SEMI-STABLE REDUCTION OF FOLIATIONS

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Abstract. The content, 1, is the minimal model theorem for foliations by curves. It continues the roll out of the various ingredients in the Green-Griffiths conjecture for algebraic surfaces, [McQ]. The minimal model theorem is, however, of an independent purely algebro-geometric interest, and is presented as such, *i.e.* a self contained theorem in complex algebraic geometry without foliation dynamics, and independent of the aforesaid motivation. A working knowledge of algebraic champs (the mis-translation stack will be eschewed) is required.

Contents

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Introduction

In a historical quirk, *cf.* [Kol96, Intro.], the study of the canonical bundle of higher dimensional varieties initiated by [Mor82], and, as such, often called Mori theory, has long ceased the original focus on rational curves in favour of a co-homological approach which would be better described as Kawamata theory. It is, therefore, not without irony that the study of rational curves on varieties foliated by curves is, arguably, Mori theory as Mori intended and leads to a complete minimal model programme.

Everything takes place in characteristic zero, so, say a projective variety $X/\mathbb{C}$, and a foliation by curves, $\mathscr{F}$, is just a (usually saturated) rank 1 sub-sheaf of the tangent sheaf, (I.19). Locally where both X and $\mathscr{F}$ are smooth this corresponds, by the classical Frobenius theorem, to a smooth fibration in the analytic topology. We therefore adopt the notation (and it's only notation) $X \to [X/\mathscr{F}]$ for foliations in order to reflect better the underlying geometry/real definition of a quotient of X by the holonomy groupoid, *cf.* II.a.2 & II.a.3. Irrespectively, there is, under mild hypothesis, *e.g.* X smooth, a well defined bundle, $K_{\mathscr{F}}$, of forms along the leaves, and corresponding notions, I.b.1, of foliated Gorenstein, resp. $\mathbb{Q}$ Gorenstein singularities. Similarly, there are, functorially with respect to the ideas, notions of foliated terminal, log-terminal, canonical and log-canonical singularities, I.b.3. Unlike their classical counterparts, however, these definitions always admit a simple description in terms of local algebra. For example, terminal (Gorenstein) is equivalent, I.b.13, to smooth along the foliation, or, equivalently given everywhere locally by a non-vanishing vector field, ∂, while a Gorenstein log-canonical singularity is a point, p, where although ∂ vanishes, the implied linearisation

$$\partial : \frac{\mathfrak{m}(p)}{\mathfrak{m}^2(p)} \to \frac{\mathfrak{m}(p)}{\mathfrak{m}^2(p)} \tag{0.1}$$

is non-nilpotent, I.b.5.

Already this local global translation is highly indicative of why Mori theory of foliations by curves is that much more tractable than that of varieties. Nevertheless, there is no free lunch, *i.e.* it transpires that from ambient dimension 3 on that there are foliations by curves which never have log-canonical singularities on any smooth bi-rational model of the ambient space. The phenomenon is quite general, [MP13, §.III.iii], and, in se, straightforward enough, *i.e.* there are certain finite group actions on vector fields whose fixed points cannot be separated from the singularities while preserving smoothness of the ambient space. In practice, however, it means that if one wants a model of a foliation $X \to [X/\mathscr{F}]$ with (foliated) log-canonical singularities, and X smooth, then one is obliged to pass from the category of varieties to the 2-category of Deligne-Mumford champs. In this context, the main theorem of [MP13] is the existence of log-canonical resolutions in ambient dimension 3, and, the reader should be aware that for the moment the existence of log-canonical resolutions in higher dimension is open.

Irrespectively, we are obviously obliged to take as our starting point smooth foliated champ $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ with log-canonical singularities- from the existence of the Gorenstein covering champ, I.b.7 & [BM97]: if there is a model with log-canonical singularities then there is one in which the ambient champ is smooth. This begins, however, to show signs of a rather pleasing loop since the natural context of the classification, [McQ08], of foliated algebraic surfaces is exactly foliated smooth bi-dimensional champs, while the universal algebraic foliation in (hyperbolic) curves $\mathscr{M}_{g,1} \to \mathscr{M}_g$ is again, naturally, a smooth Deligne-Mumford champ.

To say that this begs the question of whether the minimal model programme for foliations by curves could be run wholly inside the 2-category in which the ambient champ is smooth may, to experts in the Mori theory of varieties, seem rather absurd. It transpires, however, to be the case in a way highly reminiscent of the structure of $\mathscr{M}_{g,1} \to \mathscr{M}_g$. The precise theorem is,

1. **Main Theorem.** *(IV.e.6, IV.e.7, IV.e.8) Let* $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ *be a foliated champ which enjoys the following further properties*

(0.2) *smooth; projective moduli; log canonical, resp. canonical, foliation singularities*

then there is a sequence of contractions and flips

$$\begin{array}{ccccc} \mathscr{X}=\mathscr{X}_0 & & \mathscr{X}_1 & \cdots\cdots & \mathscr{X}_n=\mathscr{X}_{\min} \\ \downarrow & \dashrightarrow & \downarrow & \dashrightarrow & \dashrightarrow\downarrow \\ [\mathscr{X}/\mathscr{F}]=[\mathscr{X}_0/\mathscr{F}_0] & & [\mathscr{X}_1/\mathscr{F}_1] & & [\mathscr{X}_n/\mathscr{F}_n]=[\mathscr{X}_{\min}/\mathscr{F}_{\min}] \end{array} \tag{0.3}$$

such that each $\mathscr{X}_i \to [\mathscr{X}_i/\mathscr{F}_i]$ *enjoys all the (respective) properties* (0.2), *and exactly one of the following occurs*

- (a) $K_{\mathscr{F}_{\min}}$ *is nef.*
- (b) $\mathscr{X}_{\min} \to [\mathscr{X}_{\min}/\mathscr{F}_{\min}]$ *is a Mori fibre space, i.e. the locus of a single extremal ray is all of* $\mathscr{X}_{\min}$*, and the foliation is a bundle of foliated varieties where the universal cover of a fibre is the radial (supposed saturated in dimension 1) foliation on a weighted projective champ, I.d.2, whose dimension is* 1 *iff the foliation singularities are canonical.*

Here a radial foliation is just the champ/weighted projective space variant of a pencil of lines through a point of projective space, and in a further irony, the harder part of the theorem is (b) in which the use of the word flip is slightly loose since it may, when the singularities are canonical, involve "very exceptional flips", IV.e.5, *i.e.* a little invariant blowing up in the final stage, to preserve projectivity. The content of the theorem, however, should be clear: *i.e.* either we get a minimal model, or a bundle of Fano objects, and the Fano objects are particularly simple, in fact, to all intents and purposes, rational curves if the singularities are canonical.

This said, let us give a brief breakdown both of the paper and the proof.

I. The first chapter is preliminary in nature. It contains: generalities, I.a, on Deligne-Mumford champs; a revision of foliation singularities, I.b; the theory of weighted projective champs, I.c, and their radial foliations, I.d; a non-embedded variant of completion, I.e; and some remarks on the analytic topology, I.f. Technically, it's worth flagging the last 2 sections since the fact that many things fail to be an embedding for (separated) champs which are trivially so in the world of varieties, *e.g.* graphs of maps, is an issue, albeit sometimes it's true for trivial reasons, *i.e.* that the étale topology is non-classical, but in the analytic topology one can still embed.

II. The second chapter is the critical one. It first proves the cone theorem, II.d.1, in maximal generality. This was already done in [BM16] for foliated Gorenstein varieties, and its extension to foliated Gorenstein champ, II.a-II.d, may, largely, be considered technical in nature. In any case, it reveals, that the $K_{\mathscr{F}}$-negative extremal rays are invariant parabolic (*i.e.* dominated by a rational curve) champs, $\mathscr{L}$, not factoring through the singular locus. Their particularly simple intersection with the singular locus, which occurs at a unique point $p : \mathrm{pt} \to \mathscr{L}$, of the foliation is described in II.e, their normal bundle (should they have only nodes) by II.f, and their formal neighbourhoods (again for singularities no worse than nodes) in II.g. The key point here, II.g.3, is not only that the normal bundle determines the formal neighbourhood, but that everything is determined by the linearisation, (0.1), at the singularity p whose eigenvalues are, up to scaling, the slopes of the Harder-Narismhan filtration of the normal bundle. The section concludes with an examination of the functoriality of the relationship between between (0.1) and the Harder-Narismhan filtration, II.h, *i.e.* the said scaling is ambiguous in a non-trivial way up to ± 1, and this has a global manifestation; along with the necessary preliminaries, II.i, for studying extremal rays with cusps.

III. The third chapter globalises the infinitesimal information of the second to describe the sub-champs swept out by extremal rays beginning with the general discussion III.a which leads to a definition in the specific, III.a.4, of extremal champs. As such III.b-III.d is devoted to describing their structure, which, as one might imagine from 1.(b) is, III.d.7, basically that of a bundle of radially projective champs. The base of this bundle is essentially a smooth component of the singular locus, but the aforesaid issue of ± 1 in the scaling of (0.1) means that even when it has sense for it to be a Zariski bundle, it may not be.

IV. Finally we construct contractions and flips, or, better, flaps, since everything is just a question of blowing up and down. Indeed, as one might imagine, contractions, IV.a-IV.b, are easy. A critical fact, however, emerges, IV.a.4 that although a contraction renders the ambient champ less space like, *i.e.* can increase the local monodromy, it renders the foliation completely smooth about the contracted locus. As such, when one brings the full weight of the infinitesimal knowledge of §.II to bear in order to describe the formal neighbourhoods of extremal champ in a similar manner, IV.c, to that of a single ray in order to flip, IV.d, by the simple expedient of weighted blowing up and down, one concludes that flipping must terminate because it destroys a component of the singular locus at each stage. This leaves only loose ends, IV.e, to tie up related to scaling by ± 1 of (0.1), all of which can only occur when the generic leaf of the foliation is dominated by a rational curve. Consequently we conclude the demonstration of 1 in IV.e, and provide a log-variant in IV.f.

I am indebted to Bogomolov for pointing out that the language of algebraic champs was the correct setting for the main theorem; to Brunella for explaining to me the role of holonomy; to McKernan for furnishing an example that the issue of (0.1) with integer eigenvalues being only well defined up tp ± 1 is genuine; to Marie Claude for the figures; and Cécile for the original typesetting, with any subsequent flaws being the result of my own clumsy modification.

I. Preliminaries

I.a. **Normal-folds.** A normal-fold is a particularly simple kind of champ,to wit:

I.a.1. **Definition.** A normal fold is a not necessarily tame (although this will always be our context) excellent normal separated Noetherian Deligne-Mumford champ every generic point of which is scheme like.

A particularly important class of examples is given by

I.a.2. **Fact/Definition.** ([Vis89, 2.8]) Following standard usage a smooth (over an implicit base S) normal-fold will be referred to as an orbifold. In particular: a (separated) algebraic space, X, of finite type over a field k has strict (or even non-strict if the action is tame) quotient singularities iff there is an almost étale map, $\mu : \mathscr{X} \to X$, from a smooth (over k) orbifold. In this case X is the moduli, [KM97, 1.3], of $\mathscr{X}$, and conversely $\mathscr{X}$ is unique up to equivalence. As such $\mathscr{X}$ will be referred to as the *Vistoli covering champ* of X.

The following is a tiny variation on [Vis89, 2.8]'s treatment of the Vistoli covering champ

I.a.3. **Lemma.** *Let $\mu : \mathscr{X} \to X$ be the moduli of a normal-fold, with $U \to \mathscr{X}$ an étale atlas then*

$$R := \big(\textit{normalisation of}\, U \times_X U \big) \rightrightarrows U \qquad \text{(I.1)}$$

defines a groupoid and $\mathscr{X}$ is equivalent to the classifier $[U/R]$.

Proof. $U \times_X U \rightrightarrows U$ is a groupoid, so its normalisation is too. Now, let $V \hookrightarrow \mathscr{X}$ be the everywhere scheme like embedded dense Zariski open guaranteed by the definition, I.a.1, and $U' := U \times_{\mathscr{X}} V$, then V is embedded in X, so $U' \times_V U'$ is a Zariski dense open of R. It is, however, also a Zariski dense open of $R_1 := U \times_{\mathscr{X}} U$, and we have a fibre square

$$\begin{array}{ccc} U \times_X U & \longleftarrow & R_1 \\ \downarrow & & \downarrow \\ \mathscr{X} \times_X \mathscr{X} & \xleftarrow{\Delta_{\mathscr{X}/X}} & \mathscr{X} \end{array} \tag{I.2}$$

where by hypothesis the lower horizontal is finite. Consequently $R_1 \to R$ is a finite bi-rational map of excellent normal schemes so they're equal. □

Irrespective of normality we have the further simplification

I.a.4. **Lemma.** *Let $\mu : \mathscr{X} \to X$ be the moduli of a separated excellent Deligne-Mumford champ, $X' \hookrightarrow X$ the (open, possibly empty) locus where μ is an isomorphism, and $f : \mathscr{Y} \to X$ a map such that $f^{-1}(X')$ meets every generic point then f lifts to a composition $\mathscr{Y} \to \mathscr{X} \xrightarrow{\mu} X$ iff it lifts everywhere locally, i.e. for every étale neighbourhood $U \to \mathscr{X}$ of the image $f(y)$ of a geometric point y there is an étale neighbourhood V_y of y and a lifting $V_y \to U$ of f.*

Proof. Necessity is obvious. By [KM97, 1.3] and [Vis89, 2.8], there is, independently of any normal-fold hypothesis, an étale atlas $U = \coprod_\alpha U_\alpha$ of $\mathscr{X}$ and finite groups G_α acting on U_α such that $V := \coprod_\alpha V_\alpha := U_\alpha / G_\alpha$ is an étale atlas of X with $U_\alpha = \mathscr{X} \times_X V_\alpha$. Now for anything with a well defined map to X denote with a $'$ the fibre over X', so, we have open embeddings

$$\mathscr{Y}' \hookrightarrow \mathscr{Y}, \quad \mathscr{Y}'_\alpha \hookrightarrow \mathscr{Y}_\alpha := \mathscr{Y} \times_X V_\alpha \tag{I.3}$$

Consequently, by hypothesis, and refining U_α if necessary, there is an étale atlas $Y_\alpha \to \mathscr{Y}_\alpha$ and maps $f_\alpha : Y_\alpha \to U_\alpha$ such that

$$\begin{array}{ccc} Y_\alpha & \xrightarrow[f_\alpha]{} & U_\alpha \\ \downarrow & & \downarrow \\ \mathscr{Y}_\alpha & \longrightarrow & V_\alpha \end{array} \tag{I.4}$$

commutes. In particular, therefore, the G_α torsor $Y_\alpha \times_{V_\alpha} U'_\alpha$ is trivial, and we consider

$$\begin{array}{ccc} Y_0 := \coprod Y_\alpha \times G_\alpha & \xrightarrow[\mapsto \sigma . f_\alpha(y)]{y \times \sigma} & U_\alpha \\ \text{left vertical} \downarrow \text{in (I.4)} & & \\ \mathscr{Y} & & \end{array} \tag{I.5}$$

which leads (it's here, *cf.* I.a.5, we use generically scheme like) to a commutative square

$$\begin{array}{ccc} Y'_0 & \xrightarrow[\text{in (I.5)}]{\text{horizontal}} & U' \\ \text{via vertical} \downarrow \text{in (I.5)} & & \downarrow \\ X' & \longrightarrow & X \end{array} \tag{I.6}$$

As such, if we form the groupoids $R := U \times_{\mathscr{X}} U \rightrightarrows U$, and $Y_1 := Y_0 \times_{\mathscr{Y}} Y_0 \rightrightarrows Y_0$ then (I.6) ensures that $\mathscr{Y}' \to X' \hookrightarrow \mathscr{X}$ is equivalent to the composition of functors

$$Y'_1 \to R' = U' \times_X U' \hookrightarrow R \tag{I.7}$$

while by hypothesis Y'_1 is dense in Y_1 and $\mathscr{X}$ is separated, so the simple of expedient of taking the closure in (I.7) defines a functor $Y_1 \to R$. □

This is sufficiently close to optimal as to merit

I.a.5. *Remark.* One cannot replace X' by a Zariski open sub-champ $\mathscr{X}' \hookrightarrow \mathscr{X}$ in I.a.4. Indeed take $\mathscr{X}$ to be the weighted projective champ $\mathbb{P}(n,n)$, I.c.1, $n > 1$. It's moduli is $\mathbb{P}^1$, so the fibre, $\mathscr{X}'$, over a standard $\mathbb{A}^1$ is an embedded Zariski open. Moreover it's isomorphic to $\mathbb{A}^1 \times \mathrm{B}_{\mu_n}$, so in particular admits a section, and we could try to take $\mathscr{Y} = \mathbb{P}^1$. The gerbe $\mathbb{P}(n,n) \to \mathbb{P}^1$ is, however, non-trivial so the map $\mathscr{Y}' \to \mathscr{X}'$ cannot be extended to $\mathscr{Y} \to \mathscr{X}$ even though it is locally trivial, whence a fortiori without local obstruction. The problem is that if one replaces the moduli X, resp. X', by $\mathscr{X}$, resp. $\mathscr{X}'$, in (I.6) then the diagram needn't 2-commute in a slightly unusual way. Specifically, it's 2-commutative on geometric points, p say, by way of a natural transformation η_p between either possible composition, which, in the specific example, if say U_0, U_∞ are points in the standard affines around 0 and infinity is

$$\eta_p = \begin{cases} 1_p & \text{if } p \in U_0, \\ p^{-1} \xrightarrow{p^{-1/n}} p & \text{if } p \in U_\infty \backslash 0, \end{cases} \tag{I.8}$$

where the latter arrow is to be understood in the presentation (I.32). Plainly, however, $p \to \eta_p$ isn't even continuous for p in $U_\infty \backslash 0$, and (I.6) fails to be 2-commutative.

This can often be combined with

I.a.6. **Fact.** *Let $\mathscr{X}$ be a (connected) normal (or slightly more general uni-branch) excellent Deligne-Mumford champ then there is a unique normal-fold $\mathscr{X}_0$ (slightly more generally uni-branch-fold with the obvious definition of that notion) such that $\mathscr{X} \to \mathscr{X}_0$ is a locally constant gerbe under some finite group B_G.*

Proof. Since $\mathscr{X}$ is excellent and uni-branch one can insist, [EGA-IV.2, 7.6.3], that the atlas $U = \coprod_\alpha U_\alpha$ encountered at the beginning of the proof of I.a.4 consists solely of irreducible (affine) schemes U_α. Now for G_α of *op. cit.* define G'_α as the kernel of the representation $G_\alpha \to \mathrm{Aut}(U_\alpha)$ with G''_α the image, then since $\mathscr{X}$ is uni-branch $\coprod_\alpha U_\alpha \times G'_\alpha$ is a normal (groupoid sense [KM97, 7.1]) U-group scheme of the stabiliser, so for $R := U \times_{\mathscr{X}} U \rightrightarrows U$, there is, *op. cit.* 7.4, a well defined quotient $R \to R''$ where the latter is locally of the form $[U_\alpha/G''_\alpha]$. As such define $\mathscr{X}_0$ to be $[U/R'']$, and observe that all the G'_α are isomorphic. □

Finally another important application of normality. Specifically let U be the spectrum of a Noetherian local ring, A, with closed point x, and $j : U' \to U$ a Zariski open whose complement is defined by a regular sequence of length at least 2. As such, for $n \in \mathbb{N}$ the Kummer sequence,

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0, \tag{I.9}$$

applied to U and U' combine to afford a short exact sequence

$$0 \to \mathrm{H}^1(U, \mu_n) \to \mathrm{H}^1(U', \mu_n) \to \mathrm{Pic}(U')[n] \to 0 \tag{I.10}$$

In particular therefore, if A is strictly Henselian and $n^{-1} \in A$,

$$\mathrm{H}^1(U', \mu_n) \xrightarrow{\sim} \mathrm{Pic}(U')[n] \tag{I.11}$$

Now in the particular case that A is normal excellent we can take U' to be the regular locus, and identify (primitive) generators of the right hand side of (I.11) with $\mathbb{Q}$-Cartier divisors, L, on U of index $n = n(x)$, *i.e.* a Weil divisor, L, on U such that nL, but no smaller multiple, mL, $1 \le m < n$, is a line bundle, while the elements of order n on the left are just μ_n-torsors $V' \to U'$ of order exactly n, and we assert

I.a.7. **Fact/Definition.** For a $\mathbb{Q}$-Cartier divisor, L, of index n on a normal strictly Henselian U over which n is invertible, the associated *index 1-cover*, $V \to U$, is the integral closure of U in the corresponding μ_n-torsor $V' \to U'$. By construction $L \mid V$ is the trivial bundle, and, in a sense, universally so, *i.e.* if $W \to U$ is any finite map from a normal scheme W every component of which is dominant such that $L \mid W$ is trivial then it factors uniquely as $W \to V \to U$. In particular if $\Delta \to U$ is the strict Henselisation of some (scheme) point u of U of index $m|n$ then the normalisation, N, of $V \times_U \Delta$ is the trivial $\mu_{\frac{n}{m}}$-torsor over the index 1-cover, M, of Δ.

Proof. It remains to address the universal property, wherein, without loss of generality W is connected. As such all of U, V, W are the spectra of normal Henselian local rings, so they are all domains, while the function field of V over that of U is Galois by construction, so the factorisation is unique if it exists. Now let W' be the fibre over U' then by (I.11) the μ_n-torsor $W' \times_{U'} V'$ has a section, which gives the factorisation $W' \to V' \to U'$, and since everything is S_2 the simple expedient of taking global functions on these opens gives $W \to V \to U$. Applying this to the in particular: there is a map from N to M, while $V' \times_{U'} \Delta$ is a Zariski dense open of the former which is the trivial $\mu_{\frac{n}{m}}$ torsor over the pre-image of U' in the latter. □

In the category of spaces it's rather rare that index 1-covers can be glued whereas:

I.a.8. **Fact.** *Let L be a $\mathbb{Q}$-Cartier divisor on an excellent normal Deligne-Mumford champ $\mathscr{X}$ then there is a finite map, $f : \mathscr{Y} \to \mathscr{X}$, from a normal Deligne-Mumford such that f^*L is Cartier enjoying the following universal property: if $g : \mathscr{Z} \to \mathscr{X}$ is a finite map from a normal champ such that g^*L is Cartier, then there is a 2-commutative factorisation*

(I.12)
$$\begin{array}{ccccc} & & \mathscr{Y} & & \\ & {\scriptstyle h}\nearrow & \xi \Uparrow & \searrow{\scriptstyle f} & \\ \mathscr{Z} & & \xrightarrow[g]{\qquad\qquad} & & \mathscr{X} \end{array}$$

such that for any other factorisation, $\bar{\xi} : g \Rightarrow f\bar{h}$ there is a unique $\theta : h \Rightarrow \bar{h}$ for which $(g_*\theta)\xi = \bar{\xi}$.

Proof. For every closed point x of $\mathscr{X}$ let $n(x)$ be the index of L at x, and $U_x \to \mathscr{X}$ a sufficiently small étale neighbourhood such that the index 1-cover $V_x \to U_x$ of I.a.7 is well defined, with U'_x, V'_x as per *op. cit.*. Now, for $U = \coprod_x U_x$, we can without loss of generality suppose that $\mathscr{X}$ is the classifying champ of the étale groupoid $R_0 := U \times_{\mathscr{X}} U \rightrightarrows U$, and that $U' := \coprod_x U'_x$ is the locus where U is not regular. As such, the restriction, $R'_0 \rightrightarrows U'$ is a dense Zariski open of R_0 equivalent to the restrictions $R' \rightrightarrows V' := \coprod V'_x$, where $R' \to R'_0$ is both étale and finite, and we define $R \rightrightarrows V$ to be the integral closure of R_0 in R'. Consequently from the commutative diagram of fibre squares

(I.13)
$$\begin{array}{ccccc} R_0 & \longleftarrow & R'_0 & \longleftarrow & R' \\ \downarrow & & \downarrow & & \downarrow \\ U \times U & \longleftarrow & U' \times U' & \longleftarrow & V' \times V' \end{array}$$

and $V \times V \to U \times U$ finite, $R \rightrightarrows V$ defines a groupoid which by the in particular in I.a.7 has étale source and sink.

Now let $g : \mathscr{Z} \to \mathscr{X}$ be given, then, up to equivalence, we can identify this with a functor of groupoids, $g : W^1 \to R_0$, where $W^1 = W \times_{\mathscr{Z}} W \rightrightarrows W$ for some étale cover $W \to \mathscr{Z}$ finer than the pre-image of U. By I.a.7, $W \to U$ factors (uniquely) through V affording a (unique) map,

(I.14)
$$h_1 : W^1 \to R_0 \times_{U \times U} V \times V$$

and R is the normalisation of the latter, while every local ring of W^1 is finite over $U \times U$ so this actually factors as a functor (because everything is unique) $h : W^1 \to R$. As such we get a unique strictly commutative factorisation $g = hf$ given $W \to U$. This supposes, however, that all of $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ were the classifying champ of the said groupoids, whereas they may be no better than equivalent to such, and whence the uniqueness statement (I.12). $\square$

In the same vein one has

I.a.9. **Fact/Definition.** Let $\mathscr{D} \hookrightarrow \mathscr{X}$ be an effective Cartier divisor on a normal champ $\mathscr{X}$. As such for a sufficiently fine atlas $U \to \mathscr{X}$ we may identify $\mathscr{X}$ with the classifier of a groupoid $(s,t): R_0 \rightrightarrows U$ and suppose that $\mathscr{D}|_U$ is defined by $z = 0$ where $s^*z = gt^*z$ for some co-cycle $g: R_0 \to \mathbb{G}_m$. Now for $n \in \mathbb{N}$ invertible in every local ring of $\mathscr{X}$ define a groupoid with objects

$$\text{normalisation of } (T^n = z) \hookrightarrow U \times \mathbb{A}^1. \tag{I.15}$$

and arrows the normalisation, R', of the base change groupoid $R_0 \rightrightarrows V$, *i.e.* the fibre

$$\begin{array}{ccc} R_0' & \longrightarrow & V \times V \\ \downarrow & & \downarrow \\ R_0 & \xrightarrow{s \times t} & U \times U \end{array} \tag{I.16}$$

so that $R' \rightrightarrows V$ is a groupoid because $R_0' \rightrightarrows V$ is, and everything is normal. Equally R' admits the explicit description:

$$\text{normalisation of } (T_1^n = s^*z,\, T_2^n = t^*z) \hookrightarrow R_0 \times \mathbb{A}^2. \tag{I.17}$$

which is the same thing as taking normalised nth roots of s^*z and the (invertible) transition function g. By hypothesis, however, n is everywhere invertible, so $R' \rightrightarrows V$ has étale source and sink, and we define $\mathscr{X}' = [V/R'] \to \mathscr{X}$ to be the (extraction of a) nth root of $\mathscr{D}$. Observe, moreover, that a section of $s: R' \to V$ is a choice of nth root of g, so from the Cěch boundary in (I.9), the class of the fibration $\mathscr{D}' = \mathscr{D} \times_{\mathscr{X}} \mathscr{X}' \to \mathscr{D}$ in B_{μ_n}'s is exactly

$$\mathrm{c}_1(\mathscr{D}) \in \mathrm{H}^2(\mathscr{D}, \mu_n) \tag{I.18}$$

I.b. **Foliation singularities.** This section is largely a summary, for the convenience of the reader of the relevant parts of [MP13]. The one exception to this rule is the concluding digression, I.b.12-I.b.15, on how to avoid the study of boundaries altogether. Our interest is exclusively in *foliations by curves*, *i.e.* if $\mathscr{X}$ is a Deligne-Mumford champ of finite type over a field k (so $\Omega^1_{\mathscr{X}/k}$ is well defined) a torsion free quotient

$$\Omega^1_{\mathscr{X}/k} \to Q \to 0 \tag{I.19}$$

which is rank 1 at every generic point. Arguably this is not the right definition in positive or mixed characteristic since in such situations (I.19) is not likely to be locally integrable in any meaningful sense. Fortunately we never have to worry about this, so we proceed directly from (I.19) to

I.b.1. **Definition.** If $\mathscr{X}$ is normal and the double dual $Q^{\vee\vee}$ is a bundle, resp. a $\mathbb{Q}$-Cartier divisor, then we say that the foliation, $\mathscr{F}$, is Gorenstein, resp. $\mathbb{Q}$-Gorenstein, or possibly foliated Gorenstein, resp. foliated $\mathbb{Q}$-Gorenstein, if there is any danger (which there won't be) of confusion. In either case, and indeed even if $\mathscr{X}$ were only normal, we write $K_{\mathscr{F}}$ instead of $Q^{\vee\vee}$, so that in the Gorenstein case there is an ideal I_Z supported in the co-dimension 2 (schematic) singular locus Z such that

$$Q = K_{\mathscr{F}} \cdot I_Z \hookrightarrow Q^{\vee\vee} = K_{\mathscr{F}} \tag{I.20}$$

As such, even in the analytic topology, the classifying champ, $[\mathscr{X}/\mathscr{F}]$ may have no sense, albeit analytically (and with probability zero in any algebraic topology) $[\mathscr{X}\backslash Z/\mathscr{F}]$ has sense. Nevertheless to better convey the idea we write

$$\mathscr{X} \to [\mathscr{X}/\mathscr{F}] \tag{I.21}$$

as a short hand for I.19, and $\Omega^1_{\mathscr{X}/\mathscr{F}}$ for the kernel in *op. cit.*.

Unfortunately it's not technically correct to view a quasi-projective variety as a proper champ with infinite monodromy on the boundary, so we make

I.b.2. *Remark.* All of this is equally valid for champs with boundary, *i.e.* a couple $(\mathscr{X}, \mathscr{D})$, for $\mathscr{D} \hookrightarrow \mathscr{X}$ a reduced Weil divisor. Usually there'll be some further regularity, *e.g.* $\mathscr{X}$ and $\mathscr{D}$ smooth over k, but all that's a priori required is that we can give a sense to the sheaf $\Omega^1_{\mathscr{X}}(\log \mathscr{D})$, so, $\mathscr{X}$ normal is sufficient. In any case, it therefore follows that the canonical bundle of the foliation $\mathscr{F}$ may have competing definitions according as to whether a boundary is involved, $K_{\mathscr{F}}$, or not, $K_{\mathscr{F}}^{\text{nolog}}$. These are related by,

$$K_{\mathscr{F}} = K_{\mathscr{F}}^{\text{nolog}} + \sum_i \epsilon(\mathscr{D}_i)\mathscr{D}_i \tag{I.22}$$

where $\mathscr{D}_i$ are the irreducible components of $\mathscr{D}$, and for W a Weil divisor

$$\epsilon(W) = \begin{cases} 0 & \text{if } W \text{ is } \mathscr{F} \text{ (the sense of I.b.1) invariant,} \\ 1 & \text{otherwise.} \end{cases} \tag{I.23}$$

Similarly there may also be competing definitions of invariant according as to whether this is understood for a saturated sub-sheaf of $\mathscr{T}_{\mathscr{X}}$ or $\mathscr{T}_{\mathscr{X}}(-\log \mathscr{D})$ so that should there be any risk of confusion the former, equiavelently, I.b.1 will, following [MP13, I.i.2], be refered to as *strictly invariant*. Regardless, almost always our boundary will be empty, but when it isn't: $K_{\mathscr{F}}$ will, as suggested by (I.22), be reserved for the canonical with log-poles since this is more natural and the resulting formulae are cleaner.

A case in point is the following cut and paste of [MP13, I.ii.1]

I.b.3. **Definition.** Let $(U, D, \mathscr{F})$ be an irreducible local germ of a $\mathbb{Q}$-Gorenstein foliated logarithmic geometrically normal k-variety, *i.e.* the germ about the generic point of a sub-variety Y of a geometrically normal variety such that the log canonical bundle $K_{\mathscr{F}}$ is a $\mathbb{Q}$-divisor, then for v a divisorial valuation of $k(U)$ centred on Y the *log discrepancy*, $a_{\mathscr{F}}(v)$ is defined as follows:

By hypothesis there is a normal modification $\pi : \tilde{U} \to U$ of finite type, together with a divisor E on $\tilde{U}$ such that $\mathscr{O}_{\tilde{U},E}$ is the valuation ring of v. In particular, bearing in mind (I.22), there is an induced foliation $\tilde{\mathscr{F}}$ with log canonical bundle $K_{\tilde{\mathscr{F}}}$, *i.e.* whose dual is saturated in $T_{\tilde{U}}(-\log E)$. Thus there is a unique integer $a_{\mathscr{F}}(v)$ such that

$$K_{\tilde{\mathscr{F}}} = \pi^* K_{\mathscr{F}} + a_{\mathscr{F}}(v)E \tag{I.24}$$

and for ϵ as in (I.23) we say that the local germ $(U, D, \mathscr{F})$ is,

(I.25)

(1) Terminal if $a_{\mathscr{F}}(v) > \epsilon(v)$.
(2) Canonical if $a_{\mathscr{F}}(v) \geq \epsilon(v)$.
(3) Log-Terminal if $a_{\mathscr{F}}(v) > 0$.
(4) Log-canonical if $a_{\mathscr{F}}(v) \geq 0$.

Where the slightly unsettling shift of the definitions by $\epsilon(v)$ occurs as a result of the convention adopted in I.b.2 together with their correct functorial interpretation.

In contrast to this functorial framework, there is a "competing" local notion of what ought to be a good class of foliation singularities, *viz*:

I.b.4. **Set Up.** Let ∂ be a singular derivation of a local ring, $\mathscr{O}$, with residue field k. Thus, by definition, if $\mathfrak{m}$ is the maximal ideal of $\mathscr{O}$, $\partial : \mathscr{O} \to \mathfrak{m}$ and

$$\bar{\partial} : \frac{\mathfrak{m}}{\mathfrak{m}^2} \to \frac{\mathfrak{m}}{\mathfrak{m}^2} : x \mapsto \partial(x) \tag{I.26}$$

is k-linear by Leibniz's rule.

The relation between the linearisation (I.26) and (I.25) is as good as possible

I.b.5. **Revision.** [MP13, I.ii.3]. A Gorenstein foliation over the complex numbers is log-canonical iff every point is either smooth, or, its linearisation, (I.26), is non-nilpotent.

Better still, one can always reduce to the Gorenstein case thanks to the specifics of one dimensional leaves, *i.e.*

I.b.6. **Revision.** Let $(V, \tilde{D}, \tilde{\mathscr{F}}) \to (U, D, \mathscr{F})$ be the index 1-cover of the germ in I.b.3 associated to the log-canonical bundle $K_{\mathscr{F}}$ in the sense of I.a.7, or, more generally an almost étale map, then for any (n) in (I.25), $1 \leq n \leq 4$, $(U, D, \mathscr{F})$ is (n) iff $(V, \tilde{D}, \tilde{\mathscr{F}})$ is.

Proof. The easy ones are $n = 4$, [MP13, I.ii.5], and the if direction for $1 \leq n \leq 3$, [MP13, III.i.5], which also covers the subtler converse. □

Manifestly, therefore,

I.b.7. **Fact/Definition.** Let $\mathscr{X} \backslash \mathscr{D} \to [\mathscr{X} \backslash \mathscr{D} / \mathscr{F}]$ be a $\mathbb{Q}$-foliated Gorenstein logarithmic champs, then the index 1-cover, $\pi : \tilde{\mathscr{X}} \to \mathscr{X}$, defined by the log-canonical divisor $K_{\mathscr{F}}$, I.a.8, will be referred to as the Gorenstein covering champ. The map π is étale in co-dimension 2; there is an identity $K_{\tilde{\mathscr{F}}} = \pi^* K_{\mathscr{F}}$ of log-canonical divisors; $\tilde{\mathscr{X}} \backslash \tilde{\mathscr{D}} \to [\tilde{\mathscr{X}} \backslash \tilde{\mathscr{D}} / \tilde{\mathscr{F}}]$ is Gorenstein; and the cover enjoys (n), $1 \leq n \leq 4$, of (I.25) iff $\mathscr{X} \backslash \mathscr{D} \to [\mathscr{X} \backslash \mathscr{D} / \mathscr{F}]$ does.

As such, we work almost exclusively with Gorenstein foliations. Similarly the already small difference between log-canonical and canonical becomes close to irrelevant for minimal model theory, *i.e.*

I.b.8. **Definition.** Let $(U, D, \mathscr{F})$ be a germ of a normal foliated Gorenstein log-variety about a point p such that a generator (in the sense of I.b.1 vanishes along a sub-variety Y then a singularity is called *radial* iff after completion in the maximal ideal we can find a generator of the foliation of the form,

$$\partial = n_1 x_1 \frac{\partial}{\partial x_1} + \ldots + n_r x_r \frac{\partial}{\partial x_r} + \delta \tag{I.27}$$

where $x_i = 0$ defining Y are linearly independent modulo $\mathfrak{m}_{U,p}^2$, $n_i \in \mathbb{N}$, and $\delta \in \mathrm{Der}(K, I_Y)$ for some quasi-coefficient field K. In particular for U smooth: D is strictly invariant, I.b.2, iff $\mathrm{codim}(Y) = r \geq 2$.

By way of clarification let us make

I.b.9. *Remark.* This isn't quite a cut and paste from [MP13], since *op. cit.* III.i.2 insists that Y of I.b.8 has co-dimension at least 2, which, although entirely a question of convention, isn't right for doing minimal model theory. In particular, therefore, when Y has co-dimension 1, *e.g.* I.b.10.(c), $D = Y$.

Irrespectively, the above definition of a radial singularity shouldn't be confused with the closely related notion of a radial foliation I.d.2, and in any case the important point is,

I.b.10. **Revision.** [MP13, III.i.3]. For $(U, D, \mathscr{F})$ a germ of a normal foliated Gorenstein variety over a field k of characteristic 0 the following are equivalent,

(a) The singularity is radial.
(b) The singularity is log-canonical but not canonical.
(c) Y is the centre of a divisorial valuation of $k(U)$ of (log)-discrepancy zero and divisor, *cf.* I.b.9, not strictly invariant.

From which it follows that the passage from log-canonical to canonical is exactly

I.b.11. **Revision.** [MP13, III.ii.2]. If $\mathscr{X}\backslash\mathscr{D} \to [\mathscr{X}\backslash\mathscr{D}/\mathscr{F}]$ is a foliated smooth champ over a field of characteristic zero which has log-canonical but not canonical singularities then every component of sing$(\mathscr{F})$ where this occurs is smooth, and there is a smoothed weighted blow up, [MP13, I.iv.3], in each of which such that the induced log-foliation on the resulting bi-rational modification $\tilde{\mathscr{X}} \to \mathscr{X}$ has everywhere canonical logarithmic foliation singularities, which amounts to the rather strong: at every point of the exceptional divisor, $\mathscr{E}$, the induced foliation is smooth and every where transverse to $\mathscr{E}$.

Such attention to the details of the logarithmic case notwithstanding our ultimate intention is to work almost exclusively with an empty boundary. In order to do this we introduce

I.b.12. **Definition.** A foliated space with orbifold boundary is a triple $(U, \Delta, \mathscr{F})$, where $U \to [U/\mathscr{F}]$ is a foliation in the sense of I.b.1 and Δ, is a formal linear combination $\sum_i a_i \Delta_i$ of effective Weil divisors, where $a_i = 1 - n_i^{-1}$ for some positive integers $n_i < \infty$; and we say (slightly contrary to standard usage) that $(U, \Delta, \mathscr{F})$ is $\mathbb{Q}$-Gorenstein if $U \to [U/\mathscr{F}]$ is and each Δ_i is $\mathbb{Q}$-Cartier. Moreover if D is the Weil divisor $\sum_i \Delta_i$, then the discrepancy, $a^{\Delta}_{\mathscr{F}}(v)$, of $(U, \Delta, \mathscr{F})$ along a divisorial valuation v is defined to be

$$a^{\Delta}_{\mathscr{F}}(v) := a_{\mathscr{F}}(v) \; - \; \sum_i \epsilon(\Delta_i) m_i (1 - a_i) \tag{I.28}$$

where $a_{\mathscr{F}}(v)$ are the logarithmic discrepancies, (I.24), of the foliated log-variety $(U, D, \mathscr{F})$; ϵ is as (I.23); and m_i are the multiplicities of the Δ_i along the exceptional divisor E encountered in I.b.3. As such, we then say that $(U, \Delta, \mathscr{F})$ satisfies the corresponding properties (I.25) if the respective inequalities hold for $a^{\Delta}_{\mathscr{F}}(v)$ rather than $\alpha_{\mathscr{F}}(v)$.

The introduction of such orbifold boundaries is very much temporary since

I.b.13. **Revision.** [MP13, III.i.1]. Let $(U, D, \mathscr{F})$ be a foliated germ of a smooth log-variety supported at Z then the following are equivalent,

(1) $(U, D, \mathscr{F})$ is terminal.
(2) $(U, D, \mathscr{F})$ is log-terminal.
(3) D is strictly (*i.e.* in the sense of I.b.1) invariant and $\mathscr{F}$ is smooth transverse to the generic point of Z.

which in turn affords

I.b.14. **Corollary.** Let $(U, \Delta, \mathscr{F})$ be a germ of a log-canonical foliation singularity with $\mathscr{F}$-Gorenstein and non-empty orbifold boundary every component, Δ_i, of which is Cartier, then in fact it's canonical, and exactly one of the following holds

(1) Not only $(U, \mathscr{F})$ but also $(U, \Delta, \mathscr{F})$ is terminal while the non-invariant part of Δ has multiplicity 1 and is everywhere transverse to $\mathscr{F}$.

(2) $(U, \mathscr{F})$, but not $(U, \Delta, \mathscr{F})$, is terminal, the weight of every non-invariant component of Δ (of which there are at most 2) is 1/2, and the non-invariant part of D is defined by a single equation f of multiplicity 2 such that for a local generator, ∂, of the foliation $\partial^2(f)$ is a unit.

(3) As per item (2) except that f has multiplicity 1 and enjoys a simple tangency with $\mathscr{F}$, *i.e.* $\partial^2(f)$ is again a unit.

Proof. From (I.24) and (I.28), the singularity $(U, \mathscr{F})$ without boundary is log-terminal, while it is Gorenstein by hypothesis. Thus by I.b.13 it is defined by a no-where vanishing vector field ∂, and, [MP13, III.i.1] every valuation, v, centred on the singularity has $\epsilon(v) = 0$. In particular, therefore, $(U, \Delta, \mathscr{F})$ is always canonical, and it's terminal iff it's log-terminal.

Now, supposing, without loss of generality, that no component, Δ_i, is invariant consider the effect of blowing up in the maximal ideal of the germ. The discrepancy of $(U, \mathscr{F})$ is 1, so the only way for the multiplicity of D to be more than 1 is if it's 2 and all the weights $a_i = 1/2$. In this latter case the initial modification of $(U, \Delta, \mathscr{F})$ is, therefore, crepant, so the proper transform must itself be log-canonical, and whence the proper transform of D must only cut the exceptional divisor in smooth points of the induced foliation, *i.e.* $\partial^2(f)$ is a unit for f of multiplicity 2 defining D. To see that such a singularity is indeed canonical observe (proof of [MP13, III.i.1]) that in the local ring, R, of a divisorial valuation v, we can write

$$\partial = \pi^{-m}\tilde{\partial},\ f = \pi^n \tilde{f},\ \tilde{\partial}(\pi) = 0,\ v(\pi) = 1,\ m, n \in \mathbb{N} \tag{I.29}$$

for $\tilde{\partial}$ a derivation of R. As such,

$$\epsilon(v) = 0 = v(\partial^2 f) = (n - 2m) + v(\tilde{\partial}^2(\tilde{f})) \geq n - 2m \tag{I.30}$$

which is exactly the canonical condition.

Alternatively, therefore, the multiplicity of D is exactly 1, and if it's not everywhere transverse to the induced foliation then the proper transform of D must cut the exceptional divisor in the singular locus of the transformed foliation, and a blow up in this (singular) locus affords a valuation of negative discrepancy unless the weight is 1/2. As such, we're in case (1) of I.b.14 or most of case (3), *i.e.* it remains to prove that the tangency is simple. Observe, however, that D cuts the exceptional divisor in a smooth invariant sub-space, and blowing up in this not only yields a second exceptional divisor along which the discrepancy is zero, but separates the proper transform of D from the proper transform of the initial exceptional divisor. Consequently, if the tangency weren't simple, the doubly transformed D would contain an invariant subspace of the induced foliation in the second exceptional divisor, and a blow up in this would afford a valuation of negative discrepancy. Conversely a simple tangency with weight 1/2 is canonical for the same reason as (I.29)-(I.30), while an everywhere transverse divisor of any weight is log-terminal because the "weight 1 case", *i.e.* $r = 1$ in (I.27) is, I.b.10, log-canonical. □

This can be applied to reduce to an empty boundary in the obvious way, to wit:

I.b.15. **Construction.** Suppose $(U, \Delta, \mathscr{F})$ is a $\mathbb{Q}$-Gorenstein log-canonical foliated germ with orbifold boundary, with no boundary component invariant. Then composing the index 1-covers associated to $\mathscr{F}$ and the boundary components Δ_i, we find a foliated germ with orbifold boundary $(U', \Delta', \mathscr{F}')$ satisfying the hypothesis of I.b.14 such that $U' \to U$ is almost étale. By *op. cit.* and [MP13, III.i.1], the proof of [MP13, III.i.5] goes through verbatim, and the obvious variant of I.b.6 holds, *i.e.* for any (n) in (I.25), $1 \leq n \leq 4$, $(U, \Delta, \mathscr{F})$ is (n) iff $(U', \Delta', \mathscr{F}')$ is. Ignoring, for the sake of argument, the cases (2) and (3) of I.b.14, the latter boundary is, in the presence of log-canonical singularities an everywhere transverse Cartier divisor of multiplicity 1 together with a weight $1 - n^{-1}$. As such if $f = 0$ is a local equation for Δ' then we could extract a nth root $\pi : V \to U'$ to obtain a Gorenstein foliation $V \to [V/\tilde{\mathscr{F}}]$ such that,

$$K_{\tilde{\mathscr{F}}} = \pi^*(K_{\mathscr{F}} + \Delta) \tag{I.31}$$

and again the obvious variant of I.b.6 holds- for any (n) in (I.25), $1 \leq n \leq 4$, $(V, \tilde{\mathscr{F}})$ is (n) iff $(U', \Delta', \mathscr{F}')$ is- for exactly the same reason as above. Plainly all such local constructions will glue as champs by much the same argument as I.b.7, so all this is just the obvious fact that minimal model theory for foliations with orbifold boundary can be deduced from the minimal model theory of champs without boundary. The slightly subtler point, however, is that if one were to begin with a foliated champ $\mathscr{X} \backslash \mathscr{D} \to [\mathscr{X} \backslash \mathscr{D}/\mathscr{F}]$ with (integral) boundary, then extracting a $n(> 2)$th root, $\mathscr{X}_n \to \mathscr{X}$ of $\mathscr{D}$ yields a foliation $\mathscr{X}_n \to [\mathscr{X}_n/\mathscr{F}_n]$ which has log-canonical singularities iff $\mathscr{X} \backslash \mathscr{D} \to [\mathscr{X} \backslash \mathscr{D}/\mathscr{F}]$ does, so that not only the minimal model theory for foliations with orbifold boundary, but also with integral boundary, §.IV.f, can be deduced from the champs theorem without boundary.

I.c. **Weighted projective champs.** All of this section works in arbitrary generality, so over a base, say Spec(k), where k is a *ring*, with the object of interest being

I.c.1. **Definition.** For $\underline{a} = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}_{>0}$, $n > 0$, let $A_k := \mathbb{A}^{n+1}_k \backslash 0$ then by the weighted projective champ $\mathbb{P}(a_0, \dots, a_n)$, or just $\mathbb{P}(\underline{a})$, is to be understood the classifying champ $[A_k/\mathbb{G}_{m,k}]$ of the action,

$$R_k := \mathbb{G}_{m,k} \times A_k \rightrightarrows A_k : (x_0, \dots, x_n) \leftarrow\!\shortmid \lambda \times (x_0, \dots, x_n) \mapsto x^\lambda := (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n) \tag{I.32}$$

Just like any quotient space under a group there is a tautological torsor, *i.e.* $A_k \times \mathbb{G}_m$ with $\mathbb{G}_m$ action

$$\mathbb{G}_m \times \big(A_k \times \mathbb{G}_m\big) : \lambda \times (x \times z) \mapsto x^\lambda \times (\lambda z) \tag{I.33}$$

which one extends to a line bundle in the usual way, to wit:

I.c.2. **Fact/Definition.** Choose an embedding $\mathbb{G}_m \hookrightarrow \mathbb{G}_a : z \mapsto z$, then by the tautological line bundle, $\mathscr{O}(1)$, on $\mathbb{P}(\underline{a})$ is to be understood the line bundle $\mathbb{G}_a \times A_k$ with $\mathbb{G}_m$ action given by (I.33) and our aforesaid choice of embeddings. In particular, therefore, we've defined $\mathbb{V}(\mathscr{O}(1)) \mid_{A_k}$- EGA notation- whence as an equivariant $\mathscr{O}_{A_k}$-module $\mathscr{O}(1)$ has generator T where

$$T^\lambda = \lambda^{-1} T \tag{I.34}$$

so that the bundle $\omega_{A_k/k}$ of volume forms on A_k descends to the bundle $\omega := \mathscr{O}(-a_0 - \ldots - a_n)$ on $\mathbb{P}(\underline{a})$.

Unsurprisingly Serre's explicit calculation generalises to:

I.c.3. **Fact.** *The bundle $\mathscr{O}(1)$ freely generates the Picard group of $\mathbb{P}(\underline{a})$; there are, for $p \geq 0$, canonical (dual) isomorphisms of free k-modules*

$$\begin{aligned} \mathrm{H}^0(\mathbb{P}(\underline{a}), \mathscr{O}(p)) = S_p &:= \coprod_{p_0 a_0 + \cdots p_n a_n = p} k \cdot x_0^{p_0 a_0} \cdots x_n^{p_n a_n} \\ \mathrm{H}^n(\mathbb{P}(\underline{a}), \omega(-p)) = S'_p &:= \coprod_{p_0 a_0 + \cdots p_n a_n = p} k \cdot \frac{dx_0 \cdots dx_n}{x_0 \cdots x_n} \cdot x_0^{-p_0 a_0} \cdots x_n^{-p_n a_n} \end{aligned} \tag{I.35}$$

and any other co-homology of any other line bundle in any degree vanishes.

Proof. The Picard group of A_k is trivial, so a line bundle on $\mathbb{P}(\underline{a})$ is the same thing as a map $\phi : R_k \to \mathbb{G}_m$ from the groupoid (I.32) satisfying the co-cycle condition $\phi(gf) = \phi(g)\phi(f)$. There are, however, no (algebraic) maps from A_k to $\mathbb{G}_m$, so all such co-cycles are integer multiples of the tautological one. As to the second part: if $\pi : A_k \to \mathbb{P}(\underline{a})$ is the projection then for any sheaf $\mathscr{F}$ on A_k the Leray spectral sequence reads

$$\mathrm{H}^i(\mathbb{P}(\underline{a}), R^j \pi_* \mathscr{F}) \Rightarrow \mathrm{H}^{i+j}(A_k, \mathscr{F}) \tag{I.36}$$

Now the co-homology of the right hand side of (I.36) is known, *i.e.* there are canonical dual, [SGA-II, Exposé IV.5.5], isomorphisms

$$\mathrm{H}^0(A_k, \mathscr{O}_{A_k}) = \coprod_p S_p, \quad \mathrm{H}^n(A_k, \omega_{A_k/k}) = \coprod_p S'_p \tag{I.37}$$

while on the left hand side there are canonical isomorphisms

$$\pi_* \mathscr{O}_{A_k} \xrightarrow{\sim} \coprod_{q \in \mathbb{Z}} \mathscr{O}(q), \quad \pi_* \omega_{A_k/k} \xrightarrow{\sim} \coprod_{q \in \mathbb{Z}} \omega(q) \tag{I.38}$$

and all higher direct images in (I.36) vanish, whence (I.35) by identifying the weight of the action of $\mathbb{G}_m$ in the equivariant isomorphism between (I.37) and (I.38) afforded by (I.36). □

In addition the bundle ω is the bundle of volume forms on $\mathbb{P}(\underline{a})$ when this has sense, *i.e.*

I.c.4. **Claim.** The moduli of any $\mathbb{P}(\underline{a})$ is projective, in fact better there is a finite flat map

$$\mathbb{P}^n_k \to \mathbb{P}(\underline{a}) : [x_0, \dots, x_n] \mapsto [x_0^{a_0}, \dots, x_n^{a_n}] \tag{I.39}$$

and $\mathbb{P}(\underline{a})$ is Deligne-Mumford iff all the a_i are invertible in k. In addition the coordinate functions, $\partial_i = \frac{\partial}{\partial x_i}$, afford a $\mathbb{G}_m$ equivariant isomorphism

$$\coprod_i \mathscr{O}(a_i) \xrightarrow[\partial_0+\cdots+\partial_n]{\sim} T_{A_k/k} \tag{I.40}$$

leading to the Euler sequence of $\mathbb{G}_m$-modules on A_k, equivalently bundles on $\mathbb{P}(\underline{a})$

$$0 \to T_{\mathbb{G}_m} \xrightarrow{\sim} \mathscr{O} \xrightarrow{a_i x_i \partial_i} \coprod_i \mathscr{O}(a_i) \to \pi^* T_{\mathbb{P}(\underline{a})/k} \to 0 \tag{I.41}$$

whenever $\mathbb{P}(\underline{a})$ is Deligne-Mumford, so in particular

$$\Lambda^n \Omega_{\mathbb{P}(\underline{a})/k} \xrightarrow{\sim} \omega \tag{I.42}$$

Proof. The functor $\lambda \times x_i \mapsto \lambda \times x_i^{a_i}$ of the corresponding groupoids in (I.32) yields (I.39), while the stabiliser of the point with all but the ith coordinate 0 is μ_{a_i} so the Deligne-Mumford criteria is plainly necessary, and, similarly it is sufficient since slicing (I.32) along $x_i = 1$ covers $\mathbb{P}(\underline{a})$ by affines with μ_{a_i}-action. The rest just amounts to λ acting on ∂_i by λ^{-a_i}. □

The triviality of (I.39) notwithstanding we have

I.c.5. **Corollary.** If k is simply connected, then every $\mathbb{P}_k(\underline{a})$ is simply connected, *i.e.* irrespectively of any Deligne-Mumford criteria, there are no non-trivial Γ-torsors over $\mathbb{P}(\underline{a})$ for every finite group Γ.

Proof. By hypothesis $\mathbb{P}^n_k$ is simply connected, so it's sufficient by (I.39) to prove vanishing of a suitable Cěch group, *i.e.* that the groupoid

$$R := \mathbb{P}^n_k \times_{\mathbb{P}(\underline{a})} \mathbb{P}^n_k \rightrightarrows \mathbb{P}^n_k \tag{I.43}$$

doesn't admit any non-trivial functors to Γ. The space R may, by I.32, be expressed as the classifier of the $\mathbb{G}_m$ action $(x_i, y_i) \mapsto (\lambda x_i, \lambda y_i)$ on the product of affine curves

$$(x_i)^{a_i} = (y_i)^{a_i} \subset \mathbb{A}^2_k \tag{I.44}$$

complemented in 0×0. Now the curves in (I.44) are geometrically connected, so their product is connected. It's also l.c.i. of dimension at least 2, so it's homotopy depth is at least 2, whence the complement in 0 of the product is connected, and we're done a fortiori- the fact that projections in (I.44) are the source and sink in (I.43) isn't even needed. □

Of which we will require the following variant

I.c.6. **Corollary.** If k is simply connected, and $\pi : \mathscr{P} \to \mathbb{P}_k(\underline{a})$ is a fibration in locally constant gerbes B_G for some finite group G such that $\mathscr{P}$ is simply connected, then G is a cyclic group of order a (invertible in k) and $\mathscr{P} \xrightarrow{\sim} \mathbb{P}_k(a\underline{a})$ in such a way that π is just $\lambda \mapsto \lambda^a$ in I.c.1.

Proof. The right way to prove this is the long exact sequence of homotopy groups of a fibration, which may be done wholly algebraically [McQ15, III.g]. However, for convenience here is an ad hoc argument.

From I.c.5, $\mathbb{P}_k(\underline{a})$ is simply connected, so by [Gir71, IV.3.4] the locally constant gerbes up to isomorphism in B_G's over $\mathbb{P}_k(\underline{a})$ are canonically isomorphic to

$$\mathrm{H}^2(\mathbb{P}_k(\underline{a}), Z) \tag{I.45}$$

where Z is the centre of G. In particular if $\mathscr{P} \to \mathscr{P}'$ is $\mathscr{P}$ modulo the centre, *cf.* I.a.6, then $\mathscr{P}' \xrightarrow{\sim} \mathbb{P}_k(\underline{a}) \times \mathrm{B}_{G/Z}$, which isn't simply connected. As such, without loss of generality $G = Z$ is abelian, and the Leray spectral for π affords an isomorphism

$$\mathrm{E}_2^{0,1} = \mathrm{End}(Z) \xrightarrow[d_2^{0,1}]{\sim} \mathrm{H}^2(\mathbb{P}_k(\underline{a}), Z) = \mathrm{E}_2^{2,0} \tag{I.46}$$

If, however, p is the characteristic of k then from inductive application of the Artin-Schrier sequence

$$0 \to \mathbb{F}_p \to \mathbb{G}_a \to \mathbb{G}_a \to 0 \tag{I.47}$$

the latter group in (I.46) is the prime to p part of Z, so our initial G is cyclic of some order a prime to p. We have however a fibration,

$$\mathbb{P}_k(a\underline{a}) \to \mathbb{P}_k(\underline{a}) \tag{I.48}$$

in B_{μ_a}'s by the simple expedient of sending λ to λ^a in I.c.1, which is the generator of (I.45). □

Another very important fact which generalises is

I.c.7. **Fact.** *Let $n = 1$ and E a vector bundle on $\mathbb{P}(\alpha)$ then there are unique integers b_j such that (non-canonically)*

$$E \xrightarrow{\sim} \coprod_j \mathscr{O}(b_j) \tag{I.49}$$

Proof. We've done the rank 1 case in I.c.3, and we go by induction on the rank, $r > 1$. The push-forward of E to the moduli of $\mathbb{P}(\underline{a})$ is coherent, so there are plenty of meromorphic sections. As such, choose one of maximal degree to get a short exact sequence of bundles

$$0 \to \mathscr{O}(b_r) \to E \to E'' \to 0 \tag{I.50}$$

Now by the induction hypothesis and I.c.3 this is split unless there is some $b_j > b_r$, $j < r$, such that

$$\mathrm{H}^0(\mathbb{P}(\underline{a}), \mathscr{O}(b_j - b_r - a_0 - a_1)) \neq 0 \tag{I.51}$$

Consequently if we twist (I.50) by $\mathscr{O}(-b_r - a_0)$ then the kernel has no co-homology by I.c.3, while the co-kernel has a direct summand $\mathscr{O}(b_j - b_r - a_0)$ which has a non-trivial section given by tensoring anything in (I.51) with $X_1^{a_1}$, and we contradict the maximality of b_r. □

We've passed over the unicity since

I.c.8. *Remark.* The uniqueness of the integers b_j in (I.49) is just an easy version of the uniqueness of the Harder-Narismhan filtration which, for β_j a complete repetition free list of the b_j ordered by $\beta_1 < \beta_2 < \cdots < \beta_m$ takes the form

$$E = E^0 \supset E^1 = \coprod_{b_j > \beta_1} \mathscr{O}(b_j) \supset \cdots \supset E^{m-1} = \coprod_{b_j > \beta_{m-1}} \mathscr{O}(b_j) \supset E^m = 0 \tag{I.52}$$

I.d. **Radial foliations.** In this section we work over $\mathbb{C}$, and, unfortunately we'll need

I.d.1. *Notation.* The vector $\underline{a} \in \mathbb{Z}_{>0}^{n+1}$ will be written (at least for this section) as the n-tuple of positive integers $(a_0, aa_1, \ldots, aa_n)$, $n \geq 1$, where $a_1, \ldots, a_n$ are relatively prime, and $a \in \mathbb{N}$.

The lack of symmetry in the notation is in the nature of

I.d.2. **Definition.** The radial foliation, $\mathscr{R}$, on $\mathbb{P}(\underline{a})$ is equivalently

(a) The foliation defined by the 0th coordinate $\mathscr{O}(a_0) \to T_{\mathbb{P}(\underline{a})}$ in the Euler sequence (I.41).
(b) The foliation defined by the (rational) projection $\mathbb{P}(\underline{a}) \dashrightarrow \mathbb{P}(aa_1, \ldots, aa_n)$.

In the particular case that $n = 1$ there is a certain ambiguity in the definition according as to whether one saturates (a) at the centre of the projection in (b), albeit, fortunately this tends to be clear according to context.

To which one can add a bunch of properties which will aid in radial foliation recognition

I.d.3. **Facts.** Given a radial foliation $\mathbb{P}(\underline{a}) \to [\mathbb{P}(\underline{a})/\mathscr{R}]$,

(a) It's canonical bundle, $K_{\mathscr{R}}$ (understood logarithmically if $n = 1$) is $\mathscr{O}(a_0)$.
(b) On the étale neighbourhood of the (unique) singular point given by $x_0 = 1$, $x_i = 0$, $i \geq 1$ in (I.32), $\mathscr{R}$ is generated by the vector field $a_1 x_1 \frac{\partial}{\partial x_1} + \cdots a_n x_n \frac{\partial}{\partial x_n}$
(c) The ith coordinate axis in (b) is a smooth embedded $\mathscr{R}$-invariant $\mathbb{P}(a_0, aa_i)$ with $K_{\mathscr{R}}$ degree $-1/aa_i$, while the degree of the generic invariant champ is $-1/a$.
(d) The smoothed weighted blow up, [MP13, I.iv.3], $\mathscr{P} \to \mathbb{P}(\underline{a})$ in the singularity with weights $a_1, \ldots, a_n$ resolves I.d.2.(b). Indeed, *cf.* I.b.11, the induced foliation $\mathscr{P} \to [\mathscr{P}/\tilde{\mathscr{R}}]$ is a bundle of $\mathbb{P}(a_0, a)$'s over a $\mathbb{P}(aa_1, \ldots, aa_n)$, and $K_{\tilde{\mathscr{R}}}(+\mathscr{E}) = K_{\mathscr{R}}$ for $\mathscr{E}$ the exceptional divisor.

Proof. Of these only (d) is meritorious of comment. Specifically smoothed weighted blow ups in [MP13, I.iv.3] are understood to have weights without a common divisor, so in the first place by the formulae of [MP13, pg. 89] and I.a.4, we have a resolution

$$\begin{array}{ccc} \mathscr{P}_0 & \xrightarrow[\rho_0]{} & \mathbb{P}(a_1, \ldots, a_n) \\ {\scriptstyle \pi_0}\big\downarrow {\scriptstyle \text{weighted blow up with weights } a_i} & & \\ \mathbb{P}(a_0, aa_1, \ldots, aa_n) & & \end{array} \tag{I.53}$$

in which the exceptional divisor $\mathscr{E}_0$ is isomorphic to $\mathrm{B}_{\mu_{a_0}} \times \mathbb{P}(a_1, \ldots, a_n)$, and the various bundles are related by

$$\rho_0^* \mathscr{O}_{\mathbb{P}(a_1, \ldots, a_n)}(1) = \pi_0^* \mathscr{O}(a) - \mathscr{E}_0 \tag{I.54}$$

All of which becomes much cleaner if, the common divisor not withstanding, one permits the weights $aa_1, \ldots, aa_n$. This is equivalent to taking an ath root of $\mathscr{E}_0$, so we get a diagram in which the square is fibred

$$\begin{array}{ccccc} \mathbb{P}(a_0, aa_1, \ldots, aa_n) & \xleftarrow[\text{with weights } aa_i]{\text{weighted blow up}} & \mathscr{P} & \xrightarrow[\text{of } \mathscr{E}_0]{\text{extract } a\text{th root}} & \mathscr{P}_0 \\ & & {\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle \rho_0} \\ & & \mathbb{P}(aa_1, \ldots, aa_n) & \xrightarrow[\text{of order } a]{\text{non-trivial gerbe}} & \mathbb{P}(a_1, \ldots, a_n) \end{array} \tag{I.55}$$

by (I.54), *i.e.* the gerbe of the bottom horizontal is the class of $\mathscr{O}(1)$ in $\mathrm{H}^2(\mathbb{P}(a_1, \ldots, a_n), \mu_a)$. In particular, therefore, if $\mathscr{E}$ is the new exceptional divisor then (I.54) becomes

$$\rho^* \mathscr{O}_{\mathbb{P}(aa_1, \ldots, aa_n)}(1) = \pi^* \mathscr{O}(1) - \mathscr{E} \tag{I.56}$$

while the fibres of ρ are identically those of ρ_0. The latter, however, are simply connected since ρ_0 has a section, so, [BN06, 1.1], a local calculation of their non-scheme like points implies that they're all $\mathbb{P}(a_0, a)$'s. $\square$

By way of disambiguation let us present the next proposition in the form

I.d.4. **Fact/Definition.** Every deformation of a radial foliation is locally trivial, *i.e.* if for a (geometrically) pointed scheme $\mathrm{pt} \xrightarrow{s} S$ we have a map $\mathscr{X} \to [\mathscr{X}/\mathscr{F}] \to S$ (equivalently of foliations indexed by the points of S) for which the special fibre $\mathscr{X}_s \to [\mathscr{X}_s/\mathscr{F}_s]$ is a radial foliation, then there is an étale neighbourhood $U \to S$ such that

$$\begin{array}{ccc} \mathscr{X} \times_S U & \xrightarrow{\sim} & \mathscr{X}_s \times U \\ \downarrow & & \downarrow \\ [\mathscr{X} \times_S U/\mathscr{F}] & \xrightarrow{\sim} & [\mathscr{X}_s/\mathscr{F}_s] \times U \end{array} \tag{I.57}$$

commutes, with the horizontal arrows isomorphisms.

Proof. By [Art69] it will suffice to replace S, resp. $\mathscr{X}$, by its completion in s, resp. the fibre, and to prove (I.57) in the formal category- so, keeping the same notation, $U \xrightarrow{\sim} S$. Consequently, if $\mathfrak{m}$ is the ideal of s and $S_n = \mathrm{Spec}(\mathscr{O}_S/\mathfrak{m}^n)$, it will even suffice to prove (I.57) with $U = S_n$, where, by way of notation, $\mathscr{X}_n := \mathscr{X} \times_S S_n$. Proceeding by induction on $n \geq 1$, the case $n = 1$ is given, while [SGA-I, Exposé III.5] applies as written to show that the obstruction to extending an isomorphism from $\mathscr{X}_n$ to $\mathscr{X}_0 \times S_n$ to the $n+1$th thickening lies in

$$\mathrm{H}^1(\mathscr{X}_0, T_{\mathscr{X}_0} \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1}) \tag{I.58}$$

By the Euler sequence, (I.41), and Serre's explicit calculation, (I.c.3), this is zero. As such, we can certainly find an isomorphism $f : \mathscr{X}_n \xrightarrow{\sim} \mathscr{X}_0 \times S_n$, but it may not be foliated, *i.e.* the composition

$$f^*\Omega_{\mathscr{X}_0 \times S_{n+1}/\mathscr{F}} \to \Omega_{\mathscr{X}_{n+1}} \to K_{\mathscr{F}} \otimes \mathscr{O}_{\mathscr{X}_{n+1}} \tag{I.59}$$

may be non-trivial. We have, however, a foliated isomorphism at the nth level, and $\mathscr{X}_0$ is S_2 so (I.59) is, equivalently, a non-trivial map

$$T_{\mathscr{F}} \mid_{\mathscr{X}_0} \xrightarrow{\sim} \mathscr{O}(a_0) \to \mathscr{T}_{\mathscr{X}_0/\mathscr{F}} \tag{I.60}$$

where the normal sheaf to the radial foliation is by (I.41) described by the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathscr{O} & = & \mathscr{O} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{O}(a_0) & \longrightarrow & \coprod_j \mathscr{O}(aa_j) & \longrightarrow & \coprod_{j>0} \mathscr{O}(aa_j) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{O}(a_0) & \longrightarrow & T_{\mathscr{X}_0} & \longrightarrow & T_{\mathscr{X}_0/\mathscr{F}} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{I.61}$$

Twisting by $\mathscr{O}(-a_0)$ an arrow (I.60) is, therefore, a quotient of the space of global sections in the middle of the rightmost column of (I.61), *i.e.* the $\mathbb{C}$-vector space of vector fields with, in the notation of (I.32), basis

$$x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} \cdot \frac{\partial}{\partial x_j}, \quad a_0 i_0 + aa_1 + \cdots aa_n = a_j - a_0 \ j > 0, \ i_k \geq 0 \tag{I.62}$$

On the other hand- [SGA-I, Exposé III.5] again- the possibilities for changing the isomorphism f are a principal homogeneous space under

$$\mathrm{H}^0(\mathscr{X}_0, T_{\mathscr{X}_0} \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1}) \tag{I.63}$$

whose effect on (I.60) is given by the Lie bracket

$$T_{\mathscr{X}_0} \to \mathrm{Hom}(T_{\mathscr{F}}, T_{\mathscr{X}_0/\mathscr{F}}) : D \mapsto [-, D] \tag{I.64}$$

which at the level of global sections has, by explicit calculation, image exactly (I.62), so a suitable twist of f under (I.63) is a foliated isomorphism. □

We will equally need a slight generalisation, to wit:

I.d.5. *Remark.* The same statement is equally true under the hypothesis that the universal cover of $\mathscr{X}_s$ is a radial foliation. Indeed since for π_1 a finite group, all modules in which the cardinality of π_1 is invertible are acyclic, and we're in characteristic zero, so the obstruction (I.58) still vanishes and (I.64) is still surjective on global sections.

I.e. **Net completion.** The entire contents of this section should be standard, but it's not in the EGA's, so we give the details. We begin with the easiest case, *viz:* a local embedding $f : Y \to X$ of (not necessarily separated) schemes. Thus by definition, [EGA-I, 4.2.1 & 4.5.1], for every $y \in Y$ there are (Zariski) open neighbourhoods $Y \supseteq U \ni y$, resp. $X \supseteq V \ni f(y)$, such that

$$f : U \hookrightarrow V \tag{I.65}$$

is a closed embedding. In particular, therefore, we have a short exact sequence

$$0 \to \mathscr{I} \to f^{-1}\mathscr{O}_X \to \mathscr{O}_Y \to 0 \tag{I.66}$$

of sheaves, for some ideal $\mathscr{I}$, and we observe

I.e.1. **Fact/Definition.** For every $n \in \mathbb{Z}_{>0}$, define $\mathscr{O}_{f_n} := f^{-1}\mathscr{O}_X/\mathscr{I}^n$, then the ringed space $Y_n := (Y, \mathscr{O}_{f_n})$ is a scheme.

Proof. The question is local on Y, so, modulo notation we can, (I.65), suppose $f : Y \hookrightarrow X$ is a closed embedding of affines. In particular, therefore, it's defined by a quasi-coherent sheaf of ideals $\mathscr{J}$. As such $\mathscr{O}_{f_n}$ is the sheaf (on Y) associated to the pre-sheaf,

$$U \mapsto \varprojlim_{V \cap Y = U} \Gamma(V, \mathscr{O}_X)/\Gamma(V, \mathscr{J})^n \tag{I.67}$$

This is, however, already not only a sheaf, but the structure sheaf, $\mathscr{O}_X/\mathscr{J}^n$, of the nth thickening of Y in X, so Y_n is a scheme. □

For the avoidance of possibly competing definitions when (without relevance to our current considerations) things fail to be Noetherian or excellent or whatever let us make

I.e.2. **Fact/Definition.** A morphism $f : \mathscr{Y} \to \mathscr{X}$ of Deligne-Mumford champs is net if it is étale locally a closed embedding, *i.e.* for every geometric point y of Y there are étale neighbourhoods $U \to Y$ of y, resp. $V \to X$ of $x = f(y)$, together with a closed embedding $U \hookrightarrow V$ such that

$$\begin{array}{ccc} U & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{f} & X \end{array} \tag{I.68}$$

commutes. Consequently if everything is Noetherian, then f is net iff the strict Henselisation $\mathscr{O}^h_{\mathscr{Y},y}$ is a quotient of $\mathscr{O}^h_{\mathscr{X},x}$ in every point, *cf.* [SGA-I, Exposé I,3.7].

Now suppose $f : Y \to X$ is a net map of algebraic spaces. Replacing X by a suitable (embedded) Zariski open, we may by I.68 find étale covers $U \to Y$, resp. $V \to X$, affording (a not necessarily fibred) square of the form (I.68) in which $U \hookrightarrow V$ is a closed embedding. As such $R_0 := V \times_Y V \rightrightarrows V$, resp. $R := U \times U \rightrightarrows U$ are (not necessarily closed unless Y, resp. X

is separated) embedded in $V \times V$, resp. $U \times U$ so that the induced functor $R_0 \to R$ is a not necessarily closed embedding, and we make

I.e.3. **Fact/Definition.** For every $n \in \mathbb{N}$, $R_n \hookrightarrow R$, resp. $U_n \hookrightarrow V$ is the nth thickening of $R_0 \hookrightarrow R$, resp. $U \hookrightarrow V$, in the sense of I.e.1. In particular $R_n \rightrightarrows U_n$ is an étale equivalence relation, and we define the nth thickening, Y_n, of Y along f to be the quotient U_n/R_n. Consequently if Y is a scheme, then Y_n is too.

Proof. Consider the diagram

$$\begin{array}{ccccc} R_0 & \longrightarrow & R_U & \longrightarrow & R \\ {\scriptstyle s}\downarrow & & \downarrow & & \downarrow{\scriptstyle s} \\ U & = & U & \longrightarrow & V \end{array} \tag{I.69}$$

where the rightmost square is fibred. Thus all the verticals are étale, the rightmost horizontals are closed embeddings, while the composition of the top row is an embedding, so $R_0 \hookrightarrow R$ is an open embedding, and whence the source and sink of $R_n \rightrightarrows U_n$ are étale. Finally for any scheme T, the sets $R(T) \rightrightarrows V(T)$ form an equivalence relation, and we can identify the T-points of R_n with those of R such that the nth power of the ideal of the fibre over R_0 is 0, which since everything is étale implies that $R_n(T) \rightrightarrows U_n(T)$ is an equivalence relation. □

This brings us to a net map, $f : \mathscr{Y} \to \mathscr{X}$, of champs, then proceeding exactly as above, ($U \to \mathscr{X}$, $V \to \mathscr{Y}$ étale covers *etc.*) we find that f is equivalent to a functor $R_0 \xrightarrow{F} R$ between groupoids, which as a map is itself net, and whence

I.e.4. **Fact/Definition.** The nth thickening of $\mathscr{Y}_n$ along f is the classifying champ $[U_n/R_n]$ of the étale groupoid $R_n \rightrightarrows U_n$ where R_n is the nth thickening of R_0 along the functor F, so, inter alia there is a natural map $f_n : \mathscr{Y}_n \to \mathscr{X}$ extending f.

Proof. The fact that $R_n \rightrightarrows U_n$ is an étale groupoid is mutatis mutandis the proof of I.e.3, and the description of the T-points therein also suffices to conclude that f_n exists. Finally, refining the covers U, V as necessary, the definition of $\mathscr{Y}_n$ is, up to equivalence, independent of the given presentation. □

It therefore only remains to make

I.e.5. **Fact/Definition.** The completion, $\mathfrak{Y}$, along a net map $f : Y \to X$ of schemes is the direct limit, $\varinjlim_n Y_n$, in the category of formal schemes of the nth thickenings $f_n : Y_n \to X$ of I.e.3. Similarly the completion, $\hat{\mathscr{Y}}$, along a net map $f : \mathscr{Y} \to \mathscr{X}$ of champ is the classifier of the étale groupoid which is the completion, $\mathfrak{R} \rightrightarrows \mathfrak{U}$, along the net functor $F : R_0 \to R$ of I.e.4. Consequently, by construction, f factors as

$$\mathscr{Y} \hookrightarrow \hat{\mathscr{Y}} \xrightarrow{\hat{f}} \mathscr{X} \tag{I.70}$$

where the former map is an embedding, and the latter is net.

I.f. **Trivial remarks on the analytic topology.** As we've observed in the proof of I.a.4 every separated Deligne-Mumford champ is étale locally the classifier, $[U/G]$, of a (not necessarily faithful) finite group action $G \times U \rightrightarrows U$. An étale neighbourhood is, however, rarely embedded, so this isn't quite as convenient as the corresponding analytic statement, *i.e.*

I.f.1. **Fact.** *If $\mathscr{X}/\mathbb{C}$ is a separated Deligne-Mumford champ of finite type, then for every geometric point, x, there is an étale neighbourhood $x \in \Delta \to \mathscr{X}$ in the analytic topology together with a finite group action $G_x \times \Delta \rightrightarrows \Delta$ of the stabiliser such that $[\Delta/G_x] \hookrightarrow \mathscr{X}$ is an open embedding.*

Proof. From the algebraic statement: the coarse moduli U/G_x is an étale neighbourhood of the moduli $\mu : \mathscr{X} \to X$ such that we have a fibre square

$$\begin{array}{ccc} \mathscr{X} & \longleftarrow & [U/G_x] \\ {\scriptstyle\mu}\downarrow & & \downarrow \\ X & \longleftarrow & U/G_x \end{array} \tag{I.71}$$

There is however an open embedding $\Delta' \hookrightarrow U/G_x$ whose composition with the lower horizontal in (I.71) is an embedding, so $\mu^{-1}(\Delta')$ is embedded in both $\mathscr{X}$ and $[U/G_x]$, while it's pre-image, Δ, in U is both embedded and G_x equivariant. □

We will only ever have to consider smooth champs in the analytic topology, but as it happens, everything works in maximal generality. We require:

I.f.2. **Lemma.** *If X is a reduced complex space then the sheaf, $\mathscr{R}_X$, of real analytic functions on X is coherent.*

Proof. The discussion is local, so we can suppose that X is a closed analytic subset of $U \subset \mathbb{C}^n$ with finitely many irreducible components $X_1, \cdots, X_r$. Each X_i has a conjugate $\bar{X}_i$ and by [Nar66, V, Prop. 8] for any $x \in X_i$ the complexification of X_i at x in the real manifold $\mathbb{R}^n \times \mathbb{R}(1)^n$ is $X_i \times \bar{X}_i$. Consequently, *op. cit.* V, Prop. 1, $\cup_i X_i \times \bar{X}_i$ contains the complexification of X at any $x \in X$; and each X_i is everywhere locally Zariski dense in $X_i \times \bar{X}_i$, so X is everywhere locally Zariski dense in $\cup_i X_i \times \bar{X}_i$. Consequently by *op. cit.*, $\cup_i X_i \times \bar{X}_i$ is everywhere the complexification of X, so by *op. cit.* V, Prop. 5, $\mathscr{R}_X$ is coherent. □

This combines with Malgrange's preparation theorem to afford:

I.f.3. **Fact/Definition.** If $\mathscr{C}_\bullet$ is the sheaf of continuous functions on a topological space, and $X/\mathbb{C}$ is a reduced complex space then, functorially in X, there is a well defined subsheaf, $\mathscr{A}_X \hookrightarrow \mathscr{C}_X$ of smooth functions. In the particular case that $\mu : \mathscr{X} \to X$ is the moduli of a separated Deligne-Mumford champ,

$$\mu_* \mathscr{A}_{\mathscr{X}} \subseteq \mathscr{A}_X \subseteq \mu_* \mathscr{C}_{\mathscr{X}} = \mathscr{C}_X \tag{I.72}$$

Proof. First pass to the real analytic functions $\mathscr{R}_X$, and for a local embedding $i : X \hookrightarrow M$ in a smooth about $x \in X$, with ideal I_X in $\mathscr{R}_M$ we have by I.f.2 and [Mal02, VI.3.10] an exact sequence

$$0 \leftarrow \mathscr{R}_X \otimes_{\mathscr{R}_M} \mathscr{A}_M \leftarrow \mathscr{A}_M \leftarrow \mathscr{A}_M \otimes_{\mathscr{R}_M} I_X \leftarrow 0 \tag{I.73}$$

wherein $\mathscr{A}_M \otimes_{\mathscr{R}_M} I_X$ is equally the ideal of smooth functions, $\mathscr{A}_M$, vanishing on X. In particular, therefore, we have an embedding

$$\mathscr{A}_X^M := \mathscr{R}_X \otimes_{\mathscr{R}_M} \mathscr{A}_M \hookrightarrow \mathscr{C}_X \tag{I.74}$$

Now observe (by way of the obvious diagram chase implied by (I.73)) that if M has the embedding dimension of X at x then for any other smooth embedding $X \hookrightarrow N$ at x, there is a unique isomorphism which fills the right hand side of

$$\begin{array}{ccc} \mathscr{R}_X & \longrightarrow & \mathscr{A}_X^N \\ \| & & \\ \mathscr{R}_X & \longrightarrow & \mathscr{A}_X^M \end{array} \tag{I.75}$$

in such a way that the diagram commutes. As such $X \mapsto \mathscr{A}_X$ is a well defined, and functorial, while (I.72) is immediate from I.f.1 and (I.74). □

In order to apply this we need another

I.f.4. **Lemma.** *Let* $\mu : \mathscr{X} \to X$ *be the moduli of a Deligne-Mumford champ and* $\coprod_{\alpha\in A} W_\alpha \to X$ *an open cover (in the classical sense) then up to passing to a locally finite refinement there are functions*

$$\rho_\alpha \in \Gamma(X, \mu_* \mathscr{A}_{\mathscr{X}}) \text{ with support in } W_\alpha \text{ such that } \sum_\alpha \rho_\alpha = 1 \tag{I.76}$$

In particular for $\mathscr{M}$ *any sheaf of* $\mathscr{A}_{\mathscr{X}}$*-modules,*

$$\mathrm{H}^q(\mathscr{X}, \mathscr{M}) = 0, \quad \forall q > 0 \tag{I.77}$$

Proof. Refining as necessary we can suppose that we have covers $\coprod_{\alpha\in A} U_\alpha$, $\coprod_{\alpha\in A} V_\alpha$ with $\bar{U}_\alpha \subset V_\alpha$; $\bar{V}_\alpha \subset W_\alpha$ and each of U_α, V_α, W_α satisfies I.f.1, *i.e.* there are étale covers $\coprod_{\alpha\in A} U'_\alpha \to \mathscr{X}$, *etc.*; finite group actions $G_\alpha \rightrightarrows U'_\alpha$ *etc.*; G_α equivariant inclusions $\bar{U}'_\alpha \subset V'_\alpha$ *etc.*; and compatible identifications of U_α with U'_α/G_α *etc.*. As such if $f_\alpha : W'_\alpha \to [0,1]$ is a smooth (in the sense of I.f.3) function which is identically 1 on U'_α, resp. identically 0 off V'_α then its trace, g_α, is a global section of $\mu_*\mathscr{A}_{\mathscr{X}}$ supported in W_α which is identically 1 on U_α, resp. identically 0 off V_α, and

$$\rho_\alpha(x) := \frac{g_\alpha(x)}{\sum_\beta g_\beta(x)} \tag{I.78}$$

does the job. Consequently any sheaf of $\mu_*\mathscr{A}_X$ modules is flasque, while μ_* is acyclic on $\mathbb{Q}$-vector spaces, and whence (I.77). □

We come therefore to the point of the discussion, by way of

I.f.5. **Fact.** *If* $\mathscr{Y} \hookrightarrow \mathscr{X}$ *is an embedding of smooth complex Deligne-Mumford champ with* $\mathscr{Y}$ *proper, then there are a family of open embeddings* $\mathscr{Y} \hookrightarrow \mathscr{U}_t \hookrightarrow \mathscr{X}$ *with* $\cap_t \mathscr{U}_t = \mathscr{Y}$ *and each* $\mathscr{Y} \xrightarrow{i_t} \mathscr{U}_t \xrightarrow{r_t} \mathscr{Y}$ *a deformation retract with* $i_t r_t$ *homotopic to the identity.*

Proof. The expedient of taking the trace under G_x in I.f.1 affords locally equivariant metrics which by (I.76) can be patched to a smooth metric, ω, on $\mathscr{X}$. As such at every geometric point x there is a G_x equivariant neighbourhood $V_x \hookrightarrow T_{\mathscr{X},x}$ of 0 such that the exponential afforded by ω yields an embedding

$$\exp : [V_x/G_x] \to \mathscr{X} \tag{I.79}$$

On the other hand by (I.77) the exact sequence

$$0 \to T_{\mathscr{Y}} \to T_{\mathscr{X}} \to N_{\mathscr{Y}/\mathscr{X}} \to 0 \tag{I.80}$$

has a smooth splitting, $n : N_{\mathscr{Y}/\mathscr{X}} \to T_{\mathscr{X}}$ so $\exp(n)$ restricted to appropriate neighbourhoods of the zero section in $N_{\mathscr{Y}/\mathscr{X}}$ gives what we want. □

This is, of course, just the usual proof of the corresponding fact for smooth manifolds so it's worth making

I.f.6. *Remark.* Slightly, but not much, more subtly if X is Kähler then so is $\mathscr{X}$.

Finally we require a baby GAGA,

I.f.7. **Fact.** *Let* $\mathscr{X}/\mathbb{C}$ *be a normal complex analytic champ, i.e. the classifier of an étale groupoid* $R \rightrightarrows U$ *in the analytic topology, whose moduli* $\mu : \mathscr{X} \to X$ *is a finite map to an algebraic space with algebraic ramification in co-dimension 1 then* $\mathscr{X}$ *is an algebraic Deligne-Mumford champ. Similarly, if* $\mathscr{Y}'_t \to \mathscr{U}_t$ *is a smooth champ finite over the neighbourhoods of I.f.5, then there is an algebraic champ* $\mathscr{Y}' \to \mathscr{Y}$ *such that (in the notation of op. cit.)* $\mathscr{Y}'_t$ *is equivalent to* $r_t^*\mathscr{Y}'$.

Proof. Without loss of generality $\mathscr{X}$ is connected, so exactly as in I.a.6, there is a map $\mathscr{X} \to \mathscr{X}_0$ expressing $\mathscr{X}$ as a locally constant gerbe in B_Γ's for some finite group Γ wherein the stabiliser of the generic point of $\mathscr{X}_0$ is trivial, and by [Art66, 5.1] $\mathscr{X}_0$ is algebraic. As to $\mathscr{X} \to \mathscr{X}_0$, we must first consider the link in the sense of Giraud, [Gir71, IV.1.1.7.3], *i.e.* the representation of $\pi_1(\mathscr{X}_0)$ in the outer automorphisms of Γ, but these are the same in the algebraic and analytic categories, so the next port of call is the obstruction to the existence of a champ with a given link. This is, [Gir71, VI.2.3], a class in $\mathrm{H}^3(\mathscr{X}_0, Z)$, where Z is the centre of the link, *i.e.* the locally constant sheaf in the centre of Γ with induced $\pi_1(\mathscr{X}_0)$ action. By [SGA-IV, Exposé XVI.4.1], étale and analytic cohomology coincide, while the obstruction vanishes analytically, so there is at least one algebraic champ, $\mathscr{X}' \to \mathscr{X}_0$ which is a locally constant gerbe in B_Γ's for the same link. Equally $\mathscr{X}'$ is an analytic champ, so, in either case the equivalence class of all possible champs with this link is, [Gir71, IV.3.4], the orbit of $\mathscr{X}'$ under $\mathrm{H}^2(\mathscr{X}_0, Z)$, and whence $\mathscr{X} \to \mathscr{X}_0$ is algebraic by another application of [SGA-IV, Exposé XVI.4.1]. The argument for the second part about the $\mathscr{U}_t$'s proceeds mutatis mutandis given I.f.5. □

II. $K_{\mathscr{F}}$ NEGATIVE CURVES

II.a. **Foliations as birational groupoids.** As we've already remarked prior to I.b.1 the point of view of a foliation as an integrable quotient of the cotangent sheaf is misleading. Rather a foliation should be considered as an infinitesimal equivalence relation outside of its singularities, and the equivalence of this definition to that involving linear 1^{st} order data as a non-trivial theorem (not withstanding the triviality of the proof) specific to characteristic zero. In any case let us begin by reviewing the equivalence, whence let X be a normal affine variety over $\mathbb{C}$ and $\mathscr{F}$ a smooth foliation on X. Notice that X may be singular, so $\mathscr{F}$ smooth means that (everywhere locally) for some (and indeed any) embedding of X in a smooth variety M the composition,

$$T_{\mathscr{F}} \to \mathscr{T}_X \to T_M \otimes \mathscr{O}_X \tag{II.1}$$

is an injection of bundles. Now consider the diagonal Δ in $X \times X$, with p_i the projections, and $p_2^* T_{\mathscr{F}}$ the foliation obtained by pull-back from the 2^{nd} direction. Dualising commutes with flat pull-back so this is notationally unambiguous, whence shrinking X as necessary we can find a local generator ∂ of $T_{\mathscr{F}}$ and $f \in I_\Delta$ such that $p_2^*\partial(f)$ is non-zero on X. We put $\delta = (p_2^*\partial(f))^{-1}\partial$, and for any function g on $X \times X$ define,

$$\tilde{g} := \sum_{n=0}^{\infty} (-1)^n \frac{f^n \delta^n(g)}{n!} \in \hat{\mathscr{O}}_\Delta := \varprojlim_n \mathscr{O}_{X\times X}/\mathscr{I}_\Delta^n \tag{II.2}$$

then $\delta\tilde{g} = 0$, and better still if $\hat{\Delta}$ is the completion of $X \times X$ in Δ then the inclusion of rings,

$$\mathscr{O}_{\mathscr{F}} := \{h \in \hat{\mathscr{O}}_\Delta : \partial h = 0\} \subset \hat{\mathscr{O}}_\Delta \tag{II.3}$$

corresponds to a relatively smooth fibration of formal schemes,

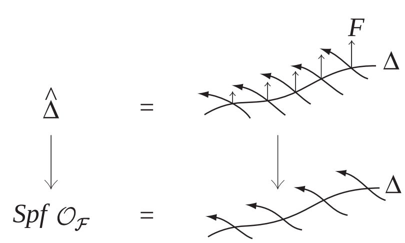

FIGURE 1. Construction of the infinitesimal groupoid

such that the pull-back of the image of Δ in $\mathrm{Spf}\,\mathscr{O}_{\mathscr{F}}$ is the corresponding infinitesimal equivalence relation, i.e. the formal sub-scheme of $\hat{\Delta}$ defined by the ideal generated by $\mathscr{O}_{\mathscr{F}} \cap \mathscr{I}_{\Delta}$ or equivalently the maximal sub-ideal of $\mathscr{I}_{\Delta}$ invariant by $\mathscr{F}$. Rather more picturesquely, fig. 1, what we have done is add a small germ in the $p_2^* T_{\mathscr{F}}$ direction for each point in the diagonal.

To extend this to champs, even separated ones, is a little delicate since unless the champ is in fact an algebraic space the diagonal will fail to be an embedding. To remedy this it suffices to observe that we've actually been working in,

$$\mathfrak{P}_X := \mathrm{Spf}\,\mathscr{P}_X\,, \quad \mathscr{P}_X = \varprojlim_n \mathscr{P}_X^{(n)} \tag{II.4}$$

where $\mathscr{P}_X^{(n)}$ is Grothendieck's sheaf of n-jets viewed as a nilpotent $\mathscr{O}_X$-algebra by way of the 1st-projection. If, however, $\mathscr{X}$, is Deligne-Mumford champ, then by definition it is equivalent to a groupoid with étale source and sink so there are well defined sheaves of nilpotent $\mathscr{O}_{\mathscr{X}}$-algebras, $\mathscr{P}_{\mathscr{X}}^{(n)}$ of n-jets, and of course idem, modulo replacing nilpotent by topologically so, for the inverse limit $\mathscr{P}_{\mathscr{X}}$. Equally the formation of the formal spectrum is a local construction, while both the projectors and the diagonal embedding patch, so we obtain an object which we summarise by way of,

II.a.1. **Definition.** The jet groupoid of a champ de Deligne-Mumford $\mathscr{X}$ is the formal champ,

$$\mathfrak{P}_{\mathscr{X}} = \mathrm{Spf}\,\mathscr{P}_{\mathscr{X}} \rightrightarrows \mathscr{X} \tag{II.5}$$

with source map p_1, sink p_2, and identity the diagonal.

Notice in particular that the diagonal is actually embedded in the jet groupoid, so its worth emphasising what's happening. Specifically for a geometric diagonal point $x \times x$ in $\mathscr{X} \times \mathscr{X}$, its automorphism group is simply $\mathrm{Aut}(x) \times \mathrm{Aut}(x)$. Inside this group we have a copy of $\mathrm{Aut}(x)$ sitting diagonally. Now any attempt to define diagonal type subgroups of automorphisms for off diagonal points, and whence define an actual étale "neighbourhood" in which $\mathscr{X}$ embeds in some sort of diagonal way, is doomed to failure. At the infinitesimal level this can, and is, achieved by II.a.1.

Turning then to champs foliated by curves, or indeed even foliated full stop, the corresponding foliations on étale neighbourhoods of the champ are again by supposition invariant by the corresponding étale groupoid so that we may once again apply the expedient of summary by way of definition, i.e.

II.a.2. **Summary/Definition.** Let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated champ, $\mathscr{Z}$ its singular locus, and $\mathscr{U} = \mathscr{X} \backslash \mathscr{Z}$ the smooth locus then the infinitesimal equivalence relation $\mathfrak{F} \rightrightarrows \mathscr{U}$ defined according to the correspondence which associates to $\mathscr{F}$ a formal subscheme of the jet groupoid, fig. 1 *et. seq.*, will be denoted the smooth infinitesimal groupoid of $\mathscr{F}$.

This construction may, however, fail catastrophically over $\mathscr{Z}$, i.e. consider:

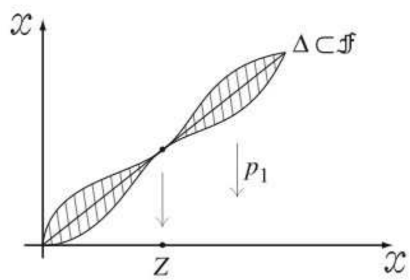

FIGURE 2. A groupoid with essential singularity.

then over $\mathscr{Z}$ we may have an essential singularity, so that the smallest closed formal sub-champ of $\mathfrak{P}_{\mathscr{X}}$ containing $\mathfrak{F}$ is $\mathfrak{P}_{\mathscr{X}}$ itself.

To remedy this latest difficulty we allow the possibility of birational groupoids, i.e. such that the identity map is simply birational. With this extra flexibility we can complete across the singularities. Specifically let,

$$\pi : \tilde{\mathfrak{P}}_{\mathscr{X}} \to \mathfrak{P}_{\mathscr{X}} \tag{II.6}$$

be the blow up of in the diagonal embedding $\Delta(\mathscr{Z})$ of $\mathscr{Z}$ understood with any implied nilpotent structure on the singular locus. Now let $U \to \mathscr{X}$ be an étale neighbourhood of a geometric point $z \in \mathscr{Z}$ with $U \hookrightarrow M$ an embedding into a smooth. Consider coordinates $x_1, \dots, x_n$ on M restricting to functions on U, then for $\mathscr{F}|_U$ Gorenstein, and shrinking U as necessary we may suppose that the foliation is defined by a vector field ∂ on U, which we write using the summation convention as,

$$\partial = a_i \frac{\partial}{\partial x_i} \tag{II.7}$$

so that $\mathscr{I}_{\mathscr{Z}}|_{\mathscr{U}} = (a_i)$. Now introduce x_i, y_i as coordinates on $U \times U$ obtained from our initial coordinates by way of 1^{st} and 2^{nd} pull-back respectively, and put $z_i = x_i - y_i$, then in z_i, x_i coordinates,

$$p_2^* \partial = p_2^* a_i \frac{\partial}{\partial y_i} = -p_2^* a_i \frac{\partial}{\partial z_i} . \tag{II.8}$$

Consequently on the blow up, II.6, around U on the $p_1^* a_i \neq 0$ patch, we have:

$$\partial \left(\frac{z_i}{p_1^* a_i} \right) = \frac{-p_2^* a_i}{p_1^* a_i} = 1 + \frac{(p_1^* a_i - p_2^* a_i)}{p_1^* a_i} . \tag{II.9}$$

On the other hand the diagonal embedding of $\mathscr{Z} \times_{\mathscr{X}} U$ has ideal $(p_1^* a_i, z_i)$ so on the proper transform $\tilde{\Delta}$ of Δ in $\tilde{\mathfrak{P}}_{\mathscr{X}}$ not only can we locate each point in some $p_1^* a_i \neq 0$ patch for an appropriate i, but indeed the function $z_i / p_1^* a_i$ in $I_{\tilde{\Delta}}$ enjoys a non-zero derivation with respect to $\pi^* \partial$. Better still we have blown up in a centre invariant by $p_2^* \mathscr{F}$ so the induced foliation $\widetilde{p_2^* \mathscr{F}}$ on $\tilde{\mathfrak{P}}_{\mathscr{X}}$ is both smooth in a neighbourhood of $\tilde{\Delta}$ and everywhere transverse to it. Whence we can just repeat our minor variant of the classical Frobenius theorem to obtain,

II.a.3. **Fact/Definition.** Let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated Gorenstein champ, then there is a formal sub-champ $\tilde{\mathfrak{F}}$ of $\tilde{\mathfrak{P}}_{\mathscr{X}}$, (II.6), together with projection maps, $p_i \circ \pi$, $i = 1$ or 2 defining a birational groupoid, i.e.

$$\tilde{\mathfrak{F}} \rightrightarrows \mathscr{X} \tag{II.10}$$

where the identity and composition are rational maps. In addition the projection $p_1 \circ \pi$ factors as,

$$\tilde{\mathfrak{F}} \xrightarrow{p} \tilde{\Delta} \to \mathscr{X} \tag{II.11}$$

with the former map in (II.11) relatively smooth of dimension 1. We call this structure the *infinitesimal birational groupoid* of the foliation.

Notice in particular,

II.a.4. **Fact.** *There is an isomorphism,* $N_{\tilde{\Delta}/\tilde{\mathfrak{F}}} \xrightarrow{\sim} \mathscr{O}_{\tilde{\Delta}}(p_2^* T_{\mathscr{F}})$.

II.b. **Chow's Lemma.** We'll confine ourselves to that which is strictly necessary for applications. Our interest centres on smooth formal champs $\mathfrak{F}$ whose trace $\mathscr{C}$ is a smooth champ of dimension 1. From our utilitarian point of view we'll confine ourselves to the case where $\dim \mathfrak{F} = 2$. Irrespectively there is a well defined normal bundle $N_{\mathscr{C}/\mathfrak{F}}$, and we make,

II.b.1. **Definition.** $\mathfrak{F}$ is a concave formal neighbourhood of $\mathscr{C}$ if $\deg(N_{\mathscr{C}/\mathfrak{F}}) > 0$.

Unsurprisingly the classical Chow lemma continues to hold, i.e.

II.b.2. **Lemma.** (Chow, Grauert et al.) *Let L be a line bundle on $\mathfrak{F}$ then there is a quadratic polynomial P_L, depending on L, such that for all $n \in \mathbb{N}$,*

$$h^0(\mathfrak{F}, L^{\otimes n}) \leq P_L(n)\,. \tag{II.12}$$

Proof. Let $\mathfrak{F}_m$ be the m^{th}-thickening of $\mathscr{C}$ then we have an exact sequence,

$$0 \to \mathrm{Sym}^m N^{\vee}_{\mathscr{C}/\mathfrak{F}} \to \mathscr{O}_{\mathfrak{F}_{m+1}} \to \mathscr{O}_{\mathfrak{F}_m} \to 0\,. \tag{II.13}$$

On the other hand if $h^0(\mathscr{C}, L^n \otimes \mathrm{Sym}^m N^{\vee}_{\mathscr{C}/\mathfrak{F}}) \neq 0$, then,

$$m \deg(N_{\mathscr{C}/\mathfrak{F}}) \leq n \deg_{\mathscr{C}}(L)\,. \tag{II.14}$$

Consequently for any $n \in \mathbb{N}$,

$$\mathrm{H}^0(\mathscr{O}_{\mathfrak{F}_{m+1}} \otimes L^n) \hookrightarrow \mathrm{H}^0(\mathscr{O}_{\mathfrak{F}_m} \otimes L^n) \tag{II.15}$$

is injective, provided $m > M := \dfrac{n \deg_{\mathscr{C}}(L)}{\deg(N_{\mathscr{C}/\mathfrak{F}})}$ and whence

$$h^0(\mathfrak{F}, L^{\otimes n}) = \varprojlim_m h^0(\mathfrak{F}_m, L^{\otimes n}) \leq \sum_{k=0}^{M} h^0(\mathscr{C}, L^n \otimes N^{-k}_{\mathscr{C}/\mathfrak{F}})\,. \tag{II.16}$$

Moreover by [BN06, 1.1] we can find a map, $\rho : C \to \mathscr{C}$ from an honest curve, while for any bundle E, $h^0(\mathscr{C}, E) \leq h^0(C, \rho^* E)$, so we conclude by Riemann-Roch. □

II.c. **Bend & Break.** We are now in a position to extend the results of [BM16], so to this end let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated Gorenstein normal champ with projective moduli space $\pi : \mathscr{X} \to X$, and H an ample bundle on the latter. As ever the basic object of study is $K_{\mathscr{F}}$ negative curves on $\mathscr{X}$, i.e., profiting once more from [BN06, 1.1], maps $f : C \to \mathscr{X}$ from a smooth curve such that $K_{\mathscr{F}} \cdot_f C < 0$. We impose further the condition that f does not factor through the singular locus $\mathscr{Z} = \mathrm{sing}(\mathscr{F})$. Consequently if we consider the infinitesimal birational groupoid as fibred over $\tilde{\mathscr{X}} = \mathrm{Bl}_{\mathscr{Z}}(\mathscr{X})$ via p in (II.11), then f admits a lifting $\tilde{f} : C \to \tilde{\mathscr{X}}$ and we may form the fibre square,

$$\begin{array}{ccc} \tilde{\mathfrak{F}} & \longleftarrow & \tilde{\mathfrak{F}}_C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{\mathscr{X}} & \underset{\tilde{f}}{\longleftarrow} & C \end{array} \tag{II.17}$$

In addition the identity map of the groupoid gives a section s of p of the left, so a fortiori of the right, vertical arrow, which is everywhere well defined since we're working with $\tilde{\mathscr{X}}$ rather than $\mathscr{X}$. Consequently, by II.a.4, $\tilde{\mathfrak{F}}_C$ is a concave, II.b.1, neighbourhood of $s(C)$, and for $\tilde{\mathfrak{P}}_{\mathscr{X}}$ as per (II.6), we have natural maps,

$$\tilde{\mathfrak{F}}_C \to C \times \tilde{\mathfrak{P}}_{\mathscr{X}} \to C \times \mathrm{Bl}_{\Delta(\mathscr{Z})}(\mathscr{X} \times \mathscr{X}) \tag{II.18}$$

where the moduli, W, of $\mathrm{Bl}_{\Delta(\mathscr{Z})}(\mathscr{X} \times \mathscr{X})$ is projective because X is, and we assert, II.c.1. **Claim.** The Zariski closure of the image of $\tilde{\mathfrak{F}}_C$ in $C \times W$ is irreducible of dimension 2.

Proof. Indeed let Y be the Zariski closure, which is irreducible since $\tilde{\mathfrak{F}}_C$ is. Moreover if L is an ample line bundle on $C \times W$ then by definition,

$$\mathrm{H}^0(Y, L) \to \mathrm{H}^0(\tilde{\mathfrak{F}}_C, L) \tag{II.19}$$

is injective by the definition of Y, so we're done by the Chow lemma, II.b.2. ☐

Now let $\mathscr{Y}$ be $Y \times_{(C\times W)} (C \times \mathrm{Bl}_{\Delta(\mathscr{X})}(\mathscr{X} \times \mathscr{X}))$ then we further assert

II.c.2. **Better Claim.** For $\mathscr{Y}/C$ viewed as a C-champ via the projection $C \times W \to C$ there is a net C-map $\tilde{\mathfrak{F}}_C \to \mathscr{Y}$. In particular, therefore, this affords a section $s : C \to \mathscr{Y}$ such that:

(a) $\mathscr{Y}$ is smooth in a neighbourhood of $s(c)$.
(b) $K_{\mathscr{Y}/C} \cdot_s C = K_{\mathscr{F}} \cdot_f C$.
(c) The 2nd projection yields a map of foliated champs $(\mathscr{Y}/C) \to (\mathscr{X} \to [\mathscr{X}/\mathscr{F}])$.

Proof. By base change in the fibre square:

$$\begin{array}{ccc} \tilde{\mathfrak{F}} \times C & \longleftarrow & \tilde{\mathfrak{F}}_C \\ {\scriptstyle p\times\tilde{f}}\downarrow & & \downarrow \\ \tilde{\mathscr{X}} \times \tilde{\mathscr{X}} & \xleftarrow{\text{diagonal}} & \tilde{\mathscr{X}} \end{array} \tag{II.20}$$

the above horizontal is net, while, *cf.* II.a.1, $\mathfrak{P}_{\mathscr{X}} \to \mathscr{X} \times \mathscr{X}$ is net, so $\tilde{\mathfrak{P}}_{\mathscr{X}} \to \mathrm{Bl}_{\Delta(\mathscr{X})}(\mathscr{X} \times \mathscr{X})$ is too, and $\tilde{\mathfrak{F}}$ is embedded in $\mathfrak{P}_{\mathscr{X}}$ by definition II.a.3. As such $\tilde{\mathfrak{F}}_C \to C \times \mathrm{Bl}_{\Delta(\mathscr{X})}(\mathscr{X} \times \mathscr{X}))$ is a composition of net maps, which, by construction, has an image embedded in $\mathscr{Y}$. ☐

The following, therefore, affords invariant rational curves through a generic point of the image of C.

II.c.3. **Fact.** *Suppose in addition to II.c.2.(a)-(b) a family $p : \mathscr{Y} \to C$ of uni-dimensional champ with a section s satisfies $K_{\mathscr{Y}/C} \cdot_s C < 0$ then there is a finite extension $\mathbb{C}(C) \to K$ such that $\mathscr{Y}_K$ is dominated by $\mathbb{P}^1_K$.*

Proof. We may, without loss of generality, suppose that $\mathscr{Y}$, and indeed any base change thereof, is normal. In particular, therefore, I.a.6, there is a fibration $\mathscr{Y} \to \mathscr{Y}_0$ expressing the former as a locally constant gerbe over a normal-fold, so that by [BN06, 1.1] we may further suppose that $\mathscr{Y} = \mathscr{Y}_0$. As such if the generic fibre of p is not dominated by a rational curve then, *op. cit.*, there is a finite extension $\mathbb{C}(C) \to K$ such that $\mathscr{Y} \times_C K$ is an orbifold of the form $[S_K/G]$ for some non-rational K-curve S_K and finite group G acting generically freely. Denoting by Y the moduli of $\mathscr{Y}$, and identifying K with the function field of a smooth curve B, we can suppose that S_K is the generic fibre over B of the integral closure S of Y in the function field of S_K. The normalisation $\mathscr{S}$ of the fibre $\mathscr{Y} \times_Y S$ is, therefore, a gerbe over S with generic fibre S_K. Consequently, by purity, $q : \mathscr{S} \to \mathscr{Y}$ is ramified only in components of fibres of $\mathscr{Y} \to C$. In addition q is étale locally Galois since $S \to Y$ is and $\mathscr{Y} \to C$ is smooth in a neighbourhood of the section $s(C)$, so by [SGA-I, Exp. XIII, Cor. 5.3], q is étale locally around $s(C)$ the extraction of roots of fibres. As such, by the simple expedient of taking $\mathbb{C}(C) \to K$ sufficiently large, we can suppose- around $s(C)$ and it's pre-image- that q is scheme like and S is smooth. Better still since q is only ramified in fibres,

$$K_{S/B} = q^* K_{\mathscr{Y}/C}, \text{ and thus, } K_{S/B} \cdot_{\tilde{s}} B < 0, \tag{II.21}$$

for any lifting $\tilde{s}$ of s. Consequently, we may from from either [BM16] or the classical theorem of Arakelov, [Szp81], conclude to the absurdity that the generic fibre of $S \to B$ is a rational curve. ☐

The fibres of p in II.c.3 may not themselves be rational curves, and it is convenient to give them a name, to wit

II.c.4. **Fact/Definition.** A smooth 1-dimensional Deligne-Mumford champ, $\mathscr{L}$, over a field k is said to be parabolic if it's geometric fibre is dominated by a rational curve. Rather conveniently this occurs, [BN06, 1.1], iff the topological Euler-characteristic $\chi(\mathscr{L}) > 0$.

From which we can proceed to our conclusion

II.c.5. **Proposition.** *Let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated normal gerbe over a projective variety X, which is foliated Gorenstein along some $K_{\mathscr{F}}$ negative curve $C_0 \subset X$ around the generic point of which $\mathscr{F}$ is a non-singular foliation of $\mathscr{X}$, then for a generic $c \in C_0$ there is an invariant parabolic champ, $g_c : \mathscr{L}_c \to \mathscr{X}$ such that for M any* nef. *$\mathbb{R}$-divisor on $\mathscr{X}$, and $||$ the moduli,*

$$M._{|g_c|}|\mathscr{L}_c| \leq 2\,\frac{M._f\, C}{-K_{\mathscr{F}}.\, C} \tag{II.22}$$

Proof. We apply II.c.2 with C a curve mapping to the normalisation of the gerbe over C_0 in $\mathscr{X}$. By II.c.3 the generic fibre of $\mathscr{Y} \to C$ is an invariant parabolic champ, so it only remains to produce the degree bound. To this end identify the image of the section s with a curve C such that $C^2 = -K_{\mathscr{F}} \cdot C$ in the normal surface which is the moduli. Whence if L is a generic fibre of the same, M is notationally confused with the restriction of the given nef. $\mathbb{R}$-divisor, and $x \in \mathbb{R}_{>0}$ then by the Hodge index theorem,

$$2x \cdot (L \cdot M)\, C^2 \leq (L + xM)^2\, C^2 \leq \{C \cdot (L + xM)\}^2 \tag{II.23}$$

so taking $x = (M \cdot C)^{-1}$ we conclude. □

The same proof works, under the weaker hypothesis that only a neighbourhood of C_0 in the moduli is projective. More interestingly, the presence of even the most mild non-scheme like structure on $\mathscr{X}$ can necessitate the precision of II.c.5 that the existence of a parabolic invariant champ $\mathscr{L} \ni c$ is only guaranteed for generic c. Indeed:

II.c.6. *Remark.* Take a section C with positive square of a Hizerbruch surface $P \to C$. In the fibre through some $c \in C$, choose some set Q of points off C, and for $q \in Q$ let $n_q \in \mathbb{N}_{>1}$ be given. Choose a germ of a smooth curve, γ transverse to the fibre P_c at q. Blowing up in q, we get the proper transform γ_1 of γ, we then blow up in the point where this crosses the exceptional divisor, and repeat this process n_q times before blowing down the first $n_q - 1$ curves. The resulting surface S then has isolated cyclic quotient singularities with monodromy $\mathbb{Z}/n_q$ at each q in the proper transform of P_c, which itself meets at each q a rational curve in the fibre, but the said proper transform is the only component of the fibre meeting the section. Passing to the Vistoli covering champ, we see the necessitate for taking $c \in C$ generic in II.c.5, since the gerbe over the proper transform fails to be parabolic as soon as, $\sum_q (1 - 1/n_q) > 2$.

II.d. **The Cone of Curves.** We may now apply the basic estimate II.c.5 to the cone of curves of a foliated Gorenstein normal champ $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ over $\mathbb{C}$. Indeed more precisely we have,

II.d.1. **Fact.** *Let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated Gorenstein normal champ with log-canonical singularities in dimension 1 and projective moduli, then there are countably many $\mathscr{F}$-invariant parabolic, champ $\mathscr{L}_i$, with, $0 < -K_{\mathscr{F}}.\, \mathscr{L}_i \leq 2$ such that,*

$$\overline{\mathrm{NE}}\,(\mathscr{X})_{\mathbb{R}} = \overline{\mathrm{NE}}\,(\mathscr{X})_{K_{\mathscr{F}} \geq 0} + \sum_i \mathbb{R}_+\, \mathscr{L}_i \tag{II.24}$$

where $\overline{\mathrm{NE}}\,(\mathscr{X})_{K_{\mathscr{F}} \geq 0}$ is the sub-cone of the closed cone of curves on which $K_{\mathscr{F}}$ is non-negative. Better still the $\mathbb{R}_+\, \mathscr{L}_i$ are locally discrete, and if $R \subset \overline{\mathrm{NE}}\,(\mathscr{X})_{\mathbb{R}}$ is an extremal ray in the half space $\mathrm{NE}_{K_{\mathscr{F}} < 0}$ then it is of the form $\mathbb{R}_+\, \mathscr{L}_i$.

This is a wholly formal consequence, as per [Kol96, III.1.2], of the following variant of II.c.5

II.d.2. **Variant.** *Let* $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ *be as above, and* $C_0 \subset X$ *a* $K_{\mathscr{F}}$*-negative curve in the moduli, then for generic* $c \in C_0$ *there is a* $\mathscr{F}$*-invariant parabolic champ* $\mathscr{L}_c \ni f(c)$ *with* $0 < -K_{\mathscr{F}} \cdot \mathscr{L}_c \leq 2$ *such that for all* nef. $\mathbb{R}$*-divisors* M *on* X*, and* $||$ *the moduli,*

$$M. |\mathscr{L}_c| \leq 2 \frac{(M. C_0)}{-K_{\mathscr{F}}. C_0} . \tag{II.25}$$

The variant requires a couple of facts of independent interest to wit

II.d.3. **Fact.** *If* $\mathscr{Z}$ *is the singular locus of a foliated Gorenstein-champ* $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ *with log-canonical singularities in dimension 1, then* $\mathscr{O}_{\mathscr{Z}}(K_{\mathscr{F}})$ *is semi-ample.*

Proof. Consider the linearisation map, i.e. the composition of,

$$D : \mathscr{I}_{\mathscr{Z}}/\mathscr{I}_{\mathscr{Z}}^2 \underset{d}{\longrightarrow} \Omega_{\mathscr{X}} \otimes_{\mathscr{O}_{\mathscr{X}}} \mathscr{O}_{\mathscr{Z}} \longrightarrow K_{\mathscr{F}} \otimes \mathscr{I}_{\mathscr{Z}}/\mathscr{I}_{\mathscr{Z}}^2 \tag{II.26}$$

By the Leibniz rule, this map is $\mathscr{O}_{\mathscr{Z}}$ linear, and since the singularities are log-canonical in dimension 1, for $z \in \mathscr{Z}$ outside a finite set, some symmetric function of D defines a section over $\mathscr{Z}$, non-vanishing at z, of some power $K_{\mathscr{F}}^{\otimes n}$, [MP13, I.ii.4], and we conclude by the Zariski-Fujita theorem. □

II.d.4. **Fact/Definition.** Let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated Gorenstein champ; $f : \mathscr{L} \to \mathscr{X}$ the normalisation of an invariant uni-dimensional champ not factoring through the singular locus $\mathscr{Z}$; $\chi(\mathscr{L})$ its topological Euler-characteristic; and $\mathrm{s}_{\mathscr{Z}}(f)$ the Segre class of f along $\mathscr{Z}$, *i.e.* the multiplicity (counted with stabilisers) of the pre-image $f^{-1}I_{\mathscr{Z}}$ of the ideal of singularities, then

$$\begin{aligned} K_{\mathscr{F}} \cdot_f \mathscr{L} &= -\chi(\mathscr{L}) - \mathrm{Ram}_f + \mathrm{s}_Z(f) \\ &\geq -\chi(\mathscr{L}) + \sum_{l \in f^{-1}(\mathscr{Z})} \frac{1}{|\mathrm{Aut}_{\mathscr{C}}(c)|} \end{aligned} \tag{II.27}$$

Proof. The image of $f^*\Omega^1_{\mathscr{X}}$ is $\Omega^1_{\mathscr{L}}$ is always $\Omega^1_{\mathscr{L}}(-\mathrm{Ram}_f)$, while in the particular, I.20, it's equally $f^*K_{\mathscr{F}}.f^{-1}I_{\mathscr{Z}}$, which proves the 1st line in (II.27). To get the second, observe that in characteristic 0 f can only ramify where it meets $\mathscr{Z}$. On the other hand if $f : \Delta \to U$ is a local branch of f meeting a singularity in $f(0)$, and

$$\partial = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n} \tag{II.28}$$

is a local generator of $\mathscr{F}$ with x_i coordinates on a smooth embedding of U then the local contribution to $-\mathrm{Ram}_f + \mathrm{s}_Z(f)$ is

$$\begin{aligned} &- \min_i \{\mathrm{ord}(\dot{x}_i(t))\} + \min_i \{\mathrm{ord}(f^*(a_i))\}, \quad f : t \mapsto x_i(t) \\ &= 1 + (\min_i \{\mathrm{ord}(f^*(a_i))\} - \min_i \{\mathrm{ord}(f^*(x_i))\}) \geq 1 \end{aligned} \tag{II.29}$$

whence the 2nd line on correcting for the order of the stabiliser. □

At which point we can return to

proof of II.d.2. By II.d.3 we need only prove the variant under the additional condition present in II.c.5 that the foliated champ is non-singular over a generic point of C_0. As such re-taking the notation of the proof of *op. cit.*, we have a bi-dimensional champ $p : \mathscr{Y} \to C$ whose fibres map invariantly by g to $\mathscr{X}$, which is the normalisation of its image. The said image, $\mathscr{A}$, say, admits a possibly non-saturated, injection $T_{\mathscr{F}} \to \mathscr{T}_{\mathscr{A}}$. Every component of the singular locus is invariant by every vector field, so by [BM97], normalisation (in characteristic 0) can be realised in co-dimension 1 by a sequence of blow ups in $\mathscr{F}$-invariant centres. Thus $g^*T_{\mathscr{F}}$ maps to $\mathscr{T}_{\mathscr{Y}/C}$

in co-dimension 1, whence, everywhere since $\mathscr{Y}$, and therefore $\mathscr{T}_{\mathscr{Y}/C}$, is S_2. Consequently for generic $c \in C$,

$$-K_{\mathscr{F}} \cdot_{g_c} C \leq \mathscr{T}_{\mathscr{Y}/C} \cdot C \tag{II.30}$$

while by the adjunction formula, II.d.4, and smoothness of $\mathscr{Y}$ in co-dimension 2, we have,

$$\mathscr{T}_{\mathscr{Y}/C} \cdot \mathscr{Y}_c = -\chi(\mathscr{Y}_c) \leq 2 \tag{II.31}$$

so, indeed $-K_{\mathscr{F}} \cdot_{g_c} C \leq 2$ for generic c as required. □

In particular, under the hypothesis of log-canonical singularities in dimension 1, $K_{\mathscr{F}}$-negative curves are never contained in the singular locus of the foliation, and we proceed to examine the possibilities for $K_{\mathscr{F}}$ negative invariant parabolic champs outside the same. Whence let $f : \mathscr{L} \to \mathscr{X}$ be the normalisation of such, which we express as a locally constant gerbe, $\pi : \mathscr{L} \to \mathscr{L}_0$, over a champ without generic stabiliser, then by (II.27)

$$0 > K_{\mathscr{F}} \cdot_f \mathscr{L} \geq (\mathscr{L} : \mathscr{L}_0) \cdot \left\{ -2 + \sharp f^{-1}(\mathscr{Z}) + \sum_{q \notin \pi f^{-1}(\mathscr{Z})} (1 - \frac{1}{d_q}) \right\} \tag{II.32}$$

where in the sum, d_q is the order of the local monodromy, and $\sharp$ means integer valued cardinality of a set. As such,

II.d.5. **Fact/Definition.** For a Gorenstein foliation $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ in the presence of log-canonical singularities in dimension 1, an irreducible $K_{\mathscr{F}}$-negative invariant champ (or just an irreducible $K_{\mathscr{F}}$-negative invariant champ whose generic point meets the smooth locus of the foliation if there are no hypothesis on the singularities of $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$) has a normalisation, $f : \mathscr{L} \to \mathscr{X}$, with $\mathscr{L}$ parabolic, and furthermore:

(a) The pre-image under f of the singular locus $\mathscr{Z}$ is supported in at most 1 point.
(b) If this pre-image is $\neq \emptyset$, then $\mathscr{L}_0$ has at most one non-scheme like point outside it.
(c) If there is no such singular point $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ is generically a fibration in parabolic champs.

Proof. Items (a) and (b) are clear from (II.32) which leaves (c). In this case f is an embedding whose normal bundle is flat via the representation afforded by the linear holonomy, while $\pi_1(\mathscr{L})$ is finite, so, for $\tilde{f} : \tilde{\mathscr{L}} \to \mathscr{X}$ the composition with the universal cover, the deformations of $\tilde{f}$ are (locally) a smooth space of dimension $\dim(\mathscr{X})-1$, and every deformation of $\tilde{f}$ is invariant. □

II.e. **Singular structure of $K_{\mathscr{F}}$-negative curves.** Throughout this section $f : \mathscr{L} \to \mathscr{X}$ is a map from a smooth invariant $K_{\mathscr{F}}$-negative curve with the further specifications of II.d.5. In particular f is an embedding everywhere except possibly at a point $p \in f^{-1}(\mathscr{Z})$. At p, however not only may the monodromy exceed that of the generic point of $\mathscr{L}$, but f may fail to be an embedding because it has a cusp and/or because the image is not uni-branched. Nevertheless there is a certain limit to the complication, whose description is the goal of this section, i.e.

II.e.1. **Fact/Definition.** Let everything be as in II.d.5 albeit we insist that $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ has log-canonical singularities, and suppose moreover that $f^{-1}(\mathscr{Z}) \neq \emptyset$ with $p : \mathrm{pt} \to \mathscr{L}$ the resulting geometric point, then the étale local contribution, (II.29), to $-\mathrm{Ram}_f + \mathrm{s}_{\mathscr{Z}}(f)$ at p is exactly 1. As such by (II.27) and (II.32)

$$K_{\mathscr{F}}.\mathscr{L} = -1/d \tag{II.33}$$

where d is the maximum value of a stabiliser of $\mathscr{L}$ outside p, which is either attained at a unique point or is the same everywhere in the complement of p, and we refer to such curves as $-\frac{1}{d}\,\mathbb{F}$ *curves.*

We proceed by a series of lemmas beginning with

II.e.2. **Claim.** The foliation $\mathscr{F}$, by way of restriction over the generic point, affords a singular derivation of $\mathscr{L}$.

Proof. We re-take the notation of (II.28)-(II.29) in the proof of II.d.4. It therefore follows exactly as in the proof of II.d.2 that ∂ defines a derivation of $\mathscr{O}_\Delta$, and it remains to prove that it's actually singular at p. To see this observe that if $b : \tilde{\mathscr{X}} \to \mathscr{X}$ were the blow up in p then the induced foliation (understood without saturation if the singularities are not canonical, *i.e.* locally defined by $b^*\partial$) cannot (by the Frobenius theorem) be smooth where the proper transform of f crosses the exceptional divisor. On the other hand, a sequence of blow ups in singular points resolves any singularity of any branch of f, so for $b : \tilde{\mathscr{X}} \to \mathscr{X}$ now a chain of such, we can suppose that the proper transform $\tilde{f} : \mathscr{L} \to \tilde{\mathscr{X}}$ is an embedding crossing the exceptional divisor in a singular point, $\tilde{f}(p)$, of the regular derivation $b^*\partial$, *i.e.* ∂ affords a singular derivation of $\mathscr{O}_\Delta$. □

Applying II.e.2, we can, in the said notation, write the restriction to $\mathscr{L}$ of a generator étale locally as

$$\partial = y^{r+1}u(y)\frac{\partial}{\partial y}, \quad u(y) \in \mathscr{O}_\Delta^\times,\ r \in \mathbb{Z}_{\geq 0}, \tag{II.34}$$

and the content of II.e.1 is that $r = 0$. All of which is a useful, if non-essential, point of reference in establishing our next

II.e.3. **Claim.** Understanding $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ in the log-sense, I.b.2, if necessary, *cf.* I.b.10, Without loss of generality $\mathscr{X}$ in II.e.1, is a smooth champ.

Proof. By [BM97] there is a $\mathscr{F}$-equivariant resolution of singularities

$$b : \tilde{\mathscr{X}} \to \mathscr{X} \tag{II.35}$$

So that understanding $\tilde{\mathscr{X}} \to [\tilde{\mathscr{X}}/\tilde{\mathscr{F}}]$ in the log-sense if necessary the canonical bundle is unchanged. As such if b is an isomorphism at the generic point of f, there is a unique lifting $\tilde{f} : \mathscr{L} \to \tilde{\mathscr{X}}$ satisfying the hypothesis of II.e.1, and there is nothing to do. It may, however, happen that b is a modification around the image of f. Nevertheless every component of the fibre over the said image is invariant, amongst which we choose one over the generic point of f and normalise it to get a not necessarily fibred square

$$\begin{array}{ccc} \tilde{\mathscr{X}} & \xleftarrow[F]{} & \mathscr{Y} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle B} \\ \mathscr{X} & \xleftarrow{\tilde{f}} & \mathscr{L} \end{array} \tag{II.36}$$

wherein any vector field along $\mathscr{F}$ on the bottom left hand corner lifts naturally everywhere else. In particular, therefore, there is a possibly very far (even logarithmically) from saturated (*cf.* II.e.2) bundle of derivations

$$F^*b^*T_{\mathscr{F}} \to \mathscr{T}_{\mathscr{Y}} \tag{II.37}$$

whose singular locus is contained in $B^{-1}(p)$, so that the restriction

$$F^*b^*T_{\mathscr{F}}\ |_{\mathrm{sing}(F^*b^*T_{\mathscr{F}})} \tag{II.38}$$

is trivial. On the other hand b, and whence B, is relatively projective, so $\mathscr{Y}$ has projective moduli and since (II.38) provides an appropriate variant of II.d.3 we may, since it makes no other use of saturation, apply II.d.1 to conclude that there are $F^*b^*K_{\mathscr{F}} = B^*f^*K_{\mathscr{F}}$-negative invariant curves

$$\tilde{f} : \tilde{\mathscr{L}} \to \mathscr{Y} \to \tilde{\mathscr{X}} \tag{II.39}$$

lifting f. Of course, plausibly, $\tilde{\mathscr{L}} \to \mathscr{L}$ is ramified over p, but this would only cause a non-zero value of r in (II.34) to go up. □

Now, as we've said, II.d.5.(a) notwithstanding the image of f in $\mathscr{X}$ can even fail to be unibranch. However

II.e.4. **Claim.** Hypothesis as in II.e.3, then without loss of generality f is an embedding.

Proof. In an easier variant of the proof of II.e.3: given $f : \mathscr{L} \to \mathscr{X}$ with $\mathscr{X}$ smooth, we can find a composition, $b : \tilde{\mathscr{X}} \to \mathscr{X}$, of blow ups in singular points of the foliation such that the unique lifting $\tilde{f} : \mathscr{L} \to \tilde{\mathscr{X}}$ is an embedding. □

At which juncture we have a well defined normal bundle $N_{\mathscr{L}/\mathscr{X}}$ and a specialised foliation to the same. Indeed somewhat more generally

II.e.5. **Fact/Definition.** Let $f : \mathscr{Y} \to \mathscr{X}$ be net, I.e.2 albeit that much more, I.e.5, is true, descent yields a well defined normal cone $C_{\mathscr{Y}/\mathscr{X}}$. Specifically if $V \to \mathscr{X}$ is étale, then there is a sufficiently small étale neighbourhood U of any geometric point of $\mathscr{Y} \times_{\mathscr{X}} V$ such that $U \hookrightarrow V$, and the pull-back to U of the associated cone is,

$$\text{(II.40)} \qquad \operatorname{Spec} S := \bigoplus_{n=0}^{\infty} \frac{I_{U,V}^{n}}{I_{U,V}^{n+1}} .$$

In particular if the image of f is invariant, then the foliation leaves $I_{U,V}$ invariant, so a local generator ∂ of $T_{\mathscr{F}}$ passes to a graded derivation of S by way of applying it to any lifting of an element in the n^{th}-graded piece, and then reducing modulo $I_{U,V}^{n+1}$. This process may not immediately lead to a foliation, but only a pre-foliation, i.e. the specialisation may not be saturated. Nevertheless, for ease of notation, *cf.* II.e.3, we continue to ignore such a distinction, which, in any case, we'll clear up in II.f.1. Irrespectively, if $\mathscr{Y}$ is a smooth invariant curve not factoring through the singular locus, $\mathscr{Z}$, for y a coordinate along U around a point of $f^{-1}(\mathscr{Z})$, and x_i normal coordinates the specialisation of ∂ takes, by (II.34), the form,

$$\text{(II.41)} \qquad \partial : y \mapsto b(y) = y^{r+1} u(y) \partial y \,(\operatorname{mod} I_{U,V}) , \qquad x_i \mapsto a_{ij}(y)\, x_j = \partial x_i \,(\operatorname{mod} I_{U,V})$$

where the summation convention is employed, so, equivalently the specialisation may be viewed as a connection on $N_{\mathscr{Y}/\mathscr{X}}$ with singularities.

By way of II.e.3 and II.e.4 this may be applied to the case in point via

II.e.6. **Fact.** *Let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated smooth champ, and $f : \mathscr{L} \to \mathscr{X}$ an invariant net map from a (smooth) parabolic champ not factoring through the singularities such that $K_{\mathscr{F}} \cdot_f \mathscr{L} < 0$ then either $r = 0$ or the linearisation,* (I.26)*, $\bar{\partial}$ of a generator at the singular point is nilpotent.*

Proof. Without loss of generality, $\mathscr{L}$ is simply connected so $\mathscr{L} \xrightarrow{\sim} \mathbb{P}(d, e)$ for some $d, e \in \mathbb{N}$, [BN06, 1.1]. We have, therefore, a rather explicit description of $\mathscr{L}$, to wit:

$$\text{(II.42)} \qquad \begin{array}{ccccc} & & \mathbb{G}_m & \xrightarrow[\beta]{t \mapsto t^{-e}} & U' \xrightarrow{\sim} \mathbb{A}^1 \to [\mathbb{A}^1/\mu_d] \hookrightarrow \mathscr{L} \\ & & {\scriptstyle t \mapsto t^d} \big\downarrow {\scriptstyle \alpha} & & \\ \mathscr{L} \hookleftarrow [\mathbb{A}^1/\mu_e] \leftarrow \mathbb{A}^1 & \xrightarrow{\sim} & U & & \end{array}$$

Furthermore, by II.d.5, we may suppose that the pre-image of the singular locus is a point p which we identify with 0 (the origin in U) while ∞ will denote the origin in U'. Consequently by I.c.7 there is a longitudinal coordinate y, resp. η, and normal coordinates x_i, resp. ξ_i in neighbourhoods of 0, resp. ∞ such that

$$\beta^*\eta = t^{-e}, \quad \alpha^* y = t^d, \; \beta^*\xi_i = t^{-a_i}\alpha^* x_i \tag{II.43}$$

where the integers a_i are afforded by the Harder-Narismhan filtration

$$N_{\mathscr{L}/\mathscr{X}} \xrightarrow{\sim} \coprod_i \mathscr{O}_{\mathscr{L}}(a_i) \tag{II.44}$$

so, the basis x_i, resp ξ_i may even be supposed μ_e, resp μ_d invariant, *i.e.*

$$(\epsilon, x_i) \mapsto \epsilon^{-a_i} x_i\,, \quad (\delta, \xi_i) \mapsto \delta^{-a_i}\xi_i\,, \quad \epsilon \in \mu_e, \delta \in \mu_d \tag{II.45}$$

Irrespectively, $T_{\mathscr{F}} \xrightarrow{\sim} \mathscr{O}_{\mathscr{L}}(e - dr)$, where by hypothesis $e > dr$, and we normalise generators around 0 and ∞ according to

$$\partial_0(y) = dy^{r+1}, \quad \partial_\infty(\eta) = -e \tag{II.46}$$

so that for a specialised foliation described, *cf.* (II.41), by matrices A, resp. B, over U, resp. U',

$$A(t^d) \, - \, t^{dr}\Delta \, = \, \frac{1}{t^{e-dr}} DB(t^{-e})D^{-1} \tag{II.47}$$

where Δ, resp. D, is the diagonal matrix with entries a_i, resp t^{a_i}. Consequently if we order the a_i to be decreasing in i, then every $i \leq j$th entry of DBD^{-1} on the right of (II.47) is a polynomial in t^{-1}, so from $e > dr$, $A(t^d)$ is an upper semi-triangular matrix with diagonal $a_i t^{dr}$, and whence the said linearisation is nilpotent if $r > 0$. □

Manifestly this completes the proof of II.e.1 by I.b.5, and merits

II.e.7. *Remark.* The difficulty in II.e.1 comes from the fact that if $\mathfrak{X}$ were the completion in the singularity p,

$$\mathrm{H}^0(\mathfrak{X}, T_{\mathscr{F}}) \tag{II.48}$$

may not contain a generator, ∂. Indeed supposing f an embedding (just to fix ideas since it's of no importance) so that the monodromy, G, at p acts on the coordinate y of (II.34) by a character, γ, then

$$\partial^\sigma = \partial \Rightarrow \gamma(\sigma)^r = 1 \quad \sigma \in G \tag{II.49}$$

On the other hand from the adjunction formula, II.d.4, in the notation of (II.32)

$$K_{\mathscr{F}} \cdot \mathscr{L} = (\mathscr{L} : \mathscr{L}_0)\Big(\frac{r}{\mathrm{ord}(\gamma)} - \frac{1}{d_q}\Big) \tag{II.50}$$

which from (II.49) is non-negative as soon as $r > 0$. There is, however, not only no way to guarantee that (II.48) contains a generator, but this may well be impossible on every birational model with log-canonical singularities since this is the root cause, [MP13, III.iii.3.bis], of why log-canonical resolutions need not exist in the category of varieties.

II.f. **Linear Holonomy of, at worst, nodal $-\frac{1}{d}\mathbb{F}$ Curves.** Throughout $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ is a (saturated) foliation of a smooth complex Deligne-Mumford champ; $f : \mathscr{L} \to \mathscr{X}$ is a $-\frac{1}{d}\mathbb{F}$ curve, with f net, and $\mathscr{L}$ smooth. As such we have a specialised foliation, II.e.5, to the normal bundle, $N_{\mathscr{L}/\mathscr{X}}$, and we assert

II.f.1. **Claim.** The specialised foliation is in fact saturated.

Proof. Suppose $\mathscr{L}$ is simply connected (which one can always reduce to by [SGA-I, Exposé I, 8.3], or [McQ15, IV.a.2] in a slightly more appropriate generality, and I.e.5) then by the definition of a $-\frac{1}{d}\mathbb{F}$ curve we have $r = 0$ in (II.47), while lack of saturation is equivalent to the matrix A of *op. cit.* being divisible by t which can only happen if the matrix Δ therein is 0, *i.e.* the normal bundle is trivial. Thus, exactly as in the proof of II.d.5, f moves in a covering family, f_t, of disjoint invariant parabolic champ each of which must meet sing($\mathscr{F}$) for numerical reasons- *i.e.* $K_{\mathscr{F}} \cdot_f \mathscr{L} = K_{\mathscr{F}} \cdot_{f_t} \mathscr{L}$, (II.d.4), and generic f_t a generic embedding- so the singular locus of $\mathscr{F}$ must be a divisor. □

As such we can cease to worry about whether the foliation is a saturated or not, and

II.f.2. **Set Up.** We further suppose that $\mathscr{L}$ is simply connected, *i.e.* it is a weighted projective champ $\mathbb{P}^1(d, e)$, I.c.1.

In particular therefore we can describe the total space, $N := N_{\mathscr{L}/\mathscr{X}} \to \mathscr{L}$, of the normal bundle as the classifying champ $[E/\mathbb{G}_m]$ of the action

$$E := \big((\mathbb{A}^2\backslash 0) \times \mathbb{A}^n)\big) \times \mathbb{G}_m : (y_0, y_1) \times (x_1, \cdots, x_n) \times \lambda \mapsto (\lambda^d y_0, \lambda^e y_1) \times (\lambda^{a_i} x_i) \tag{II.51}$$

where as in II.e.6 the weights

$$a_1 \geq a_2 \cdots \geq a_n \tag{II.52}$$

are those of the Harder-Narismhan filtration, I.c.7 of the normal bundle. Consequently if $\pi : E \to N$ is the projection then the tangent space to the normal bundle is described by an Euler sequence of $\mathbb{G}_m$-equivariant, *cf.* I.c.2, bundles

$$0 \to \mathscr{O} \xrightarrow{1 \mapsto \rho} T_E = \mathscr{O}(d) \amalg \mathscr{O}(e) \amalg_i \mathscr{O}(a_i) \to \pi^* T_N \to 0 \tag{II.53}$$

where ρ is the radial, *cf.* I.d.2, vector field

$$\rho := d y_0 \frac{\partial}{\partial y_0} + e y_1 \frac{\partial}{\partial y_1} + a_1 x_1 \frac{\partial}{\partial x_1} + \cdots + a_n x_n \frac{\partial}{\partial x_n} \tag{II.54}$$

Now by II.e.1 the canonical bundle of the specialised foliation is $\mathscr{O}(-e)$, while for any $\mathbb{G}_m$-equivariant coherent sheaf $\mathscr{E}$ we have, in the notation of I.c.1, a Höschild-Serre spectral sequence

$$\mathrm{H}^p(\mathrm{B}_{\mathbb{G}_m}, \mathrm{H}^q(A_k, \mathscr{E})) \Rightarrow \mathrm{H}^{p+q}(\mathbb{P}^1(d,e), \mathscr{E}) \tag{II.55}$$

and whence by I.c.3

$$\mathrm{H}^1(\mathrm{B}_{\mathbb{G}_m}, \pi^* K_{\mathscr{F}}) = 0 \tag{II.56}$$

Combining this with (II.53) implies that the specialised foliation on the normal bundle is defined by a vector field ∂ on the total space E such that

$$\partial^\lambda = \lambda^{-e} \partial, \quad \lambda \in \mathbb{G}_m \tag{II.57}$$

At the same time, by construction, (II.41), there are functions F_p, A_{ij} in $\mathbb{C}[\mathbb{A}^2]$ such that

$$\partial = F_0 \frac{\partial}{\partial y_0} + F_1 \frac{\partial}{\partial y_1} + A_{ij} x_j \frac{\partial}{\partial x_i}, \quad 1 \leq i, j \leq n \tag{II.58}$$

where as per *op. cit.* we employ the summation convention. As such from (II.52), (II.57), and our normalisation, (II.42), that the singularity is at $(0, 1)$,

$$F_0 = 0, \; F_1 \in \mathbb{C}^\times, \; A_{ij}, \text{ is } a_i - a_j - e \text{ weighted homogeneous.} \tag{II.59}$$

In particular therefore, by (II.52), A_{ij} is an upper semi-triangular matrix with 0 diagonal. We can, however, do better, to wit:

II.f.3. **Fact.** *For a possibly different splitting of the Harder-Narismhan filtration, I.c.8, of* $N_{\mathscr{L}/\mathscr{X}}$ *and after a trivial renormalisation by a constant*

$$\partial = -e\frac{\partial}{\partial y_1} \tag{II.60}$$

Proof. Consistent with the notation of (II.52) the Harder-Narismhan filtration may be written as

$$[0] = N_0 \subsetneqq N_1 \subsetneqq N_2 \subsetneqq \cdots \subsetneqq N_k = N_{\mathscr{L}/\mathscr{X}} \tag{II.61}$$

where the normal bundle of N_i in N_{i+1} restricted to the zero section is a trivial bundle twisted by some $\mathscr{O}_{\mathscr{L}}(\alpha_i)$ for α_i a complete repetition free list of the a_i's; thus strictly decreasing as one proceeds up the chain. By (II.59) this is equally a filtration by $\mathscr{F}$-invariant sub-bundles, so, understanding the induced foliation on a sub-bundle logarithmically, I.b.2, if necessary (*i.e.* $a_1 = 0$) we prove (II.60) by induction on the length of the chain (II.61). The case $k = 1$ is immediate by (II.59), so by induction the matrix A_{ij} is an idempotent of the form

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}, \quad A \in \mathrm{Hom}_{\mathbb{G}_m}(N_k/N_{k-1}, N_{k-1})(-e) \tag{II.62}$$

Plainly, we aim for (II.59) via a change of coordinates of the form

$$\begin{bmatrix} \tilde{x}_{a_i<\alpha_k} \\ \tilde{x}_{a_i=\alpha_k} \end{bmatrix} = \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{a_i<\alpha_k} \\ \tilde{x}_{a_i=\alpha_k} \end{bmatrix}, \quad B \in \mathrm{Hom}_{\mathbb{G}_m}(N_k/N_{k-1}, N_{k-1}) \tag{II.63}$$

so that what we have to solve (in matrices of function in $\mathbb{C}[\mathbb{A}^2]$) is:

$$e\frac{\partial B}{\partial y_1} = A, \tag{II.64}$$

in a way that respects the $\mathbb{G}_m$-equivariance of (II.62)-(II.63), which, (II.59), is clear. □

To re-interpret this in terms of the standard affine patches U, U' of (II.42)-(II.43) one simply splits (II.53) along the inclusion of the respective (quasi) transversals $y_p = 1$, *i.e.*

II.f.4. **Summary.** Suppose the (embedded) $-\frac{1}{d}\mathbb{F}$ curve has at worst nodes, equivalently that its normalisation is net over $\mathscr{X}$, and that the universal cover, $\mathscr{L}$, of the same is a $\mathbb{P}^1(d, e)$, then $\mathscr{L} \to \mathscr{X}$ is net with a well defined normal bundle $N_{\mathscr{L}/\mathscr{X}}$ such that after pulling back, II.f.2, to the universal cover we have in the étale description, (II.42)-(II.43), of the normal bundle

(1) On $U \xrightarrow{\sim} \mathbb{A}^1$ an étale neighbourhood of the singularity a μ_e invariant generator of the specialised foliation, and $\zeta \in \mu_e$-action given by,

$$\partial = dy\frac{\partial}{\partial y} + a_i x_i \frac{\partial}{\partial x_i}, \quad y \mapsto \zeta^d y \quad x_i \mapsto \zeta^{a_i} x_i, \ a_i \in \mathbb{N} \tag{II.65}$$

(2) On $U' \xrightarrow{\sim} \mathbb{A}^1$ a complementary neighbourhood of the singularity, a basis ξ_i of functions invariant by the specialised foliation, on which $\zeta \in \mu_d$ acts via $\xi_i \mapsto \zeta^{a_i}\xi_i$.

(3) A patching $t^{a_i}\beta^*\xi_i = \alpha^* x_i$ in the notation of (II.42), and whence, an isomorphism

$$N_{\mathscr{L}/\mathscr{X}} \xrightarrow{\sim} \coprod_i \mathscr{O}_{\mathscr{L}}(a_i) \tag{II.66}$$

In particular the canonical or Harder-Narismhan filtration of $N_{\mathscr{L}/\mathscr{X}}$, (II.61), is a filtration by $\mathscr{F}$-invariant sub-bundles whose slopes and rank may be read directly from the generator (II.65) at the singularity.

II.g. **Formal Holonomy.** We wish to extend the previous discussion of linear holonomy of smooth $-\frac{1}{d}\mathbb{F}$ champs to the rather more delicate case of formal holonomy. Plainly when the curve, $\mathscr{L}$, is smooth and simply connected, the calculations are easier, and we denote by $\mathfrak{X} \to [\mathfrak{X}/\mathscr{F}]$ a foliated smooth formal champ whose trace $\mathscr{L}$ is a $-\frac{1}{d}\mathbb{F}$ champs isomorphic to $\mathbb{P}^1(d,e)$. In practice $\mathfrak{X}$ will, by [SGA-I, Exposé I, 8.3] or [McQ15, IV.a.2], be the universal cover of the the completion of a smooth foliated algebraic champ $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ along the net map, I.e.5, afforded by the normalisation of an at worst nodal $-\frac{1}{d}\mathbb{F}$ curve. For the moment, however, this is logically irrelevant. Supposing no risk of confusion with the notation of (II.40)-(II.41), we replace U' by V in (II.42), and take $\mathfrak{U} \to \mathfrak{X}$, resp. $\mathfrak{V} \to \mathfrak{X}$, to be formal étale neighbourhoods in the analytic topology with trace the $\mathbb{A}^1$'s U, resp. V of (II.42). In particular, therefore, $\mathfrak{U}$, resp. $\mathfrak{V}$, has a μ_e, resp. μ_d, action and there are open analytic embeddings $[\mathfrak{U}/\mu_e] \hookrightarrow \mathfrak{X}$, resp. $[\mathfrak{V}/\mu_d] \hookrightarrow \mathfrak{X}$, extending $[U/\mu_e] \hookrightarrow \mathscr{L}$, resp. $[V/\mu_d] \hookrightarrow \mathscr{L}$. Whence $\mathfrak{V}$ is simply connected, and, in the analytic topology, there is a certain strengthening of II.f.4, *viz:* the foliation may be supposed trivial over $\mathfrak{V}$, *i.e.* we have analytic coordinate functions ξ_i, η normal and parallel to our $\mathbb{A}^1$ respectively such that in $\mathfrak{V}$ the foliation is just the formal fibration $\xi_1 \times \cdots \times \xi_n : \mathfrak{V} \to \hat{\Delta}^n$, where the latter space is a n-polydisc completed in the origin. The algebra $\mathbb{C}[[\xi_1, \ldots, \xi_n]]$ comes equipped with a μ_d action- the formal holonomy representation- which, modulo the maximal ideal, is nothing other than that of the linear holonomy, (II.45). The said algebra is, however, an inverse limit of finite dimensional vector spaces over a field in which d is invertible, so the action may be written $\xi_i \to \zeta^{-a_i}\xi_i$ without prejudice to II.f.4.(1)-(3).

Now, we can choose ∂ on $\mathfrak{U}$ to be μ_e invariant, and, inductively we further suppose: for $m \in \mathbb{N}$ given, and a possibly different μ_e-invariant generator, ∂, on $\mathfrak{U}$ that there is a coordinate function y restricting to that of II.f.4.(1), such that,

$$\partial y \equiv dy(\mathscr{I}_{\mathscr{L}}^m), \quad (\zeta, y) \mapsto \zeta^d y\,(\mathscr{I}_{\mathscr{L}}^m), \quad (\zeta, \partial) \mapsto \partial \in \mathrm{Der}(\mathscr{O}_U), \quad \zeta \in \mu_e \tag{II.67}$$

The space $\mathfrak{U}$, unlike its trace U, has non-trivial units, so, a priori this isn't equivalent to the weaker

$$\partial y \equiv duy(\mathscr{I}_{\mathscr{L}}^m), \quad (\zeta, y) \mapsto \zeta^d y\,(\mathscr{I}_{\mathscr{L}}^m), \quad (\zeta, \partial) \mapsto \partial \in \mathrm{Der}(\mathscr{O}_U), \quad \zeta \in \mu_e \tag{II.68}$$

for u invertible modulo $\mathscr{I}_{\mathscr{L}}^m$. Nevertheless, we're in characteristic zero, so, in fact

II.g.1. **Claim.** The conditions (II.67) and (II.68) are equivalent.

Proof. Supposing (II.68), we have

$$\partial(y) - duy = f, \; y^\zeta - \zeta^d y = g \quad f, g \in \Gamma(U, \mathscr{I}_{\mathscr{L}}^m) \tag{II.69}$$

from which the invariance of ∂ affords,

$$d\zeta^d y(u^\zeta - u) \;=\; \zeta^d f - f^\zeta + \partial(g) - du^\zeta g \in \Gamma(U, \mathscr{I}_{\mathscr{L}}^m) \tag{II.70}$$

and we conclude that $u^\zeta - u \in \mathrm{H}^1(\mu_e, \mathscr{I}_{\mathscr{L}}^m)$. Since everything is tame, however, such a cohomology group vanishes, so we can find a μ_e-invariant unit v equal to u modulo $\mathscr{I}_{\mathscr{L}}^m$, and replacing ∂ by $v^{-1}\partial$ we deduce (II.67) from (II.69). □

Denoting by $\mathfrak{X}_m$, U_m, V_m, etc. the reduction of whatever modulo $\mathscr{I}_{\mathscr{L}}^m$, observe, by II.f.4.(3), that for y as in (II.67) there is a function t_0 on $U_m \times_{\mathfrak{X}} V_m$ such that yt_0^{-d} is congruent to 1 modulo nilpotents. We are, however, in characteristic 0, so, from the power series of the logarithm, yt_0^{-d} has a dth root. Thus

$$\exists\, t \in \Gamma(U_m \times_{\mathfrak{X}} V_m) \;\ni\; y\,|_{U_m \times_{\mathfrak{X}} V_m} = t^d, \tag{II.71}$$

and we further assert,

II.g.2. **Claim.** Suppose that (II.67) holds, then for a possibly different μ_d linear basis ξ_i of the algebra $\mathbb{C}[[\xi_1, \ldots, \xi_n]]$ compatible with any previous choice of the same modulo $\mathscr{I}_{\mathscr{L}}^{m'}$ for $m' < m$,

(1) There are coordinates x_i normal to $\mathscr{L}$ on U such that in $\mathscr{I}_{\mathscr{L}}/\mathscr{I}_{\mathscr{L}}^{m+1}$,

$$\partial(x_i) = a_i x_i \quad x_i \mapsto \zeta^{a_i} x_i,\ \zeta \in \mu_e,\ a_i \in \mathbb{N},\ 1 \le i \le n \tag{II.72}$$

(2) The x_i glue to the ξ_i via $\alpha^* x_i = t^{a_i} \beta^* \xi_i$ as global sections of the $\mathscr{O}_{\mathfrak{X}_m}$ module

$$\mathscr{I}_{\mathscr{L}}/\mathscr{I}_{\mathscr{L}}^{m+1}(a_i) \tag{II.73}$$

where $\mathscr{O}_{\mathfrak{X}_m}(1)$ is the bundle with transition function t on $U_m \times_{\mathfrak{X}} V_m$.

Proof. We proceed inductively on m, the case $m = 1$ being II.f.4, so, by the first item of the induction hypothesis for $m-1$, $m \ge 2$, we can find coordinate functions x_i normal to $\mathscr{L}$ whose reduction modulo $\mathscr{I}_{\mathscr{L}}^2$ are a basis of the normal bundle over U such that,

$$\partial(x_i) - a_i x_i = a_{iJ}(y)\, x^J \in \Gamma(U_m, \mathscr{I}_{\mathscr{L}}^m/\mathscr{I}_{\mathscr{L}}^{m+1}), \tag{II.74}$$

where x^J is the monomial $x_1^{j_1} \ldots x_n^{j_n}$, $j_1 + \cdots + j_n = m$, the summation convention is employed, and $a_{iJ}(y)$ is an entire function. Similarly by the second part of the inductive hypothesis:

$$t^{a_i}\xi_i - x_i \,|_{U_m \times_{\mathfrak{X}} V_m} = b_{iJ}(t)\, x^J \in \Gamma(U_m \times_{\mathscr{X}} V_m,\ \mathscr{I}_{\mathscr{L}}^m/\mathscr{I}_{\mathscr{L}}^{m+1}) \tag{II.75}$$

with the same conventions, but where, now, $b_{iJ}(t)$ are only holomorphic for $t \in \mathbb{G}_m$. Combining (II.74) & (II.75), we obtain,

$$t\,\dot{b}_{iJ} + b_{iJ}(a_J - a_i) = -a_{iJ}(t^d) \in \mathscr{O}_{\mathbb{G}_m} \tag{II.76}$$

where $a_J = \sum_i j_i\, a_i$, and no summation is implied. Again we can integrate this, by way of

$$\frac{d}{dt}\,(t^{(a_J - a_i)}\, b_{iJ}) = -\, a_{iJ}(t^d)\, t^{a_J - a_i - 1}\,. \tag{II.77}$$

A priori, however, the b_{iJ} are holomorphic for $t \in \mathbb{G}_m$, so the b_{iJ} are, in fact, meromorphic, and no $a_{iJ}\, t^{a_J - a_i - 1}$ has a residue, whence:

$$b_{iJ} = h_{iJ}(t^d) + \frac{\lambda_{iJ}}{t^{a_J - a_i}} \tag{II.78}$$

where h_{iJ} is entire, and λ_{iJ} is a constant. In particular,

$$\tilde{x}_i := x_i + h_{iJ}(t^d) x^J \text{ satisfies } \partial(\tilde{x}_i) = a_i \tilde{x}_i \ (\operatorname{mod} I_{\mathscr{L}}^{m+1}) \tag{II.79}$$

and defines n normal coordinate functions on $\mathfrak{U}$, such that,

$$\tilde{x}_i = t^{a_i} \tilde{\xi}_i,\ \text{where, } \tilde{\xi}_i := \xi_i - \lambda_{iJ}\xi^J. \tag{II.80}$$

The far left hand side of (II.79) is entire in t^d, so $\tilde{\xi}_i$ is still a μ_d-linear basis of $\mathscr{I}_{\mathscr{L}}/\mathscr{I}_{\mathscr{L}}^{n+1}\,|_{V_m}$ (compatible with our previous choices), and $a_J \equiv a_i(d)$ if $\lambda_{iJ} \neq 0$ by the coincidence of the formal holonomy with the linear holonomy (II.45). It therefore only remains to guarantee the μ_e linearity, (II.45). To this end, supposing the change of basis in (II.79) & (II.80) already made so as to momentarily drop the ˜ from the notation, we have for $\zeta \in \mu_e$ a generator:

$$x_i^\zeta - \zeta^{a_i} x_i = g_{iJ}(y) x^J \in \Gamma(U_m, \mathscr{I}_{\mathscr{L}}^m/\mathscr{I}_{\mathscr{L}}^{m+1}) \tag{II.81}$$

Applying the invariance of ∂ in (II.67), the right hand side of (II.81) must belong to the eigenspace of a_i for ∂ viewed as a $\mathbb{C}$-linear map. As such,

$$g_{iJ}(y) = \sum_{nd + a_J = a_i} g_{iJn} y^n \tag{II.82}$$

and we can suppose that the x_i have been rendered μ_e-linear by a coordinate change,

$$\tilde{x}_i \,=\, x_i + \sum_{nd+a_J=a_i} f_{iJn} y^n x^J \tag{II.83}$$

which yields new functions over ∞,

$$\tilde{\xi}_i := t^{-a_i}\tilde{x}_i = \xi_i + \sum_{nd+a_J=a_i} f_{iJn}\xi^J \tag{II.84}$$

and since J now has cardinality at least 2, this is also a μ_d linear coordinate change. $\square$

Let us now observe how to boot strap in the presence of II.g.2, by finding some y satisfying (II.67) modulo $\mathscr{I}_{\mathscr{L}}^{m+1}$, $m \geq 1$. Over U we have, in the notation/spirit of the proof of II.g.2,

$$\partial y = dy + c_J\, x^J (\mathscr{I}_{\mathscr{L}}^{m+1})\,, \quad \partial x_i = a_i\, x_i + c_{iK}\, x^K (\mathscr{I}_{\mathscr{L}}^{m+2}) \tag{II.85}$$

where the summation convention is back in force, with respect to multi-indices J and K of degrees m, $m+1$ respectively, all the c_*'s are regular functions of y, and by tameness of the monodromy $(\zeta, y) \mapsto \zeta^d y$, $\zeta \in \mu_e$ in all of $\mathscr{O}_U$. We know that the holonomy of the system (II.85) is a quotient of μ_d, so, we again take t as in (II.71), and at a presumably negligible risk of notational confusion let,

$$\xi_i = t^{-a_i}\, x_i + b_{iK}(t)\, x^K \,(\operatorname{mod} \mathscr{I}_{\mathscr{L}}^{m+2}) \tag{II.86}$$

be a basis of invariant functions on an analytic étale neighbourhood of $\mathbb{G}_m$, with summation over the multi-index K of degree $m+1$ being implied. Combining these, yields for any i,

$$c_{iK}\, t^{(a_K - a_i - 1)} + \frac{d}{dt}\,(t^{a_K}\, b_{iK}) \,=\, \begin{cases} \frac{a_i}{d}\cdot c_J\, t^{a_J-(d+1)} & \text{if } x^K = x^J x_i \text{ for some } J, \\ 0 & \text{otherwise} \end{cases} \tag{II.87}$$

By II.f.1, we known there is some i with $a_i \neq 0$, while b_{iK} must be holomorphic in $\mathbb{G}_m$, so

$$\text{if } a_J = d \text{ then } c_J(0) = 0, \tag{II.88}$$

since in such an eventuality the exponent of the leftmost term, $a_J - 1$, is non-negative. Similarly, if much more straightforwardly, the μ_e invariance of ∂, and our insistence that $y \mapsto \zeta^d y$ implies,

$$c_J^\zeta = \zeta^{d-b_J} c_J, \quad b_J = \sum_i b_i j_i, \text{ for, } J = (j_1, \dots, j_n) \tag{II.89}$$

with b_i as per II.g.2.(1), and whence,

$$\text{if } c_J(0) \neq 0, \text{then } b_J \equiv d(e). \tag{II.90}$$

On the other hand consider the obstruction to finding a coordinate $\tilde{y}$ over U restricting to the same on $\mathscr{L}$ such that,

$$\partial\tilde{y} = d(1+\lambda)\,\tilde{y}\; (\mathscr{I}_{\mathscr{L}}^{m+1})\,, \quad \lambda = \lambda_J\, x^J \in \mathscr{I}_{\mathscr{L}}^{m}\,, \quad \lambda_J \in \mathscr{O}_{U \cap \mathscr{L}}\,. \tag{II.91}$$

If we look for such a $\tilde{y}$ in the form, $y + \Lambda_J\, x^J$, with Λ_J constants, then we require to solve,

$$(a_J - d)\Lambda_J - d\lambda_J\, y = -c_J \tag{II.92}$$

for all J. However if $a_J \neq d$, then $\Lambda_J = -c_J(0)(a_J - d)^{-1}$, and λ_J whatever, will do, while if $a_J = d$, then by (II.88) we can take $\Lambda_J = 0$, and $\lambda_J = c_J\, y^{-1}$. Whether trivially in the latter case, or by (II.89)-(II.90) in the former case, $\tilde{y}^\zeta \equiv \zeta^d \tilde{y}\,(\mathscr{I}_{\mathscr{L}}^{m+1})$, so we obtain (II.68), and whence (II.67) by II.g.1.

As per II.g.2, the coordinate $\tilde{y}$ also restricts to the previous choice modulo $\mathscr{I}_{\mathscr{L}}^{m}$, so we obtain in the limit an extension of the canonical/Harder-Narismhan filtration to the whole neighbourhood, *i.e.*

II.g.3. **Proposition/Summary.** *Let* $\mathfrak{X} \to [\mathfrak{X}/\mathscr{F}]$ *be a foliated smooth formal champ whose trace is a smooth simply connected* $-\frac{1}{d}\mathbb{F}$ *curve,* $\mathscr{L} \xrightarrow{\sim} \mathbb{P}^1(d,e)$*, then,*

(1) There is a bundle $\mathscr{O}_{\mathfrak{X}}(1)$ *lifting* $\mathscr{O}_{\mathscr{L}}(1)$ *and a smooth formal invariant divisor* D*, with* $\mathscr{O}_{\mathfrak{X}}(D) = \mathscr{O}_{\mathfrak{X}}(d)$ *transverse to* $\mathscr{L}$ *which restricts to the unique point* z *of* $\mathscr{L} \cap \mathrm{sing}\,(\mathscr{F})$*.*

(2) There is a filtration of formal invariant sub-champs,

$$\mathscr{L} = \mathfrak{X}_0 \subsetneq \mathfrak{X}_1 \subsetneq \cdots \subsetneq \mathfrak{X}_k = \mathfrak{X} \tag{II.93}$$

such that if $\alpha_1 > \cdots > \alpha_k$ *are the distinct eigenvalues of* ∂ *considered linearised in* $\mathrm{End}\,(N_{\mathscr{L}/\mathfrak{X}} \otimes \mathbb{C}(z))$*, and normalised by II.f.4.(1) with* $n_1, \ldots, n_k$ *the dimensions of the corresponding eigenspaces then* $\mathfrak{X}_i$ *is defined by* $\mathscr{F}$*-invariant global sections* γ_j *of* $\mathscr{O}_{\mathfrak{X}}(\alpha_j)$*,* $j > i$*, and* n_j*-sections for each* j*. In particular,*

$$N_{\mathscr{L}/\mathfrak{X}_i} \xrightarrow{\sim} \coprod_{j \le i} \mathscr{O}_{\mathscr{L}}(\alpha_j)^{n_j} \tag{II.94}$$

(3) All of this is encoded in a particular μ_e *linear coordinate system,* y, x_i*,* $y \mapsto \zeta^d y$*,* $x_i \mapsto \zeta^{a_i} x_i$ *of an étale neighbourhood* $\mathfrak{U} \to \mathfrak{X}$ *with trace* $\mathbb{A}^1$ *containing the singular point over which we have a* μ_e *invariant generator,*

$$\partial = dy\frac{\partial}{\partial y} + a_i\, x_i \frac{\partial}{\partial x_i} \tag{II.95}$$

summation convention in force, so that the α_j*, are a complete repetition free list of the* a_i*.*

II.h. **Jordan Decomposition.** We briefly interrupt our discussion of $K_{\mathscr{F}}$-negative invariant champs to recall some salient facts on Jordan decomposition which will be relevant both to our study of cusps, and the local uniqueness of the Harder-Narismhan filtration. The situation is entirely local and, initially, scheme-like, i.e. $\mathscr{O}$ is the ring of formal power series $\mathbb{C}\,[[x_1, \ldots, x_n]]$, $\mathfrak{m}$ its maximal ideal, and ∂ a $\mathbb{C}$-derivation of $\mathscr{O}$ with a singularity at the origin. Recall that since $\mathscr{O}$ is an inverse limit of finite dimensional vector spaces ∂ admits a Jordan decomposition, i.e. $\partial = \partial_S + \partial_N$, where the semi-simple part ∂_S acts as a semi-simple matrix on each $\mathscr{O}/\mathfrak{m}^n$, $n \in \mathbb{N}$, ∂_N is nilpotent, and of course $[\partial_S,\, \partial_N] = 0$. In particular if $\partial_S = \lambda_i\, x_i \frac{\partial}{\partial x_i}$, summation convention, then a conventional choice of basis for the nilpotent fields commuting with ∂_S is,

II.h.1. **Revision.** (cf. [Mar81]) Notations as above then $\partial_N = \sum_{i=1}^{n} \sum_{Q_i} a_{Q_i} x^{Q_i} x_i \frac{\partial}{\partial x_i}$, $a_{Q_i} \in \mathbb{C}$, where either,

(i) $Q_i = (q_1, \ldots, q_n)$, $q_j \in \mathbb{N} \cup \{0\}$, $x^{Q_i} = x_1^{q_1} \ldots x_n^{q_n}$, $\Lambda \cdot Q_i = 0$, or
(ii) $Q_i = (q_1, \ldots, q_n)$, $q_i = -1$, $q_j \in \mathbb{N} \cup \{0\}$, $j \ne i$, $x^{Q_i} = x_1^{q_1} \ldots x_n^{q_n}$, $\Lambda \cdot Q_i = 0$.

Now the Jordan decomposition of a vector field is certainly unique, and whence the property of semi-simplicity of a vector field is wholly unambiguous. For a foliation however the situation is rather more delicate since there is a question of rescaling by units. Whence suppose our field ∂ is semi-simple, and consider a field $\tilde{\partial} = u\partial$, where $u \equiv 1(\mathfrak{m})$ to avoid stupidity. Furthermore let's say, without loss of generality, that $\partial = \partial_S = \lambda_i\, x_i \frac{\partial}{\partial x_i}$ then we assert,

II.h.2. **Claim.** Notations as above, there is a change of coordinates of the form, $\xi_i = u_i\, x_i$, $u_i \equiv 1(\mathfrak{m})$, and $\varepsilon \equiv 0(\mathfrak{m})$ with $\partial\varepsilon = 0$ such that the Jordan decomposition of $\tilde{\partial}$ is,

$$\tilde{\partial} = \lambda_i\, \xi_i \frac{\partial}{\partial \xi_i} + \varepsilon\, \lambda_i\, \xi_i \frac{\partial}{\partial \xi_i} \tag{II.96}$$

i.e. $\tilde{\partial}$ may not be semi-simple, but the extent to which it is not is very particular.

Proof. Consider the following inductive proposition for $k \in \mathbb{N}$,

there are coordinates $x_{ik} = u_{ik}\, x_i$, $u_{ik} \equiv 1(\mathfrak{m})$, $\tilde{\partial} = u_k\, \partial_k$, $\partial_k = \lambda_i\, x_{ik} \frac{\partial}{\partial x_{ik}}$, $u_k \equiv 1(\mathfrak{m})$ such that $u_k^{-1} = 1 + \varepsilon_k + \delta_k$, where ε_k, δ_k are defined by way of the Jordan decomposition of $\mathfrak{m}$ as $\operatorname{Ker} \partial_k \oplus \operatorname{Im} \partial_k$, and $\delta_k \in \mathfrak{m}^k$.

The case $k = 1$ is simply our given data. Otherwise consider trying to improve the situation by putting, $x_{ik+1} = v_{ik}\, x_{ik}$, $v_{ik} \equiv 1(\mathfrak{m})$ to be chosen. If such a change were to actually render $\tilde{\partial}$ semi-simple then we would have to solve,

$$\partial_k \log v_{ik} = \lambda_i \left(\frac{1}{u_k} - 1 \right) = \lambda_i (\varepsilon_k + \delta_k) \tag{II.97}$$

which plainly may not be possible if $\lambda_i \neq 0$, and $\varepsilon_k \neq 0$. However we can solve $\partial_k \log v_{ik} = \lambda_i\, \delta_k$, so that in particular, $v_{ik} \equiv 1(\mathfrak{m}^k)$, while in the new coordinates,

$$\tilde{\partial} = \frac{1 + \delta_k}{1 + \varepsilon_k + \delta_k}\, \lambda_i\, x_{ik+1} \frac{\partial}{\partial x_{ik+1}} \tag{II.98}$$

which is indeed what we're looking for, since putting $u_{k+1} = (1 + \delta_k)\, u_k$ then,

$$u_{k+1}^{-1} = 1 + \varepsilon_k (1 + \delta_k)^{-1} = 1 + \varepsilon_k + \sum_{n=1}^{\infty} (-1)^n\, \varepsilon_k\, \delta_k^n \tag{II.99}$$

so that $\delta_{k+1} \in \mathfrak{m}^{k+1}$.

Certainly therefore the $\delta_k \to 0$, but the proof also shows that for each i the infinite product, $\prod_k v_{ik}$ converges to some u_i, so putting $\xi_i = u_i\, x_i$ we're certainly done on observing that $\partial \varepsilon = 0$ obliges,

$$\left[\lambda_i\, \xi_i \frac{\partial}{\partial \xi_i}\, ,\ \varepsilon\, \lambda_i\, \xi_i \frac{\partial}{\partial \xi_i} \right] = 0\,. \tag{II.100}$$

□

The consequence of the fact that not only can Jordan decomposition of a rescaling of semi-simple only fail in a very controlled way, but also that Jordan decompositions of rescalings are related in such a straight forward way suggests that we introduce,

II.h.3. **Definition.** A germ of a foliation $(\hat{\Delta}^n, \mathscr{F})$ on a formal disc, *i.e.* $\mathrm{Spf}(\mathbb{C}[[t_1, \cdots, t_n]])$, with a not necessarily isolated singularity at the origin is said to be semi-simple, if $T_{\mathscr{F}} = \mathscr{O}_{\hat{\Delta}^n} \partial$ for some semi-simple vector field ∂.

As an important example/application consider the situation of blowing up in the origin, i.e. $\rho : (X, \tilde{\mathscr{F}}) \to (\hat{\Delta}^n, \mathscr{F})$ is the said modification with induced foliation and X is the completion in the exceptional divisor of the blow up of $\operatorname{Spec} \mathscr{O}$. Denoting by, $\partial = \partial_S + \partial_N$ a Jordan decomposition of any generator $T_{\mathscr{F}}$ we have,

II.h.4. **Fact.** *Suppose $\partial_S \neq 0$ and $(X, \tilde{\mathscr{F}})$ is not everywhere smooth (which in any case could only happen if in suitable coordinates $\partial = x_i \frac{\partial}{\partial x_i}$) then the following are equivalent,*

1. *$(\hat{\Delta}^n, \mathscr{F})$ is semi-simple.*
2. *$(X, \tilde{\mathscr{F}})$ is semi-simple at all of its singular points.*
3. *$(X, \tilde{\mathscr{F}})$ is semi-simple at one of its singular points, and $(\hat{\Delta}^n, \mathscr{F})$ is semi-simple modulo $\mathfrak{m}^2$.*
4. *$(X, \tilde{\mathscr{F}})$ is semi-simple at one of its singular points.*

Before proceeding, we will require a lemma, to wit:

II.h.5. **Lemma.** *Notations as above, then at every point of its singular locus, $\rho^*\partial_N$ is nilpotent.*

Proof. Without loss of generality we can suppose the projective coordinates of some singular point, p, in the exceptional divisor to be $[1, 0, \ldots, 0]$. Thus if $\partial_N = a_{ij}x_j\frac{\partial}{\partial x_i}$, summation convention in force, then $a_{i1}(0) = 0$ for every $i \geq 2$. This is equivalent, however, to the column vector defined by p being an eigenvector, so $a_{11}(0) = 0$ too. Now, observe that a square matrix, $[c_{ij}]_{i,j\geq 1}$ with a zero first column is nilpotent iff the matrix $[c_{ij}]_{i,j\geq 2}$ is nilpotent, while the linearisation of $\rho^*\partial_N$ in p is,

$$\text{(II.101)} \qquad \begin{bmatrix} a_{11}(0) & 0 & \ldots & 0 \\ \frac{\partial a_{21}}{\partial x_1}(0) & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots & \ldots & a_{ij}(0) & \ldots \\ \frac{\partial a_{n1}}{\partial x_1}(0) & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

which has a zero first row, so it's also nilpotent. □

proof of II.h.4. Since $(X, \tilde{\mathscr{F}})$ is not everywhere smooth the induced foliation is given everywhere by $\rho^*\,\partial$ (cf. I.b.10) so trivially (1) implies everything else, while both (2) & (3) trivially imply (4). Consider therefore (4) $\Rightarrow$ (1). As in the above proof of II.h.5, a singular point of the singular locus of $\rho^*\partial$ is an eigenvector of its linearisation, whence an eigenvector of the linearisations of $\rho^*\partial_S$ & $\rho^*\partial_N$, and thus a singularity of both $\rho^*\partial_S$ & $\rho^*\partial_N$. We know, however, that every singularity of the former is semi-simple, so by II.h.5, $\rho^*\partial = \rho^*\partial_S + \rho^*\partial_N$ remains a Jordan decomposition at every point of the singular locus of $\tilde{\mathscr{F}}$. By hypothesis, at such a point p, there is some semi-simple generator $\tilde{\partial}$, so an application of II.h.2 yields $\varepsilon \in \hat{\mathscr{O}}_{X,p}$ such that $\rho^*\,\partial(\varepsilon) = 0$, and,

$$\text{(II.102)} \qquad \varepsilon\,\rho^*\,\partial_S = \rho^*\,\partial_N\,.$$

As such, if, x_1 is an eigenvector of ∂_S, with eigenvalue $\lambda_1 \neq 0$, then for $f = \partial_N(x_1)$, $\varepsilon = \rho^*(f/\lambda_1 x_1)$, while:

$$\text{(II.103)} \qquad 0 = x_1\partial(f/x_1) = \partial f - \frac{f}{x_1}\cdot(\lambda_1\,x_1 + f)$$

so $x_1 \mid f$, and ε is actually a function on $\hat{\Delta}^n$, from which we conclude. □

A further question which we may reasonably address here is the uniqueness, or lack thereof, of the Jordan decomposition. Even without rescaling the particular choice of coordinates in which we may write a semi-simple field as $\lambda_i\,x_i\,\partial/\partial x_i$ may be catastrophically non-unique. Plainly the worst possible case is when all the λ_i are rational, or equivalently up to a harmless rescaling integers. Even this is of course not unique but it's not too bad since of course any rational point in some $\mathbb{P}^N(\mathbb{Q})$ is up to multiplication by ± 1 uniquely represented by a tuple of relatively prime integers, consequently let's establish some notation,

II.h.6. *Notation.* Let ∂ be a semi-simple derivation of $\mathscr{O}$ with integer eigenvalues $a_1, \ldots, a_r$, $-b_1, \ldots, -b_t$, a_i, $b_j \in \mathbb{Z}_{>0}$, s zeroes, $r \geq 1$, although possibly $t = 0$, i.e. no negatives, and $(a_1, \ldots, a_r, b_1, \ldots, b_t) = 1$, then we will suppose these ordered by decreasing size, i.e.

$$\text{(II.104)} \qquad a_1 \geq a_2 \geq \cdots \geq a_r > 0 > -b_t \geq \cdots \geq -b_1$$

and by $\alpha_1, \ldots, \alpha_k$, $k \leq r$, $\beta_1, \ldots, \beta_l$, $l \leq t$ a complete repetition free list of the same, so that,

$$\text{(II.105)} \qquad \begin{gathered} a_1 = \alpha_1 > \alpha_2 > \cdots > \alpha_k > 0 \\ 0 > -\beta_l > \cdots - \beta_2 > \cdots > -\beta_1 = -b_1\,. \end{gathered}$$

Now for a given choice of basis of a semi-simple derivation ∂ with the said eigenvalues i.e. a particular way of writing it as $a_i\,y_i\,\frac{\partial}{\partial y_i} - b_j\,x_j\,\frac{\partial}{\partial x_j}$, with say z_k the null vectors, we can introduce, II.h.7. **Definition.** The Harder-Narismhan pair of $(\hat{\Delta}^n, \mathscr{F})$ with respect to the data (∂, y_i, x_j) is the invariant formal sub-schemes, X_+, X_- whose ideals are generated by the non-positive, respectively non-negative, eigenvectors of ∂. If instead we take strictly negative, respectively strictly positive, eigenvectors then the resulting subschemes, denoted $X_+^{\geq 0}$, $X_-^{\leq 0}$, will be called the non-strict Harder-Narismhan pair.

Manifestly, apart from abbreviating Harder-Narismhan to H-N, what's important is that the H-N pairs are well defined up to ± 1, i.e.

II.h.8. **Fact.** *Fix a choice of semi-simple ∂ with integer eigenvalues normalised as per II.h.6, then the following are equivalent,*

1. $\{X_+, X_-\}$, *respectively* $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, *is the H-N, resp. non-strict H-N, pair with respect to ∂ in the basis* $\{x_i, y_j\}$.
2. $\{X_+, X_-\}$, *resp.* $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, *is the H-N, resp. non-strict H-N, pair with respect to ∂ in any semi-simple basis.*
3. $\{X_+, X_-\}$, *resp.* $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, *is the H-N, resp. non-strict H-N, pair of any semi-simple $\tilde{\partial} = u\partial$ in any semi-simple basis for the same, where $u \equiv 1(\mathfrak{m})$.*

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are all trivial, so consider (1) $\Rightarrow$ (3). By II.h.2, we know that we can find units u_i, $v_j \equiv 1(\mathfrak{m})$ such that if $\eta_i = u_i\, y_i$, $\xi_j = v_j\, x_j$ then $\tilde{\partial} = a_i\, \eta_i\, \frac{\partial}{\partial \eta_i} - b_j\, \xi_j\, \frac{\partial}{\partial \xi_j}$. As such $\{X_+, X_-\}$, resp. $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, is the H-N, resp. strict, pair of $\tilde{\partial}$ in the basis $\{\xi_i, \eta_j\}$. Now suppose $\tilde{\partial} = a_i\, f_i\, \frac{\partial}{\partial f_i} - b_j\, g_j\, \frac{\partial}{\partial g_j}$ in some other basis f_i, g_j. At the mod $\mathfrak{m}^2$ level this is just a question of the uniqueness of diagonalisation/the commutator of a diagonal matrix, so without loss of generality let's say $f_i \equiv \xi_i$, and $g_j \equiv \eta_j(\mathfrak{m}^2)$. For higher order terms, consider the Taylor expansion,

$$f_i = \xi_i + \sum_{\#J + \#K \geq 2} c_{iJKL}\, \xi^J\, \eta^K\, \zeta^L\,, \tag{II.106}$$

where, as ever, ξ^J *etc.* is the monomial $\xi_1^{j_1} \ldots \xi_r^{j_r}$ *etc.*, and $\zeta_1, \ldots, \zeta_s$ are the null vectors. Now $\tilde{\partial}\, f_i = a_i\, f_i$ so,

$$c_{iJKL} \neq 0 \Rightarrow \sum_\alpha a_\alpha\, j_\alpha - \sum_b b_\beta\, k_\beta = a_i\,. \tag{II.107}$$

Consequently if $f_i \notin (\xi_1, \ldots, \xi_r)$, then we have a manifest absurdity, and so conclude by symmetry. ∎

The dependence on ± 1 is, however, unavoidable. Indeed let, $\hat{\Delta}^n \to [\hat{\Delta}^n/\mathscr{F}]$ be a germ of a singular foliation invariant by a finite group G, or, equivalently for ∂ a generator,

$$\partial^\sigma = \sigma\partial\sigma^{-1} = u(\sigma)\partial, \quad u : G \to \mathscr{O}_{\hat{\Delta}^n}^\times \tag{II.108}$$

where u is a group co-cycle, so, better, by the acylicity of B_G on torsion free abelian groups, a character χ on replacing ∂ by $v\partial$ for a suitable unit. At which point, however, if $\partial = \partial_S + \partial_N$ is a Jordan decomposition of ∂, then $\partial^\sigma = \partial_S^\sigma + \partial_N^\sigma$ is a Jordan decomposition of ∂^σ, so by unicity of the same,

$$\partial_S^\sigma = \chi(\sigma)\partial_S, \text{ and, } \partial_N^\sigma = \chi(\sigma)\partial_N \tag{II.109}$$

As such, if in addition $\hat{\Delta}^n \to [\hat{\Delta}^n/\mathscr{F}]$ is semi-simple, then, by II.h.2, ∂ and ∂_S generate the same foliation, so,

II.h.9. **Fact.** *If $\hat{\Delta}^n \to [\hat{\Delta}^n/\mathscr{F}]$ is a germ of a singular semi-simple foliation invariant by a finite group G, then there is a character $\chi : G \to \mathbb{Q}(1)/\mathbb{Z}(1)$ of G and a semi-simple generator*

∂ of the foliation such that, $\partial^\sigma = \chi(\sigma)\partial$, for all $\sigma \in G$. In particular, if the eigenvalues of a linearisation in $\mathfrak{m}/\mathfrak{m}^2$ are in $\mathbb{P}^{n-1}(\mathbb{Q})$ then χ takes values in $\{\pm 1\}$, and,

(a) If χ is trivial, all of X_+, X_-, $X_+^{\geq 0}$, $X_-^{\leq 0}$ are G invariant, and there is a H-N pair, respectively non-strict H-N pair, of embedded $\mathscr{F}$-invariant formal sub-champs, $\{[X_+/G], [X_-/G]\}$, respectively $\{[X_+^{\geq 0}/G], [X_-^{\leq 0}/G]\}$, in $[\hat{\Delta}^n/G]$.

(b) Otherwise, in the notation of II.h.6, $a_i = b_i$, $r = t$, etc., and $[X_+/\mathrm{Ker}_\chi]$, respectively $[X_+^{\geq 0}/\mathrm{Ker}_\chi]$, is isomorphic to $[X_-/\mathrm{Ker}_\chi]$, respectively $[X_-^{\geq 0}/\mathrm{Ker}_\chi]$, but is only net in $[X_+ \cup X_-/G]$, respectively $[X_+^{\geq 0} \cup X_-^{\geq 0}/G]$, which in turn are embedded in $[\hat{\Delta}^n/G]$, being defined by the G-invariant ideal (x_iy_i, z_k), respectively (x_iy_i).

Proof. If x is an eigenvector of ∂ with eigenvalue λ, then for any $\sigma \in G$, x^σ is an eigenvector of ∂ with eigenvalue $\lambda\chi(\sigma)^{-1}$, so when the eigenvalues are rational, χ must take values in rational roots of unity. □

Consequently, even in a purely scheme like situation, we have two canonical pairs rather than two pairs of canonical sub-schemes, and we make:

II.h.10. **Remark/Definition.** Let $\hat{\Delta}^n \to [\hat{\Delta}^n/\mathscr{F}]$ be a germ of a singular semi-simple foliation such that the eigenvalues of a linearisation in $\mathfrak{m}/\mathfrak{m}^2$ are in $\mathbb{P}^{n-1}(\mathbb{Q})$ then there are two canonical pairs of invariant formal subschemes, the H-N pair, $\{X_+, X_-\}$, and the non-strict H-N pair $\{X_+^{\geq 0}, X_-^{\leq 0}\}$, where the former intersect in the origin, the latter in the whole singular locus. If no-confusion is likely, the suffices may be dropped.

In the particular case of II.g.3, the trace of the formal neighbourhood $\mathfrak{X}$ affords a distinguished eigenvector, so the character appearing in II.h.9 around the singularity, p, is trivial. As such, by op. cit., the H-N pair, respectively non-strict H-N pair, extends from a formal neighbourhood of p to a pair of embedded invariant formal sub-champs $\{\mathfrak{X}_+, \mathfrak{X}_-\}$, respectively, $\{\mathfrak{X}_+^{\geq 0}, \mathfrak{X}_-^{\leq 0}\}$ of $\mathfrak{X}$. An important further task will be to extend this to cusps.

II.i. **Cusps.** We consider the consequences of the previous discussion for cuspidal $-\frac{1}{d}\,\mathbb{F}$ curves, $f : \mathscr{L} \to \mathscr{X}$, where, as ever, $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ is a foliated smooth champ. In the first instance the discussion is purely local, so, say, $f : \hat{\Delta}^1 \to \hat{X}$, the map between completions in the singularity $0 \in f^{-1}(\mathscr{Z})$, for $\mathscr{Z} = \mathrm{sing}(\mathscr{F})$. By, for example [BM97], the cusp may, *cf.* II.e.4, be resolved by the étale local operation of blowing up in the sequence of closed points,

$$\tilde{\mathscr{X}} = \mathscr{X}_N \to \ldots \to \mathscr{X}_1 \to \mathscr{X}_0 = \mathscr{X} \tag{II.110}$$

of which the first is $z := f(0)$, and subsequently where the proper transform of f meets the exceptional divisor until such time that f becomes an embedding, $\tilde{f}$, say, meeting the proper transform in $\tilde{z}$. Necessarily each blow up in (II.110) is in a point where the foliation is singular, so $K_{\tilde{\mathscr{F}}} \leq K_{\mathscr{F}}|_{\tilde{\mathscr{X}}}$, and $\tilde{f}$ can only fail to be a $-\frac{1}{d}\,\mathbb{F}$ curve if $\tilde{\mathscr{F}}$ is smooth everywhere around $\tilde{f}$. Now although such an occurrence is highly simplifying, *e.g.* $\mathscr{F}$ is algebraic in conics, II.d.5.(c), the foliation has a first integral in a (finite) étale neighbourhood of $\mathscr{L}$ *etc.*, it's preferable to avoid a separation of cases by viewing such a final situation as a $-\frac{1}{d}\,\mathbb{F}$ curve for $K_{\tilde{\mathscr{F}}} + E$, equivalently working logarithmically, I.b.2, around the final exceptional divisor, E, in (II.110). In this way, II.g.3.(3) and II.h.4 are always valid, from which:

II.i.1. **Lemma.** *Let $f : \mathscr{L} \to \mathscr{X}$ be a $-\frac{1}{d}\,\mathbb{F}$ curve meeting the singular locus in z, then around z the foliation is semi-simple.*

Consequently, let's say, $\partial = \lambda_i x_i \frac{\partial}{\partial x_i}$ a semi-simple generator of the foliation in the complete local ring $\mathscr{O}_{\hat{X},z}$, with $f : t \mapsto x_i(t) = t^{v_i}u_i(t)$ an expression for the cusp in terms of some local parameter t, with $v_i \in \mathbb{Z}_{>0}$, and u_i units whenever f^*x_i is not identically zero. As such, for any pair of indices i, j for which $f^*(x_ix_j)$ is not identically zero,

$$\text{(II.111)} \qquad \frac{\dot{x}_i(t)}{\lambda_i x_i} = \frac{\dot{x}_j(t)}{\lambda_j x_j}$$

Whence, if we re-label the coordinate system as y_i for those non-zero on the curve, x_j for those identically zero, and $y_1(t) = t^{v_1}$, then:

$$\text{(II.112)} \qquad \frac{\dot{y}_i(t)}{y_i} = \frac{v_i}{t} + \text{holomorphic} = \frac{\lambda_i v_1}{\lambda_1 t}$$

so, $y_i(t) = \eta_i t^{v_i}$, for some constant η_i, thus, without loss of generality $\eta_i = 1$ and $\lambda_i = v_i$. Proceeding thus, there may be some mild redundancy. Indeed, the cusp has an embedding dimension k, and re-labelling so that v_1 is minimal, then if $v_1|v_i$ one can replace any x_i by something in the same eigenspace (of ∂ qua operator on $\mathscr{O}$) which vanishes identically, *viz*: $y_i - y_1^{v_i/v_1}$, and in general, one can achieve,

$$\text{(II.113)} \qquad v_1 < v_2 < \cdots < v_k\,, \quad \text{and } v_{i+1} \notin \mathbb{Z}_{\geq 0}\, v_1 + \cdots + \mathbb{Z}_{\geq 0}\, v_i\,.$$

for each $1 \leq i \leq k$, so we get exactly k y_i's, the v_i have gcd 1 since f is bi-rational, and every other coordinate is a x_j vanishing identically.

Now, by hypothesis the local monodromy group, G, preserves the foliation on the formal completion, $\hat{X}$, of $\mathscr{X}$ at the singular point. Appealing to (II.109), we may suppose that it acts on the above ∂ by a character χ, and we denote by H the stabiliser of the image C of (the irreducible branch) $f: L \to \hat{X}$ obtained by completing the local ring of $\mathscr{L}$ at p. Consequently there is a factorisation,

$$\text{(II.114)} \qquad f: [L/H] \xrightarrow{\nu} [C/H] \xrightarrow{\phi} [\hat{X}/G]$$

and since everything is convergent in the étale topology, this can be glued to a global factorisation,

$$\text{(II.115)} \qquad f: \mathscr{L} \xrightarrow{\quad\nu\quad} \mathscr{C} \xrightarrow{\quad\phi\quad} \mathscr{X}$$

where the first map is the normalisation of $\mathscr{C}$, ϕ is net, and $\mathscr{C}$ is uni-branch. As such, outwith the unique singular point p, ν is an isomorphism, and ϕ a closed embedding. Equally, the wholly general I.e.5 applies, so there is a formal champ $\mathfrak{X}$ with trace $\mathscr{C}$ such that $\mathfrak{X} \to \hat{\mathscr{X}}$ onto the completion of $\mathscr{X}$ in the image of f is étale representable, and,

II.i.2. **Fact.** *Let $f: \mathscr{L} \to \mathscr{C} \hookrightarrow \mathfrak{X} \to \mathscr{X}$ be the above factorisation of the normalisation, $f: \mathscr{L} \to \mathscr{X}$ of a $-1/d\mathbb{F}$ cusp, with v_i as (II.113), and y_i, x_j, as above, suitable formal coordinates (on $\hat{X}$) about the singular point, then there are $a_j \in \mathbb{Z}$ such that the foliation is generated by,*

$$\text{(II.116)} \qquad \partial = dv_i\, y_i\, \frac{\partial}{\partial y_i} + a_j\, x_j\, \frac{\partial}{\partial x_j}$$

Proof. If there is a divisorial valuation of negative discrepancy passing through the closed singular point, then the proposition follows from I.b.10, (II.113), and the fact that the v_i have g.c.d. 1, so we may suppose that the singularity is canonical rather than just log-canonical.

Now we require a certain re-appraisal of (II.111)-(II.113) in the presence of the action of H in (II.114). To this end let I be the ideal of the image, C, of the cusp in the completion $\hat{X}$ in the singular point p whose maximal ideal we denote by $\mathfrak{m}$, then we have a H-equivariant exact sequence

$$\text{(II.117)} \qquad 0 \to I/I \cap \mathfrak{m}^2 \to \Omega_{\hat{X}} \otimes \mathbb{C}(p) \to \Omega_C \otimes \mathbb{C}(p) \to 0$$

which is equally equivariant under a semi-simple generator ∂ of the foliation. In particular, therefore, the induced endomorphism

$$\partial : \Omega_C \otimes \mathbb{C}(p) \to \Omega_C \otimes \mathbb{C}(p) \tag{II.118}$$

may be supposed to have eigenvalues the (distinct) v_i of (II.113) with multiplicity (both geometric and algebraic) equal to 1. As such, although H acts on ∂ a priori by a character, (II.109), such an action must, *cf.* II.h.9, be trivial. Consequently the $\mathbb{C}$-linear decomposition of $\mathfrak{m}$ into eigenspaces of ∂ is also H-equivariant. On the other hand all exact sequences of $\mathbb{C}[H]$-modules are split exact, so from

$$0 \to (I(\lambda) \cap \mathfrak{m}^2)/(I(\lambda) \cap \mathfrak{m}^k) \to I(\lambda)/(I(\lambda) \cap \mathfrak{m}^k) \to (I(\lambda) \cap \mathfrak{m}^2) \to 0,\ k \geq 2,\ \textit{etc.} \tag{II.119}$$

for any eigenvalue λ of ∂, we can write the H-action as blocks of $\mathbb{C}$-linear actions

$$H \ni \sigma : x_{i,\lambda} \mapsto A_{ij}(\sigma) x_{j,\lambda}, \quad y_i \mapsto \chi_i(\sigma) y_i \tag{II.120}$$

for a coordinates system $\{x_{i,\lambda}, y_i\}$ in which $x_{i,\lambda} \in I(\lambda)$, the y_i's afford eigenvectors of (II.118) with eigenvalues v_i, and the χ_i are characters. In particular there is a filtration which is both H and $\mathscr{F}$ equivariant

$$F^p = \Big(\prod_i y_i^{b_i} \prod_{j,\lambda} x_{j,\lambda}^{c_{j,\lambda}} : \sum_i (b_i v_i) + \sum_{j,\lambda} c_{j,\lambda} \geq p\Big) \tag{II.121}$$

of the complete local ring. Plainly, however, the filtration (II.121) is actually the completion of a bi-equivariant filtration of the Henselian local ring of $\mathfrak{X}$ (in fact even that of $\mathscr{X}$, albeit here, (II.114), the invariance under the possibly larger local monodromy may fail) so it affords, [MP13, I.iv.3], a smoothed $\mathscr{F}$-invariant weighted blow up

$$\rho : \tilde{\mathfrak{X}} \to \mathfrak{X} \tag{II.122}$$

which is an isomorphism off p. In particular, therefore, the unique lift $\tilde{f} : \mathscr{L} \to \tilde{\mathfrak{X}}$ of f of (II.115) is a $-\frac{1}{d}\,\mathbb{F}$ curve with smoothly embedded image, and II.g.3 holds. By direct calculation, (II.124), *cf.* [MP13, pg. 180], however, the eigenvalues (in an étale patch) of ∂ and $\rho^*\partial$ along the proper transforms of the $x_{i,\lambda}$'s differ by 1, so (II.116) follows from (II.95) applied to $\rho^*\partial$. □

Of course, we also proved that not just the linear holonomy, but actually all of the holonomy is cyclic of order dividing d, so although II.i.2 is sufficient for applications, we can actually do better thanks to,

II.i.3. **Fact.** *Let $\mathfrak{X} \to [\mathfrak{X}/\mathscr{F}]$ be a foliated smooth formal champ whose trace has an étale neighbourhood the invariant affine cusp, C, i.e. image of $t \mapsto (t^{v_1}, \dots, t^{v_k})$, for v_i as per (II.113), $t \in \mathbb{A}^1$, with the origin the unique point where C meets the foliation singularities, then the formal holonomy is cyclic of order at most d iff we can find formal holomorphic functions, $y_1, \dots, y_k, x_1, \dots, x_\ell$, restricting to a coordinate system on an analytic neighbourhood in $\mathfrak{X}$ of the singular point with the y_i's embedding coordinates respecting (II.113), x_j vanishing on the cusp and a generator ∂ for the foliation all of which are holomorphic on an étale neighbourhood (in the analytic topology) of $\mathfrak{X}$ with trace C such that for some $a_j \in \mathbb{Z}$,*

$$\partial = d v_i\, y_i \frac{\partial}{\partial y_i} + a_j\, x_j \frac{\partial}{\partial x_j} \tag{II.123}$$

holds on an any analytic étale neighbourhood of the singularity where the y_i, x_j form a system of coordinates.

Proof. The if direction is trivial, and for smooth curves this is II.g.3, or, more accurately a slight re-phrasing thereof. In any case, the affine cusp has no (holomorphic) Picard group, so a global holomorphic generator, ∂, of the foliation on $\mathfrak{X}$ exists, and we proceed to combine II.g.3 with II.i.2 to achieve the required form. In particular, by the latter, and II.h.2, we can find coordinates y_i, x_j and an invariant function ε, all in the completion in the singularity, 0, which render $\tilde{\partial} := (1+\varepsilon)^{-1}\partial$ in the given form. A local coordinate system for the weighted blow up (II.122) is given by:

$$y_1 = \tilde{y}_1^{v_1},\ y_i = \tilde{y}_i \tilde{y}_1^{v_i},\ x_j = \tilde{x}_j \tilde{y}_1 \tag{II.124}$$

in which $\tilde{y}_i = 1$, $i > 1$ where our cusp crosses the exceptional divisor, p, say, so the v_1th roots of unity act without fixed points in a neighbourhood of p, and $\tilde{y}_1$, $\tilde{z}_i := \tilde{y}_i - 1$, $\tilde{x}_j$ furnish coordinates in which $\rho^*\tilde{\partial}$ is semi-simple at p.

Now we appeal to II.g.3, to find a possibly different generator, $D = v\rho^*\partial$ for v a (holomorphic) unit on an étale neighbourhood of $\tilde{\mathfrak{X}}$ with trace the resolution $\tilde{C}$ of C, such that,

$$D = d\eta\frac{\partial}{\partial\eta} + (a_j - d)\xi_j\frac{\partial}{\partial\xi_j} \tag{II.125}$$

along with some other coordinate ζ_i, such that, η, ζ_i, ξ_j agree with $\tilde{y}_1$, $\tilde{z}_i$, $\tilde{x}_j$ modulo $\mathfrak{m}(p)^2$, but the former are defined on all of $\tilde{\mathfrak{X}}$. Both coordinate systems are semi-simple, so II.h.2 applies to yield units u, u_i, $i > 1$, w_j in the completion at p such that $\tilde{\eta} = u\tilde{y}$, $\tilde{\zeta}_i = u_i\tilde{z}_i$, $\tilde{\xi}_j = w_j\tilde{x}_j$ are a semi-simple coordinate system for D in the complete local ring at p. Effecting an appropriate linear change, this latter coordinate system is related to that in the η, ζ_i, ξ_j by,

$$\tilde{\eta} = \eta + \sum_{m\geq 2}\sum_{|I|+|J|=m} c_{IJ}(\eta)\zeta^I\xi^J \tag{II.126}$$

and similarly, employing the notation of (II.74) *et seq.*, for the $\tilde{\zeta}_i$, and the $\tilde{\xi}_j$. Both the left and right hand sides in (II.126) have the same eigenvalue, *viz:* d, so for all I, J we must have,

$$d\eta c'_{IJ}(\eta) + (a_J - d|J|)c_{IJ}(\eta) = 0 \tag{II.127}$$

and a_J takes only finitely many values for $|I| + |J|$ bounded. Consequently, for every m such that, $|I| + |J| = m$, c_{IJ} is a polynomial in η, from which $\tilde{\eta}$ converges not just in the completion at p, but in the full étale neighbourhood with trace $\tilde{C}$.

Arguing similarly for the $\tilde{\zeta}_i$, $\tilde{\xi}_j$'s, and ε, we may, without loss of generality, suppose, $\tilde{\eta} = \eta$, $\zeta_i = \tilde{\zeta}_i$, $\xi_i = \tilde{\xi}_j$, and that ε is defined in a neighbourhood with trace C. Thus we may suppose that $\varepsilon = 0$, and whence

$$\frac{Du}{u} = d(1-v),\ \frac{Du_i}{u_i} = 0,\ \frac{Dw_j}{w_j} = (a_j - d)(1-v) \tag{II.128}$$

where, without loss of generality, all of u, u_i, w_j are congruent to 1 modulo $\mathscr{I}_{\tilde{C}}$. Thus, for example, we can write,

$$u = \exp(\sum_{m\geq 2}\sum_{|I|+|J|=m} u_{IJ}(\eta)\zeta^I\xi^J) \tag{II.129}$$

so if $\tilde{v}_{IJ}(\eta)$ are the coefficients of a similar Taylor expansion for $v(1+\epsilon)$, then from (II.128)-(II.129),

$$d\eta u'_{IJ}(\eta) + a_J u_{IJ}(\eta) = -d\tilde{v}_{IJ}(\eta) \tag{II.130}$$

where the right hand side is holomorphic in η, while, a priori, the left hand side is formal, whence, a postiori, holomorphic. Consequently u is well defined on our étale neighbourhood with trace $\tilde{C}$, so, idem for $\tilde{y}_1$, and by an identical argument, all of the $\tilde{z}_i$, $\tilde{x}_j$ are equally so defined on the said neighbourhood. The relation of these to the original coordinates y_i, x_j defined on completing $\mathfrak{X}$ in the singular point is given by (II.124), so, not just the normalising factor $(1+\varepsilon)$, but also the y_i, x_j, are defined on an étale neighbourhood of $\mathfrak{X}$ with trace C. By construction, however, y_i, x_j are already a formal coordinate system at the singularity, so they are in fact coordinates on at worst an analytic neighbourhood of the same, while on any such (II.123) holds by construction. □

The role of the analytic topology in II.i.3 and its proof merits clarification by way of

II.i.4. *Remark.* Profiting from the Euclidean algorithm to solve $c_1 v_1 + \cdots + c_k v_k = 1$, for some integers c_i, one would might like to make a more strict analogue of II.g.3. Indeed in the above notation, $\phi = y_1^{c_1} \ldots y_k^{c_k}$ is a meromorphic function on the affine cusp which, close to the singularity, restricts to a coordinate function on the normalisation, and one might hope to form an explicit patch with an étale affine neighbourhood of the non-scheme like point at infinity according to the relation $\phi^e = s^{-d}$, for s a coordinate on the $\mathbb{A}^1 \ni \infty$, *cf.* (II.42) & (II.124). In principle, however, ϕ so constructed has an essential singularity at infinity. Plainly the problem is the intervention of v in (II.128), which unlike the smooth case cannot be avoid. Specifically, as in the smooth case, for t the unique (up to scaling by a constant) coordinate on the normalisation of the affine cusp, one would like to normalise, *cf.* (II.46), a generator ∂ of the foliation restricted to the cusp according to

$$\partial(t) = t \tag{II.131}$$

so that a postiori $\phi = t$ and everything is meromorphic over ∞. It can, however, happen under the hypothesis of II.i.3 that (II.131) doesn't admit a solution. If one follows the proof of II.i.3 and takes ∂ to be holomorphic then this is equivalent to asking that the unit v which appears restricts to a unit defined on the affine cusp rather than just its normalisation. Similarly, if one works algebraically this is equivalent to $K_{\mathscr{F}}$ restricting to an algebraically (rather than just holomorphically) trivial bundle on the affine cusp. Consequently a counter example where (II.131) cannot be solved is

$$\partial = 2(x+y)\frac{\partial}{\partial x} + 3(y+x^2)\frac{\partial}{\partial y}, \quad y^2 = x^3 \subset \mathbb{A}^2. \tag{II.132}$$

Since for $t = \sqrt{x}$, $\partial(t) = t(1+t)$, and $T_{\mathscr{F}}$ defined by gluing this to the unique (up to scaling by a constant) nowhere vanishing field, ∂_∞, on $V \xrightarrow{\sim} \mathbb{A}^1 \ni \infty$ along the open set $V \backslash \{-1, \infty\}$ by way of

$$\partial = \frac{(1+t)}{t} \cdot \partial_\infty \tag{II.133}$$

defines a bundle whose restriction to the affine cusp is algebraically non-trivial. As such:

II.i.5. *Warning.* Formal neighbourhoods of cusps, even though the problem is wholly at the level of the bundle of derivations defined by restricting the foliation to the reduced cuspidal curve, do not admit a description comparable to II.g.3. Ultimately, therefore, our treatment of cusps, §III.c, requires global hypothesis, III.c.1, rather than the local hypothesis of II.i.3

On the bright side, however:

II.i.6. *Remark.* In the course of the proof, we've complemented II.g.3 even in the smooth case, since, in principle, even if a generator ∂ on the étale neighbourhood U of II.g.3.(3) were semi-simple at the singular point, there might have been an obstruction to expressing ∂ in terms of semi-simple coordinates on an analytic neighbourhood of $0 \in U$, as found in op. cit., due to a possible re-scaling by a unit implicit in (II.91). We see, however, from the proof of II.i.3, that there is no such obstruction.

III. Extremal Subvarieties

III.a. **Generalities.** Unless specified otherwise, throughout this chapter $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ will be a foliated non-singular champ, with log-canonical foliation singularities. We switch our attention from $K_{\mathscr{F}}$ negative curves, to $K_{\mathscr{F}}$ negative extremal rays R. The moduli X is of course supposed projective so if H_R is a nef. Cartier divisor supporting the ray, i.e. $H_R.\, \alpha = 0$, and α in the closed cone of curves iff $\alpha \in R$, then for sufficiently large $m \in \mathbb{N}$, $A_R := m\, H_R - K_{\mathscr{F}}$

is ample. In any case following Kollàr, Mori, et al., cf. [Kol96] III.1, we introduce our main object of study, by way of,

III.a.1. **Definition.** The locus of R, $\mathrm{Loc}\,(R)$ is the set of closed points $x \in X(\mathbb{C})$ such that there is a curve $x \in C \subset X$ with $[C] \in R \subset \mathrm{NS}_1(X)$.

Observe that a priori $\mathrm{Loc}\,(R)$ is not a subvariety of X. Indeed for $m \in \mathbb{N}$, we can filter $\mathrm{Loc}\,(R)$ by sub-schemes $\mathrm{Loc}_m\,(R)$ on demanding that $x \in \mathrm{Loc}_m\,(R)$ if we can take the curve C of the definition to have $A_R.\, C \leq m$. That $\mathrm{Loc}_m\,(R)$ is a sub-scheme is immediate from the existence of the Hilbert scheme. To remedy this let us consider,

III.a.2. **Definition.** A R-pre-extremal subvariety is an irreducible subvariety $Y \subset \mathrm{Loc}\,(R)$ maximal amongst the set of irreducible varieties contained in the locus.

Trivially, the dimension in chains of proper inclusions of irreducible varieties must increase so R-pre-extremal subvarieties exist; any $x \in \mathrm{Loc}\,(R)$ is contained in one; and $\mathrm{Loc}\,(R)$ is the a priori countable union of all of them. Now if Y is R-pre-extremal, and $y \in Y$ then there is a C_y with $[C_y] \subset R$ containing y. However applying II.d.2, we know, for y generic, there is an invariant parabolic champ $f_y : \mathscr{L}_y \to \mathscr{X}$ through y with moduli L_y such that,

$$H_R.\, L_y \leq 2\, \frac{H_R.\, C_y}{-K_{\mathscr{F}}.\, C_y} = 0\,. \tag{III.1}$$

So in fact $L_y \in R$, and $A_R.\, L_y \leq 2$. Additionally L_y cannot be contained in $\mathrm{sing}\,(\mathscr{F})$ since it has $K_{\mathscr{F}}$-negative degree, so we can make a $\mathscr{F}$-invariant subvariety W by adding to generic points of Y an appropriate L_y. On the other hand Y is by hypothesis R-pre-extremal, so $W = Y$, i.e. Y is $\mathscr{F}$ invariant, with the induced foliated variety $Y \to [Y/\mathscr{F}]$ being a pencil of rational curves of A_R degree at most 2. Hilbert schemes, however, exist, and being invariant is a closed condition so in fact there are at most finitely many R-pre-extremal subvarieties for a given R. Better still the Hilbert scheme yields for any R-pre-extremal subvariety Y a flat family, $L \to T$, for some irreducible sub-scheme T of the Hilbert scheme such that the projection of L to X factors as a generically finite map over Y. An awkward case occurs when X is itself a R-pre-extremal subvariety, i.e. $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ is a pencil in parabolic champs. As a result we introduce,

III.a.3. **Definition/More Terminology.** A R-extremal subvariety Y is a subvariety of a R-pre-extremal subvariety Y' which is maximal amongst the subvarieties of Y' which are covered by invariant curves passing through at least one point of the image in X of the singular locus of $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$.

So indeed unless $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ is a pencil in parabolic champs then extremal and pre-extremal coincide, while in the awkward case an extremal variety will be specified by taking the invariant curves passing through an appropriate component of the singular locus. Now pulling everything back by the moduli map, $\pi : \mathscr{X} \to X$, define a *R-extremal champ* as the fibre over an extremal sub-variety, idem whether for pre-extremal or the locus, denoted $\mathscr{L}\mathrm{oc}\,(R)$, and observe,

III.a.4. **Fact.** *The locus $\mathscr{L}\mathrm{oc}\,(R)$ of an extremal ray, is a finite union of R-pre-extremal champs. Denote by $\mathscr{L}\mathrm{oc}'\,(R)$ the subvariety which is the union of R-extremal champs, then any $\mathscr{Y} \subset \mathscr{L}\mathrm{oc}'\,(R)$ making up this union is covered by $-1/d\,\mathbb{F}$ curves, where d may vary from curve to curve. There is however a family $\mathscr{L} \to T$ of champs, possibly non-flat at the non-scheme like points, such that, $(\mathscr{L} \to T) \to (\mathscr{Y} \to [Y/\mathscr{F}])$ is a generically finite map of foliated champs.*

In the same, albeit more refined, vein we will also employ:

III.a.5. **Fact.** *Suppose $\mathscr{X}$ is a smooth separated champ (over a field for ease of exposition) and $f : Y \to \mathscr{X}$ a map from a proper algebraic space then there is a separated (Deligne-Mumford)* *champ $\mathscr{T}$ and a deformation $F : Y \times \mathscr{T} \to \mathscr{X}$ of f such that if $G : Y \times M \to \mathscr{X}$ is any deformation of f parametrised by an algebraic space M, then there is a map $g : M \to \mathscr{T}$ and a natural transformation $\gamma : G \Rightarrow F(\mathrm{id}_Y \times g)$ such that if $h : M \to \mathscr{T}$ is any other map for which there is a natural transformation $\theta : G \Rightarrow F(\mathrm{id}_Y \times h)$ then there is a unique natural transformation $\alpha : g \Rightarrow h$ for which $\theta = F_*(\mathrm{id} \times \alpha)\gamma$. In addition the dimension of $\mathscr{T}$ at the point afforded by the trivial deformation and the above universal property is at least,*

$$h^0(f^* T_{\mathscr{X}}) - h^1(f^* T_{\mathscr{X}}) \tag{III.2}$$

Proof. The existence of $\mathscr{T}$ is a special (if key) case of the main theorem of [Ols06]. As such the dimension computation is infinitesimal and wholly space like in nature, *cf.* II.a.1, *i.e.* deformations of the trace of the formal space

$$\mathfrak{P} := \mathrm{Spf}(f^* \mathscr{P}_{\mathscr{X}}) \tag{III.3}$$

so we can replace $\mathscr{X}$ in III.2 by $\mathfrak{P}$ and appeal to [Kol96, I.2.16]- we only need the case Y projective. □

III.b. **Finding Weighted Projective Spaces.** As ever let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated smooth champ with log canonical foliation singularities, albeit with projective moduli, and $f : \mathscr{L} \to \mathscr{X}$ the normalisation of a $-\frac{1}{d}\mathbb{F}$ curve with at worst nodes, and, in the notation of II.g.3, eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_n$ of a generator ∂, in the normal directions, at the unique point p where f meets the singular locus. If $a_1 \leq 0$, then we simply have nothing to say for the moment. Otherwise, consider the net completion, $q : \mathfrak{X} \to \hat{\mathscr{X}}$, I.e.5, of $\mathscr{X}$ along the composite of f with the the universal cover, $q : \tilde{\mathscr{L}} \to \mathscr{L}$. By II.g.3, cf. II.h.10, there is a unique invariant closed formal sub-champ, $\mathfrak{X}_+ \hookrightarrow \mathfrak{X}$ such that,

$$N_{\tilde{\mathscr{L}}/\mathfrak{X}_+} \xrightarrow{\sim} \coprod_{a_i > 0} \mathscr{O}_{\tilde{\mathscr{L}}}(a_i) \tag{III.4}$$

By the Chow lemma, II.b.2, there is an irreducible sub-variety X_+ of the moduli X of $\mathscr{X}$ of the same dimension as $\mathfrak{X}_+$ obtained by taking the Zariski closure of the image of this in X. We therefore have maps,

$$\begin{array}{ccc} \mathfrak{X}_+ & \longrightarrow & X_+ \\ \downarrow & & \downarrow \\ \mathscr{X} & \longrightarrow & X \end{array} \tag{III.5}$$

so the leftmost vertical factors through the gerbe $\mathscr{X}_+ := \mathscr{X} \times_X X_+ \to X_+$, and even through the normalisation, $\tilde{\mathscr{X}}_+ \to \mathscr{X}_+$, since $\mathfrak{X}_+$ is smooth. The said vertical arrow is, however, net so $\mathfrak{X}_+ \to \tilde{\mathscr{X}}_+$ is étale. Indeed the assertion is local, and everything is excellent, so it suffices to work with the corresponding complete local rings in geometric points, but then $\mathfrak{X}_+$ can be identified with an irreducible component of $\mathscr{X}_+$, from which its isomorphic to its image in the normalisation, and we assert:

III.b.1. **Claim.** There is a smoothed weighted, [MP13, I.iv.3], blow up $\beta : \mathscr{X}_b \to \tilde{\mathscr{X}}_+$ supported in the point p such that the induced (after saturation) foliation $\mathscr{X}_b \to [\mathscr{X}_b/\mathscr{F}_b]$ is smooth and everywhere transverse to the exceptional divisor.

Proof. Since p is isolated and, as above, $\mathfrak{X}_+$ and $\tilde{\mathscr{X}}_+$ have isomorphic complete local rings it will suffice to prove that there is a smoothed weighted blow up of the complete local ring, $\hat{\mathscr{O}}$, of $\mathfrak{X}_+$ completed in p which is independent of any automorphism, σ, of $\hat{\mathscr{O}}$ preserving the foliation.

Now by II.g.3.(3) there are coordinates $y_0, y_1, \cdots, y_r$ in $\hat{\mathscr{O}}$; positive integers $a_i > 0$, $0 \le i \le r$; and a generator ∂ of the foliation such that

$$\partial = a_0 y_0 \frac{\partial}{\partial y_0} + a_i\, y_i \frac{\partial}{\partial y_i} \tag{III.6}$$

wherein $y_i = 0$, $i > 0$ define $\mathscr{L}$, so that for $i > 0$, a_i are as in (III.4), while $a_0 = d$ in the notation of (II.95). As such if σ is an automorphism of $\hat{\mathscr{O}}$ preserving the foliation, then there is a unit u_σ such that

$$\partial^\sigma = \sigma \partial \sigma^{-1} = u_\sigma \partial \tag{III.7}$$

and y_i^σ is an eigenvector of the linearisation of ∂ with eigenvalue $u_\sigma(0)^{-1} a_i$, for all $0 \le i \le r$, so $u_\sigma(0) = 1$. Consequently, by II.h.2 and $a_i > 0$, ∂^σ is not only semi-simple but

$$\partial = a_0 \eta_0 \frac{\partial}{\partial \eta_0} + a_i\, \eta_i \frac{\partial}{\partial \eta_i} \tag{III.8}$$

for a coordinate system of the form $\eta_i = u_i y_i$, u_i a unit, $0 \le i \le r$. If, therefore, we define a filtration of $\hat{\mathscr{O}}$ by the ideals

$$I_n = (y_0^{t_0} \cdots y_r^{t_r} \mid a_0 t_0 + \cdots + a_r t_r \ge n) \tag{III.9}$$

then this is independent of the choice of y_i in (III.6) since a basis of the eigenvectors of ∂ with eigenvalue a_i are monomials $y_0^{t_0} \cdots y_r^{t_r}$ with $a_0 t_0 + \cdots + a_r t_r = a_i$, and it is independent whether of σ, resp. the choice of ∂, by (III.8), resp. mutatis mutandis. The filtration, (III.9), defines a weighted blow up exactly as in (III.18) with smoothing as per (III.19). □

Now let us apply this to a qualitative description of $\mathscr{X}_+$, *i.e.*

III.b.2. **Corollary.** If $\mathscr{L}$ corresponds to an extremal ray R in Néron-Severi, with supporting function H_R, and ample bundle $A_R = mH_R - K_{\mathscr{F}}$, then for all $x \in \mathscr{X}_+$, there is a $-1/d(x)\,\mathbb{F}$-so, by definition, II.d.5-an invariant parabolic champ $\mathscr{L}_x \ni x$ in $\mathscr{X}$ which, in addition, meets the singular locus in the same singular point p as $\mathscr{L}$; and every invariant curve is not only of this form, but is parallel to R in Néron-Severi. In particular the singular locus of the induced foliation in $\mathscr{X}_+$ is the isolated point p.

Proof. The in particular follows from the antecedents. Otherwise, without loss of generality, we can replace $\mathscr{X}_+$ by $\tilde{\mathscr{X}}_+$; form the weighted blow up $\beta : \mathscr{X}_b \to \tilde{\mathscr{X}}_+$ of III.b.1; lift f to $\tilde{f} : \tilde{\mathscr{L}} \to \mathscr{X}_b$, for a possibly different but still parabolic $\tilde{\mathscr{L}}$ by II.d.5.(b), and argue as in *op. cit.* (c) to find a deformation $\mathscr{M}/T$, T proper, of $\tilde{f}$ composed with the universal cover of $\tilde{\mathscr{L}}$ which covers $\tilde{\mathscr{X}}_b$, so, equivalently the push-forward of which covers $\tilde{\mathscr{X}}_+$.

If, however, $\sum a_i C_i$ is some effective invariant 1-cycle numerically equivalent to a rational multiple of $\pi_*[\mathscr{L}]$ then every C_i generates R, so the gerbe $\mathscr{C}_i$ over any such C_i is a $K_{\mathscr{F}}$ negative invariant curve. Consequently, we require, in the first instance, to show that every $K_{\mathscr{F}}$ negative invariant curve, with $f : \mathscr{C} \to \tilde{\mathscr{X}}_+$ it's normalisation, is a $-1/d\,\mathbb{F}$ curve for some d, so, equivalently, avoiding the possibilities,

(a) $f(\mathscr{C}) \subseteq \mathrm{sing}(\mathscr{F}) \cap \mathscr{X}_+ \subseteq \mathrm{sing}(\mathscr{F})$, which is impossible by the definition of log canonical singularities as encountered in the proof of II.d.2.

(b) $f(\mathscr{C}) \cap \mathrm{sing}(\mathscr{F}) \neq \emptyset$. Should this occur then f is an embedding, and for $\tilde{\mathscr{C}} \to \mathscr{C}$ the universal cover, another application of II.d.5.(c) affords a finite étale neighbourhood $\tilde{V} \to V$ of the completion in $\mathscr{C}$ with trace $\tilde{\mathscr{C}}$, such that the induced foliation in $\tilde{V}$ is a smooth fibration. From which, the generic invariant curve misses p, which is absurd.

Now, a fortiori, the singularities of the induced foliation in $\mathscr{X}_+$ are contained in $\mathrm{sing}(\mathscr{F}) \cap \mathscr{X}_+$, and by construction this has at least the isolated point p. The leaves of $\mathscr{F}$ in $\mathscr{X}_+$ afford, however, a family of connected curves $C \to T$ in X over an irreducible base T, the gerbes over each component of each fibre of which have been seen to be $-1/d\,\mathbb{F}$ curve for some d. As such, suppose there is another singular point q, then there is a $-1/d\,\mathbb{F}$ curve through it, and this must be the gerbe over some component C_i of some fibre C_t. By definition, however, a $-1/d\,\mathbb{F}$ curve cannot meet $\mathrm{sing}(\mathscr{F})$ in any other point, while meeting p is a closed condition, so there is a different curve C_j in the fibre C_t through p. The fibre is, however, connected, so there must be a third curve C_k meeting the singular locus twice, which is nonsense. $\square$

From which we deduce a series of corollaries,

III.b.3. **Corollary.** The champ $\tilde{\mathscr{X}}_+$, but, cf. the pre-amble to §II.e, maybe not $\mathscr{X}_+$, is smooth.

Proof. The singular locus, $\mathscr{B}$, of $\mathscr{X}_+$ is invariant by every vector field, so, a fortiori by $\mathscr{F}$, while every leaf meets p, so $\mathscr{B}$ must meet it, yet, by construction the complete local rings at p of $\tilde{\mathscr{X}}_+$ and $\mathfrak{X}_+$ coincide, while the latter is smooth. $\square$

III.b.4. **Corollary.** The moduli Y_+ of any representable étale cover $\mathscr{Y}_+ \to \tilde{\mathscr{X}}_+$ has exactly one point over p, so, in particular if $\mathscr{C} \hookrightarrow \tilde{\mathscr{X}}_+$ is any embedded $-1/d\,\mathbb{F}$, then the natural map, $\pi_1(\mathscr{C}) \twoheadrightarrow \pi_1(\mathscr{X}_+)$, be it of analytic or algebraic fundamental groups, is surjective.

Proof. Any étale cover $\mathscr{Y}_+ \to \tilde{\mathscr{X}}_+$ still has étale neighbourhoods around a cover of $\mathscr{L}$ satisfying II.g.3 with $\tilde{\mathscr{X}}_+$ instead of $\mathscr{X}$ in op. cit. As such the proof of III.b.2 certainly applies to deduce that $\mathscr{Y}_+$, or, more correctly Y_+ has foliation singularities supported in an isolated point whenever Y_+ is algebraic. It applies, however, even if Y_+ were a priori analytic since the deformations of smooth parabolic invariant champs in the weighted blow up guaranteed by III.b.1 are certainly open, but they're also closed by the simple expedient of taking the limit algebraically and lifting to the universal cover. As to the in particular, otherwise, $\mathscr{C} \times_{\tilde{\mathscr{X}}_+} \mathscr{Y}_+$ is disconnected, and $\mathscr{C} \to \tilde{\mathscr{X}}_+$ is supposed an embedding, so there would be at least two singular points in $\mathscr{Y}_+$. $\square$

III.b.5. **Corollary.** For each eigendirection $\frac{\partial}{\partial x_i}$ of the linearisation of a foliation generator in $\mathrm{End}\,(N_{\mathscr{L}/\tilde{\mathscr{X}}_+} \otimes \mathbb{C}(p))$ there is an at worst nodal $-1/d_i\,\mathbb{F}$ invariant champ $f_i : \mathscr{L} \to \mathscr{X}$ through p with a branch parallel to $\frac{\partial}{\partial x_i}$ and a rational multiple of R in Néron-Severi.

Proof. There is a formal invariant curve in the said direction in the formal étale neighbourhood $\mathfrak{X}_+$, but every leaf is a $-1/d\,\mathbb{F}$ curve for some d, and all branches of the embedded image are isomorphic. $\square$

Additionally points in $\mathbb{P}^t(\mathbb{Q})$, $t \in \mathbb{N}$, are, up to ± 1, uniquely represented by $t+1$ tuples of integers with $\gcd = 1$, so if we change to a more homogeneous notation, viz:

III.b.6. **New Notation.** Linearise a local generator ∂ of $T_{\mathscr{F}}$ in the completion of $\hat{\mathscr{O}}_{\mathscr{X},p}$ of $\mathscr{O}_{\mathscr{X},p}$ in $\mathfrak{m}_{\mathscr{X}}(p)$ by way of, $\partial = a_1\, y_1 \frac{\partial}{\partial y_1} + \cdots + a_r\, y_r \frac{\partial}{\partial y_r} - b_i\, x_i \frac{\partial}{\partial x_i}$, $a_i \in \mathbb{N}$, $b_i \in \mathbb{N} \cup \{0\}$, $(a_1, \ldots, a_r, b_1, \ldots, b_t) = 1$, with $x_i = 0$ local equations for $\tilde{\mathscr{X}}_+$, the summation convention in the obvious way, and t the codimension of $\mathscr{X}_+$. As such in the above situation, III.b.5, $a_i \mid d_i$.

By III.b.4 we can (since otherwise I.c.5 will do) conclude that $\tilde{\mathscr{X}}_+$ has finite analytic, and whence finite algebraic, fundamental group on establishing,

III.b.7. **Claim.** Let $\mathscr{C} \to \mathbb{P}^1$ be a gerbe with at most 2 points whose monodromy exceeds that of the generic point, and which has a unique singular point, p, every branch of which is smooth, then the topological fundamental group $\pi_1(\mathscr{C})$ is finite.

Proof. The local model, C, of $\mathscr{C}$ is b-smooth branches through p on which a finite group acts G transitively on the branches branches while fixing p. In particular, the monodromy of the generic point is isomorphic to the stabiliser of any point other than p, which, in turn, is a proper sub-group of G since its image in the permutation representation on branches fixes at least one such. Consequently, p is a point of $\mathscr{C}$ with non-generic monodromy, and we denote by q the other such, should it exist, or some point distinct from p otherwise. In either case, let $U \ni p$ be the complement of q in $\mathbb{P}^1$.

Now, observe, that if $\mathscr{L} \to \mathscr{C}$ is the normalisation, and $B \hookrightarrow C$ a branch whose stabiliser in the permutation representation is H, then $[B/H]$ is a local model for $\mathscr{L}$, and $\mathscr{L}_U := \mathscr{L} \times_{\mathbb{P}^1} U$ has fundamental group H, and universal cover isomorphic to U, with H acting linearly. In particular, if we identify a branch with a disc, Δ, in U, embed C in V where the latter is b copies of U through the point p, and for good measure observe that all of this is necessarily compatible with a linearisation of G in appropriate coordinates, we find a commutative diagram of fibre squares with vertical embeddings,

$$\begin{array}{ccccc} C & \longrightarrow & [C/G] & \longrightarrow & \Delta \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & \mathscr{C} \times_{\mathbb{P}^1} U & \longrightarrow & U \end{array} \tag{III.10}$$

The upper left horizontal arrow is, however, the universal cover, and all the verticals are homotopy equivalences since the rightmost is, so the lower left is a universal covering. As in (II.42), the mapping $U \to \mathscr{L}_U$ may not extend over q as a map from $\mathbb{P}^1$ to $\mathscr{L}$, but this holds over some cyclic Galois cover $\tilde{U} \to U$ ramified exactly in p which respects the commutativity of,

$$\begin{array}{ccc} \tilde{U} & \longrightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathscr{L} \end{array} \tag{III.11}$$

Better still, taking b copies $\tilde{V}$ of $\tilde{U}$, the resulting composition $\tilde{V} \to \mathscr{C} \times_{\mathbb{P}^1} U$ with the lower left map in III.10, now admits an extension, $\bar{V} \to \mathscr{C}$, over b copies of $\mathbb{P}^1$ meeting in a single point since the upper horizontal in (III.11) is an embedding. By construction, $\bar{V} \to \mathscr{C}$ is open in the origin, and everywhere else it's flat, so it's open everywhere. As such if $\mathscr{M} \to \mathscr{L}$ is any (not necessarily finite) representable connected étale covering with group Γ, then $\mathscr{M} \times_{\mathscr{L}} \bar{V} \xrightarrow{\sim} \bar{V} \times \Gamma$, and the image of any $\bar{V} \times \gamma \to \mathscr{M}$, $\gamma \in \Gamma$ is open and closed, so it's all of $\mathscr{M}$. ☐

Now let $\mathscr{Y} \to \tilde{\mathscr{X}_+}$ be the finite universal cover assured by III.b.4 and III.b.7, then we further assert,

III.b.8. **Claim.** $\mathrm{Pic}(\mathscr{Y}) \xrightarrow{\sim} \mathbb{Z}$.

Proof. By construction $\pi : \mathscr{Y} \to Y$ is a gerbe over a projective variety, and the proof of [DI87] that the Hodge-De Rham spectral sequence degenerates at E_1 is valid mutatis mutandis since it only requires local smoothness and the co-homological criteria for ampleness both of which hold on $\mathscr{Y}$. As such, since $\mathscr{Y}$ is simply connected and π is acyclic,

$$\mathrm{H}^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}) = \mathrm{H}^1(Y, \pi_* \mathscr{O}_{\mathscr{Y}}) = \mathrm{H}^1(Y, \mathscr{O}_Y) = 0 \tag{III.12}$$

Now quite generally we have that $\mathrm{Pic}\,(\mathscr{Y})_{\mathbb{Q}} = \mathrm{Pic}\,(Y)_{\mathbb{Q}}$, and by (III.12), these are equally their respective Néron-Severi groups with $\mathbb{Q}$-coefficients. The Néron-Severi group, $\mathrm{NS}_1(Y)_{\mathbb{Q}}$, of Y is, however, known, e.g. [Kol96] II.4.21, to be of rank 1, so: $\mathrm{Pic}\,(\mathscr{Y})_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}$, which is equally the image of the Picard group under $(c_1)_{\mathbb{Q}}$ in $H^2(\mathscr{Y}, \mathbb{Q}(1))$ as deduced from the exponential sequence,

$$\text{(III.13)} \qquad H^1(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}) = H^1(Y, \mathscr{O}_Y) \to \mathrm{Pic}(\mathscr{Y}) \xrightarrow{c_1} H^2(\mathscr{Y}, \mathbb{Z}(1))$$

while the remaining possibility of torsion is excluded by $\mathscr{Y}$ simply connected, and the exact sequence,

$$H^1(\mathscr{Y}, \mathbb{Q}(1)/\mathbb{Z}(1)) \to H^2(\mathscr{Y}, \mathbb{Z}(1)) \to H^2(\mathscr{Y}, \mathbb{Q}(1)) \qquad \square$$

We will need some auxiliary constructions, so, initially, $\mathscr{Y}_0$, *i.e.* $\mathscr{Y}$ modulo its generic stabiliser, I.a.6, and their moduli, Y. At the singular point p identified with the origin in the notation of III.b.6, we have, therefore, its stabiliser G in $\mathscr{Y}$, of which the stabiliser G_0 in $\mathscr{Y}_0$ is a quotient acting faithfully on the local ring. Furthermore in a minor variation of III.b.6 we have, étale locally at p, a foliation generator ∂ with co-prime positive integer eigenvalues a_i which is invariant by the action G. Under this action, however, eigenvectors must go to eigenvectors, so the linear representation, ρ of G, which is equally its local action, splits as a direct sum of ρ_α's, where $\alpha \in A$ is a complete repetition free list of the a_i's, and ρ_α permutes the eigenvectors of ∂ with eigenvalue α. In particular, therefore, in the notation of III.b.6, the action of G commutes with the action of $\mathbb{G}_m$ defined by,

$$\text{(III.14)} \qquad \lambda \times (y_1, \ldots, y_r) \mapsto \underline{y}^\lambda = (\lambda^{a_1} y_1, \ldots \lambda^{a_r} y_r)$$

while the leaves may be identified with the images of,

$$\text{(III.15)} \qquad \phi_{\underline{c}} : t \mapsto (c_1 t^{a_1}, \ldots, c_r t^{a_r}), \text{ where, } \underline{c} \in \mathbb{A}^r \backslash 0$$

with two such functions $\phi_{\underline{c}}, \phi_{\underline{c'}}$ defining the same leaf in Y iff,

$$\text{(III.16)} \qquad \underline{c'} = (\rho(g)\underline{c})^\lambda, \quad g \in G, \, \lambda \in \mathbb{G}_m$$

with $\mathbb{G}_m$ action as per (III.14), which, as we've said, commutes with G, so if H is the image of the representation

$$\text{(III.17)} \qquad G \to \mathrm{Aut}(P(a_1, \ldots, a_r))$$

in automorphisms of the moduli of the weighted projective champ $\mathbb{P}(a_1, \ldots, a_r)$, then the leaf space is $P(a_1, \ldots, a_r)/H$.

Similarly, if we consider the weighted blow up,

$$\text{(III.18)} \qquad \mathscr{Y}_1 := \mathrm{Proj}(\coprod_n I_n) \to \mathscr{Y}_0, \quad I_n = (y_1^{t_1} \ldots y_r^{t_r} : a_1 t_1 + \ldots a_r t_r \geq n)$$

then the moduli, E, of the exceptional divisor is equally the said leaf space, so we have a map $\mathscr{Y}_1 \to E$. In addition $\mathscr{Y}_1$ has only quotient singularities, so we can form the smoothed weighted blow up $\mathscr{Y}_2 \to \mathscr{Y}_1$, [MP13, I.iv.3], or if one prefers not to cross reference, replace $\mathscr{Y}_1$ by what is locally its Vistoli covering champ, I.a.2. In particular $\mathscr{Y}_2$ is smooth, with smooth connected exceptional divisor $\mathscr{E}_2$. Certainly the moduli of $\mathscr{E}_2$ is E, but it's usually false that $\mathscr{Y}_2$ maps to $\mathscr{E}_2$ because the latter is highly non-scheme like. Indeed since $\rho|_{G_0}$ is faithful, the stabiliser of a generic point is the kernel, K of $G_0 \to H$, which by (III.16) and (III.14) is isomorphic under the restriction of ρ to some finite group of roots of unity μ_{a_0} acting according to (III.14), albeit for $\lambda \in \mu_{a_0}$. Alternatively: in the stabiliser of every geometric point of $\mathscr{E}_2$, K may be identified with the normal sub-group of pseudo-reflections in $\mathscr{E}_2$, and killing such reflections affords a map $\mathscr{Y}_2 \to \tilde{\mathscr{Y}}$, where $\tilde{\mathscr{Y}}$ is smooth, still a gerbe over the moduli of $\mathscr{Y}_1$, and $\mathscr{Y}_2 \to \tilde{\mathscr{Y}}$ is the extraction of a a_0th root of a smooth divisor $\mathscr{E} \xrightarrow{\sim} [\mathbb{P}(a_1, \ldots, a_r)/H]$- this latter notation being absolutely unambiguous since H acts on $\mathbb{P}(a_1, \ldots, a_r)$ because of the commutativity of

G with (III.14). Consequently we have a diagram,

$$\begin{array}{ccc} (\mathscr{Y}_1, \mathscr{E}_1) & \xleftarrow{\text{Vistoli covering}} & (\mathscr{Y}_2, \mathscr{E}_2 = \frac{1}{a_0}\cdot\tilde{\mathscr{E}}) \\ \text{Weighted blowup}\downarrow & & \downarrow a_0\text{th root} \\ \mathscr{Y}_0 \ni p & \xleftarrow[\text{if } a_0>1.]{\text{not defined at } p} & (\tilde{\mathscr{Y}}, \tilde{\mathscr{E}} \xrightarrow{\sim} [\mathbb{P}(a_1,\dots,a_r)/H]) \end{array} \tag{III.19}$$

where, to be precise, the final arrow is an isomorphism off $\tilde{\mathscr{E}}$ and is defined over p iff $a_0 = 1$. This final auxiliary pair is the good one for extending the map $\mathscr{Y}_1 \to E$, to wit:

III.b.9. **Claim.** The moduli of the $\mathscr{Y}_1 \to E$ lifts to a map $\pi : \tilde{\mathscr{Y}} \to \tilde{\mathscr{E}}$, and better still, not only is this the quotient $\tilde{\mathscr{Y}} \to [\tilde{\mathscr{Y}}/\tilde{\mathscr{F}}]$ but there is an $a \in \mathbb{N}$ such that this expresses that foliation as a fibration in $\mathbb{P}(1, a)$'s in the étale site of $\tilde{\mathscr{E}}$, while the identity,

$$K_{\mathscr{F}}|_{\mathscr{Y}_2} = K_{\mathscr{F}_2} + \mathscr{E}_2 = K_{\tilde{\mathscr{F}}} + \tilde{\mathscr{E}}|_{\mathscr{Y}_2} \tag{III.20}$$

with implied pull-backs those in (III.19) not only gives sense to $K_{\mathscr{F}}$ on $\tilde{\mathscr{Y}}$, but is a well defined tautological bundle, *i.e.* of degree $1/a$ on geometric fibres.

Proof. We will prove the statement in the analytic topology, since by [Gir71, IV.3.4] and [SGA-IV, XVI.4.1], *cf.* [McQ15, IV.a.3], it is equivalent, and trying to avoid this just leads to repeating variations on the steps in op. cit.

The smoothed weighted blow up operation- left vertical followed by top horizontal in III.19- smooth the foliation, and dropping to $\tilde{\mathscr{Y}}$ it remains smooth since $\mathscr{E}$ is everywhere transverse. Now let q be a geometric point of $\tilde{\mathscr{E}}$, with S_q its stabiliser in $\tilde{\mathscr{Y}}$, then we can find a polydisc Δ^r centred on q with coordinates y_i, $y_1 = 0$ an equation for $\tilde{\mathscr{E}}$, $\tilde{\partial} = \frac{\partial}{\partial y_1}$ generating the foliation, and S_q acting linearly via,

$$y_1 \times \sigma \mapsto \chi(\sigma) y_1, \quad y_i \times \sigma \mapsto \theta_{ij}(\sigma) y_j \tag{III.21}$$

From which, we can naturally identify $\theta : S_q \to \mathrm{GL}(r-1, \mathbb{C})$ with the full (not just linear) holonomy of the piece- $[\Delta/S_q]$- of the leaf $\mathscr{L}_q \ni q$ through q in $\tilde{\mathscr{Y}}$, and θ is faithful because there are no pseudo-reflections in $\tilde{\mathscr{E}}$.

The foliation is smooth with proper leaves, so their universal cover is constant, and since the leaves are $-\frac{1}{d}\,\mathbb{F}$-curves in $\mathscr{Y}_0$ without generic monodromy, and the generic point of $\tilde{\mathscr{E}}$ has no-monodromy, this is $\mathbb{P}(1, a)$ for some $a \in \mathbb{N}$, and the monodromy representation extends to,

$$S_q = \pi_1([\Delta/S_q]) \to \pi_1(\mathscr{L}_q) \to \mathrm{GL}(r-1) \tag{III.22}$$

so the first arrow in (III.22) is an injection. By either the long exact sequence of a fibration or, more algebraically [McQ15, III.c.3], $\pi_1(\mathscr{L}_q)$ is an extension of the fundamental group of the orbifold over which it is a locally constant gerbe by a quotient of the generic monodromy by a central element, so S_q is also surjective by II.d.5(b). As such, the holonomy covering of $\mathscr{L}_q$ is its universal covering, so that for S_q acting diagonally, we have an embedding,

$$[\mathbb{P}(1, a) \times \Delta^{r-1}/S_q] \hookrightarrow \tilde{\mathscr{Y}} \tag{III.23}$$

for some possibly smaller transversal polydisc, and the natural projection,

$$[\mathbb{P}(1, a) \times \Delta^{r-1}/S_q] \to [\Delta^{r-1}/S_q] \hookrightarrow \tilde{\mathscr{E}} \tag{III.24}$$

is the unique analytic continuation of our initial projection $[\Delta^r/S_q] \to \tilde{\mathscr{E}}$. This latter exists everywhere in a neighbourhood of $\tilde{\mathscr{E}}$- in fact everywhere in a formal neighbourhood would be enough which follows from the normal form III.b.6- so the projections (III.24) glue by I.a.4 to a projection on all of $\tilde{\mathscr{Y}}$. The final assertion, (III.20), is an easy local calculation at the singularity. $\square$

The fibration in III.b.9 has connected and simply connected fibres, so,

$$\pi_1(\tilde{\mathscr{Y}}) \xrightarrow{\sim} \pi_1(\tilde{\mathscr{E}}) \xrightarrow{\sim} \pi_1([\mathbb{P}(a_1, \ldots, a_r)/H]) \tag{III.25}$$

and by I.c.5, a weighted projective space is simply connected, so this latter group is H, which in turn affords a connected H-covering of $\tilde{\mathscr{Y}}\backslash\tilde{\mathscr{E}}$ since this is embedded as a representable Zariski open of $\tilde{\mathscr{Y}}$. Further the diagram, (III.19) can be formed locally with $\mathscr{Y}_0$ either $[\Delta^r/G_0]$, or $[\Delta^r/K]$, yielding a pair of diagrams with the obvious commutativity between them. Consequently the above H-covering of $\tilde{\mathscr{Y}}\backslash\tilde{\mathscr{E}}$ implied by (III.25) glues to the H-covering $[\Delta^r/K] \to [\Delta^r/G_0]$, and since $\mathscr{Y}$ is simply connected, we must have $H = 1$, and we further assert,

III.b.10. **Claim.** The foliation $\mathscr{Y}_0 \to [\mathscr{Y}_0/\mathscr{F}]$ is isomorphic to the radial foliation, $\mathscr{R}$, on the weighted projective champ $\mathbb{P}(a_0, aa_1, \ldots, aa_n)$. In particular, since $\mathscr{Y}_0$ is generically scheme like, a, and a_0 are relatively prime.

Proof. The start of the Leray spectral sequence applied to the fibration π of III.b.9 yields an exact sequence,

$$0 \to \mathrm{H}^1(\tilde{\mathscr{E}}, \mathbb{G}_m) \xrightarrow{\pi^*} \mathrm{Pic}(\tilde{\mathscr{Y}}) \to \mathrm{H}^0(\tilde{\mathscr{E}}, R^1\pi_*\mathbb{G}_m) \xrightarrow{d_2^{0,1}} \ldots \tag{III.26}$$

and by (III.20) this latter group is generated by the image of $K_{\mathscr{F}}$, so $d_2^{0,1} = 0$, and for a as per III.b.9 we can write,

$$\mathscr{O}_{\tilde{\mathscr{Y}}}(\tilde{\mathscr{E}}) = T^a_{\mathscr{F}} \otimes \mathscr{O}_{\tilde{\mathscr{E}}}(-m) \tag{III.27}$$

for some $m \in \mathbb{N}$, with the latter bundle the tautological bundle, I.c.2, on our weighted projective space. Forming, the exact sequence,

$$0 \to \mathscr{O}_{\tilde{\mathscr{Y}}}(aT_{\mathscr{F}} - \mathscr{E}) \to \mathscr{O}_{\tilde{\mathscr{Y}}}(aT_{\mathscr{F}}) \to \mathscr{O}_{\tilde{\mathscr{E}}} \to 0 \tag{III.28}$$

and pushing forward by π, affords,

$$0 \to \mathscr{O}_{\tilde{\mathscr{E}}}(m) \to \pi_*\mathscr{O}_{\tilde{\mathscr{Y}}}(aT_{\mathscr{F}}) \to \mathscr{O}_{\tilde{\mathscr{E}}} \to 0 \tag{III.29}$$

which by I.c.3 is a split rank 2 vector bundle, V, with the splitting even being canonical if $a > 1$. Indeed, we already know by III.b.9 that if there were extra monodromy at ∞ then it forms a smooth divisor on $\tilde{\mathscr{Y}}$ admitting a group of reflections of order a, so, equivalently if we killed these pseudo reflections, then all of the above is equally valid for some $\mathscr{Y}_{0,a}, \tilde{\mathscr{Y}}_a$, *etc.*, and $\tilde{\mathscr{Y}} \to \tilde{\mathscr{Y}}_a$ is an extraction of an ath root of a section, ∞, of the $\mathbb{P}^1$ bundle, $\mathbb{P}(V) = \tilde{\mathscr{Y}}_a$.

Now, by (III.25) *et. seq.* $G_0 = K$, and the important thing to observe is that because of the commutativity of the action of G_0 with the $\mathbb{G}_m$-action (III.14), the locally constant gerbe $\mathscr{E}_2 \to \tilde{\mathscr{E}}$ of (III.19) in B_K's is in fact trivial, so $a_0|m$ by (I.18). If, however, a_0 and a were to have a non-trivial gcd, $\alpha > 1$, then the leafwise universal cover,

$$\mathbb{P}(\frac{a_0}{\alpha}, \frac{a_1}{\alpha}) \to \mathbb{P}(a_0, a) \tag{III.30}$$

of the fibres of $\mathscr{Y}_2 \to \tilde{\mathscr{E}}$ is globally well defined, *i.e.* by (III.29): raising to the power α on the $\mathbb{G}_m$ torsor $\mathscr{O}_{\tilde{\mathscr{E}}}(\frac{m}{\alpha})$ and extending over 0 and ∞. The resulting covering $\tilde{\mathscr{Y}}_2 \to \mathscr{Y}_2$ is étale representable, and locally about the singularity, patches to the μ_α covering $[\Delta^r/\mu_{\frac{a_0}{\alpha}}] \to [\Delta^r/\mu_{a_0}]$, and whence the absurdity that $\mathscr{Y}$ isn't simply connected.

Having thus established the in particular, everything else follows quickly. The fact that a_0 and a are relatively prime imply that in an embedded neighbourhood (formal will do) of the singularity p, $\mathscr{Y} \to [\mathscr{Y}/\mathscr{F}]$ is isomorphic to the radial foliation, $\mathscr{R}$, I.d.2, on the said weighted projective champ, $\mathscr{P}$. All of the above, and specifically (III.19), apply, *cf.* I.d.3, if our starting points is $\mathscr{P} \to [\mathscr{P}/\mathscr{R}]$. The fact that we have an isomorphism at p, and the same monodromy at infinity, obliges us to have the same $\mathbb{P}(1, a)$ bundle, so $\tilde{\mathscr{Y}} \to [\tilde{\mathscr{Y}}/\tilde{\mathscr{F}}]$ and $\tilde{\mathscr{P}} \to [\tilde{\mathscr{P}}/\tilde{\mathscr{R}}]$

are isomorphic in a way compatible with the initial isomorphism at p, and whence, I.d.3.(d), $\mathscr{Y}_0 \to [\mathscr{Y}_0/\mathscr{F}]$ is isomorphic to $\mathscr{P} \to [\mathscr{P}/\mathscr{R}]$. □

Our initial $\mathscr{Y}$ is simply connected, and a locally constant gerbe over $\mathscr{Y}_0$, so by I.c.6, it is again a weighted projective champ, and only the notation changes,

III.b.11. **Fact.** *The foliation $\mathscr{Y} \to [\mathscr{Y}/\mathscr{F}]$ is isomorphic to the radial foliation on the weighted projective space $\mathbb{P}(a_0, aa_1, \dots, aa_n)$, where a_0 is the order of the stabiliser of the singularity p, and the generic leaf is a $-\frac{1}{a}$-curve. In particular the generic stabiliser is cyclic of order the gcd of a_0 and a.*

III.c. **Ignoring Cusps.** So far we haven't discussed what may happen if our extremal ray R is represented by an invariant champ $f : \mathscr{L} \to \mathscr{X}$ which has a cusp at the unique singular point z where f meets $\mathrm{sing}\,(\mathscr{F})$. This is, however, easily reduced to the previous case by way of

III.c.1. **Claim.** Let z be a geometric point of the singular locus of a foliated smooth champ, $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$, with log-canonical foliation singularities, and projective moduli, then if there is a $K_{\mathscr{F}}$-negative extremal ray, R, represented by a $-1/d\,\mathbb{F}$ curve through z there exists a $-1/d'\,\mathbb{F}$ curve through z with at worst nodes.

Proof. Let $\pi : \tilde{\mathscr{X}} \to \mathscr{X}$ be the blow up in z then the exceptional divisor, $\mathscr{E}$ is invariant and $\pi^* K_{\mathscr{F}}$ is again the foliated canonical bundle unless perhaps all the eigenvalues in say, III.b.6, are equal, but then there are no $-1/d\,\mathbb{F}$ cusps through z by II.i.2, and we're done. As such, by the cone theorem, II.d.1, there is a $-1/d'\,\mathbb{F}$ curve whose class, $\tilde{R}$, in $\mathrm{NE}_1(\tilde{\mathscr{X}})$ is extremal and $\pi^* R \gg \tilde{R}$. Consequently, without loss of generality, we may suppose that there is a $-1/d\,\mathbb{F}$ curve $f : \mathscr{L} \to \mathscr{X}$ which has a cusp at z, and whose class, resp. that of its proper transform $\tilde{f}$, is extremal in $\mathrm{NE}_1(\mathscr{X})$, resp. $\mathrm{NE}_1(\tilde{\mathscr{X}})$. The local structure of a branch of a cusp is described by (II.113) and II.i.2, and, in the notation of *op. cit.* $\tilde{f}$ meets the exceptional divisor with a (local) multiplicity v_1 in a scheme like chart. Now consider, $\pi' : \mathscr{X}' \to \mathscr{X}$ where $\mathscr{X}'$ is the extraction of a v_1th root of $\mathscr{E}$, then the induced map $f' : \mathscr{L} \to \mathscr{X}'$ has at worst nodes. On the other hand $\mathscr{E}$ is invariant so the canonical class is the same and f' is still extremal, and, somewhat superfluously, the singularities $\mathscr{X}' \to [\mathscr{X}'/\mathscr{F}']$ are still log-canonical since $\mathscr{E}$ is smooth. In any case, at the point z' where f' crosses the exceptional divisor we can apply III.b.5 to find $-1/d_i\,\mathbb{F}$ curves with smooth branches parallel to every axis afforded by the embedding dimensions of the original cusp, any of which represent the extremal ray. In particular if one takes the $-1/d'\,\mathbb{F}$ curve in an eigendirection normal to the exceptional divisor in the local coordinates at z' implied by those of II.i.2 at z, then the projection of this curve to $\mathscr{X}$ has at worse nodes. □

III.d. **Structure of Extremal Champs.** We begin with exactly the same preliminaries as III.b prior to (III.4) except that in the notation of *op. cit.* our interest is the unique formal champ $\mathfrak{X}_{\geq 0} \hookrightarrow \mathfrak{X}$ with normal bundle

$$N_{\tilde{\mathscr{L}}/\mathfrak{X}_{\geq 0}} \xrightarrow{\sim} \coprod_{a_i \geq 0} \mathscr{O}_{\tilde{\mathscr{L}}}(a_i) \qquad \text{(III.31)}$$

Now $\mathfrak{X} \to \mathscr{X}$ is net so the tangent space to the deformation space (wherein we insist that the deformation meets $\mathrm{sing}(\mathscr{F})$) whether of $\tilde{f} : \tilde{\mathscr{L}} \to \mathscr{X}$ or any composition with $\mathbb{P}^1 \to \tilde{\mathscr{L}}$ is the tangent space to the deformation space whether of $\tilde{\mathscr{L}} \hookrightarrow \mathfrak{X}_{\geq 0}$, or such a composition. The latter are however un-obstructed, (III.2), so by way of $\mathfrak{X}_{\geq 0} \hookrightarrow \mathfrak{X} \to \mathscr{X}$ the former are too. Consequently there is a Zariski closed sub-variety, $X_{\geq 0}$, of the moduli- the variety swept out by the deformations of $\tilde{f}$ or compositions thereof with $\mathbb{P}^1 \to \tilde{\mathscr{L}}$- of the same dimension as $\mathfrak{X}_{\geq 0}$

and containing its image. Exactly as in (III.5) we therfore get maps

$$\begin{array}{ccccccc} \mathfrak{X}_{\geq 0} & \longrightarrow & \tilde{\mathscr{X}}_{\geq 0} & \longrightarrow & \mathscr{X}_{\geq 0} & \longrightarrow & X_{\geq 0} \\ & & & & \downarrow & & \downarrow \\ & & & & \mathscr{X} & \longrightarrow & X \end{array} \tag{III.32}$$

wherein the square is fibred, $\tilde{\mathscr{X}}_{\geq 0} \to \mathscr{X}_{\geq 0}$ is normalisation, and the top leftmost arrow is an étale cover over its image. As ever we normalise a local generator ∂ of the foliation in the complete local ring $\hat{\mathscr{O}}_{\mathscr{X},p}$, for $p = f^{-1}(\operatorname{sing}\mathscr{F})$, according to III.b.6 with $d_1 = a_1 d$, $d \in \mathbb{N}$ albeit with the refinement

III.d.1. **New Notation.** Linearise a local generator ∂ of $T_{\mathscr{F}}$ in the completion of $\hat{\mathscr{O}}_{\mathscr{X},p}$ of $\mathscr{O}_{\mathscr{X},p}$ in $\mathfrak{m}_{\mathscr{X}}(p)$ by way of, $\partial = a_1 y_1 \frac{\partial}{\partial y_1} + \cdots + a_r y_r \frac{\partial}{\partial y_r} - b_i x_i \frac{\partial}{\partial x_i}$, $a_i \in \mathbb{N}$, $b_i \in \mathbb{N}$, $(a_1, \ldots, a_r, b_1, \ldots, b_t) = 1$, with $x_i = 0$ local equations for $\tilde{\mathscr{X}}_{\geq 0}$, and $z_1, \ldots, z_s$ the additional (formally invariant) functions which cut out $\tilde{\mathscr{X}}_{+}$, (III.5), so that t is now the codimension of $\mathscr{X}_{\geq 0}$, and $s + t$ the co-dimension of $\mathscr{X}_{+}$.

Now let us suppose that the $-1/d\,\mathbb{F}$ curve $f : \mathscr{L} \to \mathscr{X}$ affording (III.31) is an extremal ray, R, then we have constructed an integral invariant sub-champ $\mathscr{X}_{\geq 0}$ of $\mathscr{X}$ through every point of which there is a $-1/e\,\mathbb{F}$ champ, for varying e, parallel to R in Néron-Severi, and we assert

III.d.2. **Claim.** Let $\mathscr{Z}$ be the intersection of $\mathscr{X}_{\geq 0}$ with the singular locus of $\mathscr{F}$, then $\mathscr{Z}$ is smooth and connected.

Proof. Firstly, suppose $\mathscr{Z}$ is a disjoint union of components $\mathscr{Z}'$, $\mathscr{Z}''$, then we may consider the sub-champs $\mathscr{Y}'$, $\mathscr{Y}''$ whose moduli is covered by $K_{\mathscr{F}}$-negative extremal 1-dimensional champs parallel to R through $\mathscr{Z}'$ and $\mathscr{Z}''$ respectively. Consequently if $y \in \mathscr{Y}' \cap \mathscr{Y}''$ it is a singular point of some extremal 1-dimensional invariants champs $\mathscr{L}'$, $\mathscr{L}''$, so in $\mathscr{Z}' \cap \mathscr{Z}''$ by II.d.5.(a), which is nonsense, and $\mathscr{Z}$ is connected. Better still at the singularity, p, of the initial curve f, we know, II.i.2, that $\mathscr{Z}$ is irreducible and smooth of dim $= s$, so there is some irreducible component $\mathscr{Z}_0$ of $\operatorname{sing}(\mathscr{F})$ of dimension s contained wholly in $\mathscr{Z}$. However for any $\zeta \in \mathscr{Z}$, there is a $-1/e(\zeta)\,\mathbb{F}$ champ $\mathscr{L}_\zeta \ni \zeta$ contained in $\mathscr{X}_{\geq 0}$, so $\operatorname{sing}(\mathscr{F})$ is smooth at ζ by another application of II.i.2. Consequently $\zeta \mapsto \dim_\zeta \operatorname{sing}(\mathscr{F})$ is not just upper semi-continuous but continuous, i.e. constant $= s$ in the notation of III.d.1, , and since $\mathscr{Z}$ is connected: $\mathscr{Z}_0 = \mathscr{Z}$ is smooth irreducible of dimension s. □

Now consider the ideal $I_{\mathscr{Z}}$ of $\mathscr{Z}$ in $\mathscr{X}$, then the composition

$$I_{\mathscr{Z}} \xrightarrow[d]{} \Omega_X \longrightarrow K_{\mathscr{F}}.I_{\mathscr{Z}}\,. \tag{III.33}$$

affords an $\mathscr{O}_{\mathscr{Z}}$-linear map

$$I_{\mathscr{Z}}/I_{\mathscr{Z}}^2 \xrightarrow[D_Z]{} I_{\mathscr{Z}}/I_{\mathscr{Z}}^2 \otimes K_{\mathscr{F}} \tag{III.34}$$

of which the trace gives a global section of $\mathscr{O}_{\mathscr{Z}}(K_{\mathscr{F}})$. Plausibly this is zero, but by II.i.2 it may, on normalising in the direction of some smooth branch guaranteed by III.c.1 and (III.31), be identified locally with a $\mathbb{Q}$-valued function, so, by III.d.2, it's non-zero iff the trace of the normalisation III.d.1 is non-zero at some $p \in \mathscr{Z}$. Similarly the 2nd symmetric function is a global section of $\mathscr{O}_{\mathscr{Z}}(2K_{\mathscr{F}})$ which may locally be identified with a $\mathbb{Q}$-valued function, whose expression in the notation of *op. cit.* is

$$\frac{1}{2}\left(\sum_i a_i - \sum_j b_j\right)^2 - \frac{1}{2}\left(\sum_i a_i^2 + \sum_j b_j^2\right) \tag{III.35}$$

so if the trace doesn't define a nowhere vanishing section of $\mathscr{O}_{\mathscr{Z}}(K_{\mathscr{F}})$ there is at worst an étale double cover $\mathscr{Z}^{+-} \to \mathscr{Z}$ such that $K_{\mathscr{F}} \mid_{\mathscr{Z}^{+-}}$ is trivial. As a result the eigenvalues of $D_{\mathscr{Z}}$ are well defined constant functions up to a choice of generator of $\mathscr{O}_{\mathscr{Z}}(K_{\mathscr{F}})$ when this is possible, and otherwise they're well defined on $\mathscr{Z}^{+-}$. Thus if necessary we choose a lifting p^+ of the singularity p of the curve of (III.31) to the double cover, and subsequently choose our local generator in such a way to have compatibility with our formal linearisation at p (identified locally with p^+ if necessary), i.e. the eigenvalues of $D_{\mathscr{Z}}$ are everywhere $a_1, \dots, a_r, -b_1, \dots, -b_t$, with $a_i, b_j \in \mathbb{N}$, and $\gcd(a_1, \dots, a_r, b_1, \dots, b_t) = 1$. In any case, for every $\zeta \in \mathscr{Z}$, there is a well defined pair of eigenspaces, $\{T_{+(\zeta)}, T_{-(\zeta)}\}$ of $T_{\mathscr{X}} \otimes \mathbb{C}(\zeta)$, and every $K_{\mathscr{F}}$-negative 1-dimensional invariant champ has tangent space at ζ contained in precisely one of these. To fully profit from this we will have to extend from the normal bundle to a formal neighbourhood of $\mathscr{Z}$, which probably shows that being lazy about convergence wasn't perhaps an optimal use of time. The discussion is local over affine neighbourhoods U of $\mathscr{Z}$ over which the normal bundle and $K_{\mathscr{F}}$ trivialise, and which we consider centred on a point ζ of $\mathscr{Z}$. To momentarily simplify the notations let λ_i denote the necessarily non-zero eigenvalues of the normal bundle, and consider the following inductive proposition,

III.d.3. **Claim.** Let $\hat{\mathscr{O}}_U$ be the completion of $\mathscr{O}_U$ in $\mathfrak{m}_{\mathscr{X}}(\zeta)$, then for $k \in \mathbb{N}$, we have coordinates x_i normal to Z (evidently giving a basis for $N^\vee_{Z/x}$) and a generator ∂ of $\mathscr{F}$ over U such that,

1. $\partial x_i = \lambda_i \, x_i \, (\mathrm{mod} I_Z^k)$
2. There is a semi-simple generator $\hat{\partial}$ of $T_{\mathscr{F}} \otimes \hat{\mathscr{O}}_{U,\zeta}$ of the form $\lambda_i \, \xi_i \, \frac{\partial}{\partial \xi_i}$, for $\xi_i \in \hat{\mathscr{O}}_{U,\zeta}$ and $\xi_i = x_i \, (\mathrm{mod} I_Z^k)$.

Proof. The case $k = 2$ trivially follows from the previous discussion, so consider going from k to $k+1$, which evidently we wish to be compatible with restriction so that things converge. In any case, in terms of our usual notations about monomials and summation conventions we have, mod I_Z^{k+1},

(III.36)
$$\partial x_i = \lambda_i \, x_i + a_{iJ} \, x^J \, , \; a_{iJ} \in \mathscr{O}_U \, , \; \xi_i = x_i + b_{iJ} \, x^J \, , \; b_{iJ} \in \hat{\mathscr{O}}_{U,\zeta} \, .$$

Furthermore, $\hat{\partial} = u\partial$, $u \in \hat{\mathscr{O}}_{U,\zeta}$, and, $u = 1 + u_{iK} \, x^K$, $u_{iK} \in \hat{\mathscr{O}}_{U,\zeta}$, with $\# J = k$, $\# K = k - 1$. Now if we just put these equations together then we obtain,

(III.37)
$$a_{iJ} = \begin{cases} (\lambda_i - \lambda_J) \, b_{iJ} - u_{iK} \, \lambda_i & \text{if } x^K \, x_i = x^J, \\ (\lambda_i - \lambda_J) \, b_{iJ} & \text{otherwise} \end{cases}$$

without any summations. The second case is rather good since if $\lambda_i \neq \lambda_J := j_p \, \lambda_p$ we conclude that the b_{iJ} are algebraic, so if without loss of generality we replace x_i, by,

(III.38)
$$x_i \mapsto x_i + \sum_{\substack{\lambda_i \neq \lambda_J \\ x_i \nmid x^J}} b_{iJ} \, x^J$$

then in fact we conclude that $a_{iJ} = 0$ if $x_i \nmid x^J$. As for the 1st-case we do what we can. Specifically, again without loss of generality we can replace x_i by,

(III.39)
$$x_i \mapsto x_i + \sum_{\lambda_i \neq \lambda_J} \frac{a_{iJ}}{\lambda_i - \lambda_J} \, x^J$$

so that $a_{iJ} = 0$ if $\lambda_i \neq \lambda_J$, while if $\lambda_i = \lambda_J$ we conclude that u_{iK} is algebraic. Thus if we replace ∂ by,

(III.40)
$$\partial \mapsto \left(1 + \sum_{\lambda_K = 0} u_{iK} \, x^K \right) \partial$$

then $u_{iK}=0$ if $\lambda_K=0$, so in fact we can suppose $a_{iJ}=0$ for all J. Consequently, $\hat{\partial}$ has the form,

$$\left(1+\sum_{\lambda_K\neq 0} u_{iK}\,x^K\right)\partial\,. \tag{III.41}$$

However if we replace $\hat{\partial}$ by,

$$\tilde{\partial}=\left(1+\sum_{\lambda_K\neq 0}\tilde{u}_{iK}\,\xi^K\right)^{-1}\hat{\partial} \tag{III.42}$$

for $\tilde{u}_{iK}$ appropriate functions of coordinates z in $\hat{\mathscr{O}}_{Z,\zeta}$ which restrict from coordinates in $\hat{\mathscr{O}}_{U,\zeta}$ annihilated by ∂, and of course $\tilde{u}_{iK}=u_{iK}\ (\mathrm{mod}\, I_Z)$, then by II.h.2 $\tilde{\partial}$ is still semi-simple, with respect to a possibly different basis $\tilde{\xi}_i$ of the form $v_i\,\xi_i$, $v_i\equiv 1\,(I_Z^{k-1})$. To complete the induction, therefore, it suffices to observe, on supposing without loss of generality that $\xi_i=\tilde{\xi}_i$, that,

$$\xi_i\mapsto\xi_i-\sum_{\substack{\lambda_K=0\\ x^J=x_i\,x^K}}\tilde{b}_{iJ}(z)\,\xi^J \tag{III.43}$$

for $\tilde{b}_{iJ}$ satisfying much the same prescriptions as the $\tilde{u}_{iK}$ is still a trivialising basis for $\hat{\partial}$. $\square$

Consequently over an appropriately small affine U containing ζ, and bearing in mind that for any $\zeta'\in Z$ we know we can find appropriate coordinates in $\hat{\mathscr{O}}_{\mathscr{X},\zeta'}$ annihilated by ∂, we obtain formal subschemes U_+, U_- of the completion $\hat{U}$ of U in Z, whose subsequent completion at any $\zeta'\in Z\cap U$ is the non-strict Harder-Narismhan pair of II.h.10. The monodromy of the pair $\{U_+,U_-\}$ is precisely the monodromy of the pair $\{T_{X_+},T_{X_-}\}$, so either these patch to formal subchamps, $\mathfrak{X}_+^{\geq 0}$, $\mathfrak{X}_-^{\leq 0}$ of the completion of $\mathscr{X}$ in $\mathscr{Z}$, which completed at any point is the non-strict H-N pair, and of course we normalise so that $\forall\,\zeta\in Z$, $T_+(\zeta)=T_{X_+}\otimes\mathbb{C}(\zeta)$, $T_-(\zeta)=T_{X_-}\otimes\mathbb{C}(\zeta)$, or we get the same conclusion on a double covering of the completion.

With this out of the way we can quickly proceed to a conclusion. To begin with complete $\mathscr{X}_{\geq 0}$ in $\mathscr{Z}$, call it $\mathfrak{Y}$. By III.c.1 there is, for every $\zeta\in\mathscr{Z}$, a $-1/d\,\mathbb{F}$ curve through ζ with at worst nodes and parallel to the given extremal ray. By the unicity of III.d.1 up to $\pm$ such a curve must factor through $\mathfrak{X}_+^{\geq 0}\cup\mathfrak{X}_-^{\leq 0}$, which is always well defined even if $\mathfrak{X}_+^{\geq 0}$, $\mathfrak{X}_-^{\leq 0}$ are only well defined on a cover. In addition, exactly as post (III.31), the deformation space of the universal cover of the normalisation of such a curve is un-obstructed, so locally, it covers whichever of $\mathfrak{X}_+^{\geq 0}$, $\mathfrak{X}_-^{\leq 0}$ it factors through, and we've normalised so that our initial curve factors through $\mathfrak{X}_+^{\geq 0}$, so $\mathfrak{Y}$ is either $\mathfrak{X}_+^{\geq 0}$ or, $\mathfrak{X}_+^{\geq 0}\cup\mathfrak{X}_-^{\leq 0}$. In particular, there is a smooth Zariski open $\mathscr{U}\hookrightarrow\mathscr{X}_{\geq 0}\backslash\mathscr{Z}$, which close to $\mathscr{Z}$ is just the complement of the same, so, all leaves in $\mathscr{X}_{\geq 0}$ meet $\mathscr{U}$. On the other hand the singular locus of $\mathscr{X}_{\geq 0}$ is invariant by the induced foliation, so it's at worst contained in $\mathscr{Z}$, and indeed it's either empty or all of $\mathscr{Z}$ according to whether its completion $\mathfrak{Y}$ is smooth or not, *i.e.* iff the H-N pair is without monodromy or not. In the latter case, $\mathfrak{Y}=\mathfrak{X}_+^{\geq 0}\cup\mathfrak{X}_-^{\leq 0}$ so the normalisation $\tilde{\mathscr{X}}_{\geq 0}$ is smooth, and indeed $\tilde{\mathscr{X}}_{\geq 0}\to\mathscr{X}_{\geq 0}$ is everywhere an isomorphism except over $\mathscr{Z}$ where it's the double cover $\mathscr{Z}^{+-}\to\mathscr{Z}$, and for the unity of notation we put $\tilde{\mathscr{Z}}\hookrightarrow\tilde{\mathscr{X}}_{\geq 0}$ to be $\mathscr{Z}^{+-}$ or $\mathscr{Z}$ as appropriate.

We next wish to consider the operation of "projecting to $\mathscr{Z}$", by sending an invariant 1-dimensional champ to its unique singular point. To this end, we introduce the moduli, $X_{\geq 0}$, and the orbifold $\tilde{\mathscr{X}}_{\geq 0}^{0}$ associated, I.a.6, to the normalisation $\tilde{\mathscr{X}}_{\geq 0}$. Again the issue is that we have to be careful about the gerbe structure on $\mathscr{Z}$, so, say

$$\text{(III.44)} \quad \begin{array}{l} \mathscr{Z}_0' \hookrightarrow \tilde{\mathscr{X}}_{\geq 0}^0 \text{ the embedded sub-champ over the pre-image of the moduli } Z \text{ of } \mathscr{Z}, \\ \text{and } \mathscr{Z}_0 \text{ the associated orbifold, so that } \mathscr{Z}_0' \to \mathscr{Z}_0 \text{ is a locally constant gerbe.} \end{array}$$

We now proceed as in III.b.9. In the first instance (III.18) again affords a (well defined by III.d.3) weighted blow up $\tilde{\mathscr{X}}_{\geq 0}^1 \to \mathscr{X}_{\geq 0}^0$, whose exceptional divisor, $\mathscr{E}$, is the projectivisation of the graded $\mathscr{O}_{\mathscr{Z}_0'}$-algebra

$$\text{(III.45)} \qquad A := \coprod A_n := I_n \otimes \mathscr{O}_{\mathscr{Z}_0'}, \quad I_n = (y_1^{t_1} \dots y_r^{t_r} : a_1 t_1 + \dots a_r t_r \geq n)$$

In particular therefore the automorphism group of any geometric point of $\mathscr{Z}_0'$ has a projective representation in the automorphisms of $\mathrm{Proj}(A)$, and better still

III.d.4. **Claim.** The kernel, $\mathscr{S}'$, of the representation of the stabiliser, $\mathscr{S} \to \mathscr{Z}_0'$, in automorphisms of $\mathrm{Proj}(A)$ is locally constant, and the operation of quotienting by the stabiliser, *cf.* I.a.6, affords a factorisation $\mathscr{Z}_0' \to \mathscr{Z}' \to \mathscr{Z}_0$ of locally constant gerbes.

Proof. Let $U \to \tilde{\mathscr{X}}_{\geq 0}^0$ be a small étale neighbourhood of $\zeta \in \mathscr{Z}_0'$, with G the local monodromy, then by definition any $\sigma \in G$ which acts trivially on $\mathrm{Proj}(A)$ acts trivially on the pre-image $Z \hookrightarrow U$ of $\mathscr{Z}_0'$, *i.e.* σ is a well defined element of every stabiliser $G_z \hookrightarrow G$ of every $z \in Z$, which stabilises $\mathrm{Proj}(A)$ around z by the uniform definition of the y_i's in (III.45), *i.e.* III.d.3. □

Now, modulo notation, the diagram (III.19) and the proof (which doesn't employ the simple connectedness of $\mathscr{Y}$ of *op. cit.*) of III.b.9 are valid as stated, so "projection along a leaf" certainly yields, in the notation of *op. cit.*

$$\text{(III.46)} \qquad \tilde{\mathscr{Y}} \to \tilde{\mathscr{E}}$$

On the other hand $\tilde{\mathscr{E}}$ maps, *cf.* (III.14) *et seq.*, to $\mathrm{Proj}(A)$ understood as a cone over $\mathscr{Z}'$, and whence (III.46) affords a composition

$$\text{(III.47)} \qquad \tilde{\mathscr{X}}_{\geq 0}^0 \backslash \mathscr{Z}_0' \to \mathscr{Z}' \to \mathscr{Z}_0$$

which may, plainly, be extended everywhere locally around $\mathscr{Z}_0'$ while $\mathscr{Z}_0$ itself is an orbifold so by I.a.4 we finally get a projection

$$\text{(III.48)} \qquad \pi_0 : \tilde{\mathscr{X}}_{\geq 0}^0 \to \mathscr{Z}_0 \text{ and a composition } \pi : \tilde{\mathscr{X}}_{\geq 0} \to \tilde{\mathscr{X}}_{\geq 0}^0 \xrightarrow{\pi_0} \mathscr{Z}_0.$$

Before proceeding, let us emphasise the need for caution by way of

III.d.5. *Warning.* In general (III.48) needn't extend to a map to $\mathscr{Z}_0'$ or even $\mathscr{Z}'$. As such the extent to which one can profit from III.b.11 is limited according to whether we can glue together the universal covers of the fibres of π in (III.48), or some variant thereof for a different champ structure over the base, which *de facto* requires that π, or the said variant has a section.

Consequently we confine our description of π to

III.d.6. **Claim.** Let $\tilde{\mathscr{X}}_{\geq 0} \to [\tilde{\mathscr{X}}_{\geq 0}/\tilde{\mathscr{F}}]$ be the induced foliation then π is a smooth $\tilde{\mathscr{F}}$-equivariant (foliated) fibre bundle (in the étale topology) with fibre a foliated champ whose (finite) universal cover is described by III.b.11, *i.e.* a weighted projective champ in its radial foliation.

Proof. By construction, (III.47), functions on $\mathscr{Z}_0$ are invariant, *i.e.* π is certainly a $\tilde{\mathscr{F}}$-equivariant morphism of smooth champs. As such the map

$$\text{(III.49)} \qquad d\pi : \Omega^1_{\mathscr{Z}_0} \to \Omega^1_{\tilde{\mathscr{X}}_{\geq 0}}$$

is given, locally, by a $s \times (r+s)$ matrix, P, say such that for ∂ a local generator of the foliation there is a $(r+s) \times (r+s)$ matrix B for which $\partial P = PB$, so the locus where $d\pi$ fails to have full rank is $\tilde{\mathscr{F}}$-invariant. By definition, however, every leaf of $\tilde{\mathscr{F}}$ meets $\tilde{\mathscr{Z}}$, and, III.d.3, π is smooth in a formal neighbourhood of $\tilde{\mathscr{Z}}$, whence the co-kernel of (III.49) is a vector bundle of rank s everywhere, and a surjective map of smooth varieties is flat as soon as the fibres are equidimensional, so π is smooth. As such the condition, (I.58), for π to be a bundle of champs is is true by III.b.11, I.c.3 and (because we're in characteristic zero) the Höschild-Serre spectral sequence. Consequently, by III.b.11, for $V \to \mathscr{Z}_0$ a sufficiently small étale neighbourhood, $\pi^{-1}(V)$ is of the form $[V \times_{\mathbb{C}} \mathbb{P}_{\mathbb{C}}(\underline{a})/G]$ for G a finite group of automorphisms of a weighted projective champ $\mathbb{P}_{\mathbb{C}}(\underline{a})$. Again, however, by he Höschild-Serre spectral sequence, the representation of G cannot be deformed, so the only obstruction to having a bundle of foliated champs is that radial foliations on weighted projective champs might deform. This is, however, excluded by I.d.4. $\square$

We have, therefore, established

III.d.7. **Large Fact.** *Given a $-1/d\mathbb{F}$ curve $f : \mathscr{L} \to \mathscr{X}$ parallel to an extremal ray R in Néron-Severi meeting* $\mathrm{sing}(\mathscr{F})$ *with p the unique geometric point of their intersection, then after multiplication by a suitable constant, a linearisation in* $\mathrm{End}(\Omega_{\mathscr{X}} \otimes \mathbb{C}(p))$ *of a generator ∂ of the foliation is a diagonal matrix* $\mathrm{diag}\{a_1, \dots, a_r, 0, -b_1, \dots, -b_t\}$, $a_i, b_j \in \mathbb{N}$ *without common divisor and s zeroes. Better still, normalising so that the tangent space to $f(\mathscr{L})$ lies in the positive eigenspace, there is an R-extremal champ $\mathscr{X}_{\geq 0} \hookrightarrow \mathscr{X}$ containing f such that,*

(a) *$\mathscr{X}_{\geq 0}$ contains a unique, smooth s-dimensional component $\mathscr{Z}$ of the singular locus of $\mathscr{F}$.*
(b) *The normalisation $\tilde{\mathscr{X}}_{\geq 0}$ retracts onto $\mathscr{Z}_0$ where the pre-image $\tilde{\mathscr{Z}} \hookrightarrow \tilde{\mathscr{X}}_{\geq 0}$ of the singular locus is a locally constant gerbe over $\mathscr{Z}_0$,* (III.44)*, via π of* (III.48)*, and we have exactly one of,*
   (i) $K_{\mathscr{F}} \mid_{\mathscr{Z}}$ is trivial, and $\tilde{\mathscr{X}}_{\geq 0} \xrightarrow{\sim} \mathscr{X}_{\geq 0}$.
   (ii) $K_{\mathscr{F}}^{\otimes 2} \mid_{\mathscr{Z}}$ is trivial, but $K_{\mathscr{F}} \mid \mathscr{Z}$ is not, then $\tilde{\mathscr{Z}} \to \mathscr{Z}$ is an étale μ_2 covering which is exactly where $\tilde{\mathscr{X}}_{\geq 0} \to \mathscr{X}_{\geq 0}$ fails to be an isomorphism.
(c) *The fibration π is actually an étale bundle of foliated varieties where the transition functions are automorphisms of a foliated variety $\mathscr{Y} \to [\mathscr{Y}/\mathscr{F}]$ whose (finite) universal cover is the radial foliation on some $\mathbb{P}(a_0, aa_1, \dots, aa_n)$ for a_0, a as per III.b.11.*
(d) *Every extremal champ meeting* $\mathrm{sing}(\mathscr{F})$ *is of this form.*

There are a few loose ends here which we'll tidy up via

III.d.8. *Remark.* All of the above includes the case that $\mathrm{sing}\mathscr{F}$ has dimension zero at z but non-trivial monodromy, *cf.* II.h.9. By II.d.5.(c), the only way that an extremal ray can fail to meet $\mathrm{sing}(\mathscr{F})$ is if the foliation is generically a fibration in in parabolic champs. This is also the only way not just that (b).(ii) (so inter alia an isolated singularity with monodromy switching the H-N pair) can occur, but that (possibly different) extremal rays can factor through both the positive and negative parts of the H-N pair. This is, however, more subtle, so its proof is postponed. It is, therefore, not unreasonable to paraphrase III.d.7 as "every" extremal sub-champ is a smoothly embedded bundle of radially foliated weighted projective spaces.

Irrespectively, however, of clarifying when b.(ii) does occur, we have

III.d.9. **Corollary.** The number of extremal rays in the half space, $\mathrm{NE}_{K_{\mathscr{F}}<0}$ is finite.

Proof. An extremal ray which meets a singularity is described by III.d.7, and by II.h.9 it must factor through either X_+ or X_- of the H-N pair, so every connected component of $\mathrm{sing}(\mathscr{F})$ meets at most two such sub-champs which themselves are maximal amongst those covered by extremal rays meeting $\mathrm{sing}(\mathscr{F})$. By II.d.5.(c) we're therefore done unless $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ is generically a fibration in rational curves, but in this case, *cf. op. cit.*, the component of the deformation space of an invariant curve which doesn't meet $\mathrm{sing}(\mathscr{F})$ cover $\mathscr{X}$ with leaves, so all such rays are equivalent. $\square$

IV.a. **Contractions.** We will profit from a number of simplifications afforded by the analytic topology. As such we spell out our

IV.a.1. **Set Up.** Let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated champ with projective moduli, and $\mathscr{Y} \hookrightarrow \mathscr{X}$ be an embedded invariant sub-champ equal to $\tilde{\mathscr{X}}_{\geq 0}$ of III.d.7 in case b.(i) for some extremal ray. Fix a (not necessarily scheme like) point $z \hookrightarrow \mathscr{Z}_0$, *i.e.* the pre-image of a point in the moduli, for $\mathscr{Z}_0$ as in *op. cit.*, and let $\mathscr{X}' \hookrightarrow \mathscr{X}$ be an embedded analytic open neighbourhood of $\mathscr{Y}_z$ whose intersection $\mathscr{Y}' \hookrightarrow \mathscr{Y}$ with $\mathscr{Y}$ is (as a foliated variety) of the form $\pi^{-1}(\mathscr{Z}')$, III.d.7.(c), for $z \in \mathscr{Z}' \hookrightarrow \mathscr{Z}_0$ a small embedded analytic neighbourhood. Such data admits, therefore, arbitrarily small shrinkings around $\mathscr{Y}_z$ which, in order to employ I.f.5, we 'll make without warning. This is equally the good set up for constructing flips, so in our immediate context we add the precision that uniquely for this section, IV.a, $\mathscr{Y}$ is a divisor.

Profiting from III.b.11, and shrinking as necessary, we have, from III.d.7.(c) and I.f.5, that for some polydisc V there is a fibred square

$$\text{(IV.1)}\qquad \begin{array}{ccc} \mathscr{D} := \mathbb{P}(a_0, aa_1, \ldots, aa_r) \times V & \longrightarrow & \mathscr{X}^1 \\ \downarrow & & \downarrow \\ \pi^{-1}(\mathscr{Z}') & \longrightarrow & \mathscr{X}' \end{array}$$

where the horizontal arrows are embeddings; the vertical arrows étale Galois coverings under $\pi_1(\mathscr{Y}_z)$; a_i as in III.d.1; and a, a_0 as in III.b.11. In particular, therefore, for $\mathscr{O}(1)$ the tautological bundle on the weighted projective space in the left hand corner of (IV.1), II.g.3 implies

$$\text{(IV.2)}\qquad N_{\mathscr{D}/\mathscr{X}^1} \xrightarrow{\sim} \mathscr{O}(-ab)$$

for $b = b_1$ of III.d.1. Now consider the operation, I.a.9, of extracting a dth root of the Cartier divisor $\mathscr{D}$,

$$\text{(IV.3)}\qquad \begin{array}{ccc} \mathscr{D}'' & \longrightarrow & \mathscr{X}'' \\ \downarrow & & \downarrow \\ \mathscr{D} & \longrightarrow & \mathscr{X}^1 \end{array}$$

then, for any d the left hand vertical is a locally constant gerbe under B_{μ_d} and if, moreover $d = ab$ this gerbe is trivial, so by I.f.5 again, after appropriate shrinking there is a fibre square,

$$\text{(IV.4)}\qquad \begin{array}{ccc} \mathscr{D} & \longrightarrow & \tilde{\mathscr{X}} \\ \downarrow & & \downarrow \\ \mathscr{D}'' & \longrightarrow & \mathscr{X}'' \end{array}$$

where, once more, the horizontals are embeddings, and the verticals étale coverings, but now under μ_{ab}. This construction has a number of convenient properties, to wit:

IV.a.2. **Claim.** The complement $X^* := \tilde{\mathscr{X}} \backslash \mathscr{D}$ is everywhere space like, and an étale Galois covering of $\mathscr{X}' \backslash \mathscr{Y}'$ with group an extension of the form

$$\text{(IV.5)}\qquad 1 \to \mu_{ab} \to E_z \to \pi_1(\mathscr{Y}_z) \to 1, \quad \textit{i.e. } \mathscr{X}' \backslash \mathscr{Y}' \xrightarrow{\sim} [X^*/E_z]$$

Proof. That we have a covering with the said group is immediate from (IV.1), (IV.3), and (IV.4), while by (IV.2) $N_{\mathscr{D}/\tilde{\mathscr{X}}}$ is isomorphic to $\mathscr{O}(1)$. As such the local monodromy acts faithfully on the complement of the zero section $\mathscr{D} \hookrightarrow N_{\mathscr{D}/\tilde{\mathscr{X}}}$, so, a fortiori X^* is space like. □

Before profiting from this let us make,

IV.a.3. *Remark.* One could certainly take an abth root, even globally of $\mathscr{Y} \hookrightarrow \mathscr{X}$, say

$$\mathscr{X}_{ab} \to \mathscr{X} \tag{IV.6}$$

with $\mathscr{X}'_{ab} \hookrightarrow \mathscr{X}$ the resulting neighbourhood of $\mathscr{Y}'$. This does not imply, however, that (IV.5) is split since there may be torsion effects in $\mathrm{Pic}(\mathscr{Y})$- *cf.* (I.18). Similarly, if one is prepared to assume that $\mathrm{Pic}(\mathscr{X}^1) \xrightarrow{\sim} \mathrm{Pic}(\mathscr{D})$ then one can do the steps (IV.3)-(IV.4) in a single move, *viz:* extract the abth root of the section of $\mathscr{O}(-ab)$ defined by $\mathscr{D}$. This is easy if one completes in $\mathscr{D}$, *i.e.* the exponential sequence for the nth thickening

$$0 \to I^n/I^{n+1} \xrightarrow{x \mapsto 1+x} \mathscr{O}^{\times}_{n+1} \to \mathscr{O}^{\times}_n \to 0 \tag{IV.7}$$

and I.c.3, but otherwise would requires a little analysis that can reasonably be avoided, via I.f.5, by confining ourselves to purely topological statements.

Having arrived to this juncture, however, we can complete $\tilde{\mathscr{X}}$ in $\mathscr{D}$ to a formal champ, $\hat{\mathscr{X}}$, with trace $\mathscr{D}$, and argue as in (IV.7) to deduce

$$\mathrm{Pic}(\hat{\mathscr{X}}) \xrightarrow{\sim} \mathrm{Pic}(\mathscr{D}) = \mathbb{Z}\mathscr{O}(1) \tag{IV.8}$$

As such the $\mathbb{G}_m$-torsor $\mathfrak{X} \to \hat{\mathscr{X}}$ defined by $\mathscr{O}(1)$ has trace a product with V of the $\mathbb{G}_m$-torsor, (I.33), in the definition of a weighted projective space. The latter is, however, space like, so $\mathfrak{X}$ is a formal space, which can be described wholly explicitly, *i.e.*

$$\mathfrak{X} \xrightarrow{\sim} (\mathbb{A}^{r+1}\backslash\{0\}) \times \hat{\Delta} \times V, \quad \hat{\Delta} := \mathrm{Spf}\mathbb{C}[[x]] \tag{IV.9}$$

on which $\lambda \in \mathbb{G}_m$ acts according to

$$(\mathbb{A}^{r+1}\backslash\{0\}) \times \hat{\Delta} \times V \ni (y_0, y_1, \dots, y_r) \times x \times z \mapsto (\lambda^{a_0} y_0, \lambda^{aa_1} y_1, \dots, \lambda^{aa_n} y_r) \times \lambda^{-1} x \times z \tag{IV.10}$$

Now observe that the ring, A, of $\mathbb{G}_m$ invariant functions affords maps

$$\mathfrak{X} \to \hat{\mathscr{X}} \xrightarrow{\text{moduli}} \hat{X} \to V \times \mathrm{Spf}A\,(\xrightarrow{\sim} \mathbb{C}[[x^{a_0} y_0, x^{aa_1} y_1, \dots, x^{aa_n} y_r]]\,) \tag{IV.11}$$

where, by definition, A is equally the ring of formal functions on $\hat{X}$. Consequently the final map in (IV.11) is a formal contraction in the sense of [Art70], and whence by *op. cit.* is the completion in V of the contraction of analytic spaces

$$\begin{array}{ccccc} \mathscr{D} & \longrightarrow & \tilde{\mathscr{X}} & \xrightarrow{\text{moduli}} & \tilde{X} \\ {\scriptstyle \text{projection } \pi \text{ of (III.48)}}\downarrow & & \rho\downarrow & & \rho_0\downarrow{\scriptstyle \text{contraction}} \\ V & \longrightarrow & X_z & = & X_z \end{array} \tag{IV.12}$$

In particular therefore by (IV.11) X_z is smooth, and we're well advanced in proving

IV.a.4. **Proposition/Summary.** *There is a Galois covering $\tilde{\mathscr{X}} \to \mathscr{X}'$ under E_z,* (IV.5)*, ramified uniquely over $\mathscr{D} \to \mathscr{Y}'$, and there (in the notation of III.d.1 and III.b.11) to order exactly ab such that*

(a) *The contraction, X_z, of* (IV.12) *is smooth.*
(b) *It's E_z equivariant, and although $\mathscr{X}' \to [X_z/E_z]$ may not be defined at $\mathscr{Y}$, $\mathscr{X}'_{ab}$,* (IV.6)*, to $[X_z/E_z]$ is everywhere defined.*
(c) *The contraction is birational, i.e. $\mathscr{X}'\backslash\mathscr{Y} \xrightarrow{\sim} [X_z\backslash V/E_z] = [X^*/E_z]$.*

Better still, all of this globalises, i.e. there is a foliated smooth champ $\mathscr{X}_0 \to [\mathscr{X}_0/\mathscr{F}_0]$ fitting into a diagram (the contraction of $\mathscr{Y}$) -described locally by (a)-(c)- and an isomorphism off $\mathscr{Y}$,

$$\begin{array}{ccc} \mathscr{X}_{ab} & \xrightarrow[\rho]{} & \mathscr{X}_0 \\ {\scriptstyle K_{\mathscr{F}} \textit{ unramified}}\downarrow{\scriptstyle \text{(IV.6)}} & & \\ \mathscr{X} & & \end{array} \tag{IV.13}$$

Proof. We've done (a) & (c), and as per IV.a.3 we have from the construction, (I.15), of extracting roots a map $\mathscr{X}'' \to \mathscr{X}_{ab}$. If, however, $\mathscr{Y}'_{ab} \hookrightarrow \mathscr{X}'_{ab}$ is the reduced fibre over $\mathscr{Y}$ then

$$\begin{array}{ccc} \mathscr{D}'' & \longrightarrow & \mathscr{D} \\ \downarrow & & \downarrow \\ \mathscr{Y}'_{ab} & \longrightarrow & \mathscr{Y}' \end{array} \tag{IV.14}$$

is not just commutative but the top horizontal is the pull back of the locally constant gerbe defined by the bottom horizontal. As such, the square is fibred so the left vertical is a representable étale cover, and whence $\mathscr{D}$ is the universal cover of $\mathscr{Y}'_{ab}$, so that shrinking as necessary, $\tilde{\mathscr{X}} \to \mathscr{X}'_{ab}$ is equally the universal cover. In particular, therefore, we have a diagram

$$\begin{array}{ccc} \tilde{\mathscr{X}} & \longrightarrow & X_z \\ \downarrow & & \downarrow \\ \mathscr{X}'_{ab} & & [X_z/E_z] \end{array} \tag{IV.15}$$

wherein the left hand vertical is an E_z-torsor. Better still the pull-back of $\mathscr{O}_{\mathscr{X}'_{ab}}(\mathscr{Y}'_{ab})$ to $\mathscr{D} \hookrightarrow \tilde{\mathscr{X}}$ is $\mathscr{O}(-1)$, so there is an E_z action on the torsor $\mathfrak{X}$ commuting with the $\mathbb{G}_m$-action (IV.10). Consequently, the top horizontal is E_z equivariant, so, by the definition of the bottom right hand corner, the square can be completed along the bottom horizontal, *i.e.* (b) holds.

Turning to globalisation, the unicity of contractions ensures that the contraction of the subspace of the moduli, X, of $\mathscr{X}$ defined by the moduli of $\mathscr{Y}$ to that of $\mathscr{Z}_0$ is an algebraic space X_0. Now denote by $*$ the complement of $\mathscr{Y}$, or the contracted locus as appropriate, then for ζ another point of $\mathscr{Z}_0$ the normalisation of $X_z^* \times_{X_0} X_\zeta^*$ is equally that of $X_z^* \times_X X_\zeta^*$ so by I.a.3, either projection of

$$R := \big(\text{ normalisation of } U \times_{X_0} U\big) \rightrightarrows U, \quad U = \coprod_z X_z \tag{IV.16}$$

is unramified in co-dimension 1. Consequently, by purity, they're unramified everywhere, and since $R^* \rightrightarrows U^*$ is both a groupoid and dense in R, (IV.16) defines an étale groupoid, or, equivalently, an orbifold $\mathscr{M}_0 \to X_0$ with atlas U. At the same time, we can express $\mathscr{X}_{ab}$ as a locally constant gerbe in B_Γ's over an orbifold $\mathscr{M}$ for some finite group Γ. Thus $\mathscr{M}$ and $\mathscr{M}_0$ agree on an open dense set, so by I.a.4 and (IV.12), there is a map $\rho : \mathscr{M} \to \mathscr{M}_0$. Next observe that the contracted locus is an embedded smooth sub-champ of real co-dimension at least 4, whence the homotopy depth about the same, [SGA-II, Exposé XIII.6], is also at least 4, so the locally constant gerbe $\mathscr{X}^* \to \mathscr{M}^*$ extends uniquely to a locally constant gerbe $\mathscr{X}_0 \to \mathscr{M}_0$. On the other hand locally the universal cover is generically scheme like, IV.a.2, so from the long exact sequence of a fibration we must have

$$1 \to \Gamma \to E_z = \pi_1(\mathscr{X}'_{ab}) \to \pi_1(\mathscr{M}') \to 1 \tag{IV.17}$$

for $\mathscr{M}'$ a small neighbourhood of $\mathscr{Y}'_{ab}$. On the other hand in the diagram

$$\begin{array}{ccc} X_z & \longleftarrow & \tilde{\mathscr{X}} \\ \downarrow & & \downarrow \\ \mathscr{M}_0 & \longleftarrow & \mathscr{M}' \end{array} \tag{IV.18}$$

the left hand is the universal cover of it's image under the group E_z/Γ, so by (IV.17), the diagram (IV.18) is a pull-back of a covering along the bottom horizontal. In particular, therefore, (IV.18) is fibred so for a locally constant sheaf, Λ, $R^1\rho_*\Lambda = 0$, and the Leray spectral sequence yields an exact sequence

$$0 \to \mathrm{H}^2(\mathscr{M}_0, \Lambda) \to \mathrm{H}^2(\mathscr{M}, \Lambda) \to \mathrm{H}^0(\mathscr{M}_0, R^2\rho_*\Lambda) \tag{IV.19}$$

In addition $\pi_1(\mathscr{M}) \xrightarrow{\sim} \pi_1(\mathscr{M}_0)$, so $\mathscr{X}$ and $\rho^*\mathscr{X}_0$ are locally constant gerbes for the same link in the sense of Giraud, [Gir71, IV.1.1.7.3], and their difference, *op. cit.* IV.3.4, defines a class in $\mathrm{H}^2(\mathscr{M}, \Lambda)$ for Λ the centre of the link- so locally the centre of the aforesaid group Γ. Since $\mathscr{X}_0 \to \mathscr{M}_0$ is locally trivial by definition, the image of this class in the rightmost group of (IV.19) is zero by (IV.18) and (IV.12), while the resulting class in the leftmost group is trivial because this is the same as $\mathrm{H}^2(\mathscr{M}^*, \Lambda)$. □

It follows that we've actually proved a little more, to wit:

IV.a.5. **Remark/Definition.** From (IV.15), the fibre of the horizontal arrow in (IV.13) has fibre $\tilde{\mathscr{X}}$ over X_z, which, in turn is the smooth weighted blow up (composition of left vertical and top horizontal in (III.19)) with weights $a_0, aa_1, \dots, aa_n$ in the obvious coordinates suggested by (IV.11) while by purity the left vertical in (IV.13) is exactly the same as the rightmost vertical in (III.19), *i.e.* killing a group (here μ_{ab}) of pseudo reflections. Moreover, since $\rho : \mathscr{X} \to \mathscr{X}_0$ needn't be everywhere defined it's more technically correct to call the birational map ρ a flip, which, in turn has the very specific structure of (IV.13), which might reasonably be described as a *flap*.

The resulting foliation on $\mathscr{X}_0$ is described by

IV.a.6. **Corollary.** The canonical bundles of the various foliations are related by

(IV.20) $$K_{\mathscr{F}_{ab}} = K_{\mathscr{F}} \mid_{\mathscr{X}_{ab}} = \rho^* K_{\mathscr{F}_0} + a_0 \mathscr{Y}_{ab}$$

so, in particular, $\mathscr{F}_0$ is smooth and everywhere transverse to the contracted locus.

Proof. The first identity in (IV.20) is just that the left vertical in (IV.13) is unramified along the foliation because $\mathscr{Y}$ is invariant, while the 2nd identity follows, for purely numerical reasons, from (IV.2) and III.b.11. Now say D is a local generator of $\mathscr{F}_0$ on X_z, and s_0 is the coordinate function of weight a_0 in (IV.11), then, by (IV.20), $\rho^*(s_0 D)$ is an everywhere regular derivation which coincides with a local generator of $\mathscr{F}_{ab}$ at every point where $\rho^*(s_0)$ only vanishes along the exceptional divisor. In particular, therefore, it coincides by III.b.11 with a local generator close to $\mathrm{sing}(\mathscr{F}_{ab})$, where by *op. cit.* a local equation $x = 0$ for the exceptional divisor may be supposed of the form $x^{a_0} = \rho^* s_0$. Now the exceptional divisor is invariant, and by (II.g.3) defines a non-zero eigenspace at the singularity so $\rho^*(D(s_0))$ is non-zero everywhere, whence, idem $D(s_0)$, □

IV.b. **Projectivity of the contraction.** By way of a rather general projectivity criteria

IV.b.1. **Lemma.** *Let X be a proper algebraic space over a field k, then X is projective iff both of the following conditions hold*

(a) *for every irreducible subspace $Y \hookrightarrow X$*

$$\big(\bar{\mathrm{NE}}_1(Y) \ni \alpha \mapsto 0 \in \bar{\mathrm{NE}}_1(X)\big) \Rightarrow \alpha = 0$$

(b) *The cone $\mathrm{NE}_1(X) \subseteq \mathrm{NS}_1(X)_{\mathbb{R}}$ doesn't contain a line.*

Proof. The conditions are clearly necessary. The second condition is equivalent to the existence of a Cartier divisor H non-negative on $\mathrm{NE}_1(X)$ such that

(IV.21) $$\big(\bar{\mathrm{NE}}_1(X) \ni \alpha \mapsto H.\alpha = 0\big) \Rightarrow \alpha = 0$$

Thus if (a) & (b) hold for X they hold for every sub-variety, so, by induction we can suppose $H^{\dim(Y)}.Y > 0$ for every non-trivial sub-variety of dimension smaller than that of X. Consequently, by the Nakai-Moishezon criteria, [Kol90, 3.11], we require to prove for every irreducible component of X of maximal dimension the top power of H is positive. As such, say, without loss of generality, X irreducible of dimension $d+1$ and $p : X' \to X$ a projective modification, then p^*H is nef. Better still some Zariski open of X is a scheme, whence it contains sub-varieties of all possible dimensions, thus $H^d = p_*(p^*H^d)$ is a non-zero class in $\bar{\mathrm{NE}}_1(X)$, and so by (b) $H^{d+1} > 0$. □

A less general, but more relevant variation of the same is

IV.b.2. **Corollary.** Let $p : X' \to X$ be proper; an isomorphism off $Z \hookrightarrow X$; with X' projective and X a $\mathbb{Q}$-factorial algebraic space over a field k, then X is projective iff both of the following conditions hold

(a) for every irreducible subspace $Y \hookrightarrow Z$

$$\big(\bar{\mathrm{NE}}_1(Y) \ni \alpha \mapsto 0 \in \bar{\mathrm{NE}}_1(X)\big) \Rightarrow \alpha = 0$$

(b) The cone $\mathrm{NE}_1(X) \subseteq \mathrm{NS}_1(X)_{\mathbb{R}}$ doesn't contain a line.

Proof. Again necessity is obvious and (b) affords a Cartier divisor H non-negative on $\mathrm{NE}_1(X)$ satisfying (IV.21) which we prove satisfies *op. cit.* (and whence IV.b.1.(a)) for all sub-varieties $Y \hookrightarrow X$ by induction on their dimension. In dimension 1, there are two cases a curve, Y, factors through Z so $H \cdot Y > 0$ by IV.b.2.(a), or it doesn't. In the latter case, however, $Y \backslash Z$ is a non-empty curve in the quasi-projective variety $X\backslash Z$, so it certainly intersects non-trivially some divisor $D \hookrightarrow X\backslash Z$ without being contained in it. By hypothesis, however, the closure $\bar{D} \hookrightarrow X$ of D is $\mathbb{Q}$-Cartier so $\bar{D} \cdot Y \neq 0$ and IV.b.1.(a) holds. Similarly for Y of dimension $d+1 \leq \dim(X)$ we again distinguish 2-cases. If Y factors through Z we're done by hypothesis, otherwise we prove $H\,|_Y$ is ample. In the latter case, by Nakai-Moishezon and our induction hypothesis it's sufficient to prove $H^{d+1} \cdot Y > 0$. As before, however, there is a Cartier divisor, $\bar{D}$ on X intersecting Y non-trivially, so $H^d \cdot \bar{D} \cdot Y > 0$, while: for all $\epsilon > 0$ sufficiently small, $H - \epsilon\bar{D}$ satisfies (IV.21), so $p^*(H - \epsilon\bar{D})$ is nef., and $(H - \epsilon\bar{D})^{d+1} \cdot Y \geq 0$, whence $H^{d+1} \cdot Y > 0$. □

Of which a corollary to the corollary is

IV.b.3. **Corollary.** Let everything be as in IV.b.2 then we can replace condition (a) by

$$Z \text{ is projective and } \big(\bar{\mathrm{NE}}_1(Z) \ni \alpha \mapsto 0 \in \bar{\mathrm{NE}}_1(X)\big) \Rightarrow \alpha = 0 \tag{IV.22}$$

Which may be applied to the case in point, *i.e.*

IV.b.4. **Fact.** *The moduli of the contraction IV.a.4 is projective.*

Proof. Observe that under the hypothesis of IV.a.1 the locus of the extremal ray R must be the connected smooth divisor $\mathscr{Y}$ because $\mathscr{Y} \cdot R < 0$. Now, let $\rho : X \to X_0$ be the moduli of the contraction (IV.13), with Z the moduli of the singular locus in X meeting the extremal ray, then since X_0 is $\mathbb{Q}$-factorial, $\rho^* : \mathrm{NS}^1(X_0) \to \mathrm{NS}^1(X)$ is injective with image classes in the latter annihilated by R. Consequently, by duality there is an exact sequence

$$0 \to R \to \mathrm{NS}_1(X) \xrightarrow{\rho_*} \mathrm{NS}_1(X_0) \to 0 \tag{IV.23}$$

while $\bar{\mathrm{NE}}_1(X) \twoheadrightarrow \bar{\mathrm{NE}}_1(X_0)$, so IV.b.2.(b) holds because R is extremal. Now although there may be ambiguity, III.d.5, about the champ structure on the singular locus and the base of the contraction, there is no such ambiguity at the level of the moduli, *i.e.* Z is a section of the locus where ρ fails to be an isomorphism, so by (IV.22) and (IV.23) we need only check that a non-zero class in $\bar{\mathrm{NE}}_1(Z)$ cannot belong to R, which is clear, *e.g.* $K_{\mathscr{F}}\,|_Z$ is nef. □

IV.c. **The H-N Filtration again.** We will require knowledge of the normal bundle of the extremal smooth sub-champ $\mathscr{Y} \hookrightarrow \mathscr{X}$ of IV.a.1, akin to II.g.3 so, without loss of generality $\dim(\mathscr{Y}) > 1$. Our primary interest is the local variation of $N_{\mathscr{Y}/\mathscr{X}}$ over a small embedded analytic open, $\mathscr{Z}'$, of the base/singular locus, so to begin with, and essentially without loss of generality, we'll restrict attention to the case $s = 0$, III.d.7. As ever we first carry out our analysis at the level of the universal cover of $\mathscr{Y}$, *i.e.* a radially foliated weighted projective champ, III.b.11, and so abuse notation slightly, *i.e.* replace $\mathscr{Y}$ by its universal cover, $\mathscr{X}$ by a small neighbourhood of the former *etc.*. Naturally there are two tautological bundles of relevance, namely, $\mathscr{O}(1)$, on the weighted projective champ $\mathscr{Y}$, which, *op. cit.* is related to the radial foliation, $\mathscr{R}$, by

(IV.24) $$K_{\mathscr{R}} \xrightarrow{\sim} \mathscr{O}(-a_0)$$

and the relative tautological bundle of $\rho : \mathscr{P} := \mathbb{P}(N^\vee_{\mathscr{Y}/\mathscr{X}}) = P(N_{\mathscr{Y}/\mathscr{X}}) \to \mathscr{Y}$ which we'll denote H, while P, Y etc. will be the corresponding moduli. Now say $\mathscr{F}$ is the specialisation, *cf.* II.e.5 & II.f.1, of our original foliation to the projective normal cone, then $K_{\mathscr{F}} = \rho^* K_{\mathscr{R}}$ so a $K_{\mathscr{F}}$-negative extremal ray, R, of $\mathscr{P}$, has to be, II.d.1, an invariant curve of $\mathscr{F}$ lying over an invariant curve of $\mathscr{R}$. By III.c.1 we may suppose that the former has at worst nodes, and whence also the latter from our explicit knowledge, II.i.2, of the singularity. The moduli of such a champ is the moduli of its normalisation, so, without loss of generality, R is an extremal ray of $P(f^* N_{\mathscr{Y}/\mathscr{X}})$ for $f : \mathscr{L} \to \mathscr{Y}$ some coordinate axis of the radial foliation $\mathscr{R}$- all of which are smooth and embedded on a weighted projective space. By II.g.3 we know exactly what these are, and in terms of II.i.2 & III.d.1 we may describe them as follows: the local monodromy at the singularity, p, of the radial foliation is μ_{a_0}, and by hypothesis, III.d.7.(b).(i), there exists a local generator, ∂, of the ambient foliation on $\mathscr{X}$ which is μ_{a_0}-invariant so that the eigenvectors of Jordan decomposition of ∂ at p afford a μ_{a_0} equivariant decomposition,

(IV.25) $$N_{\mathscr{Y}/\mathscr{X}} \otimes \mathbb{C}(p) = \coprod_{1 \le i \le l} V_i$$

for V_i the subspace generated by the eigenvectors of weight $-\beta_i$ for β_i a complete repetition free list of the b_i, amongst which, II.h.6, $-\beta_l$ is largest. The decomposition (IV.25) then describes the singular locus of the specialised foliation exactly, *i.e.* it is a disjoint union

(IV.26) $$\operatorname{sing}(\mathscr{F}) = \coprod_{1 \le i \le l} P(V_i) \times \mathrm{B}_{\mu_{a_0}}$$

and the extremal ray in question is any invariant section over $\mathscr{L}$ which cuts $P(V_l)$, or, to be more precise, cuts $P(V_l) \times \mathrm{B}_{\mu_{a_0}} \hookrightarrow \mathscr{P}$ which is the embedded component of the singular locus. We can, therefore, apply III.d.7 to conclude that the extremal rays define a sub-champ $\mathscr{Y}_l \hookrightarrow \mathscr{P}$ together with a projection

(IV.27) $$\mathscr{Y}_l \to P(V_l)$$

whose fibres have universal cover a weighted projective champ $\mathbb{P}(c_0, c_1, \dots, c_r)$ for some weights c_i to be determined, radially foliated by $\mathscr{R}'$, say. Now, by (IV.26), (IV.27) has a section so $\mathbb{P}(\underline{c}) \times P(V_1)$ is the universal cover of $\mathscr{Y}_l$. We have, however, by II.g.3, $\mathscr{F}$-invariant embeddings $\mathscr{L}_i \to \mathscr{Y}_l$ lifting any coordinate axis $f_i : \mathscr{L}_i \hookrightarrow \mathscr{Y}$, and each $\mathscr{L}_i$ is simply connected, so there are $\mathscr{R}'$-invariant embeddings $f'_i : \mathscr{L}_i \hookrightarrow \mathbb{P}(\underline{c})$ of every $\mathscr{L}_i \xrightarrow{\sim} \mathbb{P}(a_0, aa_i)$, and whence $\mathbb{P}(\underline{c}) \xrightarrow{\sim} \mathscr{Y}$. Better still,

(IV.28) $$K_{\mathscr{R}'} = K_{\mathscr{F}} |_{\mathbb{P}(\underline{c}) \times P(V_1)} = K_{\mathscr{R}} |_{\mathbb{P}(\underline{c}) \times P(V_1)} \text{ and } K_{\mathscr{R}'} \cdot_{f'_i} \mathscr{L}_i = K_{\mathscr{R}} \cdot_{f_i} \mathscr{L}_i \text{ by II.d.5}$$

so $\mathbb{P}(\underline{c}) \to [\mathbb{P}(\underline{c})/\mathscr{R}']$ is, unsurprisingly, the radial foliation $\mathscr{Y} \to [\mathscr{Y}/\mathscr{R}]$ that we started with. Consequently the map

(IV.29) $$\mathscr{Y}_l \to \mathscr{Y} \times P(V_l)$$

afforded by the structural projection ρ and (IV.27) is an étale cover. As such (IV.29) exhibits the former as a locally constant gerbe over the latter. By explicit local calculation, however, *cf.* (IV.26), (IV.29) is an isomorphism in a neighbourhood of the fibre over the singularity p, so it's an isomorphism everywhere. We have, therefore, proved most of

IV.c.1. **Fact.** *Suppose, as above, that $\mathscr{Y}'$ of IV.a.1 is simply connected (whence isomorphic to the product of a polydisc with a radially foliated weighted projective space) then there is a filtration of $N_{\mathscr{Y}/\mathscr{X}}\mid_{\mathscr{Y}'}$ by invariant sub-bundles for the induced foliation,*

$$\mathscr{O} = N_l \subsetneqq N_{l-1} \subsetneqq N_{l-2} \subsetneqq \cdots \subsetneqq N_0 = N_{\mathscr{Y}/\mathscr{X}}$$

such that if $\beta_1 > \cdots > \beta_l$ is a complete repetition free list of the $b_1, \dots, b_t$ of III.d.1, and q_j, $1 \le j \le k$ the corresponding multiplicities, then for a as per III.b.11, locally over $\mathscr{Z}$:

$$N_{j-1}/N_j \xrightarrow{\sim} \mathscr{O}_{\mathscr{Y}}(-a\beta_j)^{\oplus q_j} .$$

Proof. $\mathscr{Y}_l$ of (IV.27) is, III.d.7, the image of a deformation space of extremal rays, which is constant on taking products with a small polydisc, whence this addition changes nothing, and for notational convenience we'll continue to ignore it. In any case, the embedding $\mathscr{Y}_l \hookrightarrow \mathscr{P}$ affords a sub-bundle

$$\big(\rho_* H \mid_{\mathscr{Y}_l}\big)^{\vee} \hookrightarrow N_{\mathscr{Y}/\mathscr{X}} \tag{IV.30}$$

which is the N_{l-1}th term in the above filtration. Moreover there is a canonical isomorphism

$$N_{\mathscr{Y}_l/\mathscr{P}} \xrightarrow{\sim} \rho^*\big(N_{\mathscr{Y}/\mathscr{X}}/N_{l-1}\big) \otimes H \tag{IV.31}$$

and so we conclude by induction. □

Unsurprisingly we continue to refer to this as the H-N filtration, and observe

IV.c.2. **Corollary.** Let $\mathscr{Y}' \hookrightarrow \mathscr{X}'$ be simply connected, then there is a non-canonical splitting

$$N_{\mathscr{Y}/\mathscr{X}} \mid_{\mathscr{Y}'} \xrightarrow{\sim} \coprod_j \mathscr{O}_{\mathscr{Y}}(-a\beta_j)^{\oplus q_j} \tag{IV.32}$$

and, better still, any section over $\mathscr{Y}'$ of $I_{\mathscr{Y},\mathscr{X}}/I^2_{\mathscr{Y}/\mathscr{X}} \otimes \mathscr{O}_{\mathscr{Y}'}(-a\beta_j)$ can be lifted to a (formal) section of $I_{\mathscr{Y},\mathscr{X}} \hat{\otimes} \mathscr{O}_{\hat{\mathscr{X}}'}(-a\beta_j)$ over the completion $\hat{\mathscr{X}}'$ of $\mathscr{X}'$ in $\mathscr{Y}'$.

Proof. The non-trivial case, given IV.c.1, is when the fibres of $\mathscr{Y} \to \mathscr{Z}'$ have dimension 1. This is, however, II.g.3, and otherwise it's immediate by IV.c.1 and I.c.3. □

The apparently arbitrary choice of such sections notwithstanding, choose some, say

$$\underline{\xi} := \xi_j : \mathscr{O}_{\hat{\mathscr{X}}'}(a\beta_j) \to \hat{I} := I_{\mathscr{Y},\mathscr{X}} \hat{\otimes} \mathscr{O}_{\hat{\mathscr{X}}'},\ 1 \le j \le t \tag{IV.33}$$

and define, *cf.* (III.18), a filtration on $\hat{I}$ by way of:

$$F^p_{\underline{\xi}} \hat{I} := \big(\xi_1^{j_1} \cdots \xi_t^{j_t} \mid b_1 j_1 + \cdots + b_t j_t \ge p\big)\ b_j := a\beta_j \tag{IV.34}$$

i.e. the ideal generated by the images of the $\mathscr{O}_{\hat{\mathscr{X}}'}(j_1 b_1 + \cdots j_t b_t)$ under (IV.33), and observe

IV.c.3. **Claim.** The filtration (IV.34) is algebraic, *i.e.* shrinking as necessary, there is a filtration $F^p I_{\mathscr{Y},\mathscr{X}} \mid_{\mathscr{X}'}$ whose completion is (IV.34). Better still this is independent of the choice (IV.33), and $\mathscr{F}$ invariant.

Proof. Plainly F^p contains some power, say q, of $\hat{I}$, so the first part just amounts to the coherence of $F^p/\hat{I}^q$ on the qth thickening of $\mathscr{Y}'$. As to the first part of the better still: say η_j is another choice, then either this is the same as ξ_j, or there is a smallest $p>0$ such that

$$0 \neq \xi_j - \eta_j \in \left(F^p_{\underline{\xi}}/F^{p+1}_{\underline{\xi}}\right) \otimes \mathscr{O}_{\mathscr{Y}'}(-b_j) \xrightarrow{\sim} \coprod_{b_1 j_1 + \cdots + b_t j_t = p} \mathscr{O}_{\mathscr{Y}'}(p - b_j) \tag{IV.35}$$

so $p \geq b_j$, whence $\eta_j : \mathscr{O}_{\hat{\mathscr{X}}'}(b_j) \to F^{b_j}_{\underline{\xi}}$, and we're done by symmetry. Similarly, suppose the composition

$$\mathscr{O}_{\hat{\mathscr{X}}'}(b_j) \xrightarrow{\xi_j} F^{b_j}\hat{I} \to \mathscr{O}_{\hat{\mathscr{X}}'} \to K_{\mathscr{F}} \tag{IV.36}$$

doesn't factor through $K_{\mathscr{F}} \otimes F^{b_j}\hat{I}$, then there is a smallest $b_j > p \geq 0$ through which it does factor, so (IV.36) affords a non-zero $\mathscr{O}_{\mathscr{Y}'}$-linear map

$$\mathscr{O}_{\hat{\mathscr{Y}}'}(b_j) \to K_{\mathscr{F}} \otimes \coprod_{b_1 j_1 + \cdots + b_t j_t = p} \mathscr{O}_{\mathscr{Y}'}(p) \xrightarrow{\sim} \coprod_{b_1 j_1 + \cdots + b_t j_t = p} \mathscr{O}_{\mathscr{Y}'}(p - a_0) \tag{IV.37}$$

which is nonsense. □

Putting this all together we have therefore

IV.c.4. **Fact/Definition.** Let $\mathscr{Y} \hookrightarrow \mathscr{X}$ be as in IV.a.1 then there is a $\mathscr{F}$-invariant filtration

$$\cdots \subset F^p \cdots \subset F^{>0} = I_{\mathscr{Y},\mathscr{X}} \subset \mathscr{O}_{\mathscr{X}} \tag{IV.38}$$

such that

(a) The restriction of (IV.38) to a small embedded analytic neighbourhood $\mathscr{X}'$ as defined in IV.a.1 pulls back to (IV.34) on the universal cover of $\mathscr{X}'$.
(b) For $f : \tilde{\mathscr{L}} \hookrightarrow \mathfrak{X} \to \mathscr{X}$ the normalisation of an extremal ray with at worst nodes embedded in its net completion, the pull back of IV.38 is the filtration defined by the invariant divisors II.g.3.(2) combined in the (obvious) way suggested by (IV.34).

Proof. The filtration has already been defined on the universal cover, say $\mathscr{X}'' \to \mathscr{X}'$ with Galois group π_1. As such, it descends to $\mathscr{X}'$ provided (IV.34) admits a π_1 action, which is clear from the proof of IV.c.3 because the H-N filtration, IV.c.1, is π_1-equivariant. Similarly: to compare the filtrations on 2-small analytic open embeddings $\mathscr{X}'_\alpha \hookrightarrow \mathscr{X}$, $\mathscr{X}'_\beta \hookrightarrow \mathscr{X}$ we only need to compare them on any (faithfully flat) étale covering of $\mathscr{X}'_\alpha \cap \mathscr{X}'_\beta$, so again this is just IV.c.3 and the definition (IV.34) as is (b). □

IV.d. **Existence of flips.** Let $\mathscr{Y} \hookrightarrow \mathscr{X}$ be as in IV.a.1 then by (IV.38) there is a $K_{\mathscr{F}}$-invariant smoothed weighted blow up, [MP13, I.iv.3], defined as in (III.19), to wit:

$$\begin{array}{ccc} \mathscr{X}_1 := \mathrm{Proj}\left(\coprod_p F^p\right) & \xleftarrow{\text{Vistoli covering}} & \mathscr{X}_2,\ K_{\mathscr{F}_2} = K_{\mathscr{F}}\mid_{\mathscr{X}_2} \\ {\scriptstyle \text{weighted blowup}}\Big\downarrow{\scriptstyle \text{Everything } \mathscr{F} \text{ invariant}} & & \\ \mathscr{X} & & \end{array} \tag{IV.39}$$

Before progressing let us make a clarifying

IV.d.1. *Remark.* The implied weights in (IV.39) are not the $a\beta_j$ of IV.34 but $b'_i := b_i/b$ where b_i are as per III.d.1 and b is their gcd. Following (IV.13), however, we'll be taking the covering

$$\mathscr{X}_2 \xleftarrow{ab\text{th root of } \mathscr{E}_2} \mathscr{X}_{ab}, \quad \mathscr{Y}_{ab} := \frac{1}{ab} \cdot \mathscr{E}_2 \tag{IV.40}$$

and the totality, *i.e.* the horizontal in (IV.39) composed with (IV.40), is, functorially with respect to the ideas *the smoothed weighted blow up with weights ab_i* where a is given by III.b.11

and the b_i by III.d.1. Consequently there's a certain convenience in doing both steps at once, or, at least, referring, as we will, to their totality in terms of the unifying idea.

This said the exceptional divisor $\mathscr{E}_2$ on $\mathscr{X}_2$ is described by

IV.d.2. **Claim.** The weighted projective bundle $\mathscr{E}_2 \to \mathscr{Y}$ enjoys the following triviality property: for $\mathscr{Y}' \hookrightarrow \mathscr{Y}$ as per IV.a.1 ($\mathscr{Z}' \hookrightarrow \mathscr{Z}_0$ understood sufficiently small) and $\mathscr{Y}'' \to \mathscr{Y}'$ its finite universal cover

$$\mathscr{E}_2 \mid \mathscr{Y}'' \xrightarrow{\sim} \mathscr{Y}'' \times \mathbb{P}(b'_1, \ldots, b'_t) \tag{IV.41}$$

Moreover the induced foliation (understood either logarithmically, I.b.2, or, equivalently, without saturation if the fibres $\mathscr{Y} \to \mathscr{Z}_0$ have dimension 1) has canonical bundle the restriction of $K_{\mathscr{F}}$ and singular locus the fibre over the unique connected component $\mathscr{Z}$ of $\mathrm{sing}(\mathscr{F})$ contained in $\mathscr{Y}$.

Proof. The pth factor of the graded algebra associated to (IV.34) is $\mathscr{O}_{\mathscr{Y}''}(p)$ tensored with the pth factor of the trivial graded algebra freely generated by generators of weights b_i, $1 \le i \le t$, *cf.* (IV.35) & (IV.37), which has the same Proj as that which is freely generated after cancelling the common factors, IV.d.1, whence (IV.41). As to the moreover: the exceptional divisor of a (weighted) blow up in an invariant centre is always smooth in the foliation direction, so we only have to compute what happens over the singular locus which we can do explicitly using IV.c.4 by way of its relation, II.g.3.(3), with the Jordan decomposition, and appropriate local coordinates, *cf.* (II.124). □

Now, irrespectively of whether $\mathscr{E}_2$ is extremal in $\mathscr{X}_2$, the cone theorem applies to $\mathscr{E}_2$ in it's induced foliation, while extremal rays in $\mathscr{Y}$ with at worst nodes lift (*cf.* the preamble to the proof of IV.c.1) to the same in $\mathscr{E}_2$ by II.g.3. As such III.d.7 applies to $\mathscr{E}_2$ in se (*i.e.* as the locus of its own extremal ray) to imply

IV.d.3. **Fact/Definition.** The champ $\mathscr{E}_2$ is a bundle of foliated varieties (whose fibres have universal covers radial foliations on a $\mathbb{P}(a_0, aa_1, \ldots, aa_n)$) over an orbifold $\mathscr{Z}_0$ which (for good measure) is itself a bundle, IV.d.2, of $\mathbb{P}(b'_1, \ldots, b'_t)$'s over the orbifold structure on the singular locus of $\mathscr{Y}$. Consequently for $\mathscr{X}_{ab}$ as in (IV.40) there is a contraction $\rho : \mathscr{X}_{ab} \to \mathscr{X}_0$ of $\mathscr{Y}_{ab}$ to a locally constant gerbe over $\mathscr{Z}_0$ such that the induced foliation $\mathscr{X}_0 \to [\mathscr{X}_0/\mathscr{F}_0]$ is smooth and everywhere transverse to the locus where ρ is not an isomorphism. The bi-rational map $\rho : \mathscr{X} \to \mathscr{X}_0$ will, irrespectively of whether the moduli of $\mathscr{X}_0$ is projective, be referred to as a *flip*, and the more precise data

$$\begin{array}{ccc} \mathscr{X}_{ab} & \xrightarrow[\text{(IV.13)}]{\text{Blow down, } \rho_+, \text{ with weights } aa_i} & \mathscr{X}_0 \\ {\scriptstyle \text{Blow up, } \rho_-, \text{ with weights}} \big\downarrow {\scriptstyle ab_j, \text{ (IV.39) \&(IV.40)}} & & \\ \mathscr{X}_- := \mathscr{X} & & \end{array} \tag{IV.42}$$

of a weighted blow up followed by a weighted blow down as a *flap*.

Proof. As observed the structure of $\mathscr{E}_2$ is implied by III.d.7 given the structure, IV.d.2, of the singular locus. This is, however, the sum total of what we need to deduce the existence of the contraction ρ from IV.a.4, *i.e.* the condition that $\mathscr{Y}_{ab}$ is covered by extremal rays of the ambient space is necessary for the projectivity of the moduli of the contraction, but not for its existence as an algebraic space. □

To examine the projectivity of this construction let us suppose in addition to IV.a.1, IV.d.4. **Set Up.** Fix an extremal ray R and suppose that every $-1/d\mathbb{F}$ curve equivalent to R belongs to a connected smooth embedded sub-champ, $\mathscr{Y}_p \hookrightarrow \mathscr{X}$, of the form IV.a.1, and all such sub-champs are disjoint, equivalently none of the following occur

(a) For some smooth connected component $\mathscr{Z} \to \mathrm{sing}(\mathscr{F})$ there are 2 such sub-champs (for the same R) meeting in $\mathscr{Z}$, II.h.9.
(b) For some smooth connected component $\mathscr{Z} \to \mathrm{sing}(\mathscr{F})$, and, again, the same R, III.d.7.(b).(ii) occurs.
(c) There is a representative of R avoiding the singular locus.

Observe that the criteria for the projectivity of the flip is particularly simple, *i.e.*

IV.d.5. **Claim.** In the context of (IV.42), the following are equivalent

(a) The moduli of the flipped champ $\mathscr{X}_+$ is projective.
(b) The cone $\bar{N}E_1(\mathscr{X}_+)$ does not contain a line.
(c) The $-1/d\mathbb{F}$ curve contracted by ρ_+ is extremal.

Proof. Plainly (a) implies (b), and (IV.22) always holds- same argument as end of the proof of IV.b.4- whence, conversely, IV.b.3, (b) implies (a), while (b) iff (c) is the general duality considerations of (IV.23). □

The same applies, a little more generally, if one flips several sub-champs in $\mathscr{X}$ at the same time, provided, as is our context, IV.d.4, the champs being flipped are all disjoint, which we'll employ without further comment in

IV.d.6. **Claim.** The flip, (IV.42), of any of the $\mathscr{Y}_p$ has projective moduli.

Proof. Since the horizontal arrows in (IV.42) are (étale locally) weighted blow downs it will suffice to do everything at once, which is all we need anyway. As such, consider the totality, at the level of the moduli, of the flaps (IV.42) performed in all of the $\mathscr{Y}_p$, *i.e.*

(IV.43)
$$\begin{array}{ccc} X(R) & \xrightarrow[\rho_+]{} & X_+ \\ {\scriptstyle\rho_-}\big\downarrow & & \\ X_- & & \end{array}$$

with E^p the exceptional divisors; C^p_- curves in the same contracted by ρ_-; and $C^p_+ \hookrightarrow E^p$ a $K_{\mathscr{F}}$-negative invariant curve contracted by ρ_+. Fix p, then by the cone theorem, II.d.1, there are finitely many extremal rays represented by (multiples of) $K_{\mathscr{F}}$-negative invariant curves, R_i, and a (pseudo) effective class Z_p on which $K_{\mathscr{F}}$ is non-negative such that on $X(R)$

(IV.44)
$$C^p_+ = \sum_i R_i + Z_p$$

Now, by construction, (IV.12), $(\rho_-)_*(C^p_+)$ is parallel to R, so $(\rho_-)_*(Z_p)$ is too. However, ρ_- is unramified in the foliation direction, so $(\rho_-)_*(Z_p) = 0$. Consequently, by the projectivity of X_-, Z_p is a sum

(IV.45)
$$\sum_q c^q_- C^q_-, \quad c^q_- \geq 0$$

On the other hand all the R_i lie over R, so by our hypothesis IV.d.4 and (IV.12) every R_i is parallel to some C^q_+ for some q. Thus we equally have

(IV.46)
$$\sum R_i = \sum_q c^q_+ C^q_+, \quad c^q_+ \geq 0$$

Combining all of (IV.44)-(IV.46) we have therefore

(IV.47) $$C_+^p = \sum_q \left(c_+^q C_+^q + c_-^q C_-^q \right), \text{ every } C_+^q \text{ extremal by (IV.46)},$$

while all the divisors E_q are disjoint and strictly negative on both C_+^q, C_-^q, so the only index that can occur on the right of (IV.47) is p. Consequently, C_+^p is extremal and we're done by IV.d.5. ☐

IV.e. **Exceptional flips and termination.** The first case to be considered is

IV.e.1. **Set Up.** $\mathscr{Y} \hookrightarrow \mathscr{X}$ is an extremal sub-champ satisfying III.d.7.(b).(ii), with $\mathscr{Z} \hookrightarrow \mathscr{Y}$ the unique (smooth) connected component of $\mathrm{sing}(\mathscr{F})$ contained in it.

Now observe that by the unicity and local uniformity, III.d.3, of Jordan decomposition there is a well defined (smoothed) weighted blow up supported in $\mathscr{Z}$ whose weights in the notation of III.d.1 are

(IV.48) y_i, resp. x_i, has weight a_i, resp. b_i, where $a_i = b_i$ and $r = t$.

and whose effect is described by

IV.e.2. **Claim.** Let $\mathscr{X}_1 \to \mathscr{X}$ be the smoothed weighted blow up defined by (IV.48) with $\mathscr{E}_1$ its exceptional divisor and $\mathscr{Y}_1$ the proper transform of $\mathscr{Y}$ then

(a) The singular locus of $\mathscr{F}_1$ over $\mathscr{Z}$ is the intersection of $\mathscr{E}_1$ and $\mathscr{Y}_1$. It is smooth connected, and, for good measure, a $\mathbb{P}(a_1, \dots, a_r)$-bundle over the μ_2 covering of $\mathscr{Z}$ defined in III.d.7.(b).(ii).
(b) The embedded sub-champ $\mathscr{Y}_1 \hookrightarrow \mathscr{X}_1$ is the locus of (not just a connected component of) an extremal ray R_1 satisfying III.d.7.(b).(i).
(c) The exceptional divisor $\mathscr{E}_1$ is covered by $K_{\mathscr{F}}$-nil invariant parabolic champs.

Proof. To calculate the singular locus we use the Jordan coordinates of III.d.1, so, [MP13, I.iv.3], on, say the $y_1 \neq 0$ chart we have local coordinates η_i, ξ_j defined by

(IV.49) $$y_1 = \eta_1^{a_1},\, y_2 = \eta_2 \eta_1^{a_2}, \cdots, y_r = \eta_r \eta_1^{a_r},\, x_1 = \xi_1 \eta_1^{b_1}, \cdots, x_r = \xi_r \eta_1^{b_r}$$

which gives that étale locally there are 2 smooth component of the singular locus in the fibre of $\mathscr{E}_1$ over $\mathscr{Z}$, which in turn are the intersection of $\mathscr{Y}_1$ and $\mathscr{E}_1$. Plainly (paragraph prior to (III.45)) the local system defined by these components is the same as the μ_2 cover $\mathscr{Z}^{+-} \to \mathscr{Z}$, so the singular locus is connected, and the good measure part is clear. As to (b) this is just an easy variation on (IV.44)-(IV.45). Specifically suppose the proper transform, L_1, of an invariant curve isn't extremal then *op. cit.* and $\mathscr{E}_1 \cdot L_1 > 0$ imply the absurd. Finally (c) follows from the explicit coordinates (IV.49) and the fact that the canonical, $K_{\mathscr{F}_1}$ is just $K_{\mathscr{F}} |_{\mathscr{X}_1}$. ☐

We can, therefore, combine this with (IV.42) to make

IV.e.3. **Fact/Definition.** By an *exceptional flip* (or, better, flap) is to be understood, for $\mathscr{Y} \hookrightarrow \mathscr{X}$ as in IV.e.1, the diagram

(IV.50) $$\begin{array}{ccc} \mathscr{X}_{ab} & \xrightarrow[\text{(IV.42)}]{\rho_+} & \mathscr{X}_+ \\ {\scriptstyle \text{flip of } \mathscr{Y}_1 \text{ in IV.e.2}} \Big\downarrow {\scriptstyle \rho_- of \text{(IV.42)}} & & \\ \mathscr{X}_- = \mathscr{X}_1 \hookleftarrow \mathscr{E}_1 & & \\ {\scriptstyle \text{Weighted blow up}} \Big\downarrow {\scriptstyle \text{IV.e.2}} & & \\ \mathscr{X} & & \end{array}$$

Better still

(a) The moduli of $\mathscr{X}_+$ is projective.
(b) The image $\mathscr{E}_+$ of $\mathscr{E}_1$ is covered by invariant parabolic champs (in fact it's a bundle of such over a $\mathbb{P}(a_i)\times\mathbb{P}(b_j)$-bundle over $\mathscr{Z}^{+-}$) none of which meet the singular locus, so the generic fibre of $\mathscr{X}_+ \to [\mathscr{X}_+/\mathscr{F}_+]$ is a smooth parabolic champ.

Proof. Part (a) follows from IV.d.5 and IV.e.2.(b), while $\mathscr{E}_+$ is contained in the smooth locus of $\mathscr{F}_+$ by IV.d.3 whence (b) by IV.e.2.(c) and II.d.5.(c). □

Similarly if IV.d.4.(a) occurs or slightly more generally

IV.e.4. **Claim.** If there are 2 extremal (not necessarily for the same ray) champ meeting in the same component of sing($\mathscr{F}$) then the generic fibre of $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ is a parabolic champ. Moreover if both varieties arise from the same extremal ray, *i.e.* IV.d.4.(a), then the flip (IV.42) does not have projective moduli, and there are invariant parabolic champ in (the original $\mathscr{X}$) which do not meet sing($\mathscr{F}$) and are parallel to the given extremal rays, *i.e.* IV.d.4.(c) also holds.

Proof. Choose one, say $\mathscr{Y}'$, of the extremal varieties, flip it, and irrespectively of whether the moduli is projective IV.d.3 and II.d.5.(c) still apply. Furthermore, if both rays are extremal then as in the proof of IV.e.2 the proper transform, R_1, of an invariant curve in the other, say, $\mathscr{Y}''$, is an extremal ray. Plainly, however, the invariant curves, L, in the fibre over $\mathscr{Y}'$ have the form $R_1 + C_-$ where C_- is contracted by ρ_-, while the exceptional divisor, $\mathscr{E}_1$, of ρ_- is negative on L, and positive on R_1, whence it's negative on C_-, so C_- is effective; L isn't extremal, and the moduli of $\mathscr{X}_+$ isn't projective. On the other hand $(\rho_-)_*R_1$ is an invariant parabolic champ missing sing($\mathscr{F}_+$), so it can be moved off the flipped locus to some R_+. As such the proper transform $\tilde{R}_+$ (in $\mathscr{X}_{ab}$ of (IV.42)) is a linear combination of L and R, so $(\rho_-)_*(\tilde{R}_+)$ is parallel to the original extremal ray. □

Given the well defined way in which it occurs, the loss of projectivity in IV.e.4 is very far from deadly. Nevertheless there are several obvious reasons for avoiding it so we make

IV.e.5. **Fact/Definition.** By a *very exceptional flip* (or, better, flap) is to be understood, for $\mathscr{Y}' \hookrightarrow \mathscr{X}$ and $\mathscr{Y}'' \hookrightarrow \mathscr{X}$ a pair of extremal varieties meeting in the same component of the singular locus, IV.d.4, of $\mathscr{F}$ and parallel to the same extremal ray, IV.e.4, the diagram (IV.50) with the further proviso

(IV.51) The arrow ρ_-, resp. ρ_+, is the weighted blow up, resp. down, in both $\mathscr{Y}'$ and $\mathscr{Y}''$.

The moduli of the resulting champ $\mathscr{X}_+$ is projective, while the resulting foliation $\mathscr{F}_+$ is smooth and everywhere transverse to the locus where ρ_+ is not an isomorphism for exactly the same reasons that the corresponding statements hold for the exceptional flips of IV.e.3.

Now flipping, be it exceptional or otherwise, manifestly terminates for the simple reason that the number of connected components of the singular locus decreases by at least 1 with the flip of any extremal ray, and so in increasing order of difficulty we have,

IV.e.6. **Proposition/Summary.** *Let $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ be a foliated champ which is not a foliation in parabolic champs and which enjoys the following further properties*

(IV.52) *smooth; projective moduli; log canonical, resp. canonical, foliation singularities*

then there is a sequence of contractions and flips in the sense of IV.a.4 and IV.d.3 (or alternatively just flaps (IV.13) *&* (IV.42) *),*

$$\begin{array}{ccccc} \mathscr{X} = \mathscr{X}_0 & & \mathscr{X}_1 & \cdots\cdots & \mathscr{X}_n = \mathscr{X}_{\min} \\ \downarrow & \dashrightarrow & \downarrow & \dashrightarrow \quad \dashrightarrow & \downarrow \\ [\mathscr{X}/\mathscr{F}] = [\mathscr{X}_0/\mathscr{F}_0] & & [\mathscr{X}_1/\mathscr{F}_1] & & [\mathscr{X}_n/\mathscr{F}_n] = [\mathscr{X}_{\min}/\mathscr{F}_{\min}] \end{array} \tag{IV.53}$$

such that each $\mathscr{X}_i \to [\mathscr{X}_i/\mathscr{F}_i]$ *enjoys all the (respective) properties* (IV.52)*, and* $K_{\mathscr{F}_{\min}}$ *is nef.*

Proof. The hypothesis that the foliation isn't in parabolic champs implies, IV.e.3.(b) & IV.e.4, that we must, at every stage, be in the situation of (IV.d.4), *i.e.* III.d.7.(b).(i). Consequently we eventually run out of components of the singular locus through which a $-1/d\,\mathbb{F}$-curve can pass, and we terminate with $K_{\mathscr{F}}$ nef. by the cone theorem, II.d.1. □

The alternative to which is

IV.e.7. **Proposition/Summary.** *Let everything be as in IV.e.6 with the exception of the hypothesis "not a foliation in parabolic champs" which we replace by "no model has nef. (foliated) canonical bundle" then after a sequence of contractions and flips in the sense of IV.a.4, resp. IV.d.3, as described in* (IV.53) *all of* (IV.52) *continues to hold (i.e. we're still excluding the exceptional cases IV.e.3 and IV.e.5) and exactly one of the following happens*

- (a) $\mathscr{X}_n \to [\mathscr{X}_n/\mathscr{F}_n]$ *is a Mori fibre space, i.e. the locus of a single extremal ray is all of* $\mathscr{X}_n$ *and the foliation is a bundle of foliated varieties where the universal cover of a fibre is the radial (supposed saturated in dimension 1) foliation on a weighted projective space whose dimension is* 1 *iff the foliation singularities are canonical.*
- (b) *At least one of IV.d.4.(a) or (b) occurs at every connected component of the singular locus. In particular, therefore, all of the foliation singularities are canonical.*

Proof. If we exclude (b), then the only other thing that can happen is that the locus of an extremal ray is everything with the champ itself described by III.d.7.(b).(i), *i.e.* (a), while the various facts about canonical vs. log-canonical singularities are just the definitions. □

This leaves us to elaborate the final case

IV.e.8. **Proposition/Summary.** *Should case (b) of IV.d.3 occur then, without loss of generality, there are no occurrences of either contractions, IV.a.4, or the flips of IV.d.3, and should there be any exceptional flips we continue by*

$$\Big(\mathscr{X}_n \to [\mathscr{X}_n/\mathscr{F}_n]\Big) \dashrightarrow \Big(\mathscr{X}_{n+1} \to [\mathscr{X}_{n+1}/\mathscr{F}_{n+1}]\Big) \tag{IV.54}$$

wherein all possible exceptional flips IV.e.3 are performed at once with all of (IV.52) *being preserved. If we're still not done, i.e.* $\mathscr{F}_{n+1}$ *isn't smooth, then IV.d.4.(a) occurs, and we have the following choices for* $\big(\mathscr{X}_{n+1} \to [\mathscr{X}_{n+1}/\mathscr{F}_{n+1}]\big) \dashrightarrow \big(\mathscr{X}_{n+2} \to [\mathscr{X}_{n+1}/\mathscr{F}_{n+2}]\big)$*,*

- (a) *For each component of the singular locus of* $\mathscr{F}_{n+1}$ *choose an extremal sub-champs and flip it according to IV.d.3. This will necessarily result in the loss of projectivity, IV.e.4, but otherwise the list* (IV.52) *is conserved.*
- (b) *Perform at the same time all possible very exceptional flips, IV.e.5, and thus preserve the list* (IV.52) *in its entirety.*

In either case $\mathscr{X}_{n+2} \to [\mathscr{X}_{n+2}/\mathscr{F}_{n+2}]$ *is a bundle of* 1*-dimensional parabolic champs which is identically its own Mori fibre space.*

Proof. All exceptional or very exceptional flips can only occur at smooth connected components of the singular locus so the extremal sub-champs that they determine cannot intersect (except, of course, in a very exceptional flip wherein $\mathscr{Y}' \cup \mathscr{Y}''$ of IV.e.5 should be thought of as a single entity) so, without loss of generality, all these operations can be combined into one. Better still both the extremal champ, $\mathscr{Y}$, of an exceptional flip, IV.e.3, or $\mathscr{Y}' \cup \mathscr{Y}''$ in the case of a very exceptional flip IV.e.5 are the only invariant sub-champs meeting their respective components of the singular locus, whence the two exceptional cases commute with contractions, IV.a.4, and (non-exceptional) flips IV.d.3, so there's no loss of generality in supposing that all such operations have already been exhausted. □

IV.f. **Logarithmic remarks.** In order to reference it we spell out our

IV.f.1. **Set Up.** By hypothesis $\mathscr{D} \hookrightarrow \mathscr{X}$ will be a divisor, no generic point of which is invariant, in a connected smooth proper champ, and $\mathscr{X}\backslash\mathscr{D} \to [\mathscr{X}\backslash\mathscr{D}/\mathscr{F}]$ a foliation with log-canonical singularities.

As such, by I.b.10, $\mathscr{D}$ is smooth and everywhere transverse to $\mathscr{F}$. In particular, therefore, for every $e \in \mathbb{Z}_{>1}$, the extraction $\epsilon : \mathscr{X}^e \to \mathscr{X}$ of a eth root, I.a.9, of $\mathscr{D}$ is smooth, and the induced foliation $\mathscr{X}^e \to [\mathscr{X}^e/\mathscr{F}^e]$ has, I.b.15, log-canonical singularities which, I.b.13, are terminal around the pre-image of $\mathscr{D}$. Furthermore we assert,

IV.f.2. **Claim.** Let everything be as above with $f : \mathscr{C} \to \mathscr{X}$ a map from a (smooth irreducible) curve such that $(K_{\mathscr{F}} + \mathscr{D}) \cdot_f \mathscr{C} < 0$ then f does not factor through $\mathscr{D}$. In particular, therefore, there is a lifting $f^e : \mathscr{C}^e \to \mathscr{X}^e$ and $K_{\mathscr{F}^e} \cdot_{f^e} \mathscr{C}^e < 0$.

Proof. The tangency between $\mathscr{D}$ and $\mathscr{F}$ always yields a section of $\mathscr{O}_{\mathscr{D}}(K_{\mathscr{F}} + \mathscr{D})$, which by hypothesis is trivial, *i.e.* in a highly degenerate case of II.d.3 the trace is a constant section over$\mathscr{D}$, so f certainly cannot factor through it. As such there is certainly a lifting $f^e : \mathscr{C}^e \to \mathscr{X}^e$, while

$$K_{\mathscr{F}^e} \cdot_{f^e} \mathscr{C}^e \le (K_{\mathscr{F}^e} + \mathscr{D}^e) \cdot_{f^e} \mathscr{C}^e = (K_{\mathscr{F}} + \mathscr{D}) \cdot_f \mathscr{C} < 0 \tag{IV.55}$$

where $\epsilon^*\mathscr{D} = e\mathscr{D}^e$, and $\mathscr{D}^e$ is smooth. □

It certainly therefore follows that if $K_{\mathscr{F}^e}$ is nef. then $K_{\mathscr{F}} + \mathscr{D}$ is nef., but, plausibly in running the minimal model programme for $\mathscr{X}^e \to [\mathscr{X}^e/\mathscr{F}^e]$ we could loose the hypothesis of IV.f.1. Observe, however, that the operations of flipping and extracting roots commute, *i.e.*

IV.f.3. **Fact.** *For any any contraction, resp. flip,*

$$\Big(\mathscr{X}^e \to [\mathscr{X}^e/\mathscr{F}^e]\Big) \dashrightarrow \Big(\mathscr{X}^e_+ \to [\mathscr{X}^e_+/\mathscr{F}^e_+]\Big) \tag{IV.56}$$

in the sense of IV.a.4, resp. IV.d.3, there is a contraction, resp. flip,

$$\Big(\mathscr{X} \to [\mathscr{X}/\mathscr{F}]\Big) \dashrightarrow \Big(\mathscr{X}_+ \to [\mathscr{X}_+/\mathscr{F}_+]\Big) \tag{IV.57}$$

such that the proper transform, $\mathscr{D}_+ \hookrightarrow \mathscr{X}_+$ satisfies IV.f.1, and $\mathscr{X}^e_+ \to \mathscr{X}_+$ is the extraction of an eth root of $\mathscr{D}_+$.

Proof. That a contraction, resp. flip, of $\mathscr{X}^e \to [\mathscr{X}^e/\mathscr{F}^e]$ determines the same of $\mathscr{X} \to [\mathscr{X}/\mathscr{F}]$ is immediate from IV.f.2 and the definitions if $\mathscr{X}$ has projective moduli. However, even without this, it still follows since projectivity is only used, *cf.* III.d.2, to ensure that the contracted, resp. flipped, sub-champ $\mathscr{Y}$ meets a unique component of sing($\mathscr{F}$) through which each of the $-1/d\,\mathbb{F}$ curves which cover $\mathscr{Y}$ must pass. Irrespectively, what we need to do in the first instance is to prove that there is a map,

$$\mathscr{X}^e_+ \to \mathscr{X}_+ \tag{IV.58}$$

To this end observe, exactly as in the final steps of the proof, (IV.19) *et seq.*, of IV.a.4, the expression of other side of (IV.58) as a locally constant gerbe over an orbifold, I.a.6, is determined in co-dimension 2, so, without loss of generality, there is no generic stabiliser. Furthermore, flips are actually flaps, so by the unicity of contraction both sides of (IV.58) have the same moduli X_+, and whence they equally factor through the same Vistoli covering champ $\mathscr{X}^v_+$, I.a.2. Now, to go from any smooth champ to the Vistoli covering champ of its moduli one kills, [Vis89, 2.8], pseudo reflections. A pseudo reflection, however, of a foliated champ stabilises exactly one of an invariant divisor or a generically transverse divisor, so we have further factorisations such as

$$\mathscr{X}^e_+ \xrightarrow[\text{reflections}]{\text{kill transverse}} \mathscr{X}^i_+ \xrightarrow[\text{reflections}]{\text{kill invariant}} \mathscr{X}^v_+ \tag{IV.59}$$

and similarly for $\mathscr{X}_+ \to \mathscr{X}_+^v$. Now let x be a geometric point of the proper transform $\mathscr{D}_+^e \hookrightarrow \mathscr{X}_+^e$ of $\mathscr{D}$; G_x its stabiliser; and $U \to \mathscr{X}_+^e$ an étale neighbourhood then there is a non-trivial normal sub-group, S_x, generated by pseudo-reflections fixing smooth branches of $\mathscr{D}_+^e$, while $\mathscr{X}_+^e \to [\mathscr{X}_+^e/\mathscr{F}_+^e]$ is smooth at x by our hypothesis IV.f.1 and IV.a.4. As such by I.b.13 and (the non-subtle) part of I.b.6, the induced foliation, $\mathscr{G}$, on $V := U/S_x$ is also smooth. Equally $U \to V$ is ramified uniquely in the image, Δ, of $\mathscr{D}$, to order e so, there is a factorisation

$$U \to V^e \to V \tag{IV.60}$$

through an eth root of Δ in which the first map is almost étale, so by *op. cit.* V^e in the induced foliation $\mathscr{G}^e$ is log-terminal. Consequently, $V \to [V/\mathscr{F}]$ with the orbifold boundary $(1-1/e)\Delta$ is also log-terminal, whence by I.b.14 Δ is smooth and everywhere transverse to $\mathscr{G}$, and so $\mathscr{D}_+^e$ is too. This is, however, equivalent to: S_x is a cyclic normal sub-group of G_x and the restriction of the character, $\chi_x : G_x \to \mathbb{G}_m$ afforded by $\mathscr{D}_+^e$ to S_x is an isomorphism, so every sub-group of S_x is normal. The monodromy of every generic point of $\mathscr{D}_+^e$ is, moreover, of the form

$$0 \to \mu_e \to \mu_{ee'} \to \mu_{e'} \to 0 \tag{IV.61}$$

where e' is the order of the corresponding stabiliser in the original $\mathscr{X}$. Consequently, the μ_e in (IV.60) afford a well defined normal sub-group scheme of the stabiliser $\mathscr{S} \to \mathscr{D}_+^e$ which just as in (IV.59) can be killed to yield a factorisation

$$\mathscr{X}_+^e \xrightarrow[\text{in } \mu_e]{\text{kill reflections}} \mathscr{X}_+^j \xrightarrow[\text{reflections}]{\text{kill all further}} \mathscr{X}_+^v \tag{IV.62}$$

in which the image in $\mathscr{X}_+^j$ of $\mathscr{D}_+^e$ is smooth everywhere transverse to the foliation, and the first map in (IV.62) is just the extraction of an eth root. By definition, however, $\mathscr{X}_+^j$ and $\mathscr{X}_+$ coincide in co-dimension 1, and since they're both smooth they're equal by purity and I.a.4. □

Next observe that we equally have a log cone theorem, *i.e.*

IV.f.4. **Fact.** *Let $\mathscr{X}\backslash\mathscr{D} \to [\mathscr{X}\mathscr{D}/\mathscr{F}]$ be a logarithmic foliated normal champ with both $K_{\mathscr{F}}$ and $\mathscr{D}$ Cartier; log-canonical singularities in dimension 1 and projective moduli, then there are countably many $\mathscr{F}$-invariant parabolic, champ $\mathscr{L}_i$, with, $0 < -(K_{\mathscr{F}}.+\mathscr{D})\cdot\mathscr{L}_i \le 2$ such that,*

$$\overline{\mathrm{NE}}\,(\mathscr{X})_{\mathbb{R}} = \overline{\mathrm{NE}}\,(\mathscr{X})_{K_{\mathscr{F}}+\mathscr{D}\ge 0} + \sum_i \mathbb{R}_+\,\mathscr{L}_i \tag{IV.63}$$

where $\overline{\mathrm{NE}}\,(\mathscr{X})_{K_{\mathscr{F}}+\mathscr{D}\ge 0}$ is the sub-cone of the closed cone of curves on which $K_{\mathscr{F}}+\mathscr{D}$ is non-negative. Better still the $\mathbb{R}_+\,\mathscr{L}_i$ are locally discrete, and if $R \subset \overline{\mathrm{NE}}\,(\mathscr{X})_{\mathbb{R}}$ is an extremal ray in the half space $\mathrm{NE}_{K_{\mathscr{F}}+\mathscr{D}<0}$ then it is of the form $\mathbb{R}_+\,\mathscr{L}_i$.

Proof. By I.b.14, IV.f.2 is independent of any smoothness hypothesis, so, II.d.1, we have a cone theorem for $K_{\mathscr{F}^e}$. On the other hand, if $R \subset \overline{\mathrm{NE}}\,(\mathscr{X})_{\mathbb{R}}$ is an extremal ray in the half space $\mathrm{NE}_{K_{\mathscr{F}}+\mathscr{D}<0}$ then it's an extremal ray in the half space $\mathrm{NE}_{K_{\mathscr{F}^e}<0}$ for all $e \gg 0$, whence by II.d.1 there is an invariant parabolic champ $f : \mathscr{L} \to \mathscr{X}$ with $K_{\mathscr{F}^e}\cdot_f \mathscr{L} \ge -2$ parallel to it. In particular, therefore, the extremal rays in this half space are locally discrete. Similarly if ρ is the dimension of Néron-Severi, with $\alpha \in \overline{\mathrm{NE}}_1(\mathscr{X})$, then there are a sequence of classes $\alpha_e \in \overline{\mathrm{NE}}\,(\mathscr{X})_{K_{\mathscr{F}^e}\ge 0}$, and generators R_{ei} of extremal rays, $0 \le i \le n_e \le \rho$, in the half space $\mathrm{NE}_{K_{\mathscr{F}^e}<0}$ such that

$$\alpha = \alpha_e + \sum_{i=1}^{n_e} R_{ei} \tag{IV.64}$$

Subsequencing in e as necessary, we may suppose $n = n_e$ is independent of n, and all of α_e, R_{ei} converge. Plainly, however, the α_e converge to a class in the half space $K_{\mathscr{F}^e} \ge 0$, which, equally is either true of a given R_{ei}, or it belongs to a half space $K_{\mathscr{F}}+\mathscr{D}+\epsilon H < 0$- H is ample, $\epsilon > 0$- in which, as noted, extremal rays are discrete, so $\mathbb{R}_+ R_{ei}$ is independent of e. □

Which can be combined with IV.f.3 to yield

IV.f.5. **Proposition/Summary.** *Let $\mathscr{X}\backslash\mathscr{D} \to [\mathscr{X}\backslash\mathscr{D}/\mathscr{F}]$ be as in IV.f.1 with projective moduli, and non-empty boundary $\mathscr{D}$; $\mathscr{X}^2 \to [\mathscr{X}^2/\mathscr{F}^2]$ the square root of $\mathscr{D}$; $\mathscr{X}^2_{\text{final}} \to [\mathscr{X}^2_{\text{final}}/\mathscr{F}^2_{\text{final}}]$ the result of a maximal sequence of contractions and flips in the sense of IV.a.4, resp. IV.d.3, as described in* (IV.53) *(i.e. we exclude the exceptional cases IV.e.3 and IV.e.5) then there is is a foliated logarithmic champ $\mathscr{X}_{\text{final}}\backslash\mathscr{D}_{\text{final}} \to [\mathscr{X}_{\text{final}}\backslash\mathscr{D}_{\text{final}}/\mathscr{F}_{\text{final}}]$ satisfying IV.f.1 with projective moduli, and non-empty boundary of which $\mathscr{X}^2_{\text{final}} \to \mathscr{X}_{\text{final}}$ is the square root of $\mathscr{D}_{\text{final}}$, and exactly one of the following happens*

(a) *$K_{\mathscr{F}^2_{\text{final}}}$, so, IV.f.2, a fortiori $K_{\mathscr{F}_{\text{final}}} + \mathscr{D}_{\text{final}}$, is nef.*
(b) *The foliation $\mathscr{X}^2_{\text{final}} \to [\mathscr{X}^2_{\text{final}}/\mathscr{F}^2_{\text{final}}]$ is a bundle of foliated varieties where the universal cover of a fibre is the radial foliation on a weighted projective space of dimension at least 2. As such the same is true of $\mathscr{X}_{\text{final}}\backslash\mathscr{D}_{\text{final}} \to [\mathscr{X}_{\text{final}}\backslash\mathscr{D}_{\text{final}}/\mathscr{F}_{\text{final}}]$; $\mathscr{D}$ is the hyperplane at infinity, i.e. up to change of weighted projective coordinates $x_0 = 0$ on the universal cover in the notation of I.d.2; and $K_{\mathscr{F}_{\text{final}}} + \mathscr{D}_{\text{final}}$ is torsion.*
(c) *Idem as item (b) except that the fibres of the bundle are weighted projective space of dimension one, and the implied Mori fibre space is exactly the foliation $\mathscr{X}_{\text{final}}\backslash\mathscr{D}_{\text{final}} \to [\mathscr{X}_{\text{final}}\backslash\mathscr{D}_{\text{final}}/\mathscr{F}_{\text{final}}]$, i.e. on each parabolic fibre $K_{\mathscr{F}_{\text{final}}} + \mathscr{D}_{\text{final}}$ is negative.*

Proof. By II.d.5 the structure of a $K_{\mathscr{F}} + \mathscr{D}$ negative invariant champ $f : \mathscr{L} \to \mathscr{X}$ is rather particular, *i.e.* either it misses $\mathscr{D}$ completely, or it misses the singular locus completely, and cuts $\mathscr{D}$ in one point. If, however, IV.e.7.(b) were to occur for $\mathscr{X}^2 \to [\mathscr{X}^2/\mathscr{F}^2]$, then the foliation is in parabolic champ; the generic champ must meet $\mathscr{D}$; but none of the smooth invariant champ in the exceptional flipped locus- $\mathscr{E}_+$ in IV.e.3.(b)- can meet $\mathscr{D}$ because an extremal subvariety satisfying IV.e.1 must meet the singularities. Consequently by IV.f.2, IV.f.3 and IV.e.6 it remains to show that IV.e.7.(a) implies IV.f.5 (b) or (c), but this is clear since by I.d.2.(a) and I.c.3 the only divisors everywhere transverse to the radial foliation are, in the notation of *op. cit.*, defined by a weighted homogeneous function, F, of weight a_0 such that $\frac{\partial F}{\partial x_0} \neq 0$. ☐

Finally, let us conclude with

IV.f.6. *Remark.* While it's true, I.22, that the only part of a divisor which is relevant to minimal model theory are the components whose generic points are transverse to the foliation, it may well be case that one starts with a divisor $\mathscr{D} = \mathscr{D}' + \mathscr{D}''$ where, say, $\mathscr{D}'$ satisfies IV.f.1, $\mathscr{D}''$ is invariant, and whether $\mathscr{D}$, or just $\mathscr{D}''$ is simple normal crossing, and, for whatever reason, one wants to have a similar situation on $\mathscr{X}_{\text{final}}$ after running the minimal model programme IV.f.5. Now, certainly, hypothesis such as $\mathscr{D}''$ simple normal crossing are nothing to do with the definitions of log-canonical singularities, so there's no reason for them to be conserved by IV.f.5. On the other hand, simple normal crossings whether of $\mathscr{D}$ or $\mathscr{D}''$ can, by [BM97] and the definition of log-canonical singularities, be restored by invariant blowing up without prejudice to the $K_{\mathscr{F}} + \mathscr{D}$ nefness conclusion of IV.f.5.(a) or the smooth fibration in parabolic champ statement IV.f.5.(b).

References

[Art66] M. Artin, *Etale coverings of schemes over Hensel rings*, Amer. J. Math. **88** (1966), 915–934. MR 0207716

[Art69] ———, *The implicit function theorem in algebraic geometry*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 13–34. MR 0262237

[Art70] ———, *Algebraization of formal moduli. II. Existence of modifications*, Ann. of Math. (2) **91** (1970), 88–135. MR 0260747 (41 #5370)

[BM97] Edward Bierstone and Pierre D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302. MR 1440306

[BM16] Fedor Bogomolov and Michael McQuillan, *Rational curves on foliated varieties*, Foliation theory in algebraic geometry, Simons Symposia, vol. 2, Springer Intl. Publishing, 2016, pp. 21–51.

[BN06] Kai Behrend and Behrang Noohi, *Uniformization of Deligne-Mumford curves*, J. Reine Angew. Math. **599** (2006), 111–153. MR 2279100 (2007k:14017)

[DI87] Pierre Deligne and Luc Illusie, *Relèvements modulo p^2 et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), no. 2, 247–270. MR 894379

[Gir71] Jean Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin-New York, 1971, Die Grundlehren der mathematischen Wissenschaften, Band 179. MR 0344253 (49 #8992)

[EGA-I] A. Grothendieck, *Éléments de géométrie algébrique. I. Le langage des schémas*, Inst. Hautes Études Sci. Publ. Math. (1960), no. 4, 228. MR 0217083

[EGA-IV.2] Alexander Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 24, 231, Rédigés avec la collaboration de Jean Dieudonné. MR 0199181 (33 #7330)

[SGA-I] ———, *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61, Augmenté de deux exposés de M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. MR 2017446 (2004g:14017)

[SGA-II] ———, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 4, Société Mathématique de France, Paris, 2005, Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d'un exposé de Michèle Raynaud. With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original. MR 2171939 (2006f:14004)

[SGA-IV] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin-New York, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. MR 0354654 (50 #7132)

[KM97] Seán Keel and Shigefumi Mori, *Quotients by groupoids*, Ann. of Math. (2) **145** (1997), no. 1, 193–213. MR 1432041 (97m:14014)

[Kol90] János Kollár, *Projectivity of complete moduli*, J. Differential Geom. **32** (1990), no. 1, 235–268. MR 1064874

[Kol96] ———, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180

[Mal02] B. Malgrange, *On nonlinear differential Galois theory*, Chinese Ann. Math. Ser. B **23** (2002), no. 2, 219–226, Dedicated to the memory of Jacques-Louis Lions. MR 1924138 (2003g:32051)

[Mar81] Jean Martinet, *Normalisation des champs de vecteurs holomorphes (d'après A.-D. Brjuno)*, Bourbaki Seminar, Vol. 1980/81, Lecture Notes in Math., vol. 901, Springer, Berlin-New York, 1981, pp. 55–70. MR 647488

[McQ] Michael McQuillan, *Foliated Mori Theory & Hyperbolicity of Algebraic Surfaces*, http://www.mat.uniroma2.it/~mcquilla.

[McQ08] ———, *Canonical models of foliations*, Pure Appl. Math. Q. **4** (2008), no. 3, part 2, 877–1012. MR 2435846 (2009k:14026)

[McQ15] ———, *Elementary topology of champs*, arXiv:1507.00797, 2015.

[Mor82] Shigefumi Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. (2) **116** (1982), no. 1, 133–176. MR 662120

[MP13] Michael McQuillan and Daniel Panazzolo, *Almost étale resolution of foliations*, J. Differential Geom. **95** (2013), no. 2, 279–319. MR 3128985

[Nar66] Raghavan Narasimhan, *Introduction to the theory of analytic spaces*, Lecture Notes in Mathematics, No. 25, Springer-Verlag, Berlin-New York, 1966. MR 0217337

[Ols06] Martin C. Olsson, Hom*-stacks and restriction of scalars*, Duke Math. J. **134** (2006), no. 1, 139–164. MR 2239345

[Szp81] L. Szpiro, *Séminaire sur les Pinceaux de Courbes de Genre au Moins Deux*, Astérisque, vol. 86, Société Mathématique de France, Paris, 1981. MR 642675

[Vis89] Angelo Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), no. 3, 613–670. MR 1005008