Foliations in characteristic p

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Singularities

By a foliation (by curves) on a scheme, or better algebraic stack, X over a locally noetherian base S we'll mean a rank 1 quotient of $\Omega_{X/S}$, *i.e.* a short exact sequence,

$$0 \to \Omega_{X/\mathcal{F}} \to \Omega_{X/S} \to K_{\mathcal{F}}.I_Z \to 0$$

This definition supposes a ceratin amount of regularity. If we're working with stacks then X should be Deligne-Mumford, otherwise $\Omega_{X/S}$ isn't defined, while to write the quotient as $K_{\mathcal{F}}.I_Z$, where $K_{\mathcal{F}}$ is a bundle and Z the singular locus supposed of co-dimension at least 2 amounts to supposing that the foliation is Gorenstein, *i.e.* given everywhere by a vector field.

If we further suppose that S is a field, k, say, and X is normal irreducible, then we can functorially extend the definitions of Mori theory. The basic reference is [M4] I.6-7 which I have no intention of repeating. I think this reference is extremely clear, but unfortunately a lot of it is only valid in characteristic zero. In particular I.6.6-11 are false in char p. Nevertheless, I.6.12 is probably true in all characteristics, albeit that the proof only works with some mild additional hypothesis, *e.g.* $p > \dim X$. To state things, we'll need,

Revision cf. [Ma]. We let A be a complete regular local ring containing a coefficient field, $\overline{k}$, supposed algebraically closed. Next let $\partial \in \mathfrak{m}\,\mathrm{Der}_{\overline{k}}(A)$. For every $n \in \mathbb{N}$ we have an exact sequence,

$$0 \longrightarrow \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \longrightarrow \frac{A}{\mathfrak{m}^{n+1}} \longrightarrow \frac{A}{\mathfrak{m}^n} \longrightarrow 0\,.$$

We can consider ∂ as a $\overline{k}$-linear endomorphism, ∂_n, of $A_n = A/\mathfrak{m}^n$ for each n. Consequently ∂_n has a Jordan decomposition $\partial_{S,n} \oplus \partial_{N,n}$ into a semi-simple and nilpotent part. These are compatible with the restriction maps $A_{n+1} \to A_n$, and so on taking limits give a Jordan decomposition $\partial_S \oplus \partial_N$ of ∂. We observe:

Fact ∂ is semi-simple iff there is a choice of generators $x_i \in \mathfrak{m}$, together with $\lambda_i \in \overline{k}$ such that,

$$\partial = \sum_i \lambda_i\, x_i\, \frac{\partial}{\partial x_i}\,.$$

Describing the nilpotent part is worth the trouble. Notice: $[\partial_S, \partial_N] = 0$, so given ∂_S as above we just compute a basis for fields which commute with it.

Putting $\Lambda = (\lambda_1, \dots, \lambda_n)$ and $\Lambda \cdot -$, to be the usual inner product, albeit with values in $\overline{k}$, these are easily seen to be, cf. op.cit.

(a) $x^Q\, x_i\, \frac{\partial}{\partial x_i}$, $\Lambda \cdot Q = 0$, $Q = (q_1, \dots, q_n)$, $q_j \in \mathbb{N} \cup \{0\}$

(b) $x^Q\, x_i\, \frac{\partial}{\partial x_i}$, $\Lambda \cdot Q = 0$, $Q = (q_1, \dots, q_n)$, $q_j \in \mathbb{N} \cup \{0\}$ for $j \neq i$, $q_i = -1$

where of course $x^Q = x_1^{q_1} \dots x_n^{q_n}$. As such,

Divertimento Let ∂ be a non-singular derivation of a complete regular local ring A over an algebraically closed field $\overline{k}$ of characteristic $p > 0$ isomorphic to its residue field then there is a choice of coordinates $x, y_1, \dots, y_n$ in the maximal ideal such that up to multiplication by a unit,

$$\partial = \frac{\partial}{\partial x} + \sum_{i=1}^{n} x^{p-1}\, f_i(x^p, \underline{y})\, \frac{\partial}{\partial y_i}\,.$$

Proof. We can certainly multiply ∂ by a unit, in such a way that for some $x \in \mathfrak{m}$, $\partial x = 1$. Now consider $\widetilde{\partial} = x\partial$, and its Jordan decomposition $\partial_S \oplus \partial_N$. Trivially $\partial_S = x\,\frac{\partial}{\partial x}$, in some coordinate system $x, y_1, \dots, y_n$. Observe that in our formulae for the nilpotent part we must have an exponent of x in the monomial $(x^Q\, x_i)$ at least 1, since $x \mid \widetilde{\partial}$, whence the claim.$\Box$

This is best possible, and so could reasonably be called the characteristic p Frobenius theorem. One can only do better if,

Definition [E] The foliation is p-closed if for some, and infact any, local generator ∂ of the foliation the fields ∂^p and ∂ are parallel.

The special coordiantes of the divertimento can be used to 'compute' the p curvature, *i.e.* the ideal where $\partial^p \wedge \partial$ vanishes is the ideal cut out by the f_i. As such,

Fact The following are equivalent for a foliated smooth connected scheme or stack,

(I) The foliation is p closed.

(II) There exists a closed point ξ such that in the complete local ring $\hat{\mathcal{O}}_{X,\xi}$ there are coordinates $(x, y_1, \dots, y_n)$, and the foliation has the form,

$$(x, y_1, \dots, y_n) \mapsto (x^p, y_1, \dots, y_n)$$

(III) For every point ξ there are coordinates in the complete local ring $\hat{\mathcal{O}}_{X,\xi}$ such that the foliation has the form,

$$(x, y_1, \dots, y_n) \mapsto (x^p, y_1, \dots, y_n)$$

Consequently, even if Z is empty, it's far from true that curves invariant by a foliation by curves have to be smooth, and this is the case even if the foliation isn't p closed, For example,

$$\partial = \frac{\partial}{\partial x} + x^{p-1}(y^2 + x^p)\frac{\partial}{\partial y}$$

then the invraiant curve $y^2 + x^p$ certainly isn't smooth, even though the foliation isn't p closed.

Anyway, back at the statement of [M4] I.6.12,

Fact ($p > \dim X$ to be safe) A foliation singularity is log-canonical iff any field ∂ defining it is non-nilpotent.

The proof of op. cit. doesn't appeal to the singularities being in co-dimension 2, so is equally valid on permitting divisorial singularities, *i.e.* log-triples $(X, B, \mathcal{F})$ for B a boundary of infinite weight in the sense of op. cit. I.6.1. Whence,

Corollary (again $p > \dim X$) At any point outside Z the foliation singularities are canonical.

The hypothesis $p > \dim X$ is probably un-necessary, but for $p = \dim X$, the corollary cannot be improved, *i.e.* one cannot get terminal rather than canonical.

Resolutions

Singularities which are canonical rather than log-canonical can be described in the strict Hensilisation A_h of a complete local ring as follows. Taking a coefficent field K we can write $A_h = K[[x_1, \ldots, x_m]]$, and decompose the field as, $\partial = \partial_K + \delta$, where $\delta \in \mathrm{Der}_{K/k}$, and $\partial_K = \lambda_i \frac{\partial}{\partial x_i}$, with $\lambda_i \in A$. As such the field is log-canonical if some λ_i modulo the maximal ideal is non-zero, and,

Fact It only fails to be canonical if after multiplication by a unit the λ_i take values in $\mathbb{F}_p$.

It follows from this that for surfaces in positive characteristic, all the problems that one find regarding resolution on 3-folds in characteristic 0, [M5] I.1, are already present, *e.g.*

Example The log-canonical singularity, $\partial = \frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$ in characteristic 5 does not admit a canonical resolution by a sequence of blow ups in smooth centres.

Log-canonical singularities on surfaces do, however, admit resolutions after weighted blowing up. This requires, however, the ambient space to be replaced by a stack and/or an increase in the local monodromy if one starts on a stack should one wish to keep the ambient space smooth. The construction is straightforward,

Triviality Let $\partial = \frac{\partial}{\partial x} + n\frac{\partial}{\partial y}$ be a log-canonical singularity in characteristic p. Denote by η a nth root of y, and blow up the (x, η) plane in the origin. On the $x \neq 0$ patch the action of $\mathbb{Z}/n\mathbb{Z}$ is fixed point reflecting, and admits a smooth quotient. On the $\eta \neq 0$ patch there is an honest quotient singularity for $n > 1$. Glueing these patches leads to a smooth stack $\mathcal{X}$ on which there is a single non-scheme like point with stabiliser $\mathbb{Z}/n\mathbb{Z}$, and the induced foliation has canonical singularities.

It follows from this discussion,

Warning If $(X, \mathcal{F})/\mathbb{Z}$ is a smooth scheme or algebraic stack foliated by curves, it is in general impossible to blow up or for that matter pass to satcks and take roots in order to achieve that modulo p the singularities are canonical for p sufficently large. Indeed such a caveat already holds in chracteristic zero on replacing Spec$\mathbb{Z}$ by a curve, and p by a closed point, albeit that the number of 'bad' closed points is countable.

Refined tautology

Let X be a scheme or algebraic stack over a locally noetherian base S. For S a field k, and C a curve (=1 dimensional scheme or algebraic stack, which from now on means in the Deligne-Mumford sense) we consider a separable map $f : C \to X$. This admits a derivative, $f' : C \to \mathbb{P}(\Omega_{X/k})$. Denoting by L the tautological bundle, we have,
Tautology $L._{f'}C = -\chi_C - \mathrm{Ram}_f \leq -\chi_C$

This is immediate from the definitions. In the case that C is invariant by a foliation by curves $\mathcal{F}$, with X smooth to fix ideas, this can be re-wriiten as follows: outside of Z, the foliation defines a section of $\mathbb{P}(\Omega_{X/k}) \to X$, whose completion over Z is the blow up $\widetilde{X}$ of X in Z, including any implied nilpotent structure. As such there is an exceptional divisor E, and $L \mid_{\widetilde{X}} = K_{\mathcal{F}}(-E)$. The intersection $E._{f'}C$ can be identified with the segre class $s_Z(f)$, and the tautology becomes,

$$K_{\mathcal{F}}._{f}C = -\chi_C - \mathrm{Ram}_f + s_Z(f)$$

We'd like to refine this by 'removing' the segre class term. This can be done for S of finite type over $\mathbb{Z}$, and $(X, \mathcal{F}) \to S$ a family of foliations by curves with CANONICAL SINGULARITIES AT EACH OF ITS GENERIC POINTS. The precise statement is,
Fact Hypothesis as above, (OTHERWISE THINGS ARE WHOLLY FALSE) then for every $\epsilon > 0$ there is a closed Z_ϵ containing none of the generic points of X such that every $f : C \to X \otimes k(s)$ invariant by $\mathcal{F} \otimes k(s)$ and not factoring through Z_ϵ satisfies,

$$K_{\mathcal{F}}._{f}C \leq -\chi_C + \epsilon H._{f}C$$

where H is big at every generic point.

This is stated in [M4] V.6.1 for S a field of characteristic zero. Infact the situation there even involves invariant discs, which don't have so much sense in mixed characteristic. The proof though, which is all of op. cit. §V, continues to have perfect sense. As such, I have no intention of repeating it. I will though, as an example, sketch the key points for a family of smooth and geometrically connected surfaces over some localisation of $\mathbb{Z}$, with each singularity a section. Plainly the difficulty is to estimate $s_Z(f)$. This is, however, subordiante to a local question: how does a local invariant curve meet Z, so we can suppose everything complete in Z. We first consider the case modulo p, so complete in $Z \otimes \mathbb{F}_p$. Here everything is as in characteristic 0, for $p \gg 0$, with the exception of those singularities which are log-canonical rather than canonical. These look like,

$$\partial = \frac{\partial}{\partial x} + \lambda(p) \frac{\partial}{\partial y}$$

for $\lambda(p) \in \mathbb{F}_p$. This implies that an invaraint curve has an order of tangency with $xy = 0$ of order at least $m+n$ where m, n are the smallest positive integers whose ratio mod p is $\lambda(p)$. Since the singularities are generically canonical, we can for $p \gg 0$ suppose that $\lambda(p) = \lambda(\mathrm{m}od\ p)$, where λ is not a positive rational, so the tangency in question increases linearly in p. Constructing sections of H which vanish to high order along $xy = 0$ at Z is, however, cheap. Indeed to be in either of the ideals (x, y^d) or (x^d, y) is about d conditions, and this is uniform in p, whereas $H^{\otimes d}$ has about d^2 global sections, so for a f invariant in characteristic p,

$$\min\{p, d\}(s_Z(f) - \mathrm{Ram}_f) \ll \sqrt{d}H._f C$$

provided f doesn't factor through some divisor determined by a global section of $H^{[\sqrt{d}]}$, so we conclude by choosing p and d appropriately.

Applications

Before doing a mixed characteristic application, let's do one in characteristic zero, *i.e.*

Aplication 1 A generic hypersurface in characteristic 0 in $\mathbb{P}^3$ of sufficently high degree (c. 30) is hyperbolic.

By the main results of [M3] or [DE], this reduces by [M1] and [C] to knowing in characteristic 0 that a family $(X, \mathcal{F}) \to S$ of surfaces of general type foliated by curves admits a uniform bound on rational and elliptic curves invariant by $\mathcal{F}$. One knows, *e.g.* [B] or [J], that there are finitely many invariant curves on each $X \otimes \mathbb{C}(s)$, but this doesn't come with the necessary effective quantification. Plainly we can suppose everything is flat over S, and that the curves f_n in question are dense in X, with degrees δ_n with respect to an ample H going to ∞ in n. Denote by K the function field of S, then there is an exact sequence,

$$\coprod_s \mathrm{N}E^1(X_s) \otimes \mathbb{R} \to \mathrm{N}E^1(X) \otimes \mathbb{R} \to \mathrm{N}E^1(X \otimes K) \otimes \mathbb{R} \to 0$$

By duality each f_n, or better $\frac{1}{\delta_n} f_n$, is a functional on the middle group, and on subsequencing these functionals converge to a limit Φ which annihilates the kernel. Consequently we obtain a class, again denoted Φ, in $\mathrm{NE}_1(X \otimes K)$. Under the hypthesis that the singularities are generically canonical, this class satisfies $K_{\mathcal{F}}.\Phi \leq 0$. Since the curves f_n are Zariski dense, Φ is nef. while the foliated variety admits a minimal model $(X \otimes K, \mathcal{F}) \to (X_0 \otimes K, \mathcal{F}_0)$, so, infact, $K_{\mathcal{F}_0}.\Phi = 0$. The index theorem excludes $K^2_{\mathcal{F}_0} > 0$, so infact by the classification theorem for foliations, [M2], and the fact that X has general type, $(X \otimes K, \mathcal{F})$ is a bi-disc quotient in one of its natural foliations. Such surfaces have no moduli, so we're done.

Application 2 Let $(X, \mathcal{F}) \to S$ be a family of geometrically irreducible surfaces of general type foliated by curves over the ring of integers then either,

(a) There exists a constant $-\kappa < 0$, and a proper subscheme $Z \subset X$ such that any separable map $f : C \to X$ from a curve in positive characteristic not factoring through Z and invariant by $\mathcal{F}$ satisfies,

$$H._f C \leq -\kappa \chi_C$$

for H ample.

or

(b) The generic point is a bi-disc quotient in one of its natural foliations.

Everything proceeds exactly as in Application 1. In application 2, however, it is possible for bi-disc quotients to have Zariski dense sets of rational curves in mixed characteristic. So let's do,

Application 3 Suppose a family $X \to S$ of geometrically irreducible surfaces over the ring of integers of a number field K has many symmetric tensors, *i.e.* $\Omega_{X \otimes K/K}$ is big, then for all sufficiently large primes either,

a) There is a constant $-\kappa < 0$, independent of p, and a proper subscheme $Z \subset X$ such that any separable map $f : C \to X$ from a curve in positive characteristic not factoring through Z satisfies,

$$H._f C \leq -\kappa \chi_C$$

for H ample.

or

b) There is a sequence of curves $f_i : C_i \to X$ defined at possibly different, but nevertheless sufficently large p, such that,

$$\limsup \frac{\chi_{C_i}}{H._{f_i} C_i} = 0$$

Should this occur, then, after base extension, there is a dominant map $\rho : \widetilde{F} \to X$ from a bi-rational modfication of a bi-disc quotient F. Moreover the maps $f_i : C \to X$ lift to $\tilde{f}_i : C_i \to \widetilde{F}$, and $\tilde{f}$ is invariant by at least one of the natural foliations on $\widetilde{F}$.

Indeed as in [B] one fixes a global section of $\mathrm{Sym}^n \Omega_{X \otimes K/K} \otimes H^\vee$, for H ample on X, and appropriately large n. As such condition (a) is immediate for sufficently large p unless the derivatives of the curves factor through the horizontal components of the section considered as divisors in $\mathbb{P}(\Omega_{X/S})$. The said divisors may be identified with foliated surfaces, so we conclude by Application 2. We may further clarify by way of,

Application 3 bis. Suppose we are indeed in the exceptional situation (b) of the above application then, as noted, we base extend so that S maps to the spectrum of the ring of integers T of the real quadratic field of definition of F. This allows us to divide S into good primes S_g, and bad primes S_b, where the good primes are exactly those lying over primes in T whose residue field is some $(\mathbf{p})$, *i.e.* p splits in T, and bad otherwise. In particular (a) continues to hold for all separable maps $f : C \to X$ defined at good primes p.

By way of proof, suppose that we're in the bad situation 3(b), then we can suppose that all of the maps $\tilde{f}_i$ are invariant by one of the natural foliations on a modified bi-disc quotient, $\widetilde{F}$. The said quotient, however, has many symmetric tensors, so we can repeat the reasoning of 3(a) to conclude that there is a birational modification $\widetilde{G}$ of a possibly different Hilbert modular surface G, such that further liftings of the f_i are invariant by one of the natural foliations. In reality, however, $G = F$. Better still *since the quotient of F by either of the natural foliations at a good prime p is isomorphic to F and such foliations are actually integrable over $\mathbb{Z}_p$* we may even take the height n quotient of F by one of the natural foliations, and reason as above, to deduce that curves violating 3(a) at good primes actually map inseparably under every height n quotient, whence they may be identified with the reduction of the invariant curves over $\mathbb{Z}_p$, of which there are finitely many. As such violation of 3(a) does not take place at good primes. It may, however, take place at bad primes. Indeed at bad primes, neither of the natural foliations on the bi-disc quotient are p-closed, so infact there are only finitely many invariant curves for any given bad p. Nevertheless, the said invariant curves at a given bad p are cut out by modular forms of weight $(2p, -2)$ (or $(-2, 2p)$ depending on which foliation we're looking at) and their degree is **NOT** bounded in p. Worse still there are infact smooth bi-disc quotients with rational curves of unbounded degree defined at bad p. Let us summarise this discussion by way of,

Corollary Let $X \to S$ be a family of geometrically irreducible surfaces over a localisation of a ring of algebraic integers with many symmetric tensors at its generic point then for sufficently large p there is a bound, uniform in p on the degree of any rational or elliptic curve on the reduction of X mod p **UNLESS** after base change X is dominated by a birational modification of a bi-disc quotient and the primes p are inert in the real quadratic field of definition of the said quotient. In this exceptional case the number of rational or elliptic curves is bounded for a given bad p, but the bound is not uniform.

Remarks on bi-disc quotients

The classification theorem is easy if the foliation is smooth, so let's suppose that it's singular to fix ideas. A singularity on a bi-disc quotient has the form,

$$\partial = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}$$

for λ real quadratic irrational. So infact it's p closed whenever $\lambda \in \mathbb{F}_p$, or, similarly if the quotient arises from a representaion of $\mathrm{SL}_2(K)$ for $K/\mathbb{Q}$ quadratic irrational its p closed whenever p splits in K. When it's not p-closed there is a tangency divisor between $\mathcal{F}^p$, and $\mathcal{F}$. This is given by a global section of $K_{\mathcal{F}}^{p-1} \otimes K_X$, and Shepherd-Barron told me that this divisor will contain lots of rational curves of degree around p. Whence, application 2 has some optimality. It's plainly suggestive that (a) of the said application is false for algebraic points over $\mathbb{Q}$, and we really should be able to prove this. Maybe it could even be done by mathematica if someone knew what buttons to press. This would be a counterexample to quantative Vojta conjectures, [V].

There is of course the question as to whether one can prove the classification theorem in positive characteristic. The only positive result that I have is,
Fact Something of foliated kodaira dimension $-\infty$ which has a singularity whose semi-simple part has an irrational eigenvalue admits an infinite number of primes p at which it is p-closed.

Unfortunately the condition on the eigenvalue is non-trivial, and while true for bi-disc quotients seems to be impossible to guarantee. What's worse, even if this could be guaranteed, what to do next ? Szpiro's argument in the algebraically integrable situation would need p^n closed in the sense of [E], and this doesn't even seem to be related to numerical conditions. Comunque, seems to me that neither the above fact or further discussion of bi-disc quotients would be that relevant to the current considerations. It may though be worth adding some remarks about holonomy and p-closure earlier on.

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