Almost étale resolution of foliations

Michael McQuillan & Daniel Panazzolo

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Introduction

The ubiquity of canonical singularities goes far beyond the narrow confines in which they were first conceived in [R]. Quite generally, given the 'smooth objects' within a given category, those with canonical singularities could be defined as the largest sub-category containing the former for which the ramification is well defined and non-negative. As such, they bear a tight relation with the notion of isoperimetric dimension introduced in [G], and are the natural category in which theorems of "uniformisation type" take place. By way of examples,

- A complex-projective variety admits a metric of negative Ricci curvature -1 iff it is the canonical model (the original context of [R]) of a variety of general type, [EGZ].
- A complex projective variety foliated in curves admits a leafwise metric of negative Ricci curvature -1 and continuous non-trivial transverse variation iff it is the (foliated) canonical model of a foliation of general type, [M3].

Where, as a not unimportant precision, one should, more correctly, state the above, in either case, on the smallest algebraic stack, with moduli as given, on which the dualising sheaf is a bundle, *i.e.* the Gorenstein covering stack, I.ii.5.

The main known existence theorems in characteristic zero are: the trivial foliation over a point (*i.e.* a variety) and co-dimension 1 in ambient dimension 2, [Se], or 3, [C2]. This paucity of results has a reason,

Proposition (§III.iii) *Already in characteristic zero, in all dimensions greater than or equal to 3, there are varieties $(X, \mathcal{F})$ foliated by curves such that no proper bi-rational modification $\pi : (\tilde{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ with $\tilde{X}$ smooth has canonical singularities.*

Whence, plainly, blowing up an ambient smooth space in smooth centres can never lead to a canonical resolution. The immediate reason for this phenomenon is that canonical singularities in relative dimension 1 understood functorially according to the general rules of [SGA] I.exposé VI, so $X \to [X/\mathcal{F}]$ is better notation, have the uniquely peculiar property, III.i.5, that the quotient of canonical by a finite group is canonical. A deeper reason is that there is no a priori relation between $X \to [X/\mathcal{F}]$ and meromorphic functions on X as there is in the case $X \to \mathrm{pt}$. For example, the indeterminacy of any meromorphic function may trivially be resolved by taking it's graph, and were there any similar procedure to resolve $X \to [X/\mathcal{F}]$ by way of a modification $\tilde{X} \to X$ which was canonical and Gorenstein, then irrespective of the singularities of $\tilde{X}$, the proposition would be false by [BM].

This deeper reason has many manifestations at the level of trying to establish a suitable resolution procedure. In light of recent developments, [K1], the case of $X \to \mathrm{pt}$ is fairly trivial, *i.e.* by decreasing induction on the multiplicity and embedding dimension, with some routine technicalities because differentiation doesn't quite commute with blowing up. In the easier problem of local uniformisation for rational rank 1 valuations (which is inductively the non-trivial one) even this technicality disappears, and one has a very simple induction on multiplicity alone. So far, however, even an appropriate local uniformisation of foliations by curves on 3-folds has resisted all efforts, and although it is understood, [CRM], that in the difficult rational rank 1 case one should be looking for a formal invariant curve of maximal contact, finding it has proved elusive because unlike resolutions of $X \to \mathrm{pt}$ sub-varieties of maximal contact are very rare, and, in general, are defined by differential rather than algebraic equations.

As a result it becomes necessary to look at the singularities of the foliation with more care than one might employ when studying resolutions of ideals. In this context there is a very useful local description of the closely related notion of log-canonical whenever the foliation is Gorenstein. Thus, by hypothesis, the foliation is defined by a vector field ∂, vanishing at some maximal ideal $\mathfrak{m}$, so there is an associated, quite possibly zero, linearisation,

$$\overline{\partial} \in \mathrm{End}\left(\mathfrak{m}/\mathfrak{m}^2\right)$$

and the singularity is log-canonical iff $\overline{\partial}$ is non-nilpotent. Again, peculiar to relative dimension 1, this is also canonical unless the singularity is *radial*, III.i.2. As such, from the point of view of the Proposition one could equally have said log-canonical rather than canonical, and, III.ii.2, there is no practical difference in the resolution problem between the two. In any case, the basic difficulty is exactly occasioned by nilpotence.

These difficulties not withstanding the second author in [P] proposed a series of definitions to address the problem of nilpotence, and effectively solved the problem of the existence of canonical resolutions in dimension 3 when he constructed log-canonical resolutions in the 1-category of real manifolds with boundary. The difference between this category and the 1-category of smooth algebraic or complex spaces is that it is preserved by weighted blowing up. Smoothness of the ambient space is, however, preserved by weighted blowing up in the 2-category of algebraic or complex Deligne-Mumford stacks. This remark is sufficient to import mutatis mutandis the proof of [P] to the algebraic or complex setting whenever an additional structure referred to as an axis, I.iii.1, exists at the first stage of the algorithm. Unfortunately, I.iii.2, such an additional structure can fail to exist without additional hypothesis such as projectivity. On the other hand, an axis provides a particular coordinate system in which to calculate the Newton polyhedron, and one might reasonably suspect that any invariants of the singularities will achieve their minimum value for a generic choice. This is made precise in §II.i-ii, allowing us to completely dispense with this additional structure, and so permit the algorithm of op. cit. to work in maximum generality, *viz:*

Theorem *Let $\mathcal{C}$ be the 2-category of triples $\mathbf{X} = (X, D, \mathcal{F})$ consisting in a foliated 3 dimensional stack in complex spaces with boundary, relatively compact in a triple of the same form, then there is an étale local modification functor,*

$$\mathfrak{M}_{\log}(\mathbf{X}) : \mathbf{X} = \mathbf{X}_0 \leftarrow \mathbf{X}_1 \leftarrow \ldots \leftarrow \mathbf{X}_m = \widetilde{\mathbf{X}}$$

consisting of a sequence of either smoothed weighted blow ups, I.iv.3, or algorithmic modifications, II.iii.5, in invariant centres in the 2-category of smooth logarithmic Deligne-Mumford stacks, such that $\widetilde{\mathbf{X}}$ has canonical singularities and normal crossing boundary.

The notions of étale local modification functor, and algorithmic modification are defined in II.iii.7 and II.iii.5 respectively. In practice the former means that each step is determined by the maximum strata of an invariant, here Inv of II.iii.1, with values in a discrete ordered group which is wholly independent of coordinates, *i.e.* it is an étale local algorithm. Almost certainly a better algorithm exists in which each step is a smoothed weighted blow up, but the fact that one can preserve étale locality without this has its own interest. For the avoidance of doubt let us observe,

Corollary *Let $\mathbf{X} = (\mathcal{X}, \mathcal{D}, \mathcal{F})$ be a foliated stack in 3-dimensional complex spaces with boundary then there is a proper bi-rational modification $\widetilde{\mathbf{X}} \to \mathbf{X}$ such that $\widetilde{\mathbf{X}}$ has canonical singularities and simple normal crossing boundary. The assignment, $\mathbf{X} \mapsto \widetilde{\mathbf{X}}$ is functorial and étale local. Under the additional hypothesis that $\mathbf{X}$ is relatively compact in a triple of the same form, e.g. algebraic of finite type, one can realise, non-functorially, any $\mathbf{X}_{k-1} \leftarrow \mathbf{X}_k$ which is an algorithmic modification by the composition of a sequence,*

$$\mathbf{X}_{k-1} = \mathbf{X}_{k,0} \leftarrow \mathbf{X}_{k,1} \leftarrow \ldots \leftarrow \mathbf{X}_{k,n} = \mathbf{X}_k$$

of smoothed weighted blow ups in smooth weighted centres invariant by the induced foliation at each stage.

The functor $\mathfrak{M}_{\log}$ is made up from the modification functor $\mathfrak{M}$, II.iii.9, based on [P], which yields log-canonical singularities under mild hypothesis on the boundary, an easy blow up functor $\mathfrak{B}$, III.ii.2, to go from there to canonical, together with the algorithmic blow up functor $\mathfrak{B}\mathfrak{M}$ of [BM]. Individually these functors always have the property that the centres at each stage have normal crossings with the exceptional divisor, whence a simple normal crossing exceptional divisor at each stage. In combination, however, perhaps this may only be restored at the end. This results from the limitation II.ii.1 on the boundary in the definition of $\mathfrak{M}$, and is an inessential technical lacuna. Wholly essential however, given the proposition and theorem, is that the resolution takes place in the 2-category of smooth stacks. By [V] smooth stacks almost étale over their moduli are in 1-1 correspondence with varieties with quotient singularities, so we may spell out what's going on in some detail:

Scholion *Alternatively, in the 1-category of algebraic or complex spaces with quotient singularities, there is a sequence of weighted blow ups,*

$$(X, D, \mathcal{F}) = (X_0, D_0, \mathcal{F}_0) \leftarrow (X_1, D_1, \mathcal{F}_1) \leftarrow \ldots \leftarrow (X_n, D_n, \mathcal{F}_n) = (\tilde{X}, \tilde{D}, \tilde{\mathcal{F}})$$

which is initially the moduli of the sequence of the Theorem/Corollary, but is then continued by way of centres which are either strictly invariant or everywhere transverse to the induced foliation. The resulting final situation has canonical singularities, and ambient singularities at worst isolated $\mathbb{Z}/2$ quotient singularities. The form of the quotient singularity that may occur is fully described, III.iii.3, and examples are provided where it does occur. In dimension 3 (as opposed to higher dimensions) it is this singularity, and this singularity alone that is the cause of the Proposition. In particular the Theorem is best possible, and there does not even exist a proper bi-rational modification of this singularity in the 1-category of spaces which is both (foliated)-Gorenstein and log-canonical.

As such, smoothness, from the point of view of the geometrisation properties of canonical singularities with which we introduced them above, is a functorial chimera. Already there were strong indications of this in the canonical model theory of foliated surfaces [M1], and, indeed the proposition already holds for foliated surfaces in characteristic at least 3, albeit in a less interesting way since there is always a log-canonical resolution with ambient smooth space. It should, therefore, not surprise that the Theorem is well adapted to the minimal model programme for foliations by curves in any dimension, [M2], [M3] and is a key ingredient in the proof, [M4], of the Green-Griffiths' conjecture for algebraic surfaces with enough 2-jets.

The root of the Proposition is an original intuition of Felipe Cano based on his work [C1], elaborated by Fernando Sanz in [S]. Both authors are indebted to them for sharing this particular knowledge, and more generally to Felipe Cano for having shared his entire expertise in the whole discipline. They are also indebted to the organisers Erwan Rousseau and Gianluca Pacienza of the conference Algebraic Varieties and Hyperbolicity at IRMA, Strasbourg, which occasioned the fortunate meeting of the authors, and subsequent writing of this article. The first author, however, considers that this is in stark contrast to a long list of people from France to Brazil who were criminally negligent in drawing the work of the second author to his attention. The first author would be more than happy to name and shame those involved. Being late though, is better than never, so Jorge Vitório Pereira escapes this criticism (just) and the first author is happy to acknowledge his role in preparing the fortuitous meeting in Strasbourg by bringing the work of the second author to his attention a couple of months earlier. Fortunately, Cécile is radically more efficient than Jorge, and so many more thanks to her for the web copy.

I. Generalities

I.i Ordering

As in all resolution problems it will be necessary to introduce a slight variation of the principle object of interest in order to define a suitable invariant that will decrease under blowing up. To this end, we introduce,

I.i.1 Definition A *smooth foliated stack with ordered s.n.c. boundary* is a 4-tuple $(\mathcal{X}, \mathcal{D}, \Upsilon, \mathcal{F})$ such that,

(1) $\mathcal{X}$ is a smooth (in general, but not always, connected) Deligne Mumford stack in finite dimensional analytic spaces. So finite type over a field of characteristic zero if our context were purely algebraic, which could be supposed algebraically closed, but since this is subordinate to the analytic case, we will always work over the complex numbers $\mathbb{C}$.

(2) A simple normal crossing divisor $\mathcal{D}$ each of whose irreducible components $\mathcal{D}_i$ is a smooth co-dimension 1 sub-stack of $\mathcal{X}$.

(3) For each closed point p a total ordering, $\Upsilon : p \mapsto \iota_p$, on the boundary components through p such that if $\iota_p(D) < \iota_p(E)$ for some p, then $\iota_q(D) < \iota_q(E)$ for all q whenever both sides are defined.

(4) A foliation by curves $\mathcal{F}$ leaving $\mathcal{D}$ invariant, *i.e.* a line bundle $T_{\mathcal{F}}$ on $\mathcal{X}$ together with an injection,

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_{\mathcal{X}}(-\log \mathcal{D})$$

with torsion free co-kernel.

The related data of the 3-tuple $(\mathcal{X}, \mathcal{F}, \mathcal{D})$ will be referred to as a *smooth foliated stack with s.n.c. boundary*. In the event that neither the smoothness of $\mathcal{X}$ nor of $\mathcal{D}$ nor even that $T_{\mathcal{F}}$ is anything better than reflexive rank 1 is supposed then we will call such a triple a *foliated logarithmic stack*. Consequently, the following remark is relevant,

I.i.2 Caution/Definitions A foliation is often defined as a saturated sub-sheaf of $T_{\mathcal{X}}$, and since $T_{\mathcal{X}}(-\log \mathcal{D}) \subset T_{\mathcal{X}}$ such a definition may well not be compatible with I.i.1.(4). Indeed, $T_{\mathcal{F}}$ in the sense of I.i.1.(4) is saturated in $T_{\mathcal{X}}$ iff it is saturated at each generic point of $\mathcal{D}$, which in turn is true iff the foliation considered as a saturated sub-sheaf of $T_{\mathcal{X}}$ fixes every generic point of $\mathcal{D}$. Consequently, and functorially with respect to the ideas, the dual $K_{\mathcal{F}}$ of $T_{\mathcal{F}}$ as defined in I.1.i.(4) is the *log canonical* bundle of the foliated logarithmic stack $(\mathcal{X}, \mathcal{D}, \mathcal{F})$. Furthermore, should a component of $\mathcal{D}$ be invariant by the foliation viewed as a saturated sub-sheaf of $T_{\mathcal{X}}$ then it will be called *strictly invariant*.

I.ii Log canonical singularities

These are defined functorially with respect to the ideas in the usual way, *i.e.*

I.ii.1 Definition Let $(U, D, \mathcal{F})$ be an irreducible local germ of a $\mathbb{Q}$-Gorenstein foliated logarithmic normal variety or analytic space, *i.e.* the germ about the generic point of a sub-variety Z of a normal variety or complex space X such that the log canonical bundle $K_{\mathcal{F}}$ is a $\mathbb{Q}$-divisor, then for v a divisorial valuation of $\mathbb{C}(U)$ centred on Z the *log discrepancy*, $a_{\mathcal{F}}(v)$ is defined as follows:
By hypothesis there is a normal modification $\pi : \tilde{U} \to U$ of finite type, together with a divisor E on $\tilde{U}$ such that $\mathcal{O}_{\tilde{U},E}$ is the valuation ring of v. In particular, bearing in mind I.i.2, there is an induced foliation $\tilde{\mathcal{F}}$ with log canonical bundle $K_{\tilde{\mathcal{F}}}$, *i.e.* whose dual is saturated in $T_{\tilde{U}}(-\log E)$, and,

$$K_{\tilde{\mathcal{F}}} = \pi^* K_{\mathcal{F}} + a_{\mathcal{F}}(v)E$$

If furthermore we define $\epsilon(v)$ to be zero if E is strictly invariant, and 1 otherwise, then provided the following hold for all divisorial valuations centred on Z we say that the local germ $(U, D, \mathcal{F})$ is,

(1) *Terminal* if $a_{\mathcal{F}}(v) > \epsilon(v)$.

(2) *Canonical* if $a_{\mathcal{F}}(v) \geq \epsilon(v)$.

(3) *Log-Terminal* if $a_{\mathcal{F}}(v) > 0$.

(4) *Log-canonical* if $a_{\mathcal{F}}(v) \geq 0$.

Where the slightly unsettling shift of the definitions by $\epsilon(v)$ occurs as a result of the convention adopted in I.i.2 together with their correct functorial interpretation.

The discussion of definitions (1)-(3) will be postponed till §III.1, and a priori what is of relevance is the definition of log-canonical singularities. A priori it depends on the topology in which it is defined, and the number of flavours of such depends on the nature of X. Irrespectively, in all cases the completion $\hat{U}$ in the maximal ideal maps to the germ U, and by [K2] VI.1.4, a divisorial valuation $\hat{v}$ of $\mathbb{C}(\hat{U})$ determines via the inclusion $\mathbb{C}(U) \hookrightarrow \mathbb{C}(\hat{U})$, a divisorial valuation v on U. In addition $\hat{U} \to U$ is unramified, so any anxiety as to ambiguity in the definition is eliminated by way of,
I.ii.2 Fact Notations as above then $(U, D, \mathcal{F})$ has a log-canonical singularity if and only if $(\hat{U}, \hat{D}, \hat{\mathcal{F}})$ has a log canonical singularity.
Proof: $a_{\hat{\mathcal{F}}}(\hat{v}) = a_{\mathcal{F}}(v)$. $\square$

Rather more usefully, however, we can explicitly describe log-canonical singularities as soon as the germ $(U, D, \mathcal{F})$ is *Gorenstein*, *i.e.*, $K_{\mathcal{F}}$ (equivalently for U normal $T_{\mathcal{F}}$) is a line bundle. Consequently, the foliation is defined by a vector field ∂ and we have,
I.ii.3 Possibilities Exactly one of the following occurs,

(a) ∂ is smooth, *i.e.* there exists $f \in \mathcal{O}_{X,Z}$ such that $\partial(f) \neq 0 \in \mathbb{C}(Z)$.

(b) Otherwise, so ∂ not only leaves $\mathfrak{m}_{X,Z}$ invariant but descends to a $\mathbb{C}(Z)$ linear endomorphism of the Zariski tangent space,

$$\bar{\partial} : \frac{\mathfrak{m}_{X,Z}}{\mathfrak{m}^2_{X,Z}} \longrightarrow \frac{\mathfrak{m}_{X,Z}}{\mathfrak{m}^2_{X,Z}}$$

With this in mind, one observes,

I.ii.4 Fact ([M2] §I.6) Suppose the germ (U, D, F) is Gorenstein, then it is log canonical if and only if a local generator ∂ is either smooth or $\bar{\partial}$ is a non-nilpotent endomorphism of the Zariski tangent space.

Proof: The only if definition is straightforward, since otherwise we're in I.ii.3.b, and $\bar{\partial}$ is nilpotent. As such, it may very well be zero, in which case blowing up in Z, and normalising yields divisorial valuations with negative discrepancy, while for an arbitrary nilpotent field one can explicitly construct a valuation with negative discrepancy by way of blowing up, and normalisation, in suitable centres defined by the Jordan blocks of $\bar{\partial}$. Conversely, with the notation of I.ii.1, let $\pi : \tilde{U} \to U$ be a modification of U associated with a divisorial valuation, ∂, $\tilde{\partial}$ generators of the foliation on U, and $\tilde{U}$ respectively, with x a uniformising parameter of the divisor E, then for a the discrepancy of the valuation v,

$$\partial = x^{-a} u \tilde{\partial}$$

for u a unit. Now suppose that ∂ is smooth, then there is a function f on U such that, $v(\partial f) = 0$. As such,

$$av(x) = v(\tilde{\partial}(\pi^* f)) \geq 0$$

so $a \geq 0$. More generally, observe that since $\tilde{\partial}$ leaves E invariant, then for any $q \in \mathbb{N}$,

$$x^{qa} \partial^q$$

is a regular differential operator on $\tilde{U}$ leaving E invariant. Furthermore if ∂ is non-nilpotent then for any $n \in \mathbb{N}$ there is a $q \in \mathbb{N}$ and a function $f_n \in \mathfrak{m}_{X,Z} \backslash \mathfrak{m}_{X,Z}^2$ on either an analytic or strictly Henselian germ, U_*, say, such that,

$$\partial^q(f_n) = \lambda f_n \bmod \mathfrak{m}_{X,Z}^n$$

for some unit λ. Indeed this is just a consequence of Jordan decomposition, and since v is divisorial no formal function has infinite order, so we can actually ensure that f_n is fixed modulo $\mathfrak{m}_{X,Z}^2$. for v_* any divisorial valuation of U_* lying over v, whence $v_*(f_n)$ is bounded independently of n. Consequently, for n sufficiently large,

$$v_*(f_n) = v(\partial^q f_n) \geq -aqv_*(x) + v_*(Df_n)$$

where D is a regular differential operator leaving E invariant, *i.e.* $v_*(Df_n) \geq v(f_n)$, so again $a \geq 0$. □

To cover the $\mathbb{Q}$-Gorenstein case, one observes,

I.ii.5 Fact/Definition Let $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ be a normal foliated $\mathbb{Q}$-Gorenstein log stack then there is a normal foliated Gorenstein log stack, $\nu : (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{D}, \mathcal{F})$ with $\nu : \tilde{\mathcal{X}} \to \mathcal{X}$ finite and étale in co-dimension 1, which is in fact universal for the said properties, and whence will be referred to as the *Gorenstein covering stack*. In particular, $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ has log canonical singularities if and only if it's Gorenstein covering stack does.

Proof: Let $U \to \mathcal{X}$ be an atlas such that for some $n \in \mathbb{N}$, $\mathcal{O}_U(nK_{\mathcal{F}})$ is trivial, and n is minimal. Consequently, we can define a finite covering $V \to U$ of degree at most n such that $\mathcal{O}_V(K_{\mathcal{F}})$ is trivial, and V is normal. By hypothesis the diagonal of $\mathcal{X}$ is representable so $V \times_{\mathcal{X}} V$ is a space, and by the construction of V, its normalisation R yields an étale groupoid,

$$R \rightrightarrows V$$

and the classifying stack $[V/R]$ is the required universal widget.

Certainly therefore, $\nu^* K_{\mathcal{F}} = K_{\tilde{\mathcal{F}}}$, while if $\tilde{\mathcal{E}}$ is any divisor on a normal modification of $\tilde{\mathcal{X}}$ lying over $\mathcal{E}$, then the log-discrepancy of $\tilde{\mathcal{F}}$ around $\tilde{\mathcal{E}}$ is just $\mathrm{ord}_{\tilde{\mathcal{E}}}(\mathcal{E})$ that of $\mathcal{F}$ around $\mathcal{E}$. $\square$

This leads to,

I.ii.6 Definition/Summary For $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ a normal $\mathbb{Q}$-Gorenstein foliated log stack with Gorenstein cover $\nu : \tilde{\mathcal{X}} \to \mathcal{X}$ the singular locus, $\mathrm{sing}(\mathcal{F})$, of $\mathcal{F}$ is the reduced image of the closed substack where a local generator of $\tilde{\mathcal{F}}$ is not smooth in the sense of I.ii.3.(b). In particular the locus of non-log canonical points, $\mathrm{NLC}(\mathcal{F})$, which, with reduced structure, we may identify with $\mathrm{NLC}(\tilde{\mathcal{F}})$, is a closed sub-stack of $\mathrm{sing}(\mathcal{F})$. Given I.i.2, however, $\mathrm{sing}(\mathcal{F})$, can be of co-dimension 1. Nevertheless, this only occurs at generic points of $\mathcal{D}$ which are not invariant by the foliation viewed as a saturated sub-sheaf of $T_{\mathcal{X}}$, and since $\mathcal{X}$ is regular in co-dimension 1, such points are log-canonical. Consequently, $\mathrm{NLC}(\mathcal{F})$ is closed, and of co-dimension at least 2.

I.iii Axes

In [P] the need to work outwith the natural 2-category of foliated log-stacks was not limited to ordering, but involved an additional structure, called an *axis*. For ease of reference we recall its definition, *viz:*

I.iii.1 Definition Let $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ be a normal $\mathbb{Q}$-Gorenstein foliated log-stack, with $\mathfrak{N}$ its completion in $\mathrm{NLC}(\mathcal{F})$, then by an axis for $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ is to be understood a Gorenstein foliated log-stack, $(\mathfrak{N}, \mathcal{D}|_{\mathfrak{N}}, \mathcal{A})$, such that,

(a) There is a (not necessarily connected) étale neighbourhood U of $\mathrm{NLC}(\mathcal{F})$ such that for some $\mathcal{A}_U$, $\mathcal{A}$ is the restriction to $\mathfrak{N}$ of the same.

(b) The foliation $\mathcal{A}$ is smooth, *i.e.* as per I.ii.3.(a) defined everywhere locally by a non-vanishing vector field.

(c) The foliation $\mathcal{A}$ is transverse to every generic point of $\mathrm{NLC}(\mathcal{F})$ contained in the boundary.

(d) No $f : \mathrm{Spf}\mathbb{C}[[x]] \to \mathfrak{N}$ which is $\mathcal{A}$ invariant, with trace a non-boundary geometric point, is $\mathcal{F}$ invariant.

As such the definition of an axis comes a priori, by way of I.i.1.(3), with a foliation that leaves $\mathcal{D}|_{\mathfrak{N}}$ invariant, and plainly, the existence of an axis imposes conditions on $(\mathcal{X}, \mathcal{D}, \mathcal{F})$, for example even if $(\mathcal{X}, \mathcal{D})$ were as per I.i.1.(1) & (2)

then as soon as a point of NLC($\mathcal{F}$) meets dim$\mathcal{X}$ components of $\mathcal{D}$ an axis cannot exist. There are, however, much worse when problems associated with existence of an axis which we summarise by way of,

I.iii.2 Remark If, for example, $\mathcal{X}$ were a projective variety with at most one boundary component through any point of NLC($\mathcal{F}$), one could obtain an axis in a Zariski open around NLC($\mathcal{F}$)- standard Serre vanishing arguments with an ample line bundle. Beyond this it may simply be impossible to find such a thing even on $\mathfrak{N}$ once NLC($\mathcal{F}$) has compact components. Already for an algebraic space, there may be very few line bundles on $\mathfrak{N}$, in fact, quite possibly only the trivial one, and, in such a situation, it will not always be possible to realise I.iii.1 (a)-(e). The case of analytic spaces with few meromorphic functions in neighbourhoods of NLC($\mathcal{F}$) is worse again, and, when $\mathcal{X}$ is not a space around $p \in$ NLC($\mathcal{F}$), there are even local obstructions. More precisely, at a geometric point with non-trivial monodromy G, the local situation is of the form $[V/G]$ for V strictly Henselian, or germ of a poly-disc, according to whether we're working algebraically or analytically. In either case, we may identify the action of G with its linearisation on,

$$\mathrm{Aut}(\mathfrak{m}/\mathfrak{m}^2)$$

for $\mathfrak{m}$ the maximal ideal at p. The condition I.iii.1 (a) implies that this action must admit a common eigenvector for all $g \in G$, which cannot be guaranteed unless G is abelian. Finally even supposing that all such local obstructions were overcome, and suitable eigenvectors were available, then they determine characters χ which may be identified with elements of the local Picard group. Whence there is a further global obstruction, beyond those for complex or algebraic spaces, that these characters must belong to the image of,

$$\mathrm{Pic}(\mathfrak{N}) \longrightarrow \mathrm{Pic}([V_N/G]) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \ni \chi$$

Consequently the existence of an axis outwith hypothesis such as Stein or projective cannot be guaranteed.

I.iv Weighted Blowing Up

The weighted blow up of a stack $\mathcal{X}$ is defined as follows,

I.iv.1 Definition A weighted blow up of weight $\omega = (\omega_1, \ldots, \omega_r)$, $\omega_i \in \mathbb{N}$ without common divisor, with smooth centre, of co-dimension r, is the projectivisation,

$$\pi : \mathcal{X}' = \mathrm{Proj}(\mathcal{O}_{\mathcal{X}} \oplus \mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \ldots) \rightarrow \mathcal{X}$$

where $\mathcal{I}_k$ are sheaves of ideals on $\mathcal{X}$ such that locally in the étale site of $\mathcal{X}$ there are smooth coordinate functions $x_1, \ldots, x_r$, defining a smooth co-dimension r centre, and $\mathcal{I}_k$ is generated by the monomials,

$$(x_1^{a_1} \ldots x_r^{a_r} | a_1\omega_1 + \ldots + a_r\omega_r \geq k)$$

Now, manifestly the minor difficulty with weighted blowing up, even in a smooth centre, is that as soon as some $\omega_i > 1$, it may very well lead to singularities. Nevertheless, these are no worse than quotient singularities, and so we may appeal to,

I.iv.2 Fact/Definition (characteristic 0) Let $\mathcal{X}$ be an algebraic stack with quotient singularities, so by definition normal, then there is a smooth stack $\mu : \tilde{\mathcal{X}} \to \mathcal{X}$, such that μ is finite and étale in co-dimension 1. Further μ may be taken as being universal with respect to the above properties, in which case we will refer to it as the *Vistoli covering stack*.

Proof: Following [V], let $U \to \mathcal{X}$ be an étale atlas, then refining U as necessary, we may realise each connected component U_i of U as the coarse moduli of some V_i/G_i, where V_i is smooth, and G_i acts freely in co-dimension 1. As such, we put $V = \coprod_i V_i$, and R the normalisation of,

$$V \times_{\mathcal{X}} V$$

which as ever is a space, or indeed a scheme if our context were algebraic, by the representability of the diagonal, so, by purity, we obtain an étale groupoid,

$$R \rightrightarrows \mathcal{X}$$

and the required universal widget is the classifying stack $[V/R]$. $\Box$

Naturally, this leads to,

I.iv.3 Definition A *smoothed weighted blow up* $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ of weight ω in a smooth centre is the Vistoli covering stack $\tilde{\mathcal{X}} \to \mathcal{X}'$ of a weighted blow up $\mathcal{X}' \to \mathcal{X}$ of weight ω in a smooth centre.

An explicit description of charts for smoothed weighted blow ups of smooth stacks will be helpful, so to this end take a sufficiently small étale neighbourhood V of the moduli X such that $\mathcal{X} \times_X V = [U/G]$ for some finite group acting on a smooth affine U on which the $\mathcal{I}_k$ admit a description as per I.iv.1. Thus, although the coordinates in this description may not be G invariant, we may, and will, identify the sheaves of ideals $\mathcal{I}_k|_{[U/G]}$ with G equivariant ideals of functions I_k on U. Consequently, the weighted blow up is, locally, a classifying stack of the form,

$$\mathcal{X}'_V = [\mathrm{Proj}(\Gamma(\mathcal{O}_U) \oplus I_1 \oplus I_2 \oplus \ldots)/G]$$

Now étale neighbourhoods V_p of a closed point, p, on a weighted Proj, can after a suitable re-ordering of the indices, be described as follows: there are non-zero constants $\eta_2, \ldots, \eta_s$, $s \leq r$, and coordinate functions $y_1, \ldots, y_n$ on a smooth affine W_p related to the original coordinate functions $x_1, \ldots, x_n$ by way of,

$$x_1 = y_1^{\omega_1},\, x_i = y_1^{\omega_i}(y_i + \eta_i),\, 2 \leq i \leq s,\, x_i = y_1^{\omega_j} y_j,\, s < j \leq r,\, x_k = y_k,\, r < k$$

together with an action of $\mathbb{Z}/d_p$ for d_p the gcd of $\omega_1, \ldots, \omega_s$ given for θ a primitive d_pth root of unity by,

$$y_1 \mapsto \theta y_1,\, y_i \mapsto \theta^{-\omega_i} y_i,\, 2 \leq i \leq r,\, y_k \mapsto y_k,\, r < k$$

such that the coarse moduli of $W_p/(\mathbb{Z}/d_p)$ may be identified with V_p, which may usefully be compared with [P] 2.4. In any case such a V_p may not be G stable, but for G_p the stabiliser of p, and further étale localisation as necessary,

$[V_p/G_p] \to \mathcal{X}'_V$ is not only étale, but so too is the induced map on coarse moduli. Furthermore the strict Henselisation of W_p around p is unique with respect to the property that it is strictly local smooth connected and almost étale over the strict Henselisation of V_p, so, modulo further localisation we can lift the action of $\sigma \in G_p$ to an action of some $\tilde{\sigma}$ on W_p. Such liftings need not, however, respect the group structure of G_p, so we can, in general, do no better than,

I.iv.4 Summary Not only is W_p with coordinate functions as described above an étale neighbourhood of a smoothed weighted blow up, $\tilde{\mathcal{X}}$, of a smooth stack, $\mathcal{X}$, but there is an action of a finite group $\tilde{G}_p$, given as an extension,

$$1 \to \mathbb{Z}/d_p \to \tilde{G}_p \to G_p \to 1$$

such that both the map, $[W_p/\tilde{G}_p] \to \tilde{\mathcal{X}}$ and the induced map on moduli are open embeddings.

Furthermore the above explicit description also yields,

I.iv.5 Fact The exceptional divisor, $\mathcal{E}$ of a smoothed weighted blow up, $\tilde{\mathcal{X}}$, of a smooth stack $\mathcal{X}$ in a connected centre is itself smooth and connected.

Finally let us offer a small precision on the dependence of the weighted blow up on the local coordinates x_i. Specifically say m is the maximum of the ω_i, and J the ideal of the support, *i.e.* $(x_1, \dots, x_r)$, and suppose,

$$\xi_i = x_i \mod J^m$$

then for $I_k(x)$, $I_k(\xi)$ the ideals generated by the weighted monomials in x_i and ξ_i, an explicit calculation reveals for any k,

$$I_k(x) \subset I_k(\xi)$$

and conversely. As such $I_k(x) = I_k(\xi)$, so the weighted blow ups coincide, and, whence, by the universal property, also the smoothed weighted blow ups, *i.e.*

I.iv.6 Remark Whether smoothed or not a weighted blow up with supporting ideal J is wholly determined by its structure modulo $J^{\max_i\{\omega_i\}}$.

I.v Modifying orders and axes

It remains to discuss how orders change and axes were changed during the resolution procedure. All steps of the procedure will be by smoothed weighted blow ups in smooth $\mathcal{F}$ invariant (albeit not necessarily strictly) centres. Indeed the stronger condition of being an *invariant weighted blow up, i.e.* the sheaf of graded algebras occurring in I.iv.1 is $\mathcal{F}$ invariant will even hold. As such,

I.v.1 Definition Let $\mathbf{X} = (\mathcal{X}, \mathcal{D}, \Upsilon, \mathcal{F})$ be a smooth foliated stack with ordered smooth boundary and $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ a smoothed invariant blow up in a not necessarily connected centre with exceptional divisor $\mathcal{E}$ then the associated smoothed weighted blow up,

$$\pi : \tilde{\mathbf{X}} \to \mathbf{X}$$

is the 4-tuple $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\Upsilon}, \tilde{\mathcal{F}})$, where

1. $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ is the aforesaid smoothed weighted blowing-up;
2. The divisor $\tilde{\mathcal{D}}$ is the total transform of $\mathcal{D}$, thus the proper transform $\bar{\mathcal{D}}$ together with the exceptional divisor $\mathcal{E}$. This of course, implicitly supposes, as will be true, that each resulting component of $\tilde{\mathcal{D}}$ is smooth and the total divisor has normal crossings;
3. The ordering $\widetilde{\Upsilon}$ restricted to $\bar{\mathcal{D}}$ is the pullback of Υ, while $\mathcal{E}$ is regarded as greater than every component of the proper transform whenever this has sense;
4. The proper transform $\tilde{\mathcal{F}}$ of the foliation $\mathcal{F}$, *i.e.* by the hypothesis of the invariance of the weighted blow up, and the almost étale nature of the Vistoli cover the foliation lifts to a map,

   $$\pi^* T_{\mathcal{F}} \to T_{\tilde{\mathcal{X}}}(-\log \tilde{\mathcal{D}})$$

   which a priori may not be saturated, and so we saturate it to a map,

   $$\pi^* T_{\tilde{\mathcal{F}}} \to T_{\tilde{\mathcal{X}}}(-\log \tilde{\mathcal{D}})$$

   whence, in particular, $K_{\tilde{\mathcal{F}}} \leq \pi^* K_{\mathcal{F}}$.

Now suppose the foliated log-stack $\mathbf{X} = (\mathcal{X}, \mathcal{D}, \mathcal{F})$ admits an axis $\mathcal{A}$. Plainly, associated to the modified data $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}})$ there is a new non-log canonical locus, together with the completion $\tilde{\mathfrak{N}}$ of $\tilde{\mathcal{X}}$ at the same. Furthermore, since the weighted blow up is invariant, $\mathrm{NLC}(\tilde{\mathcal{F}}) \subset \pi^{-1}(NLC(\mathcal{F}))$, by the very definition II.ii.1.(4), so there is an induced map of formal schemes, $\pi : \tilde{\mathfrak{N}} \to \mathfrak{N}$ factoring through a projective modification of $\mathfrak{N}$, so at the price of correcting for poles, the original axis yields a saturated rank 1 sub-sheaf of $T_{\tilde{\mathfrak{N}}}(-\log \tilde{\mathcal{D}})$, *i.e.* we have an induced foliated log-stack $(\tilde{\mathfrak{N}}, \tilde{\mathcal{D}}|_{\tilde{\mathfrak{N}}}, \tilde{\mathcal{A}})$, and so:

I.v.2 Definition Notations as above, then should it occur that $\tilde{\mathcal{A}}$ is an axis for the foliated log stack $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}})$ then it will be called the proper transform of the axis.

I.v.3 WARNING Although we do not employ an axis as encountered in [P], it is useful to remember that the invariant, inv, of the next section is locally equivalent to that of op. cit. with a generic axis. In particular the above non-trivial condition on the proper transform continues to be implicitly satisfied.

II. The Algorithm

II.i. Equireducibility

Possibly the algorithm starts with a smooth foliated stack $(\mathcal{X}, \mathcal{F})$ without boundary, and we concentrate our initial attention on the generic point of a supposed positive dimensional component c of $\mathrm{NLC}(\mathcal{F})$. The discussion is local, so we identify c with its schematic generic point, and denote by $\mathcal{O}_{\mathcal{X},c}$ the strict Henselisation at c, with K the algebraically closed residue field. We assert:

II.i.1 Lemma There is a $\mathcal{F}$-invariant weighted filtration on the maximal ideal $\mathfrak{m}(c)$ of $\mathcal{O}_{\mathcal{X},c}$ depending only on $\mathcal{F}$. In particular it is preserved by every automorphism of $\mathcal{O}_{\mathcal{X},c}$ which preserves $\mathcal{F}$. Furthermore it is edge stable in the sense of [P] §5.4.

Proof Let I be the scheme theoretic singular locus of the foliation, with ν its multiplicity along $\mathfrak{m} = \mathfrak{m}(c)$. In addition for ∂ a generator of $\mathcal{F}$ there is a minimal μ such that we have a non-zero $\mathcal{O}_{X,c}/\mathfrak{m}^{\mu}$ linear map,

$$\bar{\partial} \in \mathrm{End}\big(\mathfrak{m}/\mathfrak{m}^{\mu+1}\big)$$

null on $\mathfrak{m}^2/\mathfrak{m}^{\mu+1}$. There are cases to consider according as whether this map is what is termed di-critical or not, *i.e.* multiplication by a function of multiplicity $\mu - 1$ or not, and whether $\mu = \nu$ or $\nu + 1$.

Let us first suppose that $\mu = \nu$ and we're not in the di-critical situation, with y, z a generic basis of $\mathfrak{m}$, then in the notation of [P] §5.3 which we freely adapt without any reference to an axis, the generic higher vertex is $(-1, \mu)$. Most of the time this will also be the main vertex, and the weight filtration is just the trivial one, *i.e.* by powers of $\mathfrak{m}$. The exception is that there is a unique change of coordinates $z \mapsto z + \chi y$, $\chi \in K = \mathcal{O}_{X,c}/\mathfrak{m}$, such that up to a homothety,

$$\bar{\partial} = (z^{\mu} + az^{\mu-1}y)\frac{\partial}{\partial y} + bz^{\mu}\frac{\partial}{\partial z}$$

for some $a, b \in K$.

Here we must distinguish sub-cases. The first is a and b zero, so by virtue of the definition of μ, the main vertex, written (m_2, m_3) for consistency with [P], will always be $(-1, \mu)$. We may, furthermore, canonically express our initial situation by observing that ∂ induces an operator, again denoted ∂, in,

$$\mathrm{Der}(\mathfrak{m}/\mathfrak{m}^{\mu+1}, (z)^{\mu}/\mathfrak{m}^{\mu+1})$$

and since ∂ cannot vanish on a hypersurface there is a maximal $k \in \mathbb{N}$ such that for a possibly different z,

$$\partial : \mathfrak{m}/\mathfrak{m}^{\mu+k} \to (z)^{\mu}/\mathfrak{m}^{\mu+k} \text{ and } \partial : \mathcal{O}_{\mathcal{X},c}/\mathfrak{m}^{\mu+k-1} \to (z)^{\mu}/\mathfrak{m}^{\mu+k-1}$$

while such an ideal (z) is uniquely determined by the first condition modulo $\mathfrak{m}^{k+1}$. Following [P] we must now look for the main edge of the Newton polygon, or equivalently a vertex (m_2+p, m_3-q) such that $\Delta_2 := p/q$ is minimal. In the cases of the trivial filtration this is, of course, equal to 1, whereas in the present sub-case it must satisfy,

$$1 < \Delta_2 \leq k+1$$

Irrespectively of the actual value, this edge is preserved by all coordinate changes of the form,

$$z \mapsto z + f, \quad y \mapsto y + g, \quad f \in \mathfrak{m}^{\lfloor \Delta_2 \rfloor},\ g \in \mathfrak{m}$$

since (z) is unique modulo $\mathfrak{m}^{k+1}$, and we have an intrinsic weighted filtration in which z has weight p, and y weight q.

The case where at least one of a or b is non-zero implies that $\mu \geq 2$. It is treated exactly as above, with the only differences being that it is a so called nilpotent configuration so the main vertex, (m_2, m_3) is at $(0, \mu-1)$, and k is maximal for the single property,

$$\partial \in \mathrm{Der}(\mathfrak{m}/\mathfrak{m}^{\mu+k}, (z)^{\mu-1}/\mathfrak{m}^{\mu+k})$$

We therefore turn to the dicritical case, so necessarily $\mu \geq 2$. For generic y the higher vertex is $(-1, \mu+1)$, and the configuration is nilpotent with main vertex $(0, \mu-1)$. Once more, most of the time the filtration is trivial, and the only case of a non-trivial filtration occurs when the function occurring as the multiplier in $\bar{\partial}$ may be written as $z^{\mu-1}$, and everything is as per the previous exceptional and nilpotent configuration.

This leaves the case $\mu - 1 = \nu \geq 1$, and one should take y so that it is not one of the co-vectors in $\mathfrak{m}/\mathfrak{m}^2$ which divide the function in $\mathrm{Sym}^{\nu}\mathfrak{m}/\mathfrak{m}^2$ affording the multiplier. For an initial generic z we have a regular configuration, and the filtration will again be trivial unless a unique change of coordinates $z \mapsto z + \chi y$, $\chi \in K$ yields,

$$\partial \in \mathrm{Der}(\mathcal{O}_{X,c}/\mathfrak{m}^{\mu}, z^{\nu}/\mathfrak{m}^{\mu})$$

The unicity of z modulo $\mathfrak{m}^2$ implies that we will still get the trivial filtration unless for the same z it is also true that,

$$\partial \in \mathrm{Der}(\mathfrak{m}^{\mu}/\mathfrak{m}^{\mu+1}, (z)^{\mu-1}/\mathfrak{m}^{\mu+1})$$

At which point we're necessarily in a nilpotent configuration with main vertex $(0, \nu) = (0, \mu-1)$, and the discussion is identical to the previous nilpotent configurations. $\square$

As it happens, we have not bothered to check that the above filtration is $\mathcal{F}$ invariant, since, this is a corollary of the relation between the above assertion and the related notion of filtration defined by an axis in [P]. Specifically:

II.i.2 Remark Given an axis, op. cit. defines a *smoothly adapted* coordinate system x, y, z at a point p where $\mathrm{NLC}(\mathcal{F})$ is positive dimensional and outwith the boundary as one which satisfies,

- $\frac{\partial}{\partial z}$ is a local generator of the axis;
- $y = z = 0$ defines the reduced structure of $\mathrm{NLC}(\mathcal{F})$ at p.

In particular, the possible coordinate changes which respect the axis are necessarily more rigid. Nevertheless the above schema of identifying the main vertex, and finding a stable main edge from which one defines a $\mathcal{F}$-invariant filtration is adapted mutatis mutandis from op. cit., but done as in II.i.1 is fully independent of choices, whereas op. cit. is only independent up to the choice of axis. As such the edge stable filtration of op. cit. is not always the same as that given by I.i.2. It is, however, the same for a generic axis. The genericity criteria on the axis are as follows: I.i.2 identifies an ideal (z) unique modulo $\mathfrak{m}^{k+1}$, and a generator A of the axis should certainly satisfy $A(z) \neq 0$. Further the axis determines a unique A-invariant ideal (y) normal to the component, so there are additional genericity hypothesis of the form y not parallel to finitely many co-vectors in $\mathfrak{m}/\mathfrak{m}^2$, and in the di-critical case $A(V) \neq 0$ for V the co-dimension 3 sub-space of $\mathfrak{m}/\mathfrak{m}^3$ defined by the unique $\mathcal{F}$-invariant pencil of smooth planes through c.

The advantages of II.i.1 over II.i.2 lead to considerable simplification in the definitions of [P] beginning with,

II.i.3 Definitions A point $p \in \mathrm{NLC}(\mathcal{F})$ where the latter is positive dimensional and smooth in its reduced structure will be called *equireducible* if there is a coordinate system x, y, z with $y = z = 0$ defining the reduced structure of $\mathrm{NLC}(\mathcal{F})$ at p such that,

(a) Generically on the implied component c, y, z are as per the proof of II.i.1.

(b) If $(\alpha, m_2+\beta, m_3-\gamma)$ is a vertex of the Newton polygon, $\gamma > 0$, (m_2, m_3), or better $(0, m_2, m_3)$, the main vertex encountered in II.i.1, then, $\beta/\gamma \geq \Delta_2$.

The weight vector $(0, \omega_2, \omega_3)$ is the smallest integer point on the positive half line through $(0, 1, \Delta_2)$, and the invariant is,

$$\mathrm{inv}(\mathcal{X}, \mathcal{F}, p) = (m_3, \underline{0}) \in \mathbb{Z}^5_{\geq 0}$$

understood in the standard lexicographic ordering. The displacement vector is given by $\Delta = (\Delta_1, \Delta_2) = (0, \Delta_2)$.

II.ii Boundary and Ordering

We next discuss the local theory of a smooth foliated log-stack $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ with non-empty boundary at a point $p \in \mathcal{D} \cap \mathrm{NLC}(\mathcal{F})$. As implied by I.iii.1 we suppose,

II.ii.1 Set Up There are at most two irreducible components of $\mathcal{D}$ through p, and should there be two components their intersection will be supposed to not be a component of $\mathrm{NLC}(\mathcal{F})$. In particular we take coordinates x, y, z such that,

- If $p \in \mathcal{D}$ and $\iota_p = [i]$ then $\mathcal{D}_i = \{x = 0\}$;
- If $p \in \mathcal{D}$ and $\iota_p = [i, j]$ (with $i > j$) then $\mathcal{D}_i = \{x = 0\}$ and $\mathcal{D}_j = \{y = 0\}$.

In what follows one may usefully bear in mind that in the presence of an axis as employed in [P], the coordinates are as per II.ii.1 with the additional proviso that the axis is $\frac{\partial}{\partial z}$. In any case, our main reference for this section is [P] §3, which we'll follow in all things except for the choice of an axis therein.

To begin with, let us consider the case where there is only one component of the boundary through p. This is, in fact, the most difficult case since it's the only one in which nilpotent configurations can occur, and, it is closely related to II.i.1. More precisely following [P] we must first find the *higher vertex, i.e.*, the lexicographic minimum of the vertices in the Newton polyhedra. To this end observe that we have a canonical diagram with an exact row,

$$\begin{array}{ccccccccc} & & & & T_{\mathcal{F}}|_{\mathcal{D}_i} & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{D}_i} & \longrightarrow & T_{\mathcal{X}}(\log \mathcal{D}_i)|_{\mathcal{D}_i} & \longrightarrow & T_{\mathcal{D}_i} & \longrightarrow & 0 \end{array}$$

Consequently the multiplicity $\nu \in \mathbb{N}$ of the vertical arrow, and of its composition μ with projection to the tangent space of the boundary is well defined. The latter, however, is the multiplicity of the induced foliation in $\mathcal{D}_i$, so it is finite iff the divisor is strictly invariant, and unlike II.i.1 we can only assert that $\nu \leq \mu$. Nevertheless the difference is slight, and we have:

II.ii.2 Fact In the notation of the proof of II.i.1 the higher vertex, main vertex (expressed via ν rather than μ) are $(0, h_2, h_3)$, $(0, m_2, m_3)$. In general the k occurring in the proof of II.i.1 need not be finite, however, when it is the displacement vector is $\Delta = (0, \Delta_2)$, and the main edge is parallel to $(\Delta, -1)$. Further, as per II.i.2 these are equally the values of the same for a generic axis at p in the sense of [P], and they form an *edge stable* Newton data as per op. cit. §4.1.

The situation that the above k is infinite means that the horizontal arrow in the above diagram vanishes on a smooth divisor of the form z^{ν} or $z^{\nu-1}$ according to the configurations encountered in II.i.1. Although straightforward, it leads to a rather different weighted filtration, whence the discussion of it is postponed. As such, we have a well defined weighted ideal I modulo $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i)$ with weights p, q on z and y respectively with unicity as per II.i.1, or equivalently every vertex of the Newton polyhedron of the form $(0, m, n)$ has $m + n\Delta_2 \geq m_2 + m_3\Delta_2$. In any case, there is a largest $r \in \mathbb{N} \cup \{\infty\}$ such that this continues to hold modulo $\mathcal{O}_{\mathcal{X}}(-r\mathcal{D}_i)$. Should r be infinite, then we'll be in the situation of a weight vector, and implied weighted blowing up as per II.i.3. On the other hand, such a centre would only be well defined in the completion in $\mathcal{D}_i$, whereas by I.iv.6 we require it to be well defined in the completion of a component of NLC($\mathcal{F}$). We could appeal to op. cit. 4.23 to remedy this, but, we do actually have a finite time procedure for deciding whether r is infinite or not by way of,

II.ii.3.(a) Fact Notations as above then r is infinite implies that there is a unique smooth component of NLC($\mathcal{F}$) transverse to $\mathcal{D}_i$. We denote the ideal of this component by J.

In the presence of the necessary condition II.ii.3, we can *adapt* the coordinate system so that $J = (y, z)$. Consequently, the unicity conditions of II.i.1 hold modulo powers of J, and,

II.ii.3.(b) Fact Furthermore r is infinite iff for the coordinate system adapted to J as above II.i.3.(b) holds, and in this case the weight vector and invariant are as per II.i.3.

Notice, in particular, that the limitation on possible changes of coordinates implied by being adapted to J means not only is II.i.3.(b) an intrinsic condition, but since the Newton polyhedron has finitely many vertices the condition can be verified or otherwise modulo a finite, albeit perhaps not effectively computable, power of $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i)$, so such a centre is well defined after completion in J.

We thus turn to the consideration of r finite. Necessarily this will imply a non-trivial main face in the sense of [P] §3.4, and a weighted blow up supported in a point. By hypothesis, and irrespectively of coordinate changes, there will always be a vertex $(r, m_2 + a, m_3 - b)$, $b > 0$ of the Newton polyhedron such that, $a/b < \Delta_2$, and for a given coordinate a weighted ideal I with weights p, q, $\Delta_2 = p/q$, on z and y respectively which is well defined modulo $\mathcal{O}_{\mathcal{X}}(-r\mathcal{D}_i)$. Consequently there is a face generated by the main edge and the lattice point of the polyhedron (which need only be a vertex if we compute the polyhedron modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$) such that,

$$C_r := \frac{r}{b\Delta_2 - a}$$

is minimal under the further conditions $b > 0$, $a/b < \Delta_2$. This gives a well defined, *i.e.* independent of choices of coordinates, weighted ideal,

$$I_{r,C_r} := (x^i y^j z^k : i + jC + kC\Delta_2 \geq Cm_2 + C\Delta_2 m_3) \mod \mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$$

containing every vertex of the polyhedron computed modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$. A priori this depends on the coordinates, albeit the only possibilities for changing this filtration are,

$$y \mapsto y + \kappa x^r, \quad z \mapsto z + \lambda x^r y^i,\ i < \Delta_2, \quad \lambda, \kappa \in \mathbb{C}$$

which span faces in 3-space, which may or may not be faces of the polyhedron with values of C equal to r and $r/\Delta_2 - i$ respectively, and we order the possible changes by way of increasing C, whence we may change both y and z simultaneously at the maximum of such C if Δ_2 were an integer. Before progressing, let us make,

II.ii.4 Warning The above coordinate changes appear to be identical to the adapted coordinate of changes of [P] §3.6. The latter, however, depend on the choice of axis, the former do not, and are intrinsically determined modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$ with increasing C.

Now there are cases to consider. The easy one is C_r strictly less than the minimum value C^0 of C amongst coordinate changes, *i.e.* r/Δ_2. In this case I_{r,C_r} modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$ is well defined and independent of any coordinate changes. Should $C_r = C^0$, the face is either stable or not, in the sense of op. cit. If it is stable then I_{r,C_r} is well defined modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$. Otherwise there is a unique, cf. op. cit. 4.13, coordinate change such C_r increases and the resulting weighted ideal I in y, z is well defined modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)\mathfrak{m}(p)$, or, indeed I_{r,C_r} is well defined modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$ if C^0 is also the maximum value of C amongst all coordinate changes. We now apply the same reasoning for the next relevant value C^1 that coordinate changes may afford. Again if $C_r < C^1$ everything is well defined, and there is nothing to do, while in the case of equality we pose the stability question, and either stop or continue depending on the result. Continuing in this way we find a value of C_r such that I_{r,C_r} modulo

$\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$ is independent of any coordinate changes. Apart from being finite by the definition of r the resulting value of C_r is fairly arbitrary unless $\Delta_2 \in \mathbb{N}$, in which case $C_r \leq r$, while if C_r exceeds the maximum C amongst coordinate changes for $\Delta_2 \notin \mathbb{N}$, *i.e.* $r/(\Delta_2 - \lfloor \Delta_2 \rfloor)$, then the resulting weighted ideal I in y, z is also well defined modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$.

Having thus defined everything modulo $\mathcal{O}_{\mathcal{X}}(-(r+1)\mathcal{D}_i)$, we proceed inductively to carry out the same procedure modulo $\mathcal{O}_{\mathcal{X}}(-s\mathcal{D}_i)$ for $s \geq r+1$. Thus, for example, nothing changes except to conclude well definedness modulo $\mathcal{O}_{\mathcal{X}}(-s\mathcal{D}_i)$ should $C_s = C_{s-1}$, and otherwise we argue as above to find a $C_s \leq C_{s-1}$ such that I_{s,C_s} is well defined modulo $\mathcal{O}_{\mathcal{X}}(-s\mathcal{D}_i)$. This process eventually terminates, however, since it cannot continue unless there is an integer vector $(s, m_2 + a, m_3 - b)$, $b > 0$, $a/b < \Delta_2$, with,

$$\frac{s}{(b\Delta_2 - a)} < C_r < \infty$$

from which s is certainly bounded, and we obtain a stable value $C = C_s$ for $s \gg 0$ together with a well defined weight vector $(\omega_1, \omega_2, \omega_3)$ which is the smallest tuple of natural numbers on the line through $(1, C, C\Delta_2)$. The resulting weighted filtration,

$$I_n = (x^i y^j z^k | i\omega_1 + j\omega_2 + k\omega_3 \geq n), \quad n \in \mathbb{N}$$

is independent of choices of coordinates, and is equally the filtration in a stable system of coordinates in the sense of [P] §4.1 provided the axis is generic.

There remains to discuss the situation of k infinite in II.ii.2. The support of the filtration $z = x = 0$ is already determined by the reduction modulo $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i)$, and it remains to find the weight vector $(\omega_1, 0, \omega_3)$. By II.ii.2 the reduction modulo $\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i)$ determines both the higher and main vertices. Writing the main vertex as $(0, m_2, m_3)$, there is a maximum $l \in \mathbb{N}$ such that a generator ∂ of the foliation vanishes to order m_3 along a principle ideal (z) modulo $\mathcal{O}_{\mathcal{X}}(-l\mathcal{D}_i)$, and this determines (z) uniquely modulo $\mathcal{O}_{\mathcal{X}}(-(l+1)\mathcal{D}_i)$. Computing the Newton-polyhedron in coordinates x, y, z, for z determined as above, and the coordinate system otherwise arbitrary up to I.ii.1, we find vertices $(a, m_2 + b, m_3 - c)$, $c > 0$ such that $\Delta_1 := a/c$ is minimal. The value of (ω_1, ω_3) is then the smallest pair of natural numbers on the line through $(1, \Delta_1)$. Necessarily there is some $1 \leq j \leq m_3 + 1$ such that $\Delta_1 \leq j/l$ so the unicity of (z) modulo $\mathcal{O}_{\mathcal{X}}(-(l+1)\mathcal{D}_i)$ ensures that the filtration,

$$I_n = (x^i z^k | i\omega_1 + k\omega_3 \geq n), \quad n \in \mathbb{N}$$

is independent of coordinates. Again, this is the same filtration that one encounters in [P] in a stable system of coordinates with respect to a generic axis. In the above notation the displacement vector is $\Delta = (\Delta_1, \Delta_2)$ with $\Delta_2 = b/c$ minimal amongst all vertices affording Δ_1.

The situation of two components through the point p is similar, and, in fact, easier, since there are no nilpotent configurations in the sense of [P] §3.4. Consequently we obtain,

II.ii.5 Fact/Definitions Let things be as in the set up II.ii.1 then, up to the fixed ordering in the presence of two components, there is a weighted filtration I_n of $\mathcal{O}_{\mathcal{X}}$ depending only on $\mathcal{F}$ and the ordering. This filtration is identical with the weighted filtration associated to a stable coordinate system arising from an axis in the sense of [P] §4.8 provided that the axis is sufficiently generic. In particular this intrinsic filtration is $\mathcal{F}$-invariant, it has a weight vector, $(\omega_1, \omega_2, \omega_3)$ (where possibly one of ω_1 or ω_2 is zero), and a displacement vector (Δ_1, Δ_2). We put $\delta_p = 1$ if $\Delta_1 > 0$, 0 otherwise, d_p the dimension of the centre, and extend II.i.3, for a quadruple $\mathbf{X} = (\mathcal{X}, \mathcal{D}, \Upsilon, \mathcal{F})$ by way of,

$$\mathrm{inv}(\mathbf{X}, p) = (m_3, \sharp\iota_p - 1, \delta_p, (1 - d_p)(m_2 + m_3), (1 - d_p)(m_3 + 1)! \max\{\Delta_2, 1\})$$

Modulo a change in lexicographic ordering, if we dropped the factor $(1-d_p)$, and instead of δ_p we took $(m_3 + 1)!\Delta_1$, then up to taking into account that the axis free filtration always has $\Delta_2 \geq 1$ whenever this is relevant, the invariant is exactly the same as [P] 4.6. However, in op. cit. only the difference between $\Delta_1 = 0$ or not is used in the resolution procedure, while the final terms only intervene at 0 dimensional centres, so we reflect this in the invariant. The first term never increase under the implied smoothed weighted blowing up, while m_3 decreases if, ordering as per II.ii.1, the centre has the form $y = z = 0$. Otherwise should neither m_3 nor $\sharp\iota_p$ decrease, δ_p will decrease in a blow up supported in $z = x = 0$, while at least one of the latter two decreases when the centre is supported in a point. The above invariant is not just the value on a generic axis but the minimal value amongst all such, so by op. cit. 4.29 we have,
II.ii.6 Proposition For p a closed point of a smooth foliated stack with ordered smooth boundary $\mathbf{X}$ of dimension 3 which is either equireducible or satisfies the conditions of the set up II.ii.1, there exists an embedded open sub-stack $\mathcal{U}$ around p such that if $\mathbf{U}; = \mathbf{X} \times_{\mathcal{X}} \mathcal{U}$ is the restriction, *i.e.* base change every element of the quadruple by $\mathcal{U}$, there is a well defined $\mathcal{F}$-invariant smoothed weighted blow up,

$$\pi : \widetilde{\mathbf{U}} \to \mathbf{U}$$

Better still, for every closed point q of $\tilde{\mathcal{U}}$ lying over p which is not log-canonical (necessarily lying in the boundary) the conditions of II.ii.1 are satisfied, and:

$$\mathrm{inv}(\widetilde{\mathbf{U}}, q) \ <_{\mathrm{lex}} \ \mathrm{inv}(\mathbf{U}, p)$$

II.iii Algorithmic resolution

While the invariant determines an étale local resolution algorithm, it is not quite as simple as blowing up in the maximal stratum. As such, let us introduce,
II.iii.1 Definition Let everything be as in II.i.3 or II.ii.5, with D_p the dimension of the germ at p of locus in NLC($\mathcal{F}$) where $m_3 = m_3(p)$ then,

$$\mathbb{Z}^5_{\geq 0} \ni \mathrm{Inv}(\mathbf{X}, p) := \begin{cases} (m_3, 0, \sharp\iota_p - 1, m_2 + m_3, (m_3 + 1)!\Delta_1) & D_p = 0, \\ (m_3, 1, \underline{0}) & D_p = 1. \end{cases}$$

Of course, neither inv nor Inv need be everywhere defined, so let us remedy this by the preparatory step, *viz:*

II.iii.2 Points 1 Let $\mathbf{X}$ be as per II.ii.5, then if there exists points outwith the boundary which are not equireducible, we prepare the situation by way of the usual blow up, $\mathbf{X}_1 \to \mathbf{X}_0 = \mathbf{X}$, in the discrete set of such.

Before proceeding let us note,

II.iii.3 Remark In [P] §5.6 there is a more complicated first stage, *distinguished vertex blowing up*. This was required to preserve the axis, and is redundant.

This stage is plainly étale local, so without loss of generality we may suppose that in a neighbourhood of every closed point p there is a weighted centre determined by either II.i.3 or II.ii.5, whose support we denote by Y_p. We assert,

II.iii.4 Claim Notations as above, with $q \in Y_p$ then the weighted blow ups determined by p and q at q coincide iff $Y_q = Y_p$.

Proof: We may suppose both weighted centres are supported on a curve. The case $\delta_p = \delta_q = 1$ is clear. Now consider the possibility that the support is in the boundary, and say $\delta_p = 0$, then the assertion is true as soon as $\delta_q = 1$ by II.ii.3, and the construction of the filtration for k infinite. Similarly, we're again done by II.ii.3 if the generic point of the centre is not in the boundary. $\square$

Now, let us discuss the procedure for lowering Inv. If $D_p = 0$, and $\iota_p = 1$ then this is clear from II.ii.6. Otherwise it is implicit: if after the smoothed weighted blow up in Y_p there developed a centre Y_q with $D_q = 1$ and the same m_3, then it must meet the proper transform of the divisor at another point r, and at r, inv would go up, so every centre in the exceptional divisor at which m_3 does not go down is a point, so again by II.ii.6,

II.iii.2.bis Points 2 If $D_p = 0$, then Inv goes down, *i.e.* II.ii.6 holds with inv replaced by Inv.

Alternatively, $D_p = 1$, and we have the following possibilities,

(a) $\iota_p = 0$, *i.e.* Y_p is an equireducible centre; or p is a point of discontinuity of ι_p and $\delta_p = 0$; or $\iota_p = \delta_p = 1$.

(b) p is a point of discontinuity of ι_p, $d_p = 0$, and $\mathrm{NLC}(\mathcal{F})$ is smooth at p.

(c) p is a point of discontinuity of ι_p, $\mathrm{NLC}(\mathcal{F})$ is a plane node at p on which m_3 is constant. Necessarily $\delta_p = 1$.

In the first two cases of (a), m_3 decreases under weighted blowing up in Y_p. in the final one it may not, but by II.ii.6 the locus where it does not is supported in points, so again we can replace inv by Inv in II.ii.6. The other cases are more subtle and such a p is termed a *bad point* in [P] 5.9. In either case there is a unique component C of $\mathrm{NLC}(\mathcal{F})$ through p which is distinct from the centre Y_p. In the former case, as per Points 2, on making the smoothed weighted blow up, the stratum where m_3 doesn't decrease consists of the proper transform, $\widetilde{C}$ of C, and possibly some points. On the latter Inv has decreased, at the former either $\widetilde{C}$ becomes the centre or inv decreases, so after a chain of smoothed weighted blow ups supported in points the proper transform of C eventually satisfies one of the first two sub-cases of (a). Similarly in case (b): on making the smoothed

weighted blow up in Y_p we either reduce to the said sub-cases of (a) for the proper transform of C, or we go back to case (b). Let us summarise this via,
II.iii.5 Definition/Fact Let $\mathbf{U} \ni p$ be a small neighbourhood on which $D_p = 1$, then on performing the smoothed weighted blow up in Y_p, Inv goes down. Otherwise, there is a finite, possibly empty, chain of smoothed weighted blow ups supported on points at which the proper transform of C meets the exceptional divisor and Inv is unchanged, after which the weighted centre of II.ii.5 is supported on the proper transform of C and under this final weighted blow up Inv goes down. In either case the resulting total modification, $\pi : \mathbf{U}_1 \to \mathbf{U}_0 = \mathbf{U}$ will be referred to as the *algorithmic modification* at p, and for $\mathbf{U}$ sufficiently small, any $q \in U_1$ satisfies,

$$\mathrm{Inv}(\mathbf{U}_1, q) <_{\mathrm{lex}} \mathrm{Inv}(\mathbf{U}, p).$$

Let us clarify the relation between blowing up and algorithmic modification,
II.iii.6 Fact/Warning Supposing it exists, *e.g.* algebraic case, algorithmic modifications in a maximum stratum of Inv of some smooth foliated stack with smooth ordered boundary, $\mathbf{X}$, glue to a proper modification $\pi : \mathbf{X}_1 \to \mathbf{X}_0$, with relatively projective moduli. Whenever the stratum has finitely many components this modification may be realised by a sequence of smoothed weighted blow ups supported in smooth centres invariant by the induced foliation at each stage. Such a realisation is not, and cannot, be made functorial in the sense of II.iii.7 below. Worse if an infinite chain of irreducible components were a connected component of the stratum, then there is no smoothed weighted blow up $\mathbf{W} \to \mathbf{X}$ in a smooth centre through which π factors.
Proof: Following [P] §5.9, we consider the dual intersection graph of the stratum, with a direction on the edges arising from the ordering. Without loss of generality this is connected, and the definition of ordering implies that it is a tree. The structure of this tree is in no way affected by performing the locally finite weighted blow ups at points encountered in (b) above, so we may suppose that bad points are in 1-1 correspondence with its edges. As soon as this tree is finite, it must have vertices of valency 1. The totality of vertices of valency 1 defines a smoothed weighted blow up in a smooth centre, and the tree of the stratum on the proper transform with the same value of Inv is the previous tree minus its vertices of valency 1. Repeating, yields a total modification obtained by continuing until the tree is empty and this is the same as gluing the local algorithmic modifications. The former procedure is not, however, functorial since if $\coprod \mathbf{U} \to \mathbf{X}$ is a sufficiently fine étale atlas and the tree on $\mathbf{X}$ has a vertex of valency 2, then we can ensure that there is some connected component $\mathbf{V}$ of $\mathbf{U}$ which meets no vertex of valency 1 in $\mathbf{X}$, while the vertex of valency 2 restricted to $\mathbf{V}$ has valency 1. Consequently, the restriction of the first stage of this procedure on $\mathbf{X}$ to $\mathbf{U}$ yields an empty blow up over $\mathbf{V}$, but when carried out on $\mathbf{U}$ it is non-trivial over $\mathbf{V}$. Similarly, in the presence of an infinite chain in the tree, should $\mathbf{W} \to \mathbf{X}$ exist, it must be the weighted blow up in some vertex. Every vertex has valency 2, and at the edge where the direction is away from the vertex, one wishes to perform the modification of case (c) above with the opposite ordering. The operation, however, is not commutative. $\square$

Let us, therefore, introduce a functor in the spirit of [K1], *i.e.*

II.iii.7 Definition Let $\mathcal{C}$ be a subcategory of smooth foliated space with smooth ordered boundary satisfying II.ii.1 such that the underlying foliation $(U, \mathcal{F})$ of $\mathbf{U} \in \mathrm{ob}(\mathcal{C})$ is a foliated affine schemes of finite type, or a relatively compact open in a foliated Stein space, according as to whether our context is algebraic or analytic then a modification functor $\mathfrak{M}$ associates functorially a sequence of smoothed weighted blow ups or algorithmic modifications in not necessarily connected, but non-empty, centres,

$$\mathfrak{M}(\mathbf{U}) : \mathbf{U} = \mathbf{U}_0 \leftarrow \mathbf{U}_1 \leftarrow \ldots \leftarrow \mathbf{U}_k = \widetilde{\mathbf{U}}$$

such that each weighted centre is invariant by the induced foliation at each stage with its support crossing the boundary normally and contained in the non-log canonical locus of the said stage, II.ii.1 holds, while the foliation singularities of $\widetilde{\mathbf{U}}$ are everywhere log-canonical. A modification functor is said to be étale local if for $\mathfrak{M}(\mathbf{U})_+$ the part of the above sequence with positive indices, and any surjective étale (not necessarily quasi finite in the analytic case) map $h : \mathbf{V} \to \mathbf{U}$, the square:

$$\begin{tikzcd} \mathfrak{M}(\mathbf{V})_+ \arrow[r, "\mathfrak{M}(h)"] \arrow[d] & \mathfrak{M}(\mathbf{V})_+ \arrow[d] \\ \mathbf{V} \arrow[r, "h"] & \mathbf{U} \end{tikzcd}$$

is Cartesian.

Once one has an étale local modification functor defined on a sufficiently large category several things are true for general nonsense reasons, *viz:*

II.iii.8 Fact Suppose every foliated smooth 3-dimensional affine scheme of finite type, or Stein space, according to the context, has both an atlas and gluing relations in $\mathcal{C}$ then for any quasi separated smooth algebraic or analytic foliated stack with smooth ordered boundary of dimension 3, $\mathbf{X}$, satisfying II.ii.1 at every point of the (possibly empty) boundary,

(a) There is a proper bi-rational modification $\widetilde{\mathbf{X}} \to \mathbf{X}$ with relatively projective moduli such that $\widetilde{\mathbf{X}}$ has log-canonical singularities.

(b) $\mathbf{X} \mapsto \widetilde{\mathbf{X}}$ is a functor such that for any étale $h : \widetilde{\mathbf{W}} \to \widetilde{\mathbf{X}}$ we have a Cartesian square,

$$\begin{tikzcd} \widetilde{\mathbf{X}} \arrow[d] & \widetilde{\mathbf{W}} \arrow[l, "\tilde{h}"'] \arrow[d] \\ \mathbf{X} & \mathbf{W} \arrow[l, "h"'] \end{tikzcd}$$

(c) If $\mathbf{X}$ has finite type or is relatively compact in a possibly larger ambient stack, there is a finite sequence of smoothed weighted blow ups and algorithmic modifications in not necessarily connected centres,

$$\mathfrak{M}(\mathbf{X}) : \mathbf{X} = \mathbf{X}_0 \leftarrow \mathbf{X}_1 \leftarrow \ldots \leftarrow \mathbf{X}_k = \widetilde{\mathbf{X}}$$

with all the same properties of $\mathfrak{M}|_{\mathcal{C}}$ in II.iii.7.

(d) By II.iii.6, any map in the above sequence $\mathbf{X}_k \to \mathbf{X}_{k-1}$ which is an algorithmic modification rather than a weighted blow up, is (non-functorially) a sequence of weighted blow ups in centres as enunciated in II.iii.7.

Proof: Let, $\mathbf{R} \overset{t}{\underset{s}{\rightrightarrows}} \mathbf{U}$ be an étale groupoid in $\mathcal{C}$ then we have a diagram,

$$\begin{tikzcd} s^*\mathfrak{M}(\mathbf{U}) \arrow[d] & \mathfrak{M}(\mathbf{R}) \arrow[l, "\sim"', "\mathfrak{M}(s)"] \arrow[r, "\sim", "\mathfrak{M}(t)"'] \arrow[d] & t^*\mathfrak{M}(\mathbf{U}) \arrow[d] \\ \mathbf{U} & \mathbf{R} \arrow[l, "s"'] \arrow[r, "t"] & \mathbf{U} \end{tikzcd}$$

which, since $\mathfrak{M}$ is a functor, could be diagram chased to a descent datum, but since the vertical arrows are bi-rational we can legitimately write $s^*\mathfrak{M}(\mathbf{U}) = t^*\mathfrak{M}(\mathbf{U})$ for any clivage, and everything is trivial by definition. $\Box$

It therefore only remains to establish,

II.iii.9 Theorem Let $\mathcal{C}$ of II.iii.7 be the sub-category in which each object has at most one bad point or is the source of a surjective étale map to such, then $\mathcal{C}$ admits a modification functor.

Proof: The preparatory stage II.iii.2 is manifestly functorial and étale local, so we may suppose that Inv is everywhere defined. Fix $M \in \mathbb{Z}^5_{\geq 0}$. The set of $\alpha \leq_{\text{lex}} M$ is totally ordered, and gives an ascending chain of categories $\mathcal{C}_\alpha$ where the maximum value of the invariant is at most α. Proceeding by induction in α, we can suppose $\mathfrak{M}$ defined on $\mathcal{C}_{<M} = \varinjlim_{\alpha<M} \mathcal{C}_a$. Otherwise let $\mathbf{U} \in \mathcal{C}_M$ but not in any smaller $\mathcal{C}_\alpha$. If the second entry of M is 0 we apply II.iii.2.bis, and II.iii.5 otherwise to obtain a weighted blow up or algorithmic modification $\mathbf{U}_1 \to \mathbf{U}_0 = \mathbf{U}$ respecting surjective étale maps such that $\mathfrak{M}(\mathbf{U}_1)$ is defined by induction and II.iii.8.(c). As such,

$$\mathfrak{M}(\mathbf{U}) := \begin{cases} \mathfrak{M}(\mathbf{U}) & \mathbf{U} \in \mathrm{ob}(\mathcal{C}_{<M}), \\ \mathfrak{M}(\mathbf{U}_1)_+ \to \mathbf{U}_1 \to \mathbf{U} & \text{otherwise.} \end{cases}$$

does the job on $\mathcal{C}_M$ and is compatible with direct limits. $\Box$

III. Complements

III.i Canonical singularities

The deepest known applications of the main theorem II.iii.9 arise from a better understanding of the relation between canonical and log-canonical singularities. As such, we re-visit §I.ii, with the notations therein, and observe,

III.i.1 Fact ([M2] I.6.11) Let $(U, D, \mathcal{F})$ be a foliated germ of a smooth variety supported at Z, cf. I.ii.1, then the following are equivalent,

(1) $(U, D, \mathcal{F})$ is terminal.

(2) $(U, D, \mathcal{F})$ is log-terminal.

(3) D is strictly invariant and $\mathcal{F}$ is smooth transverse to the generic point of the support Z.

Proof: The implication (1) $\Rightarrow$ (2) is trivial. Next let I be the ideal of Z, ∂ a local generator of the foliation, and m the multiplicity of $\partial(I)$ along Z. As such, bearing in mind I.i.2, with, $\pi : \tilde{U} \to U$ the blow up in Z and v the valuation associated to the exceptional divisor, E, the canonical bundle of the induced foliation $\tilde{\mathcal{F}}$ satisfies,

$$K_{\tilde{\mathcal{F}}} = \pi^* K_{\mathcal{F}} - (m-1)E$$

Whence, if this were log-terminal, then, $m < 1$, thus, in fact $m = 0$, so, indeed (2) $\Rightarrow$ (3). For the remaining implication, notice that by the convention I.i.2 we may, without loss of generality, suppose that D is empty, and we take x to be a function with $\partial(x) = 1$. Now, let E be the exceptional divisor associated to some valuation v centred on Z, with π a uniformiser, and $\tilde{\partial}$ a generator of the induced foliation around E in the usual sense, *i.e.* not following the convention I.i.2. Consequently, after multiplication by units, we may suppose, étale locally, that for some $n \in \mathbb{Z}$,

$$\partial = \pi^n \tilde{\partial}$$

while for an appropriate $m \in \mathbb{N}$, $x = \pi^m$. Whence,

$$0 \leq v(\tilde{\partial}(\pi)) = 1 - m - n$$

On the other hand since v is divisorial, it is also a valuation of the local ring completed in Z. Equivalently, no element of the completion in Z has infinite valuation with respect to v. The Frobenius theorem (or more correctly its proof) yields, however, that there is a non-zero, non-unital, function y in the completion with $\partial(y) = 0$. As such, if l is the valuation of this function, there is a unit, u, around E, such that,

$$0 = \pi^l \tilde{\partial}(u) + l\pi^{l-1} u \tilde{\partial}(\pi)$$

So in fact, $v(\tilde{\partial}(\pi)) \geq 1$, $\epsilon(v) = 0$, and $n \leq -m \leq -1$. From, which,

$$a_{\mathcal{F}}(v) = -n + \epsilon(v) \geq 1 > \epsilon(v) = 0 \qquad \square$$

There is a related and somewhat more subtle proposition which will require,
III.i.2 Definition Let $(U, D, \mathcal{F})$ be a germ of a normal foliated Gorenstein variety with Z of co-dimension at least 2 then a singularity is called *radial* iff after completion in the maximal we can find a generator of the foliation of the form,

$$\partial = n_1 x_1 \frac{\partial}{\partial x_1} + \ldots + n_r x_r \frac{\partial}{\partial x_r} + \delta$$

where modulo $\mathfrak{m}_{U,Z}^2$ the x_i afford a basis of the Zariski tangent space, $n_i \in \mathbb{N}$, and $\delta \in \mathrm{Der}(K, \mathfrak{m}_{U,Z})$ for some quasi-coefficient field K. In particular D is strictly invariant.

One could argue that Z co-dimension 1, and D not strictly invariant is also radial, but this would be notationally inconvenient. Notice also that radiality is not just a property at the level of the linearisation, but in the full completion of the local ring. Its functorial content is expressed by,

III.i.3 Fact ([M2] I.6.12) For $(U, D, \mathcal{F})$ a germ of a normal foliated Gorenstein variety the following are equivalent,

(a) The singularity is radial.

(b) The singularity is log-canonical but not canonical.

(c) Z is the centre of a divisorial valuation of $\mathbb{C}(U)$ of (log)-discrepancy zero and exceptional divisor not strictly invariant.

Proof: (a)⇒(b) is trivial- make the weighted blow up supported on Z with weights $n_1, \dots, n_r$.

(b)⇒(c). By hypothesis there is a divisorial valuation v satisfying $a(v) \geq 0$, but not $a(v) \geq \epsilon(v)$, so $\epsilon(v)$ must be 1, equivalently the maximal ideal is not strictly invariant, and $a(v) = 0$.

(c)⇒(a). Let v be the valuation in question, R the valuation ring, $\pi \in R$ a uniformiser, and ∂ a local generator of the foliation. By hypothesis there is some $\tilde{\partial} \in \mathrm{Der}(R)$ such that $\partial = \pi\tilde{\partial}$, and $u := \tilde{\partial}\pi$ a unit.

The image under v of $\mathcal{O}_{U,Z}$ in $\mathbb{Z}_{\geq 0}$ is a semi-group, whose element n_i we arrange to be strictly increasing in $i \in \mathbb{Z}_{\geq 0}$. This defines a filtration,

$$F^i := \left\{ f \in \mathcal{O}_{U,Z} : v(f) \geq n_i \right\}$$

which by a theorem of Chevalley is equivalent to the $\mathfrak{m}_{U,Z}$ adic topology. Observe that we have a commutative diagram,

$$\begin{array}{ccccc} F^i/F^{i+1} & \longrightarrow & \mathfrak{m}_R^{n_i}/\mathfrak{m}_R^{n_i+1} & \xrightarrow{\sim} & k(v)\pi^{n_i} \\ \big\downarrow \partial & & \big\downarrow \pi\tilde{\partial} & & \big\downarrow \times n_i u \\ F^i/F^{i+1} & \longrightarrow & \mathfrak{m}_R^{n_i}/\mathfrak{m}_R^{n_i+1} & \xrightarrow{\sim} & k(v)\pi^{n_i} \end{array}$$

with the leftmost arrows embeddings. Now let x_i, $1 \leq i \leq r$ be a $k(Z)$ basis of F^1/F^2 then we can write,

$$\partial(x_i) = a_{ij}x_j, \; x_i = \pi^{n_1}u_i, \quad a_{ij} \in k(Z), \, u_i \in k(v)$$

where we employ the summation convention, and so:

$$a_{ij}u_j = n_1 u u_j$$

From which, $n_1 u$ is an eigenvalue of the matrix a_{ij}, so, at worst we may suppose u is in the algebraic closure of of $k(Z)$. However, strict Henselisation doesn't alter the trace of ∂ restricted to F^1/F^2, so it's actually in $k(Z)$. Whence after rescaling we may suppose that ∂ on F^i/F^{i+1} is multiplication by n_i.

Now we can decompose ∂ as $\partial_K + \delta$ for δ as per III.i.2, with ∂_K K-linear, and we have exact sequences,

$$0 \longrightarrow F^i/F^{i+1} \longrightarrow \mathcal{O}_{U,Z}/F^{i+1} \longrightarrow \mathcal{O}_{U,Z}/F^i \longrightarrow 0$$

while by induction ∂_K is semi-simple on the rightmost term with eigenvalues strictly less than n_i, and a homothety with eigenvalue n_i on the leftmost, so ∂ is radial. $\square$

This has the important corollary,

III.1.4 Corollary If $(U, D, \mathcal{F})$ is a foliated $\mathbb{Q}$-Gorenstein germ with canonical singularities, then for every divisorial valuation, v, centred on it, $\epsilon(v) = 0$.

Proof: For the moment we only need the Gorenstein case, while the general case will follows from this and the diagram in the next corollary. Regardless, quite generally a divisorial valuation may be resolved by a chain of blow ups in its centres on the successive elements of the chain. Now either such centres remain invariant, and we're done by III.i.3, or they're generically transverse, and we conclude by III.i.1. $\square$

This brings us to a property of terminal and canonical singularities that is specific to foliations in curves, *viz:*

III.1.5 Corollary Let $\nu : (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{D}, \mathcal{F})$ be the Gorenstein covering stack of a normal $\mathbb{Q}$-Gorenstein foliated log-stack then the singularities of the former are terminal, log-terminal, canonical, respectively log-canonical iff they are so of the latter.

Proof: Identify a divisorial valuation w of the covering lying over some v with divisors E, F on germs of modifications $(\tilde{\mathcal{Y}}, \tilde{\mathcal{G}})$, and $(\mathcal{Y}, \mathcal{G})$ about the same. This affords a diagram,

$$\begin{array}{ccc}
(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}}) & \xleftarrow{\pi} & (\tilde{\mathcal{Y}}, E, \tilde{\mathcal{G}}) \\
\Big\downarrow \nu & & \Big\downarrow \nu \\
(\mathcal{X}, \mathcal{D}, \mathcal{F}) & \xleftarrow{\pi} & (\mathcal{Y}, F, \mathcal{F}))
\end{array}$$

where without loss of generality $\mathcal{D}$ is not strictly invariant, and by definition,

$$K_{\tilde{\mathcal{G}}} + \epsilon(E)E = \pi^*(K_{\tilde{\mathcal{F}}} + \tilde{\mathcal{D}}) + a(E)E, \; K_{\mathcal{G}} + \epsilon(F)F = \pi^*(K_{\mathcal{F}} + \mathcal{D}) + a(F)F$$

The rightmost vertical arrow is extraction of a root of F, from which $\epsilon(E) = \epsilon(F)$, so, en passant III.i.4 in general, while the leftmost is almost étale, thus:

$$K_{\tilde{\mathcal{G}}} + \epsilon(E)E = \nu^*(K_{\mathcal{G}} + \epsilon(F)F), \text{ and, } K_{\tilde{\mathcal{F}}} + \tilde{\mathcal{D}} = \nu^*(K_{\mathcal{F}} + \mathcal{D})$$

whence the generally valid formula,

$$a(E) = (E : F)a(F)$$

Consequently the if direction is always valid, and has nothing to do with foliations. Similarly the only if direction is always valid for log-terminal and log-canonical singularities. The subtle one is only if be it for terminal or canonical singularities which follows from the Gorenstein case of III.i.4. $\square$

III.ii Canonical resolutions

We will require to extend III.1.3 from generic points of log-canonical singularities to their closure. To this end let I be any sheaf of ideals containing the singular ideal of the foliation, then any local generator of the foliation lies in,

$$I \otimes_{\mathcal{O}_{\mathcal{X}}} T_{\mathcal{X}}$$

Whence for any $p \in \mathbb{N}$ there is a map,

$$T_{\mathcal{F}}^{\otimes p} \longrightarrow \Lambda^p(I/I^2) \otimes_{\mathcal{O}_{\mathcal{X}}/I} \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}/I}(\Lambda^p(I/I^2), \mathcal{O}_{\mathcal{X}}/I)$$

and so, in particular, we may take the trace to obtain sections,

$$\sigma_p \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/I \otimes_{\mathcal{O}_{\mathcal{X}}} K_{\mathcal{F}}^{\otimes p})$$

Equivalently the symmetric functions of the linearisation of the foliation along I are well defined irrespectively of whether I is l.c.i. or not. As such for n the dimension of $\mathcal{X}$, the sub-linear system of $|n!K_{\mathcal{F}}|$ defined by powers of symmetric functions is well defined and, by definition, without base locus as soon as the singularities are log-canonical. In particular if we take I to be the ideal of the closure Z of the centre appearing in III.1.3, this linear system is generically constant, whence constant, and σ_1 yields an isomorphism,

$$\mathcal{O}_Z \xrightarrow{\sim} K_{\mathcal{F}}|_Z$$

which we identify with a global generator ∂ of the foliation modulo I. Whence there is a commutative diagram,

$$\begin{array}{ccccccc} I/I^2 & \longrightarrow & \Omega^1_{\mathcal{X}}|_Z & \longrightarrow & \Omega^1_Z & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle \partial} & & \big\downarrow{\scriptstyle d(\partial\omega)} & & \big\downarrow{\scriptstyle 0} & & \\ I/I^2 & \longrightarrow & \Omega^1_{\mathcal{X}}|_Z & \longrightarrow & \Omega^1_Z & \longrightarrow & 0 \end{array}$$

Consequently the symmetric functions σ_p vanish generically for $p > r$, the co-dimension of Z, and the characteristic polynomial of the map in the middle may be supposed of the form,

$$T^{n-r}(T^r + \sigma_1 T^{r-1} + \ldots + \sigma_r) \in \mathbb{Z}[T]$$

The corresponding eigenspaces, $\mathcal{E}_{n_i}$, $n_i \in \mathbb{Z}_{\geq 0}$ define saturated coherent sub-sheaves of $\Omega^1_{\mathcal{X}}|_Z$, and we wish to prove that they are vector bundles. Completion in any point p is, however, faithfully flat, so it suffices to compute the Jordan decomposition, cf. III.iii.1, of ∂ in a point modulo I. On the other hand the rank of each $\mathcal{E}_{n_i}$ as an $\mathcal{O}_{\mathcal{X},p}$ module is whatever it is generically, so, indeed, at each point ∂ is semi-simple, and we've established the initial steps of,

III.ii.1 Lemma Let Z be a component of the singular locus of a foliated log-stack $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ with log-canonical singularities such that the generic point of Z

is not a canonical singularity then Z is a smooth connected component of the singular locus, and at each point p of Z we may find a coordinate system in the completion of $\mathcal{X}$ in Z such that III.i.2 holds.
Proof: By III.i.3 we must have a singularity of the foliation on ignoring $\mathcal{D}$. On the other hand at every point of Z the singularity is log-canonical, so every component of $\mathcal{D}$ that meets Z must be strictly invariant. Consequently, taking $\mathcal{D}$ to be empty has no effect on the definitions, and we go by induction modulo I^m in a neighbourhood of p. The case of $m = 2$ has already been done, so for $n_i \in \mathbb{Z}_{\geq 0}$ the eigenvalues, and ∂ a local generator, the situation is that for coordinates x_i,

$$X_i := \partial(x_i) - n_i x_i \in I^m,\ I = (x_1, \ldots, x_r),\ m \geq 2,\ 1 \leq i \leq n$$

As such we seek a coordinate change, $x_i \mapsto x_i + \xi_i$, for $\xi \in I^m$. This leads to the equations,

$$(n_i - \partial)\xi\ =\ X_i$$

By the initial case, we known that the co-kernel of,

$$(n_i - \partial) : I^m/I^{m+1} \xrightarrow{\sim} \mathrm{Sym}^m(I/I^2) \longrightarrow I^m/I^{m+1}$$

is a vector bundle in which X_i is generically zero, so its zero everywhere, and the equations can be solved. $\square$

Now we can apply this in the obvious way to obtain,
III.ii.2 Resolution Let $\mathcal{C}_{\mathrm{lc}}$ be as per II.iii.7, but with every object $\mathbf{U}$ enjoying log-canonical singularities, and without prescription on dimension, then there is an étale local blow up functor with at most one stage,

$$\mathfrak{B}(U) = \begin{cases} \mathbf{U} & \text{if } \mathbf{U} \text{ has canonical singularities,} \\ \mathbf{U} = \mathbf{U}_0 \leftarrow \mathbf{U}_1 = \widetilde{\mathbf{U}} & \text{otherwise.} \end{cases}$$

consisting, in the latter case, of a single smoothed weighted blow up in a strictly invariant smooth centre crossing the boundary normally and with resulting total boundary simple normal crossing. In particular, in dimension 3, the composition $\mathfrak{M}_{\mathrm{can}} : \mathfrak{B} \to \mathfrak{M}$ yields an étale local modification functor such that all of II.iii.6-8 holds with log-canonical replaced by canonical.
Proof: The only relevant singularity is described by III.ii.1, and it's support is smooth, so it suffices to check that there is an étale local weighted filtration which does as claimed. By I.iv.6 it will suffice to do this after completion in the non-canonical locus Z. Plainly, in the above notation, the obvious candidate is to place weights n_i on the x_i for $1 \leq i \leq r$. As such let $h : \mathbf{V} \to \mathbf{U}$ be a surjective étale map with sink a formal affinoid in $\mathcal{C}_{\mathrm{lc}}$ with non-canonical singularities, and $\mathbf{U}$ connected. On $\mathbf{U}$ we may suppose that we have coordinates x_i, respectively y_i in a neighbourhood of $\mathbf{V}$ such that,

$$\partial_U(x_i) = n_i x_i,\ \partial_V(y_i) = n_i y_i\ h^*\partial_U = f\partial_V$$

and f is 1 modulo the ideal I of the non-canonical locus. These relations suffice to imply,

$$y_i \in \big(h^*(x_1^{j_1} \dots x_r^{j_r}) | n_1 j_1 + \dots n_r j_r = n_i\big)$$

Indeed after taking n_ith roots ξ_i of $h^* x_i$, to obtain a covering $\pi : \mathbf{W} \to \mathbf{V}$, the above ideal is $\pi_*(J^{n_i})$ for $J = (\xi_1, \dots, \xi_r)$. The derivation $\pi^* h^* \partial_U$ acts on J^m / J^{m+1} as multiplication by m, so inducting on $m < n_i$, $\pi^* y_i \notin J^m$ implies,

$$(m - n_i)\pi^* y_i = 0 \in J^m / J^{m+1}$$

The discussion is local on $\mathbf{V}$, whence it applies in the other direction, so the weighted filtrations agree. $\square$

This is particularly useful in establishing the more general,

III.ii.3 Log-Resolution Let $\mathcal{C}'$ be as per II.iii.7 but with neither ordering nor simple normal crossing hypothesis on the boundary, and $(U, \mathcal{F})$ of dimension 3 with canonical singularities, then there is an étale local modification functor,

$$\mathfrak{M}'(U) : \mathbf{U} = \mathbf{U}_0 \leftarrow \mathbf{U}_1 \leftarrow \dots \mathbf{U}_k = \widetilde{\mathbf{U}}$$

in the 2-category of log-stacks such that at each stage the weighted centres are invariant by the induced logarithmic foliation with support in the non-canonical locus, and $\widetilde{\mathbf{U}}$ has canonical singularities. On the resulting category $\mathcal{C}''$, *i.e.* as per $\mathcal{C}'$ but $(U, D, \mathcal{F})$ with canonical singularities, there is an étale local blow up functor, $\mathfrak{B}'$ with centres smooth reduced and strictly invariant at each stage such the total exceptional divisor, $\tilde{\mathcal{E}}$, and final boundary, $\tilde{\mathcal{E}} + \tilde{\mathcal{D}}$, have simple normal crossings. In particular the composition,

$$\mathfrak{M}_{\log} : \mathfrak{B}' \to \mathfrak{M}' \to \mathfrak{B} \to \mathfrak{M}$$

is an étale local modification functor on the category $\mathcal{C}_{\log}$ of foliated 3-folds with arbitrary boundary, so that all of II.iii.6-8 holds in this generality, albeit with the caveat that the centres may not always have normal crossings with the exceptional divisor, so a normal crossing exceptional divisor may be lost during the resolution procedure even though it is restored in the final situation via the application of $\mathfrak{B}'$.

Proof: We divide our boundary into strictly invariant components, $\mathcal{E}_0$, say, and the rest $\mathcal{B}_0$. Only the non-strictly invariant components can occasion non-log canonical singularities, which for f a local equation of $\mathcal{B}_0$, and ∂ a generator of a foliation with canonical singularities are defined by the ideal,

$$(f, \partial(f))$$

or, equivalently, the tangencies of $\mathcal{F}$ to $\mathcal{B}_0$, so, inter alia the singularities of $\mathcal{B}_0$. Consequently, in the first instance consider trying to smooth $\mathcal{B}_0$. By III.1.4 any exceptional divisor that we introduce will be strictly invariant, and so at the k-th stage the non-strictly invariant part $\mathcal{B}_k$ of $\mathcal{D}_k$ will always be the proper, as opposed to total, transform of $\mathcal{B}_0$. On the other hand, by [BM] there is an étale local blow up functor, $\mathfrak{B}\mathfrak{M}$, whose final situation yields a total transform of $\mathcal{B}_0$

which is simple normal crossing, with centres in the singularities in the proper transform of $\mathcal{B}_k$ at each stage. A fortiori these lie in the non-log canonical locus, whence we may suppose for $k \gg 0$ that $\mathcal{B}_k$ has at worst simple normal crossings, and by a further sequence of blow ups in triple, and double intersections we may even suppose $\mathcal{B}_k$ is smooth. At this point II.ii.1 holds, and we apply the étale local modification functor $\mathfrak{M}$ of II.iii.9 to an initial situation with smooth boundary $\mathcal{B}_k$. The composition of this with the initial application of $\mathfrak{BM}$ yields $\mathfrak{M}'$, and a final situation in which the components, $\mathcal{B}'$, of the boundary which are not strictly invariant are smooth everywhere transverse to the induced foliation. The initial situation for constructing $\mathfrak{B}'$ is, therefore, a boundary $B' + E'$, with B' as above, every component of E' strictly invariant, but with no simple normal crossing hypothesis. The functor $\mathfrak{BM}$ also respects smooth maps, so if we apply it to E' the resulting centres are smooth and strictly invariant. Such centres preserve log-canonicity of the triple, so we obtain a final situation $(\tilde{\mathcal{U}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}})$ in which $\tilde{\mathcal{D}}$ decomposes as a strictly invariant part $\tilde{\mathcal{E}}$ with simple normal crossing, together with a remaining part $\tilde{\mathcal{B}}$ which is smooth everywhere transverse to $\tilde{\mathcal{F}}$, which, in turn, implies that $\tilde{\mathcal{D}} = \mathcal{E} + \mathcal{B}$ has simple normal crossing, *i.e.* $\mathfrak{B}'$ is just an appropriate application of $\mathfrak{BM}$. $\square$

III.iii Reduction of Monodromy

We will require a classification of $\mathbb{Q}$-Gorenstein log-canonical singularities of foliations which are not actually Gorenstein. To this end, recall,

III.iii.1 Revision Let A be a complete local ring with algebraically closed residue field k and maximal ideal $\mathfrak{m}$ such that, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) < \infty$ then for $\partial \in \mathrm{Der}_k(A)$ singular there is a Jordan decomposition,

$$\partial = \partial_S + \partial_N$$

into semi-simple and nilpotent parts. In particular, if A is regular, there are functions $x_1, \ldots, x_r \in \mathfrak{m}$ forming a k-basis mod $\mathfrak{m}^2$ such that,

- $\exists \Lambda = (\lambda_1, \ldots, \lambda_r) \in k^r$ for which,

$$\partial_S = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \ldots + \lambda_r x_r \frac{\partial}{\partial x_r}$$

  .

- The nilpotent field may be written as,

$$\sum_{1 \le i \le r} x_i \frac{\partial}{\partial x_i} \sum_{Q_i = (q_{i1}, \ldots, q_{ir})} a_{Q_i} x_1^{q_{i1}} \ldots x_r^{q_{ir}}$$

  where in the inner sum, $q_{ij} \in \mathbb{Z}_{\ge 0}$ unless $j = i$ in which case -1 is also permitted, while for the standard inner product, $\Lambda . Q_i = 0$, and all i.

Now suppose a finite group G acts tamely on A, (*i.e.* the cardinality of G is prime to the characteristic of k) fixing k together with the foliation defined by a singular field ∂, then, without loss of generality, we may suppose that the group acts as,

$$\partial^\sigma := \sigma\partial\sigma^{-1} = \chi(\sigma)\partial,\ \sigma \in G$$

for χ a character of G. Consequently, with the notations of III.iii.1, $\partial_S^\sigma + \partial_N^\sigma$ is a Jordan decomposition of ∂^σ with eigenvectors x_i^σ, and, indeed,

$$\partial_S^\sigma = \chi(\sigma)\partial_S,\ \text{and}\ \partial_N^\sigma = \chi(\sigma)\partial_N$$

for all $\sigma \in G$. From which, x_i^σ is again an eigenvalue of ∂_S but with eigenvector $\chi(\sigma)^{-1}\lambda_i$. Whence we have an induced permutation action by a cyclic group on the set of distinct linear eigenspaces, with non-zero eigenvalues, so, in particular, the order of the character is at most r. There is no difficulty in proceeding from here to an explicit description of all possibilities. An example is, however, of more immediate relevance, *viz:*

III.iii.2 Example A The character is faithful, and has order exactly r. The group action is then necessarily of the form,

$$(x_1, x_2, \ldots, x_r) \mapsto (x_r, x_1, \ldots, x_{r-1})$$

while the semi-simple part of a generator of the foliation may be taken as,

$$\partial_S = \sum_{i=1}^{r} \zeta^i x_i \frac{\partial}{\partial x_i}$$

for ζ a primitive r-th root of unity, together with, say r prime for simplicity, a nilpotent part of the form,

$$\partial_N = a(x_1 \ldots x_r)\partial_S$$

for a a formal function of one variable.

Proof: Just apply the above considerations in conjunction with III.iii.1. $\square$

As a consequence, observe,

III.iii.2.bis Corollary Let $\tilde{\mathfrak{X}} \to \mathfrak{X} = \mathrm{Spf}A$ be the blow up in the origin, then the induced $\mathbb{Z}/r$ action for a faithful character is transitive at the foliation singularities of the modification- indeed it cyclically permutes the singularities among themselves. In particular, the modification,

$$[\mathfrak{X}/\mathbb{Z}/3] \longleftarrow [\tilde{\mathfrak{X}}/\mathbb{Z}/3]$$

is smooth and space like at every foliation singularity in the exceptional divisor.

Proof: In the coordinates $x_1, \ldots, x_r$ of III.iii.2, the foliation singularities on the exceptional $\mathbb{P}^{r-1}$ are the r points with all but one entry zero. $\square$

In dimension 3, the only other possibility for the order of the character is,

III.iii.3 Example B The character has order 2. Should it be faithful the group action may be written as,

$$(x, y) \mapsto (y, x),\ z \mapsto -z$$

so that the semi-simple part of a generator of the foliation is of the form,

$$\partial_S = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

together with a nilpotent part,

$$\partial_N = a(xy, z)x\frac{\partial}{\partial x} - a(xy, -z)y\frac{\partial}{\partial y} + c(xy, z)\frac{\partial}{\partial z}$$

where a, c are formal functions of two variables with a arbitrary, and c even, *i.e.* $c(xy, -z) = c(xy, z)$, in z, non-unital.
Proof: Again, by the previous considerations in conjunction with III.iii.1. $\square$

This case is rather more interesting. The singularity comes equipped with an invariant curve, $x = y = 0$, and, should it fail to converge, there is absolutely no analogue of III.iii.2.bis,
III.iii.3.bis Possibility A is the completion in a closed point of the germ of a foliated algebraic or complex space $(X, \mathcal{F})$ with $\mathbb{Z}/2$ action, and the curve $x = y = 0$ is not algebraic, or even just not analytically convergent. Should this occur, the singularity is isolated, and for $(Y, \mathcal{G})$ the foliated quotient variety under the $\mathbb{Z}/2$ action, there is no bi-rational modification which is Gorenstein and canonical around the proper transform of the curve, so, a fortiori there is no smooth modification of Y with canonical singularities.
Proof: The singular locus of the completion is the completion of the singular locus, so if the singularity is not isolated, by III.iii.3 $x = y = 0$ is the singular locus, and the curve must converge. Now suppose to the contrary there were such a modification $\pi : \tilde{Y} \to Y$, then the rational map $X \dashrightarrow \tilde{Y}$ has a certain ideal I of indeterminacy. On the other hand for ∂ a generator of the foliation around the singularity, there is some $n \in \mathbb{N}$ such that,

$$J := (I, \partial(I), \partial^2(I), \ldots, \partial^n(I))$$

is ∂ invariant. Whence by [BM], there is a resolution $\rho : \tilde{X} \to X$ of J by a sequence of blow ups in smooth ∂ invariant centres. By hypothesis the Zariski closure of the curve has co-dimension at least one, so it has a well defined lifting to $\tilde{X}$ which cuts the exceptional divisor in some closed point p, and we assert,
Claim The above resolution $\tilde{X} \to X$ is a resolution of I at p.
Sub-proof Indeed, $\rho^{-1}(J)$ is a Cartier divisor with simple normal crossings, defined by $f = 0$ for some $\rho^* f$, and $f \in \partial^n(I)$ for some n, and we require to show that $n = 0$. The easy case is if the curve is Zariski dense, so that the order of vanishing, v, at the origin upon restriction to the curve is a valuation, and for any function g, $v(\partial g) > v(g)$, so certainly $n = 0$. Otherwise, the Zariski closure of the curve is an invariant divisor, so, a fortiori invariant by ∂_S, whence, without loss of generality $x = 0$, with say, $\xi = 0$ an equation for the proper transform of the same in $\tilde{X}$. Should f not vanish on $x = 0$ then the previous argument applies without change. If not it has some order $d \in \mathbb{N}$, $f = x^d f_1$, and we're done unless $f = \partial(g)$ for some function g. Now suppose that the order e of g

along $x = 0$ is less than d, then $g = x^e g_1$, and,

$$\partial(g_1) + e\frac{\partial(x)}{x}g_1 = 0 \mod (x = 0)$$

This is the same as, $eg_1 + Dg_1 = 0$ mod $(x = 0)$, for D the operator,

$$-y\frac{\partial}{\partial y} + c(0, z)(1 + a(0, z))^{-1}\frac{\partial}{\partial z}$$

which is a saddle node in normal form, so $g_1|_{x=0} = y^e$. By hypothesis however, g_1 and $x = 0$ are convergent, so the curve would converge. This is absurd, so $e = d$, and,

$$f_1 = d\frac{\partial(x)}{x}g_1 + \partial(g_1)$$

from which $v(f_1) = v(g_1)$. On the other hand, $f_1|g_1$, so $g = x^d g_1$ is equally a generator of $\rho^{-1}(J)$. Continuing by induction on n we conclude. □

Putting this together we have around p a diagram,

$$\begin{tikzcd}
(X, \mathcal{F}) \arrow[d, "\lambda"] & (\tilde{X}, \tilde{\mathcal{F}}) \arrow[l, "\rho"'] \arrow[d, "\lambda"] \\
(Y, \mathcal{G}) & (\tilde{Y}, \tilde{\mathcal{G}}) \arrow[l, "\pi"']
\end{tikzcd}$$

in which every entry has canonical singularities, while the left and upper arrows are unramified in the foliation direction, whence so are the other two. In particular ∂ lifts to $\tilde{X}$, and there is a field D on $\tilde{Y}$ such that D also lifts to $\tilde{X}$ with $\rho^*\partial$ a unit times $\lambda^* D$. Now we may take our original function z as a uniformiser of the curve at p in $\tilde{X}$, so that $\zeta = z^2$ is a uniformiser of its image at $q = \lambda(p)$ in $\tilde{Y}$. As such if the restriction of D to the image curve vanishes to order m at q, then ∂ restricted to the curve vanishes to order $2m - 1$, and this contradicts the fact that c in III.iii.3 is even. □

Now let us make various applications beginning with:

III.iii.4 Reduction of Monodromy A Let $(X, D, \mathcal{F})$ be a foliated smooth log-3-fold then in the 1-category of foliated log-3-folds with quotient singularities there is a sequence of modifications,

$$(X, D, \mathcal{F}) = (X_0, D_0, \mathcal{F}_0) \leftarrow (X_1, D_1, \mathcal{F}_1) \leftarrow \ldots \leftarrow (X_k, D_k, \mathcal{F}_k) = (\tilde{X}, \tilde{D}, \tilde{\mathcal{F}})$$

where each modification is a weighted blow up in a foliated log-invariant centre, and the final model has canonical singularities. In addition, at points of the final model where the ambient space is not smooth, we have exactly one of the following possibilities,

(a) The foliation singularity is terminal, and on the Vistoli covering stack $\tilde{\mathcal{X}} \to \tilde{X}$ the induced foliation is everywhere transverse to the corresponding non-scheme like points.

(b) The singular point is precisely the $\mathbb{Z}/2$ quotient singularity of III.iii.3.bis.

Proof: At the price of allowing quotient singularities on X_0 we may by III.ii.3 and III.i.4 suppose $(X, D, \mathcal{F})$ has canonical singularities, and we denote by $\mathcal{X}^g \to X_0$, and $\mathcal{X}^v \to X_0$ its Gorenstein and Vistoli covering stack respectively. To obtain the former from the latter around a closed point x, observe that the monodromy of the former is in fact a character of the latter. Indeed, it is precisely the character occurring in III.iii.1 at singular points of $\mathcal{F}$, since being an eigenfunction for the group action is a functor of finite type, and everything is étale local. Now by [BM] $\mathcal{X}^g$ admits a smooth strictly invariant resolution, so, without loss of generality, we may suppose that the Vistoli covering stack has at most cyclic monodromies. Better still, if,

$$x \mapsto g(x),\ x \mapsto v(x)$$

are the upper semi-continuous functions on closed points (identified with the same on the moduli) corresponding to the orders of the Gorenstein and Vistoli covering stacks then at every point $\tilde{x}$ of the above resolution of $\mathcal{X}^g$ over x,

$$g(\tilde{x}) \leq v(\tilde{x}) \leq g(x)$$

Consequently, we may by sufficient repetition of the above, eventually suppose that the Vistoli and Gorenstein covering stacks coincide, *i.e.* in the local description the character is faithful.

The only possibility for an order 3 character is III.iii.2 with $r = 3$, and the only possibility for a non-isolated order 2 character is III.iii.3 with $c(xy, z) \in (x, y)$. This latter case is just a variant of III.iii.2 with $r = 2$. In either case we apply III.iii.2.bis, and (a) holds. The case of isolated $\mathbb{Z}/2$ monodromies, with the curve $x = y = 0$ of III.iii.3 divergent has been discussed, and, should it occur, we know by III.iii.3.bis that (b) holds, and can never be eliminated. Conceivably, however, this curve may converge, or, better, form a centre that is admissible according as to whether the context is analytic or algebraic. Necessarily, it is $\mathbb{Z}/2$ invariant, so, we may, after globally smoothing it, legitimately blow up in it to kill the monodromy at the foliation singularities.

This leaves the possibility of monodromy at smooth points. Here the coincidence of the Vistoli and Gorenstein covering stacks about a geometric point p yields a local generator, ∂, such that for G_p the local monodromy,

$$\partial^\sigma = \chi(\sigma)\partial,\ \sigma \in G_x$$

for some faithful character χ. On the other hand, we can find a function ξ on an étale analytic neighbourhood of p such that $\partial(\xi) = 1$, so, G_p also acts faithfully on the divisor $\xi = 0$. Whence, ξ pertains to the ideal of non-scheme like points, and so the foliation is, indeed, everywhere transverse to the same. $\square$

While the wisdom of going beyond this is questionable, we do have:

III.iii.4.bis Reduction of Monodromy B Everything as per Reduction of Monodromy A, but now with further blow ups in centres which are everywhere transverse to the induced foliation then on the final model only possibility (b) may occur at non-smooth points.

Proof: This is simply a discussion as how to resolve quotient singularities of type (a) in III.iii.4. It is valid in any dimension n, and is as follows: on the Gorenstein covering stack at a not necessarily closed point p where the quotient fails to be smooth, we can identify G_p with a sub-group of $\mathrm{Aut}(\mathfrak{m}(p)/\mathfrak{m}(p)^2)$. This representation has a certain number r of non-zero eigenvalues, and is cyclic of some order $n(p) > 1$. As a result we can find coordinates x, y_i, z_j, $1 \le i \le r-1$, $1 \le j \le n-r$, such that the representation and a generator ∂ of the foliation are given by,

$$x^\sigma = \chi(\sigma)x,\ y_i^\sigma = \chi(\sigma)^{a_i} y_i,\ z_j^\sigma = z_j,\ \partial = \frac{\partial}{\partial x}$$

for $\chi : G_p \to \mathbb{G}_m$ the faithful representation afforded by $K_{\mathcal{F}}$, and $(a_1, \dots, a_{r-1}, n(p)) = 1$. If r is minimal, then this defines a smooth substack of $\mathcal{X}^g$ of co-dimension r through p which is everywhere smooth along its closure, and the weighted blow up $\tilde{\mathcal{X}}^g \to \mathcal{X}^g$ with weights $1, a_i$ on x, y_i respectively is also globally well defined. The exceptional divisor, $\mathcal{E}$, of this weighted blow up has pseudo reflecting monodromy of order $n(p)$, so $\tilde{\mathcal{X}}^g \to \tilde{\mathcal{X}}^v$ where the latter is the Vistoli covering stack of the moduli of the former is ramified in $\mathcal{E}$ to order $n(p)$. By I.iv.4, the monodromy at any geometric point $q \in \tilde{\mathcal{X}}^g$ is at worst a cyclic extension of $\mathbb{Z}/n(p)$ by $\mathbb{Z}/a_i$ for some a_i, whence $\tilde{\mathcal{X}}^v$ has strictly smaller cyclic monodromy at every q in the exceptional divisor, and by the formulae pre I.iv.4, or more correctly on replacing y_1 by $y_1^{n(p)}$, condition (a) of III.iii.4 continues to hold on $\tilde{\mathcal{X}}^v$. Whence, by decreasing induction in r and the maximum order of the monodromy, we conclude. $\square$

Like most of the results of §III, as opposed to §II, reduction of monodromy be it A or B is a general structure theorem about canonical and log-canonical singularities valid in some generality. One should, however, be careful to interpret them correctly, so let us make:

III.iii.5 Remarks The object of Reduction of Monodromy B is simply to show that the failure to be able to make a canonical or log-canonical resolution with an ambient smooth space is uniquely attributable to (b) of III.iii.4. It is furthermore an amusing exercise to enumerate all counterexamples of type (b) in all dimensions, and in dimension n there are always examples with monodromy $\mathbb{Z}/d$ for any $2 \le d \le n-1$. Consequently, there are no such examples on surfaces, and a key point in the nature of canonical singularities was long overlooked. Nevertheless, even on surfaces there are already indications. In the first place the passage from Reduction of monodromy A to Reduction of monodromy B is valid in all characteristics with the same proof provided that the monodromy is tame. The conclusion, however, is subtly different since the procedure of III.iii.4.bis will, in general, replace non-space like canonical singularities by radial singularities which are log canonical but not canonical. Thus, for example, already on surfaces in characteristic 3, one cannot make monodromy free canonical resolutions with ambient smooth space, cf. [M1] I.2.10/11. Similarly the minimal, whence functorial, canonical resolution of surface singularities of op.

cit. §III usually has singularities of type (a), while in all dimensions applying the minimal model algorithm of [M2] to the final situation of Reduction of Monodromy B, will simply undo all the extra steps between III.iii.4 and III.iii.4.bis, so the latter ought to be reckoned as having no practical value whatsoever. In addition, by way of a final structural observation, starting from from canonical singularities with tame monodromy at the first stage, Reduction of Monodromy A is valid in all characteristics by [B], and, of course, all dimensions modulo an enumeration of the examples of type (b).

It therefore only remains to show that III.iii.3.bis really occurs. To this end we first translate the problem into its manifestation on a smooth model of the $\mathbb{Z}/2$ quotient variety. Plainly we only need to do this about the proper transform of the curve $x = y = 0$ that occurs therein, which we'll slightly abusively denote by v even though it may not quite be a valuation. By a single blow up in the fixed locus of the group action we resolve the quotient singularity, and we find a priori formal coordinates ξ, η, ζ around the centre of v on the smoothed quotient such that the foliation is given by a generator of the form,

$$D = (\zeta\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}) + B(w,\zeta)\zeta(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}) + C(w,\zeta)(2\zeta\frac{\partial}{\partial\zeta} - \eta\frac{\partial}{\partial\eta})$$

where $\zeta = 0$ is the equation of the exceptional divisor, $w = \xi^2 - \eta^2\zeta$ is the defining equation of a Whitney umbrella, $\xi = \eta = 0$ the defining equations of v, and B, C are arbitrary formal functions of two variables, except that C is a non-unit not divisible by w. Indeed in the original notation of III.iii.2 Example B, and up to diving w by 4, we put: $a = \alpha + z\beta$, for α and β even in z, then $B(1+\alpha) = \beta$, and $C(1+\alpha) = c$. Now consider the particular choice of $B = \beta \in \mathbb{C}\backslash(1/2)\mathbb{Z}_{<0}$, $C = -\zeta$, giving rise to a family of fields, D_β, and suppose, as we quite legitimately may that the coordinate system ξ, η, ζ is defined in the Henselisation of the local ring, or even just analytically convergent, so that we have a perfectly algebraic perturbation,

$$D_{\beta,\lambda} = D_\beta + \lambda\zeta\frac{\partial}{\partial\eta}$$

for $\lambda \in \mathbb{C}$. To such a field with $\lambda \neq 0$ there is a purely formal coordinate change whereby the perturbation term disappears. Specifically, $\hat{\xi} = \xi - \lambda X(\zeta)$, and $\hat{\eta} = \eta - \lambda Y(\zeta)$, where,

$$X(\zeta) = \sum_{n\geq 1} c_n\zeta^n,\ Y(\zeta) = \sum_{n\geq 1}(\beta+n)c_n\zeta^n,\ c_n = \prod_{i=0}^{n}(\beta+i-1)(\beta+i-\frac{1}{2})$$

gives $D_{\beta,\lambda}$ as a field of the form D_β in $\hat{\xi}, \hat{\eta}, \zeta$ coordinates, so, perhaps better $\hat{D}_\beta$. Consequently, to summarise,

III.iii.6 Fact Possibility III.iii.3.bis really occurs, so in particular it is in general impossible to have a canonical or even log-canonical resolution of a foliated 3-fold in the 1-category of varieties or algebraic spaces without the $\mathbb{Z}/2$ quotient singularity described therein, equivalently for the ambient object to be smooth one must work in the 2-category of stacks with $\mathbb{Z}/2$ monodromy, so that at foliation singularities III.iii.4 and III.iii.4.bis are absolutely optimal from the point of view of reduction of monodromy.

To which it is appropriate to adjoin,

III.iii.7 Historical remarks The above example, and indeed the entire discussion of this section traces itself to F. Sanz, [S], prompted by an intuition of F. Cano resulting from his formal local uniformisation theorem, [C1]. While lacking the general statement of III.iii.3.bis, [S] describes a 3-complex parameter family of examples that cannot be resolved by blowing up in smooth convergent invariant centres, of which the example pre III.iii.6 is a co-dimension 1 subspace. This example has created much confusion, nevertheless it is a wholly functorially phenomenon, which is completely explained by III.i.5, and III.iii.3.bis.

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