \text{Ext}_X^{n-1}(\mathcal{I}_Y, K_{X/\mathcal{F}}) \times \text{H}^0(K_{\mathcal{F}} \otimes \mathcal{O}_Y) : (\phi, \tau) \mapsto \text{RES}_Y(\phi, \tau) = \text{RES}(\phi)(\tau)$$

and this pairing is independent of the neighbourhood U of Y .

The principle utility of this pairing is for X compact, albeit its calculation is local around Y . More precisely, we have a diagram with exact rows:

$$\begin{array}{ccccccc} \text{H}^0(X, K_{\mathcal{F}}) & \rightarrow & \text{Ext}_X^1\left(\frac{\Omega_X^{n-1}}{K_{X/\mathcal{F}}}, K_X\right) & \longrightarrow & \text{H}^{1,1}(X) & \longrightarrow & \text{H}^1(X, K_{\mathcal{F}}) \\ \uparrow & & \uparrow & & \parallel & & \uparrow \\ \text{H}^0(X, K_{\mathcal{F}} \mathcal{I}_Y) & \rightarrow & \text{H}^1(X, \Omega_{X/\mathcal{F}}) & \longrightarrow & \text{H}^{1,1}(X) & \longrightarrow & \text{H}^1(X, K_{\mathcal{F}} \mathcal{I}_Y) \end{array}$$

which is in duality with,

$$\begin{array}{ccccccc} \text{H}^n(X, K_{\mathcal{F}}) & \leftarrow & \text{H}^{n-1}\left(X, \frac{\Omega_X^{n-1}}{K_{X/\mathcal{F}}}\right) & \leftarrow & \text{H}^{n-11, n-1}(X) & \leftarrow & \text{H}^{n-1}(X, K_{X/\mathcal{F}}) \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ \text{Ext}_X^n(\mathcal{I}_Y, K_{X/\mathcal{F}}) & \leftarrow & \text{Ext}_X^{n-1}(\Omega_{X/\mathcal{F}}, K_X) & \leftarrow & \text{H}^{n-1, n-1}(X) & \leftarrow & \text{Ext}_X^{n-1}(\mathcal{I}_Y, K_{X/\mathcal{F}}) \end{array}$$

In particular by way of their images in $H^{1,1}$, respectively $H^{n-1,n-1}$, we have a pairing,

$$\mathrm{Ext}_X^1\left(\frac{\Omega_X^{n-1}}{K_{X/\mathcal{F}}}, K_X\right) \times \mathrm{Ext}_X^n(\mathcal{I}_Y, K_{X/\mathcal{F}}) \longrightarrow \mathbb{C}$$

The functoriality of duality ensures that the pairing depends only on the images (defined via the obvious diagram chase) in:

$$\mathrm{Coker} : H^0(X, K_{\mathcal{F}}) \rightarrow H^0(K_{\mathcal{F}} \otimes \mathcal{O}_Y), \quad \mathrm{Ker} : \mathrm{Ext}_X^n(\mathcal{O}_Y, K_{X/\mathcal{F}}) \rightarrow H^n(X, K_{\mathcal{F}})$$

Consequently, irrespectively of how we lift, say:

$$\tilde{c} : \mathrm{Ext}_X^1\left(\frac{\Omega_X^{n-1}}{K_{X/\mathcal{F}}}, K_X\right) \longrightarrow H^0(K_{\mathcal{F}} \otimes \mathcal{O}_Y)$$

from the co-kernel to H^0 , the functoriality of duality ensures that this latter pairing may be expressed in terms of I.1.2 by way of,

$$\mathrm{Ext}_X^1\left(\frac{\Omega_X^{n-1}}{K_{X/\mathcal{F}}}, K_X\right) \times \mathrm{Ext}_X^n(\mathcal{I}_Y, K_{X/\mathcal{F}}) : (L, \phi) \mapsto \sum_Y \mathrm{RES}_Y(\phi \tilde{c}(L))$$

where the sum is taken over all connected components of the singular locus to emphasise the local nature of the latter. The mild ambiguity in the definition of \tilde{c} is genuine as the following example should illustrate:

I.1.3 Example Away from the singularities an infinitesimal descent datum for a coherent sheaf \mathcal{E} on X to the champ classifiant $[X/\mathcal{F}]$ of the foliation is simply a connection along the leaves, *i.e.* a map,

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} K_{\mathcal{F}}$$

satisfying the Leibniz rule with respect to $\partial : \mathcal{O} \rightarrow K_{\mathcal{F}}$. For any vector bundle the obstruction to the existence of such a connection on all of X lies in a relative Atiyah class,

$$at_{\mathcal{F}}(E) \in H^1(X, \mathrm{End}(E) \otimes K_{\mathcal{F}})$$

As such a line bundle L admits a leafwise connection as soon as the image from $H^{1,1}$ to $H^1(X, K_{\mathcal{F}})$ of its chern class vanishes. Given such a connection we obtain a global section,

$$\tilde{c}(L, \nabla) \in H^0(X, K_{\mathcal{F}} \otimes \mathcal{O}_Y) : \ell_\alpha \mapsto \frac{\nabla \ell_\alpha}{\ell_\alpha}$$

where ℓ_α is a local generator of L around a geometric point y in some opens U_α which cover U . Of course the connection is only unique up to,

$$H^0(X, \mathrm{End}(E) \otimes K_{\mathcal{F}})$$

for a vector bundle, whence the above ambiguity modulo $H^0(X, K_{\mathcal{F}})$.

Since X is smooth, general local co-homology considerations imply that to give such a connection is equivalent to giving it in co-dimension 2, whence:

I.1.4 Definition/Fact Denote by $\text{Pic}^0(\mathcal{F})$ the group of line bundles on X with infinitesimal descent data to $[X/\mathcal{F}]$ away from the singularities, *i.e.* bundles with leafwise connection as per I.1.3. In particular while such bundles are flat along the leaves they may descend only locally rather than globally to $[X/\mathcal{F}]$. In addition let $M(X/\mathcal{F})$ be the (possibly empty) space of invariant measures without support on Y , then the images of these groups in \mathbb{H}^2 , respectively \mathbb{H}_2 , admit the residue formula,

$$(L, d\mu) \mapsto \int_X c_1(L)d\mu = \sum_Y \text{RES}_Y(\tilde{c}(L, \nabla), d\mu) = - \sum_Y \lim_{\epsilon \rightarrow 0} \int_{\partial Y_\epsilon} \tilde{c}(L, \nabla)d\mu$$

which is independent of the connection ∇ .

To which we may conclude with the most pertinent example,

I.1.5 Further Example Away from the singularities $K_{X/\mathcal{F}}$ is genuinely invariant. Dually to I.1.1 the fact that $[X/\mathcal{F}]$ will most likely have no étale neighbourhoods at the singularities implies that even for this bundle it is ambiguous as to whether it really is a bundle on $[X/\mathcal{F}]$ (depending on how we define this at the singularities) or not. It does, however, have a canonical connection,

$$d = \nabla : K_{X/\mathcal{F}} \longrightarrow K_X = K_{X/\mathcal{F}} \otimes K_{\mathcal{F}}$$

induced by exterior differentiation. In turn, around the singularities $\partial : \mathcal{O}_X \rightarrow K_{\mathcal{F}}\mathcal{I}_Y$ affords a linear map,

$$\Omega_X \otimes \mathcal{O}_Y \longrightarrow \Omega_X \otimes \mathcal{O}_Y(K_{\mathcal{F}})$$

whence $\text{Tr}\partial$ is a section of $\mathbb{H}^0(X, K_{\mathcal{F}} \otimes \mathcal{O}_Y)$ and,

$$\tilde{c}(K_{X/\mathcal{F}}, \nabla) = -\text{Tr}(\partial)$$

.

I.2 Local vs Global

There follows a series of remarks intended to illustrate the difficulty of calculating the residue I.1.2 as soon as the singular locus is positive dimensional. Of themselves they are in-essential, but should aid understanding. To this end recall, [SGA2], the particularly satisfactory description of the duality theorem for a complete regular local ring $\hat{\mathcal{O}}$ over a field k . Supposing the dimension is n we may suppose $\hat{X} = \text{Spf}\hat{\mathcal{O}}$ is the completion of $X = \mathbb{A}_k^n$ in the origin x , and we have a duality,

$$\mathbb{H}^0(\hat{X}, \mathcal{O}_{\hat{X}}) \times \mathbb{H}_x^n(X, \omega_X) \longrightarrow k$$

The latter group is,

$$\lim_{n \rightarrow \infty} \text{Ext}_X^n(\mathcal{O}/\mathfrak{m}^n, \omega_X)$$

and the pairing is the Grothendieck residue. In terms of coordinates, z_1, \dots, z_n over \mathbb{C} its elements may be identified with,

$$f(z_1^{-1}, \dots, z_n^{-1})dz_1 \dots dz_n \otimes \gamma_1 \otimes \dots \otimes \gamma_n$$

for f a polynomial in n indeterminants, and γ_i the δ -function of a loop around x in the i th direction. As such the duality is the Cauchy residue,

$$(g, f) \mapsto \oint_{\gamma_1 \times \dots \times \gamma_n} \tilde{g}(z_1, \dots, z_n) f(z_1^{-1}, \dots, z_n^{-1}) dz_1 \dots dz_n$$

where \tilde{g} is any convergent (*e.g.* polynomial) function which agrees with the formal function g up to the maximal degree of f . Alternatively one may express this as an integral over a co-dimension 1 real hypersurface, *cf.* [GH] §5.

In contrast to this the first non-local case of I.1.2 occurs in dimension 3. As it happens (and we'll see in the next section), one can quite quickly reduce to a slightly different sub-scheme of Y which is LCI, albeit since we're just sketching the difficulty one may as well assume this for Y . In any case if Y is LCI of co-dimension d , then the only local Ext groups are in dimension d , *i.e.*

$$\mathcal{E}xt^q(\mathcal{O}_Y, E) = 0, \quad q \neq d$$

for any locally free E . In particular, the local global spectral sequence for Ext is degenerate, and for U as per the last section:

$$\text{Ext}_U^n(\mathcal{O}_Y, K_{X/\mathcal{F}}) = \mathbb{H}^{n-d}(U, \mathcal{E}xt^d(\mathcal{O}_Y, K_{X/\mathcal{F}}))$$

In dimension $n = 3$, $d = 2$ this is particularly simple, and doesn't involve more than a couple of lines diagram chasing, *i.e.* by the injectivity of distributions we have a diagram with exact rows:

$$\begin{array}{ccccc} \text{Hom}_U(\mathcal{I}_Y, K_{X/\mathcal{F}} \otimes \mathcal{D}_X^{0,2}) & \leftarrow & \Gamma(U, K_{X/\mathcal{F}} \otimes \mathcal{D}_X^{0,2}) & \leftarrow & \text{Hom}_U(\mathcal{O}_Y, K_{X/\mathcal{F}} \otimes \mathcal{D}_X^{0,2}) \\ \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} \\ \text{Hom}_U(\mathcal{I}_Y, K_{X/\mathcal{F}} \otimes \mathcal{D}_X^{0,3}) & \leftarrow & \Gamma(U, K_{X/\mathcal{F}} \otimes \mathcal{D}_X^{0,3}) & \leftarrow & \text{Hom}_U(\mathcal{O}_Y, K_{X/\mathcal{F}} \otimes \mathcal{D}_X^{0,3}) \end{array}$$

where one starts with $d\mu$ in the top left, lifts to some T , so that $\bar{\partial}T$ is the global class in Ext^3 . On the other hand, there is no local $\mathcal{E}xt^3$, so on a cover $\coprod_\alpha U_\alpha \rightarrow U$, we find on U_α distributions S_α supported in Y such that $T_\alpha = T + S_\alpha$ is $\bar{\partial}$ closed, and $S_{\alpha\beta} = S_\alpha - S_\beta$ is the desired class in $\mathbb{H}^1(U, \mathcal{E}xt^2(\mathcal{O}_Y, K_{X/\mathcal{F}}))$. In particular, there is absolutely no local obstruction to lifting $d\mu$ in a way which is $\bar{\partial}$ -closed. As such, it is not at all a question of how one does the integral (in an indefinite sense) locally, but rather how these liftings patch, *i.e.* in the notation of the last section, the residue, supposing $\text{spt}\rho_\alpha \subset U_\alpha$ compact, is:

$$\sum_\alpha \bar{\partial}\rho_\alpha \omega_\alpha T = \sum_\alpha \bar{\partial}\rho_\alpha (\omega_\alpha T_\alpha - S_\alpha) = - \sum_\alpha \omega_\alpha \bar{\partial}(\rho_\alpha S_{\alpha\beta})$$

Alternatively, let us bring the foliation into play. A non-trivial case, albeit extremely special and improbable, *cf.* IV.1, occurs when on some cover U_α we can find coordinates such that the foliation has the form,

$$\partial_\alpha = z_\alpha \frac{\partial}{\partial z_\alpha} + x_\alpha^p \frac{\partial}{\partial y_\alpha}$$

and we suppose (actually it follows) for further simplicity that the z_α patch to an invariant centre manifold Z , so $\omega_\alpha = z_\alpha^{-1} dz_\alpha$ is a perfectly good lifting of the trace of I.1.5. Observe that while the sum $\tilde{\tau}$ over α of the $\bar{\partial}\rho_\alpha\omega_\alpha$ is integrable, in fact its even smooth, the individual $\bar{\partial}\rho_\alpha\omega_\alpha$ are not necessarily integrable, and one cannot interchange the order of summation with integration. As such, to fix ideas, say with respect to a different index set i (possibly the same) subordinate to a cover as above so that we have good coordinates x_i, y_i, z_i we triangulate by way of V_i along Y rather than use bump functions, then we have a triangulation \tilde{V}_i of U , and:

$$\int_U \tilde{\tau} d\mu = \sum_i \int_{\tilde{V}_i} \tilde{\tau} d\mu = \sum_i \lim_{\epsilon_i \rightarrow 0} \int_{\substack{|z_i| = \epsilon_i \leq |x_i| \\ y_i \in V_i}} \omega d\mu + \int_{\substack{y_i \in \partial V_i \\ \epsilon_i \leq |x_i|, |z_i|}} \omega d\mu$$

The first of the integrals on the rights is easily seen to be bounded by the Segre class around the singularities. The latter integral “should” cancel according to,

$$\int_{\substack{y_i \in \partial V_i \\ \epsilon_i \leq |x_i|, |z_i|}} \omega d\mu = \sum_j \int_{\substack{y_i \in \partial V_i \cap \partial V_j \\ \epsilon_i \leq |x_i|, |z_i|}} \omega d\mu$$

with the sum being taken over edges of the triangulation, so that $(\partial V_i) \cap V_j$ and $(\partial V_j) \cap V_i$ have the opposite orientation. One cannot, however, a priori arrange that the domains $|x_i| \geq \epsilon_i$ and $|x_j| \geq \epsilon_j$ are in fact the same. Plainly this is the difficulty of interchanging integration with summation for the $\bar{\partial}\rho_\alpha\omega_\alpha$ that we’ve already noted in different clothes. It is also, and more helpfully, a problem of the holonomy around Y of the induced foliation in the centre manifold.

Now while the appearance of an essentially topological obstruction such as the holonomy may be rather encouraging, one should observe:

- The coordinate x_α may be supposed to define an algebraic hypersurface, in fact an exceptional divisor after a blow up. The existence domain of [MP2], however, for such coordinates is only in sectors of the argument of x_α up to π/p , and this is shown to be best possible.
- Sectors of width π/p are uselessly small, *i.e.* the centre manifold is hopelessly non-unique in such sectors, and irrespectively of how small the neighbourhood even the local behaviour of leaves is utterly out of control as the argument goes from $\pi/p - \epsilon$ to $\pi/p + \epsilon$.
- In particular, the holonomy about the induced foliation in the centre manifold has nothing more than a formal sense.

In terms of co-equalisers here is another description of the same phenomenon. Given that we’re only describing the difficulty we’ll be a bit informal, but cf. [M7] for a precise discussion of invariant measures and co-equalisers. In any case by [MP2], and the current §2, one may reduce to the case where $U \setminus E - E$ the exceptional divisor resulting from blowing up- is covered (this will almost certainly employ sectors as above) by finitely many opens W_a such that on each W_a the foliation is a fibration $W_a \rightarrow T_a$ whose fibres do not adhere in E , whence,

a priori, not in the singular locus either. This implies that on every fibre $\pi_a^{-1}(t)$, ω is integrable, while $d\mu$ descends to a measure $d\mu(t_a)$. Consequently:

$$\int_{W_a} \tilde{\tau} d\mu = \int_{T_a} d\mu(t_a)(\pi_a)_*(\tilde{\tau}) = \int_{T_a} d\mu(t_a)(\pi_a)_*(\omega|_{\partial W_a})$$

At which point taking $T = \coprod_a T_a$ one may show that the function,

$$w = \prod_a (\pi_a)_*(\omega|_{\partial W_a}) \in C^0(T)$$

is a co-equaliser, *i.e.* there is a Borel function f on the arrows of $(s, t) : R \rightrightarrows T$ such that,

$$w = (s_* - t_*)(f)$$

Now in principle, and by definition, the value of an invariant measure on co-equalisers should be zero, but this is only true if s_*f and t_*f are integrable. Again this is a manifestation that ω may not be integrable on every face of a triangle, even though it is so leaf by leaf, and the sum over the faces is integrable. Of course if Y were \mathbb{P}^1 , and the singularities really as simple as this toy example, f would be integrable, but such a case is not only trivial, but, by many ways, may be shown not to occur.

As a final caveat to trying to do the problem locally, one should bear in mind the simple sub-case where the centre manifold is everywhere convergent along Y . Whence, locally, there are absolutely no obstructions on an invariant measure supported in the same, and no reason whatsoever for any unbounded function of x_α to be integrable, let alone x_α^{-p} . Of course, there are global obstructions for the existence of such an invariant measure, which in a certain sense is what we'll use, but, as we've said, by [MP2] II.1.5, we start from the situation where the centre manifold simply fails to exist on a sector which is sufficiently large to be useful.

I.3 Isolated Residues

Supposing resolution of singularities (a theorem in dimension 3, [MP1]), one may show that one never needs to tackle an isolated residue which is more complicated than,

I.3.1 Set Up Let U be a neighbourhood of an isolated foliation singularity, such that on completion in the singularity a (convergent) generator of the foliation has a Jordan decomposition,

$$\partial = \partial_S + \partial_N, \quad \partial_S = \sum_{i=1}^n \lambda_i \hat{z}_i \frac{\partial}{\partial \hat{z}_i}, \quad \lambda_1 \dots \lambda_n \neq 0$$

and the remaining (formal) coordinate \hat{x} may be supposed such that $\hat{x} = 0$ is not only convergent, but the exceptional divisor after appropriate blowing up, and $\hat{x}^{-p-1} \partial \hat{x}$ is invertible, even $1 + \mathfrak{m}$, for some $p \in \mathbb{N}$.

Before proceeding, let us note,

I.3.2 Caveat Already in dimension 3, even starting from a foliated projective 3-fold X , the resolution $\pi : \mathcal{X} \rightarrow X$ takes place in the 2-category of champs de Deligne-Mumford. This means that the above U will only be an étale neighbourhood of \mathcal{X} , or equivalently for some finite group G (actually $\mathbb{Z}/2$ in dimension 3) the champ classifiant $[U/G]$ is open in \mathcal{X} . Manifestly, however, this is irrelevant to the residue calculation which may be safely performed in U .

Now take coordinates z_i, x such that the Jordan decomposition holds modulo \mathfrak{m}^N (in fact even modulo $\mathcal{O}(-NE)$ by way of blowing up) for some large N to be chosen, and suppose that we have an invariant measure with no Segre class around the exceptional divisor, or, even, just the point, then:

$$\oint_{|x|=\epsilon, |z_i|\leq\delta} \frac{dx}{x} d\mu + \sum_i \oint_{\substack{|z_i|=\delta, |z_j|\leq\delta \\ |x|\leq\epsilon}} \left(\frac{\partial x}{x} \frac{z_i}{\partial z_i} \right) \frac{dz_i}{z_i} d\mu = 0$$

Next observe that on the face $|z_i| = \delta$,

$$|\partial z_i| \gg \delta - \epsilon^N C$$

for some constant C provided ∂_N vanishes to first order, which can always be achieved by blowing up. Whence $\epsilon = \delta$ is a good choice, albeit we make:

I.3.3 Remark The important thing here is to have the modulus of $(\partial z_i)^{-1} z_i$ bounded above, and $\delta = \epsilon^\alpha$ any $\alpha \in \mathbb{R}_{>0}$ will work provided N is chosen sufficiently large. In the context of Segre classes, here Lelong numbers, this amounts to the same with weights, but the vanishing of the Segre class without weights implies the vanishing of that with weights. Alternatively blow ups in weighted centres may be resolved by a sequence of modifications in smooth centres. Irrespectively, this kind of estimate is extremely robust, so for ease of notation we will often simply take $\delta = \epsilon$ even though we occasionally use $\delta = \epsilon^\alpha$ for appropriate α in §IV.3/IV.4, and do so without comment.

This said, we therefore have that,

$$\oint_{|x|=\epsilon, |z_i|\leq\epsilon} \frac{dx}{x} d\mu \ll \epsilon^p s_{Z, d\mu}(\epsilon)$$

where the symbol $s_{Z, d\mu}(\epsilon)$ is the part of the Segre class/Lelong number arising from the faces $|z_i| = \epsilon$, at radius ϵ , which is plainly the same thing as the Segre class/Lelong number up to $o(\epsilon) \rightarrow 0$. The residue of any 1-form ω that we have to calculate has the property,

$$\omega d\mu = f_i \frac{dz_i}{z_i} d\mu, \quad \omega d\mu = g \frac{dx}{x^{p+1}} d\mu = 0$$

on the $|z_i| = \epsilon$, respectively $|x| = \epsilon$, faces with $|f_i|, |g|$ bounded, and so:

I.3.4 Fact Let things be as in I.3.1, I the ideal of the singularity, and suppose (as we may after further blowing up) that ∂_N is zero to first order, *i.e.* $\partial \in \mathfrak{m}^2 \text{Der}(\mathcal{O})$, then,

$$\text{RES} : \text{Ext}_U^{n-1}(I, K_{X/\mathcal{F}}) \xrightarrow{\sim} \text{Ext}_U^n(\mathcal{O}/I, K_{X/\mathcal{F}})$$

vanishes on $d\mu$.

I.4 Logarithmic holonomy

Like §1.2 the current section is in-essential, but it explains the holonomy content of the final section, §IV.4. It is, therefore, useful reading either pre or post the same. It also shows how super attracting dynamics occurs for foliations. Our set up is as follows:

I.4.1 Set Up Let Y be a reduced connected 1 dimensional analytic space whose components Y_i are smooth, and whose singularities are at worst nodes. Around Y we have an analytic surface S containing Y as a tubular neighbourhood, and \hat{S} will be the completion of S in Y . The surface S is foliated by way of $S \rightarrow [S/\mathcal{F}]$, all the components of Y are supposed invariant, so, certainly, there are singularities at the intersections of components. Not just these singularities, however, but, for convenience, any other singularities on Y will be supposed reduced in the sense of Seidenberg, albeit this hypothesis is really only necessary at the intersections of components.

Now take a component C of Y with C^* the complement of the foliation singularities on the same. For T a 1-dimensional germ of an analytic space we have the holonomy representation, and its completion,

$$h : \pi_1(C^*) \rightarrow \text{Aut}(T), \quad \hat{h} : \pi_1(C^*) \rightarrow \text{Aut}(\hat{T})$$

Consider the exponential, $\exp : \mathfrak{H} \rightarrow T : \tau \mapsto \exp(\tau) = t$ from the germ of the former around $-\infty$. Understanding by $\text{Aut}_{-\infty}(\mathfrak{H})$ germs of diffeomorphisms preserving $-\infty$ we can define liftings of the above representations as follows: in the first place at the singularities on Y that aren't nodes there is a further branch of the foliation, and this may be supposed convergent, so, say, an invariant divisor D on S . This may fail at nodes, but remains true formally, and, since the latter is our real interest we'll suppose it analytically for ease of exposition. Consequently for Y_C the part of Y omitting C , with S_C a tubular neighbourhood of C , C^* is homotopic to $S_C \setminus (D + Y_C)$, and for $V \rightarrow C^*$ the cover afforded by the holonomy representation inducing Σ over $S_C \setminus (D + Y_C)$ we have a diagram,

$$\begin{array}{ccc} V \subset \Sigma & \xrightarrow{t} & T \\ \downarrow & & \\ C^* \subset S_C \setminus (D + Y_C) & & \end{array}$$

with horizontal map an étale covering of the pair. This implies that the fundamental group of $S_C \setminus (D + Y)$ is canonically an extension,

$$1 \longrightarrow \mathbb{Z}(1) \longrightarrow \pi_1(S_C \setminus (D + Y)) \longrightarrow \pi_1(C^*) \longrightarrow 1$$

or alternatively, if we make no hypothesis on the singularities around C , and take C^\times to be a suitable neighbourhood of C^* punctured in the same, then we have,

$$1 \longrightarrow \mathbb{Z}(1) \longrightarrow \pi_1(C^\times) \longrightarrow \pi_1(C^*) \longrightarrow 1$$

In either case there are liftings,

$$\log h : \pi_1(C^\times) = \pi_1(S_C \setminus (D + Y)) \longrightarrow \text{Aut}_{-\infty}(\mathfrak{H})$$

Formally the situation is more problematic than is desirable, since the “fundamental group of a punctured formal disc” has no sense except in a pro-finite way, even though one has a perfectly good non-profinite theory of fundamental groups of analytic formal schemes. Nevertheless, in the first instance, elements of \hat{h} admit logarithms of the form,

$$\log \hat{h} = \tau + \lambda + f(e^\tau)$$

for f a formal series, $\lambda \in \mathbb{C}$, and these can be composed according to the obvious rules, so, what is true, is that we get an extension,

$$1 \longrightarrow \mathbb{Z}(1) \longrightarrow \widehat{\text{Log}}(C^\times) \longrightarrow \widehat{\text{Hol}}(C^*) \longrightarrow 1$$

for $\widehat{\text{Hol}}$ the image of \hat{h} . In the situation being discussed where S is convergent this subtlety is a bit irrelevant, but if only a formal surface were given (which would have been the case had we intended to really use the present discussion) it's rather critical since $\pi_1(S_C \setminus (D + Y_C))$ would have its obvious sense, but $\pi_1(S_C \setminus (D + Y))$ would be rather problematic.

In any case, these definitions have so far done nothing except replace the action of the holonomy group $\text{Hol}(C^*)$ on T by that of $\text{Log}(C^\times)$ on \mathfrak{H} , and the germ of the champ classifiant $[S/\mathcal{F}]$ at the geometric point determined by C^* , *i.e.* $[T/\text{Hol}(C^*)]$, by the groupoid $[\mathfrak{H}/\text{Log}(C^\times)]$ which is equivalent to $[T^\times/\text{Hol}(C^*)]$, T^\times being T punctured in the origin. Now consider passing from C to D at some singularity, say:

$$\partial = x \frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y}, \quad \text{Re}(\lambda) > 0, \quad D = (y = 0), \quad C = (x = 0)$$

and everything convergent to fix ideas, with transversals T_C, T_D to C and D respectively at y , respectively x , equal to 1. In a minor abuse of notation, put:

$$t = xy^{1/\lambda}, \quad \tau = \log t, \quad s = yx^\lambda, \quad \sigma = \log s$$

and observe that we have inclusions,

$$\text{Log}(D^\times) \xleftarrow{j} \mathbb{Z}(1)^2 \xrightarrow{i} \text{Log}(C^\times)$$

where for generators $a, b \in \mathbb{Z}(1)$,

$$\begin{aligned} i(a) &= \tau + a & j(a)(\sigma) &= \sigma + \lambda a \\ i(b) &= \tau + b/\lambda & j(a)(\sigma) &= \sigma + b \end{aligned}$$

or equivalently one conjugates the representation in τ to that in σ by way of $\sigma = \lambda\tau$, and the condition $\text{Re}(\lambda) > 0$ ensures that neighbourhoods of $-\infty$ in τ go to the same in σ . As such we may extend through the singularity in the obvious Seifert-Van Kampen-esque way, *i.e.*

$$\text{Log}(C^\times \cup D^\times) := \text{Log}(C^\times) *_{\mathbb{Z}(1)^2} \text{Log}(D^\times)$$

which of course is perfectly consistent with,

$$\pi_1(C^\times \cup D^\times) := \pi_1(C^\times) *_{\mathbb{Z}(1)^2} \pi_1(D^\times)$$

and the representation h extends, or, similarly at the level of $\widehat{\text{Log}}$ alone if our context were purely formal. It is furthermore true that this correctly reflects the dynamics of the foliation, *i.e.* the composition $\mathfrak{H}_C \rightarrow T_C \rightarrow [X/\mathcal{F}]$ is étale, and we get a diagram,

$$\begin{array}{ccc} H_C & \longleftarrow & R \\ \downarrow & & \downarrow \\ [C^\times \cup D^\times]/\mathcal{F} & \longleftarrow & H_C \end{array}$$

such that the germ of the étale groupoid $R \rightrightarrows \mathfrak{H}_C$ around $-\infty$ is,

$$\text{Log}(C^\times \cup D^\times) \times \mathfrak{H}_C \rightrightarrows \mathfrak{H}_C$$

with the above action. Plainly this doesn't make sense if $\text{Re}(\lambda) \leq 0$, or the singularity is a node. Equally plainly the hypothesis that the singularity is analytically linearisable isn't actually necessary and just the given form to 1st order will do, provided we take the actual conjugation between \mathfrak{H}_C and \mathfrak{H}_D given by the foliation. The formal situation is both easier, and more difficult, *i.e.* for λ irrational the singularity is formally linearisable and everything is as above, otherwise in the notation post III.4.1(c),

$$s^l = t^k \exp\left(\sum_{m=1}^{\infty} t^{krm} R_m(\log t)\right)$$

for R_m polynomial, and we can continue to give a formal sense to maps preserving $-\infty$, and how to compose them.

Next we form the dual graph G of Y , or more correctly the dual graph of the sub-curve were all the singularities at intersections of components is as above, *i.e.* a vertex for each component and an edge whenever the singularity has the above form at the intersection of components. In particular for each edge e there are mutually inverse automorphisms, $\phi_{+-}(e)$, $\phi_{-+}(e)$, in $\text{Aut}_{-\infty}(\mathfrak{H})$ such that,

$$\tau_- = \phi_{-+}(\tau_+), \phi_{+-}(e) : v_{\text{longleftarrow}} \text{ftrightarrow} v_+ : \phi_{-+}(e), \tau_+ = \phi_{+-}(\tau_-)$$

describes the conjugation from a logarithmic transversal at one vertex to the other. A loop γ in G is simply a series of directed edges bringing us back to the same vertex, so, in a minor abuse of notation, we have a representation,

$$\theta : \pi_1(G) \longrightarrow \text{Aut}_{-\infty}(\mathfrak{H}) : \gamma \mapsto \prod_{e \in \gamma} \phi(e)$$

Let us abuse notation a bit further to put things together, *i.e.* suppose Y , G connected and let Y^\bullet be the union over components, C , of the C^* together with

the singularities corresponding to the edges of G , and Y^\times a neighbourhood of Y^\bullet punctured in the same, then we have a diagram of representations,

$$\begin{array}{ccc} \pi_1(Y^\times) & \xrightarrow{\log h} & \text{Aut}_{-\infty}(\mathfrak{H}) \\ \downarrow & & \parallel \\ \pi_1(G) & \xrightarrow{\theta} & \text{Aut}_{-\infty}(\mathfrak{H}) \end{array}$$

and this action accurately reflects the holonomy groupoid defining the foliation $S \rightarrow [S/\mathcal{F}]$, *i.e.* the germ of the induced étale groupoid $R \rightrightarrows \mathfrak{H}$ about $-\infty$ arising from the restriction to Y^\times is the action of the image $\text{Log}(Y^\times)$ of $\log h$ on \mathfrak{H} . Similarly in the formal case, albeit purely in terms of a subgroup $\widehat{\text{Log}}(Y^\times)$ of automorphisms. By way of clarity let us offer,

I.4.2 Example Take for Y an elliptic gorenstein foliation singularity, *cf.* [M1], with more than one components, with say $-n_i = Y_i^2$ the self intersection of the components. The components Y_1, \dots, Y_d , each isomorphic to \mathbb{P}^1 , are ordered as a cycle: Y_i meeting only Y_{i-1}, Y_{i+1} at \circ_i , respectively $\infty_{i-d+1} = 1-$ and for y_i a coordinate along Y_i at \circ_i with x_i the normal the foliation has a generator ∂_i around the affine line through \circ_i ,

$$\partial_i = y_i \frac{\partial}{\partial y_i} - \lambda_i x_i \frac{\partial}{\partial x_i}$$

where the above are related by,

$$x_{i+1} = y_i^{-1}, \quad y_{i+1} = y_i^{n_i} x_i, \quad \lambda_{i+1} = (n_i - \lambda_i)^{-1}, \quad \partial_{i+1} = \lambda_i \partial_i$$

As such, for $t_i = x_i y_i^{\lambda_i}$ we have a cycle of transformations of half planes,

$$\mathfrak{H}_{i-1}, \tau_{i-1} = \log t_{i-1} \xrightarrow{\lambda_{i-1}} \mathfrak{H}_i, \tau_i = \log t_i \xrightarrow{\lambda_i} \mathfrak{H}_{i+1}, \tau_{i+1} = \log t_{i+1}$$

while each $\pi_1(Y_i^*)$ is (canonically) $\mathbb{Z}(1)^2$. The topology being as simple as possible the cycle $\mathbb{Z} = \pi_1(G)$ acts on $\mathbb{Z}(1)^2$ and $\pi_1(Y^\times)$ is a semi-direct product,

$$1 \longrightarrow \mathbb{Z}(1)^2 \longrightarrow \pi_1(Y^\times) \longrightarrow \mathbb{Z} \longrightarrow 1$$

the so called “isotropy group of the cusp”, whose representation in $\text{Aut}_{-\infty}(\mathfrak{H})$ is faithful, and in terms of generators a, b of $\mathbb{Z}(1)^2$ is given by the matrices,

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \lambda_1 b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 \dots \lambda_d & 0 \\ 0 & 1 \end{bmatrix}$$

and $\lambda_1 \dots \lambda_d$ is an automorphism of the image of $\mathbb{Z}(1)$ in this representation. Indeed, each λ_i is a quadratic irrationality, *e.g.* we have the continued fraction,

$$\lambda_1 = \frac{1}{n_d - \frac{1}{n_{d-1} - \frac{1}{\dots - \frac{1}{n_2 - \lambda_1}}}}$$

In particular, to see that the representation is faithful one observes, [V] II.2.1, that while the implied basis $a, b\lambda_1$ of the kernel to the cycle is not canonical, there is an infinite set of such choices, which is canonical, and on which \mathbb{Z} acts faithfully by way of $\lambda_1 \dots \lambda_d$. Furthermore, identifying, as above, the representation of $\pi_1(G)$, whence of $\pi_1(Y^\times)$, with a cyclic subgroup of $\mathbb{R}_{>0}^\times \subset \mathbb{G}_m$ this is also the representation afforded by the log canonical bundle $K_{S/\mathcal{F}}$ which, while trivial on each Y_i is globally not so. Indeed dually to the ∂_i one has 1-forms,

$$\omega_i = \lambda_i \frac{dy_i}{y_i} + \frac{dx_i}{x_i}, \quad \omega_i = \lambda_{i+1} \omega_{i+1}$$

As such, elliptic gorenstein is a slight misnomer, *i.e.* although $K_S + Y$ is trivial on the cycle and descends to a bundle on the contraction, since,

$$K_S = K_{\mathcal{F}} + K_{S/\mathcal{F}}$$

and the latter term isn't even trivial as a bundle on Y , let alone S , by way of the above, the canonical $K_{\mathcal{F}}$ is never \mathbb{Q} -Gorenstein on the contraction despite being torsion on a neighbourhood of each component Y_i .

While straightforward this example is rather instructive. Specifically: the champ classifiant $[Y^\times/\mathcal{F}]$ has the logarithmic holonomy not just as a germ, but it is $[\mathfrak{H}/\Gamma]$ for Γ a finite sub-group of $\text{Aut}(\mathfrak{H})$ (not just $\text{Aut}_{-\infty}(\mathfrak{H})$) with, say, generators as above. As such, it certainly admits at least one invariant measure, *i.e.* the Poincaré metric on \mathfrak{H} . This, in turn, admits the potential, $\log |\log |t_1|^2|$ which, while invariant by the action of the $\mathbb{Z}(1)^2$ is not so by the cycle of the graph which acts in a super attracting way,

$$\log |t_1|^2 \mapsto (\lambda_1 \dots \lambda_d) \log |t_1|^2$$

In particular the assertion, valid for the usual holonomy, that, at least formally, it must take values in S^1 in the presence of an invariant measure is false for the logarithmic since in the logarithmic variable the cycle $\pi_1(G)$ even has, already at first order, the faithful representation in \mathbb{G}_m ,

$$\gamma \mapsto \prod_{e \in \gamma} \lambda(e)$$

for (directed) multipliers $\lambda(e)$ at the edges as above. Let us, therefore, conclude by way of,

I.4.3 Remark/Summary The problem of the non-triviality of the representation of $\pi_1(G)$ appears in a closely related form in the final section §IV.4. The difficulty that it poses is not dealt with by way of the relevant (formal) logarithmic holonomy, but by a more algebraic alternative. Ideally it would have been treated as an extension of the basic construction III.1.1 to the logarithmic case (notice, by the way, there being no contradiction between III.1.1 and any of the above since logarithmic co-equalisers must be calculated logarithmically, *i.e.* with subsets of half planes). Unfortunately, unlike the current ideal case which

is akin to the measure being supported in the centre manifold, §III-IV are properly “almost holonomy” rather than holonomy. In particular there is always an error associated with the multiplicities $p(v)$ at the vertices of G as encountered in §IV.4, and this error is potentially very non-uniform from vertex to vertex. As a result, I couldn’t get what should have been the right estimate, *i.e.* up to the error, cf. §IV.4, $\epsilon^{p(v)}o(\epsilon)$, the measure is radially symmetric. Such an estimate would have rendered the global residue calculation purely topological, and achieved wholly by the method of “almost holonomy”, as opposed to the more algebraic considerations which are employed to treat the potential difficulty of many singularities of type III.4.1 (b)/(c).

II. Particular Singularities

II.1 Generically Transverse

We begin with an investigation of certain singularities where the relation between the formal and analytic structure is particularly good. In the first instance so much so that there is no need to restrict the dimension, *i.e.*

II.1.1 Set Up In a neighbourhood U of a $2+n+m$ dimensional complex space, $m, n \in \mathbb{N} \cup \{0\}$, about a singular point of the foliation in the completion \hat{U} in the singular locus the foliation admits a formal generator of the form,

$$z \frac{\partial}{\partial z} + \frac{y_1^{q_1} \dots y_n^{q_n} x^{p+1}}{p(1 + \nu(y, w)x^p)} \frac{\partial}{\partial x}$$

where x, y_i, z, w_j are formal coordinates, $p, q_i \in \mathbb{N}$, albeit $1 \leq i \leq n, 1 \leq j \leq m$, so the y_i , respectively w_j , and whence the q_i , are suppressed should m or n be zero.

Under these hypothesis, and possibly after blowing up in the singular locus, a sufficiently small open neighbourhood of the singularity admits for any value of the arguments of x or y_i an open neighbourhood of the form, $V = S \times S_1 \times \dots \times S_n \times \Delta \times \Delta^m$, *i.e.* discs for each of the coordinates z, w_j , and sectors $x \in S, y_i \in S_i$ such that $\xi = x^{-p} y_1^{-q_1} \dots y_n^{-q_n}$ has an argument up to 2π and a branch within $\pi/2$ of -1 such that: on the same we may find an analytic generator of the foliation together with a conjugation to the normal form II.1.1, [MP2] VI.3.3(c). Given this we may easily integrate the foliation. In the first place by way of a conformal change of variable in S we may suppose $\nu = 0$, and so obtain invariant functions,

$$s = z \exp(\xi), y_i, w_j \quad 1 \leq i \leq n, 1 \leq j \leq m$$

By way of notation, let us denote the invariant functions y_i, w_j by a vector \mathbf{t} of such. As a result if Z_ζ is the transversal $z = \zeta \neq 0$, then \mathbf{t} fibres Z_ζ over T , say, with fibre embeddable on the omission of the exceptional divisor $E, x = 0$ or $y_i = 0$, in \mathbb{C} by ξ , *i.e.*

$$\begin{array}{ccc} Z_\zeta^* := Z_\zeta \setminus E & \xrightarrow{\xi \times \mathbf{t}} & \mathbb{C} \times T \\ \downarrow \mathbf{t} & & \\ T & & \end{array}$$

where all the maps are open, and the horizontal one an embedding. On \mathbb{C} we have the action of $\mathbb{Z}(1)$ by translations, and the groupoid,

$$R \rightrightarrows Z_\zeta^*$$

induced by the foliation is the pull-back of the groupoid given by the action of $\mathbb{Z}(1)$ on $\mathbb{C} \times T$. As such while far from essential in the immediate context let

us reduce the question of the existence of invariant measures to purely fibrewise considerations in \mathbf{t} , to wit:

II.1.2 Lemma Let $\pi : X \rightarrow B$ be a map of metric spaces, and $d\mu$ a measure on X of finite mass, then there is a measure $d\nu$ on B and a family of measures $b \mapsto d\lambda_b$ with support in $X_b = \pi^{-1}(b)$ such that for any bounded continuous function f on X , $b \mapsto d\lambda_b$ is bounded and,

$$\int_X f d\mu = \int_B d\nu \int_{X_b} f d\lambda_b$$

proof (for want of a reference) Define $d\nu$ by the obvious formula,

$$d\nu(g) = \int_X (\pi^* g) d\mu$$

then for any bounded continuous function f ,

$$g \mapsto \int_X (\pi^* g) f d\mu$$

is absolutely continuous with respect to $d\nu$ and admits a $L_\infty(d\nu)$ derivative, $D(f)$. By [F] 2.9.8 there is a formula for this, *viz*: let $b \in B$, $B_\epsilon(b)$ a ball of radius ϵ about it and $\nu_b(\epsilon)$ its ν measure, then,

$$D(f)(b) = \lim_{\epsilon \rightarrow 0} \frac{1}{\nu_b(\epsilon)} \int_{\pi^{-1}(B_\epsilon(b))} f d\mu$$

which, here, is in fact defined and bounded not just ν -a.e. but everywhere. \square

We can, and will, therefore, think of $d\mu$ as $d\nu(b)d\lambda_b$, and, of course, modulo the obvious change of notation $b \mapsto \mathbf{t}$, we have:

II.1.3 Corollary/Definition For a $R \rightrightarrows Z_\zeta^*$ invariant measure $d\mu$ described fibrewise by $d\nu(\mathbf{t})d\mu_{\mathbf{t}}$ as per II.1.2, $d\mu$ is invariant iff the fibres $d\mu_{\mathbf{t}}$ are invariant for ν almost all \mathbf{t} .

There are, of course, lots of measures on \mathbb{C} invariant by translations in $\mathbb{Z}(1)$, so the only way to exclude the existence of $d\mu$ is by considerations of mass, and this is plainly the case. Indeed each point in each fibre admits a neighbourhood, all translations of which are disjoint, and there are infinitely many such. Whence every $d\mu_{\mathbf{t}}$ is zero, and since every leaf off the centre manifold $z = 0$ meets Z_ζ we conclude,

II.1.4 Fact Let things be as in II.1.1, modulo blowing up in the singular locus to ensure that $x = 0$ or $y_i = 0$ is the exceptional divisor and an analytic conjugation on the domain V to the normal form, then an invariant measure for the foliation on V lies either in the exceptional divisor or the centre manifold.

In the particular case that $m = n = 0$, one may appeal to [S] to conclude that the centre manifold exists in all of U . In general, however, such reasoning is not valid, and we must appeal to the Segre class around the exceptional divisor $x = 0$. The definition of Segre class is only relevant to the part of the measure off the divisor, so, henceforth this precision may be omitted. For any

sub-variety, and in some generality, [M4] §III, it is defined as soon as the sub-variety is compact, but, post factum may be localised. As such, supposing U a neighbourhood of some X in which the singular locus is compact, and letting $f = 0$ be an equation for the divisor $x = 0$ in all of U with E the exceptional divisor, the Segre class $s_{E,d\mu}^X$ dominates:

$$s_{E,d\mu}^U := \lim_{\epsilon \rightarrow 0} s_{E,d\mu}^U(\epsilon) := \oint_{|f|=\epsilon} \frac{df}{f} d\mu$$

Now, the centre manifold in V embeds in $\mathbb{C} \times T$ by way of $f \times \mathbf{t}$, and $d\mu$ descends to a measure $d\mu(\mathbf{t})$ in \mathbf{t} - not to be confused with $d\mu_{\mathbf{t}}$ - whence:

$$s_{E,d\mu}^U = \frac{|S|}{2\pi} \int_T d\mu(\mathbf{t})$$

where $|S|$ is the aperture of the sector S . Consequently:

II.1.5 Corollary Suppose U is a neighbourhood in a complex space or champ de Deligne-Mumford analytique in which the Segre class of an invariant measure around the hypothesised compact singular locus in the ambient space is zero, then modulo appropriate blowing up, an invariant measure for the foliation $U \rightarrow [U/\mathcal{F}]$ of implied type found in II.1.1 is zero (remember, everything in the exceptional divisor is ignored).

One should observe that in terms of our goal of relating residues to Segre classes, the hypothesis of zero Segre class is unavoidable, *i.e.*

II.1.6 Remark Already in dimension 3, even supposing the centre manifold perfectly analytic one might encounter a residue of the form,

$$\lim_{\epsilon \rightarrow 0} \int_{|xy|=\epsilon} (1 + \nu x^p) \frac{dx}{x^{p+1}y^q} d\mu$$

for say $\nu \in \mathbb{C}^\times$, which in the above notation would become,

$$\lim_{\epsilon \rightarrow 0} \nu \int_{T,|y|=\epsilon} y^{-q} d\mu(\mathbf{t})$$

which admits no a priori reduction in terms of the Segre class about $y = 0$ or $x = 0$.

II.2 Linear and unbounded

Amongst all singularities in the centre manifold those that are linearisable are rather generic. Nevertheless, the behaviour in the ambient 3-fold is dependent on the eigenvalue of the linearisation. A case where this behaviour is as simple as possible is given by,

II.2.1(a) Set Up In a neighbourhood U of foliated 3-fold of a point in the singular locus after completion in the same, the foliation admits a formal generator of the form,

$$z \frac{\partial}{\partial z} + \frac{x^p}{p(1 + \nu x^p)} \left(x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \right)$$

where x, y, z are formal coordinates, $p \in \mathbb{N}$, $\lambda, \nu \in \mathbb{C}$, $\lambda \neq 0$, $\operatorname{Re}(\lambda) \geq 0$. Again, after blowing up if necessary, we may suppose that $x = 0$ defines the exceptional divisor E .

Here by [MP2] VI.4.3 the relation between the formal and analytic structure is straightforward, *viz*: there is an open neighbourhood of the form, $V = S \times \Delta^2$, with x in a sector S of aperture up to $2\pi/p$ and $x \mapsto \xi = x^{-p}$ branching within $\pi/2$ of the negative real axis, while y, z vary in discs, such that over V one may achieve an analytic conjugation to the normal form. Again, after a conformal change of variable in S we may suppose $\nu = 0$, and so obtain invariant functions,

$$s = z \exp(\xi), \quad t = y \xi^{\lambda/p}$$

As before, for fixed ζ we introduce the transversal $Z_\zeta : z = \zeta$, and its complement. Z_ζ^* by the exceptional divisor, which we fibre by way of t over T to obtain,

$$\begin{array}{ccc} Z_\zeta^* := Z_\zeta \setminus E & \xrightarrow{\xi \times t} & \mathbb{C} \times T \\ \downarrow t & & \\ T & & \end{array}$$

where, again, all maps are open, and the horizontal one an embedding. Once again the groupoid,

$$R \rightrightarrows Z_\zeta^*$$

induced by the action of the foliation in V on Z_ζ^* is that obtained by pull-back of the groupoid in which $\mathbb{Z}(1)$ acts on $\mathbb{C} \times T$ by translations on the first factor.

Now we can argue as per II.1.4. Every leaf outside the centre manifold and the exceptional divisor meets Z_ζ^* . The structure of the fibres is straightforward, [MP2] §III.1, and every point in every fibre over t not only has an infinite orbit under $\mathbb{Z}(1)$ but admits a neighbourhood all of whose orbits are disjoint. Whence by II.1.2 we obtain,

II.2.2 Fact Let the set up be as per II.2.1.(a), then on the domain $V = S \times \Delta^2$, any invariant measure for the foliation on V lies either in the exceptional divisor, $x = 0$, or the centre manifold, $z = 0$.

Our next task is to consider the structure inside the centre manifold. This is trickier than one might think since apart from the fact that we can only conjugate to the normal form for x in a sector, the invariant manifold $y = 0$ need not exist in all of U but only in a sector of width up to $3\pi/p$, [MP2] §V.2 & §VI.4. Nevertheless, as might be expected, we assert:

II.2.3 Claim If $\lambda \notin \mathbb{R}_+$ then in addition the support of any invariant $d\mu$ off the exceptional divisor is the curve $z = y = 0$, whence the latter is convergent in all of U by [S].

proof Cover U (more correctly $U \setminus E$) by domains $V_\alpha = S_\alpha \times \Delta_\alpha^2$ where we can conjugate to normal form, and denote by s_α, t_α the invariant functions previously constructed. Plainly for any leaf ℓ in the centre manifold W_α of V_α , $s_\alpha(\ell) = 0$.

For convenience choose the V_α such that α goes from 1 to $n + 1$, V_α meets only $V_{\alpha-1}$ and $V_{\alpha+1}$, the overlaps are simply connected, and we identify 1 with $n + 1$. A priori on $V_\alpha \cap V_{\alpha+1}$, t_α is a function of $t_{\alpha+1}$ and $s_{\alpha+1}$. If, however, we consider t_α restricted to the centre manifold $W_{\alpha+1}$ then it is a function of t_α alone. Better still if $\text{Re}(\lambda) > 0$, then $t_{\alpha+1}$ maps $W_{\alpha+1} \setminus E$ onto \mathbb{C} , and every leaf in $W_{\alpha+1}$ meets V_α , so $t_\alpha|_{W_{\alpha+1}}$ is an entire function $h_{\alpha,\alpha+1}$ of $t_{\alpha+1}$. The evident composition $h_{1,2}h_{2,3} \dots h_{n,n+1}$ leads to an entire function h of t_1 which generates (a possibly complicated, and, a priori no better than flat) groupoid,

$$H \rightrightarrows \mathbb{C}$$

The fact that the W_α need not patch implies that this groupoid may have little to do with the action of the foliation in U . However, by II.2.2, we may not only descend a hypothesised invariant measure to \mathbb{C} by way of t_1 , but it must also be left invariant by H .

Now by [MP2], one may suppose that the formal and analytic conjugations coincide modulo arbitrary, but fixed, powers of the exceptional divisor, *i.e.* not quite a full asymptotic expansion since *op. cit.* is a bit lazy on this score. In terms of the t_α this translates into asymptotics at ∞ , so, in the first place each $h_{\alpha,\alpha+1}$ sends ∞ to itself, so, in fact, all of these, and whence h is a polynomial. Furthermore h is étale at ∞ , whence it's the first term in the asymptotics, *i.e.* H is the action of $\mathbb{Z}(1)$ by $\exp(\lambda \bullet)$. Finally, we can cover by transversals T_i in V_1 of finite mass, whose image under t_1 is surjective, whence the hypothesised invariant measure is one of finite mass on \mathbb{C} invariant by the given action of $\mathbb{Z}(1)$, which, except for the δ -function at the origin, is nonsense.

It remains to treat the case of $\lambda \in \mathbb{R}(1)$, which is a little easier since there is actual holonomy about $x = 0$ in the sectors themselves. As a function of the orientation, *i.e.* $\text{Im}(\lambda) > 0$, or $\text{Im}(\lambda) < 0$, its either contracting or expanding, so, modulo a change of orientation, without loss of generality contracting. Consequently, for a sector $S \ni x$ transversal to the said curve, the measure is invariant by a groupoid,

$$H \begin{array}{c} \xrightarrow{\tau} \\ \xrightarrow{\sigma} \end{array} S$$

generated by a map $h : S \rightarrow S$ with Taylor series,

$$h(x) = e^{-2\pi|\lambda|x}(1 + O(x^N))$$

for $N \in \mathbb{N}$, which we could take to be infinite, but we'll suppose N finite to indicate the robustness of such a situation. Indeed for ϵ small we find a co-equaliser (*i.e.* $\sigma_* - \tau_*$ of compactly supported) of the form,

$$\mathbb{1}_{|x|>q\epsilon} - \mathbb{1}_{|x|>\epsilon}$$

for q as close to $e^{-2\pi|\lambda|}$ as we please- *i.e.* the closer the smaller the disc. Whence there is no measure in the annular sector $q\epsilon > x > \epsilon$, $x \in S$ for ϵ small, but, otherwise, arbitrary, so, the transversal has no measure off the exceptional divisor. In this case every T_α is a disc, which, unlike the previous case may

be shrunk, so, without loss of generality such transverse sectors cover each T_α minus the origin.

Of course, in either case, the δ -function at the origin, equivalently the curve $y_\alpha = 0$ is only defined in V_α , so we use [S] to conclude. \square

In the case that $\lambda \in \mathbb{R}_+ \setminus \mathbb{Q}_+$, a so called reduced singularity, the above discussion implies that any invariant measure is a radially symmetric invariant measure on \mathbb{C} pulled back by t_1 . Nevertheless, such a measure could still be inconvenient, so we again appeal to the Segre class, *viz*:

II.2.4 Fact Suppose the U of II.2.1.(a) is actually a neighbourhood in a complex space or champ de Deligne-Mumford analytique in which the Segre class of an invariant measure around the hypothesised compact singular locus in the ambient space is zero, then modulo appropriate blowing up, an invariant measure for the foliation $U \rightarrow [U/\mathcal{F}]$ of the implied type is zero.

proof For $\lambda \notin \mathbb{R}_+$ this follows a fortiori from II.2.3. Otherwise, with notations as per II.1.5,

$$s_{E,d\mu}^U \geq \lim_{\epsilon \rightarrow 0} \int_{t \in T} d\mu(t) \oint_{V_t, |f|=\epsilon} \frac{df}{f}$$

The condition $|f| = \epsilon$ is as near $|\xi| = \epsilon^{-p}$ as makes no difference for a sufficiently high a priori approximation to the normal form, so the integrand in t (really t_1) is an increasing function of ϵ which gives the aperture of S over 2π if, $\log |t| < \text{const.} - \log \epsilon$, and close to zero otherwise. As such, the dominated convergence theorem applies, so (shrinking T a little if $\lambda \in \mathbb{R}(1)$),

$$s_{E,d\mu}^U \geq \frac{|S|}{2\pi} \int_T d\mu(t)$$

where, as before, we've profited from the support being in the centre manifold to descend the measure $d\mu$ to T . \square

We require to extend this rather satisfactory discussion to

II.2.1.(b) Set Up Everything as per II.2.1.(a) but now the normal form of a generator is,

$$(p + q\lambda)z \frac{\partial}{\partial z} + \frac{x^p}{1 + \nu x^p} \left(x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \right)$$

where x, y, z are formal coordinates, $p, q \in \mathbb{N}$, $\lambda, \nu \in \mathbb{C}$, $\lambda, p + q\lambda \neq 0$, $\text{Re}(p + q\lambda), \text{Re}(p/\lambda + q) \geq 0$, with suitable a priori blowing up so that $x = 0, y = 0$ are exceptional divisors, E_1, E_2 , and $E = E_1 + E_2$.

Here the best existence domains are obtained by passing to logarithmic coordinates, or, better still, viewing $\exp \times \exp : \mathfrak{H} \times \mathfrak{H} \rightarrow \Delta^2$ as an étale neighbourhood of $\Delta^2 \setminus E$, for $\Delta^2 \ni (x, y)$. In particular, the most symmetric analogue of the previous variable t is to introduce,

$$\tau := \frac{1}{p + q\lambda} \log y - \frac{\lambda}{p + q\lambda} \log x$$

then fibre $\mathfrak{H} \times \mathfrak{H}$ over T by τ , which can subsequently be embedded in $\mathbb{C} \times T$

by way of,

$$\begin{array}{ccc} \mathfrak{H} \times \mathfrak{H} & \xrightarrow{(p \log x + q \log y) \times \tau} & \mathbb{C} \times T \\ \tau \downarrow & & \\ T & & \end{array}$$

at which point one may describe the existence domain V for a conjugation to the normal form as the pull-back to $\mathfrak{H} \times \mathfrak{H}$ of a domain in which $\log \xi := -(p \log x + q \log y)$ varies in a strip of width up to 2π resulting in a branch in the ξ plane within $\pi/2$ of the real axis, and, of course, multiplied by a disc in the variable z , cf. [MP2] §III.2.

Consequently, again after a conformal mapping in ξ , we now have invariant functions,

$$s = z \exp(\tau), \quad \tau$$

As such, we again look to the transversal, $Z_\zeta^* := Z_\zeta \setminus E$, factor it as,

$$\begin{array}{ccc} Z_\zeta^* := Z_\zeta \setminus E & \xrightarrow{\xi \times \tau} & \mathbb{C} \times T \\ \downarrow t & & \\ T & & \end{array}$$

and observe, once more, that every leaf outside the centre manifold and the exceptional divisor meets Z_ζ^* , so the fibres $d\mu_\tau$ of a hypothesised invariant measure are, again, invariant by the groupoid induced from translations by $\mathbb{Z}(1)$ in the ξ -variable.

Whence, we are again reduced to studying the possible structure inside the centre manifold. Observe that the hypothesis exclude the possibility that $\lambda \in \mathbb{R}$, and the analogue of $\lambda \in \mathbb{R}(1)$ encountered in II.2.1.(a) is $p + q\lambda \in \mathbb{R}(1)$ or $p/\lambda + q \in \mathbb{R}(1)$, albeit at most one of these can occur. Now we argue as in the proof of II.2.3, descending the measure to a measure $d\mu(\tau)$ which is invariant by a groupoid,

$$H \rightrightarrows T$$

generated by two mappings, A, B , - *i.e.* as per op. cit. but coverings indexed for sectors in x and y - entire if neither $p + q\lambda \notin \mathbb{R}(1)$ nor $p/\lambda + q \notin \mathbb{R}(1)$, while the domain of τ and the said mappings are right, respectively left, half planes otherwise. In either case, as before, we know by [MP2] the form of the mappings modulo some large, but fixed, power of the exceptional divisor, *i.e.* for a, b generators of $\mathbb{Z}(1)^2$ they have the form,

$$\begin{aligned} A : T &\longrightarrow T : \tau \mapsto \tau + \frac{a}{p + q\lambda} + O((x^p y^q)^N) \\ B : T &\longrightarrow T : \tau \mapsto \tau + \frac{b\lambda}{p + q\lambda} + O((x^p y^q)^N) \end{aligned}$$

Unfortunately this error doesn't quite translate into an expansion at ∞ in the τ variable. As such, put, $Y = (p + q\lambda)\tau$, then for $\text{Re}(Y) \rightarrow \infty$ in strips with

$\text{Im}(Y)$ bounded, the induced groupoid has asymptotically the same generators as that induced by translations in $\mathbb{Z}(1)$ and $\mathbb{Z}(1)\lambda$. Whence taking the strip to have width in the imaginary direction at least 2π , and $\text{Re}(Y) < -R$ for some sufficiently large (determined by $\text{Im}(\lambda) \neq 0$) R each point in such a strip has an open neighbourhood with an orbit enjoying infinitely many connected components, each with mass bounded below if the measure were supported at the point in question. The strip has, however, finite mass, so this is nonsense. Arguing similarly in the x -variable, and possibly shrinking the initial neighbourhood U if necessary, we cover all of T by points with orbits meeting such strips, and deduce:

II.2.5 Fact Suppose we are in the situation II.2.1.(b) then, without any hypothesis on the Segre class, any invariant measure must be supported in the exceptional divisor.

II.3 Exceptional nodes

A saddle node singularity inside the exceptional divisor is an obvious source of concern, and, again, like linearisable singularities the relation between formal and analytic theory is not always straightforward. A particular case where it is not too complicated is when the centre manifold of the node is an exceptional divisor, *i.e.*

II.3.1 Set Up In a neighbourhood U of foliated 3-fold of a point in the singular locus after appropriate blowing up, we have a singularity such that on completion in the point (not, as before, completion in the singular locus) we find a formal generator of the form,

$$qz \frac{\partial}{\partial z} + \frac{x^p y^q}{1 + y^q(R(x) + \lambda x^{p+r})} \left(y \frac{\partial}{\partial y} + \frac{x^r}{1 + \nu x^r} (qx \frac{\partial}{\partial x} - py \frac{\partial}{\partial y}) \right)$$

where x, y, z are formal coordinates, $p, q, r \in \mathbb{N}$, $\lambda, \nu \in \mathbb{C}$, $\deg R \leq r - 1$, and the exceptional divisor E has 2 components $E_1 : x = 0$, $E_2 : y = 0$, and we may even assume that the normal form is defined after completion in the former, albeit not the latter.

The analytic theory has a number of complications. In the first place it is convenient to have all of λ, ν, R equal to zero, a so called *monomialised* form. This may be achieved, [MP2], VI.2.2.(g), once and for all, *i.e.* depending only on the said parameters, in a sub-domain of $S \times \Sigma \times \Delta$ where x varies in a sector of aperture up to $2\pi/r$, z in the disc Δ , and y , or better $\log y$, in a so called spiralling domain: $\log y$ in a sector of width up to (which means strictly less than) π wholly contained in the left half plane. Having effected this change of variables on some coordinates affording a sufficiently large approximation to the normal form, we restrict our attention to the domain away from the negative real-axis in $w = x^{-r}$, *i.e.* $|\text{Im}(w)| > R$, or $\text{Re}(w) > R$, R large- otherwise leaves may be bounded, and invariant measures may exist. Now the further restrictions we must make to achieve a conjugation to the monomialised form are, for ξ , again equal to $(x^p y^q)^{-1}$, we must, necessarily, restrict it to a sector S_ξ

(or, better $\log \xi$ is restricted to a strip) of apertures up to 2π branched within $\pi/2$ of the negative real axis, in addition the parameter w can vary through at most $3\pi/2$, or, equivalently, away from the negative real axis in the same, from above it we can go to within the lower imaginary axis, while from below to within the upper imaginary axis. This yields, [MP2] VI.4.5, a domain V such that after the formal to analytic conjugation of the variables,

$$s = z \exp(\xi), \quad \tau = w - \log \xi$$

are invariant functions. Now we can argue exactly as before: restrict to a transversal, fibre in τ observe, cf. [MP2] §IV.4, that the condition of being away from the negative real axis in w is exactly what guarantees that the fibres in τ embedded by ξ in \mathbb{C} contain, apart from the branch, full neighbourhoods of ∞ , and the discrete action by translation of $\mathbb{Z}(1)$ on the same leaves the measure invariant, apply II.1.2, and conclude,

II.3.2 Fact Suppose we are in a domain V as above, of which the critical feature is being bounded away from the negative real axis in x^{-r} , then the support of any invariant measure is contained in the centre manifold or the exceptional divisor.

Now we must aim to study the situation inside the centre manifold, which risks being a little complicated since we have as many of these as we have determinations of $\log \xi$. As such fix analytic functions x, y in U defining the exceptional divisor, and for $\log(x^p y^q)^{-1}$ in a strip α with branching as described above, denote by V_α the corresponding existence domain where the conjugation to monomial/normal form is valid, with Z_α its centre manifold and τ_α as above. The domain T_α of τ_α is all of \mathbb{C} , and as before if $\alpha \cap \beta \neq \emptyset$ then every leaf in Z_α meets V_β , and conversely. Whence we may write,

$$\tau_\alpha|_{Z_\beta} = h_{\alpha\beta}(\tau_\beta)$$

for $h_{\alpha\beta}$ entire. Identifying this function is easy since an a priori choice of approximation to the normal form translates into a bound at ∞ with knowledge of a finite number of terms in the Taylor expansion. Whence, say for convenience, all strips of the same width $|S|$, then there are $\lambda_\alpha \in \mathbb{R}(1)$ such that,

$$h_{\alpha\beta} = \tau_\beta + (\lambda_\beta - \lambda_\alpha) + O(R^{-N}),$$

N as large as we like, so $h_{\alpha\beta}$ is translation by $\lambda_\beta - \lambda_\alpha$. Observe that the intersection W_α of V_α with Z_α is embedded via:

$$(w, \log(x^p y^q)^{-1}) : W_\alpha \hookrightarrow S \times L$$

where $S \subset \mathbb{C}$ is defined by way of our sectorial and modular restrictions in w , with L the surface of the logarithm, and the above procedure allows us to extend say τ_0 from W_0 , 0 a strip, to τ on all of $S \times \mathbb{C}$. Again this map would have little or nothing to do with the ambient foliated 3-fold were it not for the fact that the hypothesised invariant measure $d\mu$ is necessarily supported in Z_α , while the existence domain V_α is naturally a sub-domain of $\Delta \times S \times L$, and the pull-back

of the measure to the latter is invariant by translations under $\mathbb{Z}(1)$ on the last factor. As such, pulling back τ by the projection $\Delta \times S \times L \rightarrow S \times L$, it follows that $d\mu$ not only descends to a measure $d\mu(\tau)$ on T_0 , but it is also invariant by translations in $\mathbb{Z}(1)$. Finally, the image under τ of a transversal, say $S \times 0$ has finite mass, and each point an open neighbourhood with an infinite disjoint orbit, so the said image has zero mass, while every point in T_0 has a point of this image in its orbit, so, indeed the measure on T_0 is zero, and we deduce:

II.3.3 Fact Suppose we are in the set up II.3.1, and, in a minor notational confusion, suppose x is an analytic function in U defining the exceptional divisor E_1 , then for R sufficiently large, outwith the domain: $|\text{Im}(w)| < R, \text{Re}(w) < -R$, any invariant measure is supported in the total exceptional divisor $E_1 + E_2$.

While similar to the argument of II.2.4/5 for excluding support in the centre manifold, the above is more complicated again, so let us make,

II.3.4 Remark Say for simplicity, even if it can be excluded by blowing up, $p = 0$, and the centre manifold converges in U , then normally one excludes an invariant measure outside $x = 0, y = 0$, the so called strong and weak branches, by examining the holonomy around the strong branch which in an appropriate Fatou coordinate is translation by $\mathbb{Z}(1)$. On the other hand, before even beginning we made a spiralling restriction in $\log y$, so it might seem that there should be insufficient holonomy to conclude. However, this isn't how it works. More precisely under the above sectorial restrictions, say $x^{-r} \in S$ one finds a first integral t , of which τ is the logarithm, and the implied étale covering of $S \times \Delta^\times$ is equally that given by $\log y$. On this covering, $S \times L$, irrespective of spiralling restrictions, no leaf has a return map on any $S \times \ell$. On the other hand the restriction of τ to any such is Schlicht, and we have a natural action of $\mathbb{Z}(1)$ on τ which leaves the descended measure invariant. All of which remains true even under the spiralling restriction on $\log y$, or, if one prefers the image in $S \times \Delta^\times$ of a $\mathbb{Z}(1)$ orbit of leaves in $S \times L$ is pretty much the same leaf as before, *i.e.* contains the full leaf in $S \times \Delta'$, for Δ' a slightly smaller disc.

III. Almost holonomy

III.1 Local Set Up

As before U will be a small open neighbourhood, say a polydisc, in a 3-fold foliated by curves, about a singular point, say the origin, and the singular locus will be denoted Y . In particular, modulo blowing up, Y is either a smooth curve, or a plane curve with a node, and in the completion \widehat{U} of U in Y there is a well defined formal centre manifold \widehat{Z} provided the singularity is not a beast in the sense of [M4] §I.5, [MP2] II.2, so for our present purposes, essentially for notational convenience, we'll omit this case. As such, by way of appropriate blowing up we may further suppose that there is an invariant simple normal crossing divisor E , all of whose components are smooth and in 1-1 correspondence with components of Y by way of,

$$E_i \in |E| \longrightarrow E_i \cap \widehat{Z} \subset Y$$

whence there will always be at least one component of E meeting U , and a local equation for it will always be given by the variable x equal to zero. As per the convention already implicit in §II if there is a further component it will be given by the variable y equal to zero, albeit the latter may, otherwise, just be another variable. This convention is followed throughout [M4] and [MP2], and the above reduction is explained in [M4] §I.4, or [MP2] §V.1. In any case there is a well defined (formal) foliation in the formal centre manifold induced from U and we'll suppose further that our point of interest is a singular point of this formal foliation with Y being invariant by the same. Consequently for any simple closed curve γ in Y there is well defined formal holonomy of the said induced foliation in the centre manifold, or, equivalently an inverse system of representations,

$$\rho_n : \pi_1(\gamma) \longrightarrow \text{Aut}(\mathbb{C}[t]/t^n)$$

for $n \in \mathbb{N}$. Plainly this representation can be read off from the normal form. A complete list is provided in [MP2] VI.1.1, and we always have the right to choose a generator of the foliation (excluding beasts, and the case already encountered in II.3.1) without any loss of domain in the form,

$$\partial = z \frac{\partial}{\partial z} + x^p y^q u(x, y) D \text{ mod } \mathcal{O}(-NE)$$

for u a unit, $p, q \in \mathbb{N} \cup \{0\}$, $p \neq 0$, $N \in \mathbb{N}$ as large as we like, x, y, z honest coordinates, and D some 2-dimensional normal form.

In so much as planar normal forms consist of partial derivatives and at worst fractions in polynomials, we can define real co-dimension 1 subsets Γ_ϵ as follows: **III.1.1 Basic Construction** Obviously a simple closed curve γ in Y lies in a unique irreducible component of the same, and we can solve the equation $Dt = 0$ around γ in the normal direction to the component- *i.e.* t is a unit times x or y according to the case. There will, however, most likely be a discontinuity arising from the holonomy of the approximating plane foliation generated by D

as we turn through 2π . Consequently for $\epsilon > 0$ we have a real co-dimension 1 variety Γ_ϵ (strictly $\Gamma_{\gamma,\epsilon}$ since often estimates will have dependence on γ) defined by y (respectively x) in γ , $|z| \leq \epsilon$, $|t| \leq \epsilon$, with a possible discontinuity in the t variable at 2π - alternatively, if one prefers things smoother, say tubes $\Gamma_{\epsilon,s}$ around intervals I_s in γ with $I_s \rightarrow \gamma$. Now suppose that we have an invariant measure $d\mu$ for the foliation in U , then modulo the precision (which will frequently be omitted) of excluding sets of zero Lebesgue measure be it in ϵ or perturbations of γ , $d\mu|_{\Gamma_\epsilon}$ is well defined, and for any function $f(t)$ we can apply Stokes to obtain:

$$\begin{aligned} \int_{\Gamma_\epsilon} df(t)d\mu = \\ \int_{|z|=\epsilon, |t|\leq\epsilon} f(t)d\mu + \int_{|t|=\epsilon, |z|\leq\epsilon} f(t)d\mu + \int_{\mathbf{p}, |t|\leq\epsilon, |z|\leq\epsilon} (f(t^h) - f(t)) d\mu \end{aligned}$$

The last integral being at the discontinuity, and $t \mapsto t^h$ being the (convergent) holonomy of the approximating plane foliation defined by D around γ . In particular one recognises the final integrand as the co-equaliser of an approximation to the formal holonomy groupoid in the centre manifold. It is, however, an approximation to an object whose action on $d\mu$ has no sense. Nevertheless, amongst the above terms, the integral over Γ_ϵ , and that over $|t| = \epsilon$ can invariably be made $O(\epsilon^N)$, for any $N \in \mathbb{N}$. For example, the choice of f the characteristic function \mathbb{I}_F of some set is usually best, so the formula becomes,

$$\int_{\mathbf{p}, |t|\leq\epsilon, |z|\leq\epsilon} (\mathbb{I}_F(t) - \mathbb{I}_F(t^h)) d\mu = \int_{t \in \partial F, |z|\leq\epsilon} d\mu + \int_{|z|=\epsilon, t \in F} d\mu$$

There will always be a plane meromorphic 1-form ω with poles on E such that $\partial(\omega) = 1$, and γ will never be allowed to shrink arbitrarily. As such, only the pole around the irreducible component containing γ is ever relevant, and in a way that depends on γ this is bounded by some a priori fixed power $|t|^{-p}$. On the other hand ∂t vanishes to some high order, say, in a minor abuse of notation, $N + p$, so $|dt|$ wedged against $d\mu$ will be no worse than that of $|\omega|$ times $|t|^N$. Presently we'll work through some examples in detail, but, already we can reasonably say that the dominating term on the right of this formula is that on the face $|z| = \epsilon$. This will, however, prove (s , Lebesgue a.e allowing F to move in some family F_s) to be bounded by ϵ^p times the part, $s_{Z,d\mu}(\epsilon)$, of the Segre class around the singular locus in the normal direction to the centre manifold, so we'll have,

III.1.2 Almost holonomy estimate Let things be indicatively as above with p the order of vanishing of ∂ restricted to the formal centre manifold around the component of Y containing γ , then for a sufficiently good approximation indexed by $N \in \mathbb{N}$ to the formal foliation in the formal centre manifold,

$$\left| \int_{\mathbf{p}, |t|\leq\epsilon, |z|\leq\epsilon} (\mathbb{I}_F(t) - \mathbb{I}_F(t^h)) d\mu \right| \leq \epsilon^p s_{Z,d\mu}(\epsilon) + O(\epsilon^N)$$

An evident variant, indeed corollary, of the above reasoning occurs on varying the branch point \mathbf{p} . Typically, in practice for, say, loops γ of the form $|y| = r$, $r \in I$, and the branch the interval I itself viewed in \mathbb{R}_+ inside the domain of y in \mathbb{C} , or, for that matter $\mathbb{R}_+\lambda$ for any direction λ . In such a situation we have, **III.1.3 Variant** Notations as above, then:

$$\int_{|t| \leq \epsilon, |z| \leq \epsilon, \mathbf{p} \in b} \frac{d|y|}{|y|} (\mathbb{I}_F(t) - \mathbb{I}_F(t^h)) d\mu = \int_{t \in \partial F, |z| \leq \epsilon, |y| \in I} \frac{d|y|}{|y|} d\mu + \int_{|z| = \epsilon, t \in F, |y| \in I} \frac{d|y|}{|y|} d\mu$$

which in turn could have been viewed at Stokes applied to the region, $t \in F$, $|y| \in I$, $|z| \leq \epsilon$ since $d|y|$ vanishes on the boundary of I . In any case, the important point is that $d|y|$ has a sign on the branch, so, without loss of generality we may suppose that it is an actual length form.

III.2 Linear and bounded

We proceed to use the basic construction III.1.1 to get estimates at the singularities not encountered in §II beginning with,

III.2.1(a) Set Up In a neighbourhood U of foliated 3-fold of a point in the singular locus after completion in the same, the foliation admits a formal generator of the form,

$$z \frac{\partial}{\partial z} + \frac{x^p}{p(1 + \nu x^p)} \left(x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \right)$$

where x, y, z are formal coordinates, $p \in \mathbb{N}$, $\lambda, \nu \in \mathbb{C}$, $\lambda \notin \mathbb{R}$, $\text{Re}(\lambda) < 0$, and as per III.1, after blowing up if necessary, we may suppose that $x = 0$ defines the exceptional divisor E .

Plainly we take an approximation modulo a large, to be decided, power $\mathcal{O}_U(-NE)$ of the exceptional divisor as discussed pre III.1.1, so that, in a minor notational confusion, x, y, z are honest holomorphic coordinates. The holonomy of the approximating plane foliation is, therefore, always determined by its first order part, and given by,

$$\pi_1(\gamma) \xrightarrow{\sim} \mathbb{Z}(1) \longrightarrow \mathbb{C}^\times : a \mapsto \exp(-a/\lambda)$$

Even though the above map is wholly independent of a choice of square root of -1 , there is an orientation convention in Stokes' theorem, so as we follow an oriented loop there is an implied choice of such, and a contraction if $\text{Im}(\lambda^{-1}) > 0$, expansion otherwise, so either changing the orientation, or replacing ϵ by $\epsilon \exp(-2\pi \text{Im}(\lambda^{-1}))$ if $\text{Im}(\lambda^{-1}) < 0$, there is $q \in \mathbb{R}_+$ of modulus less than 1 such that,

$$\int_{|z| \leq \epsilon, q\epsilon' \leq |t| \leq \epsilon'} \frac{d\mu}{\mathbf{p}} = \int_{|z| = \epsilon, |t| \leq \epsilon'} \frac{d\mu}{y \in \gamma} + \int_{|t| = \epsilon', |z| \leq \epsilon} \frac{d\mu}{y \in \gamma}$$

which leads to the formula,

$$\int_{|z| \leq \epsilon, |t| \leq \epsilon'} d\mu = \sum_{n=0}^{\infty} \left(\int_{|z|=q^n \epsilon, |t| \leq q^n \epsilon'} d\mu + \int_{|t|=q^n \epsilon', |z| \leq q^n \epsilon} d\mu \right)$$

Now, say γ is the loop, $|y| = Y$ for Y in a range $I = (a, b)$, $a > 0$, then for constants c_I, C_I depending on I ,

$$c_I |x| < |t| < C_I |x|$$

so for possibly different constants c, C again depending on I we get an estimate,

$$\int_{|z|, |x| \leq \epsilon} d\mu \leq \sum_{n=0}^{\infty} \left(\int_{|z|=q^n \epsilon, |x| \leq cq^n \epsilon} |d\mu| + \int_{|t|=Cq^n \epsilon', |z| \leq q^n \epsilon} |d\mu| \right)$$

where here, and elsewhere. $|d\mu|$ is the total variation of the sliced normal distribution with measure regularity, cf. [F] 4.3.2, whereas on the left $d\mu$ is naturally a measure, and indeed the left hand side is the measure of a transversal. To proceed further let us suppose that the Segre class is well defined around Y , *e.g.* U a neighbourhood in a complex space or champ de Deligne-Mumford analytique with compact singular locus. Every choice of distance function to Y gives rise to different measures depending on ϵ that limit on $s_{Y, d\mu}$, but the difference is $o(\epsilon)$, so let us omit this from the notation, and write:

$$\begin{aligned} s_{Y, d\mu}(\delta) &\geq \oint_{\substack{|z|=\delta, |x| \leq c\delta \\ |y| \leq A}} \frac{dz}{z} + \oint_{\substack{|x|=c\delta, |z| \leq \delta \\ |y| \leq A}} \frac{dz}{z} \\ &\geq \oint_{\substack{|z|=\delta, |x| \leq c\delta \\ |y| \leq A}} \frac{dz}{z} := s_{Z, d\mu}(\delta) \end{aligned}$$

where $\delta > 0$ is anything less than some fixed constant outside a set of nil Lebesgue measure, A is just the upper limit of the size of our disc in the variable y , and c is a constant to be chosen, *i.e.* here our distance function to Y is $\max\{|z|, c^{-1}|x|\}$ in the neighbourhood U . Consequently,

$$\begin{aligned} \int_a^b dr \int_{\substack{|z|=\delta, |x| \leq c\delta \\ |y|=r}} d\mu &= \\ 2\pi \oint_{\substack{|z|=\delta, |x| \leq c\delta \\ a \leq |y| \leq b}} \left| \text{Im} \left\{ \frac{y\bar{z}\partial\bar{y}}{|y|\partial\bar{z}} \right\} \right| \frac{dz}{z} d\mu &\leq C\delta^p s_{Z, d\mu}(\delta) \end{aligned}$$

where the implied constant depends on the interval (a, b) , and, of course, the level N of approximation that we're working to. As such one could say more precisely that C is of the form $C_1 c^p$ for C_1 depending only on A , and $\delta \leq C_2 c^{-1}$,

again C_2 depending only on A , and a further dependence on the interval built into the choice of distance function. In any case, this yields an estimate,

$$\int_a^b dr \sum_{n=0}^{\infty} \int_{\substack{|z|=\delta, |x|\leq c\delta \\ |y|=r}} d\mu \leq \frac{C\epsilon^p}{1-q^p} \sup_{\delta\leq\epsilon} s_{Z,d\mu}(\delta)$$

Consider, therefore, the set of $r \in (a, b)$ where the integrand on the left exceeds B times the estimate on the right. This has Lebesgue measure at most B^{-1} , and so we deduce,

III.2.2 (a) Estimate Let γ of the basic construction III.1.1 be taken of the form $|y| = r$ for r in the interval $I = (a, b)$, then there is a constant C and limiting Segre classes $s_{Z,d\mu}(\delta)$ depending on I such that for every ϵ , and r in a set (depending on ϵ) of measure at least $(b-a) - B^{-1}$ we have,

$$\sum_{n=0}^{\infty} \int_{\substack{|z|=\delta, |x|\leq c\delta \\ |y|=r}} d\mu \leq CB\epsilon^p \sup_{\delta\leq\epsilon} s_{Z,d\mu}(\delta)$$

Next for $\epsilon < \delta$ say, and $|y| \in I$ as before with $m \in \mathbb{N}$ to be chosen, consider,

$$\int_0^\delta \frac{d\epsilon}{\epsilon^{m+1}} \int_a^b \sum_{n=0}^{\infty} \int_{\substack{|t|=Cq^n\epsilon, |z|\leq q^n\epsilon \\ |y|=r}} d\mu \leq \sum_{n=0}^{\infty} \int_{\substack{|t|\leq Cq^n\delta, |z|\leq\delta \\ |y|\in I}} \left| \frac{dtd|y|}{t^{m+1}} \right| d\mu$$

As a function of the level N of approximation modulo powers of (x^N) the exceptional divisor, the integrand on the right is at most,

$$C|x|^{N-2(p+1)-m} dx \otimes d\bar{x}d\mu$$

again for C depending on the interval (a, b) . As such, if $J(\epsilon)$ is the (necessarily non-negative) value of the integrand on the left with respect to ϵ it satisfies a bound,

$$\int_0^\delta \frac{J(\epsilon) d\epsilon}{\epsilon^m \epsilon} \leq \frac{C\delta^n}{1-q^n}$$

provided $n = N - 2(p+1) - m > 0$. Choosing N appropriately large, implies, that the set of ϵ where $J(\epsilon) > \epsilon^m$ has finite measure with respect to $\epsilon^{-1}d\epsilon$ and we obtain,

III.2.2 (b) Estimate Again let γ be of the form $|y| = r$ for r in $(a, b) = I$, with C (possibly bigger) and $s_{Z,d\mu}(\delta)$ as in III.2.2 (a), together with an exceptional set of ϵ (again depending on I) of finite $\epsilon^{-1}d\epsilon$, then for non-exceptional ϵ and r in a set (again depending on ϵ) of measure at least $(b-a) - 2B^{-1}$ the estimate III.2.2 (a) holds along with the estimate,

$$\sum_{n=0}^{\infty} \int_{\substack{|t|=Cq^n\epsilon, |z|\leq q^n\epsilon \\ |y|=r}} d\mu \leq CB\epsilon^N$$

where, in what will be a repeated notational confusion, $N \in \mathbb{N}$ is just some a priori choice of an integer as large as we please, which may be the order of

approximation along E or some finite shift of the same by irrelevant and very fixed constants, albeit that other implied constants, sets of exceptional measure, *etc.*, may depend on N .

Putting all of this together implies that on the transversal $T_{\mathbf{p}}$ at \mathbf{p} we have, **III.2.2 (c) Conclusion** Taking N sufficiently large, with quantifies as above in III.2.2 (b) then the measure of the transversal $T_{\mathbf{p}} : |x| \leq \epsilon, |z| \leq \epsilon, y = \mathbf{p}$ satisfies,

$$\int_{T_{\mathbf{p}}} d\mu \leq B\epsilon^p \max\{\sup_{\delta \leq \epsilon} s_{Z,d\mu}(\delta), \epsilon\}$$

At which point, one should recognise that this is a variant on the principles that yielded the much stronger II.2.3. The results of [MP2], which are nevertheless optimal, cf. op. cit. §III.3, are nowhere close to permitting II.2.3 in this case. One particular (and by no means the worst) problem is the divisor $y = 0$ which, formally, may be taken invariant, cannot be supposed such, whence, in our immediate context, the emphasis on the interval I which is bounded away from 0. Nevertheless the formal invariant curve $y = z = 0$ might converge, and it could then support an invariant measure. On the other hand, the above conclusion obviously cannot have any relevance to such a measure, so, once again, the Segre class must intervene. To this end, suppose the Segre class around the exceptional divisor is zero, then we have the identity,

$$\oint_{\substack{|x|=\epsilon \\ |z|\leq\epsilon \\ |y|\leq r}} \frac{dx}{x} d\mu + \oint_{\substack{|z|=\epsilon \\ |x|\leq\epsilon \\ |y|\leq r}} \operatorname{Re} \left(\frac{z\partial x}{\partial z x} \right) \frac{dz}{z} d\mu + \oint_{\substack{|y|=r \\ |z|\leq\epsilon \\ |x|\leq\epsilon}} \operatorname{Re} \left(\frac{y\partial x}{\partial y x} \right) \frac{dy}{y} d\mu = 0$$

for any $r \leq A$, albeit outside a set of zero Lebesgue measure. Now the integral over the face $|z| = \epsilon$ plainly admits the estimate $\epsilon^p s_{Z,d\mu}(\epsilon)$. The integral over the face $|y| = r$ is just the average of the previous discussion over $\mathbf{p} \in \gamma$ with respect to circle measure, and, at the price of repeating everything *mutatis mutandis* with an extra integral thrown in, all of III.2.2 (a),(b),(c) remain valid with the same quantification when averaged over $\mathbf{p} \in \gamma$. Alternatively one can homotope between different \mathbf{p} (as we'll do later) and use the estimates as they stand. In any case, the integral over the $|x| = \epsilon$ face is strictly positive, and we have,

III.2.3 Corollary Suppose further that the Segre class around the exceptional divisor is zero, then for a constant C , and an exceptional set of finite $\epsilon^{-1}d\epsilon$ measure both depending on the interval (a, b) as also the classes $s_{Z,d\mu}(\epsilon)$ we have,

$$\oint_{\substack{|x|=\epsilon, |z|\leq\epsilon \\ |y|\leq a}} \frac{dx}{x} d\mu \leq \epsilon^p s_{Z,d\mu}(\epsilon)$$

Alternatively for $r \in (a, b)$ one has the same statement with the more convoluted quantification encountered in III.2.2.

The zero Segre class hypothesis, so $s_{Z,d\mu}(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ permits an inessential, but convenient improvement of the quantification encountered in III.2.2 (a) to something more similar to III.2.2 (b). Specifically, put,

$$\Theta_Z(\epsilon) := \sup_{\delta \leq \epsilon} s_{Z,d\mu}(\delta)$$

The latter, understood globally for added convenience, decreases to zero, so the measure,

$$d\theta_Z(\epsilon) = \frac{\Theta'_Z}{\Theta_Z} d\epsilon$$

has infinite mass, while, $\Theta_Z d\theta$ has finite mass. Consequently we can on dividing by ϵ^p , take a supremum over ϵ of the integrand, then integrate against $d\theta_Z(\epsilon)$ to obtain,

III.2.2 (a)/(c) bis. Alternative Let g, I be as before, then there is a function $o(\epsilon)$ going to zero as $\epsilon \rightarrow 0$ such that for r in a set (depending on ϵ) of measure at least $(b-a) - B^{-1}$ the estimated quantity be it III.2.2 (a) or (c) is bounded by,

$$B\epsilon^p o(\epsilon)$$

This is a somewhat more convenient way to proceed when addressing,
III.2.1 (b) Set Up Everything as before in III.2.1 (a), except that now $\lambda \in \mathbb{R}$. Furthermore if $\lambda \in \mathbb{Q}$ we insist that its big height is sufficiently large compared to p . About $(p+1)$ will do, but one can achieve anything by blowing up which certainly improves the situation in the formal centre manifold since at other singularities in its proper transform one actually has 2 rather than just 1 component of the exceptional divisor.

In this situation there is absolutely no obstruction to the existence of a non-trivial invariant measure, and we proceed by computing meromorphic residues around the exceptional divisor. Specifically for $i > 0$, $r \leq A$, one has,

$$\left| \int_{\substack{|x|=\epsilon, |z|\leq\epsilon \\ |y|\leq r}} dx^{-i} d\mu \right| \leq \epsilon^{-i} \int_{\substack{|x|=\epsilon, |z|=\epsilon \\ |y|\leq A}} |d\mu| + \epsilon^{-i} \int_{\substack{|x|=\epsilon, |z|\leq\epsilon \\ |y|=r}} |d\mu|$$

The estimates of the terms on the right is very much similar to III.2.2 (a), respectively (b). Indeed for χ in say $(e^{-1}, 1)$ consider,

$$\int_{e^{-1}}^1 \frac{\chi}{\chi} \int_{\substack{|x|=\chi\epsilon, |z|=\epsilon \\ |y|\leq A}} |d\mu| = \int_{\substack{e^{-1}\epsilon \leq |x| \leq 1 \\ |z|=\epsilon, |y|\leq A}} \left| \frac{dx}{x} \right| |d\mu| \leq C\epsilon^p s_{Z, d\mu}(\epsilon)$$

for a constant depending only on A , so that for say J the range of ϵ ,

$$\int_J d\theta_Z(\epsilon) \sup_{\delta \leq \epsilon} \left(\delta^{-p} \int_{e^{-1}}^1 \frac{\chi}{\chi} \int_{\substack{|x|=\chi\delta, |z|=\delta \\ |y|\leq A}} |d\mu| \right) \leq C$$

For a possibly different constant C , and we put ourselves in the situation of zero Segre class, equivalently $d\theta_Z(\epsilon)$ has infinite measure to deduce for an $o(\epsilon)$ as per III.2.2 (a)/(c) bis, but actually no dependence on I (III.1.1 giving no information here) albeit certainly depending on A ,

$$\int_{e^{-1}}^1 \frac{\chi}{\chi} \int_{\substack{|x|=\chi\delta, |z|=\delta \\ |y|\leq A}} |d\mu| \leq \epsilon^p o(\epsilon)$$

From which we obtain,

III.2.4 (a) Estimate For $o(\epsilon)$ as above and every $B > 0$ there is subset of $(-1, 0)$ (depending on ϵ) of measure at least $1 - B^{-1}$ such that for $\log \chi$ belonging to the same we have the estimate,

$$\int_{\substack{|x|=\chi\delta, |z|=\delta \\ |y|\leq A}} |d\mu| \leq B\epsilon^p o(\epsilon)$$

The other term, like III.2.2 (b), is much more robust. Indeed, introducing once more our interval I , bounded away from 0, in which $|y|$ varies, for $m > 0$ to be chosen, we have for $\epsilon \in (0, \delta)$,

$$\int_0^\delta \frac{d\epsilon}{\epsilon^{m+1}} \int_a^b dr \int_{\substack{|x|=\epsilon, |z|\leq\epsilon \\ |y|=r}} |d\mu| \leq \int_{\substack{|z|, |x|\leq\delta \\ |y|\in I}} \left| \frac{d|x|d|y|}{|x|^{m+1}} \right| d\mu$$

Furthermore under the hypothesis III.2.1 (b) we have for some large N , possibly not arbitrary if $\lambda \in \mathbb{Q}$, but the big height condition says that its large enough,

$$\left| \frac{d|x|d|y|}{|x|^{m+1}} \right| d\mu \leq C|x|^N dx \otimes d\bar{x}d\mu$$

for C depending only on I , and $\lambda \in \mathbb{R}$ is used in an essential way. Integrating against χ one therefore obtains,

III.2.4 (b) Estimate For ϵ outside a set with finite $\epsilon^{-1}d\epsilon$ measure, there is a constant C , depending on I , such that for every $B > 0$ and $(r, \log \chi)$ in $I \times (-1, 0)$ outside a set (depending on ϵ) of measure B^{-1} we have,

$$\int_{\substack{|x|=\epsilon \\ |z|\leq\epsilon, |y|=r}} |d\mu| \leq CB\epsilon^{p+1}$$

or indeed ϵ^N rather than ϵ^{p+1} , N only limited by the big height proviso in the case $\lambda \in \mathbb{Q}$, but all constants, exceptional sets, *etc.* depending on it.

As such combining III.2.4 (a), (b) we obtain,

III.2.4 (c) Suppose things are as per the set up III.2.1 (b), and that the Segre class of our invariant measure around the singularities is zero, then taking B sufficiently large so that III.2.4 (a), (b) hold for every $\epsilon > 0$, and $(r, \log \chi)$ varying in a set (depending on ϵ) which is as close to full as we please,

$$\lim_{\epsilon \rightarrow 0} \int_{\substack{|x|=\epsilon\chi^{-1} \\ |z|\leq\epsilon, |y|\leq r}} \frac{dx}{x^j} d\mu = 0, \quad 1 \leq j \leq p+1$$

III.3 Nodes

Next on our to do list is to consider a node in the centre manifold, *i.e.*,

III.3.1(a) Set Up In a neighbourhood U of foliated 3-fold of a point in the singular locus after completion in the same, the foliation admits a formal generator of the form,

$$(p+r)z \frac{\partial}{\partial z} + x^p \left(R(x)y \frac{\partial}{\partial y} + \frac{x^{r+1}}{1 + \lambda x^{p+r}} \frac{\partial}{\partial x} \right)$$

where x, y, z are formal coordinates, $p, r \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $\deg R \leq r$, $R(0) \neq 0$, and as per III.1, after blowing up if necessary, we may suppose that $x = 0$ defines the exceptional divisor E .

In addition to immediately applying the previous conventions to take a generator ∂ convergent in analytic functions x, y, z which agrees with the normal form modulo $\mathcal{O}(-NE)$ for some $N \gg 0$, define functions of a complex variable x by way of,

$$\zeta'(x) = -\frac{R(x)}{x^{r+1}}(1 + \lambda x^{p+r}), \quad \xi'(x) = -(p+r) \frac{1 + \lambda x^{p+r}}{x^{p+r+1}}$$

where it is convenient to view ζ, ξ as functions of $X = x^{-1}$, *i.e.* in neighbourhoods of infinity. As such, $\xi(X)$ is asymptotically X^{p+r} , and $\zeta(X)$, $c_0 X^r$ for some constant $c_0 \neq 0$ (equals $R(0)/r$) whose value is important. Were the normal form to be convergent, we would then have first integrals,

$$s = z \exp(\xi), \quad t = y \exp(\zeta)$$

Unlike the cases encountered in §II.1-II.3, the existence domain for a conjugation to the normal form are complicated, [MP2], §IV.2-IV.3, VI.4.5, nevertheless for $\operatorname{Re}(\zeta) > 0$ they share many of the salient features already encountered, and by way of an illustrative, albeit logically irrelevant to the proof of the main lemma, complement we may deduce,

III.3.2 Fact Provided imaginary rays in the plane of ζ do not go to imaginary rays in the plane of ξ , *i.e.* no j satisfies $\operatorname{Re}(c_0 j^r) = \operatorname{Re}(j^{p+r}) = 0$, there are open sectors strictly (at both ends) containing $\operatorname{Re}(\zeta) > 0$ such that any invariant measure with zero Segre class must be supported in the exceptional divisor. Otherwise, idem, but for any open sectors strictly contained in $\operatorname{Re}(\zeta) > 0$.

proof This is basically an appendix to [MP2] §IV, and one should probably have a copy to hand, the notation being the same except that ζ here is z in op. cit. This said, one is basically trying to bring the field into the form $-\frac{\partial}{\partial \xi}$ in ξ, s, t variables, which results in a fibration,

$$\begin{array}{ccc} U & \xrightarrow{b \times \xi} & B \times \mathbb{C} \\ & \downarrow b=(s,t) & \\ & B \subset \mathbb{C}^2 & \end{array}$$

with, as ever, all maps open and the horizontal arrow an embedding. Whence, modulo connectedness issues, the leaves may be identified with the fibres U_b . Supposing for the sake of argument y, z varying in unit discs, the fibres have the form,

$$\log |s| < \operatorname{Re}(\xi), \quad \log |t| < \operatorname{Re}(\zeta)$$

Now for $\operatorname{Re}(\zeta) > 0$ the leaves are, on the whole, unbounded, but some cases are more unbounded than others, *viz*: if in the plane of ξ the restriction imposed by t is also open to $\operatorname{Re}(\xi) \rightarrow +\infty$ then we'll say that we're in a thick sector, otherwise,

it will be called thin. Observe furthermore that for zero Segre class the technique affording I.3.4 implies that $|x|^{-(r+2)}dx \otimes d\bar{x}$ is absolutely integrable. Indeed,

$$\oint_{\substack{|x|=\epsilon \\ |y|\leq A \\ |z|\leq B}} \frac{dx}{x} d\mu + \oint_{\substack{|y|=A \\ |x|\leq \epsilon \\ |z|\leq B}} \operatorname{Re}\left\{\frac{\partial xy}{x\partial y}\right\} \frac{dy}{y} d\mu + \oint_{\substack{|z|=B \\ |x|\leq \epsilon \\ |y|\leq A}} \operatorname{Re}\left\{\frac{\partial xz}{x\partial z}\right\} \frac{dz}{z} d\mu = 0, \quad \epsilon > 0$$

the latter 2 terms are bounded by the measure on the appropriate face in an ϵ neighbourhood of the exceptional divisor times ϵ^r and ϵ^{p+r} respectively, so the initial term is at worst $\epsilon^r o(\epsilon)$, and applying Stokes yields the assertion. It follows that the set of leaves where the integral of the density $|x|^{-(r+2)}dx \otimes d\bar{x}$ is infinite has zero measure. This is certainly true of leaves in thick sectors with $\operatorname{Re}(\zeta) > 0$, but there may be a doubt about it for $p \gg r$ in thin sectors. As such we need to add some comment to [MP2] around the imaginary axis in ζ when the imaginary axis is thick. In this situation, op. cit. IV.3.2 (a)-(d) apply without change to construct a sector around the imaginary axis where the existence domain can be taken to be a bi-disc in (y, z) and a sector in x , *i.e.* thick sectors in $\operatorname{Re}(\zeta) > 0$ bordering on the imaginary axis in ζ continue to admit a conjugation into the half space $\operatorname{Re}(\zeta) < 0$ of the analytic field to its normal form without taking a logarithm in y , albeit perhaps on quite a small sector if $p \gg r$.

This implies that the only possibility for the measure to be supported in $\operatorname{Re}(\zeta) > 0$ are thin sectors which meet an imaginary line, since, otherwise, the leaf may be analytically continued from a thin region to a thick one where there is no measure. Now, there are cases to consider. The generic one is that the imaginary axis in ζ is strictly thin, *i.e.* $\operatorname{Re}(\xi)$ goes to plus or minus ∞ along it with the asymptotics of some nearby non-imaginary ray. Here one could solve the centre manifold problem of op. cit. §IV with an existence domain sectorial in x and a bi-disc in (y, z) but maybe not the conjugation to normal form in a way better than op. cit. IV.2.5. Fortunately the difference is slight. In the case where it can be done, *i.e.* the imaginary axis has $\operatorname{Re}(\xi) \rightarrow +\infty$, the angle between the s and t level curves never goes to zero, so, in fact, $|x|^{-2}dx \otimes d\bar{x}$, and not just $|x|^{-(r+2)}dx \otimes d\bar{x}$, has infinite measure even in leaves in leaves in sectors around the imaginary axis. In the other case the only leaves which can't be analytically continued into the neighbouring thick sector must be in a bounded region for the variable X , actually an annulus of bounded moduli with inner boundary the boundary for the X variable itself, so, shrinking x as necessary, all leaves in this sector will continue to a thick sector where there is zero measure.

This leaves the possibility that purely imaginary in ζ goes to purely imaginary in ξ , equivalently the imaginary axis is critical in the notation of op. cit., so for j the point of S^1 in X -space we have $c_0 j^r$ and j^{p+r} purely imaginary. Obviously rare, but not impossible. If this situation doesn't occur we could have taken the asserted sector of zero measure to include not just $\operatorname{Re}(\zeta) > 0$, but also extend around both imaginary axis. When it does occur, one must just accept a small loss, move slightly (sectorial sense) into the domain of $\operatorname{Re}(\zeta) > 0$, and argue as above in the cases where the imaginary axis is strictly thin. \square

This is somewhat different as to how we proceeded in §II, so let us offer:

III.3.3 Remark To the extent that one is prepared to invoke [MP2] one could reasonably pursue a somewhat stronger proposition without hypothesis on the Segre class à la II.3.2/3. This can be done with some variation on the above. The problem in either case is that while elementary the existence domains in the region $\operatorname{Re}(\zeta) > 0$ depend on p, r and even $R(x)$ in a rather fastidious manner, and may be well short of width 2π in the ξ -variable. As such, op. cit. takes the point of view that there is a finite division of the ζ -plane with a conjugation to the normal form in every region, albeit possibly with further restriction on $\log y$ if $\operatorname{Re}(\zeta) < 0$. Consequently to compute the holonomy of a transversal $z = \text{constant}$, one has to patch around to find the infinite orbit which would eliminate the possibility of an invariant measure. Whence, the best way to prove this stronger proposition is to employ an approximate holonomy estimate by way of the evident variant of III.1.1 for the holonomy around a loop in the curve $x = y = 0$, albeit this would repeat much of [MP2] §IV.3 in another guise. In what follows, however, although we will not use III.3.2, we will employ the fact encountered in the proof that, in the presence of zero Segre class, $|x|^{-(r+2)} dx \otimes d\bar{x}$ is absolutely integrable against $d\mu$.

Now, in applying the basic construction III.1.1 around loops in the singular locus $x = z = 0$, with, as before $|y| = \rho$ for ρ in some interval I bounded away from 0, it's helpful to have another normal form at our disposition, *viz*:

III.3.1 (b) Alternative As per III.3.1 (a) but with the normal form,

$$z \frac{\partial}{\partial z} + x^p (\tilde{R}(x) + \tilde{\lambda} x^{p+r}) \left(y \frac{\partial}{\partial y} + \frac{x^{r+1}}{r(1 + \nu x^r)} \frac{\partial}{\partial x} \right)$$

for $\tilde{\lambda}, \nu \in \mathbb{C}$ and $\deg \tilde{R} \leq r - 1$.

Each normal form has its own role, and III.3.1(b) is more convenient for the almost holonomy around γ when $|y| = \rho$, as opposed to the actual, or even approximate, holonomy around $|z| = \rho$ which underlies III.3.2. In particular we now suppose that a convergent generator ∂ in functions x, y, z approximates III.3.1 (b) to some large order modulo $\mathcal{O}(-NE)$, $N \in \mathbb{N}$ to be specified. Denoting by t the resulting approximately invariant function envisaged by III.1.1, and γ the loop, or its canonical image in $\mathbb{Z}(1)$, for any function of x we may express it as a function of t , and vice versa, by way of $x(t) = x|_T$, for T our plane transversal. As such for ζ (basically the same function as before in different clothes) equal to $x^{-r} + \nu \log x^{-r}$ the approximate holonomy is given by,

$$h^* \zeta = \zeta + \gamma$$

for any determination of $\log x$ in a sector of width $2\pi/r$ in the argument of x .

Observe that a convenient norm to work with is,

$$\|x\|^{-r} = \min\{|\operatorname{Re}(x^{-r})|, |\operatorname{Im}(x^{-r})|\}$$

In particular $R = \epsilon^{-r}$, modulo the minor notational confusion with III.3.1 (a), is often easier to work with, while, in addition, $t(x) = x(1 + O(x^r))$. Certainly there is a branching issue in ζ but in practice it will pose no problem, while

if we suppose III.3.2, it may be taken in the domain of null measure. Either way, $|\operatorname{Re}(\zeta)| \geq R$ and $|\operatorname{Im}(\zeta)| \geq R$ is often the most convenient domain to work with. There may also be an orientation issue as to whether we have to worry about there being measure in the upper or lower half planes in ζ , so, say the former, albeit this implies that we'll be doing III.3.1 with loops oriented in the opposite direction to that implied by the choice of an imaginary part function.

All of which said, we have the following co-equaliser estimate,

$$\int_{\substack{R > \operatorname{Im}(\zeta) > R + 2\pi \\ |z| \leq \epsilon, y = \mathbf{P}}} d\mu \leq \int_{\substack{\operatorname{Im}(\zeta(t)) = R \\ |z| \leq \epsilon, |y| = \rho}} |d\mu| + \int_{\substack{|z| = \epsilon, |y| = \rho \\ \operatorname{Im}(\zeta) < R}} |d\mu|$$

Evidently, we've already encountered how to estimate the terms in question and choosing the order of the approximation to the normal form to be sufficiently high, we have the estimates,

III.3.4 (a) Estimate For a constant C depending on the interval $I = (a, b)$ in which $|y|$ varies, and R outside a set of finite $R^{-1}dR$ measure,

$$\begin{aligned} \int_a^b d\rho \left(\sum_{n=0}^{\infty} \int_{\substack{|\operatorname{Im}(\zeta(t))| = R+n \\ |z| \leq (R+n)^{-1/r} \\ |y| = \rho}} |d\mu| + \sum_{m=-\lceil R \rceil}^{\lceil R \rceil} \left[\int_{\substack{\operatorname{Im}(\zeta(t)) = m \\ |z| \leq R^{-1/r}, |y| = \rho}} |d\mu| \right. \right. \\ \left. \left. + \int_{\substack{|\operatorname{Re}(\zeta(t))| = R, |\operatorname{Im}(\zeta(t))| \geq m \\ |z| \leq R^{-1/r}, |y| = \rho}} |d\mu| \right] \right) \leq CR^{-(2+r)} \end{aligned}$$

where the implied possibility to extend the approximate holonomy estimate over strips with $\operatorname{Im}(\zeta)$ between $-R$ and R results from the fact that $\operatorname{Re}(\zeta(t))$ is holonomy invariant. This is akin to III.2.2 (b), whereas the analogue of III.2.2 (a) is,

III.3.4 (b) Estimate For a constant C depending only on the maximum value A of $|y|$, we have,

$$\int_a^b d\rho \int_{\substack{\|x\| \leq R^{-1/r}, |z| = R^{-1/r} \\ |y| = \rho}} |d\mu| \leq CR^{-p/r} \sup_{R_0 > R} s_{Z, d\mu}(R_0^{-1/r})$$

Before applying these estimates we need a consequence of the fact encountered in the proof of III.3.2 that $|x|^{-(r+2)} dx \otimes d\bar{x}$ is absolutely integrable against $d\mu$ whenever the Segre class is zero. More precisely for any $R < B < \infty$,

$$\begin{aligned} \int_{\substack{|y| = \rho \\ R \leq \operatorname{Im}(x^{-r}) \leq B \\ |\operatorname{Re}(x^{-r})| \leq R, |z| \leq R^{-1/r}}} d\operatorname{Re}(x^{-r}) d\mu + \int_{\substack{|z| = R^{-1/r} \\ R \leq \operatorname{Im}(x^{-r}) \leq B \\ |\operatorname{Re}(x^{-r})| \leq R, |y| \leq \rho}} d\operatorname{Re}(x^{-r}) d\mu = \\ \int_{\substack{\operatorname{Im}(x^{-r}) = B \\ |\operatorname{Re}(x^{-r})| \leq R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} d\operatorname{Re}(x^{-r}) d\mu - \int_{\substack{\operatorname{Im}(x^{-r}) = R \\ |\operatorname{Re}(x^{-r})| \leq R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} d\operatorname{Re}(x^{-r}) d\mu \end{aligned}$$

The integral, $I(B)$, say, over the $\operatorname{Im}(\zeta) = B$ face may be integrated against $B^{-1}dB$, and by the aforesaid absolute integrability of $|x|^{-(r+2)} dx \otimes d\bar{x}$ the

result is finite, so the liminf of $I(B)$ as $B \rightarrow \infty$ must be zero. Whence,

$$\begin{aligned} & \int_{\substack{|\operatorname{Re}(x^{-r})| \leq R, \operatorname{Im}(x^{-r}) = R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} d\operatorname{Re}(x^{-r}) d\mu = \\ & -4\pi \oint_{\substack{|y| = \rho, |z| \leq R^{-1/r} \\ |\operatorname{Re}(x^{-r})| \leq R, |\operatorname{Im}(x^{-r})| \geq R}} \operatorname{Im} \left(\frac{x^{-(r+1)} \partial xy}{\partial y} \right) \frac{dy}{y} d\mu - \\ & 4\pi \oint_{\substack{|z| = R^{-1/r}, |y| \leq \rho \\ |\operatorname{Re}(x^{-r})| \leq R, |\operatorname{Im}(x^{-r})| \geq R}} \operatorname{Im} \left(\frac{x^{-(r+1)} \partial xz}{\partial z} \right) \frac{dz}{z} d\mu \end{aligned}$$

At which point, observe that there is a considerable tangency along the foliation between the curves $|y| = \text{const.}$, and $\operatorname{Re}(x^{-r}) = \text{const.}$, *i.e.* for $\|x\| \leq R^{-1/r}$, and $|y|$ confined to I there is a constant C depending on I such that,

$$\oint_{\substack{|y| = \rho \\ \|x\|, |z| \leq R^{-1/r}}} \left| \operatorname{Im} \left(\frac{x^{-(r+1)} \partial xy}{\partial y} \right) \right| d\mu \leq C \oint_{\substack{|y| = \rho \\ \|x\|, |z| \leq R^{-1/r}}} |x^r| \frac{dy}{y} d\mu$$

and, of course, a sufficiently high order of approximation to the normal form. Similarly, for such an approximation, and another C depending only on the same,

$$\oint_{\substack{|z| = R^{-1/r} \\ \|x\| \leq R^{-1/r}, |y| = \rho}} \left| \operatorname{Im} \left(\frac{x^{-(r+1)} \partial xz}{\partial z} \right) \right| d\mu \leq CR^{-p/r} s_{Z, d\mu}(R^{-1/r})$$

The estimates III.3.4 (a)-(b) do not depend on the point \mathbf{p} in γ where we take the discontinuity, so summing the inequality arising from the basic construction III.3.1 with quantification implied by III.3.4 (a)-(b), and averaging over \mathbf{p} , we have,

$$\int_a^b d\rho \oint_{|y| = \rho, \|x\|, |z| \leq R^{-1/r}} |x|^r \frac{dy}{y} d\mu \leq CR^{-p/r} \max\left\{ \sup_{R_0 > R} s_{Z, d\mu}(R_0^{-1/r}), R^{-1} \right\}$$

Arguing similarly for $\operatorname{Re}(x^{-r}) = -R$, and combining all of which, we obtain,
III.3.5 Estimate There is a constant C depending on the interval I in which $|y|$ varies such that for R outside a set of finite $R^{-1}dR$ measure, for any $B > 0$ and ρ excluded from a set (depending on R) of measure at most B^{-1} in I ,

$$\int_{\substack{|\operatorname{Re}(x^{-r})| \leq R \\ |\operatorname{Im}(x^{-r})| = R \\ |y| = \rho, |z| \leq R^{-1/r}}} |d\operatorname{Re}(x^{-r})| d\mu \leq BCR^{-p/r} \max\left\{ \sup_{R_0 > R} s_{Z, d\mu}(R_0^{-1/r}), R^{-1} \right\}$$

The integrand above being positive for the given choice of orientation, we have an evident improvement in the quantification in ρ at our disposition. We now require a similar shape of estimate around the boundary parallel to the

imaginary axis, to wit, for χ to be chosen:

$$\begin{aligned} \int_{\substack{|\operatorname{Im}(x^{-r})| \leq R, \operatorname{Re}(x^{-r}) = -\chi R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} d\operatorname{Im}(x^{-r}) d\mu &= \int_{\substack{|\operatorname{Im}(x^{-r})| = R, \operatorname{Re}(x^{-r}) = -\chi R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} \operatorname{Im}(x^{-r}) d\mu + \\ \int_{\substack{|\operatorname{Im}(x^{-r})| \leq R, \operatorname{Re}(x^{-r}) = -\chi R \\ |y| = \rho, |z| \leq R^{-1/r}}} \operatorname{Im}(x^{-r}) d\mu &+ \int_{\substack{|\operatorname{Im}(x^{-r})| \leq R, \operatorname{Re}(x^{-r}) = -\chi R \\ |y| \leq \rho, |z| = R^{-1/r}}} \operatorname{Im}(x^{-r}) d\mu \end{aligned}$$

To deal with the first of the integrals on the right we need to use III.3.5, *i.e.*

$$\begin{aligned} \int_{e^{-1}}^1 \frac{d\chi}{\chi} \int_{\substack{|\operatorname{Im}(x^{-r})| = R \\ \operatorname{Re}(x^{-r}) = -\chi R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} |d\mu| &= \int_{\substack{|\operatorname{Im}(x^{-r})| = R \\ R \leq \operatorname{Re}(x^{-r}) \leq -e^{-1}R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} \left| \frac{d\operatorname{Re}(x^{-r})}{\operatorname{Re}(x^{-r})} \right| d\mu \\ &\leq \frac{e}{R} \int_{\substack{|\operatorname{Im}(x^{-r})| = R, |\operatorname{Re}(x^{-r})| \leq R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} d\operatorname{Re}(x^{-r}) d\mu \end{aligned}$$

Integrating both sides against $d\rho$, $\rho \in I$ is a little more convenient than applying III.3.5 directly, and leads to:

III.3.6 (a) Estimate Quantifiers as per III.3.5 but now with $\log \chi \times \rho$ varying in $(-1, 0) \times (a, b)$ excluded from a set depending on R of measure at most B^{-1} , then:

$$R \int_{\substack{|\operatorname{Im}(x^{-r})| = R, \operatorname{Re}(x^{-r}) = -\chi R \\ |y| \leq \rho, |z| \leq R^{-1/r}}} |d\mu| \leq BCeR^{-p/r} \max\left\{ \sup_{R_0 > R} s_{Z, d\mu}(R_0^{-1/r}), R^{-1} \right\}$$

Of the two remaining terms the easier is that over the face $|z| = R^{-1/r}$, which integrates against $\chi^{-1} d\chi$ to give,

$$\begin{aligned} \int_{e^{-1}}^1 \frac{d\chi}{\chi} \int_{\substack{|\operatorname{Im}(x^{-r})| \leq R \\ \operatorname{Re}(x^{-r}) = -\chi R \\ |y| \leq \rho, |z| = R^{-1/r}}} d\mu &= \int_{\substack{|\operatorname{Im}(x^{-r})| \leq R \\ -R \leq \operatorname{Re}(x^{-r}) \leq -e^{-1}R \\ |y| \leq \rho, |z| = R^{-1/r}}} \left| \frac{d\operatorname{Re}(x^{-r})}{\operatorname{Re}(x^{-r})} \right| d\mu \\ &\leq \frac{eC}{R^{1+p/r}} s_{Z, d\mu}(R^{-1/r}) \end{aligned}$$

for a constant C depending only on the order of approximation, so:

III.3.6 (b) Estimate Quantifiers as above, *i.e.* nothing to worry about except the order of approximation, then for $\log \chi \in (-1, 0)$ excluded from a set (depending on R) of measure at most B^{-1} for any B ,

$$\left| \int_{\substack{|\operatorname{Im}(x^{-r})| \leq R, \operatorname{Re}(x^{-r}) = -\chi R \\ |y| \leq \rho, |z| = R^{-1/r}}} \operatorname{Im}(x^{-r}) d\mu \right| \leq \frac{eCB}{R^{p/r}} s_{Z, d\mu}(R^{-1/r})$$

The final term will again employ the tangency along the foliation between $|y| = \text{const.}$ and $\operatorname{Re}(x^{-r}) = \text{const.}$, *viz:*

$$\int_a^b \frac{d\rho}{\rho} \int_{e^{-1}}^1 \frac{d\chi}{\chi} \int_{\substack{|\operatorname{Im}(x^{-r})| \leq R \\ \operatorname{Re}(x^{-r}) = -\chi R \\ |y| = \rho, |z| \leq R^{-1/r}}} |d\mu| = \int_{\substack{|\operatorname{Im}(x^{-r})| \leq R \\ -R \leq \operatorname{Re}(x^{-r}) \leq -e^{-1}R \\ |y| \in I, |z| \leq R^{-1/r}}} \left| \frac{d|y|}{|y|} \frac{d\operatorname{Re}(x^{-r})}{\operatorname{Re}(x^{-r})} \right| d\mu$$

While the integrand on the right may be bounded by,

$$C \left| \frac{\text{Im}(\nu x^r)}{\text{Re}(x^{-r})} \right| \frac{dy \otimes d\bar{y}}{|y|^2}$$

for a constant C depending on I and the order of approximation, so that:

$$\begin{aligned} & \int_a^b \frac{d\rho}{\rho} \int_{e^{-1}}^1 \frac{d\chi}{\chi} \left| \int_{\substack{|\text{Im}(x^{-r})| \leq R \\ \text{Re}(x^{-r}) = -\chi R \\ |y| = \rho, |z| \leq R^{-1/r}}} \text{Im}(x^{-r}) d\mu \right| \\ & \leq \frac{eC\nu}{R} \int_{\substack{|\text{Im}(x^{-r})| \leq R \\ -R \leq \text{Re}(x^{-r}) \leq -e^{-1}R \\ |y| \in I, |z| \leq R^{-1/r}}} \frac{dy \otimes d\bar{y}}{|y|^2} d\mu \end{aligned}$$

As before to estimate the term on the right one again uses co-equalisers, about $2R$ of them, of the form $\text{Im}(x^{-r}) > m$, m between $-[R]$ and $[R]$, and $\text{Re}(x^{-r}) \leq -e^{-1}R$, or, better, idem for $\text{Im}(\zeta)$ and $\text{Re}(\zeta)$ which are as near to the same thing as makes no difference, and whence:

III.3.6 (c) Estimate For quantifiers as per III.3.6 (a),

$$\begin{aligned} & \int_a^b \frac{d\rho}{\rho} \int_{e^{-1}}^1 \frac{d\chi}{\chi} \left| \int_{\substack{|\text{Im}(x^{-r})| \leq R \\ \text{Re}(x^{-r}) = -\chi R \\ |y| = \rho, |z| \leq R^{-1/r}}} \text{Im}(x^{-r}) d\mu \right| \\ & \leq \frac{BeC\nu}{R^{p/r}} \max \left\{ \sup_{R > R_0} s_{Z, d\mu}(R^{-1/r}), R^{-1} \right\} \end{aligned}$$

In order to lighten the notation a little, for every χ between e^{-1} and 1, put,

$$\|x\|_{\chi}^{-r} = \min\{(\chi)^{-1}|\text{Re}(x^{-r})|, |\text{Im}(x^{-r})|\}$$

and profit from the positivity of the integrands around the real and imaginary boundaries to improve the quantification in ρ by way of replacing I by an interval $(a', b') \subset (0, a)$ to obtain on repeating the above for $\text{Re}(x^{-r}) = R$,

III.3.7 Estimate There is a constant C depending on the interval I , such that in the presence of zero Segre class, for R outside a set of finite $R^{-1}dR$ measure, $\rho \in I$, and given B a set (depending on R) of $\log \chi \in (-1, 0)$ of measure B^{-1} outwith which,

$$\begin{aligned} & \int_{\substack{\|x\|_{\chi} = R^{-1/r} \\ |z| \leq R^{-1/r} \\ |y| \leq \rho}} (|d\text{Im}(x^{-r})| + |d\text{Re}(x^{-r})|) d\mu \leq \\ & \frac{BC}{R^{p/r}} \max \left(\sup_{R_0 > R} s_{Z, d\mu}(R_0^{-1/r}), R^{-1} \right) \end{aligned}$$

Plainly this computes any residue dx^{-i} , $1 \leq i \leq p+r$, *i.e.*

III.3.8 Corollary Everything as above in III.3.7, then for χ understood to depend on R in the way implied by III.3.7, and $1 \leq i \leq p + r + 1$,

$$\lim_{R \rightarrow \infty} \left| \int_{\substack{\|x\|_\chi = R^{-1/r} \\ |z| \leq R^{-1/r}, |y| \leq \rho}} \frac{dx}{x^i} d\mu \right| = 0$$

It also has the further important corollary for globalisation,

III.3.9 Corollary At the risk of a certain notational confusion for $\| \cdot \|$ any norm in the x -variable, $|y|$ limited to I as above, $o(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ a small function depending on the limiting Segre classes (with limit supposed zero) and the norm,

$$\oint_{\substack{|y|=\rho \\ \|x\|, |z| \leq \epsilon}} \frac{dy}{y} d\mu = \epsilon^p o(\epsilon)$$

proof Under the zero Segre class hypothesis, and, of course $d\mu$ without support in the exceptional divisor, we have as per the proof of III.3.2,

$$\int_{\substack{|y|=\rho \\ \|x\|, |z| \leq \epsilon}} \omega d\mu + \int_{\substack{\|x\|=\epsilon \\ |y| \leq \rho, |z| \leq \epsilon}} \omega d\mu + \int_{\substack{|z|=\epsilon \\ |y| \leq \rho, \|x\| \leq \epsilon}} \omega d\mu = 0$$

For ω the 1-form,

$$(1 + \nu x^r) \frac{dx}{x^{r+1}}$$

The last integral obviously has modulus $\epsilon^p o(\epsilon)$, and the middle one too by III.3.7 provided that $\| \cdot \|$ is taken to be $\| \cdot \|_\chi$. Meanwhile, on the face $|y| = \rho$ identified with $\gamma \in \mathbb{Z}(1)$,

$$\frac{dy}{y} d\mu = -(1 + O(x^N)) \omega d\mu$$

so dividing out the first integral by $-\gamma$ it is the asserted quantity up to an irrelevant error. The asserted quantity is, however, the average measure of an ϵ -ball in a translate, so, the estimate for some norm implies it for any. \square

Even were we to make use of III.3.2, the above complication would not be lessened because of the possibility that, under very particular conditions, imaginary ζ lines could go to imaginary ξ lines, so, we have to worry about (a probably imaginary) measure around the boundary $|\text{Im}(x^{-r})| = R$. In the case of exceptional nodes we have no such problem, and things are a lot easier, *i.e.*

III.3.1 (c) Set Up As per III.1.1 (a), but as per II.3.1 a formal generator in the completion at the point is given by,

$$qz \frac{\partial}{\partial z} + \frac{x^p y^q}{1 + y^q (R(x) + \lambda x^{p+r})} \left(y \frac{\partial}{\partial y} + \frac{x^r}{1 + \nu x^r} (qx \frac{\partial}{\partial x} - py \frac{\partial}{\partial y}) \right)$$

where x, y, z are formal coordinates, $p, q, r \in \mathbb{N}$, $\lambda, \nu \in \mathbb{C}$, $\deg R \leq r - 1$, and the exceptional divisor E has 2 components $E_1 : x = 0$, $E_2 : y = 0$, and we may even assume that the normal form is defined after completion in the former, albeit not the latter.

In our immediate context, take a sufficiently good approximation modulo a large power of E_1 in convergent variables x, y, z , and consider for $\chi \leq 1$,

$$\int_{\substack{\operatorname{Re}(x^{-r})=-R\chi \\ |\operatorname{Im}(x^{-r})|\leq R \\ |y|\leq\rho, |z|\leq R^{-1/r}}} d\operatorname{Im}(x^{-r})d\mu = \\ \int_{\substack{\operatorname{Re}(x^{-r})=-R\chi \\ |\operatorname{Im}(x^{-r})|\leq R \\ |y|=\rho, |z|\leq R^{-1/r}}} \operatorname{Im}(x^{-r})d\mu + \int_{\substack{\operatorname{Re}(x^{-r})=-R\chi \\ |\operatorname{Im}(x^{-r})|\leq R \\ |y|\leq\rho, |z|=R^{-1/r}}} \operatorname{Im}(x^{-r})d\mu$$

which is much simpler than before thanks to II.3.3, and we can argue as previously, in fact with quite a lot of simplification, to conclude:

III.3.10 Fact As per III.3.9, $\|\cdot\|$ any norm in the x -variable, and supposing zero Segre class, with $|y|$ limited to I as above, there is a function $o(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ such that,

$$\oint_{\substack{|y|=\rho \\ \|x\|, |z|\leq\epsilon}} \frac{dy}{y} d\mu = \epsilon^p o(\epsilon)$$

At the risk of a certain notational confusion we need a similar estimate on faces $|x| = \rho$, for $|x|$ confined to an interval I bounded away from zero. The quantity in question is simply the average measure of transversals at \mathbf{q} in the loop $\gamma : |x| = \rho$, and we already know that this is zero for \mathbf{q} outside of a strip around the negative real axis in the x^{-r} plane by II.3.3. Evidently for $\mathbf{p}, \mathbf{q} \in \gamma$ such that the measure at \mathbf{p} is zero we'd like to employ a variant of the basic construction III.1.1 by way of an approximation of the form:

III.1.1 bis Variant Define a co-dimension 1 real manifold Γ_ϵ by,

$$x \in (\mathbf{p}, \mathbf{q}), |z| \leq \epsilon, |t| \leq \epsilon$$

where t is a function of x, y which is an approximately invariant function defining E_2 .

Now, while generically a fairly trivial variant, here we should be cautious because III.3.1 (c) is not valid modulo arbitrary powers of E_2 . The reason for this problem, [MP2] VI.2.1 (g) & VI.2.2 (g) is because the operators,

$$\mathbb{I} - \alpha x^{r+1} \frac{\partial}{\partial x}$$

for α a constant, cannot be inverted in convergent power series. They can, however, be inverted in domains of argument up to $3\pi/r$ in the x -variable. To achieve the normal form one has to invert several of these with α going through finitely many positive and negative integers, which implies a certain incompatibility in the branching, so that ultimately one achieves

III.3.11 Fact For $N \in \mathbb{N}$, x confined to a sector S_N of aperture up to $2\pi/r$ branched within $\pi/2$ of the negative real axis, and y, z varying in a bi-disc Δ_N^2 , the normal form III.3.1 (b) can be realised modulo $\mathcal{O}(-NE_2)$.

Plainly given II.3.3 this will be adequate for our current considerations and we apply it as follows: take $N \in \mathbb{N}$ to be chosen, restrict ourselves to $U_N =$

$S_N \times \Delta_N^2$, take t to be our approximation (basically $y \exp(x^{-r})$) such that $\partial t \in (y^N)$, confine $|x|$ to an interval I_N bounded away from 0 with $\gamma_{\mathbf{p}, \mathbf{q}}$ a simple path in γ joining \mathbf{p}, \mathbf{q} , then:

$$\int_{\substack{T_{\mathbf{p}} \\ |t|, |z| \leq \epsilon}} d\mu \leq \int_{\substack{|z|=\epsilon, |t| \leq \epsilon \\ x \in \gamma_{\mathbf{p}, \mathbf{q}}}} |d\mu| + \int_{\substack{|z| \leq \epsilon, |t| = \epsilon \\ x \in \gamma_{\mathbf{p}, \mathbf{q}}}} |d\mu|$$

We estimate the integrals on the right, or, better, their averages in the problematic region exactly as in III.2.2 (a)/(b) respectively, and obtain:

III.3.12 Fact There is an interval I bounded away from zero such that in the presence of zero Segre class, there is a function $o(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ for which if $\rho \in I$ is excluded from a set of measure B^{-1} , then,

$$\oint_{\substack{|x|=\rho \\ |y|, |z| \leq \epsilon}} \frac{dx}{x} d\mu = B\epsilon^q o(\epsilon)$$

Here the question of zero Segre class is very much a hypothesis of convenience, which for $\xi = (x^p y^q)^{-1}$ we may apply to compute residues as follows,

$$\begin{aligned} 0 &= \oint_{\substack{|\xi|=\epsilon^{-1}, |z| \leq \epsilon \\ |x| \leq X |y| \leq Y}} \frac{d\xi}{\xi} d\mu + \oint_{\substack{|\xi| \leq \epsilon^{-1}, |z| = \epsilon \\ |x| \leq X |y| \leq Y}} \operatorname{Re} \left(\frac{\partial \xi z}{\xi \partial z} \right) \frac{dz}{z} d\mu \\ &+ \oint_{\substack{|x|=X, |z| \leq \epsilon \\ |y|^q \leq \epsilon X^{-p}}} \operatorname{Re} \left(\frac{\partial \xi x}{\xi \partial x} \right) \frac{dx}{x} d\mu + \oint_{\substack{|y|=Y, |z| \leq \epsilon \\ |x|^p \leq \epsilon Y^{-q}}} \operatorname{Re} \left(\frac{\partial \xi y}{\xi \partial y} \right) \frac{dy}{y} d\mu \end{aligned}$$

where the coordinates are chosen so that $x = 0, y = 0$ are the invariant exceptional divisors, the approximation is as good as we like modulo E_1 , and we use [MP2] V.1.9 to guarantee $\partial z = z$ modulo $(x, y)^N$ for $N \in \mathbb{N}$ as large as we please, while X, Y are just things in the intervals appearing in III.3.9/12. In particular, by the above, the final two integrals are $\epsilon o(\epsilon)$, while $\partial z = z \bmod \xi^N$, as ever N large, so the term on the $|z| = \epsilon$ face goes like ϵ times the Segre class, whence:

III.3.13 Corollary Let things be as in III.3.1 (c) with X, Y arising from the intervals of III.3.9 and III.3.12 respectively, then for $o(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$,

$$\oint_{\substack{|\xi|=\epsilon^{-1}, |z| \leq \epsilon \\ |x| \leq X |y| \leq Y}} \frac{d\xi}{\xi} d\mu = \epsilon o(\epsilon)$$

and (locally at least) all residues of interest are zero.

III.4 Other linear(ish) singularities

Let us gather together the remaining possibilities for the singularities that we have to deal with starting with the most straightforward,

III.4.1(a) Set Up In a neighbourhood U of foliated 3-fold of a point in the singular locus after completion in the same, the foliation admits a formal generator of the form,

$$z \frac{\partial}{\partial z} + \frac{x^p y^q}{1 + \nu x^p y^q} \left(x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \right)$$

where x, y, z are formal coordinates, $p, q \in \mathbb{N}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $(\operatorname{Re}(p+q\lambda), \operatorname{Re}(p/\lambda+q))$ not both negative if we avail ourselves of II.2.1 (b)), and sufficient a priori blowing up having been performed to guarantee that $x = 0$, $y = 0$ are local equations for exceptional divisors E_1 , and E_2 respectively, with $E = E_1 + E_2$.

Here with exactly the same proof (and, in fact, somewhat easier since the convergence of $y = 0$ is given) III.2.2 (c) or III.2.2 (c) bis. holds. The residue calculation is, therefore, particularly straightforward since, for, say $|x| \leq X$, $|y| \leq Y$ in the the domain of the appropriate coordinates, and again $\xi = (x^p y^q)^{-1}$,

$$\begin{aligned} 0 &= \oint_{\substack{|\xi|=\epsilon^{-1}, |z|\leq\epsilon \\ |x|\leq X |y|\leq Y}} \frac{d\xi}{\xi} d\mu + \oint_{\substack{|\xi|\leq\epsilon^{-1}, |z|=\epsilon \\ |x|\leq X |y|\leq Y}} \operatorname{Re}\left(\frac{\partial\xi z}{\xi\partial z}\right) \frac{dz}{z} d\mu \\ &+ \oint_{\substack{|x|=X, |z|\leq\epsilon \\ |y|^q \leq \epsilon X^{-p}}} \operatorname{Re}\left(\frac{\partial\xi x}{\xi\partial x}\right) \frac{dx}{x} d\mu + \oint_{\substack{|y|=Y, |z|\leq\epsilon \\ |x|^p \leq \epsilon Y^{-q}}} \operatorname{Re}\left(\frac{\partial\xi y}{\xi\partial y}\right) \frac{dy}{y} d\mu \end{aligned}$$

under the hypothesis of zero Segre class. Again, the second term is $\epsilon s_{Z, d\mu}(\epsilon)$, the other two are governed by III.2.2 (c), so:

III.4.2 Fact Shrinking X, Y a little to simplify the quantification, then for ϵ outwith a set of finite $\epsilon^{-1}d\epsilon$ measure, and zero Segre class around E ,

$$\oint_{\substack{|\xi|=\epsilon^{-1}, |z|\leq\epsilon \\ |x|\leq X |y|\leq Y}} \frac{d\xi}{\xi} d\mu = \epsilon o(\epsilon)$$

Whence, any residue that we need to calculate is not just zero locally in a natural way, but the measure admits best possible estimates of the form III.2.2 (c) for its mass on transversals.

The next cases are quite different, *i.e.*

III.4.1(b) Set Up Exactly as per III.4.1 (a) with $\lambda \in \mathbb{R}_{<0} \setminus \mathbb{Q}$, or, $k, l \in \mathbb{N}$ relatively prime,

$$z \frac{\partial}{\partial z} + \frac{x^i y^j (x^k y^l)^n}{1 + x^i y^j \nu(x^k y^l)} \left(lx \frac{\partial}{\partial x} - ky \frac{\partial}{\partial y} \right)$$

where x, y, z are formal coordinates, $n, i, j \in \mathbb{Z}_{\geq 0}$, ν a formal function in one variable. In the latter case, when choosing an approximation, it is, therefore, to be understood that ν is truncated to some appropriately high order.

Plainly the approximately invariant function $t = yx^{-\lambda}$ may have holonomy, but $|t|$ does not- one could use $x^k y^l$ in the rational case, but that only leads to a simplification globally if $i = j = 0$. In any case since $\lambda \in \mathbb{R}_{<0}$, the boundary $\max\{|z|, |t|\} = \epsilon$ is a perfectly good way to calculate the residue. Further, for $N \in \mathbb{N}$ as large as we like, we have ∂t at worst of the form $t^{N+1} \partial x/x$, while any 1-form ω of which we have to compute the residue is no worse than $t^{-m} \partial x/x$ for some fixed m . More precisely, locally the residue has the form,

$$\int_{|z|=\epsilon, |t|\leq\epsilon} \omega d\mu + \int_{|t|=\epsilon, |z|\leq\epsilon} \omega d\mu$$

where the first term will never be worse than the Segre class, while for $m \in \mathbb{N}$ large, albeit smaller than say $N/2$,

$$\int_0^\epsilon \frac{d\delta}{\delta^{m+1}} \left| \int_{\substack{|t|=\delta \\ |z|\leq\delta}} \omega d\mu \right| \leq \int_{\substack{|t|\leq\epsilon \\ |z|\leq\epsilon}} |t|^{-m} \left| \frac{dt}{t} \omega \right| d\mu \leq o(\epsilon)$$

and whence,

III.4.3 Fact For ω a 1-form of which we require to calculate the residue, for ϵ outwith a set of finite $\epsilon^{-1}d\epsilon$ measure,

$$\left| \int_{\max\{|t|,|z|\}} \omega d\mu \right| \leq \max\{s_{Z,d\mu}(\epsilon), \epsilon\}$$

One should, however, bear in mind that this is a local estimate, which lacks the good bound on transversals occurring in III.4.1 (a), so, patching it to the global procedure is a little more delicate.

This leaves us with the general rational case to tackle, *i.e.*

III.4.1(c) Set Up Exactly as per III.4.1 (b) but with normal form,

$$z \frac{\partial}{\partial z} + \frac{x^i y^j (x^k y^l)^n}{1 + x^i y^j (R(x^i y^j) + \lambda (x^k y^l)^{n+r})} \left(\left(l x \frac{\partial}{\partial x} - k y \frac{\partial}{\partial y} \right) + \frac{(x^k y^l)^r}{1 + \nu (x^k y^l)^r} \left(j x \frac{\partial}{\partial x} - i y \frac{\partial}{\partial y} \right) \right)$$

where x, y, z are formal coordinates, $n, i, j \in \mathbb{Z}_{\geq 0}$, $\lambda, \nu \in \mathbb{C}$, R of degree at most $r-1$, $r \in \mathbb{N}$, and one replaces the fields $jx \frac{\partial}{\partial x} - iy \frac{\partial}{\partial y}$ by, say $x \frac{\partial}{\partial x}$ if $i = j = 0$. Alternatively, should this occur, one may have recourse to the normal form,

$$z \frac{\partial}{\partial z} + (x^k y^l)^n \left(R(x^k y^l) \left(l x \frac{\partial}{\partial x} - k y \frac{\partial}{\partial y} \right) + \frac{(x^k y^l)^r}{1 + \nu (x^k y^l)^{n+r}} x \frac{\partial}{\partial x} \right)$$

where, now, R is of degree at most r , $R(0) \neq 0$, and all cases, $p = i + kn$, $q = j + ln$ are in \mathbb{N} .

Taking a convergent approximation to an order to be decided, we have for the approximating foliation, invariant formal functions of the form,

$$t = x y^{l/k} \exp\left(\sum_{m=1}^{\infty} (x^k y^l)^{mr} P_m(\log y)\right), \quad s = y x^{k/l} \exp\left(\sum_{m=1}^{\infty} (x^k y^l)^{mr} Q_m(\log x)\right)$$

for P_m, Q_m polynomials of degree at most m , and,

$$P_1(\log y) = k^{-2}(k\tilde{j} - \tilde{l}i) \log y, \quad Q_1(\log x) = -l^{-2}(k\tilde{j} - \tilde{l}i) \log x$$

where $(\tilde{i}, \tilde{j}) = (i, j)$ if one of i, j in III.4.1 (c) is non-zero, and $(1, 0)$ otherwise. As such, say $P_1(\log y) = p_1/k \log y$, and $Q_1(\log x) = q_1/l \log x$ to lighten the notation, and, observe, that these have formal holonomy of the form,

$$\begin{aligned} t &\mapsto \exp\left(\frac{l\gamma}{k}\right) t \left(1 + \frac{p_1\gamma}{k} t^{kr} + O(t^{2kr})\right) \\ s &\mapsto \exp\left(\frac{k\gamma}{l}\right) s \left(1 + \frac{q_1\gamma}{l} s^{kr} + O(s^{2lr})\right) \end{aligned}$$

for γ a loop in $x = 0$, respectively $y = 0$, identified to its canonical image in $\mathbb{Z}(1)$. Truncating t and s to an appropriately high order, we therefore have approximately invariant functions of the form,

$$\begin{aligned} t^k &= x^k y^l (1 + p_1 (x^k y^l)^r \log y + O((x^k y^l)^{2r} \log^2 y)) \\ s^l &= x^k y^l (1 + q_1 (x^k y^l)^r \log x + O((x^k y^l)^{2r} \log^2 x)) \end{aligned}$$

and an appropriate function to use in constructing a boundary around the singularity is $\max\{|z|, |w|\}$, where,

$$|w| := \max\{|t|^k, |s|^l\}$$

Clearly w is discontinuous, so an appropriate estimate for the mass will come from the almost holonomy estimate. To this end, observe that the approximating holonomy has the form,

$$t^{-kr} \mapsto t^{-kr} - rp_1 \gamma + o(t)$$

whence for $F(\epsilon)$ the region $|t|^k < \epsilon$, the sub-region in which, $\operatorname{Re}(t^{-kr}/p_1 \gamma) \leq 0$ maps into itself, while that where $\operatorname{Re}(t^{-kr}/p_1 \gamma) \geq 0$ maps out of itself. Consequently, we need both regions, say, $F_-(\epsilon)$ and $F_+(\epsilon)$ to estimate the discontinuity, *i.e.* their signs are opposite in the basic construction, albeit:

$$|\mathbb{I}_{F(\epsilon)}(t) - \mathbb{I}_{F(\epsilon)}(t^h)| \leq |\mathbb{I}_{F_+(\epsilon)}(t) - \mathbb{I}_{F_+(\epsilon)}(t^h)| + |\mathbb{I}_{F_-(\epsilon)}(t) - \mathbb{I}_{F_-(\epsilon)}(t^h)|$$

Notice also, that $|t|$ doesn't change much in the region of discontinuity, more precisely,

$$\frac{|\mathbb{I}_{F(\epsilon)}(t) - \mathbb{I}_{F(\epsilon)}(t^h)|}{|t|^p} \leq \frac{(1 + O(\epsilon^r))}{\epsilon^{p/k}} |\mathbb{I}_{F(\epsilon)}(t) - \mathbb{I}_{F(\epsilon)}(t^h)|$$

With this in mind, say $\log y$, $\log x$, are branched along b_y , b_x in \mathbb{R}_+ respectively, for $|y| \leq Y$, respectively $|x| \leq X$, in the spirit of the variation III.1.3, so that $d|y|$ may be taken as a length form along the branch. Now divide through by $\epsilon^{p/k}$, and integrate both sides of the basic construction III.1.1 against $|y|^{-1-q+pl/k} d|y|$ to obtain,

$$\begin{aligned} \int_{\substack{|z|, |t|^k \leq \epsilon \\ y \in b_y, |y| \leq Y}} \frac{|\mathbb{I}_{F(\epsilon)}(t) - \mathbb{I}_{F(\epsilon)}(t^h)|}{|t|^p |y|^{1+q-pl/k}} d|y| d\mu &\leq 2' \int_{\substack{|z|=\epsilon, |t|^k \leq \epsilon \\ |y| \leq Y}} \left| \frac{d|y|}{|t|^p |y|^{1+q-pl/k}} \right| d\mu + \\ \epsilon^{-p/k} \left| \int_{\substack{t \in \partial F_+(\epsilon) \\ |z| \leq \epsilon, |y| \leq Y}} \frac{d|y|}{|t|^{p/k} |y|^{1+q-pl/k}} d\mu \right| &+ \epsilon^{-p/k} \left| \int_{\substack{t \in \partial F_-(\epsilon) \\ |z| \leq \epsilon, |y| \leq Y}} \frac{d|y|}{|t|^{p/k} |y|^{1+q-pl/k}} d\mu \right| \end{aligned}$$

For $2'$ anything bigger than 2 provided ϵ is sufficiently small. The choice of integrand was cooked up to be commensurate with $|x^p y^q|^{-1} d \log |y|$, so the first integral on the left is certainly no worse than $s_{Z, d\mu}(\epsilon)$. As for the other two, ∂t is at worst $|t|^N \partial y / y$ for N as large as we like, so arguing in the previous way, these terms are as irrelevant as we wish to make them, and whence,

III.4.4 Fact For ϵ outside a set of finite $\epsilon^{-1}d\epsilon$ measure we have the following estimate for the discontinuity in t (and similarly for s),

$$\int_{\substack{y \in b_y \\ |y| \leq Y}} \frac{d|y|}{|y|^{q+1}} \int_{|z|, |t|^k \leq \epsilon} |x|^{-p} |\mathbb{I}_{|t|^k \leq \epsilon}(t) - \mathbb{I}_{|t|^k \leq \epsilon}(t^h)| \leq 2' \max\{s_{Z, d\mu}(\epsilon), \epsilon\}$$

Consequently we have estimated the difference between the current case, III.4.1 (c), and the previous one III.4.1 (b). By way of notation, to emphasise this call the discontinuity $w(\epsilon)$ and interpret the integral over $w(\epsilon)$ as an integral against an elementary function in t - *i.e.* $\mathbb{I}_{w(\epsilon)}$ is a difference $\mathbb{I}_{w_+(\epsilon)} - \mathbb{I}_{w_-(\epsilon)}$ of characteristic functions, where $w_*(\epsilon) \subset F_*(\epsilon)$. As such for ω a 1-form of which we must calculate the residue, this will be the limit in ϵ of,

$$\int_{\substack{|z|=\epsilon \\ |w| \leq \epsilon}} \omega d\mu + \int_{\substack{|w|=\epsilon \\ |z| \leq \epsilon}} \omega d\mu + \int_{\substack{w(\epsilon) \\ |w|, |z| \leq \epsilon}} \omega d\mu$$

with the implicit restrictions $|x| \leq X$, respectively $|y| \leq Y$. As such, we have an extra term not appearing in III.4.1 (b) that we've seen how to bound, whence for $3' > 3$,

III.4.5 Fact Everything as per III.4.3 then,

$$\left| \int_{\max\{|z|, |w|\}=\epsilon} \omega d\mu + \int_{\max\{|z|, |w|\} \leq \epsilon}^{w(\epsilon)} \omega d\mu \right| \leq 3' \max\{s_{Z, d\mu}(\epsilon), \epsilon\}$$

Furthermore, while unlike III.4.3 there are estimates on the mass of transversals- which we've basically seen and we'll quantify in III.4.7- one should be aware that they're well short of those found in III.4.2, so, again, there is need for caution when going from local to global.

Before proceeding to the said mass estimates, let us note that a small, and necessary, variation is possible, *viz.*,

III.4.6 Remark Globally one cannot quite use the normal form, and one must make a coordinate change of the form,

$$x \mapsto \phi x(1 + a(x^k y^l)), \quad y \mapsto \theta y(1 + b(x^k y^l))$$

for ϕ, θ constants and a, b functions of a single variable. Such changes do not quite preserve the normal form, since the nilpotent part of the plane field in III.4.1 (c) becomes,

$$(x^k y^l)^r \left(\tilde{a}(x^k y^l) x \frac{\partial}{\partial x} + \tilde{b}(x^k y^l) y \frac{\partial}{\partial y} \right)$$

again for \tilde{a}, \tilde{b} functions of a single variable. On the other hand this is an irrelevant perturbation, *e.g* the formal power series for t has the form,

$$xy^{l/k} \exp\left(\sum_{m=r}^{\infty} (x^k y^l)^m P_m(\log y)\right)$$

where P_m is a polynomial of degree at most $\lfloor m/r \rfloor$, and up to homotheties $x \mapsto \phi x$, $y \mapsto \theta y$ the leading term is exactly as before. In particular such coordinate changes, together with the implied change in w , change absolutely nothing whether in III.4.1 (c), or, easier, the rational case of III.4.1 (b).

Finally let us give the mass bound proper to the set up III.4.1 (c). Specifically to keep ourselves in the notation of §III.3, let us write the approximating holonomy as,

$$\zeta := t^{-kr} \mapsto \zeta - \gamma + o\left(\frac{1}{|\zeta|}\right)$$

then we can bound strips, $R < \text{Im}(\zeta) < R + 2\pi$, or $C < \text{Im}(\zeta) < C + 2\pi$, should $\text{Re}(\zeta) > R, C < R$ in the desired way, *i.e.* $R^{-p/kr}$ times the Segre class. Plainly this is inadequate to get the same bound for the whole transversal, but arguing as per III.3.4 it does give,

III.4.7 Fact Quantifiers as per III.2.2 (c), so, in particular $|y|$ in some interval I , then transversals $T_{\mathbf{p}}$ satisfy the bound,

$$\int_{|z|, |t| \leq \epsilon} |t|^{kr} d\mu \leq B\epsilon^p \max\{\sup_{\delta \leq \epsilon} s_{Z, d\mu}(\delta), \epsilon\}$$

and similarly for the transversals $|z|, |s| \leq \epsilon$. Indeed, one could achieve better quantification, but as above it also applies to the rational case of III.2.1(b), where one could use this bound to remove the big height condition.

III.5 The Beast

While in a certain sense straightforward, there is a risk of complication as one passes along the singular locus and the number of eigenvalues jumps from 1 to 2. This occurs at,

III.5.1 Set Up In a neighbourhood U of the singular locus of a foliated 3-fold, after sufficient blowing up we have exceptional divisors E_1, E_2 such that after completion in the former, but not the latter, the foliation has a formal generator,

$$z \frac{\partial}{\partial z} + x^p y \frac{\partial}{\partial y}$$

for x, y, z formal coordinates, E_1 given by $x = 0$, E_2 , $y = 0$, and $E = E_1 + E_2$.

The point here is that even the centre manifold need not be defined around the completion in E_2 , cf. [M4] §I.5, [MP2] II.2. In any case, take convergent coordinates x, y, z such that $x = 0, y = 0$ define E_1 , and E_2 respectively, and the singular locus is the x and y axis, with generically 2 eigenvalues on the former versus 1 on the latter. We can, therefore, suppose that there are functions a, b, c, d and some large $N \in \mathbb{N}$ such that,

$$\partial = (z + ax^N y) \frac{\partial}{\partial z} + x^p (1 + bx^N) \frac{\partial}{\partial y} + (cy + dz) x^{N+1} \frac{\partial}{\partial x}$$

is a convergent generator. In particular blowing up in the 2-eigenvalue component, we have $\partial x \in (x^N y)$ and otherwise the singularity becomes log-flat (*i.e.*

trivial from the point of view of computing residues). We can also improve the order of vanishing of $\partial z - z$ along E_2 albeit at the price of a loss of domain in x . Regardless, for ω a 1-form of which we must calculate the residue, *i.e.* $\partial(\omega)$ a function, against an invariant measure without support in E , and zero Segre class around the same, consider the following strategy for $\epsilon, \delta > 0$,

$$\int_{\substack{\max\{|z|,|x|\}=\epsilon \\ |y|\geq\delta}} \omega d\mu + \int_{\substack{\mathbb{I}_{E_2}=\delta \\ \max\{|z|,|x|\}\geq\epsilon}} \omega d\mu$$

where \mathbb{I}_{E_2} is a distance function to E_2 coinciding with $|y|$ locally. The former of these two integrals divides up as,

$$\int_{\substack{|z|=\epsilon,|x|\leq\epsilon \\ |y|\geq\delta}} \omega d\mu + \int_{\substack{|x|=\epsilon,|z|\leq\epsilon \\ |y|\geq\delta}} \omega d\mu$$

Of which, the first integrand is plainly absolutely integrable even as $\delta \rightarrow 0$, and the integral itself is bounded by the Segre class. As to the second, we may write,

$$\omega = f \frac{dy}{x^p y}$$

for f some function, so that for some m to be chosen,

$$\int_0^\epsilon \frac{dr}{r^{m+1}} \left| \int_{\substack{|x|=r,|z|\leq r \\ |y|\geq\delta}} \omega d\mu \right| \leq \int_{\substack{|z|,|x|\leq\epsilon \\ |y|\geq\delta}} \left| f \frac{d|x|}{|x|^{m+1}} \frac{dy}{x^p y} \right| d\mu$$

while by the above, after blowing up, $\partial x \in (x^{N+1}y)$, so up to a constant the above right hand side is at worst,

$$\epsilon^{N-(m+p)} \int_{\substack{|z|,|x|\leq\epsilon \\ |y|\geq\delta}} dy \otimes d\bar{y} d\mu$$

which does not depend on δ . Whence, keeping $\epsilon > 0$ we can let $\delta \rightarrow 0$ first. By the hypothesis on the Segre class, there is no residue around E_2 for $\epsilon > 0$, so our residue is, in fact,

$$\lim_{\epsilon \rightarrow 0} \int_{\max\{|z|,|x|\}=\epsilon} \omega d\mu$$

with no restriction on $|y|$, and whence zero for zero Segre class. Consequently, **III.5.2 Remark/Summary** Notwithstanding the rather complicated formal structure around E_2 the beast poses, in the presence of zero Segre class, it is no more or less of a problem than any other point in the singular locus where the induced foliation is smooth. If, however, we wished to get quantification of a given residue in terms of a possibly non-vanishing Segre class, it would be a different story. Fortunately, this isn't our situation, and the only further point to remark is that for globalisation, we'll have to apply the above not for x but for $h(x) = \phi x(1 + O(x))$ some convergent automorphism, but, plainly this is a non-diffculty. As such, in practice, the beast poses no problem, and it will be passed over without comment, *i.e.* it is to be understood that in collapsing the boundary to the singular locus the above strategy of first letting $\delta \rightarrow 0$ for $\epsilon > 0$ fixed, then taking $\epsilon \rightarrow 0$ is being observed.

Globalisation

IV.1 Warm up case

As the title suggests we first do an example in order to understand where the difficulties lie, to wit:

IV.1.1 Set Up Let U be a foliated 3-dimensional tubular neighbourhood of a smooth compact curve Y , such that:

- (a) Y is the singular locus of the foliation.
- (b) The foliation has canonical singularities along Y , but only 1-eigenvalue at each point.
- (c) In the formal centre manifold \widehat{Z} obtained on completing in Y , the induced foliation is smooth and Y is invariant.
- (d) A priori blowing up has been performed so that for some smooth connected exceptional divisor E , $Y = E \cap \widehat{Z}$ and a hypothesised invariant measure $d\mu$ has neither support on E nor Segre class around the same.

As it happens, one can show under these hypothesis that Y is an elliptic curve, but we'll eschew this so as to illustrate some general features. Around every point in Y one may (usual conventions) find formal coordinates x, y, z such that the foliation is given by,

$$z \frac{\partial}{\partial z} + x^p \frac{\partial}{\partial y}$$

where formal means completion in Y , or, indeed E , so the function y is actually convergent, and, of course, p is fixed. Now there are two problems,

- (i) The existence domain for a conjugation from analytic to such a normal form while a bi-disc in (y, z) may be a sector of aperture no more than π/p in x , and this is too small to be useful, [MP2] II.1.5.
- (ii) Even if one had a conjugation on a larger sector, there would still be the problem of how these coordinates patch as one moves whether in Y or the argument of x , with the latter being rather bad.

Let us therefore circumvent these difficulties by an appropriate approximation procedure, beginning with the formal centre manifold. On an open cover $\coprod_{\alpha} U_{\alpha} \rightarrow U$, we have functions z_{α} such that $z_{\alpha} = 0$ is as good an approximation, say modulo $\mathcal{O}(-NE)$ to the centre manifold as we please, and so,

$$z_{\alpha} = g_{\alpha\beta} z_{\beta} \text{ mod } \mathcal{O}(-NE)$$

where $g_{\alpha\beta} \in H^1(U_N, \mathbb{G}_m)$ and U_N is defined via the exact sequence,

$$0 \longrightarrow \mathcal{O}_U(-NE) \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{O}_{U_N} \longrightarrow 0$$

so for \mathcal{A}_* smooth functions we may profit from the preparation theorem to conclude that,

$$0 \longrightarrow \mathcal{A}_U(-NE) \longrightarrow \mathcal{A}_U \longrightarrow \mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}_{U_N} =: \mathcal{A}_{U_N} \longrightarrow 0$$

is exact. Furthermore,

$$|g_{\alpha\beta}|^2 \in \mathbb{H}^1(U_N, \mathcal{A}_{U_N, +}^\times) \xleftarrow[\exp]{\sim} \mathbb{H}^1(U_N, \mathcal{A}_{U_N, \mathbb{R}})$$

where the subscripts $+$, \mathbb{R} indicate positive real valued, and real valued respectively. The above arrow being an isomorphism, we may find real valued functions ϕ_α such that,

$$e^{-\phi_\alpha}|z_\alpha|^2 - e^{-\phi_\beta}|z_\beta|^2 \in \Gamma(U_{\alpha\beta}, \mathcal{A}_U(-NE))$$

whence for some ψ_α vanishing to order N along E , we have:

$$\|z\|^2 := e^{-\phi_\alpha}|z_\alpha|^2 + \psi_\alpha = e^{-\phi_\beta}|z_\beta|^2 + \psi_\beta \in \Gamma(U, \mathcal{A}_U)$$

where, notation notwithstanding, $\|z\|^2$ might take negative values, albeit we only care about its positive values so one could take a max with zero if one prefers. For convenience, we can arrange local generators ∂_α of the foliation so that,

$$\partial_a(z_\alpha) = z_\alpha \text{ mod } \mathcal{O}(-NE)$$

and whence for some f_α vanishing to order N along E ,

$$\partial_\alpha \|z\|^2 = (1 - \partial_\alpha \phi_\alpha) \|z\|^2 + e^{-\phi_\alpha} 2\text{Re}(z_\alpha \bar{f}_\alpha) + (\partial_\alpha \psi_\alpha - (1 - \partial_\alpha \phi_\alpha) \psi_\alpha)$$

Consequently if $\|z\|^2 = \epsilon^2$ and the distance to E is also ϵ and $N \gg p$, we have:

$$\frac{\partial_\alpha \|z\|^2}{\|z\|^2} = (1 + O(\epsilon^p))$$

and similarly for $\bar{\partial}_\alpha$.

The discussion of approximation to the exceptional divisor is a little more complicated. In the first place as per §III.1 we have the holonomy representation of the formal foliation in \widehat{Z} to order N , *i.e.*

$$h : \pi_1(Y) \longrightarrow \text{Aut}\left(\frac{\mathbb{C}[t]}{t^N}\right)$$

and for $g \geq 0$ the genus of Y we choose a basis of simple closed curves, $\sigma_1, \tau_1; \dots; \sigma_g, \tau_g$ in the usual way, so that,

$$\sigma_i \cdot \tau_j = \delta_{ij}, \quad \sigma_i \cdot \sigma_j = \tau_i \cdot \tau_j = 0$$

Even though Y is compact, its technically useful (and eventually necessary in the presence of singularities of the induced foliation in \widehat{Z} , so the current notation is temporary) to view the σ_i, τ_i as homology classes, with a dual basis $\sigma_i^\vee (=$

τ_i , $\tau_i^\vee (= \sigma_i)$ in co-homology, since the holonomy around the homology class gives rise to a discontinuity in the (approximately) invariant function describing $\mathcal{O}_{\widehat{Z}}(-E)$ along the co-homology class. More precisely there is a formal function t (even on all of $\widetilde{Y} \times \mathrm{Spf}\mathbb{C}[[t]]$, \widetilde{Y} the universal cover) such that the co-equalisers satisfy,

$$t|_{\tau_i^+} = h(\sigma_i)(t|_{\tau_i^-})$$

where τ_i^+ , τ_i^- are the respective sides of τ_i , similarly for σ_i and τ_i interchanged, and is continuous otherwise. A priori this only holds in \widehat{Z} but blowing up N -times in Y it becomes true in the completion \widehat{U} of U in Y modulo $\mathcal{O}(-NE)$. Now for \widetilde{U} the universal cover of U , view $|t|^2$ as a smooth section of $\mathcal{A}_{\widetilde{U}^N}(E \otimes \bar{E})$, then it may be lifted to a global section of $\mathcal{A}_{\widetilde{U}}(E \otimes \bar{E})$ which we denote by the same letter, so that for ∂_α any local generator of the foliation,

$$\partial_\alpha |t|^2 \in \mathcal{A}_{\widetilde{U}}(-NE) + \mathcal{A}_{\widetilde{U}}(-N\bar{E})$$

while on restricting to the fundamental domain implicit in our basis for the homology we have the same defined on U with a discontinuity such that,

$$|t|^2|_{\tau_i^+} = |h(\sigma_i)(t|_{\tau_i^-})|^2 \bmod \mathcal{O}(-NE)$$

and, in addition, the discontinuity notwithstanding, $|t|^2$ is commensurable in U to an actual continuous distance function \mathbb{I}_E to the exceptional divisor, *i.e.* for constants c, C one has,

$$c \leq \frac{|t|^2}{\mathbb{I}_E^2} \leq C$$

We can now form a distance function $\max\{\|z\|^2, |t|^2\}$ which will be discontinuous along real hypersurfaces (denoted by the same letter) which cut Y in σ_i , respectively τ_i . As post III.4.4 this discontinuity will be a difference of set functions, so we put,

$$\mathbb{I}_{\tau_i(\epsilon)} := \mathbb{I}_\epsilon(t^{\sigma_i}) - \mathbb{I}_\epsilon(t)$$

for \mathbb{I}_ϵ the characteristic function of $|t|^2 \leq \epsilon$, and understand by the integral “over” $\tau_i(\epsilon)$ the integral against this elementary function, and similarly for the σ_i , and τ_i interchanged. Consequently for ω any 1-form of which we must calculate the residue, it will be the limit in $\epsilon \rightarrow 0$ of,

$$\int_{\substack{\|z\|=\epsilon \\ |t|\leq\epsilon}} \omega d\mu + \int_{\substack{|t|=\epsilon \\ \|z\|\leq\epsilon}} \omega d\mu + \sum_{i=1}^g \left(\int_{\substack{\sigma_i(\epsilon) \\ \|z\|\leq\epsilon}} \omega d\mu + \int_{\substack{\tau_i(\epsilon) \\ \|z\|\leq\epsilon}} \omega d\mu \right)$$

Much of which will be handled exactly as in the local cases, *i.e.* the first term is plainly bounded by the Segre, the second will prove negligible outside a set of finite $\epsilon^{-1}d\epsilon$ measure, and we’ll bound the final ones using the almost holonomy estimate. The only really new feature is that the holonomy group may be non-commutative.

To this end we have to consider the structure of the representation beginning with its first order behaviour, *i.e.* the character:

$$\chi : \pi_1(Y) \xrightarrow{h} \text{Aut}\left(\frac{\mathbb{C}[t]}{t^N}\right) \longrightarrow \mathbb{G}_m$$

and, rather more importantly, $|\chi|$ in \mathbb{R}_+^\times which we previously encountered in §III.2. The situation where $|\chi| \neq 1$ is particularly advantageous. Indeed $|\chi|(\sigma_i) \neq 1$, and ϵ sufficiently small imply that the co-equaliser $\mathbb{I}_{\tau_i(\epsilon)}$ is actually the characteristic function of a set (or maybe its negative). More precisely, say, without loss of generality, $|\chi|(\sigma_i) > 1$ then $\mathbb{I}_{\tau_i(\epsilon)} > 0$, and dominates the characteristic function of the annulus,

$$\{\epsilon > |t| > q\epsilon\}$$

for any $q > |\chi|(\sigma_i)^{-1}$, again for ϵ depending on q sufficiently small. This implies exactly as in III.2.2 that for \mathbf{p} a point of intersection between σ_i and its dual (more correctly between a family of perturbations of σ_i which play the role of the interval I of op. cit) we can use the basic construction III.3.1 to estimate the measure not just of $\tau_i(\epsilon)$, but the whole transversal $T_{\mathbf{p}}(\epsilon)$ given by $|t| \leq \epsilon$ at \mathbf{p} . Better still modifying the basic construction of III.1.1 according to the variant III.1.1 bis (post III.3.10) we can for \mathbf{q} in a given bounded smooth path ρ get estimates for the mass of all transversals through small perturbations of it as follows. Firstly join some small perturbations $\mathbf{p}' \in P$ of \mathbf{p} to small perturbations $\mathbf{q}' \in Q$ of $\mathbf{q} \in \rho$ then for quantifiers as per III.3.12 we'll find many many variants of the basic construction with mass on the $|t| = \epsilon$ and $\|z\| = \epsilon$ faces bounded in the fashion that we've become accustomed to. Whence we get bounds on transversals for $\mathbf{q}' \in Q$ on a set of measure that is as close to full as we please. Now perturb ρ to ρ' in a way that is parametrised by Q . At this point the bounds that we get on the $|t| = \epsilon$ and $\|z\| = \epsilon$ faces in applying the basic construction to ρ' are absolute and may be applied equally to the variant III.1.1 bis at all (actually Lebesgue almost all) points of the perturbed path where the anticipated bounds are achieved, so, in summary:

IV.1.2 Fact Suppose $|\chi| \neq 1$ and the order of approximation sufficiently large, then for ϵ outside a set of finite $\epsilon^{-1}d\epsilon$ measure, given a smooth bounded path $\rho \subset Y$ (*e.g.* σ_i or τ_i) with I a small interval in the normal direction parametrising perturbations $\rho' \in I$ there is a constant C such that for $\rho' \in I$ outside a set (depending on ϵ) of measure at most B^{-1} , and $\mathbf{q} \in \rho'$ outside a set of null Lebesgue measure, we have:

$$\int_{T_{\mathbf{q}}(\epsilon)} d\mu \leq \epsilon^p BC \max\{s_{Z,d\mu}(\epsilon), \epsilon\}$$

for $T_{\mathbf{q}}$ the transversal $|t|, \|z\| \leq \epsilon$ at \mathbf{q} .

Plainly, this is way more than we need and yields a bound on the total variation of $d\mu$ on every $\tau_i(\epsilon)$ or $\sigma_i(\epsilon)$ which is much more than adequate. Consequently, we can simplify the notation a little by supposing that χ takes values in S^1 .

At this point we must be more careful since as we've seen in §III.3-4, co-equalisers of maps tangent to the identity do not afford an estimate such as IV.1.2, while maps conjugate to irrational rotations are only conjugate to such. Consequently we distinguish two cases, beginning with the possibility that the linear/first order holonomy is torsion. We therefore make,

IV.1.3 Supposition According to this case, at the negligible cost of passing to a finite étale cover, the formal holonomy of the induced foliation in the formal centre manifold is everywhere tangent to the identity.

Under this hypothesis, we're in the situation already encountered in III.4.1 (c), *i.e* the co-equaliser $\mathbb{I}_{\tau_i(\epsilon)}$ contains both positive and negative set functions, but the co-equalisers in appropriate half planes $\tau_i(\epsilon)_+$, $\tau_i(\epsilon)_-$ may be supposed strictly positive, respectively negative, and,

$$|\mathbb{I}_{\tau_i(\epsilon)}(t)| \leq |\mathbb{I}_{\tau_i(\epsilon)_+}(t^{\sigma_i}) - \mathbb{I}_{\tau_i(\epsilon)_+}(t)| + |\mathbb{I}_{\tau_i(\epsilon)_-}(t^{\sigma_i}) - \mathbb{I}_{\tau_i(\epsilon)_-}(t)|$$

Now, while overkill from the point of view of computing residues, we can argue exactly as above to conclude,

IV.1.4 Fact Suppose we're in the situation of IV.1.3, and otherwise quantifiers as per IV.1.2 albeit we only look at small normal perturbations $\tau_i' \in I$ of the given τ_i (or for that matter σ_i), then,

$$\int_{T_{\mathbf{q}}(\epsilon) \cap \tau_i(\epsilon)} |d\mu| \leq \epsilon^p BC \max\{s_{Z,d\mu}(\epsilon), \epsilon\}$$

where we write $|d\mu|$ to emphasise that the left hand side is the total variation of $\mathbb{I}_{\tau_i(\epsilon)}$ over the transversal.

Again after passing to an étale cover, it therefore remains to discuss,

IV.1.5 Remaining Case χ is S^1 valued and the formal holonomy of the induced foliation in the formal centre manifold is non-trivial (otherwise there is nothing to do) extension of the image of χ by a group of automorphisms tangent to the identity.

The estimate will be somewhat akin to IV.1.2, in a IV.1.3'sh sort of way. We normalise the representation so that r is the smallest integer for which we find a non-trivial map, say,

$$t^a = t(1 + \gamma t^r + O(t^{r+1}))^{-1/r}$$

while, as automorphisms of, $\mathbb{C}[t]/t^r$ everything commutes and is identified to an element of S^1 . This implies that the total variation of any possible discontinuity is strictly (understood in terms of the relevant sets rather than their measure) less than that of the mass of the annuli,

$$A(\epsilon) := \{\epsilon^{-r} \leq |t|^{-r} \leq \epsilon^{-r} + c\}$$

for some absolute constant c - basically the maximum modulus of the $(r+1)$ th Taylor coefficient of a generator of the representation, normalised as above. By hypothesis, we also have irrational rotations,

$$t^b = \lambda t(1 + O(t^r))$$

for $\lambda \in S^1$ non-torsion. Consequently,

$$bab^{-1} : t \mapsto t(1 + \lambda^{-r}\gamma t^r + O(t^{r+1}))$$

The supporting hyperplanes of the regions $\tau_i(\epsilon)_-$, $\tau_i(\epsilon)_+$ encountered previously for the action of $h(\sigma_i)$ may first be considered for the action of the element a , say, $a(\epsilon)_-$, and $a(\epsilon)_+$. The supporting hyperplanes of these are perpendicular to γ . Equally we have the same for the conjugate element bab^{-1} , say, $\tilde{a}(\epsilon)_-$, and $\tilde{a}(\epsilon)_+$, but perpendicular to $\lambda^{-r}\gamma$, so together these cover an annulus,

$$\epsilon^{-r} \leq |t|^{-r} \leq \epsilon^{-r} + c'$$

where $c' > 0$, but in general less than c . However this is hardly a problem, since we have every right to iterate the basic construction III.3.1 to estimate co-equalisers- indeed, already neither a nor its conjugate need have been simple. Whence dividing the loops up into compositions of smooth (to avoid technical issues about slicing) simple ones we estimate the co-equalisers as anticipated, reason as per IV.1.2, and obtain,

IV.1.6 Fact Replacing the hypothesis $|\chi| \neq 1$ by that of IV.1.5, and otherwise everything exactly as per IV.1.2, we have:

$$\int_{A_{\mathbf{q}}(\epsilon)} d\mu \leq \epsilon^p BC \max\{s_{Z,d\mu}(\epsilon), \epsilon\}$$

for $A_{\mathbf{q}}(\epsilon)$ any annulus of the form,

$$\epsilon^{-r} \leq |t|^{-r} \leq \epsilon^{-r} + c$$

inside $T_q(\epsilon)$ with c independent of ϵ , and, implicitly C depending linearly on c .

At which point we may conclude,

IV.1.7 Fact Suppose the set up IV.1.1, let \mathcal{I}_Y be the ideal of the singular locus, and $d\mu$ an invariant measure with support outside the exceptional divisor and zero Segre class around the same, then:

$$\text{RES} : \text{Ext}_U^2(\mathcal{I}_Y, K_{X/\mathcal{F}}) \longrightarrow \text{Ext}^3(\mathcal{O}/\mathcal{I}_Y, K_{X/\mathcal{F}})$$

is zero on $d\mu$.

proof As already observed the residue of a 1-form ω that must be calculated is the limit as $\epsilon \rightarrow 0$ of,

$$\int_{\substack{\|z\|=\epsilon \\ |t|\leq\epsilon}} \omega d\mu + \int_{\substack{|t|=\epsilon \\ \|z\|\leq\epsilon}} \omega d\mu + \sum_{i=1}^g \left(\int_{\substack{\sigma_i(\epsilon) \\ \|z\|\leq\epsilon}} \omega d\mu + \int_{\substack{\tau_i(\epsilon) \\ \|z\|\leq\epsilon}} \omega d\mu \right)$$

where there is no obligation to keep the $\tau_i(\epsilon)$ or $\sigma_i(\epsilon)$ fixed, *i.e.* it is sufficient that we collapse down to Y as $\epsilon \rightarrow 0$. Consequently, observing that we may safely replace U by any finite étale cover, then perturbing the $\sigma_i(\epsilon)$, $\tau_i(\epsilon)$ in ϵ as appropriate, and appealing to IV.1.2, IV.1.4, or IV.1.6 according to the circumstances, we may handle all of these discontinuities. The leading term is

trivially bounded by the Segre class, and the approximately invariant term is easy, *i.e.* for local projections $\pi_\alpha : U_\alpha \rightarrow Y$, ω may, via a partition of unity ρ_α , be written,

$$\sum_{\alpha} \rho_{\alpha} f_{\alpha} x_{\alpha}^{-p} \pi_{\alpha}^{*} \omega_{\alpha}$$

where f_{α} is a function on U_{α} , and ω_{α} a holomorphic 1-form on $U_{\alpha} \cap Y$. As ever, x_{α} is an equation for the exceptional divisor, and for any local generator ∂_{α} of the foliation $\partial_{\alpha} t$ is at worst $x_{\alpha}^{N-p} \omega_{\alpha}$ for N as large as we like. Consequently, this term is trivially small outside of a set of finite $\epsilon^{-1} d\epsilon$ measure. \square

IV.2 Warmer Case

We treat the increase in difficulty by increments, and, whence, address:

IV.2.1 Set Up Let U be a foliated 3-dimensional tubular neighbourhood of a connected compact curve Y each component of which is smooth, the singularities are at worst plane nodes, and furthermore:

- (a) The curve Y is the singular locus of the foliation.
- (b) The foliation has canonical singularities along Y , but only 1-eigenvalue at each point.
- (c) In the formal centre manifold \widehat{Z} obtained on completing in Y , the components Y_i invariant by the induced foliation are disconnected.
- (d) A priori blowing up has been performed so that for some connected exceptional divisor E , with smooth components E_k , $E_k \mapsto E_k \cap \widehat{Z}$ is a 1-1 correspondence between components of E and components of Y .
- (e) The singularities that occur are no worse than those of II.1.1, II.2.1 (a), III.2.1 (a), III.2.1 (b), or III.3.1 (a).
- (f) A hypothesised invariant measure $d\mu$ has neither support on E nor Segre class around the same.

Here one should observe that IV.1.2 (c) basically implies IV.1.2 (e). The exceptions are rather particular things such as II.3.1 with $q = 0$, or the big height condition of III.2.1 (b) not being satisfied. As such we're not quite treating the general case implied by IV.1.2 (c), but it's perfectly sufficient for our immediate goal of understanding how to treat singularities. In this context, the occurrence of any of, II.1.1, II.2.1 (a), III.2.1 (a), or III.3.1 (a) put us more or less immediately in the good case IV.1.2, so, this is really only a discussion about III.2.1 (b).

As such, to begin with, every Y_i of IV.2.1 (c) is either all of Y , or it meets a component Y'_j where the induced foliation is everywhere transverse, so by II.1.5 the measure can have no support in a neighbourhood of Y'_j . Now all we need is some notation, *i.e.* for,

$$z_i \frac{\partial}{\partial z_i} + x^{p_i} \frac{\partial}{\partial y_i}$$

the normal form at a general point of Y_i , call $p_i = p(Y_i)$ the *multiplicity* along Y_i , then IV.1.2 applies with the following quantifiers,

IV.2.2 Fact Suppose that some Y_i of IV.2.1 (c) is not all of Y , and the order of approximation sufficiently large, then for ϵ outside a set of finite $\epsilon^{-1}d\epsilon$ measure, given $U \subset Y$ bounded away from the singularities of the induced foliation in \widehat{Z} , and a smooth bounded path $\rho \subset U$ with I a small interval in the normal direction parametrising perturbations $\rho' \in I$ there is a constant C (so, a priori depending on U) such that for $\rho' \in I$ outside a set (depending on ϵ) of measure at most B^{-1} , and $\mathbf{q} \in \rho'$ outside a set of null Lebesgue measure, we have:

$$\int_{T_{\mathbf{q}}(\epsilon)} d\mu \leq \epsilon^{p_i} BC \max\{s_{Z,d\mu}(\epsilon), \epsilon\}$$

for $T_{\mathbf{q}}$ the transversal $|t|, \|z\| \leq \epsilon$ at \mathbf{q} .

We need some more notation, say, Y_i^* the complement of Y_i by small neighbourhoods (to be specified) of the singularities of the induced foliation in \widehat{Z} , and Y_k' the components where this induced foliation is everywhere transverse. If Y_i^* is all of Y_i then we're in the situation of IV.1.1, so, without loss of generality Y_i^* is non-compact. Consequently, as we have warned, we change notation, and for each i , take σ_{ij} to be a basis of the homology, with τ_{ij} a dual basis in co-homology formed as follows: for the contribution from the genus everything is a loop as per §IV.1, for each puncture σ_{ij} is a simple loop around the same, and simple paths from puncture to puncture for the τ_{ij} , albeit for convenience we take a slit from the puncture to one one of the closed loops arising from the genus should there be only one puncture. According to the various cases specified by IV.2.1 (e) there is, by IV.2.2, nothing else to do as soon as $Y_i \neq Y$. Indeed on Y_i^* we argue as in §IV.1, but in the easy way when IV.1.2 applies, the local computations around the punctures have all been done in §II, III, and while there could be a discontinuity between the local coordinates employed in these computations, and the global coordinate on Y_i^* this is comfortably dealt with by IV.2.2. Consequently, we can lighten the notation by supposing that $Y = Y_i$, drop the i from the notation, and let σ_j, τ_j be basis in homology and its dual as above. The relevant holonomy representation is that of the induced foliation in the formal centre around Y^* , which again we denote by h , with χ its first order part, and whence,

IV.2.2 bis. Fact Suppose either $|\chi| \neq 1$, or there is a singularity of type III.3.1 (a) rather than the hypothesis $Y_i \neq Y$, then IV.2.2 holds with the same quantification.

As a result, we may argue exactly as above to reduce ourselves to the case where all the singularities have the form III.2.1 (b), and the holonomy around Y^* is S^1 -valued. Consider, therefore, for a suitable 1-form ω the form of the

residue calculation with $\epsilon \rightarrow 0$,

$$\int_{\substack{\|z\|=\epsilon \\ |t|\leq\epsilon}} \omega d\mu + \int_{\substack{|t|=\epsilon \\ \|z\|\leq\epsilon}} \omega d\mu + \sum_i \int_{\substack{\tau_i(\epsilon) \\ \|z\|\leq\epsilon}} \omega d\mu + \sum_j \left(\int_{\substack{\sigma_j^*(\epsilon) \\ \|z\|\leq\epsilon}} \omega d\mu + \text{Local}_j(\epsilon, \omega d\mu) \right)$$

where the first line are all terms around Y^* , and the new terms in j , are any possible discontinuities around the punctures, interpreted as before in a signed way, between the local coordinates employed in §III, and the global approximately invariant function t on Y^* , while Local_j simply indicates the local strategy of §III for calculating the residue.

Again, we distinguish between the cases of χ having torsion or non-torsion image in S^1 . In the latter case, we have sub-cases: the formal holonomy is linearisable to sufficiently high (basically p) order, or, it is not. In the former case there is no issue since $|t|$ may be supposed continuous. In the latter case we may normalise the representation so that for some $\lambda, \nu \in S^1$ with the latter non-torsion, we find an irrational rotation, a by ν , and any other element has the form,

$$b(t) = \lambda t(1 + \beta t^r + O(t^{r+1}))$$

for $r \in \mathbb{N} \cup \{\infty\}$, albeit there is a minimal $r \in \mathbb{N}$ for which $\beta \neq 0$. Continuing to denote this element by b , we have,

$$c = [a, b] = t(1 + \beta \lambda^{-r} (\nu^r - 1) + O(t^{r+1}))$$

. Consequently, we're in exactly the situation of IV.1.6, *i.e.* c and aca^{-1} are elements of the formal holonomy, combinations of which yield co-equalisers which, in turn, afford an estimate of the mass of any annulus of the shape indicated in op. cit., and this is plainly enough to deal with any possible discontinuities along the co-homology classes τ_i in Y^* .

The situation around the punctures is, however, more subtle. The easy case is when the eigenvalue at the puncture is irrational, *i.e.* it's not a singularity of the form III.2.1 (b) with rational eigenvalue. By the above the commutator of an irrational rotation is just the group of all rotations. Consequently, up to scaling the global t on Y^* is (more precisely modulo the order of approximation) the function $xy^{-1/\lambda}$ encountered in III.2.1 (b) *et sequel.*, or the holonomy around the puncture in the t variable has the form $b(t)$ as above, albeit with r possibly larger than the minimal one, while the local variable $xy^{-1/\lambda}$ is a function of t of the same form, *i.e.* the first non-linear term in either case occurs to the same order. As such the estimate à la IV.1.6, with quantifiers as per IV.2.2, for the mass of annuli: $\epsilon^{-r} \leq |t|^{-r} \leq \epsilon^{-r} + \text{const.}$, for the minimal r as above is more than adequate to estimate the total variation over any $\sigma_j^*(\epsilon)$.

We thus arrive to the principle pre-occupation of this section which is singularities of the form III.2.1 (b) with rational eigenvalue. The completion \widehat{U} of U in the singular locus admits a covering $\pi : \widehat{V} \rightarrow \widehat{U}$ ramified only in the

formal divisors $y = 0$ for y a local equation in the normal form of a singularity of type III.2.1 (b). In most cases, we can suppose that the ramification is only at the singularities with rational eigenvalue. The exceptional cases being $Y \xrightarrow{\sim} \mathbb{P}^1$ with at most 2 rational eigenvalues with distinct numerators when it may also be necessary to ramify in some irrational ones too. In any case, call R the set of ramification points in Y , and observe that the holonomy of the induced foliation in the centre manifold \widehat{W} in \widehat{V} has no torsion. On the other hand the only way to conserve canonical singularities after such a covering is to ramify in what may be purely formal divisors, so this is not a trivial operation like its smooth counterpart implicit in IV.1.3/5. Nevertheless the singularities in \widehat{W} with rational eigenvalue only (big height condition) contribute holonomy to a negligibly high order, so for a basis $\tilde{\sigma}_i, \tilde{\tau}_i$ of homology, respectively co-homology, of the corresponding covering $\pi : \tilde{Y} \rightarrow Y$ of Y the discontinuity is confined to co-homology classes, and homology classes around irrational punctures, say, $\tilde{\sigma}_j^*$ for some indices j . Furthermore, away from the ramification π is étale, so, it extends convergently to a covering $\pi : \tilde{U} \rightarrow U_R$ where U_R is the complement of U by polydiscs around the ramification points $R \subset Y$. Consequently if s is the invariant formal function on \widehat{V} continuous outwith the $\tilde{\tau}_i$ and possibly some loops $\tilde{\sigma}_j^*$ at irrational punctures, then we have a smooth lifting to \tilde{U} of the same modulo as large a power of the exceptional divisor as we please, which, in a minor notational confusion, we continue to denote by the same letter.

Now denote by d the degree of π , so $\pi_*|s|$ is commensurate to the d th power of the distance to the exceptional divisor, and for ω a 1-form on U we can do the residue calculation as,

$$\begin{aligned} & \int_{\substack{\|z\|=\epsilon, |\pi_*s| \leq \epsilon^d \\ y \in Y^*}} \omega d\mu + \int_{\substack{|\pi_*s|=\epsilon^d, \|z\| \leq \epsilon \\ y \in Y^*}} \omega d\mu + \sum_i \int_{\substack{\tilde{\tau}_i(\epsilon) \\ \|z\| \leq \epsilon}} \pi^* \omega d\mu \\ & + \sum_j \int_{\substack{\tilde{\sigma}_j^*(\epsilon) \\ \|z\| \leq \epsilon}} \pi^* \omega d\mu + \sum_k \int_{\substack{\sigma_k^*(\epsilon) \\ \|z\| \leq \epsilon}} \omega d\mu + \sum_{jk} \text{Local}_j(\epsilon, \omega d\mu) \end{aligned}$$

where by way of notation we distinguish the co-equalisers $\tilde{\sigma}_j^*(\epsilon), \sigma_k^*(\epsilon)$ according as π is un-ramified, or otherwise at the singularity which affords the homology class, and “local” again just means the local strategy. Now all the eigenvalues at the $\tilde{\sigma}_j^*(\epsilon)$ ’s are around singularities with irrational eigenvalue, so, we know how to estimate the total variation here. For the same reason, or arguing as per IV.1.3/4 if all the eigenvalues were rational, we have the expected mass estimate also over the $\tilde{\tau}_i(\epsilon)$. Amongst the $\sigma_k^*(\epsilon)$ ’s, again the irrational ones pose no problem, which leaves a possible discontinuity around punctures with a rational eigenvalue between π_*s , and the special coordinates of §III.2.1 (b) to discuss. To this end let x, y, z be local coordinates for the normal form at such a point, then modulo some large power, N , of the exceptional divisor π_*s is invariant and continuous, so for $k|l$ the eigenvalue, it has, up to homothety, the form,

$$(x^k y^l)^{d/k} (1 + a(x^k y^l))^d \text{ mod } \mathcal{O}(-NE)$$

for some polynomial a in a single variable. As such the change of local coordinates,

$$x \mapsto x(1 + a(x^k y^l))$$

together with rescaling in y so that we have a disc of radius 1, ensures that, we change nothing in III.2.1 (b) and, at the same time, avoid any discontinuity with π_*s . We, of course, know how to calculate all the local terms in j and k , and whence,

IV.2.3 Fact Let things be as in IV.2.1 with \mathcal{I}_Y the ideal of the singular locus, and $d\mu$ an invariant measure with support outside the exceptional divisor and zero Segre class around the same, then:

$$\text{RES} : \text{Ext}_U^2(\mathcal{I}_Y, K_{X/\mathcal{F}}) \longrightarrow \text{Ext}^3(\mathcal{O}/\mathcal{I}_Y, K_{X/\mathcal{F}})$$

is zero on $d\mu$.

IV.3 Penultimate Case

We move towards a conclusion by way of introducing the difficulties associated with connected but not irreducible components of the singular locus which are left invariant by the induced foliation in the formal centre manifold, *i.e.*,

IV.2.1 Set Up Let U be a foliated 3-dimensional tubular neighbourhood of a connected compact curve Y each component of which is smooth, the singularities are at worst plane nodes, and furthermore:

- (a) The curve Y is the singular locus of the foliation.
- (b) The foliation has not just canonical singularities, but are as convenient as possible in the sense of [MP2] V.1.8. In addition the locus Y_1 where there is exactly 1-eigenvalue (counted with multiplicity) will be supposed of pure dimension 1.
- (c) In the formal centre manifold \widehat{Z} obtained on completing in Y_1 , the sub-curve Y' formed by components Y'_i invariant by the induced foliation has a dual graph without cycles.
- (d) A priori blowing up has been performed so that for some connected exceptional divisor E , with smooth components E_k , $E_k \mapsto E_k \cap \widehat{Z}$ (strictly speaking $\widehat{E}_k \mapsto \widehat{E}_k \cap \widehat{Z}$ when the number of eigenvalues jumps from 1 to 2) is a 1-1 correspondence between components of E and components of Y .
- (e) A singularity of the induced foliation in \widehat{Z} at which there is only one component of Y is no worse than II.2.1 (a), III.2.1 (a)/(b) or III.3.1 (a)- which, in fact, already follows by (b) above.
- (f) A hypothesised invariant measure $d\mu$ has neither support on E nor Segre class around the same.

From what we've already seen in §IV.1/2, there is an increasing degree of nuisance value associated with holonomy in the centre manifold which is torsion to first order, and we'll require a parenthesis to examine this more carefully. To this end let C be a component of Y' , and C^* the complement of C by discs around any singularities of the induced foliation in \widehat{Z} . We will suppose,

IV.3.2 Hypothesis Notations as above the formal holonomy around C^* , so a fortiori around the excluded singularities, is to first order with values in S^1 , while the punctures are not nodes as encountered in III.3.1 (a)/(b).

Now we have various sub-cases to consider. The easy one we have already largely seen, *viz*:

IV.3.3 (a) Easy Case The holonomy in C^* contains a map conjugate (formally) to an irrational rotation. There may be several such, but we choose one, and normalise the representation so that, $t \mapsto \lambda t$, $\lambda \in S^1$ non-torsion, for some variable t around the loop, say, a , in question.

Now consider the holonomy $t \mapsto b(t)$ around any puncture, which, by hypothesis is conjugate to that given by one of the normal forms III.3.1 (b), III.4.1 (b)/(c). Now, say the conjugation is $\sigma b \sigma^{-1}$, for σ of the form,

$$t \mapsto \nu t(1 + \beta t^m + O(t^{m+1}))$$

and $\beta \neq 0$. Should the normal form be an irrational rotation with multiplier ρ , we therefore have,

$$b(t) = \rho t(1 + \beta(1 - \rho^m)t^m + O(t^{m+1}))$$

So, as already observed, the commutators $[a, h]$, $[a^2, h]$ afford maps which allow, as per IV.1.6, the appropriate mass estimate on annuli, $\epsilon^{-m} \leq |t|^{-m} \leq \epsilon^{-m} + \text{const}$. Of course m may be infinite, but that equates, up to homothety, to an identity between t and the coordinates affording the normal form around the puncture.

It may be, however, that the holonomy around the puncture is torsion to first order, and so the normal form of the singularity is as per III.4.1 (c), or even just torsion in the case of the second possibility in III.4.1 (b), or torsion to all intents and purposes in the rational case of III.3.1 (b). Irrespectively, the normal form of the holonomy about the puncture is to leading order,

$$n(s) = \rho s(1 + \gamma s^{kr} + O(s^{k(r+1)}))$$

where, now, ρ is a k th root of unity, and possibly r infinite in the pure torsion case, with (at a minor risk of notational confusion) s the normalised local variable employed in the local residue calculations of III.4.1 (b)/(c), or its natural extrapolation to the rational case of III.3.1 (b). As such, whenever the above conjugation σ between t and s has $m \geq kr$, we find,

$$b(t) = \rho t(1 + \gamma \nu^{kr} t^{kr} + O(t^{kr+1}))$$

and by way of commutators, we gain bound the mass of transversals of an annulus, $\epsilon^{-kr} \leq |t|^{-kr} \leq \epsilon^{-kr} + \text{const}$. in an appropriate way. In the situation

that $m < kr$, we must distinguish cases. Should k not divide m , then we're akin to the previous irrational case, *i.e.* for $b = \sigma^{-1}n\sigma$, the commutators $[a, b]$, $[a^2, b]$ again yield a bound on the annuli $\epsilon^{-m} < |t|^{-m} < \epsilon^{-m} + \text{const}$. Otherwise, $m = m_0k$, and, as per the end of the previous section, we change the local coordinates around the puncture by way of,

$$x \mapsto \nu^{-1}x(1 + \alpha(x^k y^l)^{m_0}), \quad y \mapsto y$$

for $\sigma^{-1}t = \nu^{-1}t(1 + \alpha t^{m_0k} + O(t^{m_0k+1}))$, and x the normal coordinate to C in \widehat{Z} . This has no effect on the normal form III.4.1 (b), a meaningless (big height condition again) effect for III.3.1 (b), and a minor effect on the normal form in the case of III.4.1 (c), whose negligibility has already been noted in III.4.6. Consequently the possibility that $k|m$ may be excluded. This equally applies, and indeed a particular case was already encountered at the end of §IV.2, to the extreme case that there is no first order holonomy around C^* , *i.e.* we may simply change the local variable of III.3.1 (b), or III.4.1 (b)/(c)- in all cases $k=1$ - so that the local approximately invariant function s around the puncture agrees with the global one t (irrespective of its determination, and, indeed to any order) without prejudice to the local residue calculations of §III. As such, it remains to consider,

IV.3.3 (b) Fastidious Case The first order holonomy of C^* in \widehat{Z} is torsion, and non-trivial.

By hypothesis, therefore, the first order holonomy is cyclic of some order d , say, generated by a d th root of unity λ on a variable t , normalised by way of,

$$a(t) = \lambda t(1 + t^{nd} + O(t^{d(n+1)}))$$

for some maximal $n \in \mathbb{N} \cup \{\infty\}$ amongst all maps which are primitive d torsion to first order- so, in fact we put $n = \infty$ if we go over our large a priori order of approximation N , so, we can make the better normalisation,

$$\lambda t(1 + A(t^{nd}))$$

for A a function of a single variable. Now take t for the globally approximately invariant function, and consider the conjugation to the local variable s of §III by way of σ as above, so, plainly $k|d$. Again, if $k|m$, or for that matter m bigger than our sufficiently large N , we can just change the local variable x so that s and t coincide without prejudice to the local strategies of §III. Thus, without loss of generality, $k \nmid m$, and, a fortiori $\lambda^m \neq 1$. On the other hand,

$$a^{-1}(t) = \lambda^{-1}t(1 + B(t^{nd}))$$

again for B in a single variable. Furthermore the holonomy, around the puncture in the variable s has the form,

$$n(s) = \rho s(1 + f(s^k))$$

for f in a single variable of sub-degree r - the normal form is actually a function of s^{kr} , but we may be in the situation of III.4.6., so better than the above cannot

be guaranteed. This implies that,

$$b = \sigma^{-1}n\sigma : t \mapsto \rho t \{1 + \beta(1 - \rho^m)t^m + f(\nu^k t^k) + O(t^{m+1})\}$$

so, irrespectively of the relative sizes of m and k , the coefficient of t^m does not vanish. To get a suitable mass estimate (the possibility $kr < m < 2kr$ is where the real nuisance value lies) on transversal we consider the commutator of b with $a^{d/k}$,

$$[a^{d/k}, b] = t\{1 + \beta(1 - \rho^{-m})(1 - \lambda^{-md/k})t^m + g(t^k) + O(t^{m+1})\}$$

where, again, g is polynomial in a single variable, and, in addition it vanishes to order strictly greater than the maximum of r and dn/k . Consequently, by repeatedly forming commutators with $a^{d/k}$ we eventually obtain an element of the holonomy that has the form,

$$c(t) = t\{1 + \beta(1 - \rho^{-m})(1 - \lambda^{-md/k})e t^m + O(t^{m+1})\}$$

for some $e \in \mathbb{N}$. Now we have to distinguish two cases. The easy one is $\lambda^m \neq -1$. In this case the $(m+1)$ th Taylor coefficients of c , and $[a, c]$ (which is again tangent to the identity to order m) are linearly independent over \mathbb{R} , and we can, once more, bound the mass of an annulus, $\epsilon^{-m} < |t|^{-m} < \epsilon^{-m} + \text{const.}$ in the usual way, which in turn is a more than adequate bound for the discontinuity between t and s . Otherwise, $\lambda^m = \rho^m = -1$ and we may only be able to bound the total variation of mass between the regions, $|t|^{-m} > \epsilon^{-m}$ and $|c(t)|^{-m} > \epsilon^{-m}$. Fortunately, however, $\rho^m = -1$, so up iterating this bound, *i.e.* using c^f for f large (about 2 will do), we bound the mass of the discontinuity between s and t . We have, therefore, achieved the following dichotomy;

IV.3.4 Summary Suppose the holonomy of the induced foliation in \tilde{Z} around C^* is to first order in S^1 and that the multiplicity of the singularity around C is p . Then we may find an approximately invariant function t around C^* such that for any singularity of the form III.4.1 (b),(c) and σ a homology class around the same either we can choose the local coordinates of III.4.1 such that the local invariant function s of op. cit. (actually t in the notation therein) agrees with s or the total variation in the discontinuity $\sigma^*(\epsilon)$ between t and s around σ (strictly speaking around a small perturbation σ' of σ) admits the bound,

$$\int_{T_{\mathbf{q}} \cap \sigma^*(\epsilon)} |d\mu| \leq \epsilon^p BC \max\{s_{Z, d\mu}(\epsilon), \epsilon\}$$

for $\mathbf{q} \in \sigma$ and quantifiers, *etc.*, as per IV.2.2.

Which doesn't cover the case of III.2.1 (b), but since $s = xy^{-1/\lambda}$, $\lambda \in \mathbb{R}_{<0}$, and the local strategy there is to use the boundary $|x| = \epsilon$, since the invariant manifold $y = 0$ may only exist formally, we can just rescale so that $|y| = 1$ on the boundary, and the relevant discontinuity there, *i.e.* between $|x|$ and $|t|$, is, in fact, that between $|s|$ and $|t|$. Whence,

IV.3.4 bis. Similarly Exactly as per IV.3.4, but for singularities of type III.2.1(b) and the discontinuity understood as that between $|t|$ and the local variable $|x|$.

We can also clear up the relation between the discontinuity along cohomology classes τ occasioned by the dual homology class σ . Again, we're in the situation of IV.3.3 (a) or (b), with holonomy around σ of the form,

$$h(\sigma)(t) = \rho t \{1 + \alpha t^r + O(t^{r+1})\}$$

with t normalised as above. Whence in the easy case 4.3.3 (a), as previously observed in §IV.1/2, we always get bounds on the mass of an annulus, $\epsilon^{-r} < |t|^{-r} < \epsilon^{-r} + \text{const.}$, and, idem for IV.3.3 (b) if $\lambda^r \neq \pm 1$. Should $\lambda^r = -1$, we bound the mass of a smaller region, but as above this is the mass of the discontinuity anyway, so we're okay. Finally if $\lambda^r = 1$ then, a fortiori, $\rho^r = 1$, and again the mass is appropriately bounded, so that without recourse to finite coverings to kill the torsion,

IV.3.5 Variation Everything as per IV.3.4, with τ_i a dual basis in co-homology to the homology basis σ_i for C^* , but here we bound the total variation in mass over $\tau_i(\epsilon)$, or, more correctly a small perturbation of it.

Now let us organise the global residue computation. By II.1.5 and III.5.2 there is nothing to do except at components C of the singular locus where (counted with multiplicity) there is everywhere one eigenvalue (counted with multiplicity), the foliation is not log-flat, and C is invariant by the induced foliation in the formal centre manifold \widehat{Z} . Now as per §I.4, form the dual graph G with vertices such components C , and edges intersections of the same. By the hypothesis IV.3.1 (c), G is a tree, and, we do not prejudice the residue calculation by supposing that it is connected. As such, we may choose a root R , say, which in turn defines a unique path to each vertex, whence (depending on R) a direction on each edge according to the sense of increasing distance from R . At a directed edge e we introduce local coordinates x_e, y_e , and a multiplier ν_e , *i.e.*,

$$v_- : (x_e = 0) \xrightarrow{\nu_e} v_+ : (y_e = 0)$$

according to the rule: $\nu_e = p(v_-)/p(v_+)$, for $p(v)$ the multiplicity along the vertex v (so q/p in the notation of §III) whenever the singularity at e is not of the form III.4.1 (b) or (c), and otherwise according to the form of the first order part of the singularity in \widehat{Z} , *i.e.*,

$$y_e \frac{\partial}{\partial y_e} - \lambda_e x_e \frac{\partial}{\partial x_e}$$

$\lambda_e \in \mathbb{R}_+$, and irrespectively of whether its rational or irrational we put $\nu_e = \lambda_e$. We continue to denote by C^* the curve C minus discs about singularities in \widehat{Z} , but we also introduce a curve C^\bullet where we re-fill C^* at singularities of the form III.4.1 (b)/(c) or III.2.1 (b) where the dichotomy of IV.3.4 & bis implies that the normalised approximately invariant function t_C extends around the singularity by way of agreement with its local equivalent in III.4.1 (b)/(c) or its modulus with that of $|x|$ in III.2.1 (b). As such, continuing in the abuse of notation between t_C and a smooth lift of the same modulo a large unspecified power of the exceptional divisor, we extend the abuse to all of C^\bullet by way of the

above rule. Finally if $\{R, C\}$ is the path from R to C we introduce a multiplier λ_C according to $\lambda_R = 1$, and, otherwise:

$$\lambda_C = \prod_{e \in \{R, C\}} \lambda_e$$

together with a sub-graph G^\bullet whose edges are intersections of filled curves C^\bullet . As such, by the definition of $|w|$ in III.4.1 (c), at such an edge, $|t_{v_+}| = |t_{v_-}|^{\nu_e}$.

The function $|t| := |t_C|^{1/\lambda_C}$ has, therefore, on the curve defined by G a discontinuity, $\tau(\epsilon)$, not only at co-homology classes τ in any C^* for every vertex, but also (bearing in mind we take a slit at the puncture if the corresponding vertex has only one singularity) the extension of these up to, and around (it continues naturally from one component to another by way of $w(\epsilon)$ post III.4.4) a singularity when this has the form III.4.1 (c) which is an edge in G^\bullet . In this latter case, the mass bound comes from IV.2.2 & bis, or IV.3.4 when the former does not apply, for the part of the discontinuity in C^* and always from III.4.4 close to the singularity. Otherwise at an edge of G not in G^\bullet we have one, or two loops, σ^* , according as the singularity is filled in one or neither of the vertices. Certainly there may be a discontinuity between $|t_C|$ and the boundary employed in the appropriate local strategy of §III at $\sigma^*(\epsilon)$, or, indeed, any other σ^* which goes around a puncture in C^* that is not filled in C^\bullet , but in all such cases we may either trivially bound it by IV.2.2 & bis, and the evident extension of the same to the singularities III.3.1 (c) and III.4.1 (a), or, when necessary by the more delicate IV.3.4.

Now let's look at the form of the residue calculation for an appropriate 1-form ω , and $\epsilon \rightarrow 0$, *i.e.*,

$$\begin{aligned} & \sum_C \left(\int_{\substack{\|z\|=\epsilon, |t_C| \leq \epsilon^{\lambda_C} \\ \text{nhd. of } C^*}} \omega d\mu + \int_{\substack{\|z\| \leq \epsilon, |t_C| = \epsilon^{\lambda_C} \\ \text{nhd. of } C^*}} \omega d\mu \right) + \sum_i \int_{\tau_i(\epsilon)} \omega d\mu \\ & \sum_j \int_{\sigma_j^*(\epsilon)} \omega d\mu + \sum_k \text{Local}_k(\epsilon, \omega d\mu) \end{aligned}$$

where, by way of notation, all local strategies at singularities, irrespective of whether it's at a singularity which is an edge of G^\bullet or not, have been lumped together even if it's more correct to think of such edges globally since at them the local and global strategies coincide. The discontinuities $\tau_i(\epsilon)$, $\sigma_j^*(\epsilon)$ have been defined above. and observed to be bounded in an appropriate way, while the first two integrals are our friends the Segre class, and something negligible. Whence:

IV.3.6 Fact Let things be as in IV.3.1 with \mathcal{I}_Y the ideal of the singular locus, and $d\mu$ an invariant measure with support outside the exceptional divisor and zero Segre class around the same, then:

$$\text{RES} : \text{Ext}_U^2(\mathcal{I}_Y, K_{X/\mathcal{F}}) \longrightarrow \text{Ext}^3(\mathcal{O}/\mathcal{I}_Y, K_{X/\mathcal{F}})$$

is zero on $d\mu$.

IV.4 General Case

We now bring our calculation to a conclusion, *viz*:

IV.4.1 Set Up Everything as per IV.3.1, but without the no-cycles hypothesis IV.3.1 (c), and suppose (pro tempore) that, in addition, the neighbourhood U is projective, *i.e.* admits an embedding in \mathbb{P}^n for some $n \in \mathbb{N}$.

In the notation of the previous section, plainly the problem lies in the multipliers ν_ϵ , *i.e.* their product around a cycle may not be 1. To this end, consider a singularity of the form III.4.1 (b), but with $\lambda \notin \mathbb{Q}$, and let us explore an alternative strategy for effecting the local residue computation. Specifically, let $\alpha, \beta \in \mathbb{R}_{>0}$ and put $|f| = |x|^\alpha |y|^\beta$, with the notations of III.4.1 (b), with $g = x^a y^b$, $a, b \in \mathbb{Z}_{\geq 0}$. Observe that up to an irrelevant error given a sufficiently high order of approximation,

$$x^{-a} y^{-b} \frac{dx}{x} = (a + b\lambda)^{-1} \frac{dg}{g^2} d\mu$$

As such, since we require to calculate the residue of the left hand side for $a \leq p$, $b \leq q$, it will be sufficient to do the right hand side for all such g under the same hypothesis. On the other hand for X, Y some upper bounds for $|x|, |y|$,

$$- \int_{\substack{|f|=\epsilon, |z|\leq\epsilon \\ |x|\leq X, |y|\leq Y}} \frac{dg}{g^2} d\mu = \int_{\substack{|f|=\epsilon, |z|=\epsilon \\ |x|\leq X, |y|\leq Y}} \frac{1}{g} d\mu + \int_{\substack{|f|=\epsilon, |z|\leq\epsilon \\ |x|=X}} \frac{1}{g} d\mu + \int_{\substack{|f|=\epsilon, |z|\leq\epsilon \\ |y|=Y}} \frac{1}{g} d\mu$$

The first of these integrals may be estimated in the usual way, *viz*:

$$\int_0^1 \frac{d\chi}{\chi} \int_{\substack{|f|=\epsilon, |z|=\epsilon \\ |x|\leq X, |y|\leq Y}} \frac{1}{|g|} |d\mu| = \int_{\substack{|z|=\epsilon, |f|\leq\epsilon \\ |x|\leq X, |y|\leq Y}} \left| \frac{1}{|g|} \frac{d|f|}{|f|} \right| d\mu \leq s_{Z, d\mu}(\epsilon)$$

while the other two integrals may be seen to be negligible exactly as per III.2.4 (b), so:

IV.2.2 Fact Let things be as in III.4.1 (b) with an irrational eigenvalue, and suppose that the Segre class around the exceptional divisor of our invariant measure is zero, then for $\epsilon > 0$ outside a set of finite $\epsilon^{-1} d\epsilon$ measure, and $(X, Y, \log \chi)$ varying in a set (depending on ϵ) which as close to full with respect to Lebesgue measure as we please,

$$\lim_{\epsilon \rightarrow 0} \int_{\substack{|f|=\chi^{-1}\epsilon, |z|\leq\epsilon \\ |x|\leq X, |y|\leq Y}} x^{-a} y^{-b} \frac{dx}{x} d\mu = 0, \quad a \leq p, b \leq q$$

One should be aware that there is a hidden subtlety here. More precisely, from the perspective of switching to a local strategy where one calculates all residues by Stokes, rather than simply considerations of size on the boundary, it is a priori required that we can calculate all residues of the form,

$$\frac{f(x, y) dx}{x^p y^q x}$$

and the possibility of doing this does not exactly follow from IV.2.2, since if one expands $f(x, y)$ in a Taylor series, and applies IV.2.2 there could be a small divisors issue. On the other hand, one only has to do the residues of IV.2.2 for $1 \leq a \leq p$, and $1 \leq b \leq q$ together with,

$$x^{-a} f(y) \frac{dx}{x}, \quad 1 \leq a \leq p$$

for any convergent f , and similarly with x, y interchanged. The relevant rational combinations of $1, \lambda$ which might be a source of concern are, therefore,

$$-a + n\lambda, \quad 1 \leq a \leq p, \quad n \in \mathbb{Z}_{\geq 0}$$

while $\text{Re}(\lambda) \leq 0$, so the modulus of the combination in question is at least 1, and there is no issue. Consequently, let us note where the real problem is by way of,

IV.4.3 (a) Remark Evidently the same strategy works in the alternative rational case of III.4.1 (b) with $\lambda = -k/l$ whenever $a + b\lambda \neq 0$, *i.e.* $x^a y^b$ not a power of the approximately invariant function $x^k y^l$. In such good cases, we can employ arbitrary multipliers at the edges of our graph, and there will be no possibility of a continuity problem occasioned by cycles in the graph. In the situation that $x^a y^b = (x^k y^l)^d$, some $d \in \mathbb{N}$, there remains, however, a risk of a problem since the only choice that works is k/l .

In light of this improvement, let us look again at the singularities III.3.1 (c), and III.4.1 (a). In the latter case, the required changes to argue as above are negligible, but in the former case there are several problems resulting from the fact that the normal form doesn't converge on completion in the singular locus, albeit these can be dealt with as pre III.3.11, and a much thornier problem about getting the right estimates for a residue of the form $x^{-a} dx$. Consequently, let us adopt an expedient that is adequate for our purposes, and consider only the possibility that, $\alpha \leq p$, and $\beta \leq q$, then, supposing zero Segre class,

$$\oint_{\substack{|f|=\epsilon \\ |x| \leq X, |y| \leq Y}} \frac{df}{f} d\mu = \epsilon o(\epsilon)$$

with the same proof, *i.e.* III.3.13, & III.4.2 respectively. This also holds for α or β zero, and, in fact with the better bound $\epsilon^{p+r/\alpha}$ in the case of nodes. Now such a bound may not be adequate to compute any residue, but supposing α, β integers, it does suffice to deduce,

IV.4.4 (a) Fact Say the edge is a node or a singularity of the form III.4.1 (a), II.2.1 (b) being trivial, then for $\omega \in K_{\mathcal{F}}$ vanishing to order $p - \alpha$ along E_1 , $q - \beta$ along E_2 and ϵ outside a set of finite $\epsilon^{-1} d\epsilon$,

$$\lim_{\epsilon \rightarrow 0} \int_{\substack{|f|=\epsilon \\ |x| \leq X, |y| \leq Y}} \omega d\mu = 0$$

As such, we may return to the difficulty of rational resonances, and the last remaining possibility III.4.1 (c). Here the strategy of how to apply Stokes has

no essential difference with IV.2.2/IV.4.3 (a) beyond the estimation at the ends, which are no longer negligible. Indeed for $g = x^a y^b$, and $la \neq kb$ at the end $|x| = X$ we find the mass bound,

$$\int_{1/2}^1 \int_{1/2}^1 \frac{d\chi}{\chi} \frac{d\phi}{\phi} \int_{\substack{|f|=\chi\epsilon \\ |y|=\phi X \\ |z|\leq\epsilon}} |d\mu| \leq \frac{\tilde{l}i - k\tilde{j}}{2} \int_{\substack{|x|\leq\epsilon^{1/\alpha} Y^{-\beta/2\alpha} \\ X/2 \leq |y| \leq X \\ |z|\leq\epsilon}} \text{Im}(h) \frac{dy \otimes d\bar{y}}{|y|^2} d\mu$$

where the notation is as per the normal form III.4.1 (c), or adjusted as per III.4.6, from which one finds that h has the form,

$$h = (x^k y^l)^r + O((x^k y^l)^{r+1})$$

thus the mass bound III.4.7 is exactly what we require, so, again,

IV.4.4 (b) Fact If the edge has the form III.4.1 (c) and $ka - lb \neq 0$, then IV.4.2 continues to hold.

It is instructive to observe,

IV.4.3 (b) Remark While this situation is a bit better than the rather absolute obstruction posed by the rational case of III.4.1 (b), *i.e.* for $w = x^k y^l$ we have up to an irrelevant error, and irrelevant homotheties,

$$w^{-m} \frac{dx}{x} d\mu = \left(\frac{dw}{w^{r+m+1}} + \nu \frac{dw}{w^{m+1}} \right) d\mu$$

for $1 \leq m \leq n$, n as per III.4.1 (c). Integrating over the boundary $|f| = \epsilon$ as above, the latter term is no problem since, $|w|^m \geq |x|^p |y|^q$, and the bound III.4.7 on transversals is adequate. For the initial term, however, it is not, since this needs the stronger bound of IV.2.2, which we may, or may not, have, and would, in any case, be needed on both transversal $|x| = \text{const.}$, and $|y| = \text{const.}$

To improve the situation we need to do a little surgery. This will only be at points where the induced foliation in \hat{Z} is smooth, so introduce formal coordinates x, y, z such that the foliation is given by,

$$\partial = z \frac{\partial}{\partial z} + x^p \frac{\partial}{\partial y}$$

We wish to perform the weighted blow up $\pi : \tilde{U} \rightarrow U$ of (z, y, x^p) in the sense of champ de Deligne-Mumford so as to preserve smoothness. The only relevant étale patch of the blow up has coordinates, ξ, η, ζ , where:

$$\eta^p = y, x = \eta\xi, \zeta = \eta^p \zeta, \quad \eta \mapsto \theta\eta, \xi \mapsto \theta^{-1}\xi, \zeta \mapsto \zeta$$

and we have an open inclusion in \tilde{U} of the classifying champ $[\Delta^3/\mu_p]$ for the implied action of μ_p by the p th root of unity θ . As such the foliation is given by,

$$\partial = (1 - \xi^p) \zeta \frac{\partial}{\partial \zeta} + \frac{\xi^p}{p} \left(\eta \frac{\partial}{\partial \eta} - \xi \frac{\partial}{\partial \xi} \right)$$

while on the other patches it is smooth or log flat, so absolutely irrelevant for computing residues. As to the measure itself this extends (thanks to the

finiteness of the weighted Lelong number) by zero over the exceptional divisor in a way that has finite mass with respect to a smooth metric on \tilde{U} , so it remains an invariant measure on the same. It is, nevertheless, true that we've replaced a smooth point of the foliation in \tilde{Z} by a non-schematic singular one. This has, however, no adverse effect on the residue calculation: we work on a tubular neighbourhood V of the proper transform of Y , the singularity in the formal centre manifold in V has no extra holonomy since $\eta\xi \mapsto \eta\xi$ under μ_p , and we can treat this singularity as an extremely easy case of III.2.1 (b).

As to why we would want to do this: observe that the exceptional divisor E defining the components gets replaced by its proper transform \mathcal{E} around some not wholly schematic component \mathcal{C} of the singular locus, so in the proper transform \mathcal{Z} of the centre manifold,

$$N_{\mathcal{C}/\mathcal{Z}} = \mathcal{O}_{\mathcal{C}}(\pi^*E - F) = \pi^*N_{C/\tilde{Z}}(-F)$$

for F the exceptional divisor. As such, if we repeat this operation in enough points, the resulting \mathcal{Z} will be (formally) convex. Now let's bring the projectivity into play, and take some very large multiple nH , to be specified, of a very ample divisor such that $D \in |nH|$ cuts Y in some reduced set B of smooth points with tangency to order m_b to the divisor $y = 0$ understood in the above local coordinates around $b \in B$. Consequently for F_b the exceptional divisor over $b \in B$, the weighted blow up having been performed in all of these, and $\pi : V \rightarrow U$ again a tubular neighbourhood around the proper transform of Y ,

$$n\pi^*H = \pi^*D = \sum_{b \in B} p(C)m_b F_b$$

for $p(C)$ the multiplicity of the component C through b . The good choice of n is, therefore,

$$n \prod_C p(C)$$

and plainly $m_b = p(C)^{-1}n$, so that:

$$\pi^*H - \sum_{b \in B} F_b = \mathcal{L} \in \text{Pic}(V)$$

is at worst n -torsion, and has monodromy around each non-schematic point $\mu_{p(C)}$ for C the corresponding component. From which the étale cover $\rho : \tilde{U} \rightarrow V$ defined by \mathcal{L} is everywhere schematic, and for \tilde{Z} the centre manifold in \tilde{U} with \tilde{C} some component over C ,

$$N_{\tilde{C}/\tilde{Z}} = \rho^*\pi^*N_{C/\mathcal{Z}}(-H)$$

and, as it happens, $\rho^*\pi^*Z = \tilde{Z} + p(C)^{-1}nH$ on each component, so \tilde{U} is highly convex.

We may lose IV.3.1 (d), but in a reasonably trivial way, *i.e.* for \mathcal{E} an exceptional divisor in V defining a component $\rho^*\mathcal{E}$ will still be smooth, but

possibly disconnected. In principle this is a minor caveat, and in any case for E a component of some $\rho^*\mathcal{E}$, the multiplicity $p(E)$ is well defined by way of $p(C)$ for C the original component in U corresponding to \mathcal{E} . Next let S be the singular locus in \tilde{U} with scheme structure around the components invariant by the induced foliation in the centre manifold. We have a total exceptional divisor,

$$E_{\text{tot}} = \sum p(E)E$$

and various divisors E between 0 and E_{tot} partially ordered by increasing multiplicity. In particular, there are maps,

$$K_{\mathcal{F}} \otimes \mathcal{O}_{E'|_S}(-E) \longrightarrow K_{\mathcal{F}} \otimes \mathcal{O}_{E''|_S}(-E)$$

for $E_{\text{red}} \leq E'' \leq E' \leq E_{\text{tot}}$, $E \geq 0$ any, and the same canonical for all of U , V , \tilde{U} , which we emphasise by omitting π^* , ρ^* on the same. In addition, as per I.1.5, the trace form yields an isomorphism between \mathcal{O}_S and $K_{\mathcal{F}}|_S$, so if $E' = E'' + F$ we have an exact sequence,

$$0 \rightarrow \mathcal{O}_{F|\tilde{Z}}(-E'' - E) \rightarrow K_{\mathcal{F}} \otimes \mathcal{O}_{E'|_S}(-E) \rightarrow K_{\mathcal{F}} \otimes \mathcal{O}_{E''|_S}(-E) \rightarrow 0$$

The map $\rho\pi$ is acyclic on the reduced components of S , so taking H sufficiently large and inducting on the partial ordering, we obtain,

IV.4.5 Lemma In the above notation all of the following maps are surjections,

$$\Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_S(-E)) \rightarrow \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_{E'|_S}(-E)) \rightarrow \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_{E''|_S}(-E))$$

proof It only remains to observe that the kernel of,

$$\mathcal{O}_S \longrightarrow \mathcal{O}_{E_{\text{tot}}|_S}$$

is supported in dimension zero. \square

Now applying the same considerations in V with \mathcal{S} the singular locus, \mathcal{E} an exceptional divisor, *etc.*, we have exact sequences,

$$\begin{array}{ccccc} 0 \rightarrow \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_S(-E_{\text{red}})) & \rightarrow & \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_S) & \rightarrow & \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_{E_{\text{red}}|_S}) = \mathbb{C} \rightarrow 0 \\ & \uparrow \rho^* & & \uparrow \rho^* & \parallel \\ 0 \rightarrow \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_S(-\mathcal{E}_{\text{red}})) & \rightarrow & \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_S) & \rightarrow & \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{E}_{\text{red}}|_S}) = \mathbb{C} \rightarrow 0 \end{array}$$

such that the group in the top left hand corner has lots of sections. Nevertheless one should be cautious since although its generated by global sections, it may not, for example, separate points, albeit it separates pull-backs of points, and similar, from \mathcal{S} , and this will be enough.

More precisely let ω in the bottom right group be given. At each singularity in the induced foliation it has a Taylor expansion. Apart from III.4.1 (c), the normal forms converge after completion in the singular locus, and we're only ever interested in this modulo a large power of the exceptional divisor on restriction to the centre manifold, so the expansion is, in fact, convergent in such coordinates.

In the cases of interest, III.4.1 (b)/(c), these coordinates are actually unique up to homothety in the irrational case, and otherwise the unimportant modification III.4.6. Let us look at this expansion more closely in these cases writing it as a Laurent expansion,

$$\begin{aligned}\omega &= \sum_{\substack{1 \leq a \leq p \\ 1 \leq b \leq q}} w_{ab} x^{-a} y^{-b} \frac{dx}{x} + \sum_{\substack{1 \leq a \leq p \\ b \geq 0}} w_{ab} x^{-a} y^b \frac{dx}{x} \\ &+ \sum_{\substack{a \geq 0 \\ 1 \leq b \leq q}} w_{ab} x^a y^{-b} \frac{dx}{x} + \sum_{a, b \geq 0} w_{ab} x^a y^b \frac{dx}{x}\end{aligned}$$

Each of these four regions has its own structure. The final one is rather trivial from the point of view of using Stokes for a residue calculation along $|f| = |x|^\alpha |y|^\beta$, since, $f^{-1} \partial(f) = (\alpha + \lambda \beta) x^{-1} \partial x$, so everything here is trivially bounded by the Segre class provided $\alpha + \lambda \beta \neq 0$, and otherwise we won't be using Stokes, and, will be keeping to the previous strategy, *i.e.* using the approximately invariant perturbation of $|f|$. The two middle regions need not be in the image of ∂ , but, *cf.* post IV.2.2, they are modulo the good region already discussed. Consequently, the obstruction to being able to write ω as d of meromorphic with a suitable pole (IV.4.3 (b) being an example of what is unsuitable), and whence apply Stokes on a boundary of our choice, is wholly in the leading region, and it is finite dimensional. On the further hypothesis that ω , viewed as a section of $K_{\mathcal{F}}$, vanishes on \mathcal{E}_{red} , *i.e.* belongs to the group on bottom left of the diagram, we can find $w_E \in \mathbb{C}$, and, $\omega_E \in \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_S(-E))$ such that,

- At every singularity as above,

$$\tilde{\omega} := \rho^* \omega - \sum_{E \geq E_{\text{red}}} w_E \omega_E \in K_{\mathcal{F}}(x^p, y^q)$$

equivalently: no Taylor coefficients in the obstructed sector.

- The form of the Taylor expansion of ω_E is what we shall call *simple*, *i.e.* for E locally $x^c y^d$, and $a = p - c, b = q - d$,

$$\omega_E = \begin{cases} d(x^{-a} y^{-b}) \bmod K_{\mathcal{F}}(x^p, y^q) & \text{non-resonant,} \\ x^{-a} y^{-b} \frac{dx}{x} \bmod K_{\mathcal{F}}(x^p, y^q) & \text{otherwise} \end{cases}$$

where amongst the singularities of the form III.4.1 (b)/(c), non-resonant means: not a rational eigenvalue k/l with $ka = lb$. In either case we can effect the local calculation of the residue of ω_E using the boundary $|x|^\alpha |y|^\beta = \epsilon$ by way of Stokes in the non-resonant case, and by the approximately invariant function commensurate to it otherwise. A similar statement is not quite possible for arbitrary ω , but we can achieve,

$$\rho^* \omega = \omega_0 + \sum_{E \geq E_{\text{red}}} w_E \omega_E + \tilde{\omega} \quad \omega_E \in \Gamma(K_{\mathcal{F}} \otimes \mathcal{O}_S(-E)), w_E \in \mathbb{C}$$

where all of the above are satisfied, except possibly the simplicity of ω_0 (here $a = p, b = q$) for singularities of type III.4.1 (c), or, indeed the rational case of III.4.1 (b), should, notation as per op. cit., one of i , or j be non-zero, but not both.

In order to apply these considerations, let us introduce an easy version of something already encountered in §I.4, *viz*:

IV.4.6 Definition Let G be a graph. By a (directed) multiplier is to be understood the assignment to each possible direction, say, $+$, $-$, of an edge multipliers $\nu_e^-, \nu_e^+ \in \mathbb{R}_+^\times$ such that $\nu_e^- \nu_e^+ = 1$. The multiplier is said to be continuous if the product around every directed cycle is 1.

Alternatively, as in §I.4, loops γ in G are just sequences of directed edges returning to whence they came, so we have a representation,

$$\nu : \pi_1(G) \longrightarrow \mathbb{R}_+^\times : \gamma \mapsto \prod_{e \in \gamma} \nu_e^\pm$$

or equivalently a class $\nu \in H^1(G, \mathbb{R}_+^\times)$. Consequently, the continuity condition is the triviality of this class, or what amounts to the same thing assigning weights $v \mapsto a(v)$ at each vertex, so that for a directed edge from v to w the multiplier is $a(w)a(v)^{-1}$.

Now plainly, post §IV.3, the only thing that we have left to worry about in our residue calculation is the continuity of our multipliers. In the decomposition of an arbitrary ω as above, the term $\tilde{\omega}$ can be done in a myriad of ways. For example just take as multipliers the trivial co-cycle corresponding to taking the actual multiplicities $p(v)$ as the weights for the vertices. In this way the local strategy at singularities is unchanged except in the cases III.4.1 (b)/(c), and a non-resonant boundary, *i.e.* $p + \lambda q \neq 0$, for λ the eigenvalue of the singularity normalised as per op. cit. Should this case occur, we switch to the strategy of using Stokes. The dichotomy of IV.3.4 ensures that this causes no additional continuity problems in the case of III.4.1 (b), with the procedure to be employed for gluing the boundary outside the singularity to that close by being as per IV.3.4 bis. Pretty much the same is true in the case III.4.1 (c), but one may have to add discontinuities about extra homology classes/loops around the singularity in either component even when local/global agreement occurs in IV.3.4 to account for the change in strategy. This is, however, no worse than the total variation of $w(\epsilon)$ which we already bounded appropriately in the proof of III.4.4.

Essentially identical remarks apply to computing the residues of ω_E for $E \geq E_{\text{red}}$, or for that matter ω_0 were all the Taylor expansions at every singularity of type III.4.1 (c) to be simple, *i.e.* both i, j non-zero or both zero. Indeed for weights on the vertices, one takes the multiplicities of $E_{\text{tot}} - E$, at the vertex which corresponds to a component of the same, so, as for $\tilde{\omega}$, the multiplicities of E_{tot} in the case of ω_0 . The change in local strategy is again, in the cases of III.4.1 (b)/(c), only at non-resonant boundaries, which for a, b the weights at the $x = 0, y = 0$ vertices, means $a + b\lambda \neq 0$, which occasions no further problems beyond those already discussed. We also need to change strategy in

the cases III.3.1 (c), III.4.1 (a) when $E \neq 0$, but this just means use IV.4.4 (a), and nothing of substance changes.

This leaves us to compute ω_0 in general, or equivalently in a series of improbabilities of increasing ludicrousness. In any case we view this as the calculation of $\rho_*\omega_0$, and one improvement we can make a priori is to perform sufficient blowing up so as to have only one edge between any pair of vertices. As such \widehat{G} is the dual graph of the singular locus invariant by the induced foliation in \widehat{Z} . Irrespectively of whether only one rather than both of i, j in III.4.1 (b)/(c) is zero, such edges have only one possible multiplier that permits the computation to be done locally, *i.e.* the eigenvalue λ_e of the directed edge in the notation of §IV.3, and we'll refer to such edges as *rigid*. If both i, j are zero then this is the multiplier that would arise by taking the multiplicities in E_{tot} as weights for the vertices. Otherwise we can make further a priori improvement by blowing up in a problematic singularity when one of these is zero, say, j . Our initial situation is therefore in the normalisation post IV.3.5, a directed edge of the form,

$$v_- : (x_e = 0) \xrightarrow[e]{\lambda_e=l/k} v_+ : (y_e = 0)$$

where λ_e is the eigenvalue at the singularity, and we'll say that it is of type $(0, i_0)$, $i = i_0$ - so, slightly confusingly, " i " follows the x -axis, not $x = 0$. Plainly it is rigid. The effect of blowing this up is,

$$v \xrightarrow[e_1]{\lambda_e+1} F_1 \xrightarrow[e_0]{\lambda_e/(\lambda_e+1)} w$$

where F_1 is the exceptional divisor, and the numbers over the edges e_\bullet the new eigenvalues. Now the above edge e_0 is again of type $(0, i_0)$, and there is absolutely no improvement. The edge e_1 is more interesting. More precisely,

- (a) In the obvious notation, it is never of type $(j, 0)$ for any $j \in \mathbb{N} \cup \{0\}$.
- (b) It could be of type $(0, i_1)$. This happens if $(k+l)|i_0$, in which case,

$$i_1 = \frac{li_0}{(k+l)} < i_0$$

- (c) Otherwise it's of type (j_1, i_1) , $i_1 j_1 \neq 0$.

Evidently (b) can not repeat itself ad nauseum, so blowing up in e_1 , to get a new chain in which we always blow up in the leftmost edge, with exceptional divisors F_\bullet , we eventually obtain for some $d \in \mathbb{N}$,

$$v \xrightarrow[e_d]{\lambda_d} F_d \xrightarrow[e_{d-1}]{\lambda_{d-1}} \dots \xrightarrow[e_1]{\lambda_1} F_1 \xrightarrow[e_0]{\lambda_0} w$$

where λ_\bullet is the eigenvalue of the singularity normalised as per §IV.3 according to the direction e_\bullet , and d is the first integer for which (c) holds at e_d . We perform this operation a priori in U before doing anything else, so this is our graph G ,

and when we construct ω_0 it will be simple at e_d . Furthermore by (a), we have the non-resonance condition $p(F_d) - \lambda_d p(v)$, for p the multiplicity of E_{tot} , so by IV.2.2/4(b) we can use a local strategy of Stokes type at e_d for an arbitrary multiplier ν_d . The above has no effect on the topology of the original graph G_0 , and if an edge $e : v \rightarrow w$ is in both G and G_0 , then we take $\nu_e = p(w)p(v)^{-1}$, otherwise we replace it by the sequence of edges e_\bullet as above, on which,

$$\nu(e_m) = \begin{cases} \frac{k\lambda_d}{l} p(w)p(v)^{-1} & \text{if } m = d, \\ \lambda_m & \text{if } 0 \leq m < d \end{cases}$$

This yields a trivial co-cycle, so, for example to keep ourselves consistent with §IV.3, choose a root R in G , and define a multiplier at a component C , by way of,

$$\lambda_C = \prod_{e \in \{R, C\}} \nu_e^\pm$$

then proceed exactly as in the proof of IV.3.6 with the same minor caveats for edges where a change in local strategy takes place (which, beyond those already encountered for $\tilde{\omega}$, are uniquely those of the form e_d as found above) which were already noted for ω_E and $\tilde{\omega}$. Consequently, we have proved the essential in,

IV.4.7 Fact Let things be as in IV.4.1, but without the projectivity assumption, with \mathcal{I}_Y the ideal of the singular locus, and $d\mu$ an invariant measure with support outside the exceptional divisor and zero Segre class around the same, then:

$$\text{RES} : \text{Ext}_U^2(\mathcal{I}_Y, K_{X/\mathcal{F}}) \longrightarrow \text{Ext}^3(\mathcal{O}/\mathcal{I}_Y, K_{X/\mathcal{F}})$$

is zero on $d\mu$.

proof It only remains to remove the hypothesis of projectivity. As already observed the moduli of V , *i.e.* weighted blow up in sufficiently many points is formally convex. Indeed the exceptional divisor F , in the notation of the construction of V , satisfies the co-homological criterion for ampleness in its completion \widehat{V} in the singular locus. Whence, the singular locus in \widehat{V} is formally contractible, and by [A3] this contraction converges. Whence $F|_V$ is ample, while the schematic covering $\tilde{U} \rightarrow V$ can always be constructed for topological reasons- $V \setminus F$ homotopic to the singular locus minus B , which we may suppose to contain at least 3 points in every component. \square

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