RATIONAL CURVES ON FOLIATED VARIETIES

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Abstract

This article represents a study of ample subbundles of the tangent sheaf of a variety in a formal neighbourhood of a curve. With the added hypothesis of integrability it is best possible. A particular corollary is Mori's cone theorem for foliations by curves.

0 Introduction

In a series of papers, notably [B1] and [B2], the first author showed, amongst other things, that a surprising interplay between the classical Frobenius theorem on the integrability of vector fields closed under Lie bracket and various algebro-geometric considerations gave rise to some rather strong restrictions on the 'size' of subbundles of the cotangent bundle, with a particular corollary being inequalities for Chern numbers on algebraic surfaces. In refining this circle of ideas Y. Miyaoka, [Mi1], established that any quotient of the cotangent bundle of a surface of general type and positive index was big. Rather more remarkably, Miyaoka, [Mi2], subsequently considered the problem of subbundles of the tangent bundle with positive slope along a generic complete intersection of ample divisors, and by extending Mori's bend and break technique to 'deformations along a foliation' showed that these hypothesis implied the existence of covering families of rationally chain connected varieties in the direction of the foliation in question. The surface case of this result is particularly clean, since it asserts that either the bundle of top weight forms $K_{\mathcal{F}}$ along the leaves is pseudo-effective as a divisor, or the foliation is a fibration by rational curves. Continuing in these directions the second author in extending the results of the first author on boundedness of moduli of curves of a given genus on algebraic surfaces to curves with boundary employed Miyaoka's semi-positivity theorem in an essential way, cf. [M1]. The use of the said theorem therein was to force, under the hypothesis of a dense parabolic leaf, the existence of a global vector field defining the foliation on a rather crude version of what might be considered it's minimal model, i.e. a normal algebraic space on which the positive part of the Zariski decomposition of $K_{\mathcal{F}}$ coincides with the push forward of the same. It was natural however to examine this question more carefully, not just in terms of a more delicate structure of the minimal model but to introduce the study of the birational geometry of foliations per se, cf. [M2], [Br1], [Br2]. As ever a prerequisite for such a study is the understanding of subvarieties on which the cotangent bundle of our foliation is negative.

To fix ideas let us consider a foliation by curves, $\mathcal{F}$, on a smooth variety X. This is equivalent to giving a rank 1 torsion free quotient of Ω_X, whose Chern class we denote by $K_{\mathcal{F}}$. Miyaoka's theorem then asserts that if C is a 'sufficiently movable' curve then either $K_{\mathcal{F}}.\, C \geq 0$ or $\mathcal{F}$ is a (possibly singular) fibration by rational curves. The main problem here is the hypothesis, 'sufficiently movable' which more precisely means that C moves in a family $\{C_t \mid t \in T\}$ covering X such that generically C_t does not meet the singularities of $\mathcal{F}$, whereas one really wishes to understand the implications of the hypothesis $K_{\mathcal{F}}.\, C < 0$ for any curve C. The difficulties in extending Miyaoka's method, or its refinement by Shepherd-Barron, [SB], to this situation are formidable. Firstly one must establish that the hypothesis imply that the divided symmetric power algebra of $K_{\mathcal{F}}$ extends across a finitely generated extension of $\mathbb{Z}$, [Mi2], or rather more straightforwardly the foliation is defined by an inseparable scheme quotient in positive characteristic, [SB]. Even then there is the added complication that C may pass through the foliation singularities and so Mori's bend and break technique may not apply. The key to resolving this problem § 2 lies in finding a $\mathcal{F}$-invariant surface S containing C [1], and so reduce the study to something more tractable. In finding our surface, however, we necessarily show that the leaves through C are algebraic curves, and whence reprove Miyaoka's theorem in this case without any appeal to reduction in positive characteristic, thanks to a theorem of Arakelov, [A], all be it that the most satisfying proof of Arakelov's theorem is to proceed via positive characterictic, cf. [S].

With this example in mind let us consider a more general situation. We denote by $(X, \mathcal{F})$ any variety equipped with an integrable (i.e. closed under Lie-bracket) foliation $\mathcal{F}$. The singularities of X are not our interest, but rather those of $\mathcal{F}$. What this latter should mean is a measure of how far away the foliation is from being given everywhere locally by a relatively smooth fibration. To understand this it is convenient to introduce an ambient smooth space M. The foliation is given by a subsheaf $\mathcal{T}_{\mathcal{F}}$ of $\mathcal{T}_X$ of rank r, say, and for any $x \in X$ we have the natural residue map,

$$\mathcal{T}_{\mathcal{F}} \otimes k(x) \longrightarrow T_M \otimes k(x)\,.$$

Should this map be an injection onto a subspace of dimension r, then Nakayama's lemma forces $\mathcal{F}$ to be a bundle in a neighbourhood of x, and

[1] Added in preparation: J.-B. Bost, "Algebraic leaves of algebraic foliations over number fields", Orsay preprint 2000, uses what may be considered an arithmetic version of this trick which independently led him to discover the geometric trick, and its higher dimensional generalisations à la § 2.1.

better still the Frobenius theorem goes through verbatim to force $\mathcal{F}$ to be given locally by a relatively smooth fibration, and indeed our condition is even necessary for the latter. Naturally then we introduce the notion of weak regularity, 1.1, which requires $\mathcal{T}_{\mathcal{F}}$ to be a bundle, and put:

$$\operatorname{sing}(\mathcal{F}) = \{x \in X \mid \dim(\operatorname{Im}\{T_{\mathcal{F}} \otimes k(x) \longrightarrow T_M \otimes k(x)\}) < r\}.$$

This variety stratifies naturally according to the rank of the image map, and for C any curve in X, we denote by $r(C)$ the generic rank. We are now in a position to state our first theorem, viz:

THEOREM 0.1. (a) *Let $(X, \mathcal{F})$ be a weakly regular (integrable) foliated variety and C a curve in X with $T_{\mathcal{F}}|_C$ ample then for all $x \in C$ there is a $\mathcal{F}$-invariant rationally connected subvariety $V_x \ni x$ of dimension $r(C)$.*

Here we do use reduction modulo p, but only to resolve the corresponding problem for a foliation such that $X/\mathcal{F}$ exists as a scheme quotient in characteristic zero. We therefore not only obtain 0.1 but in fact,

THEOREM 0.1. (b) *Notations and hypothesis as above then the minimal degree of the rational curve connecting any two points in V_x is effectively computable. In particular there is a rational curve $L_x \ni x$ tangent to $\mathcal{F}$ such that for any nef. $\mathbb{R}$-divisor H,*

$$H.\,L_x \leq 2(r(C)+1)\,\frac{H.\,C}{-K_{\mathcal{F}_{r(C)}^{\cdot}}\,C}\,.$$

In this notation, $\mathcal{F}_{r(C)}$ is the induced foliation on the subscheme of X where the generic rank is $r(C)$. It is not immediately clear that intersecting with the canonical class has any sense for $r(C) < r$, but well definedness will emerge in the course of the proof. One should also note that there is no need to suppose $T_{\mathcal{F}}$ is saturated in $\mathcal{T}_X$ provided there is closure under Lie-bracket. Without closure under bracket one can of course find a foliation $\mathcal{G}$ corresponding to the minimal subsheaf $\mathcal{T}_{\mathcal{G}}$ of $\mathcal{T}_X$ closed under the same. It may happen, however, that this is not a bundle in a neighbourhood of our curve C, although it will certainly be as 'ample' as one needs. This poses serious technical problems akin to the difficulty of doing deformation theory on singular varieties, and so one only obtains 0.1 and 0.1(b) under the weaker hypothesis that $\mathcal{G}$ is weakly regular in a neighbourhood of C. This is however perfectly sufficient to recover Miyaoka's semi-positivity theorem, where X is supposed normal and C moves in a large base point free family.

Our study however has so far revealed nothing if $r(C) = 0$. For foliations by curves this is equivalent to our curve being wholly contained in the singular locus of the foliation. To address this question we have recourse to the notion of foliated canonical singularities. This class of singularities may be understood as the maximal one for which birational geometry of foliations makes sense, equally the precise definition à la Kawamata-Mori et al. is given in § 1.1. To explain its implications observe that for a curve

C with generic rank $r(C)$ contained in a component Y of the locus where the rank is the same there is an exact sequence of sheaves,

$$0 \longrightarrow \mathcal{N}_{r(C)} \longrightarrow T_{\mathcal{F}} \otimes \mathcal{O}_Y \longrightarrow T_{\mathcal{F}_{r(C)}}$$

where we take this as the definition of $\mathcal{N}$. For instance if $r(C)$ is maximal, $\mathcal{N}$ is zero while for foliations by curves if $r(C) = 0$ then $\mathcal{N}$ is $T_{\mathcal{F}}$. In any case we have,

THEOREM 0.1. (c) *Let $(X, \mathcal{F})$ be a weakly regular (but not necessarily integrable) foliated variety and C a curve with $r(C)$ the generic rank of the foliation along C then $\mathcal{N}_{r(C)} \otimes \mathcal{O}_C$ has a non-positive rank 1 quotient.*

Thus we have established that the rationally connected subvarieties guaranteed by 0.1(a) exhaust the ampleness of C, and so we may turn to some applications. Specifically,

THEOREM 0.2. *Let $(X, \mathcal{F})$ be a variety foliated by curves with foliated Gorenstien and foliated canonical singularities (cf. 1.1.1 and sequel) then there are countably many $\mathcal{F}$-invariant rational curves with $K_{\mathcal{F}} . L_i < 0$ such that if $\overline{\mathrm{NE}}(X)$ is the closed cone of effective curves and $\overline{\mathrm{NE}}(X)_{K_{\mathcal{F}} \geq 0}$ the subcone on which $K_{\mathcal{F}}$ is positive then,*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_{\mathcal{F}} \geq 0} + \sum_i \mathbb{R}_+ [L_i] \,.$$

Better still,

(a) *The rays $\mathbb{R}_+ [L_i]$ are locally discrete in the upper half space $\mathrm{NS}(X)_{K_{\mathcal{F}} < 0}$.*

(b) *If $(X, \mathcal{F})$ is not a ruling by rational curves $K_{\mathcal{F}} . L_i = -1$, $\forall\, i$, and $L_i \cap \mathrm{sing}(\mathcal{F}) \neq \varphi$.*

(c) *Otherwise, $K_{\mathcal{F}} . L_i \in \{-1, -2\}$.*

(d) *Every extremal ray in the half space $\mathrm{NS}(X)_{K_{\mathcal{F}} < 0}$ is of the form $\mathbb{R}_+ [L_i]$.*

A quasi-immediate corollary of this foliated cone theorem is that for a smooth foliation $K_{\mathcal{F}}$ is nef or it is a fibring by rational curves, and more generally the existence of supporting Cartier divisors for extremal rays. Whence, we have the first step in a minimal model programme for foliations by curves.

It remains to thank M. Spivakovsky for contributing his expertise (and patience) in explaining the nature of algorithmic desingularisation, together with CIMS and IHES for the invitations which permitted this collaboration, and as ever to Cécile whithout whom the contents of this article would have lingered indefinitely in the realm of 'in preparation'.

1 Singularities

1.1 Revision of definitions

Our objects of study are foliated varieties, i.e. a normal variety X equipped with a foliation $\mathcal{F}$, which we denote by $(X, \mathcal{F})$. On the other hand we do not wish to think of this as two separate objects, but rather as a unified whole. As such the singularities which we wish to study have a priori nothing to do with the space X, although as suggested above we will make the technically convenient, and rather mild assumption that X is normal. To proceed further let us note that the precise definition of a foliation is simply a saturated subsheaf of the tangent sheaf, i.e.

$$\mathcal{T}_{\mathcal{F}} \hookrightarrow \mathcal{T}_X .$$

If in addition the subsheaf $\mathcal{T}_{\mathcal{F}}$ is closed under Lie bracket, then we say that $(X, \mathcal{F})$ is integrable. One has of course the classical theorem of Frobenius over the complex numbers, which asserts that if X is smooth and $\mathcal{T}_{\mathcal{F}}$ is a subbundle of X at some point, then closure under Lie bracket is equivalent to the foliation being given locally as a fibration, whence the appellation.

Now in the study of singularities of a variety per se, the main protagonists are the cotangent sheaf, and the canonical bundle. The former is the more classical and its relation with local algebra rather well understood. The latter is rather more recent, but its study is the essential prerequisite for birational geometry. Since our results will also encompass the case of the trivial foliation, i.e. simply the study of curves on varieties, it is not surprising that we will encounter a similar phenomenon. However rather than the cotangent sheaf of the foliation, we will work with the tangent sheaf, via which we introduce our first definition, viz:

Definition 1.1.1. *A foliated variety $(X, \mathcal{F})$ is said to be weakly regular if it is given by a subbundle $T_{\mathcal{F}}$ sitting as a saturated subsheaf of the tangent sheaf $\mathcal{T}_X$.*

This definition is immediately deserving of comment. In the case of the trivial foliation the definition asserts that the tangent sheaf of the variety is a bundle. This is a priori strictly weaker than the assertion that the variety is regular, i.e. that the cotangent sheaf is a bundle. It is however a conjecture of Zariski and Lipmann that the two are equivalent. At the other extreme if $(X, \mathcal{F})$ is a foliation by curves, given that X is normal, weak regularity is equivalent to being foliated Gorenstien as introduced in [M2], and amounts to the foliation being given everywhere by a vector field. Given that the said vector field is itself allowed to vanish in codimension 2, every foliation by curves on a non-singular variety is weakly regular. On the other hand for a singular variety X, the condition is highly non-trivial, and can give a great deal of information about the foliation, cf. op. cit.

The role of normality of the underlying space in the above is fairly unimportant, where it really makes its appearance is in the consideration of the canonical sheaf of the foliation, i.e. the dual of the top exterior power of the tangent sheaf $\mathcal{T}_{\mathcal{F}}$. Naturally we denote this by $K_{\mathcal{F}}$, and follow Kawamata, Mori et al., in introducing a discrepancy function to measure the singularities. Specifically suppose $K_{\mathcal{F}}$ is "$\mathbb{Q}$-Cartier". For the moment let us be deliberately vague about what this may mean, and consider any proper birational map $p : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ from a foliated variety to our original variety then for some divisors E_i contracted by p, and rational numbers a_i, we must have:

$$K_{\widetilde{\mathcal{F}}} \equiv p^* K_{\mathcal{F}} + \sum_i a_i E_i$$

where "$\equiv$" is some suitable equivalence relation of divisors, such as rational or numerical. Given that $\widetilde{X}$ is necessarily normal, whence non-singular in codimension 1, the numbers a_i depend only on E_i considered as rank 1 discrete valuations of the field of functions of X, and so we may define a map:

$$a\,(\ ,X,\mathcal{F}) : \left\{ \begin{matrix} \text{rank 1 discrete} \\ \text{valuations } E \text{ of } k(X)^{\times} \end{matrix} \right\} \longrightarrow \mathbb{Q}. \qquad \text{– 1.1.2}$$

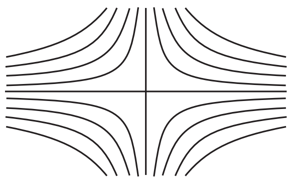
where of course we implicitly assume that the valuations have non-empty centre, so that $a\,(\ ,X,\mathcal{F})$ is even defined at the level of germs. Finally we are in a position to introduce the discrepancy of a foliated space, i.e.

$$\operatorname{discrep}(X,\mathcal{F}) := \inf_E a\,(E,X,\mathcal{F})$$

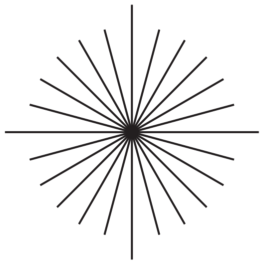
and to make,

DEFINITION 1.1.3. *A foliated space* $(X,\mathcal{F})$ *is said to have canonical singularities if* $\operatorname{discrep}(X,\mathcal{F}) \geq 0$*, and terminal singularities should this inequality be strict.*

Necessarily in the case of the trivial foliation, $K_{\mathcal{F}}$ and K_X coincide, so that the definitions of canonical and terminal do like wise. However when the rank of the foliation differs from the dimension of X this is absolutely not so. For example consider the vector field $\partial = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$ in a neighbourhood of the origin in $\mathbb{C}^2$. One may easily draw this, viz:



The underlying space X is certainly smooth, and as such has terminal singularities in the usual sense. However the pair $(X,\mathcal{F})$ is our object of study and this has discrepancy zero, i.e. it is properly canonical without being terminal. One can go further, and consider the vector field $\partial = x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y}$. Again this is easily drawn,



The underlying space is as before, but now the discrepancy is actually -1, for the foliated space. Actually these two examples are fairly representative of what can happen for an underlying smooth space X of dimension 2 together with a foliation by curves. Canonical singularities here correspond to so called reduced singularities or those of Poincaré-Dulac type, cf. [M2], and every foliation by curves on a surface may be resolved to one with canonical singularities. Nevertheless as a moment's reflection on our initial example shows this is best possible, i.e. unlike the trivial case one cannot resolve to a birational model with terminal singularities. Ultimately, however, one gains a rather better feeling for the definition and its interplay with the singularities of the underlying space by considering blow ups in foliation equivariant centres. This of course means that we specify a subvariety Y of X, together with its sheaf of ideals $\mathcal{I}_Y$ and over an affine open subset U of X ask that for all derivations $\partial \in \Gamma(U, \mathcal{T}_{\mathcal{F}})$, $\partial(\mathcal{I}_Y) \subset \mathcal{I}_Y$. This definition behaves well with respect to localisation, so it easily globalises, and we have:

LEMMA 1.1.4. *Let $p : \widetilde{X} = \mathrm{Bl}_Y(X) \to X$ be a blow up of a foliated variety $(X, \mathcal{F})$ in a $\mathcal{F}$-equivariant centre Y then we have a natural map,*

$$p^* \mathcal{T}_{\mathcal{F}} \longrightarrow \mathcal{T}_{\widetilde{\mathcal{F}}} \, .$$

Proof. The question is local, so we may assume that X is affine, say $\operatorname{Spec} A$, and $\mathcal{I}_Y$ is just the sheafication of an ideal I of A. The assertion is then simply that if $\partial \in \mathrm{Der}(A)$ lies in the tangent sheaf of the foliation then the a priori meromorphic vector field $p^* \partial$ is in fact holomorphic. This is easily verified, since locally a function f on $\widetilde{X}$ is of the form $\frac{g}{h^d}$ where d is a non-negative integer, $g \in I^d$, and h being in I, where naturally we think of ourselves as looking at functions on the $h \neq 0$ part, then of course:

$$\partial(f) = \partial\left(\frac{g}{h^d}\right) = \frac{h\,\partial g - dg\,\partial h}{h^{d+1}}$$

and since I^d is also $\mathcal{F}$ equivariant, $\partial(f)$ is a function as required. $\square$

As a simple illustration of the lemma consider a foliation by curves $(X, \mathcal{F})$ with underlying space X non-singular. Locally the foliation is given by a vector field ∂, and where ∂ vanishes is the singular locus of $\mathcal{F}$. Any point x in the singular locus is of course $\mathcal{F}$ equivariant, so in a neighbourhood of x the discrep is always less than or equal to zero. Consequently under such hypothesis a 'singularity' is 'terminal'[2] if and only if the vector field is non-zero, i.e. the foliation is locally a smooth fibration. The essence of this conclusion extends, as we shall see, to arbitrary weakly regular integrable foliations. However it is not a classification of arbitrary terminal singularities, even for foliations on surfaces, for example consider either of the classical foliations on a singular Kummer surface, i.e. an abelian surface modulo ± 1, then the 8 fixed points of this action are all terminal singularities for the corresponding foliations. In any case to justify the above remark regarding weakly regular foliations we will need a further lemma, viz:

LEMMA 1.1.5. *Let $p : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ be the blow up of a foliated variety (in characteristic zero) in a $\mathcal{F}$-equivariant centre Y as in* 1.1.4, *and let $\rho : (X^{\#}, \mathcal{F}^{\#}) \to (\widetilde{X}, \widetilde{\mathcal{F}})$ be its normalisation then in fact ρ is $\mathcal{F}$-equivariant, i.e. we have a natural map,*

$$\rho^* \, p^* \, \mathcal{T}_{\mathcal{F}} \longrightarrow \mathcal{T}_{\mathcal{F}^{\#}} \, .$$

Proof. The assertion precisely regards the locus where $X^{\#}$ is not isomorphic to $\widetilde{X}$. Better still since $X^{\#}$ is S_2 we only require to show the existence of the map in codimension 1, so let $D^{\#}$ in $X^{\#}$ be an irreducible divisor with generic point $\eta^{\#}$. If $\rho|_{D^{\#}}$ is an isomorphism generically then for any derivation ∂ in $\mathcal{T}_{\mathcal{F}}$, $\rho^* \, p^* \, \partial$ is a derivation of the local ring $\mathcal{O}_{X^{\#}, D^{\#}}$ by the previous lemma, and there is nothing to do. Otherwise let D in $\widetilde{X}$ be the image, and η a generic point of D. Since X is normal, D is necessarily a component of the exceptional divisor on $\widetilde{X}$, and as such is $p^* \, \mathcal{T}_{\mathcal{F}}$ invariant. Whence, let us consider the blow up of $\widetilde{X}$ in D around a neighbourhood of η. Using the same letter for the space and the local neighbourhood we have maps,

$$\begin{tikzcd}
\widetilde{X} & \mathrm{Bl}_D \, X \arrow[l] & X^{\#} \arrow[l] \\
D \arrow[u, hook] \arrow[ur, phantom, "\square"] & \bigcup_i D_i \arrow[l] \arrow[u, hook] & D^{\#} \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

where the first square is a fibre square. The fibre over D being generically finite to one, but definitely not an isomorphism. Now if D_j, say, is the image of $D^{\#}$ then either $D^{\#}$ is isomorphic to D_j, and applying lemma 1.1.4 forces $\rho^* \, p^* \, \partial$ to be a derivation of $\mathcal{O}_{X^{\#}, D^{\#}}$ for any ∂ in $\mathcal{T}_{\mathcal{F}}$. Otherwise it is not, and we once more make a local invariant blow up, this time

[2]'terminal' here is to be understood, somewhat incorrectly, as refering to positive discrepancy for divisors arising from a sequence of blow ups in points. Unfortunately if $\dim X \geq 3$, smooth points are not terminal.

in D_j. On the other hand since there is always a non-trivial fibre over the generic point of the centre of such a blow up, and all such blow ups are locally dominated by $X^{\#}$. This cannot continue indefinitely, from which we conclude. $\square$

Unlike the previous lemma, characteristic zero is essential here in order to deduce that the centres of the local blowing ups are equivariant under the foliation – should the exceptional divisor have multiplicity p along some component in char $= p > 0$ we'd certainly be in trouble. Applying the lemma to arbitrary Gorenstien foliations by curves implies that either the foliation is locally integrable by the usual Frobenius type procedure or it has at best a canonical singularity. The difficulty in extending such considerations to arbitrary foliations relies on identifying suitable invariant centres. We will come back to this, and a more precise discussion of the above remarks on integrability in 1.2. For the moment let us complete this introduction to singularities by way of some remarks on the condition $K_{\mathcal{F}}$ is "$\mathbb{Q}$-Cartier".

The most obvious sense of this is of course to consider the open embedding $j : (X_{sm}, \mathcal{F}_{sm}) \hookrightarrow (X, \mathcal{F})$ of the locus where say both X and $\mathcal{F}$ are smooth, and to demand that there is a positive integer m such that $j_* K_{\mathcal{F}_{sm}}^{\otimes m}$ is a Cartier divisor. This is rather strong, and so we term it $\mathbb{Q}$-foliated Gorenstien, and of course foliated Gorenstien if we can take $m = 1$. Equally for 2-dimensional normal algebraic spaces, the definition of "$\mathbb{Q}$-Cartier" can be understood in a linguistically abusive, though not mathematically abusive, sense via Mumford's intersection theory. These remarks are, however, all rather paranthetical since we will be almost exclusively concerned with foliations which satisfy the Gorenstien condition.

1.2 Towards an ideal situation

We now wish to concentrate on how to ameliorate the singularities of a foliated variety in a neighbourhood of a curve. In this section we will concentrate on foliations by curves. This not only provides some calculations essential to the general case, but illustrates the key features of what we are after without the technical complications that arise in the higher rank case. The ideal of course would be to find a neighbourhood of the curve, birationally, where the foliation is everywhere integrable. In absolute generality this is impossible, indeed it is even so on smooth surfaces. Nevertheless the impossibility only occurs for curves invariant by the foliation. Since we intend to allow arbitrary singularities on the underlying space X we will consider an embedding $X \hookrightarrow M$ of our variety into a smooth variety. In addition everything will be arbitrarily local, so let's just work in the analytic topology. Observe that for a vector field ∂ on X and $x \in X$ it is completely unclear from the definition whether $\partial \neq 0$ in $\mathcal{T}_X \otimes k(x)$ implies $\partial \neq 0$ in $T_M \otimes k(x)$. Indeed this difficulty is at the root of the Zariski-Lipmann conjecture. Whence although it depends on the embedding let's call ∂ non-singular at x if ∂ is non-zero in $T_M \otimes k(x)$, and observe the following straightforward generalisation of the Frobenius theorem,

LEMMA 1.2.1. *Let x be a non-singular point of a Gorenstien foliation by curves $(X, \mathcal{F})$ then there is a map $\pi : X \to Y$ of analytic spaces around x such that $\pi^* \Omega_Y$ is the conormal bundle of $\mathcal{F}$ in X, and π is relatively smooth.*

Proof. We proceed in the obvious way. Namely if I is the ideal of X in M we have the usual short exact sequence,

$$I/I^2 \longrightarrow \Omega_M|_X \longrightarrow \Omega_X \longrightarrow 0\,.$$

By the non-singularity hypothesis there is a vector field ∂ on M, non-vanishing at x, with $\partial(I) \subset I$ which induces our given foliation on X, where of course we permit as much localisation as we need. Now let $z_1, \ldots, z_n$ be coordinate functions on M, then without loss of generality $\partial(z_1) = 1$, as usual we may put,

$$y_i = \sum_{n=0}^{\infty} \frac{(-1)^n\, z_1^n\, \partial^n\, z_i}{n!}\,, \qquad 2 \leq i \leq n$$

and one easily checks $\partial y_i = 0$. Consequently we just put Y to be the image of X in $\mathbb{C}^{n-1}$ under the map, $(y_2, \ldots, y_n) : M \to \mathbb{C}^{n-1}$. □

Now we would like to obtain this situation around any point of a suitable curve C in X. The definition of suitable here is that C is not invariant by the foliation, so in particular the singular locus of the foliation meets C in a bunch of points. Let us concentrate on one of them, i.e. denote by X a sufficiently small neighbourhood of the point. It may of course happen that C is singular at such a point. The point is $\mathcal{F}$ equivariant so we are happy to blow up in it, so that whether self evidently or by a minor adaptation of lemma 1.1.5 we note:

FACT 1.2.2. *Notations as above, there is a sequence of varieties,*

$$X = X_0 \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_n = \widetilde{X}$$

where $X_i \to X_{i-1}$ is obtained by blowing up in a foliation equivariant centre, or by normalisation, such that the proper transform of C in $\widetilde{X}$ is non-singular.

Of course we like such sequences since they are "unramified" in the foliation direction by virtue of lemmas 1.1.4 and 1.1.5, so now we want to use a similar sequence, starting with a smooth C, to get into the local integrability situation of 1.2.1.

Consequently let $x_1, \ldots, x_n$ be coordinates on our ambient smooth space M, and suppose C is given by $x_2 = \cdots = x_n = 0$. We of course consider a vector field ∂ on M which leaves the ideal I of X invariant, and restricts to our given foliation on X which is necessarily supposed

Gorenstien. Necessarily we are supposing that ∂ is singular at the origin, we write $\partial = \sum a_i \frac{\partial}{\partial x_i}$ and consider how ∂ transforms around C under the blow up $p : \widetilde{M} \to M$ in the origin. We have a new coordinate system $x_1 = \xi_1$, $x_i = \xi_1 \xi_i$, with C transforming to $\xi_2 = \cdots = \xi_n = 0$. Putting ν_i to be the multiplicity of a_i at the origin, $a_i = \widetilde{a}_i \, x_1^{\nu_i}$, and understanding this notation in the natural way when some $a_i = 0$, we obtain:

$$p^* \partial = \widetilde{a}_i \, \xi_1^{\nu_i} \, \frac{\partial}{\partial \xi_1} + \sum_{i=2}^{n} \left(\widetilde{a}_i \, \xi_1^{\nu_i - 1} - \widetilde{a}_i \, \xi_1^{\nu_i - 1} \xi_i\right) \frac{\partial}{\partial \xi_i} \, .$$

Better still, we also have,

$$\mathrm{mult}_0(\widetilde{a}_i|_C) = \mathrm{mult}_0(a_i|_C) - \nu_i \, .$$

Now we distinguish two cases. In the first $a_1 \neq 0$, then on replacing p by a sequence of blow ups invariant in the pull-back of ∂ (so even invariant by the foliation in the case of canonical singularities) we have without loss of generality,

$$p^* \partial = \xi_1^{\nu_i} \, \frac{\partial}{\partial \xi_1} + \sum_{i=2}^{n} b_i \, \frac{\partial}{\partial \xi_i} \, .$$

Furthermore some b_i is not identically zero on C, so that possibly after some more blow ups of the same form, the pull-back of ∂ is a regular, and non-singular derivation along the proper transform of C. In the case that a_1 is identically zero, the same conclusion is even more immediate. Now let us summarise these reflections by way of a definition and proposition,

Definition 1.2.3. *Call a foliation $\mathcal{F}$ on a variety X smoothly integrable at a point $x \in X$ if it arises as in lemma* 1.2.1.

Then we have,

Proposition 1.2.4. *Let C be a non-invariant curve in a Gorenstien foliated variety $(X, \mathcal{F})$ then there is a neighbourhood $(X_0, \mathcal{F}_0)$ of C together with a proper birational map $p : (\widetilde{X}_0, \widetilde{\mathcal{F}}_0) \to (X_0, \mathcal{F}_0)$ such that $\widetilde{\mathcal{F}}_0$ is smoothly integrable around every point of the proper transform of C, and better still there is a natural map by pulling back, $p^* \, T_{\mathcal{F}_0} \to T_{\widetilde{\mathcal{F}}_0}$.*

1.3 General case

We now wish to study arbitrary integrable and weakly regular foliations around curves. Our only restriction will be that the curve is not contained in the singular locus of the foliation. The study again being local this statement is to be understood in terms of the foliation not having full rank in the tangent space of some smooth variety M into which X is embedded. The tricky thing here is that a singular point x may no longer be foliation invariant. Indeed by definition they are invariant precisely when the map at the residue field level, $T_{\mathcal{F}} \otimes k(x) \to T_M \otimes k(x)$ is zero, so we have to do a little more work to identify invariant centres. To this end, and much as before, let $x_1, \ldots, x_n$ be coordinates on M and write,

$$\partial_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}, \qquad 1 \leq i \leq r, \quad 1 \leq j \leq n$$

where r is the rank of the foliation. It goes without saying that the ∂_i leave the ideal I_X of X in M invariant, and induce our given weakly regular foliations $\mathcal{F}$. Not surprisingly the matrix $A = [a_{ij}]$ of functions on X will play a key role, let us denote by s the dimension of the image of $T_{\mathcal{F}} \otimes k(x)$ in $T_M \otimes k(x)$ around our point of study x. Observe that by row and column reduction of matrices we may find $s \times (n-s)$ and $(r-s) \times (n-s)$ matrices B, D of functions in the maximal ideal at x such that without loss of generality,

$$A = \begin{bmatrix} I & \vdots & B \\ \cdots & \cdots & \cdots \\ 0 & \vdots & D \end{bmatrix}.$$

Equally, for the same reason, there is a $r \times (n-r)$ matrix of meromorphic functions A_0, such that,

$$A = \left[I \vdots A_0 \right].$$

Now with these notations let us pause to consider the case of $r = s$ and the Frobenius theorem in this context. This time let's start with the definition, viz:

Definition 1.3.1. *A foliated variety $(X, \mathcal{F})$ is said to be smoothly integrable at $x \in X$, if there is a relatively smooth map $\pi : X \to Y$ in a neighbourhood of x such that $\pi^* \Omega_Y$ generates the conormal bundle of $\mathcal{F}$.*

Then of course we have,

Lemma 1.3.2. *Notations as above if $s = r$ at $x \in X$, then $(X, \mathcal{F})$ is smoothly integrable at x.*

Proof. Proceeding as in lemma 1.2.1 we can actually choose our coordinate functions on M, such that A_0 is not only a matrix of functions around x, but in fact the first row is identically zero. Now let us consider invariance under Lie bracket. By the invariance of I under the lifting of our various vector fields, this may simply be calculated in M and restricted to X. Consequently understanding A_0 as a $(r-1) \times (n-r)$ matrix of functions, we obtain a $(r-1) \times (r-1)$ matrix Λ of functions on X around x such that,

$$\left[0 \vdots \frac{\partial A_0}{\partial x_1} \right] = \Lambda \left[I \vdots A_0 \right]$$

which of course forces $\frac{\partial}{\partial x_1} A_0$ to be identically zero on X. Consequently if we let X_1 be the image of X in $\mathbb{C}^{n-1}$ under the map $(x_2, \ldots, x_n) : M \to$

$\mathbb{C}^{n-1}$, and π the induced map then π is relatively smooth and there is a foliation $\mathcal{F}_1$ on X_1, closed under Lie-bracket of rank $(r-1)$ which induces our given foliation, so that we may conclude by induction. $\square$

We can now turn to the situation of $s < r$ and see how for this discussion can be pushed. Allowing a meromorphic A_0, shows that at the first stage of the induction procedure (assuming of course $s \neq 0$) that there are $(s-1)$ meromorphic vector fields on X_1, which together with $\partial/\partial x_1$ generate our given foliation. Proceeding by induction gives a prefered coordinate system such that,

$$\partial_i = \frac{\partial}{\partial x_i}\ , \qquad 1 \leq i \leq s$$

$$\partial_i = \sum_{j>s} a_{ij} \frac{\partial}{\partial x_j}\ , \qquad i > s \qquad \text{– (1.3.3)}$$

with the a_{ij} holomorphic functions of $x_{s+1}, \ldots, x_n$ which all vanish at the origin. We have thus identified, locally, a suitable foliation invariant centre, viz: $x_{s+1} = \cdots = x_n = 0$. By way of our curve, C, passing through our singular point, we first consider its image under $(x_{s+1}, \ldots, x_n) : M \to \mathbb{C}^{n-s}$, and ask whether the curve itself is singular there or not. Argueing exactly as in 1.2.2 we may via a sequence of equivariant blow ups pass without loss of generality to the situation where the image of C is given by $x_{s+2} = \cdots = x_n = 0$.

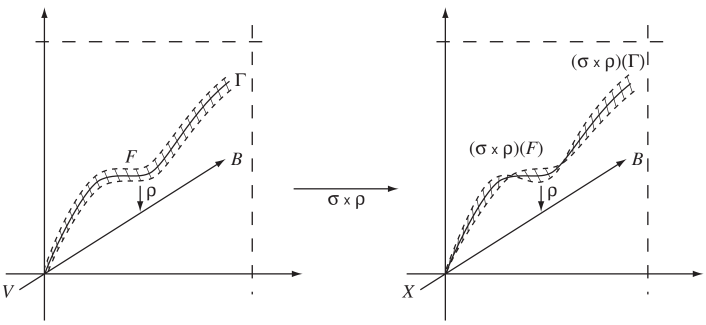
Now for a curve C whose tangent space does not generically factor through that of the foliation, we may proceed more or less as in the case of foliations by curves, namely, blow up in centres of the type we have identified until such times as the rank increases. Once the rank increases change coordinates in the obvious way, resolve any singularities on the new projection of our curve, then blow up in some foliation equivariant centres until we increase the rank again. Consequently we obtain,

PROPOSITION 1.3.4. *Let C be a curve in a weakly regular foliated variety $(X, \mathcal{F})$ whose tangent space does not generically factor through $\mathcal{F}$ and which is not contained in the singular locus of $\mathcal{F}$ then there is a neighbourhood $(X_0, \mathcal{F}_0)$ of C together with a proper birational map $p : (\widetilde{X}_0, \widetilde{\mathcal{F}}_0) \to (X_0, \mathcal{F}_0)$ such that $\widetilde{\mathcal{F}}_0$ is smoothly integrable at every point in the proper transform of C, and pull-back of derivations yields a natural map $p^* T_{\mathcal{F}_0} \to T_{\widetilde{\mathcal{F}}_0}$.*

2 Algebrisation

2.1 The Graphic neighbourhood

Let us concentrate our attention in this section on a curve C inside a foliated variety $(X, \mathcal{F})$ with the foliation of rank r, integrable, and with weakly regular singularities where additionally we will suppose that C is neither contained in the singularities nor does its tangent space factor through $T_{\mathcal{F}}$. Now by the considerations of § 1, we may find an open neighbourhood X_0 of C (either formally or in the analytic topology) together with a proper birational map (in the category of analytic spaces) $p : \widetilde{X}_0 \to X_0$ such that the induced foliation $\widetilde{\mathcal{F}}_0$ is smoothly integrable in a neighbourhood V of the proper transform $\widetilde{C}$ of C. At this point we wish to consider the induced foliation $\mathcal{F}^{\#}$ on $V \times B$, where B is the normalisation of C, in a neighbourhood of the graph Γ of the natural map from B to $\widetilde{C}$. The tangent bundle of $\mathcal{F}^{\#}$ is simply the pull-back of that of $\widetilde{\mathcal{F}}_0$, and is of course smoothly integrable around Γ. Whence let us take a union of Δ_α, $\alpha \in A$, of small open analytic sets covering Γ, and denote by $\pi_\alpha : \Delta_\alpha \to Z_\alpha$ the relatively smooth map of analytic spaces which yields $\mathcal{F}^{\#}|_{\Delta_\alpha}$. Further for each α, we have an ideal I_α of functions on Δ_α, generated by functions on Z_α vanishing on Γ. Necessarily the I_α patch and define a smooth analytic subvariety F of $\bigcup_\alpha \Delta_\alpha$ of dimension $r+1$, such that the normal bundle of Γ in F, $N_{\Gamma|F}$, is isomorphic to $T_{\widetilde{\mathcal{F}}}|_\Gamma$. Rather more intuitively what we have done is create an analytic space F by adding to each point of B the germ of the locally smooth integrable subvariety through each point of $\widetilde{C}$ guaranteed by 1.3.4, while equipping F with a map ρ to B, and σ to X, i.e.



With this in mind we can deliver the coup de grâce to the transcendental nature of our problem by way of,

FACT 2.1.1. *If $T_{\mathcal{F}}|_C$ is ample then the Zariski closure of $\sigma \times \rho(F)$ is of dimension $(r+1)$, and as such every $\mathcal{F}$-integrable subvariety through a point of $(\sigma \times \rho)(\Gamma)$ is algebraic.*

Proof. Since F comes equipped with a projection to B which pushes forward an integrable subvariety to a point, it is wholly sufficient to prove the claim on the Zariski closure. Equally both X and B are algebraic, so all we need show is that for any line bundle L on F there is a constant $C(L)$ such that $h^0(F, L^{\otimes n}) \leq Cn^{r+1}$, for all positive integers. Better still if $\widehat{F}$ is the completion of F along C, we have an injection $H^0(F, L^{\otimes n}) \hookrightarrow H^0(\widehat{F}, L^{\otimes n})$, so we might as well just consider $\widehat{F}$, and this is essentially a trivial exercise. Specifically for $m \in N$, let F_m be the $m^{\rm th}$ infinitesimal thickening, and observe by construction that there is a map $T_{\mathcal{F}}|_\Gamma \to N_{\Gamma|F}$ which is generically an isomorphism, so $N_{\Gamma|F}$ is ample. On the other hand we have the usual exact sequence,

$$0 \longrightarrow H^0(\Gamma, \mathrm{Sym}^m\ N^\vee_{\Gamma|F} \otimes L^{\otimes n}) \longrightarrow H^0(F_{m+1}, L^{\otimes n}) \longrightarrow H^0(F_m, L^{\otimes n})\,.$$

Necessarily the first group vanishes for $m \leq C(L)n$, where the constant $C(L)$ is of the form $0\,(|\deg_\Gamma L|)$, and whence,

$$h^0(\widehat{F}, L^{\otimes n}) \leq \sum_{k=0}^{Cn} h^0(\Gamma, \mathrm{Sym}^k\ N^\vee_{\Gamma|F} \otimes L^{\otimes n}) \leq Cn^{r+1}$$

where the last inequality may involve a slightly different constant, but nevertheless only depends on L as required. $\square$

Remark 2.1.2. Evidently the role of the analytic topology is only for convenience of exposition, since the above is really a proposition about formal schemes.

2.2 Cleaning up

We will continue to concentrate on the example of the previous section. Thanks to 2.1.1, we have obtained an algebraic variety W of dimension $r+1$, fibered over B by ρ, together with a section s of ρ, such that every fibre of W over B projects to a $\mathcal{F}$-invariant subvariety of X through the corresponding point of C. Even better our bundle of derivations $T_{\mathcal{F}}$ on X, lifts naturally to a bundle of $\mathcal{O}_B$-derivations, which we will continue to denote by $T_{\mathcal{F}}$, on W since after all dualising coherent sheafs is compatible with flat pull-back. Now intuitively we'd like to think of W/B as relatively smooth in a neighbourhood of $s(B)$ with relative tangent bundle $T_{\mathcal{F}}$. However, for the reasons detailed in 1.2, this is potentially rather false around points where $\mathcal{F}$ is not smoothly integrable. Whence we seek to resolve W to $\widetilde{W}$ in such a way that $T_{\mathcal{F}}$ will admit a map to $T_{\widetilde{W}/B}$ around the section, and such that in a neighbourhood of the section the fibration will be smooth. In light of lemma 1.1.4, what is therefore required is equivariant desingularisation with respect to $T_{\mathcal{F}}$ considered as a bundle of derivations. Considered with respect to a fixed smooth embedding $W \hookrightarrow M$ we have as ever a stratification of the singularities by closed subschemes W_s defined as,

$$W_s = \{w \in W \mid \dim\left(\mathrm{Im}\left\{T_{\mathcal{F}} \otimes k(w) \longrightarrow T_M \otimes k(w)\right\}\right) \leq s\}\,.$$

Now let $Y \hookrightarrow W$ be a smooth centre in which we may wish to blow up in order to carry out the desingularisation algorithm of [BM]. For $y \in Y$, there exists a unique s such that $y \in W_s \backslash W_{s-1}$, where by convention W_{-1} is empty. We may apply 1.3.3, to find a coordinate system $x_1, \ldots, x_n$ on

M in the analytic topology with respect to which a basis of $T_{\mathcal{F}}$ around y is given by,

$$\partial_i = \frac{\partial}{\partial x_i}\ , \qquad 1 \le i \le s$$

$$\partial_i = \sum_{j=s+1}^{n} a_{ij}\,\frac{\partial}{\partial x_j}\ , \qquad s < i \le r$$

where $a_{ij} = a_{ij}(x_{s+1},\ldots,x_n)$. Furthermore we have a locally smooth map, $\pi : W \to Z$ around y, given by $(x_{s+1},\ldots,x_n)$. By construction the algorithmic desingularisation procedure respects π, so we may find functions $f_1,\ldots,f_m$ of $x_{s+1},\ldots,x_n$ which generate $I_Y \otimes \mathcal{O}_{W,y}$, where the local ring is understood formally or analytically. With these preliminaries in mind for any derivation ∂ of $T_{\mathcal{F}}$ over an open subset U containing y, our explicit choice of coordinates imply,

$$\partial\,(I_Y) \subset \mathfrak{m}\,(y)\,.$$

Since this holds for all ∂ and all $y \in Y$, we see that Y is in fact $\mathcal{F}$ equivariant. Blowing up in Y therefore leaves $T_{\mathcal{F}}$ as a bundle of derivations on the blow up, and so we may continue with the algorithmic desingularisation procedure to obtain a $\mathcal{F}$-equivariant resolution $\widetilde{W}$ of W. Our section s necessarily lifts to a section $\widetilde{s}$ of $\widetilde{W}$ which immediately forces $\widetilde{W}$ to be relatively smooth in a neighbourhood of $\widetilde{s}(B)$. Changing notations slightly, we have therefore established,

Better Fact 2.2.1. *Let $(X,\mathcal{F})$ be a weakly regular foliated variety of rank r and $f : B \to X$ a map from a smooth curve such that $f^*T_{\mathcal{F}}$ is ample, and $f(B)$ is not contained in the singularities of $\mathcal{F}$ nor generically tangent to $\mathcal{F}$, then there is a smooth algebraic variety W of dimension $(r+1)$, equipped with projections σ, ρ to X and B respectively together with a section s of ρ such that if W_b denotes the smooth subvariety passing through $s(b)$ then $\sigma(W_b)$ is an $\mathcal{F}$-invariant subvariety through $f(b)$. In addition there is a natural map, $\sigma^*T_{\mathcal{F}} \to T_{W/B}$ in a neighbourhood of the section.*

2.3 Complements

Firstly we consider the case where our curve C is not contained in the singularities but may be generically tangent to $\mathcal{F}$. Again let B be the normalisation of C, and continue to denote by $\mathcal{F}$ the induced foliation on the product $X \times B$, with Γ the graph of B. We observe that Γ is now a curve which is NOT generically tangent to $\mathcal{F}$, but $T_{\mathcal{F}}|_{\Gamma}$ is ample if it were already so for $T_{\mathcal{F}}|_C$. Passing to a modification $\widetilde{X \times B}$ of our product around Γ, as in 1.3.4 we obtain a neighbourhood V of the proper transform $\widetilde{\Gamma}$ on which the induced foliation $\widetilde{\mathcal{F}}$ is smoothly integrable. Whence by 2.1.2 there is a $\mathcal{F}$-invariant subvariety W_x. Through every point x of Γ of dimension the rank of $\mathcal{F}$. Projecting forward to X gives a $\mathcal{F}$-invariant subvariety $W \supset C$ of the appropriate dimension. The equivariant desingularisation arguements of 2.2 go through verbatim to yield a resolution $\widetilde{W}$ such that $T_{\widetilde{W}}$ is ample on the proper transform $\widetilde{C}$ of C, and indeed we may even take the later isomorphic to B should we wish.

Let us finally consider the case where our curve C is contained in the singular locus, and let s be the generic rank of $\mathcal{F}$ along C. Denote therefore by Y a component of X_s containing C. The coordinate system 1.3.3 imply that Y is F-invariant so let $\mathcal{G}$ be the induced foliation of rank s. The immediate thing to note is that $\mathcal{G}$ is not necessarily weakly regular. However at a point y of C where the rank drops to $t < s$ we are guaranteed a space of derivations of Y of dimension $s - t$ if C is not tangent to $\mathcal{G}$, respectively $s - t - 1$ otherwise, which are not identically zero. The resolution procedure 1.3 therefore goes through verbatim, and we make neighbourhoods of the normalisation B of C exactly as before which are equivariant under $T_{\mathcal{F}}$. Since Y is also $T_{\mathcal{F}}$-equivariant, and the new foliation is generically a quotient of $T_{\mathcal{F}}$, we therefore have:

FINAL FACT 2.3.1. *Let s be the generic rank of $\mathcal{F}$ along C, then for every point $x \in C$ there is a smooth algebraic variety W_x of dimension s together with a map $\sigma : W_x \to X$ such that $\sigma(W_x) \ni x$ is a $\mathcal{F}$-equivariant subvariety. Better still if $f : B \to C$ is the normalisation then either,*

(a) *$T_{\mathcal{F}_s}$ is not generically tangent to C, and the W_x are the fibres of a smooth variety $\rho : W \to B$ through a section s, and there is a natural map, $\sigma^* T_{\mathcal{F}} \to T_{W/B}$, generically an isomorphism around B.*

(b) *$T_{\mathcal{F}_s}$ is generically tangent to C, W_x is independent of x and contains a copy of B such that there is a map of pairs $(W, B) \to (X, C)$ and a natural map $\sigma^* T_{\mathcal{F}} \to T_W$, generically an isomorphism around B.*

3 Deformation Theory

3.1 The Set up

We have thus reduced our initial highly transcendental problem to an algebraic one, either to produce rational curves in a smooth variety in which there is a curve on which the ambient tangent space is ample, or to find rational curves in a family of varieties over a curve in which there is a section on which the relative tangent space is ample. The latter is less standard and implies the former by the graph construction à la 2.3 so we will concentrate upon it. The former case is more or less in the literature, modulo one little trick which can easily be extracted from our discussion. We will of course be following Mori's method of reduction modulo p, so in this section we make our set up with the necessary precision, and summarise the appropriate minor variations that we require from [K], specifically § I.2 and § II.3.

To this end let B be a smooth projective curve over a field k of arbitrary characteristic, let $q : C \to B$ be a finite morphism from another curve,

$\pi : X \to B$ a projective flat family and $f : C/B \to X/B$ a morphism of B-schemes (with respect to the structure maps q and π) where X/B is supposed relatively smooth in a neighbourhood of $f(C)$. We wish to study the scheme of B-morphisms, $\mathrm{Hom}_B(C, X)$ in a neighbourhood of f. The next proposition contains all that we will need,

Proposition 3.1.1. *Notations as above then,*

(a) *The tangent space of* $\mathrm{Hom}_B(C, X)$ *at* f *is isomorphic to,* $H^0(C, f^* T_{X/B})$.

(b) *The dimension of every irreducible component of* $\mathrm{Hom}_B(C, X)$ *at* f *is at least,*

$$h^0(C, f^* T_{X/B}) - h^1(C, f^* T_{X/B}) .$$

(c) *The deformations of* f *are unobstructed if* $H^1(C, f^* T_{X/B}) = 0$. *In particular should this occur* $\mathrm{Hom}_B(C, X)$ *is smooth at* f.

(d) *If* $Z \subset X$ *has codimension at least* 2, *and* $H^1(C, f^* T_{X/B}(-c)) = 0$ $\forall$ *geometric points* c *of* C *then a generic* $g \in \mathrm{Hom}_B(C, X)$ *has image disjoint from* Z.

Proof. For part (a), we wish to consider the graph Γ of f in the relative product $C \times_B X$ and calculate its conormal bundle. To this end consider the commutative diagram,

$$\begin{tikzcd}
 & C \arrow[r, "\mathrm{id} \times_B f"] \arrow[dr, "q"'] & C \times_B X \arrow[dl, hook'] \arrow[d, "q \times_B \pi"] \\
C \arrow[ur, equal, "\mathrm{id}"] \arrow[r, "\mathrm{id} \times f"] \arrow[dr, "q \times q"'] & C \times X \arrow[d, "q \times \pi"] & B \arrow[dl, hook', "\Delta"] \\
 & B \times B &
\end{tikzcd}$$

where by definition the vertical square is Cartesian, and Γ is the image of $\mathrm{id} \times_B f$. This in turn leads to a commutative diagram of exact sequences of sheaves on Γ, viz:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & \frac{I_{\Delta,B\times B}}{I^2_{\Delta,B\times B}}\otimes\mathcal{O}_\Gamma & = & \frac{I_{\Delta,B\times B}}{I^2_{\Delta,B\times B}}\otimes\mathcal{O}_\Gamma & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \frac{I_{\Gamma,C\times X}}{I^2_{\Gamma,C\times X}} & \longrightarrow & \Omega_{C\times X}\otimes\mathcal{O}_\Gamma & \longrightarrow & \Omega_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
 & & \frac{I_{\Gamma,C\times_B X}}{I^2_{\Gamma,C\times_B X}} & \longrightarrow & \Omega_{C\times_B X}\otimes\mathcal{O}_\Gamma & \longrightarrow & \Omega_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the notation $I_{\cdot,\cdot}$ means the ideal of whatever subscheme inside the other. Now the middle row is exact because of the natural splitting of $\Omega_{C\times X}$, while the middle column is exact for general nonsense. Specifically X is by hypothesis smooth around Γ, so $\Omega_{C\times X}$ is a bundle, the conormal sheaf $I_{\Delta,B\times B}/I^2_{\Delta,B\times B} \xrightarrow{\sim} \Omega_B$ is also a bundle, and the map from Ω_B to $\Omega_{C\times X}$ is not zero, so it must be an injection of sheaves. Whence the left hand column is in fact exact, and so also is the bottom row by the 9 lemma, and/or a trivial diagram chase. From which we conclude an isomorphism via the splitting of the middle row,

$$I_{\Gamma,C\times_B X}/I^2_{\Gamma,C\times_B X} \xrightarrow{\sim} \Omega_\Gamma\backslash\Omega_{C\times_B X}\otimes\mathcal{O}_\Gamma \xrightarrow{\sim} \Omega_{X/B}\otimes\mathcal{O}_\Gamma$$

which proves (a). Better still $C\times_B X$ is non-singular around Γ by hypothesis, Γ is itself smooth, so the deformations are generically unobstructed which proves (b) and (c). To prove (d), we start by taking a smooth open neighbourhood U of f in $\mathrm{Hom}_B(C,X)$. For any geometric point $c\in C$, and for g in a possibly smaller U, we have by hypothesis, $H^1(C, g^*T_{X/B}(-c)) = 0$. Consequently if $\mathrm{Hom}_B(C,X,g|_c)$ is the space of morphisms taking c to $g(c)$, then a minor variation of the previous arguement forces $U\cap\mathrm{Hom}_B(C,X,g|_c)$ to be smooth around g of dimension

$h^0(C, g^* T_{X/B}) - \dim (X/B)$. On the other hand,

$$\{g \in U \mid g(C) \cap Z \neq \varphi\} = \bigcup_{c \in C} \bigcup_{x \in Z} \{g \in U \mid g(x) = z\}\,.$$

So that this space has dimension at most,

$$h^0(C, g^* T_{X/B}) - \dim (X/B) + \dim C \times_B Z \leq \dim U - 1$$

which completes the proof of (d). □

3.2 Bend and Break

We begin with a simple lemma,

LEMMA 3.2.1. *Let E be an ample vector bundle on a smooth curve C over a field of positive characteristic, then there is a sufficiently high power $F_n : C \to C$, of the absolute Frobenius F such that for all geometric points c of C,*

$$H^1(C, F_n^* E(-c)) = 0\,.$$

Proof. Fix an ample divisor D on C, and using F_n to denote as large a power of Frobenius as we require, we obtain by Riemann-Roch global sections of $F_n^* E(-D)$. Consequently for E of rank more than 1 we obtain a dévissage of bundles,

$$0 \longrightarrow E' \longrightarrow F_n^* E \longrightarrow E'' \longrightarrow 0$$

where E' is ample of rank 1. Moreover E'' is ample, and the proposition is clear for line bundles via Serre-duality, so for some high power of Frobenius F_m and $c \in C$ any geometric point we obtain a short exact sequence,

$$0 \longrightarrow F_m^* E'(-c) \longrightarrow F_{m+n}^* E(-c) \longrightarrow F_m^* E''(-c) \longrightarrow 0$$

where by induction the kernel and cokernel are acyclic. □

Now let us return to the notation of the previous section, viz: B, C smooth projective curves, $q : C \to B$, $\pi : X \to B$ the structure maps, with the latter family flat, and $f : C/B \to X/B$ a map of B-schemes such that X is smooth in a neighbourhood of $f(C)$. Moreover let everything be over a field of positive characteristic p, $f^* \Omega_{X/B}$ ample, and f not contained in a fibre of B. The map π trivially yields an inseparable equivalence relation π on X, with quotient $Y = X/\Pi$ factoring the geometric Frobenius on X (i.e. if locally b is a coordinate on B, and $x_1, \ldots, x_n$ the fibre coordinates, $X \to Y : (b, x_1, \ldots, x_n) \mapsto (b, x_1^p, \ldots, x_n^p)$). Denote the corresponding factorisation as below,

$$\begin{array}{ccc} X & & \\ {\scriptstyle\rho}\downarrow & \searrow^{F_X} & \\ Y = X/\Pi & \underset{\sigma}{\longrightarrow} & X^{(1)} \end{array}$$

then Y is a normal $\mathbb{Q}$-factorial variety with $\rho^* K_Y = p\,\omega_{X/B} + \pi^*\omega_B$, where we define $\omega_{X/B} = K_X - \pi^*\omega_B$. Better still since Y is normal there is a subset Z of X of codimension at least 2 such that Y is smooth outside $\rho(Z)$. Consequently for a generic point $x \in X$, proposition 3.1.1 and lemma 3.2.1 guarantee the existence of a curve $C' \ni x$, not contained in a fibre of B, algebraically equivalent to a large multiple of the geometric Frobenius composed with f, so at the numerical level, $C' \approx p^n\, C$, say.

Now let H be any nef. $\mathbb{R}$-divisor on X, and suppose $\rho^* K_Y \cdot C < 0$ then by [K] there is a rational curve M_x in Y containing x such that,

$$\begin{aligned} \sigma^* H^{(1)} \cdot M_x &\leq 2\dim X\, \frac{\sigma^* H^{(1)} \cdot \rho(C')}{-K_Y \cdot \rho(C')} \\ &= 2\dim X\, \frac{pH \cdot C}{-(p\,\omega_{X/B} \cdot C + \pi^*\omega_B \cdot C)}\,. \end{aligned}$$

Consequently if L_x is the rational curve in the pre-image of M_x under ρ, then either,

(a) $\rho|_{L_x}$ is inseparable, and:

$$H \cdot L_x \leq 2\dim X\, \frac{\rho H \cdot C}{-(p\,\omega_{X/B} \cdot C + \pi^*\omega_B \cdot C)}$$

or

(b) $\rho|_{L_x}$ is birational, and:

$$H \cdot L_x \leq 2\dim X\, \frac{H \cdot C}{-(p\,\omega_{X/B} \cdot C + \pi^*\omega_B \cdot C)}\,.$$

This is all we'll need of the bend and break technique so let's denote it by a number 3.2.2, and conclude this section.

Finding rationally connected varieties

We now consider our data of maps $q : C \to B$, $\pi : X \to B$ and $f : C/B \to X/B$ with all the previous hypothesis on smoothness ampleness etc., and proceed to show that the fibres of X/B are rationally chain connected. None of our hypothesis are changed by supposing X non-singular, so let's throw that in for good measure. As usual we take our given data over $\mathbb{C}$ and find an integral affine $\mathbb{Z}$-scheme, S, of finite type over which everything is defined and our initial characteristic zero situation corresponds to the generic fibre. So in fact we'll actually use C, B, X, q, π f, etc. for schemes and maps thereof, and rather abusively denote the generic fibre by the subscript $\mathbb{C}$. In any case given $\varepsilon > 0$ for a closed point s of S with residue field of sufficiently large positive characteristic, we may apply the considerations of the previous section to conclude the existence of a component W_ε of the Chow scheme parametrising 1-cycles with rational components of degree at most $2 \dim X \, \frac{H \cdot C}{-\omega_{X/B} \cdot C} + \varepsilon$ such that the map from the universal family $\mathcal{C}_\varepsilon$ over W_ε to X_s is dominant. Since this holds for all s in an open set, there is in fact such a component over the generic fibre. Better still since this must be true for all $\varepsilon > 0$ and there are at most finitely many such components there is a component W of the Chow scheme parametrising 1-cycles with rational components of degree at most $2 \dim X \, \frac{H \cdot C}{-\omega_{X/B} \cdot C}$ such that the universal family $\mathcal{C}$ dominates X. Moreover the universal family is proper over X, and by the inequality 3.2.2(b), these rational 1-cycles are all contained in fibres of π.

Now, we are in characteristic zero, so we can find a map well defined in codimension 1, $\nu : X_{\mathbb{C}} \dashrightarrow X_{1,\mathbb{C}}$, where $X_{1,\mathbb{C}}$ is smooth, with generic fibre a smooth rational curve in the family parametrised by $W_{\mathbb{C}}$. These fit into a diagram of $B_{\mathbb{C}}$-schemes,

$$\begin{tikzcd} X \arrow[r, dashed, "\nu"'] \arrow[d, "\pi"'] & X_{1,\mathbb{C}} \arrow[dl, "\pi"] \\ B_{\mathbb{C}} & \end{tikzcd}$$

Forgetting for a moment the subscript $\mathbb{C}$ we require to calculate $\omega_{X_1/B} := K_{X_1} - \pi^* \omega_B$. Note that since $\pi^* \omega_B$ is saturated in Ω_X over an open set which surjects onto B, the same is true for $\pi^* \omega_B$ in Ω_{X_1}, so that outside of a closed set T which is at worst the union of something of codimension at most 2 and components of some fibres which do not meet the said open neighbourhood we have a map of bundles, which is injective as a map of sheaves,

$$\nu^* \, \Omega_{X_1/B} \longrightarrow \Omega_{X/B} \, .$$

Now without loss of generality we may assume that everything is defined over the same $\mathbb{Z}$-scheme S of finite type, and consider a generic deformation $g : C_m \to X$ of a sufficiently large power of the geometric Frobenius $C_m \to C$ of our given curve composed with f. Specifically we wish to calculate the degree of $g^* \, \nu^* \, \Omega_{X_1/B}$ via our knowledge of $f^* \, \Omega_{X/B}$.

So in the first place let,

$$0 = T_0 \subset T_1 \subset \cdots \subset T_\iota = g^* \, T_{X/B}$$

be the Harder-Narasimhan filtration of $g^* \, T_{X/B}$. By definition we have,

$$\mu_{\max}\,(g^* \, T_{X/B}) = \mu(T_1) > \cdots > \mu(T_{\iota-1} \backslash T_\iota) = \mu_{\min}\,(g^* \, T_{X/B})\,.$$

In addition by lower semi-continuity in our family of deformations, and given we take g generic,

$$\mu_{\min}\,(g^* \, T_{X/B}) \geq \mu_{\min}\,(F_m^* \, f^* \, T_{X/B})$$

where $F_m : C_m \to C$ is the appropriate power of the geometric Frobenius. However by the ampleness of $T_{X/B}$ and the commutativity of Proj with base change, there is an $\varepsilon > 0$ such that $\mu(Q) \geq \varepsilon p^m$ for any line quotient Q of $F_m^* \, f^* \, T_{X/B}$. On the other hand if g is the genus of C then by Riemann-Roch we can find such a quotient with $\mu(Q) \leq \mu_{\min}(F_m^* \, f^* \, T_{X/B}) + g$, so taking $p \gg 0$ we may as well say, $\mu_{\min}(F_m^* \, f^* \, T_{X/B}) \geq \varepsilon p^m$ for a possibly smaller ε, and whence: $C_{m \cdot g} \, \omega_{X_1/B} \leq -\varepsilon p^m$. Turning now to the inseparable scheme quotient $\rho : X_1 \to Y_1$, defined via $\pi : X_1 \to B$ as before, through a generic point x of X_1 we can find for $p \gg 0$ a deformation missing the set T and any subset of codimension 2 we may introduce by passing to the quotient and so find $L_x \subset X_1$ a rational curve containing x on which ρ is inseparable, and,

$$H \cdot L_x \leq 2 \dim X \; \frac{H \cdot C}{\left(\varepsilon - \frac{\pi^* \, \omega_B \cdot C}{p^m}\right)}\,.$$

Lifting everything to characteristic zero as previously we get a ruling on X_1 by rational curves, which maps trivially to B, and a quotient $\nu : X_1 \dashrightarrow X_2$ by this ruling, well defined in codimension 2. We may thus continue by induction to establish that the fibres of X/B are rationally chain connected, with an appropriate bound on the degree of the connecting rational curves, by way of Tsen's theorem.

The entire discussion evidently goes through quasi-verbatim on replacing B by a point. The extra little idea in this case which yields a result not already in the literature is to use the deformations of f in X guaranteed a priori in positive characteristic, rather than trying to establish sufficient regularity of the quotient map $X \to X_1$ around C in characteristic zero.

4 Principal results and corollaries

4.1 The main theorem

Consider a weakly regular integrable foliated variety $(X, \mathcal{F})$. The discussion of the previous chapters shows that if C is a curve in X, and $T_{\mathcal{F}}|_C$ is ample then through every point $x \in C$ there is a rationally chain connected variety whose dimension is the generic rank along C and whence by [KMM] rationally connected, $\mathcal{F}$-invariant subvariety V_x with appropriate bounds on the degree of the connecting curves. It therefore remains to discuss the case of curves contained in the singular loss of $\mathcal{F}$ in more detail which is where the hypothesis of canonical singularities will intervene. To illustrate the principle idea let's suppose for the moment X is non-singular, and $\mathcal{F}$ is a foliation by curves. Denote by Z any subscheme contained in the singular locus of $\mathcal{F}$ considered scheme theoretically (i.e. if locally $\partial = \sum_i a_i \frac{\partial}{\partial x_i}$, then the ideal of the singular locus is that generated by the a_i). Observe that we have maps,

$$\begin{tikzcd}
I_Z \arrow[r, phantom, "\subset"] \arrow[dd] & \mathcal{O}_X \arrow[d, "d"] & & \\
& \Omega_X \arrow[r] \arrow[drr, "D"'] & K_{\mathcal{F}} \otimes I_{\mathrm{sing}(F)} \arrow[r, phantom, "\subset"] & K_{\mathcal{F}} \otimes I_Z \arrow[d, "\otimes \mathcal{O}_Z"] \\
I_Z/I_Z^2 \arrow[rrr, dashed] & & & K_{\mathcal{F}} \otimes I_Z/I_Z^2
\end{tikzcd}$$

We claim that the total composite of the maps on the right factors through I_Z/I_Z^2, and better still defines an $\mathcal{O}_Z$-linear map, which we will denote by D. The linearity is automatic by the Leibniz rule, and this in turn automatically forces the said factorisation. Let's observe some simple facts about this map. Firstly suppose $W \subset Z$ then we get maps for either subscheme, and a natural commutative diagram,

$$\begin{array}{ccc}
I_Z/I_Z^2 & \longrightarrow & I_W/I_W^2 \\
\downarrow D & & \downarrow D \\
K_{\mathcal{F}} \otimes I_Z/I_Z^2 & \longrightarrow & K_{\mathcal{F}} \otimes I_W/I_W^2
\end{array}$$

Now suppose we're considering a singular point x, with coordinate functions $x_1, \ldots, x_n$, and say ∂ a derivation defining the foliation at the point, then of course, $\partial = \sum_i a_i \frac{\partial}{\partial x_i}$, but $a_i - a_{ij}\, x_j \in \mathfrak{m}^2$, the square of the ideal at the point, with a_{ij} constants. In this case D is just the linear transform of the residue of the cotangent bundle given by,

$$D : \Omega_X \otimes k(x) \longrightarrow \Omega_X \otimes k(x) : dx_i \mapsto a_{ij}\, dx_j\ .$$

This map may of course be zero, but a moments thought shows that this cannot happen if the singularities are canonical. Indeed for general Z, we can just use the maps in the middle row together with the natural map from I_Z/I_Z^2 to the cotangent bundle to define a map, again denoted D, from $\Omega_X \otimes \mathcal{O}_Z$ to $\Omega_X \otimes \mathcal{O}_Z(K_{\mathcal{F}})$ which at every point is as above. Whence in turn for each $1 \leq n \leq \dim X$ we obtain via symmetric functions a global section, $S_n \in \Gamma(Z, \mathcal{O}_Z(K_{\mathcal{F}}))$. The issue is therefore whether S_n is zero or not. If not this contradicts the ampleness of $T_{\mathcal{F}}$ on taking Z to be our curve, so what we'll show is that if S_n is zero for all n, the singularity is not canonical. It is wholly sufficient to prove this at a singular point, so say notations as above with x the origin. If all the symmetric functions vanish then the matrix $[a_{ij}]$ is nilpotent. Linear changes of coordinates conjugate the matrix, so we can suppose:

$$\partial = \sum_{i=k}^{n-1} x_i \, \frac{\partial}{\partial x_{i+1}} + \delta$$

where $\delta \in \mathfrak{m}^2 T_{X,x}$. We blow up in the origin, and look on the $x_n \neq 0$ patch, i.e. change coordinates to $x_n = \xi_n$, $x_i = \xi_i \, \xi_n$, $i < n$. Denote the blow up by $p : \widetilde{X} \to X$, then we have:

$$p^* \, \partial = \sum_{i=k}^{n-2} \xi_i \, \frac{\partial}{\partial \xi_{i+1}} + p^* \, \delta (\mathrm{mod} \; \mathfrak{m}^2 \, T_{X,0})$$

where $\mathfrak{m}$ continues to denote the maximal ideal at the origin. Superficially it may appear that we have reduced the complexity of the matrix, this may not however be the case since we can only guarantee,

$$p^* \, \delta = \sum_{i=1}^{n-1} a_i \, \xi_n \, \frac{\partial}{\partial \xi_i} \; (\mathrm{mod} \; \mathfrak{m}^2 \, T_{X,0})$$

and so the dimension of the eigenspace may not increase. Now writing things rather more invariantly we have a new linear map,

$$\widetilde{D} : \Omega_{\widetilde{X}} \otimes k(0) \longrightarrow \Omega_{\widetilde{X}} \otimes k(0)$$

with $\widetilde{D} \, d\xi_n = 0$, and all the terms, $\widetilde{\delta}$ say, that we have lumped under the appellation mod $\mathfrak{m}^2$ enjoy the additional property of being divisible by ξ_n, with the exception of a term of the form $-\sum_{i \neq n} \xi_{n-1} \, \xi_i \, \frac{\partial}{\partial \xi_i}$. So let's blow up in the origin again, but this time look at the $\xi_{n-1} \neq 0$ patch on the blow up, i.e. put $\xi_{n-1} = \zeta_{n-1}$, $\xi_i = \zeta_i \, \zeta_{n-1}$, so that denoting the blow up morphism again by p, and the maximal ideal of the new origin still by $\mathfrak{m}$, we have,

$$p^* \, \partial = \sum_{i=k}^{n-3} \zeta_i \, \frac{\partial}{\partial \zeta_{i+1}} + \sum_{i=1}^{n-2} a_i \, \zeta_n \, \frac{\partial}{\partial \zeta_i} \; (\mathrm{mod} \; \mathfrak{m}^2 \, T_{X,0})$$

where the respective sums are understood to be zero if $k < n-3$ or $1 < n-2$, respectively. The new linear map $\overset{\approx}{D}$, say, still has characteristic polynomial zero, but has at least one more eigenvector than D, so by induction, we conclude that our singularity could not have been canonical. Evidently to conclude the main theorem in this case it is sufficient that the singularities are canonical in dimension 1, i.e. canonical outside of a bunch of points. Furthermore by embedding in a smooth manifold, and using lemma 1.1.5 to control any non-normality that the blow up procedure may introduce, we see that the hypothesis that X is smooth is not really essential, and whence arrive to our theorem in the case of foliations by curves.

The general situation is rather more delicate. We start as ever with a weakly regular foliated variety $(X, \mathcal{F})$ of rank r, and a curve C in X with s the generic rank of $\mathcal{F}$ along C. We have for Y a component of X_s containing C an exact sequence of the form,

$$0 \longrightarrow \mathcal{N} \longrightarrow T_{\mathcal{F}} \otimes \mathcal{O}_Y \longrightarrow \mathcal{T}_Y$$

where the image has rank s, and all the maps are generically of the same rank after tensoring with $\mathcal{O}_C$. Consequently if $f : B \to C$ is the normalisation, then there is a rank $r - s$ subbundle N of $f^* T_{\mathcal{F}}$ saturating the map on the left. This bundle admits a rather clean geometric description as follows, viz: we can find a neighbourhood V of the proper transform $\widetilde{C}$ of C in some $\mathcal{F}$-equivariant modification of Y such that the induced foliation $\mathcal{G}$ is smoothly integrable with tangent bundle $T_{\mathcal{G}}$, say, and so N must be the kernel of the map of bundles $f^* T_{\mathcal{F}} \twoheadrightarrow f^* T_{\mathcal{G}}$. Now let $\mathcal{I}$ be the ideal sheaf of $\widetilde{C}$ in the ambient modification $\widetilde{X}$ of X, then we have an $\mathcal{O}_{\widetilde{C}}$ linear map,

$$D : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2 \otimes \operatorname{Ker}\{T_{\mathcal{F}} \to T_{\mathcal{G}}\}^\vee .$$

This map is not in general extendable to Ω_V, but no matter, since $T_{\mathcal{F}} \to T_{\mathcal{G}}$ is a surjection of bundles around $\widetilde{C}$, should E be the double dual of $f^* \mathcal{I}/\mathcal{I}^2$ we in fact have an induced map of bundles,

$$D : E \longrightarrow E \otimes N^\vee .$$

Finally letting L be the tautological bundle on $\mathbb{P}(N^\vee)$ we have a map, $D : E \to E \otimes L$, and global sections given by symmetric functions $S_d \in H^0(\mathbb{P}(N^\vee), L^d)$. On the other hand these functions are non-zero if and only if, in our usual coordinate system, the matrix a_{ij} of functions in $x_{s+1}, \ldots, x_n$ is nilpotent for each $s < i \le r$. The identical analysis to before shows this is not possible for canonical singularities, and whence we get our desired global section over B of $\operatorname{Sym}^d N^\vee$ for some d.

4.2 Foliations by curves

In the case of foliations by curves there are some particularly beautiful corollaries of the main theorem, since the canonical bundle and cotangent bundle of the foliation now coincide. In particular let $(X, \mathcal{F})$ be a variety foliated by curves, then à la Mori we introduce the closed cone $\overline{\mathrm{NE}}(X)$ inside $\mathrm{NS}_1(X)_{\mathbb{R}}$ generated by effective curves and consider the subcone,

$$\overline{\mathrm{NE}}(X)_{K_{\mathcal{F}} \ge 0} := \{\alpha \in \overline{\mathrm{NE}}(X) \mid K_{\mathcal{F}} \cdot \alpha \ge 0\} .$$

Then we obtain,

THEOREM 4.2.1. *Let $(X, \mathcal{F})$ be a variety foliated by curves with foliated Gorenstien and foliated canonical singularities then there are countably many $\mathcal{F}$-invariant rational curves L_i with, $K_{\mathcal{F}} \cdot L_i < 0$ such that,*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_{\mathcal{F}} \geq 0} + \sum_i \mathbb{R}_+ [L_i] .$$

Better still,

(a) *The rays $\mathbb{R}_+[L_i]$ are locally discrete in the upper half space, $\mathrm{NS}_1(X)_{K_{\mathcal{F}}<0}$.*

(b) *If $(X, \mathcal{F})$ is not a ruling by rational curves $K_{\mathcal{F}} \cdot L_i = -1, \forall\, i$.*

(c) *Otherwise, $-2 \leq K_{\mathcal{F}} \cdot L_i \leq -1$.*

(d) *Every extremal ray R of $\overline{\mathrm{NE}}(X)$ (i.e. $\alpha + \beta \in R$, $\alpha, \beta \in \overline{\mathrm{NE}}(X) \Rightarrow \alpha, \beta \in R$). Lying in the half space $\mathrm{NS}_1(X)_{K_{\mathcal{F}}<0}$ is of the form $\mathbb{R}_+[L_i]$.*

Proof. Everything in the corollary except (b) and (c) follows verbatim for the corresponding theorem for K_X, as found in [K], Theorem III.1.2. To prove (b) and (c) consider an embedding of X in a smooth ambient manifold M. Observe firstly that the foliation defines a quasi-section (i.e. a section in codiemsnion 2) of $\mathbb{P}(\Omega^1_X) \to X$. Let $\widetilde{X}$ be the closure of this section, and M the tautological restricted to it. Further let L be the normalisation of any one of our L_i, then L maps to $\widetilde{X}$, by $\widetilde{f}$ say, and:

$$M \cdot_{\widetilde{f}} L = -2 - \mathrm{Ram}_f$$

where $f : L \to L_i \hookrightarrow X$ is the corresponding map. Moreover we put,

$$s_{\mathrm{sing}(\mathcal{F})}(L) = K_{\mathcal{F}} \cdot_f L - M \cdot_{\widetilde{f}} L \geq 0$$

which is precisely the contribution of the locus where $\mathcal{F}$ is singular in the sense of 1.2. Now if L_i is singular at $x \in X$, then in fact x is a singular point of f, so by a sequence of $\mathcal{F}$-equivariant blow ups we can assume that $L_i = L$ (remember $K_{\mathcal{F}}$ cannot change). Whence,

$$-1 \geq K_{\mathcal{F}} \cdot_f L = s_{\mathrm{sing}(\mathcal{F})}(L) - 2 \geq -2$$

and -2 is obtained only if L does not pass through the singularities of $\mathcal{F}$. Consequently $\mathcal{F}$ must be smoothly integrable in a neighbourhood of L, and we can apply the algorithmic decomposition procedure once more to obtain a desingularisation $X^{\#}$ of X which is $\mathcal{F}$-equivariant around L. However the classical Frobenius theorem now forces L to have flat, whence trivial normal bundle in $X^{\#}$. Consequently L moves in a dim $X - 1$ family which covers $X^{\#}$. Should C be any curve in this family, $K_{\mathcal{F}} \cdot C = K_{\mathcal{F}} \cdot L < 0$, so there is an $\mathcal{F}$-invariant rational subcurve of bounded degree through the generic point of X as required. □

We may also observe from the proof that to have an extremal ray on a foliated variety which is not a fibration is rational curves requires singularities, and so we even have,

COROLLARY 4.2.2. *Let $(X, \mathcal{F})$ be an everywhere smooth foliation by curves which is not a fibration in rational curves then $K_{\mathcal{F}}$ is nef.*

There is yet another case where usual Mori theory considerations, cf. [K], yield a theorem of some interest. Call $H_R \in \mathrm{NS}^1(X)$ a supporting function of an extremal ray R if H_R is nef., and $H_R \cdot \alpha = 0$ iff $\alpha \in R$, then:

THEOREM 4.2.3. *Hypothesis as in* 4.2.1*, and $R \subset \overline{\mathrm{NE}}(X)$ and extremal ray in the half space $\mathrm{NS}_1(X)_{K_{\mathcal{F}}} < 0$ then there is a $\mathbb{Q}$-Cartier divisor H_R which is a supporting function for R, and moreover $n\, H_R - K_{\mathcal{F}}$ is ample for $n \in \mathbb{N}$ sufficiently large.*

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