Canonical models of foliations

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Abstract

The Enriques-Kodaira classification of algebraic surfaces leaves as many questions unanswered as it solves. In order to remedy this we consider a canonical class for an ordinary differential equation on an algebraic surface, and the subsequent classification programme in the spirit of Mori. The case of 1^{st} order O.D.E.'s, and its consequences for questions of hyperbolicity and value distribution together with diophantine issues (over function fields) is worked out in detail.

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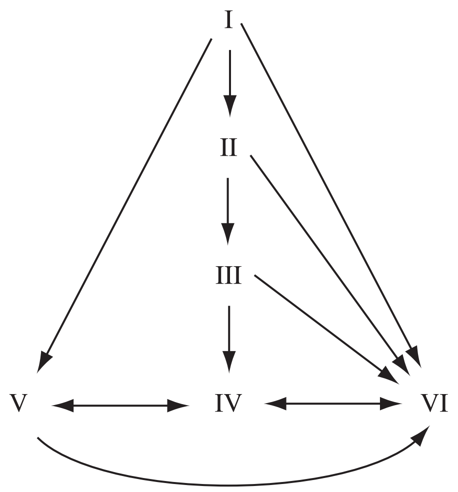


0. Introduction

One of the most striking, indeed almost teleological, aspects of the theory of algebraic curves is the remarkable equivalence between their topological, algebraic, conformal, and arithmetic structure. In the first place the topology of a smooth algebraic curve C is precisely determined by a single invariant, namely its genus, g. The genus, however, admits an algebraic interpretation as the dimension of the space of global sections of the canonical bundle K_C of the curve, and 3 distinct types of behaviour emerge. If the genus is 0 then $K_C^\vee$ is ample, and the Riemann-Roch theorem establishes an isomorphism with $\mathbb{P}^1$, while if the genus is 1 then K_C is trivial so modulo the existence of a point C is isomorphic to its Jacobian and so has the structure of a 1-dimensional complete group, i.e. it is an elliptic curve. Finally if the genus is at least 2 then K_C is ample and $K_C^{\otimes 3}$, yields an embedding of C in some projective space, i.e. its canonical model. This division into 3 types according to genus reflects itself at the conformal level in terms of curvature with the uniformisation theorem yielding a unique conformal metric of curvature 1, 0, or -1 according to whether we are in a rational, elliptic, or higher genus situation. The last block of this correspondence is the arithmetic. Should C be defined over a sufficiently large number field k, say, then the number of k-rational points of C of big height at most H grows like H^2 if and only if C is isomorphic to $\mathbb{P}^1$. For an elliptic curve the appropriate growth estimate is $\log H$, and most recently G. Faltings, [F1], established that curves of genus at least 2 are characterised by the finiteness of their rational points.

It is therefore completely natural to seek to extend this discussion to algebraic surfaces. The canonical class K_X of a smooth algebraic surface X is of course the necessary starting point, via which we have the Enriques-Kodaira classification for a minimal model of X. Should $K_X^\vee$ be ample then X is $\mathbb{P}^2$, although there is a question of non-uniqueness here, while if K_X is just not nef. X is a $\mathbb{P}^1$-bundle over a curve. For K_X or some power of it trivial there are 4 possibilities in the shape of abelian surfaces, hyperelliptic surfaces, Enriques surfaces, or $K3$ surfaces, while for K_X having many sections but not being big, X must be any elliptic fibration not already listed. Finally everything else, which corresponds to K_X big, is put under the label general type. The introduction of the Seiberg-Witten equations, and previously the work of Friedman and Qin, [FQ], shows, cf. [Do], that these distinct types of behaviour known as Kodaira dimension, $-\infty$, 0, 1, 2, respectively, are diffeomorphically invariant. Turning to the conformal geometry, $\mathbb{P}^2$ is posseses a conformal metrc with sectional curvature $2/n{+}1$, and thanks to Yau's celebrated work, [Y], the Kodaira dimension 0 surfaces all admit a metric of zero Ricci curvature, and in many ways are even flatter than that. The curvature in the elliptic fibration case is of course rather mixed, as it may be for $\mathbb{P}^1$-bundles over curves, it is on the other hand rather clear what is happening. The immediate question therefore, is what insight do we have into the conformal geometry of all these surfaces which are lumped together under the appellation of general type. Curiously, Ueno's school prefers the name hyperbolic type, but doesn't profer a conformal definition of the same, and as to what it might mean, we can certainly exclude any naive understanding of uniformisation by a bounded domain, as the example of generic hypersurfaces of large degree in $\mathbb{P}^3$ shows, cf. [M2]. Furthermore, there's every chance that a surface of general type will contain rational and elliptic curves, and indeed many of them do, albeit that they form a finite set as recently established by Miyaoka, [Mi1], and previously in a large sub-case by Bogomolov, cf. [B2] and [D]. Consequently hyperbolicity of surfaces of general type ought only to be expected modulo a boundary ∂. The basic insight of what exactly this ought to mean stems from A. Bloch, [Bl], and his principle, "nihil est in infinito quod non fuerit prius in finito". Specifically following Bloch, we call a variety X hyperbolic modulo ∂, V.3.11, if for sequences f_n of maps from the disc which don't have a convergent subsequence, the sequence is eventually arbitrarily close to ∂ in the compact open sense. It is essential to note here, however, and this is the main reason for so called counterexamples to Bloch's principle, that convergence here, and throughout, does not mean normal convergence, i.e. uniformly on compacta, but rather, following Gromov, [Gr], to a disc with bubbles, V.3.2, and indeed this is simply best possible, V.3.5, with convergence in this sense implying normal convergence if and only if there are no rational curves, or more generally, in the quasi-projective setting no affine lines, V.3.8, around which the bubbles may form. For X compact, or more generally quasi projective with hyperbolic $\mathbb{Q}$-factorial normal crossing boundary, V.3.8, and ∂ empty or contractible, the inestimable reparameterisation trick of Brody reduces this to the easier question of non-existence of maps from $\mathbb{C}$ to X, but otherwise the only known non-trivial case known appears to have been Bloch's Theorem/Cartan's Thesis, [C], for $\mathbb{P}^n \backslash \{n+2 \text{ hyperplanes}\}$, whose natural symmetry- i.e. here there are no affine lines, and whence no bubbling- perhaps served to obscure the true nature of Bloch's insight. The elegant consequence of Bloch's definition is that following Kobayashi one can define a pseudo-metric by joining two points by chains of discs, and taking the infimum over the length of all of these measured in the Poincaré metric which off ∂ is in fact not just a metric but also continuous. This metric is not quite negatively curved at the derivative level, but ought to be so at a higher jet level, cf. [De1] or [M6]. Hyperbolicity, then, in the sense of Bloch appears to be the minimal acceptable notion of the same, but in this sense there is little evidence to justify replacing general type by hyperbolic type. The Bloch definition, nevertheless, offers itself to weaker generalisations, i.e. instead of looking at maps from the disc, one looks at maps from balls, etc., and so sticking to surfaces to fix ideas, convergence of sequences of balls naturally gives an almost everywhere (in fact outside of exceptional curves of the second kind) volume form called the Yau pseudo-volume form akin to the

Kobayashi pseudo-metric. It is only in this weaker sense that the Enriques-Kodaira classification yields any conformal insight, specifically the existence of metrics with Ricci curvature -1[1]. This latter property, however, conveys little insight into the Bloch definition, whose relation to the curvature, or more correctly integrated isoperimetric inequalities is what ought to determine the arithmetic over function fields of characteristic zero, VI.1.3. Arithmetic, properly said, however, is another matter, since while there is a good notion of étale structure, and Diophantine approximation rests on comparisons of these, cf. [M5], it appears to be insufficiently fine- especially since in the compact case étale reduces to Zariski- to yield the results which might be expected from an arithmetic geometry which was properly conformal in nature. Nevertheless, we may aswell look on the bright side, and observe that our original motivation on the conformal side was the étale situation, i.e. [M3] and [M4] are strict complex étale analogues of [F2] and [V2], so perhaps returning in the opposite direction is not completely hopeless.

In any case, and manifestly, as the difference between the Kobayashi metric, and the Yau volume form evidence, the problem of Enriques-Kodaira theory is that it is too spread out, or if one prefers too averaged over all possible directions. Specifically, it is intuitively obvious, and actually true (cf. § V) that we will know much of what we may wish to know about the hyperbolicity of algebraic surfaces and many other things beside if we can understand and indeed classify, all ordinary differential equations on the same together with the evolution and curvature of their solutions. Now a priori this seems not even to enjoy the remotest algebraic sense, nevertheless, one first observes that associated to an algebraic surface there is a tower of jet bundles, as constructed originally by Green and Griffiths, [GG] but rather more satisfactorily and independently by Arrondo, Sols, and Speiser, [ASS], and Demailly, [De1], i.e.

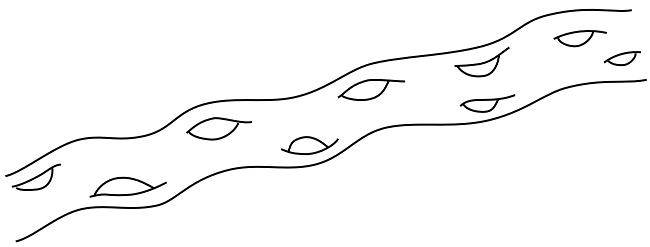
$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

where X_1 is the projective tangent space, and X_{n+1} is a $\mathbb{P}^1$-bundle over X_n with extra coordinate $\frac{d^n y}{d x^n}$, say, should x, y be local coordinates on X, so a n^{th} order O.D.E. on X just corresponds to a subvariety W in X_n. Furthermore the germ of any map j from the disc Δ, say, to X admits a mapping by n^{th} order differentiation to X_n, and j is a solution of the O.D.E. if and only if this lifting $j^{(n)}$ lies in W. Better still proceeding inductively on n we may assume W dominates X_{n-1}, so the O.D.E. induces a (probably singular) foliation by curves $\mathcal{F}$ on W, and $j^{(n)} : \Delta \to W$ defines the germ of a $\mathcal{F}$-integral sub-curve with the correspondence between solutions being one to one by just pushing down in the other direction. Desingularising W will not remove, in general,

[1] The philosophy, here, in itself ought to give a proof of the existence of the Kahler-Einstein metric in the general type case analogous to § V. It's hoped to return to this in [M6].

the singularities of the foliation but does allow us to define a canonical bundle $K_{\mathcal{F}}$ of the foliation, and whence of the O.D.E., as the chern class of the rank 1 quotient of Ω^1_W which defines $\mathcal{F}$. Whence our game begins to take shape, i.e. we will pursue our classification by way of $K_{\mathcal{F}}$.

Nevertheless all we have is a definition of the canonical class of our O.D.E. which has hardly made the problem any more algebraic. Indeed even if the quotient space $W/\mathcal{F}$ exists topologically it may have no l_2-functions (whence Connes' non-commutative description in terms of bundles of operators on the leaves, cf. [Co]) while the leaves will in general not just be Riemann surfaces of infinite genus, i.e.



but a union of such things due to the singularities. One of our first tasks therefore is to introduce suitable definitions and indeed classification of the singularities. It is a fundamental realisation of the Mori-programme that singularities of varieties are best understood via the canonical class, and so we do the same. In particular employing the obvious analogue of the discrepancy we can define foliated terminal and foliated canonical singularities (I.1.2). Further, since for fairly obvious reasons we will be obliged to work in the category of algebraic spaces, another important concept is $\mathbb{Q}$-foliated Gorenstien with the obvious definition of that notion. Unfortunately it emerges that in general terminal is not a useful notion since, rather surprisingly, in dimension at least 3, even smooth points don't have this property (I.1.6) although it should not be completely discarded, especially given our ultimate interest in foliations on surfaces, where under the foliated Gorenstien condition the equivalence of terminal and smooth holds (I.2.6). Anyway our specific interest is foliations by curves, and here we prove a structure theorem for canonical Gorenstien singularities in characteristic zero and all dimensions (I.1.11). Prior to specialising to characteristic zero, we try as far as possible to work in mixed characteristic, but it quickly emerges that the situation here is fairly hopeless. In particular the techniques of local algebra and valuation theory introduced to study foliation singularities lead to a type of Frobenius Theorem in positive characteristic (I.1.9) which easily yields examples of non-singular foliations (which are not p-closed) whose solutions are singular curves. Even worse a canonical resolution of singularities in dimension 2 and positive characteristic certainly need to exist (I.2.5). Whence, all the more reason to concentrate on characteristic zero where for surfaces the existence of a canonical resolution follows from a theorem of Seidenberg, [S], I.2.4. One should, however, bear in mind that canonical, even on non-singular surfaces, is a slightly more general notion than what are classically termed "reduced singularities". As is well known this latter class never admit a smooth resolution, and indeed this is a wholly general phenomenon for canonical singularities in all dimensions, i.e. canonical is best possible (I.2.3). The only unfortunate thing, then, in our understanding of the singularities is the difficulty in going beyond Seidenberg's Theorem, and the lack of a higher dimensional resolution theorem. A step in this direction is I.2.8, which might be phrased as "generic resolution in codimension 2", and which plays an important role in the arithmetic applications of Chapter VI. Finally, I.3, we conclude our tour of singularities with a discussion of the important principle of Baum and Bott, [B-B], that the "normal bundle of a foliation is flat along the leaves away from the singularities". This leads, of course, to various residue formulae which we put in their proper context of local cohomology. The advantage of this perspective is that it easily generalises to the somewhat non-classical situation that occurs in the aforesaid arithmetic, and even to positive characteristic provided that everything is calculated in an appropriate field.

Now while, as we've already noted, most of the well known features of foliations in characteristic zero fail to a greater or lesser extent in characteristic p, there is an important new feature known as p-closure. Specifically as observed by Rudakov and Shafarevich, [R-S], and Ekedahl, [E], the correct notion of integrability in positive characteristic is that the foliation should be given as an inseparable scheme quotient. This generalises the notion of Cartier's p-curvature, and naturally, [EST], Grothendieck's conjecture on global integrability in characteristic zero, given the same for almost all p. In light of our previous considerations we unsurprisingly consider the effects of p-closure for almost all p on the singularities, which it emerges is more or less equivalent to the holonomy about the singularity being torsion in characteristic zero (II.1.3 – II.1.5). This in turn sheds light on the phenomenon of foliations which are p-closed for infinitely many p, but not integrable in characteristic zero (II.2.3) where even in this case (regardless of dimension) the singularity must be formally linearisable. In any case, the desirability of p-closure is evident and in II.2.2 we try to give conditions for this which take into account the singularities. However, even, without this one still has Miyaoka's semi-positivity condition, [Mi2], to guarantee p-closure, which following [S-B], and a little tweaking here and there, suggests that the game of an Enriques-Kodaira classification of foliations is very much a foot by way of,

Theorem 1. (cf. II.4.3.) *Let $(X, \mathcal{F})$ be a foliated algebraic surface with underlying space X non-singular then there are finitely many rational invariant sub-curves $L_1, \ldots, L_k$ of $\mathcal{F}$ with $K_{\mathcal{F}} \cdot L_i \in \{-1, -2\}$ such that,*

$$\overline{NE}_1(X) = \overline{NE}_{K_{\mathcal{F}} \geq 0}(X) + \sum_{i=1}^{k} \mathbb{R}_+ [L_i]$$

where $\overline{NE}_1(X)$ *is the closed cone of effective curves in Néron-Severi with* $\mathbb{R}$-*coefficients and* $\overline{NE}_{K_{\mathcal{F}} \geq 0}$ *is the half space on which* $K_{\mathcal{F}}$ *is positive. Moreover* $K_{\mathcal{F}} \cdot L_i = -2$ *for some* i *iff* $(X, \mathcal{F})$ *is a smooth fibration in rational curves.*

All of which is indeed exactly as one would expect if $X \to X/\mathcal{F}$ were an algebraic fibration. In fact even the proof mimics what one might hope of a fibration, being based additionally on adjunction along integral sub-curves and the flatness of the normal bundle of a leaf outside of the foliation singularities.

At this point we wholly specialise to surfaces, and one immediately enquires as to the construction of a minimal model. However as we have noted canonical foliation singularities unlike their algebraic counterparts cannot be resolved in a terminal way. An immediate consequence of this is that for the construction a minimal model we need not only,

Minimal model characterisation.

(a) $K_{\mathcal{F}}$ nef.

but also the rather more ambitiously

(b) $K_{\mathcal{F}} \cdot C = 0 \Rightarrow C^2 \geq 0$

interpreted in Mumford intersection theory on 2-dimensional algebraic spaces, which serves to mimic eventual freeness, and as such might better be thought of as the canonical model. With these considerations in mind, § III is devoted to the proof of the following under the assumption that the foliation is not a rational pencil,

Theorem 2. (Minimal model theorem.) *Let* $(Y, \mathcal{G})$ *be a* 2*-dimensional foliated algebraic space with canonical foliation singularities*[2] *then there is a proper birational map* $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ *to a minimal model with canonical foliation singularities satisfying the defining characteristic* (a), (b) *above. The minimal model is unique except in a small number of cases, as specifically noted in the classification below. Better still the underlying space* X *has at worst rational quotient singularities or elliptic Gorenstien singularities, as such there is a unique minimal desingularisations* $\rho : (X', \mathcal{F}') \to (X, \mathcal{F})$ *whose structure over a singular fibre is one of the following mutually exclusive types,*

(i) *A chain of rational integrable sub-curves represented by its dual graph,*[3]

A_n : , $n \geq 1$

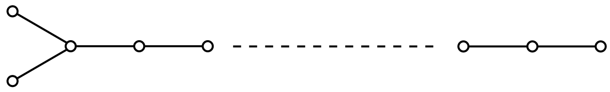
consisting either of a -1*-chain* (III.0.2), *two* -1 F*-curves of square* -2 *joined by an easy tail* (III.2.3) *or a chain of* -2 F*-curves* (III.0.2).

[2] There is no assumption a priori on the singularities of the underlying space Y except that it is normal.

[3] The use whether of A_n, D_n, etc. is intended only as regards the underlying graph, and contrary to standard usage makes no claim as to the corresponding intersection form.

(ii) *Two* $-1\,F$*-curves of square* -2 *joined by a bad tail which itself connects to a chain of* $-2\,F$*-curves* (III.2.3) *and forms the dual graph,*

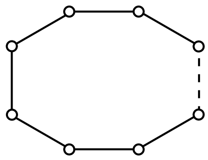
D_n :



, $n \geq 4$.

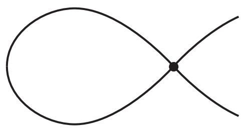
(iii) *A basic elliptic Gorenstien leaf which is not irreducible, i.e. a cycle of* $-2\,F$*-curves forming the fundamental cycle of an elliptic Gorenstien singularity* (III.0.3), *i.e. the dual graph,*

$\tilde{A}_n$:



, $n \geq 2$.

(iv) *The irreducible elliptic Gorenstien leaf, i.e. a rational integral sub-curve with a node which scheme theoretically has Segre class* (cf. II.3) *precisely two at the singular locus of* $\mathcal{F}'$, *i.e.*



Furthermore the structure of $K_{\mathcal{F}'}$ *is of the form,*

$$K_{\mathcal{F}'} = \pi^* K_{\mathcal{F}} + \sum_i E_i$$

where each E_i *is a divisor whose support is a maximal* -1*-chain as indicated in* (i) *above. In particular there is a* $\mathbb{Q}$*-factorial model on which* $K_{\mathcal{F}}$ *is nef.*

It's probably worth considering the minimal model theorem in the special case of fibrations, although here only cases (i) and (ii) can occur. In the presence of $-1\,F$ curves these contractions, indeed in general, cannot lead to a foliated Gorenstien singularity, III.1.4. In the case of fibrations, however, there is the semi-stable reduction theorem, cf. [Sz], in which case only the contraction of $-2\,F$ curves occur, (which by coincidence specific to this example correspond to the exceptional curves of the 2^{nd} kind in the usual sense contained in the fibres) and the singularities are indeed foliated Gorenstien, with $(X, \mathcal{F})$ a stable model in the sense of Deligne-Mumford, and $(X', \mathcal{F}')$ the semi-stable model. The cases (iii) and (iv), however, really do happen in general, so that, although the minimal model has been set up in direct analogy to the canonical model in order to yield eventual freeness of the canonical class, in reality it only implies,

Theorem 3. (cf. IV.2.1.) *Let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be the minimal resolution of the elliptic Gorenstien singularities as in* (iii), (iv) *above of the minimal model of a foliated 2-dimensional algebraic space with canonical singularities of numerical foliated Kodaira dimension 2 then there are effective $\mathbb{Q}$-Cartier divisors Z, whose support is the elliptic Gorenstien leaves, and H ample such that,*

$$(\pi^* K_{\mathcal{F}} = \,)\; K_{\mathcal{G}} = H + Z\,.$$

At first sight one would hope to eliminate the Z term above, but of course eventual freeness at this point is equivalent to the germ of the contraction being $\mathbb{Q}$-foliated Gorenstien. This is clear, and indeed, in a very precise way, for cases which fall into categories (i) and (ii) within the description of the minimal model. In the elliptic Gorenstien case, however, there is no analogue of the semi-stable reduction theorem, so it isn't just unclear, but actually false. Nevertheless we obtain the following clean description of this phenomenon,

Divertimento A. (cf. IV.2.2.) *Let $(X, \mathcal{F})$ be the germ of a canonical singularity on a 2-dimensional algebraic space whose minimal smooth resolution of the underlying space has an elliptic Gorenstien leaf as its fundamental cycle then $(X, \mathcal{F})$ is NEVER $\mathbb{Q}$-foliated Gorenstien, and the germ is isomorphic to the germ of the singularity at a cusp of either of the natural foliations on the Baily-Borel compactification of a Hilbert-Modular surface.*

Now it hardly seems probable that the germs of such things are unique to Hilbert-Modular surfaces (some disc bundle over a disc, with suitable group action ought to give the same germ) so any analogue of Bombieri's global generation theorem, [Bo], and nice construction of canonical models of foliations of general type appears by definition to be hopeless. The real interest of the general type case must, therefore, be reckoned to be elsewhere, and indeed this is so, as we'll subsequently see in returning to the theme of hyperbolicity.

Before returning to the said theme, the immediate question is that of classification. The aim here is not only to classify things of small Kodaira dimension, but to make canonical models of them that are as elegant and useful as possible as, for example, one has in the Enriques-Kodaira theory that a minimal rational surface is $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$, or a Hirzebruch surface, and not just something birational to $\mathbb{P}^2$. Naturally we begin with rational fibrations, where of course we permit singularities, and which by Miyaoka's aforesaid theorem correspond exactly to the foliations with numerical Kodaira dimension $-\infty$. In the spirit of the above remarks we can, however, be even more precise,

Theorem 4. (cf. V.1.1 and V.1.2.) *Let $(Y, \mathcal{G})$ be a 2-dimensional foliated algebraic space with canonical singularities then $(Y, \mathcal{G})$ is a fibration by rational curves iff there is a curve C in Y (not necessarily $\mathcal{G}$ integrable) such that $K_{\mathcal{G}} \cdot C < -1$. Better still, should this occur, there is a proper birational map $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ where $(X, \mathcal{F})$ is the natural $\mathbb{P}^1$-fibration on a $\mathbb{P}^1$-bundle over a smooth-algebraic curve. We remark that the more interesting if direction remains true provided the underlying space Y is regular.*

So amongst other things the theorem confirms the hope that $\mathbb{P}^1$-bundles ought to be the minimal models of foliations of numerical Kodaira dimension $-\infty$. Evidently the minimal model is only unique up to flipping fibres. An amusing corollary of this was already observed by M. Brunella, [Br1], in his classification of smooth foliations, i.e.

Divertimento B. *Let $(X, \mathcal{F})$ be a smooth foliation on a smooth algebraic surface then X is minimal unless $\mathcal{F}$ is the foliation associated to a pencil of lines in $\mathbb{P}^2$.*

We turn now to numerical foliated Kodaira dimension 0, which indeed is shown to correspond to foliated Kodaira dimension 0. The expectation here would be that the foliations can be classified in terms of group actions, i.e. by an element of the Lie algebra of the automorphism group, and this is indeed so. There is however a subtlety, akin to semi-stable reduction for fibrations, i.e. this is only true after taking a cover $(Y, \mathcal{G}) \to (X, \mathcal{F})$ of the minimal model étale in codimension 1, examples of the necessity being given in IV.3.0. Nevertheless in terms of the unique desingularisation $(Y', \mathcal{G}')$ the classification in terms of group actions is given by,

Theorem 5. (Classification of minimal models of Kodaira dimension 0, cf. IV.3.) *Notations as above,*

Underlying space Y'	**Group of automorphisms**	**Element of the Lie algebra**
Sesquielliptic surface[4]	*An elliptic curve*	*Any generator*
Abelian surface	*Abelian surface*	*Any irrational element, i.e. of Kronecker type that is not coming from an elliptic fibration*
Compactification of a semi-abelian surface as a $\mathbb{P}^1$-bundle over an elliptic curve by adding sections at 0 and ∞	*Semi-abelian surface*	*Any generator whose projections to either factor is non-trivial*
Idem, but now a compactification of an extension of E by $\mathbb{G}_a$ by section at ∞	*Extension of an elliptic curve by $\mathbb{G}_a$*	*Idem*
Hirzebruch surface, $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{G}_m \times \mathbb{G}_a$	*Idem*
Hirzebruch surface, $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{G}_m \times \mathbb{G}_m$	*Any irrational element, i.e. $x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$, $\lambda \notin \mathbb{Q}$*

The models here are all unique except more or less by definition in the

[4] i.e. $E \times C/G$ where E is elliptic, C an arbitrary curve and G finite.

cases governed by $\mathbb{G}_m \times \mathbb{G}_a$ or $\mathbb{G}_m \times \mathbb{G}_m$ which as indicated may be arbitrary Hirzebruch surfaces. The minimal or canonical model itself is the contraction of the unique section of the Hirzebruch surface with negative self-intersection in this case. Another notable property is that all the minimal models are either smooth, or log-smooth, i.e. there is a compact leaf D such that the foliation is defined by a locally free quotient of $\Omega_X(\log D)$.

We next turn to foliated Kodaira dimension 1, where by Kodaira's canonical bundle formula we certainly find elliptic fibrations which here play a similar role to rational fibrations in the $-\infty$ case. They also provide good examples of the difference between canonical foliated singularities, and canonical in the usual sense as can be seen in the tabulation of IV.4.5. In some sense they form a distinct subcase, which we note by way of,

Theorem 6(a). *Let $(X, \mathcal{F})$ be a minimal model of a foliation with canonical singularities which is a fibration without being isotrivial then this is so, if and only if $(X, \mathcal{F})$ is a properly elliptic fibration (IV.4.4(i)), with the exact tabulation of the singular fibres being given by IV.4.5.*

Otherwise the foliation and the Kodaira fibration are everywhere transverse, except at a finite number of fibres of the latter, which in turn are invariant by the former. Consequently, away from such fibres and singularities we can locally (in the analytic topology) construct a first integral by projecting a leaf onto a fibre – this being a special case of the graph construction of [BM] – so that the Kodaira fibration outside of this closed set is in fact a fibre bundle. Even better, the Baum-Bott principle tell us that this bundle comes from a representation of the fundamental group, and conversely suspending any such will give a foliation. The delicate question of what happens around a singular fibre of the Kodaira fibration, which may well fail to be invariant, is, as for Kodaira dimension 0, resolved by taking almost étale covers $(Y, \mathcal{G}) \to (X, \mathcal{F})$, IV.4.3, which then allows us to conclude,

Theorem 6(b). *After an almost étale cover $(Y, \mathcal{G}) \to (X, \mathcal{F})$ the minimal model of a foliation with canonical singularities which is not a non-isotrivial fibration and which enjoys Kodaira dimension 1 is one of the following,*

Ricatti Foliations. (IV.4.4(ii)) *Here the Kodaira fibration exhibits Y as a $\mathbb{P}^1$ bundle over a smooth curve B, $K_{\mathcal{G}}$ is just pulled back from the base, and there may be some invariant fibres over points $b_1, \ldots, b_r$, necessarily the fibres containing the singularities, with the foliation being given by an infinite representation of $\pi_1(B \backslash \{b_1, \ldots, b_r\})$ in $\mathrm{PGL}_2(\mathbb{C})$.*

Turbulent Foliations. (IV.4.4(iii)) *Here the Kodaira fibration exhibits Y as a product over its necessarily smooth base B, with an elliptic curve E. Again there may be invariant fibres over some $b_1, \ldots, b_r$ and the foliation is given by an infinite representation $\pi_1(B \backslash \{b_1, \ldots, b_r\})$ in $\mathrm{Aut}(E)$.*

Isotrivial Curves. (IV.4.4(iv)) *Here the Kodaira fibration exhibits Y as a product over a smooth base B of genus at least 2 with some other curve C, and the foliation is the projection to C.*

Z It's worth noting that these are not the standard definitions of Ricatti or turbulent foliations, but an a priori special case. Modulo silly things like the general definition containing foliations of smaller Kodaira dimension, the most important difference is the singularities which in the standard Ricatti definition are not even taken to be canonical. A particular assertion of the theorem is, therefore, that this more general definition can be brought into the shape of our more special one by means of blowing up, blowing down, and almost étale covers.

Once again we observe that with the exception of Ricatti foliations all of these examples are log-smooth. The generic Ricatti foliation is actually log-split, but the boundary which one adjoins is simply the sum of the fibres containing singularities, and this is only a priori a sum of integral sub-curves rather than a leaf. Regardless these cases are in fact exhaustive since if the foliated Kodaira dimension is non-negative, then it coincides with the numerical Kodaira dimension, and this seems worthy of individual note, viz:

Proposition 7. (IV.5.1) *If the foliated Kodaira dimension is positive (in particular if the underlying space is irregular or even log irregular for a boundary consisting of integrable sub-curves) then abundance holds.*

As a special sub-case it answers the following question,

Divertimento C. *Consider the blow-up of $\mathbb{P}^2$ in 9 points and let C be the proper transform of an elliptic curve through the said points then if C is a smooth leaf of an otherwise singular foliation then its normal bundle is torsion.*

The surprising thing is that the classification does not finish here. Well perhaps not so surprising since a moments thought shows that if one takes a non-singular surface X of general type with split cotangent bundle, then we get two smooth foliations $\mathcal{F}$ and $\mathcal{H}$, with $K_{\mathcal{F}}^2 = K_{\mathcal{H}}^2 = 0$. Being smooth both $(X, \mathcal{F})$ and $(X, \mathcal{H})$ are minimal models with canonical singularities, and everything is induced from a uniformisation of X by the bi-disc. However in light of the above theorems (or direct calculation) both of these examples have Kodaira dimension $-\infty$, unless X is a product, yet numerical Kodaira dimension 1. A similar example with singularities occurs on Hilbert-Modular surfaces, which in their easier manifestations are singular only at cusps, which correspond to two foliations splitting the log-cotangent bundle at mutual elliptic Gorenstien leaves.

Now it's not difficult (IV.5.3 – IV.5.5) to prove that these are the only examples which are smooth or more generally log smooth. Outside of this we have a serious problem to construct a transverse holomorphic structure to our given foliation $\mathcal{F}$ in light of the failure of the standard mechanism for doing this, i.e. abundance. On the other hand on the minimal model $K_{\mathcal{F}}$ is nef, so quite generally, [De2], admits a singular metricisation $\overline{K}_{\mathcal{F}}$, with $c_1(\overline{K}_{\mathcal{F}}) \geq 0$. The trouble is that this $(1,1)$ form cannot necessarily be said to have anything better than measure coefficients. However if they were say L_2, to fix ideas, $c_1(\overline{K}_{\mathcal{F}})^2$ would be zero, and its kernel an obvious choice for a transverse foliation $\mathcal{H}$. This may seem a rather desperate idea, since there is a whole litterature on such degenerate Monge-Ampère equations, together with no shortage of counterexamples, no matter how a priori regular, to the holomorphicity of the kernel, cf. [Mo]. The odd thing, however, is that the Bochner technique in the presence of $\mathcal{F}$ trivially implies the holomorphicity given some a priori regularity, such as C^2. Unfortunately the holomorphicity equation is non-linear, so elliptic theory cannot help, and what remains is to get to grips with the regularity. There is, of course, an obvious choice for a metricisation of $K_{\mathcal{F}}$ in the form of the Poincaré metric along the leaves, which by Riemann-Roch and the main theorem of [M1] is non-degenerate in this case outside of the invariant rational and elliptic curves. The precise definition of this metric is via the leaves on X outside of the set where the holonomy groupoid is not well defined a priori, i.e. the singularities of both the space and the foliation, excepting the contraction of $-1\,F$ chains (V.1.2). The fact that it satisfies $c_1(\overline{K}_{\mathcal{F}}) \geq 0$ (which in turn permits an extension of the definition across the singularities for general nonsense reasons) is a theorem of M. Brunella, [Br4]. Moreover a property peculiar to the Poincaré metric in this context is that the Bochner formula becomes purely local, and a regularisation arguement pushes holomorphicity through provided the Poincaré metric is continuous (V.4.2). Consequently after an almost étale cover $(Y, \mathcal{G}) \to (X, \mathcal{F})$ we have the ultimate case in the classification,

Theorem 8. (IV.5) *If $(X, \mathcal{F})$ is the minimal model of a foliation with numerical Kodaira dimension 1, and Kodaira dimension $-\infty$, then after an almost étale cover $(Y, \mathcal{G}) \to (X, \mathcal{F})$, the surface $(Y, \mathcal{G})$ is the minimal model of such a foliation with exactly elliptic Gorenstien singularities (if any), and should $(Y', \mathcal{G}') \to (X, \mathcal{G})$ be the minimal smooth resolution with Z, the contracted curves then $\Omega_{Y'}(\log Z)$ is split as $K_{\mathcal{G}} \oplus K_{\mathcal{H}}$ for some foliation $\mathcal{H}$ (with the same properties) and the whole thing arises from a uniformisation by the bi-disc respecting the natural foliations on the same compactified at finitely many cusps corresponding to Z.*

Classification achieved, it remains to complete our understanding of foliations on algebraic surfaces by returning to the theme of hyperbolicity, and general type (not to mention that this is even a pre-requisite to the proof of Theorem 8). As we've said the study of the usual canonical class K_X naturally leads to the study of measure hyperbolicity and ultimately the existence of Kahler-Einstein metrics. In the foliation context, the Bloch definition presents itself as asking whether if a sequence of invariant maps from the disc does not have a convergent subsequence then is it ultimately arbitrarily close to some invariant boundary (V.3.11). This in turn leads to an invariant Kobayashi metric, but one should note that this is not, in general, the Poincaré metric since the latter is constructed using only maps which may not pass through the non-terminal singularities, and which respect the natural orbi-fold structure on the terminal locus itself. As metrics on $T_{\mathcal{F}}$, the former is smaller than the latter, and does not enjoy the latter's upper semi-continuity. Indeed the space of maps employed to define the Poincaré metric is uniquely maximal with respect to the property that it's invariant metric is usc, and whence it seems reasonable to single it out, by calling it the Brunella space. Ultimately then, given classification, the properties of the Poincaré metric are easy consequences, modulo a lemma of Brunella, [Br4], to be recalled, of the hyperbolicity of the foliation in the Bloch sense, i.e.

Theorem 9. (V.5.1(a)) *Let $(X, \mathcal{F})$ be the minimal model of a foliation with canonical singularities and let D be the invariant algebraic curves through the singularities of X and $\mathcal{F}$. Suppose further that either $(X, \mathcal{F})$ has (foliated) general type or $\nu(\mathcal{F}) \geq 1$, and $\mathcal{F}$ is neither an elliptic fibration nor Ricatti nor turbulent with invariant fibres, then $(X, \mathcal{F})$ is hyperbolic modulo D.*

What is even better is that our method gives for free explicit quantification of the degeneration of the invariant Kobayashi metric at the boundary, by way of a wholly optimal integrated isoperimetric inequality, V.4.4, for invariant discs, i.e.

Theorem 10. (V.5.1(b)) *For $(X, \mathcal{F})$ of general type, and $\varepsilon > 0$ there are constants $\alpha > 0$, $\beta(\varepsilon) > 0$, such that if ∂ is a local generator for the foliation at a point x away from the singularities, Z, then,*

$$\|\partial\|_{\mathrm{Kob}} \geq \beta(\varepsilon) \frac{\mathrm{dist}(x, Z)^{\alpha}}{|\log \mathrm{dist}(x, D)|^{1+\varepsilon}} .$$

From this, modulo the said lemma, it immediately follows that the Lelong numbers of the Poincaré metric are zero in codimension 1, which is false for Ricatti or Turbulent foliations, in general, so basically the latter theorem is best possible. Now the lemma to which we allude, and which we recall in V.3.10 under the appeallation of no bubbling, is an identification between the the image of the natural map from the Brunella space in the compact open topology- so in particular without bubbles- to the totality of invariant maps with bubbles being closed and the nefness of the canonical bundle $K_{\mathcal{F}}$ itself. Consequently Theorem 9 already tells us a lot about convergence in the Brunella space, and when it doesn't apply we can use classification from which we easily see that the only foliations other than those of numerical Kodaira dimension 0 or less, or elliptic fibrations, for which some sort of compactness in the Brunella space is likely to fail, and indeed does fail, are Ricatti or Turbulent foliations over a rational base with one or two invariant fibres, which in turn have all leaves uniformised by $\mathbb{C}$. These latter foliations are sufficiently close (even to the point of explicitly writing them down) to being given by vector fields that they perhaps merit a name such as almost vectorial, V.1.2(e). The only other obvious obstruction to compactness in the said space is the presence of complex lines, but op. cit. already assures us that outside of these examples, these parabolic leaves must be both algebraic and finite in number. Profiting, therefore, from the extra simplicity afforded by maps in the Brunella space, we can ameliorate our isoperimetric inequality to obtain,

Divertimento D. (V.5.2) *Let $(X, \mathcal{F})$ be the minimal model of a foliation with not all leaves parabolic (in particular not in the above exceptional list) then modulo, as defined in V.3.14, the parabolic leaves, which are necessarily algebraic and finitely many, the Brunella space is compact, so that in particular the Poincaré metric is continuous outwith the said leaves, and the singularities.*

The property expressed by the divertimento is particularly vivid at the level of what may be termed the small homotopy groupoid, i.e. we first identify the foliation with it's infinitesimal groupoid, the latter is well defined away from the singularities and a formal subscheme of the completion of the product of the terminal locus with itself in the diagonal, by the Frobenius theorem this extends to a genuine analytic neighbourhood, and permits us to take a domain of holomorphy of the projection onto the first factor- up to unravelling the finite holonomy occasioned by the terminal, but non-smooth, singularities this is infact the holonomy groupoid- to which, following the author of the same, we apply the no bubbling lemma to construct by way of leafwise orbi-fold uniformisation an honest complex space, and whence,

Divertimento D. (Pictorial phrasing) *Everything as before, then equivalently, outside of the parabolic leaves, the small homotopy groupoid has the structure of a continuous disc bundle.*

Here we've deliberately used the appellation small to express the fact that the groupoid is not defined at the singularities. We can, however, remedy this by the procedure of [BM], and define a not necessarily flat groupoid, $G_{\mathcal{F}}$, with the diagonal in the small groupoid replaced by a modification (essentially the blow up in the non-terminal singularities) of the underlying space X of the minimal model, while transverse to this modified diagonal, $\widetilde{\Delta}$, we add complex lines. The no-bubbling lemma still applies, so that $G_{\mathcal{F}}$ is still a complex space, and not withstanding the potential failure of base point freeness for $K_{\mathcal{F}}$, Kodaira vanishing implies that $\widetilde{\Delta}$ can be blown down, say by $\pi : G_{\mathcal{F}} \to S_{\mathcal{F}}$, iff our foliation has general type. This, of course, corresponds to some strict psh exhaustion function ϕ on a neighbourhood of $\widetilde{\Delta}$, and the question arises as to whether ϕ may be extended to the whole of $G_{\mathcal{F}}$, i.e. is the groupoid holomorphically convex. Although we can't quite answer this, we can almost do so. Indeed if by the parabolic part of $G_{\mathcal{F}}$ we understand not only that lying over the parabolic leaves in the small groupoid, but also that which we have added in, then at a bounded distance away from this, the function ϕ

is commensurable with the square distance function on leaves computed by means of any natural Fubini-Study metric on $T_{\mathcal{F}}$ arising from global sections of sufficiently high multiples of $K_{\mathcal{F}}$. Consequently, on any domain bounded away from the parabolic part, ϕ, or something arbitrarily close to it, may certainly be extended to a strict psh function on the same. The said function may not, however, be exhuastive, but, as Brunella observes, [Br4], may easily be made so on adding in the squared distance in the Poincaré metric, and whence, in particular, $S_{\mathcal{F}}$ complemented by the divisors which are the image of the parabolic part, is an ascending union of Stein spaces in a natural way. It seems, therefore, appropriate to summarise this description as,

Divertimento D bis *A foliation is of general type iff its completed homotopy groupoid $G_{\mathcal{F}}$ is almost holomorphically convex, and a continuous disc bundle away from its parabolic part.*

Now a crucial technical point which permitted Theorem 9, although not the key isoperimetric/value distribution inequality, V.4.4, nor Theorem 10, was the replacement of Brody's reparameterisation trick by what may be termd a closed current analogue, V.2.4, which although technically more complicated than a single map from $\mathbb{C}$ cannot degenerate in an un-controlled way. The closedness is, of course, most pertinent to foliations, if we wish to apply the Baum-Bott residual intersection philosophy. Consequently the applicability of this technique encompasses the hyperbolicty of algebraic surfaces themselves, whenever the said question may be reduced to sequences of maps which are arbitrarily close to divisors and solutions of first order ODE's, or equivalently, after ramified coverings, foliations, for which we have,

Theorem 11 (V.6.2) *Let X be a projective surface, and D a simple normal crossing divisor with, $f_n : Y_n/\Delta \to X$ a sequence of maps from ramified coverings of the disc, such that the derived current T', V.2.7, associated to the derivatives $f'_n : Y_n \to \mathbb{P}(\Omega_X(\log D))$ is supported in a divisor all of whose horizontal components have degree 1. Then after a sufficiently large birational modification $\rho : (\widetilde{X}, \widetilde{D}) \to (X, D)$[5], with $\widetilde{T}'$ the derived current associated to the proper transforms of the f_n and L the tautological bundle on $\mathbb{P}(\Omega_{\widetilde{X}}(\log \widetilde{D}))$ we have,*

$$L.\,\widetilde{T}' \geq (K_{\widetilde{X}} + \widetilde{D}).\,\widetilde{T}$$

provided $\widetilde{T}'$ is closed, and its pushforward $\widetilde{T}$ satisfies, $D_1 \cdot \widetilde{T} \geq 0$ for all D_1 in the support of $\widetilde{D}$.

Unlike the occurence of Baum-Bott's philosophy in Theorem 9, this one cannot be deduced from the principle itself, but requires a precise quantification given by Brunella, [Br2], without which one could not attain the optimal

[5] Sufficiently large means determined by X, D, the said divisor etc., but not the maps f_n, while $\widetilde{D}$ is permitted to be very much bigger than $\rho^{-1}(D)$, although always with simple normal crossings.

lower bound which is indispensible for applications. The Theorem itself, under the said hypothesis, is roughly saying that the holomorphic sectional curvature of a Kahler-Einstein metric is at most -1, a fact that is false for higher order ODE's, e.g. geodiscs on ball quotients where the curvature is -2/3, so that by implication, and quite definitely as regards their extension to discs, the falsity, in their exact form, of the canonical height conjectures in [L] and [V1], V.7.2, albeit that the general philosphy, which is simply that of integrated isoperimetric inequalities remains valid. As such it is regretable that we cannot quite get such inequalities, and particularly their quasi-projective/value distribution analogues, for discs on surfaces by way of the above Theorem- the rather easier case of maps from the line having been the germ of this study, cf. [M-1], or [E-G]. Nevertheless we do have Bloch hyperbolicity, and a reasonable approximation to such inequalities by way of the quantification of the Kobayashi metric provided by,

Divertimento E (VI.1 et sequel) *Let (X, D) be the canonical model of an algebraic surface of logarithmic general type with $s_2(X, D) > 0$ (i.e. $c_{1,\log}^2 > c_{2,\log}$) then for some proper Zariski closed subset Z of $X\backslash D$, (X, D) is complete hyperbolic, V.3.11(c), modulo Z. Better still, let $\rho : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a minimal resolution such that $\widetilde{D}$ (the total reduced pull-back of D) and $\widetilde{Z}$, the proper transform of Z, have simple normal crossings, with E_i the divisors contracted by ρ and $a_i \in \mathbb{Q}_{\geq 0}$ their logarithmic discrepancies then for all $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that for any $f \in \mathrm{Hom}(\Delta, X\backslash D)$,*

$$\prod_i \|\mathbb{1}_{E_i}\|^{a_i+\varepsilon}(0)\, \|d\widetilde{f}\|_{\widetilde{\log}}(0) \leq C(\varepsilon)\|\mathbb{1}_{\widetilde{Z}}\|^{-1}$$

where $\widetilde{f} : \Delta \to \widetilde{X}$ is the proper transform, and $\| \, \|_{\widetilde{\log}}$ is any complete norm on $T_{\widetilde{X}}(-\log \widetilde{D})$.

Evidently, however, Theorem 11 is rather more general than the Divertimento, as it as an essential pre-requisite to the study of Bloch Hyperbolicity via O.D.E.s, of which the Bloch Theorem on $\mathbb{P}^2\backslash\{4 \text{ hyperplanes}\}$ as opposed to the minimum of 5 hyperplanes required to apply the Divertimento provides a good example which we discuss and prove, V.6.3. The scene thus set, we return naturally to our motivating theme, and find ourselves obliged, outwith the foliation context, to retreat into a certain conservatism with greater emphasis on entire curves, or lack thereof, and reparameterisation. After all, when applicable, studying one complex line as opposed to sequences of barely related discs is like night and day, as evidenced by,

Divertimento F (V.7.1) *Let X be a projective surface such that for some $n \in \mathbb{N}$, $h^0(X, \mathrm{Sym}^n\, \Omega_X^1) \geq 2$ then X is hyperbolic iff X does not contain any rational or elliptic curves or is covered by an abelian surface.*

While in general, we have a foliated Mori Programme, V.7.5, from which, provided dual Seshadri constants behave as they ought, will, given a genuinely canonical model of the foliation yield Bloch Hyperbolicity for foliations of general type, although only the weaker Green-Griffiths' conjecture for surfaces, provided that the curves, rather than classes, in the face $K_{\mathcal{F}}. \ = 0$ for a foliation of small numerical Kodaira dimension exhibit the type of elliptic behaviour that we have seen in dimension 2. Regardless, the said conjecture becomes in some sense irrelevant, since it could only fail for highly structural reasons, while the not unreasonable extension of Divertimento F is much closer at hand with only a minimal model being required.

On the arithmetic side, we work over function fields of characteristic 0, and so the Mordell type problem becomes a problem of compactness of spaces of sections of families of varieties, which should thus, themselves, be generically hyperbolic modulo some boundary. Supposing there is no such boundary, then it's an observation of Noguchi, [N], that complete hyperbolicity modulo the fibres which are not so suffices. Unfortunately it is also observed in op. cit. that this is not always so, albeit that it can be guaranteed for families of curves, where it is an easy consequence of reparameterisation VI.1.0., and in this case we even have a beginning of an understanding of the relation between the étale and conformal structure by an elegant work of Parshin, [P]. A substantially more quantative, and hopefully more applicable, relation between hyperbolicity and Mordellicity is provided by VI.1.3 which, morally, asserts that the latter problem is governed by the curvature on the generic fibre. Again, however, bad reduction rears it's ugly head, although the methodology of Theorem 9 easily controls it for curves, yielding the conjectured,

Divertimento G (VI.1.4) *Let X/B be a family of curves over a compact Riemann surface B then $\forall \varepsilon > 0$, there is a constant $C(\varepsilon)$ such that if $f : Y/B \to X/B$ is a map from a B-scheme of relative dimesnion 0, then,*

$$(\omega_{X/B} + D)._f\, Y \leq (1+\varepsilon)(2g(Y) - 2 + \mathrm{rad}_{f,D}) + C(\varepsilon)(Y : B)\,.$$

It ought also to yield, albeit with a constant much worse than $1+\varepsilon$, a similar statement for families of surfaces under the hypothesis of Divertimento F, but the aforesaid bad fibre problem turns our attention to the easier supposition of general type and non-negative index, where, by more classical means, we have an honest foliation by curves on an appropriate 3-fold, under which the sections corresponding to rational points are invariant. What is subsequently surprising, is that having arrived to this point, the peculiarities of working with sections, and, indeed, generically, allows, quasi verbatim, much of what we've already done in dimesnion 2, to go through , and so obtain,

Theorem 12. (cf. § VI) *Let $\pi : X \to B$ be a non-singular 3-fold fibred over a smooth curve B such that the generic fibre is a smooth geometrically integrable surface of general type and non-negative index then there is a proper closed subset Z of X and constants a, b (the former being an effectively computable function of the moduli) such that if $f : Y/B \to X/B$ is a map of B-schemes* *with Y/B of relative dimension 0, and genus g, not factoring through Z, then:*

$$K_{X \cdot_f} Y \leq a\,(2g(Y) - 2) + b\,(Y : B)$$

In particular if $X \otimes \mathbb{C}(B)$ is not isotrivial and $L/\mathbb{C}(B)$ is any finite extension then the L-rational points, $X \otimes \mathbb{C}(B)(L)$, are not Zariski dense.

We only give the proof for positive index. The non-negative case immediately reducing to this by classification, and the non-deformability of bi-disc quotients, and, indeed, everything works equally well after the addition of the ubiquitous radical in the quasi-projective setting. The strong form, however, of this theorem for algebraic points does not stand in isolation, but employs in an essential way Miyaoka's recent results [Mi1], although for rational points the more standard estimates of [Mi4] are wholly sufficient. Similar remarks apply to the following of which the above is an implicit corollary,

Theorem 13. (cf. § VI) *With notations as in Theorem 12, if we replace the hypothesis on the generic fibre of X by the hypothesis that there is a foliation by curves on X of which the maps of B-schemes, f, define integrable subcurves then the same conclusion holds.*

Ultimately this seems a good point to stop summarising, given that Theorem 13 is itself a summary of the entire paper, taking in generic resolution of singularities in dimension 3, classification for positive Kodaira dimension, and a rational point variant of the methodology of Chapter V.

This article is long, and so the thanks may be too. In the first place the ODE's as structure, rather than obstruction, for hyperbolicity on surfaces is an idea of M. Gromov, while his other contributions to this article are legion. The volunteering by M. Reid of his excellent notes, [R], kicked off § IV, and J.-P. Demailly patiently explained the problems inherent in the study of Monge Ampère equations. The ultimate shape of classification is rooted in many conversations with N. Shepherd-Barron, while the role of formal schemes in this was explained to me by F.A. Bogomolov, whose ideas for proving Theorem 8 are far preferable to the proof as it stands (V.7.6). While writing this article, I discovered that M. Brunella and G. Mendes had independently begun to pursue these questions. The former addressing the existence of minimal models in [Br3], and the latter the Kodaira fibration in [Me1]. The examples necessitating the use of almost étale covers are all theirs. Subsequently Brunella pursued the subject with vigour, and pointed out many errors in an earlier draft. His theorem on the pluri-subharmonic variation of the Poincaré metric and invitation to Dijon (leading to independent, mutually incomprehensible, but probably identical proofs of the holomorphicity of the Monge-Ampère kernel modulo continuity) were, along with his tutorials on bubbling, absolute necessities in the completion of this article. However, even more than to Brunella, is the debt to Cécile without whom the contents would just be so many hand written notes gathering dust.

I. Singularities

I.1. The definitions

Somewhat surprisingly the theory of minimal models of foliations on surfaces necessitates a consideration of singularities very much in the spirit of Mori's programme for 3-folds. Although we will ultimately be occupied only with what may occur in dimension 2 there seems to be no harm in setting out from a more general perspective. Our basic objects of study are pairs $(X, \mathcal{F})$ where X is a normal projective variety (or possibly just a proper normal algebraic space) over a field k, and $\mathcal{F}$ is an integrable foliation, i.e. a saturated subsheaf $\mathcal{T}_{\mathcal{F}}$ of the tangent sheaf $\mathcal{T}_X$ closed under lie brackets. Necessarily there is an immersion $j : (X_0, \mathcal{F}_0) \hookrightarrow (X, \mathcal{F})$ with $X \backslash X_0$ of codimension 2, X_0 non-singular and $\mathcal{T}_{\mathcal{F}_0}$ locally free so that we may define $K_{\mathcal{F}}$ to be $j_*(\Lambda^{\mathrm{top}}\, \mathcal{T}_{\mathcal{F}_0}^{\vee})$. Of course if we wish to pursue the study of such objects in the spirit of the minimal model programme then $K_{\mathcal{F}}$ better be $\mathbb{Q}$-Cartier. The strongest possible sense in which this may be understood is naturally,

Definition I.1.1. *A pair $(X, \mathcal{F})$ is said to be $\mathbb{Q}$-foliated Gorenstien if there exists $m \in \mathbb{N}$ such that $j_*\{(\Lambda^{\mathrm{top}}\, \mathcal{T}_{\mathcal{F}}^{\vee})^{\otimes m}\}$ is locally free. If in addition we can take $m = 1$, then we say that $(X, \mathcal{F})$ is foliated Gorenstien.*

Now the important thing to remark is that there is a priori no relation between the singularities of X and those of $(X, \mathcal{F})$. Indeed the curious reader may skip to IV.2.2 to see an example where the underlying space is Gorenstien but $(X, \mathcal{F})$ is not foliated Gorenstien. Conversely we suppose nothing about X beyond its normality. Regardless, as we will see shortly there are other weaker interpretations of the notion of $K_{\mathcal{F}}$ being $\mathbb{Q}$-Cartier so let us simply be a little vague about it for the moment and continue in the usual way, cf. [K2], viz:

Let $p : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ be a proper birational morphism of pairs, with the obvious definition of that notion, then in $\mathrm{NS}^1(\widetilde{X})_{\mathbb{Q}}$ we may write,

$$K_{\widetilde{\mathcal{F}}} = p^* K_{\mathcal{F}} + \sum_E a(E, X, \mathcal{F})\, E$$

where E denotes an exceptional divisor contracted by p. Should $p' : (X', \mathcal{F}') \to (X, \mathcal{F})$ be another such map then for any "reasonable" definition of $K_{\mathcal{F}}$ being $\mathbb{Q}$-Cartier (e.g. $(X, \mathcal{F})$ $\mathbb{Q}$-foliated Gorenstien) then given E in $\widetilde{X}$ with non-empty proper transform E' we have $a\,(E, X, \mathcal{F}) = a\,(E', X', \mathcal{F}')$, whence we obtain a map,

$$a\,(-, X, \mathcal{F}) : \{\text{exceptional divisors of } k(X)\} \to \mathbb{Q}\,.$$

There is obviously no harm in extending $a_{X,\mathcal{F}}$ to actual divisors on X by putting it to be identically zero on the same. Indeed this is even internally consistent since $\mathcal{T}_{\mathcal{F}}$ is saturated. At this point everything translates rather nicely into commutative algebra. Any divisor on any modification of X certainly defines a discrete rank 1 valuation on the field of rational functions, and by [ZS], Theorem 31, the converse is true if we restrict to valuations with non-empty centre (n.b. this latter condition is vacuous if X is proper). Our function $a_{X,\mathcal{F}}$ is therefore a function,

$$a_{X,\mathcal{F}} : \{\text{discrete rank 1 valuations of } k(X) \text{ with non-empty centre}\} \to \mathbb{Q}$$

and we may thus define,

Definition I.1.2. *Let Z be an irreducible subscheme of X, then,*

$$\mathrm{discrep}_{X,\mathcal{F}}(Z) = \inf\{a_{X,\mathcal{F}}(v) : \textit{ centre of } v \supset Z\}\,.$$

Furthermore we say (by analogy with the usual case) that $(X, \mathcal{F})$ has a canonical singularity at Z (respectively terminal) if the discrepancy, $\mathrm{discrep}_{X,\mathcal{F}}(Z)$, is non-negative (respectively strictly positive).

It's worth noting,

Remark I.1.3. While this definition could have been phrased more precisely as say canonical foliated singularities (resp. terminal foliated singularities) we have chosen not to do so since the usual notions of canonical and terminal make no appearance in this article. Whence the words canonical, and terminal should always be understood as canonical foliated and terminal foliated.

Now in commutative algebra terms here's what's going on. At the level of the local ring $\mathcal{O}_{X,Z}$ we have a saturated submodule $\mathcal{T}_{\mathcal{F}} \otimes \mathcal{O}_{X,Z}$ of $\mathrm{Der}_k(\mathcal{O}_{X,Z})$ $(= \mathcal{O}_{X,Z} \otimes \mathcal{T}_X$, by the way, since dualising coherent modules commutes with flat base change). A discrete rank 1 valuation v with centre containing Z corresponds to a not necessarily dominant map of local rings $\mathcal{O}_{X,Z} \to R$, where $R \subset k(X)$ is a discrete valuation ring with valuation v. There is then a unique maximal, and indeed free, submodule $T_{\widetilde{\mathcal{F}}}$ of $\mathrm{Der}_k(R)$ with $T_{\widetilde{\mathcal{F}}} \otimes_R k(X) = \mathcal{T}_{\mathcal{F}} \otimes_{\mathcal{O}_X} k(X)$ (strictly speaking we should sheafify the left hand side, but it's notationally convenient to identify modules and their sheafication, so we will). Interpreting negative powers of the maximal ideal of R in the obvious way there is an integer d with,

$$R \otimes_{\mathcal{O}_{X,Z}} \mathrm{Hom}((\Lambda^{\mathrm{top}}\, \mathcal{T}_{\mathcal{F}} \otimes \mathcal{O}_{X,Z})^{\otimes m}, \mathcal{O}_{X,Z}) = \mathfrak{m}_R^d \otimes (\Lambda^{\mathrm{top}}\, T_{\widetilde{\mathcal{F}}}^{\vee})^{\otimes m}$$

where m is the appropriate power for making $K_{\mathcal{F}}$ a bundle and the integer d is of course the discrepancy. Unfortunately this definition is not terribly practical if the foliation isn't Gorenstien, i.e. the relation between duals and tensor products is a mess. However in the Gorenstien case it's all rather tidy. Let's illustrate this for foliations by curves. Since we've assumed that X is normal, this is equivalent to $\mathcal{T}_{\mathcal{F}}$ being a bundle, i.e. the foliation is given everywhere by a possibly singular vector field. Naturally we take a vector field ∂ which generates $\mathcal{T}_{\mathcal{F}}$ generically around Z. Equally there is a derivation $\widetilde{\partial}$ of R generating $T_{\widetilde{\mathcal{F}}}$ so that if π is a uniformising parameter then, $\widetilde{\partial} = \pi^d \partial$. These observations allow us to start proving some properties of canonical singularities. In the first place we should define what we mean by a smooth point,

Definition I.1.4. *A point x of a foliation by curves $(X, \mathcal{F})$ is said to be smooth if locally around x, $\mathcal{T}_{\mathcal{F}} = \mathcal{O}_{X,x}\,\partial$ for ∂ a derivation whose residue in $T_M \otimes k(x)$ is non-zero for some ambient non-singular variety M in which X is embedded around x. The singular points,* $\mathrm{sing}(\mathcal{F})$*, is the set of non-smooth points.*

Of course it might be linguistically preferable to call a singular point a non-smooth point, since the notion of canonical singularity doesn't suppose that the point is singular in the above sense. On the other hand, this kind of convention is well established in the Mori theory framework, so no confusion seems likely. It's also worth remarking that a smooth point is Gorenstien, and this latter condition is open for general nonsense reasons, so openess of the smooth locus follows from its openess in the Gorenstien locus, which is clear. In characteristic zero smothness at x is equivalent, by the classical Frobenius theorem, to the rather more descriptive property of the foliation being given in a formal (or even analytic) neighbourhood of x by an everywhere relatively smooth fibration. This is, however, false in positive characteristic, as we shall see. In any case, our first property is:

Fact I.1.5. Let $(X, \mathcal{F})$ be a foliation by curves with underlying space X non-singular, then any smooth point of $\mathcal{F}$ is canonical.

Proof. Suppose x is not a canonical singularity then we can find a discrete rank 1 valuation ring R in $k(X)$ with $\mathcal{O}_{X,x} \hookrightarrow R$ and $\partial = \pi^e\,\widetilde{\partial}$ for some $e \in \mathbb{N}$ and $\widetilde{\partial} \in \mathrm{Der}_k(R)$. So let $f \in \mathcal{O}_{X,x}$ then $\partial f = \pi^e\,\widetilde{\partial}(f) \in \mathfrak{m}_R \cap \mathcal{O}_{X,x} \subset \mathfrak{m}(x)$. However by definition of non-singularity there is a function with $\partial f \notin \mathfrak{m}(x)$, which is absurd. □

The important thing to remark here is that the fact is not a joke. Unlike the usual K_X theory, a blow up will destroy smoothness, so you can't just compute by hand. Indeed in dimension at least 3, smooth points aren't even terminal. Here are some examples:

Example I.1.6(a). Let Z be a (smooth) invariant subvariety of a foliation by curves $(X, \mathcal{F})$ with underlying space X such that $Z \cap F_{sm} \neq \varphi$ and $\rho : \widetilde{X} = \mathrm{Bl}_Z(X) \to X$ then $K_{\widetilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$. In particular for any $x \in Z \cap F_{sm}$, $\mathrm{discrep}_{X,\mathcal{F}}(x) = 0$.

Proof. By I.1.5 we know that the discrepancy is non-negative, whence in terms of a local generator ∂ around some point $x \in Z$ it will suffice to prove that $\rho^*\partial$ has no poles. This latter is a wholly general fact, only the Gorenstien condition is important. Indeed if the ideal I_Z of Z is generated by $f_1, \ldots, f_m$, say, and we're on the $f_1 \neq 0$ patch of the blow up, then a function here is of the form $g = h/f_1^n$, $h \in I_Z^n$. On the other hand,

$$(\rho^* \partial)(g) = \frac{f_1 \, \partial \, h - n \, h \, \partial \, f_1}{f_1^{n+1}} \in \mathcal{O}_{\widetilde{X}}$$

since $\partial(I_Z^n) \subset I_Z^n$. $\square$

Such examples are perhaps somewhat rare, although they certainly exist. Here is another example of a more general character:

Example I.1.6(b). Let $(X, \mathcal{F})$ be a foliation by curves on a non-singular 3-fold, x a smooth point of the foliation and $Z \ni x$ a curve not invariant by the foliation, but tangent to it at x to order 3. Consider the sequence of modifications,

$$(X_3, \mathcal{F}_3) \underset{\rho_3}{\longrightarrow} (X_2, \mathcal{F}_2) \underset{\rho_2}{\longrightarrow} (X_1, \mathcal{F}_1) \underset{\rho_1}{\longrightarrow} (X, \mathcal{F})$$

where X_1 is the blow up of X in Z, and X_2 the blow up of X_1 in the fibre of the exceptional divisor E_1 over x with exceptional divisor E_2, and finally X_3 the blow up in the intersection of the proper transform of E_1 with E_2, then:

$$K_{\mathcal{F}_3} = \rho^* \, K_{\mathcal{F}} + \widetilde{E}_1 + \widetilde{E}_2 \, .$$

In particular $a_{X,\mathcal{F}}(E_3) = 0$.

Proof. We calculated locally about our point considered as the origin, with coordinates x, y, z where $y = z = 0$ defines Z. The supposition on tangency means, the foliation is given by a field of the form,

$$\partial = \frac{\partial}{\partial x} + f \, \frac{\partial}{\partial y} + g \, \frac{\partial}{\partial z}$$

where $f, g \in \mathfrak{m}^3$, $\mathfrak{m}$ being the maximal ideal at the origin. At a point in the $y \neq 0$ patch of the fibre the foliation is given by:

$$\partial_1 = y \, \frac{\partial}{\partial x} + y f \, \frac{\partial}{\partial y} + (g - f z_1) \, \frac{\partial}{\partial z_1}$$

where $z = y z_1$ and the discrepancy is 1. The fibre has equation $y = x = 0$ and $\partial_1 = y \, \frac{\partial}{\partial x} + \delta$ where, $\delta \in (x, y)^3 \, T_{X_2}$. So on the $x \neq 0$ patch of X_2 the foliation is given by,

$$\partial_2 = x \, y_2 \, \frac{\partial}{\partial x} + y_2^2 \, \frac{\partial}{\partial y} + x^2 \, \widetilde{\delta}$$

where $\widetilde{\delta}$ is some vector field, the discrepancy is zero relative to x_1, and $y = x y_2$. So the intersection of the proper transform of E_1 with E_2 is given by

$x = y_2 = 0$, and the blow up in this manifestly has discrepancy at most -1 relative to X_2. Whence the formula by I.1.5. $\square$

In fact tangency to order 3 in this example is relatively unimportant. Any tangency will lead to an exceptional divisor with zero discrepancy. The only difference being the length of the chain of blow ups might be a bit bigger. All of this will emerge as easy facts from an appropriately general discussion.

The first thing to note is that if Z is an integral subscheme of $\mathrm{sing}(\mathcal{F})$ and $\mathcal{F}$ is Gorenstien at Z then the foliation is given by a derivation $\partial \in \mathfrak{m}_{M,Z}\ \mathrm{Der}_k(\mathcal{O}_{M,Z})$ for $Z \subset X \subset M$, with M smooth. In particular $\partial(\mathfrak{m}_{X,Z}) \subset \mathfrak{m}_{X,Z}$ and ∂ defines a $k(Z)$ linear endomorphism $D \in \mathrm{End}_{k(Z)}(\mathfrak{m}_{X,Z}/\mathfrak{m}_{X,Z}^2)$. This may of course be zero. Indeed it is zero if $\partial \in \mathfrak{m}_{X,Z}^2\ \mathrm{Der}_k(\mathcal{O}_{X,Z})$, and in this case we have:

Fact I.1.7. Notations as above if $D = 0$ then $(X, \mathcal{F})$ is not canonical at Z.

Proof. It's sufficient to show that the discrepancy is negative for a local lifting of the foliation to some ambient smooth thing, so we might as well say that X is smooth. The question is generic on Z so we may as well say Z is smooth too. Taking coordinates $x_1 = \cdots = x_d = 0$ defining Z, and $z_1, \ldots, z_k$ along Z we can write, $\partial = \sum\limits_i f_i \frac{\partial}{\partial x_i}$, with $f_i \in (x_1, \ldots, x_d)^2$. Then just blow up in Z and the discrepancy is at most, $-\min\limits_i \mathrm{ord}_Z(f_i) + 1 \leq -1$. $\square$

The more subtle phenomenon are essentially determined by particular properties of the Jordan decomposition of D. To this end a convenient way to get to grips with things is as follows. Suppose for simplicity that X is smooth (it's always possible to reduce to this by way of ambient embeddings). We choose a quasi coefficient field K in $\mathcal{O}_{X,Z}$ (i.e. $K \to k(Z)$ is étale), then we have a non canonical isomorphism,

$$\mathrm{Der}_k(\mathcal{O}_{X,Z}) \xrightarrow{\sim} \mathcal{O}_{X,Z} \otimes_K \mathrm{Der}_k(K) \oplus \mathrm{Der}_K(\mathcal{O}_{X,Z})\,.$$

Modulo the choice of K the only canonical thing about this isomorphism is that it respects the isomorphism, $k(Z) \otimes \Omega_{K/k} \to \Omega_{k(Z)/k}$. At the level of the completion $\widehat{\mathcal{O}}_{X,Z}$ in $\mathfrak{m}_{X,Z}$ it corresponds to the structure theorem, i.e. there is a unique $\widehat{K} \xrightarrow{\sim} k(Z)$ in $\widehat{\mathcal{O}}_{X,Z}$ containing K and, $\widehat{\mathcal{O}}_{X,Z} = \widehat{K}\,[[x_1, \ldots, x_d]]$, for x_i generators of $\mathfrak{m}_{X,Z}$, and $\mathrm{Der}_K(\mathcal{O}_{X,Z})$ is generated by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}$. The important thing is that we can decompose our derivation ∂ defining the foliation as $\delta \oplus \partial_K$ with, $\delta \in \mathcal{O}_{X,Z} \otimes_K \mathrm{Der}_k(K)$ and $\partial_K \in \mathrm{Der}_K(\mathcal{O}_{X,Z})$.

At this point, the following formalism is very useful, cf. [Ma]. We let A be a complete regular local ring containing a coefficient field, L, supposed algebraically closed. Next let $\partial \in \mathfrak{m}\ \mathrm{Der}_L(A)$. For every $n \in \mathbb{N}$ we have an exact sequence,

$$0 \longrightarrow \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \longrightarrow \frac{A}{\mathfrak{m}^{n+1}} \longrightarrow \frac{A}{\mathfrak{m}^n} \longrightarrow 0\,.$$

We can consider ∂ as a L-linear endomorphism, ∂_n, of $A_n = A/\mathfrak{m}^n$ for each n. Consequently ∂_n has a Jordan decomposition $\partial_{S,n} \oplus \partial_{N,n}$ into a semi-simple and nilpotent part. These are compatible with the restriction maps $A_{n+1} \to A_n$, and so on taking limits give a Jordan decomposition $\partial_S \oplus \partial_N$ of ∂. We observe:

Fact I.1.8. ∂ is semi-simple iff there is a choice of generators $x_i \in \mathfrak{m}$, together with $\lambda_i \in L$ such that,

$$\partial = \sum_i \lambda_i \, x_i \, \frac{\partial}{\partial x_i} \,.$$

Describing the nilpotent part is worth the trouble, since it has some amusing applications in positive characteristic. In the first place $[\partial_S, \partial_N] = 0$, so given ∂_S as above we just compute a basis for fields which commute with it. Putting $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and $\Lambda \cdot -$, to be the usual inner product these are easily seen to be, cf. op.cit.[6],

(a) $x^Q \, x_i \, \frac{\partial}{\partial x_i}$, $\Lambda \cdot Q = 0$, $Q = (q_1, \ldots, q_n)$, $q_j \in \mathbb{N} \cup \{0\}$

(b) $x^Q \, x_i \, \frac{\partial}{\partial x_i}$, $\Lambda \cdot Q = 0$, $Q = (q_1, \ldots, q_n)$, $q_j \in \mathbb{N} \cup \{0\}$ for $j \neq i$, $q_i = -1$

where of course $x^Q = x_1^{q_1} \ldots x_n^{q_n}$. Our first little amusement is,

Divertimento I.1.9. Let ∂ be a non-singular derivation of a complete regular local ring A over an algebraically closed field L of characteristic $p > 0$ isomorphic to its residue field then there is a choice of coordinates $x, y_1, \ldots, y_n$ in the maximal ideal such that up to multiplication by a unit,

$$\partial = \frac{\partial}{\partial x} + \sum_{i=1}^{n} x^{p-1} \, f_i(x^p, y) \, \frac{\partial}{\partial y_i} \,.$$

Proof. We can certainly multiply ∂ by a unit, in such a way that for some $x \in \mathfrak{m}$, $\partial x = 1$. Now consider $\widetilde{\partial} = x\partial$, and its Jordan decomposition $\partial_S \oplus \partial_N$. Trivially $\partial_S = x \, \frac{\partial}{\partial x}$, in some coordinate system $x, y_1, \ldots, y_n$. Observe that in our formulae for the nilpotent part we must have an exponent of x in the monomial $(x^Q \, x_i)$ at least 1, since $x \mid \widetilde{\partial}$, whence the claim. □

The same proof gives the Frobenius theorem in characteristic zero, but also shows how it fails in char $= p > 0$. Indeed if we consider,

$$\partial = \frac{\partial}{\partial x} + x^{p-1}(y^2 + x^p) \, \frac{\partial}{\partial y}$$

then $y^2 + x^p = 0$ is an invariant curve. The foliation is not p-closed, (cf. II.1), but our invariant curve certainly isn't smooth. When we come to discuss p-closure, we'll actually find that this property is equivalent to the vanishing of

[6] The situation in op.cit. is only discussed in characteristic zero. However what is at stake is just some linear algebra which is valid in any characteristic provided $\Lambda \cdot Q$ is understood to take values in L.

the f_i in V.1.9, so that in turn a foliation in positive characteristic is globally integrable iff it is formally integrable at one point.

Anyway back at the subject of canonical singularities we note that the linearisation of ∂_K as a $\mathfrak{m}_{X,Z}/\mathfrak{m}^2_{X,Z}$ endomorphism is just our previous linearisation D, and we have the following strenghtening of V.1.7,

Fact I.1.10. If D is nilpotent then $(X, \mathcal{F})$ is non-canonical at Z.

Proof. The Jordan decomposition of D is already defined without field extension since all the eigenvalues are zero. Moreover if we only blow up in algebraic centres, then we can work in the completion. Specifically we'll blow up in $\widehat{K}$ points, which correspond to $k(Z)$ points of $\operatorname{Spec}\mathcal{O}_{X,Z}$. Better still this doesn't change things generically on Z, so if we arrive to a linearisation of ∂_K on a modification which is zero then we're done by I.1.7, so that it suffices to prove the Fact for complete local rings with $k = \widehat{K}$, not necessarily algebraically closed. This follows by direct computation. Write:

$$\partial = \sum_{i=k}^{n-1} x_i \frac{\partial}{\partial x_{i+1}} + \delta$$

where $x_1, \ldots, x_n$ are suitable coordinates, and $\delta \in \mathfrak{m}^2 T_{X,Z}$. We blow up in the origin, say, $p : \widetilde{X} \to X$, and look at the $x_n \neq 0$ patch, i.e. $x_n = \xi_n$, $x_i = \xi_i \xi_n$, $i < n$ then,

$$p^*\partial = \sum_{i=k}^{n-2} \xi_i \frac{\partial}{\partial \xi_{i+1}} + p^*\delta \pmod{\mathfrak{m}^2 T_{\widetilde{X}}}$$

where $\mathfrak{m}$ is now the maximal ideal at $\xi_1 = \cdots = \xi_n = 0$. Unfortunately the eigenspace may not go up, since a priori we can't do better than,

$$p^*\delta = \sum_{i=1}^{n-1} a_i \, \xi_n \frac{\partial}{\partial \xi_i} \pmod{\mathfrak{m}^2 T_{\widetilde{X}}}\,.$$

However everything we've stuck under the appellation of $\operatorname{mod}\mathfrak{m}^2$ is in fact divisible by ξ_n except a term of the form, $-\sum_{i\neq n} \xi_{n-1}\, \xi_i \frac{\partial}{\partial \xi_i}$. So we blow up in the origin and look at the $\xi_{n-1} \neq 0$ patch, i.e. $\zeta_{n-1} = \xi_{n-1}$, $\xi_i = \zeta_i \, \zeta_{n-1}$, $i \neq n-1$, again denoting this blow up by p and $\mathfrak{m}$ for the obvious maximal ideal, we have,

$$p^*\partial = \sum_{i=k}^{n-3} \zeta_i \frac{\partial}{\partial \zeta_{i+1}} + \sum_{i=1}^{n-2} a_i \, \zeta_n \frac{\partial}{\partial \zeta_i} \pmod{\mathfrak{m}^2 T_{\widetilde{\widetilde{X}}}}\,.$$

Understanding the sums to be zero if $k < n-3$ or $1 < n-2$, we find that we have at least one more eigenvector for the linearisation in $\mathfrak{m}/\mathfrak{m}^2$, and we're done by induction. □

We are now ready to characterise canonical singularities by way of,

Fact I.1.11. (characteristic 0) Notations as above, then modulo multiplication by a suitable unit the following are equivalent,

(1) The field ∂_K is a semi-simple $\widehat{K}$ derivation of $\widehat{\mathcal{O}}_{X,Z}$ with positive integer eigenvalues.

(2) D is not nilpotent and Z is the centre of a valuation with negative $(X, \mathcal{F})$ discrepancy.

(3) Z is the centre of a valuation with discrepancy -1 such that the maximal ideal in $k(X)$ of the said valuation is not invariant for the induced foliation.

Proof. (1) $\Rightarrow$ (2) It suffices to exhibit the centre via a chain of blow ups, and of course we can just work in the completion provided we blow up in algebraic centres. Observe that with the hypothesis, Z is in fact the singular locus of the foliation, and it's easy to construct a sequence of blow ups in the singular locus of the foliation, determined by the Euclidean algorithm for the eigenvalues, which eventually lead to a valuation of negative discrepancy centred on Z.

(2) $\Rightarrow$ (3) Suppose we have a discrete rank 1 valuation ring $R \supset \mathcal{O}_{X,Z}$ dominantly, and an induced derivation $\widetilde{\partial}$ of R with $\partial = \pi^e \widetilde{\partial}$, $e \in \mathbb{N}$, π a unformising parameter. We can certainly extend all this to the completion, where we can find an eigenfunction x in $\widehat{\mathfrak{m}}_{X,Z}$ with $\partial x = \lambda x$, $\lambda \in \widehat{K}^{\times}$. Writing $x = \pi^a u$, u a unit in $\widehat{R}$, we have:

$$\lambda x = \partial x = a \pi^{a+v(\partial\pi)-1} u + \pi^{a+e} \widetilde{\partial} u$$

so: $a = v(\partial x) \geq a + 1$ if $v(\partial \pi) = e + v(\widetilde{\partial} \pi) \geq 2$, so: $e = 1$, $v(\widetilde{\partial} \pi) = 0$, as claimed.

(3) $\Rightarrow$ (1) We might as well just start with a discrete rank 1 valuation ring $R \supset \widehat{\mathcal{O}}_{X,Z}$, with uniformiser π, where by hypothesis $\partial = \pi \widetilde{\partial}$, and $\widetilde{\partial} \pi$ is a unit. We can certainly choose π and a quasi coefficient field L of R containing $\widehat{K}$ such that if we write $\widehat{\partial} = \partial_K \oplus \delta$, with $\delta \in \widehat{\mathcal{O}}_{X,Z} \otimes \mathrm{Der}_k(\widehat{K})$, then $\partial_K = \pi^a \widetilde{\partial}_K$, $\delta = \pi^b \widetilde{\delta}$, where $a, b \in \mathbb{Z}$ and $\widetilde{\delta} \in R \otimes \mathrm{Der}_k(L)$. On the other hand,

$$\widetilde{\partial} = \pi^{a-1} \widetilde{\partial}_K + \pi^{b-1} \widetilde{\delta}$$

and $\widetilde{\partial}_K, \widetilde{\delta}$ are independent, so $a, b \geq 1$. However $\widetilde{\delta} \pi = 0$, so $a = 1$, and $\widetilde{\partial}_K \pi$ is a unit.

Now let F^n in $\widehat{\mathcal{O}}_{X,Z}$ be $\mathfrak{m}_R^n \cap \widehat{\mathcal{O}}_{X,Z}$. By a theorem of Chevalley the F^n induce the $\mathfrak{m}_{X,Z}$-adic topology, and we have an injection,

$$F^n / F^{n+1} \hookrightarrow \mathfrak{m}_R^n / \mathfrak{m}_R^{n+1}$$

the left hand side is a finite $\widehat{K}$ vector space, the right hand finite over the residue field ℓ, say, of R. In either case ∂_K is a linear map for the appropriate field, and a quick calculation shows that it's just multiplication by $n\,\widetilde{\partial}\,\pi$ on the right hand side. Consequently if $F^n/F^{n+1} \neq 0$ then all the eigenvalues of ∂_K are the same. Even better, since the characteristic is zero, taking the trace shows that they actually take values in $\widehat{K}$, so after multiplying $\widetilde{\partial}$ by a unit in $\widehat{\mathcal{O}}_{X,Z}$ we might as well say $\widetilde{\partial}\,\pi = 1$. Now consider the sequence of integers n_k defined by n_{k+1} is the minimum integer with $F^{n_{k+1}} \neq F^{n_k}$, $n_1 = 1$, and put $\mathcal{O}_k = \widehat{\mathcal{O}}_{X,Z}/F^{n_k}$. Further we have an exact sequence,

$$0 \longrightarrow F^{n_k}/F^{n_{k+1}} \longrightarrow \mathcal{O}_{k+1} \longrightarrow \mathcal{O}_k \longrightarrow 0$$

and $F^{n_k}/F^{n_{k+1}} \hookrightarrow \mathfrak{m}^{n_{k+1}-1}/\mathfrak{m}^{n_{k+1}}$, so ∂_K acts semi-simply on it with eigenvalue, $n_{k+1}-1$, where: $n_{k+1} > n_{k+1}-1 \geq n_k$. Whence, by induction, we conclude that ∂_K is semi-simple on $\mathcal{O}_{k+1}$ with positive integer eigenvalues $< n_{k+1}$. Finally for any $d \in \mathbb{N}$, $\widehat{\mathcal{O}}_{X,Z}/\mathfrak{m}^d_{X,Z}$ is a quotient of a sufficiently large $\mathcal{O}_k$, so that (1) holds. □

So to tie everything together, observe that our linearisation D is intrinsically defined at the sheaf level as a map,

$$D : \mathcal{I}_Z/\mathcal{I}_Z^2 \longrightarrow \mathcal{I}_Z/\mathcal{I}_Z^2 \otimes_{\mathcal{O}_Z} K_{\mathcal{F}}$$

even better this arises from,

$$\mathcal{I}_Z \underset{d}{\longrightarrow} \Omega_X \longrightarrow K_{\mathcal{F}} \cdot \mathcal{I}_Z \longrightarrow K_{\mathcal{F}} \otimes_{\mathcal{O}_Z} \mathcal{I}_Z/\mathcal{I}_Z^2 \,.$$

So there is a factorisation,

$$\begin{array}{ccccc}
\mathcal{I}_Z/\mathcal{I}_Z^2 & & & & \\
\downarrow & \searrow^{D} & & & \\
\Omega_X \otimes \mathcal{O}_Z & \longrightarrow & K_{\mathcal{F}} \otimes_{\mathcal{O}_Z} \mathcal{I}_Z/\mathcal{I}_Z^2 & \longrightarrow & \Omega_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(K_{\mathcal{F}})
\end{array}$$

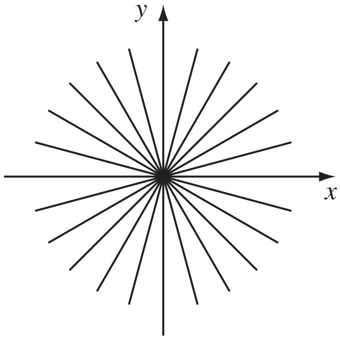
the total composite of the bottom row, just being, $\omega \mapsto d(\partial(\widetilde{\omega}))$ for any lifting $\widetilde{\omega}$ of ω. Calling this map $D_{\rm big}$, we see that D is nilpotent iff the coefficients of the characteristic polynomial of $D_{\rm big}$ all vanish, and this is a closed condition so,

Fact I.1.12. The locus of non-canonical points is closed.

I.2. Dimension 2

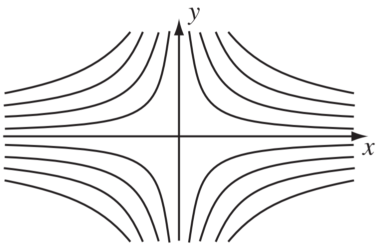
Our ultimate interest will be foliations by curves on 2-dimensional objects (in the broadest sense) so let us concentrate on this and its pecularities for the moment. To begin with it is a very reasonable place to give some examples of the preceeding discussion,

Example I.2.1(a). Consider the foliation at the origin of $\mathbb{C}^2$ given by $\partial = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$. This is a pencil of lines through the origin, and may easily be drawn,



By I.1.11 we know this is non-canonical. Of course it's trivial to study this by hand, and after blowing up in the origin we see that the discrepancy is -1, and that the induced foliation on the blow up is everywhere smooth. This may encourage the misconception that a smooth resolution is possible. However, this is easily seen to be impossible,

Example I.2.1(b). Consider the foliation at the origin of $\mathbb{C}^2$ given by $\partial = x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y}$. Again we may draw this,



Now, here, if we blow up in the origin we see the discrepancy is zero. Indeed by our characterisation of canonical singularities, we know that this example is canonical. On the exceptional divisor there are two singularities corresponding to where the proper transforms of the two branches through the origin cross it, and at both of these the singularity is much the same as before. In fact, one can blow up ad nauseam and the situation will never get any better. The basis of what's going on is that canonical is simply best possible, which we may see by way of,

Fact I.2.2. (any dimension characteristic 0) Let R be the valuation ring of the exceptional divisor on a modification of a foliated Gorenstien variety $(X, \mathcal{F})$ with canonical singularities obtained via a chain of blow ups in the singular locus of $\mathcal{F}$ then the exceptional divisor is invariant by the foliation.

Proof. Given that the modification in question arises by blowing up in invariant centres, the discrepancy is at most 0. On the other hand the singularities are canonical, so it is zero. Now let Z on X be the centre of the valuation v. This is contained in $\mathrm{sing}(\mathcal{F})$, and so is certainly foliation invariant. However if the exceptional divisor were not invariant, and ∂ defines the foliation around Z, then for any $f \in \mathfrak{m}_{X,Z}$ we would have, $v(\partial f) = v(f) - 1$, so eventually $\partial^i f \notin \mathfrak{m}_{X,Z}$ which is absurd. Indeed we can combine this with I.1.8 and sequel to establish,

Fact I.2.3. Notations as in I.2.2, and suppose the underlying space X is non-singular, then a canonical singularity does not admit a smooth resolution.

Proof. By I.2.2 it suffices to find an invariant formal branch of the foliation for any point in the singular locus. Suppose we've done this, and say the branch is C through a point z, and that $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ is a smooth resolution obtained by blowing up in singular centres with $\widetilde{X}$ non-singular. Certainly $\widetilde{X}$ cannot be isomorphic to X over z, and since it is non-singular there must be an exceptional divisor E of the type discussed in I.2.2 meeting the proper transform $\widetilde{C}$ of C. However E is invariant by $\widetilde{\mathcal{F}}$, and both E and $\widetilde{C}$ are invariant, yet transverse to each other, which is nonsense. The conclusion that this is nonsense actually only requires the weaker property that every exceptional divisor has discrepancy 0, and not even $\widetilde{X}$ non-singular, since a suitable resolution can always be constructed if $(\widetilde{X}, \widetilde{\mathcal{F}})$ is Gorenstien, cf. [BM].

Anyway, what we need to do now is find the branch. We consider a local generator ∂ of the foliation about z, and apply I.1.8 if the field ∂ is semi-simple in $\widehat{\mathcal{O}}_{X,z}$, the existence of a branch is clear. Otherwise we'll prove the existence of a branch inductively under the weaker hypothesis of a non-zero eigenvalue using the formulae for the nilpotent part of I.1.8, the case of dimension 1 being clear. So given m, let $\lambda = \max_i |\lambda_i|$ in the notations of I.1.8, and say λ is attained for $|\lambda_1|, \ldots, |\lambda_k|$, and all the other eigenvalues if any have smaller modulus. Consider the ideal $I = (x_{k+1}, \ldots, x_n)$, then this is invariant, unless for some $k+1 \leq i \leq n$, and $q_1, \ldots, q_k \in \mathbb{N} \cup \{0\}$ with $(q_1, \ldots, q_k) \neq 0$,

$$\lambda_i = q_1\lambda_1 + \cdots + q_k\lambda_k = (q_1\varepsilon_1 + \cdots + q_k\varepsilon_k)\,|\lambda|$$

where $\varepsilon_i = \frac{\lambda_i}{\lambda} \in \{1, -1\}$. Observe that in the invariant case we're done by induction unless $k = n$ while for $k < n$ some λ_j are positive, some negative for $1 \leq j \leq k$. Say $\lambda_1, \ldots, \lambda_r$ are all positive, and $\lambda_{r+1}, \ldots, \lambda_k$ negative and this time consider $I = (x_{r+1}, \ldots, x_n)$ which now certainly cuts out a positive dimensional sub-scheme. However this sub-scheme is invariant, and ∂ restricts to it with non-zero eigenvalue, so not only are we reduced to the case $k = n$, but that all the $\lambda_i = 1$. Here the Jordan decomposition in the complete local ring is precisely that modulo $\mathfrak{m}^2$, so there is an invariant hypersurface corresponding to the eigenvector of the linearisation and again we finish by induction.

At this point one might get a little bit nervous, and wonder whether even canonical resolution is possible. However for surfaces in characteristic zero, this is an old theorem of Seidenberg. Specifically:

Fact I.2.4. (characteristic 0) Let $(X, \mathcal{F})$ be a foliated algebraic surface, or even algebraic space of dimension 2 then there is a proper birational modification $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ such that the induced foliation has canonical singularities.

Proof. (sketch) Evidently without loss of generality we may assume that X is non-singular. Suppose further that $z \in \operatorname{sing}(\mathcal{F})$ is a non-canonical singularity, then there are 3 possibilities in terms of a derivation ∂ defining the foliation about z, viz:

(i) $\partial \in \mathfrak{m}_{X,z}^2 T_X$

(ii) $\partial \notin \mathfrak{m}_{X,z}^2 T_X$ and the linearisation $D \in \operatorname{End}(\mathfrak{m}/\mathfrak{m}^2)$ is nilpotent.

(iii) In formal coordinates x, y about z, after multiplication by a suitable unit, $\partial = px \frac{\partial}{\partial x} + qy \frac{\partial}{\partial y}$, $p, q \in \mathbb{N}$, relatively prime.

In case (i) the discrepancy of the exceptional divisor corresponding to blowing up in z is at most -1. Call e the negative of the discrepancy, and $I, \widetilde{I}$ the ideals of the singular sub-scheme whether on X or this particular blow up. An easy calculation in local chern classes gives, $\deg(\widetilde{I}) = \deg(I) - e(e+1)$, so after finitely many blow ups case (i) ceases to exist. Case (iii) is easily dealt with by hand by way of the Euclidean algorithm for p and q. The most fastidious case is (ii), which can be treated by direct local calculation. □

Resolution fails however in positive characteristic, essentially because positive and negative integers are indistinguishable. Here is an example,

Counterexample I.2.5 The foliation given by $\partial = x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y}$ in characteristic 5 on $\mathbb{A}^2$ does not admit a canonical resolution.

Proof. Blow up in the origin. There are two singularities, the one where the proper transform of the $y = 0$ branch crosses the exceptional divisor has eigenvalues $(1, -3)$. Blow up in this then there is a singularity at the crossing of the new exceptional divisor with the proper transform of the old, and there the eigenvalues are, after multiplication by a unit, $(1, 2)$. Modulo multiplication by another unit, there is a singularity on the blow up such that this new singularity has eigenvalues, $(1, -2)$, so the thing just repeats itself, and cannot be resolved. □

So, all the more reason to concentrate on characteristic zero, where we have:

Fact I.2.6. (characteristic 0) Let $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ be a blow up of a foliated non-singular surface with canonical singularities then $\widetilde{\mathcal{F}}$ has canonical singularities.

Proof. This is true quite generally for blow ups in invariant centres, since in this case $K_{\widetilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$. However by I.1.6(b) we know there are non-invariant centres in higher dimension where this fails. The thing about dimension 2 is that the only possibility is to blow up in a point, and in any dimension this gives rise to a canonical singularity. Indeed since this is the only way to proceed in dimension 2 we even have,

Fact I.2.6bis. (characteristic 0) Let $(X, \mathcal{F})$ be a foliated non-singular surface then a point is a terminal singularity iff it is smooth.

Another nice feature of dimension 2, or more correctly of surfaces over an algebraically closed field of characteristic 0 is that we can utilise the full force of the semi-simplicity/nilpotent considerations of I.1.8 to give a complete tabulation of canonical singularities. This can be found for example in [C-M] pg. 127. The one thing that emerges from this tabulation which we will use is,

Fact I.2.7. (characteristic 0) Let $(X, \mathcal{F})$ be a foliated non-singular surface over an algebraically closed field then through a point of $\mathrm{sing}(\mathcal{F})$ there are at most two, and at least one, formal branches of the foliation. Even better any branch is smooth, and if there are two, then they are simple normal crossing.

The next thing to note about dimension 2, is in the broad sense of 2 dimensional schemes that there is an important generalisation of I.2.5. Namely,

Fact I.2.8. Let X be a two dimensional scheme over a field k of characteristic zero, with $k(X)/k$ of finite transcendence degree, and $\mathcal{F}$ a foliation on X (i.e. everywhere locally we have a k-derivation) then there is a proper birational modification, $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ such that the induced foliation has canonical singularities.

Proof. Without loss of generality we can immediately reduce to the case that $\mathcal{O}$ is local and regular. So let's say the closed point is Z. Furthermore provided we only blow up in $k(Z)$ points we can work in the completion, so modulo a minor abuse of the term "finite transcendence degree" we can suppose without loss of generality that $X = \mathrm{Spf}\, K[[x, y]]$ for K/k of finite transcendence degree.

As usual we consider our foliation as being given by a k-derivation ∂ which we write as $\partial_K + \delta$ where $\partial_K \in \mathrm{Der}_K\, K[[x, y]]$, and $\delta \in K[[x, y]] \otimes \mathrm{Der}_k(K)$. In I.2.4 there is no supposition about algebraic closure (bear in mind the cases (ii) and (iii) which use Jordan decomposition apply over any field, since the eigenvalues are in $\mathbb{Q}$), so by a sequence of blow ups in K-points we arrive to the situation where the induced K-foliation is canonical. Specifically we have a proper birational modification $\rho : Y \to X$ via a sequence of blow ups in K-points such that locally on any affine patch of Y we have a derivation $\widetilde{\partial}_K$ generating the induced K-foliation, $\widetilde{\partial}_K$ has canonical singularities, and $\rho^* \partial_K = f \widetilde{\partial}_K$ where f is a function with zeroes, if any, on curves contracted by ρ. Moreover δ, by hypothesis, certainly lies in $\mathfrak{m}_{X,Z} \otimes \mathrm{Der}_k(K)$, so $\rho^* \delta = g\, \widetilde{\delta}$, for some function g, with $\mathcal{O}_Y \widetilde{\delta} \subset \mathcal{O}_Y \otimes \mathrm{Der}_k(K)$ saturated. So clearing common factors we have, $\rho^* \partial = h(a\, \widetilde{\partial}_K + b\, \widetilde{\delta})$ with h vanishing at most on the contractible curves, and a, b relatively prime. Now a blow up in a smooth point of the K-foliation, indices a K canonical singularity, so by the standard argument of reducing arithmetic genus (valid over any field) we can blow up in some more K-points to ensure that a and b have irreducible, simple normal crossing support. This might create poles when pulling back ∂_K, but this hardly effects the conclusion. Saturating we can say that $h = 1$, and consider various cases, for the induced field $\widetilde{\partial}$.

(i) a is a unit or b is a unit. In this case things are already canonical.

(ii) $\widetilde{\partial}_K$ has a canonical but not terminal singularity which coincides with a K-point where a/b is not defined. Blowing up in this point and looking at points on the exceptional divisor, either we're at a point where a/b is defined, so things are canonical by (i), or we're at a point with the same property, which cannot continue indefinitely since a rational map will eventually be resolved by blowing up in points, or we're reduced to:

(iii) $\widetilde{\partial}_K$ has a terminal singularity at a point where a/b is not defined. To integrate a non-zero vector field in a complete local ring only requires us to find a function f in the maximal ideal with $\partial(f) = 1$, so modulo multiplication by a suitable unit this is possible over any field of characteristic zero. Whence integrating $\widetilde{\partial}_K$ we can find functions x, y, z in the complete local ring, which are pairwise K coordinates, such that either,

$$\widetilde{\partial} = x^m \frac{\partial}{\partial x} + y^n \widetilde{\delta} \qquad \text{or} \qquad \widetilde{\partial} = y^m \frac{\partial}{\partial x} + z^n \widetilde{\delta}\,.$$

Let us do the former and easy one first. In either case the important invariant is the complexity, $\mu = (m, n)$ of the appropriate rational map. Consider the transformation $y = xt$. If $m - 1 \leq n$ then we obtain (i) and are done. If $m - 1 > n$ then we obtain,

$$(x^{m-n-1})\,(x\,\partial/\partial x - t\,\partial/\partial t) + t^n\,\partial/\partial z$$

which corresponds to a singularity precisely where the rational map $[x^{m-n-1} : t^n]$ is not defined, and this situation persists under blowing up, and so again after finitely many transformations we obtain (i). Otherwise put $x = ys$ then for $m - 1 \geq n$ there is nothing further to do. Should $m - 1 < n$ then we obtain,

$$s^m\,\partial/\partial s + y^{n-(m-1)}\,\partial/\partial z$$

i.e. $\mu \mapsto \widetilde{\mu} = (m, n - (m-1))$ with a decrease in complexity unless $m = 1$. This latter situation is however canonical. Consider now the second case under $y = xt$. At the point $x = 0$, $t = 0$ we get the situation of K-foliation singularities, and rational map singularities coinciding and so again reduce to (i). Otherwise for a suitable local coordinate w we obtain, before cancelling x, a generator of the form,

$$x^{m-1}\,\partial/\partial w + x^n\,\widetilde{z}^n\,\partial/\partial z\,.$$

If $\widetilde{z} = 0$ doesn't meet this chart or if $m - 1 \leq n$ there is nothing to do. Otherwise $\widetilde{\mu} = (m-1-n, n)$ and the complexity decreases while remaining in the same form. Now consider $x = ys$, again before cancelling y's we obtain,

$$y^{m-1}\,\partial/\partial s + y^n\,\widetilde{z}^n\,\partial/\partial z$$

so that if $\widetilde{z} \neq x$ we are done, or idem if $m - n \leq n$. Should it be otherwise then again $\mu \mapsto \widetilde{\mu} = (m-1-n, n)$ with the shape of the form remaining, whence the claim. □

The final thing to note in dimension 2, is that by way of Mumford's intersection theory we can generalise the definition of discrepancy, and whence the definitions of canonical and terminal singularities, to arbitrary foliations on normal algebraic spaces irrespective of whether they are $\mathbb{Q}$-foliated Gorenstien or not. Since we're only going to need this for X a 2-dimensional normal algebraic space over a field, we'll confine ourselves to this for ease of exposition. We have Mumford's Néron-Severi group $\mathrm{NS}^m(X)$, actually a $\mathbb{Q}$-vector space defined via,

Definition I.2.9. *Let X be a proper 2-dimensional algebraic space and $p : Y \to X$ a relatively projective birational morphism from a smooth projective variety Y (such a Y always exists by [Kn]) and E_i the curves on Y contracted by p then for C a curve on X with $\widetilde{C}$ a curve on Y there exist unique positive rational numbers a_i (by virtue of the fact that the intersection matrix $E_i \cdot E_j$ is negative) such that if $p^* C := \widetilde{C} + \sum_i a_i\, E_i$ then, $p^* C \cdot E_j = 0$ for any j. We may then define for two curves C and C', an intersection product $C \cdot C' := p^* C \cdot p^* C'$, and $\mathrm{NS}^m(X)$ is the free $\mathbb{Q}$-vector space of curves in X modulo numerical equivalence in this sense. The group $\mathrm{NS}^m(X)$ coincides with the group $\mathrm{NS}(X)$ in the usual sense if and only if X has $\mathbb{Q}$-factorial singularities.*

Consequently with this definition in mind for $(X, \mathcal{F})$ a 2-dimensional foliated algebraic space $K_{\mathcal{F}}$ always has sense in $\mathrm{NS}^m(X)$ and we may extend definition I.1.2, and the notions of being canonical and terminal to any proper 2-dimensional foliated algebraic space provided we verify,

Fact I.2.10. $(X, \mathcal{F})$ as above, E an exceptional divisor in $k(X)$ then $a_{X,\mathcal{F}}(E)$ is well-defined, and coincides with the previous definition if $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier.

Proof. Agreement with the previous definition is clear, and it suffices to consider maps,

$$(Y, \mathcal{G}) \underset{q}{\longrightarrow} (S, \mathcal{H}) \underset{p}{\longrightarrow} (X, \mathcal{F})$$

where Y and S are smooth projective, p, q are proper birational, E is contracted by p, and F_i are the curves on Y contracted by $p \circ q$, then,

$$(qp)^*\, K_{\mathcal{F}} = K_{\mathcal{G}} + \sum_i a_i\, F_i$$

say, and $q_*(qp)^* K_{\mathcal{F}} = p^* K_{\mathcal{F}} = K_{\mathcal{H}} + \sum_i a_i\,(q_* F_i)$ from which we conclude. □

I.3. Residues

This is a classical subject going back to an initial observation of Baum and Bott. Usually it is done over $\mathbb{C}$ using explicit formulae in terms of differential forms. What's at stake, however, is really just some basic facts in local cohomology, and as such, suitably interpreted is valid over any field in any characteristic. For ease of exposition we'll work with varieties $(X, \mathcal{F})$ foliated by curves over a field k, with underlying space X non-singular. We consider the foliation as given by a short exact sequence,

$$0 \longrightarrow \mathcal{N} \longrightarrow \Omega_X \longrightarrow K_{\mathcal{F}} \cdot I_Z \longrightarrow 0$$

where I_Z is the ideal of the singular subscheme of $\mathcal{F}$ and $\mathcal{N}$ is a saturated subsheaf of Ω_X. In dimension 2, $\mathcal{N}$ will be a line bundle sitting inside Ω_X as a saturated subsheaf.

Firstly let's consider the $(1,1)$ part of the cohomology. We have a diagram of exact sequences,

$$\begin{array}{ccccc} & & & & H^0(K_{\mathcal{F}} \otimes \mathcal{O}_Z) \\ & & & & \downarrow \\ H^1(X, \mathcal{N}) & \longrightarrow & H^{1,1} & \longrightarrow & H^1(K_{\mathcal{F}} \cdot I_Z) \\ & & & & \downarrow \\ & & & & H^1(K_{\mathcal{F}}) \end{array}$$

What interests us is $N^1 := \mathrm{Ker}\,\{H^{1,1} \to H^1(K_{\mathcal{F}})\}$. Similarly at the $(n-1, n-1)$ level we have dual exact sequences,

$$\begin{array}{ccccc} \mathrm{Ext}^n(K_{\mathcal{F}} \otimes \mathcal{O}_Z, K_X) & & & & \\ \uparrow & & & & \\ \mathrm{Ext}^{n-1}(K_{\mathcal{F}} \cdot I_Z, K_X) & \longrightarrow & H^{n-1,n-1} & \longrightarrow & \mathrm{Ext}^{n-1}(\mathcal{N}, K_X) \\ \uparrow & & & & \\ \mathrm{Ext}^{n-1}(K_{\mathcal{F}}, K_X) & & & & \end{array}$$

Now suppose $f : C \to X$ is an invariant curve with C smooth. This, quite generally, defines a class $[f]$ in $H^{n-1,n-1}$ via duality arising from,

$$H^{1,1} \underset{f^*}{\longrightarrow} H^1(\omega_C) = k\,.$$

However, by definition of invariance, $f^*\mathcal{N}$ maps to zero in ω_C, i.e. $[f] \in \mathrm{Ext}^{n-1}(K_{\mathcal{F}} \cdot I_Z, K_X)$. The essential then of Residue Theory is that if D is a class in N^1 the pairing induced by $H^{1,1}$ with $H^{n-1,n-1}$ factors as,

$$N^1 \otimes \mathrm{Ext}^{n-1}(K_{\mathcal{F}} \cdot I_Z, K_X) \longrightarrow H^0(K_{\mathcal{F}} \otimes \mathcal{O}_Z) \otimes \mathrm{Ext}^n(K_{\mathcal{F}} \otimes \mathcal{O}_Z, K_X) \longrightarrow k\,.$$

The local interpretation in terms of residues amounts to identifying H^0 $(K_{\mathcal{F}} \otimes \mathcal{O}_Z)$ with $H^1_Z(K_{\mathcal{F}} \cdot I_Z)$ and $\mathrm{Ext}^n(K_{\mathcal{F}} \otimes \mathcal{O}_Z, K_X)$ with $\mathrm{Ext}^{n-1}_{\widehat{X}}(K_{\mathcal{F}} \cdot I, K_X)$, where $\widehat{X}$ is the formal completion in Z, so that if Z is Cohen Macauly we have the pairing,

$$H^1_Z(K_{\mathcal{F}} \cdot I_Z) \otimes \mathrm{Ext}^{n-1}_{\widehat{X}}(K_{\mathcal{F}} \cdot I, K_X) \longrightarrow H^n_Z(K_X) \underset{\mathrm{Res}}{\longrightarrow} k\,.$$

This is however, fairly unimportant. What interests us are what are the elements of N^1, and when is the pairing over determined. One obvious class of elements in N^1 is invariant divisors, i.e. an effective divisor $\mathcal{O}_X(D)$ such that if locally it is given by $f = 0$, and ∂ defines the foliation, then $\partial(f) \in (f)$. Taking a cover by affines U_α with local equations f_α, and ∂_α for the foliation then as usual $f_\alpha = g_{\alpha\beta} f_\beta$, with $g_{\alpha\beta}$ a unit, while the chern class is $\frac{dg_{\alpha\beta}}{g_{\alpha\beta}}$. Restricting this to $K_{\mathcal{F}}$ we have,

$$\frac{dg_{\alpha\beta}}{g_{\alpha\beta}} \mapsto \frac{\partial_\alpha f_\alpha}{f_\alpha} \otimes \partial_\alpha^\vee - \frac{\partial_\beta f_\beta}{f_\beta} \otimes \partial_\beta^\vee$$

which is cohomologically trivial, so $D \in N^1$. The other important example is the chern class of $\mathcal{N}$. Since $H^1(X, K_{\mathcal{F}})$ injects into $H^1(X \backslash Z, K_{\mathcal{F}})$, we can just check this over the smooth locus. For an affine patch let $\omega_1^\alpha, \ldots, \omega_n^\alpha$ be a basis of $\mathcal{N}$, and τ^α a lifting of $\partial_\alpha^\vee$, then we have for units $h_{\alpha\beta}$,

$$\omega_1^\alpha \wedge \ldots \wedge \omega_n^\alpha = h_{\alpha\beta}\, \omega_1^\beta \wedge \ldots \wedge \omega_n^\beta\,.$$

On the other hand, $d\omega_i^\alpha = a_{ij}^\alpha\, \tau^\alpha \wedge \omega_j^\alpha \pmod{\mathcal{N} \wedge \mathcal{N}}$ for functions a_{ij}, where we employ the summation convention. Whence on differentiating we have,

$$\begin{gathered}(-1)^{i-1}(a_{ii}^\alpha)\, \tau^\alpha \wedge \omega_1^\alpha \wedge \ldots \wedge \omega_n^\alpha = \\ dh_{\alpha\beta}\, \omega_1^\beta \wedge \ldots \wedge \omega_n^\beta + h_{\alpha\beta}(-1)^{i-1} a_{ii}^\beta\, \tau^\beta \wedge \omega_1^\beta \wedge \ldots \wedge \omega_n^\beta\,.\end{gathered}$$

cancelling the $\omega_1^\beta \wedge \ldots \wedge \omega_n^\beta$ and restricting to $K_{\mathcal{F}}$, we find,

$$\frac{\partial h_{\alpha\beta}}{h_{\alpha\beta}} \otimes \partial^\vee = (-1)^{i-1}\, a_{ii}^\alpha\, \partial_\alpha^\vee - (-1)^{i-1}\, a_{ii}^\beta\, \partial_\beta^\vee$$

i.e. the chern class restricts trivially in cohomology as claimed. For amusement therefore we can note the kind of formulae which one obtains.

Example I.3.1. Let $f : C \to X$ be the normalisation of an invariant curve of a foliation by curves $\mathcal{F}$ on X over a field k. Suppose $f(C) \not\subset \mathrm{sing}(\mathcal{F})$, and for a point $c \in f^{-1}(\operatorname{sing}\mathcal{F})$ let $x_1, \ldots, x_n$ be coordinates in the completion of the local ring at $f(c)$, with the foliation being given by a field $\partial = a_i \frac{\partial}{\partial x_i}$, $a_i \in \mathfrak{m}_{f(c)}$, summation convention. Define, $\operatorname{Tr}_{f(c)}(\partial) := \frac{\partial a_i}{\partial x_i}$, and let $\mathcal{N}_{\mathcal{F}}$ be the normal bundle to the foliation then,

$$c_1(\mathcal{N}_{\mathcal{F}}) \cdot C = \sum_{c \in f^{-1}(\operatorname{sing}\mathcal{F})} \operatorname{Res}_c \left(\frac{f_* \left(\frac{\partial}{\partial \xi}\right)}{f^* \partial} \, f^* \operatorname{Tr}_{f(c)}(\partial) \, d\xi \right)$$

where ξ is a coordinate in $\widehat{\mathcal{O}}_{C,c}$, and the residue is taken in the sense of Grothendieck (= ordinary sense in characteristic zero) in the field k.

However, as we've already said, it's primarily over determination rather than explicit formulae which interests us. Over a surface, everything is quite nice. Indeed canonically N^1 is the image of $\operatorname{Ext}^{n-1}(K_{\mathcal{F}} \cdot I_Z, K_X)$ in $H^{1,1}$. Now suppose we have a collection of invariant curves C_i such that the intersection matrix $C_i \cdot C_j$ is invertible in k. Each curve C_i defines a class in N^1 and we have for each connected component z contained in $\underset{i}{\cup} C_i$ of the singular locus of $\mathcal{F}$ understood with reduced structure, residue maps,

$$\operatorname{Res}_z : N^1 \longrightarrow K_{\mathcal{F}} \otimes \mathcal{O}_z \xrightarrow{\sim} k[z]\,.$$

We put $\sum_z \dim_k k[z] = \deg(\operatorname{sing}(\mathcal{F}) \cap \underset{i}{\cup} C_i) = d$, say, and observe that if $d <$ number of curves C_i, then there are elements λ_i of k, not all zero, such that, $\sum_i \lambda_i C_i$ has no residue, and so lies in,

$$\operatorname{Im}\{H^1(X, \mathcal{N}) \longrightarrow H^{1,1}\}\,.$$

Consequently we must have $\left(\sum_i \lambda_i C_i\right) \cdot C_j = 0$ for any j. This however contradicts the invertibility of the intersection matrix, and so we have proved,

Fact I.3.2. Notations as above if the intersection matrix $C_i \cdot C_j$ is invertible in k (e.g. $\underset{i}{\cup} C_i$ contractible, and characteristic zero) then,

$$\text{The number of curves } C_i \leq \deg(\operatorname{sing}(\mathcal{F}) \cap \underset{i}{\cup} C_i)\,.$$

Finally a more substantial use of these ideas, which although not strictly necessary for our purposes does make the odd thing go through rather elegantly, is a residue theorem for transverse invariant measures T due to M. Brunella. Specifically T is a closed positive $(1,1)$ current on a non-singular surface X invariant by a foliation $\mathcal{F}$ with canonical singularities, such that along any curve the Lelong number of T is generically zero, then we have:

Fact I.3.3. (Brunella, [Br2]) Notations as above, $c_1(\mathcal{N}) \cdot T \leq 0$.

II. The cone of a foliation

II.1. p-closure

T. Ekedahl, [E], discovered that foliations in positive characteristic admit a remarkably rich structure which under mild hypothesis renders them amenable to a highly algebraic interpretation. The important new feature which renders this possible is the p^{th}-power map, i.e. one easily verifies that if ∂ is a derivation in positive characteristic then ∂^p is again a derivation. In particular with notations as in I.1, if $(X, \mathcal{F})$ is a foliation then there is a map by p^{th}-powers,

$$\mathcal{T}_{\mathcal{F}_0}^p \to \mathcal{T}_{X_0}/\mathcal{T}_{\mathcal{F}_0}$$

and we call the foliation p-closed if this map is zero. One then has,

Proposition II.1.1. (Ekedahl.) *There is a one to one correpondence between,*

(a) *p-closed foliations with rank of $\mathcal{T}_{\mathcal{F}} = r$.*

(b) *Factorisations $X \underset{\rho}{\longrightarrow} Y \underset{\sigma}{\longrightarrow} X^{(1)}$ of the geometric frobenius with* $\deg(\rho) = p^r$.

In this correspondence, given a foliation one obtains Y from the ring of functions $\{f \in \mathcal{O}_X \mid \partial(f) = 0, \partial \in \mathcal{T}_{\mathcal{F}}\}$, while given a factorisation $\mathcal{T}_{\mathcal{F}}$ is just the kernel of ρ. Sould $\mathcal{T}_{\mathcal{F}}$ be p-closed, we naturally write $X/\mathcal{F}$ for the quotient Y. There is however some need for caution regarding $X/\mathcal{F}$ even if X is non-singular. Specifically denote by $\operatorname{sing}(\mathcal{F})$ the loci where either X is singular or $\mathcal{F}$ is not locally free, then in general $X/\mathcal{F}$ is highly singular on $\rho\,(\operatorname{sing}(\mathcal{F}))$. Certainly an easy calculation shows that $X/\mathcal{F}$ is normal but it certainly need not be Gorenstien, although we do have:

Proposition II.1.2. (Ekedahl.) *If X is $\mathbb{Q}$-factorial then $X/\mathcal{F}$ is $\mathbb{Q}$-factorial, and:*

$$\rho^* K_{X/\mathcal{F}} = (p-1)\, K_{\mathcal{F}} + K_X\,.$$

Although p-closed foliations can lead to rather nasty singularities on the quotient $X/\mathcal{F}$ – for example if the foliation is given by $\partial = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$ in characteristic p, the $\mathcal{O}_Y = k[x^p, x^{p-1}y, \ldots, xy^{p-1}, y^p]$, and Y is the cone over a rational normal curve – it does actually force strong restrictions on the singularities themselves. In fact, returning to foliations by curves, we have,

Proposition II.1.3. *Let $(X_0, \mathcal{F}_0)$ be a foliation in characteristic zero with canonical singularities and $(X, \mathcal{F}) \to \operatorname{Spec}\mathcal{O}$ a model over an affine open subset of a finitely generated extension of $\mathbb{Z}$ such that for all maximal ideals $\mathfrak{m}$ of $\mathcal{O}$ outside a Zariski closed subset, $(X_{\mathfrak{m}}, F_{\mathfrak{m}})$ is $p = \operatorname{char} k(\mathfrak{m})$ closed, then at any point $z \in \operatorname{sing}(\mathcal{F}_0)$ we can find formal coordinates $x_1, \ldots, x_n$, $n \leq \dim X$, and rational integers λ_i such that the foliation is given by a field of the form,*

$$\partial = \sum_{i=1}^{n} \lambda_i\, x_i\, \frac{\partial}{\partial x_i} \in \operatorname{Der}(\widehat{\mathcal{O}}_{X_0, z})\,.$$

Proof. We use our Jordan decomposition. Indeed quite generally let ∂ be a derivation of a regular local ring $\widehat{\mathcal{O}}$ over a field k of positive characteristic. We can write down a Jordan decomposition $\partial = \partial_S + \partial_N$, of ∂ into semi-simple and nilpotent parts. An identity of Jacobson, [E] 4.2, gives:

$$\partial^p = \partial_S^p + \partial_N^p + \sum S_k(\partial_S, \partial_N) = \partial_S^p + \partial_N^p$$

where the last identity follows from the fact that the S_k are Lie polynomials, and $[\partial_S, \partial_N] = 0$. However if the foliation is p-closed then $\partial^p \wedge \partial = 0$, so:

$$0 = \partial_S \wedge \partial_S^p + \partial_S \wedge \partial_N^p + \partial_N \wedge \partial_S^p + \partial_N \wedge \partial_N^p\,. \tag{$*$}$$

Typically if $\mathfrak{m}$ is the maximal ideal of $\widehat{\mathcal{O}}$, $\partial_N(\mathfrak{m}) \subset \mathfrak{m}^2$, so: $\partial_N^p(\mathfrak{m}) \subset \mathfrak{m}^{p+1}$. However p could be close to the dimension etc., so we impose the hypothesis that our foliation is specialed from characteristic zero. In this case we're studying the completion $\widehat{\mathcal{O}}$ and a derivation ∂ of it and we already know the Jordan decomposition of ∂ mod $\mathfrak{m}^d$, say, then for p sufficiently large this is the same as that for the specialisation. Bearing this in mind and that $\partial_S \neq 0$ if the singularities are canonical we take suitable coordinates in characteristic zero such that,

$$\partial_S = \lambda_i\, x_i \, \frac{\partial}{\partial x_i} \quad \text{and} \quad \partial_N \text{ (if it exists)} = f_i \, \frac{\partial}{\partial x_i}\,,\ (f_1, \dots, f_n) \not\equiv 0 \,(\operatorname{mod} \mathfrak{m}^d)$$

where $\mathfrak{m}^d$ is some suitable power of the maximal ideal. Certainly for some e, independent of p, $\partial_N^e(\mathfrak{m}) \subset \mathfrak{m}^2$, so for p sufficiently large to avoid bad reduction, $\partial_N(\mathfrak{m}) \subset \mathfrak{m}^{\left[\frac{p}{e}\right]}$. As ever for p large, we can look at $(*)$ $\operatorname{mod} \mathfrak{m}^3$, and begin by supposing ∂ is not semi-simple $\operatorname{mod} \mathfrak{m}^2$. In particular we have a Jordan block of the form

$$D = \sum_{i=1}^{m} \lambda\, x_i \, \frac{\partial}{\partial x_i} + \sum_{i=k}^{m} x_{i-1} \, \frac{\partial}{\partial x_i}$$

for some sub-system of coordinates $x_1, \dots, x_m, m \geq 2$, and all the $\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ terms $i, j \leq m$ occuring in $*$ $\operatorname{mod} \mathfrak{m}^2$ arise from D. Plainly, therefore, $\lambda = 0$. However there is a non-zero eigenfunction x_{m+1}, say, so the nilpotent part is also zero, as claimed. Continuing to work $\operatorname{mod} \mathfrak{m}^2$, given we have no nilpotent part, the eignevalues considered as a point in projective space must be stable under Frobenius, and so certainly in $\mathbb{F}_p$. Given this for almost all p, the eigenvalues, modulo multiplication by a suitable unit, are indeed rational integers. Making such a multiplication we have $\partial_S^p = \partial_S$ and $\partial_N \wedge \partial_S = 0 \,(\operatorname{mod} \mathfrak{m}^{d+1})$ for almost all p, so in $\mathcal{O}/\mathfrak{m}^d\, \partial$, ∂_N is parallel to ∂_S. Since d was arbitrary, this proves the proposition after multiplication by a suitable formal unit.

Having singularities which are formally linearisable with rational eigenvalues is already a rather strong property, but we can easily go further, for example:

Divertimento II.1.4. Let $(X_0, \mathcal{F}_0)$ be a foliated non-singular surface in characteristic zero such that in the notations of II.1.3, $(X_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})$ is p-closed for all $\mathfrak{m}$ outside a Zariski closed set, then $(X_0, \mathcal{F}_0)$ is given by an infinitesimal formal flat groupoid.

Proof. The issue here is the flatness. Let Δ be the diagonal in $X \times X$, and $\mathcal{P}$ the completion of the latter in the former. Away from the singularities the groupoid is just the formal scheme $\mathcal{F}$, say, obtained by adding in a germ of the foliation in the direction of the second factor. If we can complete $\mathcal{F}$ over the singularities then flatness follows. Whence without loss of generality X is affine, and we're only worried about an isolated singularity z. Now let's say the foliation is given by a vector field ∂ on X, and x, y are generators of $\mathfrak{m}$ at z. We can make affine patches U_x, U_y of $X^* = X \backslash z$ defined as $\partial(x) \neq 0$ and $\partial(y) \neq 0$ respectively. Over $\mathcal{P}_{u_x}$ and $\mathcal{P}_{u_y}$ (fibres via the 1^{st} projection) the groupoid is cut out by divisors D_x and D_y, which a priori do not patch. The problem is that $\mathcal{P}_{X^*}$ has an enormous local Picard group. Call D the class so defined in $\operatorname{Pic}(\mathcal{P}_{X^*})$. Being in dimension 2, if $\mathcal{P}_n$ is the n^{th}-thickening we have an exact sequence,

$$0 \to H^1(X^*, S^n\Omega_X) \to \operatorname{Pic}(\mathcal{P}_{n+1} \otimes \mathcal{O}_{X^*}) \to \operatorname{Pic}(\mathcal{P}_n \otimes \mathcal{O}_{X^*}) \to 0\,.$$

So we need D to be zero in each $\operatorname{Pic}(\mathcal{P}_n \otimes \mathcal{O}_{X^*})$. However for fixed n, and p sufficiently large the obstruction D just specialises to the obstruction to completing the groupoid to order n in characteristic p. However, if we have p-closure and $n < p$ this is trivial – just take the Zariski closure of the inseparable groupoid over the singularities. Having patched D_x, D_y at the bundle level we now have $\mathcal{F}$ over X^* defined as the zero locus of a formal function f on $\mathcal{P}_{X^*}$, and general local cohomology nonsense extends f to all of $\mathcal{P}_X$, as required. $\square$

A few remarks are probably in order,

Remark II.1.5. (a) The proof of II.1.4 works equally well for codimension 1 foliations in general. For non-codimension 1, flatness would require more special singularities, however it ought still to be true that p-closure for almost all p forces the completion of the groupoid in characteristic zero. This shouldn't be too difficult to prove, but one has to compute an obstruction for extending coherent sheaves, which is certainly more fastidious than line bundles.

(b) Morally (modulo the flatness) II.1.3 is equivalent to completing the groupoid. In dimension 2 for example if x_1, x_2 (resp. y_1, y_2) are coordinate eigenfunctions on the 1^{st} and 2^{nd} factors of $X \times X$ and p, q the eigenvalues then the groupoid is cut out by $x_1^q x_2^p - y_1^q y_2^p$. However without knowing II.1.3 at the level of convergent power series, it seems just a bit weaker.

(c) Even p-closure at infinitely many primes is already a strong condition. Indeed II.1.4 goes through verbatim to show that under this hypothesis that the field ∂ is semi-simple. In addition, and rather evidently, p-closure can only take place when the eigenvalues considered as a point of projective space take values in $\mathbb{F}_p$.

As we noted in the sequel to I.1.9, p-closure is true globally iff it is true formally at one point. This is obvious from the definition, what II.1.9 adds to the discussion is that it permits an easy calculation of the p-curvature, i.e. $\partial^p \wedge \partial$. Specifically take a smooth point of the foliation and coordinates $x, y_1, \dots, y_n$ at this point, so that as in op.cit. the foliation is given by,

$$\partial = \frac{\partial}{\partial x} + \sum_{i=1}^{n} x^{p-1}\, f_i(x^p, y)\, \frac{\partial}{\partial y_i}$$

and the subscheme cut out by the vanishing of the p-curvature is that defined by the ideal $(f_1, \dots, f_n)$. In particular it is invariant by the foliation (although this hardly requires special coordinates $\partial(\partial^p y_i) = \partial^p(\partial y_i) = \partial^p(y_j)\, \frac{\partial}{\partial y_j}\,(\partial y_i)$, given $\partial x = 1$ will do). One should be careful however since the reduction of this subscheme may not be invariant, e.g. $f_i = x^p$. In any case, the important point, is that the invariant subscheme defined by the singular locus even when the corresponding reduced subscheme is invariant can more or less be arbitrarily complicated and certainly does not lift to characteristic zero.

II.2. Conditions for p-closure

Truly satisfactory conditions in high dimension require a lot of work in analysing the structure of the subscheme where the p-closure vanishes, so we'll restrict ourselves here to dimension 2, with underlying space X non-singular. We'll use the notations of II.1.3, i.e. $(X_0, \mathcal{F}_0)$ is a foliation in characteristic zero, and $(X, \mathcal{F}) \to \operatorname{Spec} \mathcal{O}$ a model over an affine open subset of a finitely generated extension of $\mathbb{Z}$ with $(X_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})$ the reduction at a maximal ideal. The most obvious condition is,

Lemma II.2.1. *Let D_0 be a nef. divisor on X_0 with $K_{\mathcal{F}_0} \cdot D_0 < 0$ then for p sufficiently large and* $\operatorname{char} k(\mathfrak{m}) = p$, *$(X_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})$ is p-closed.*

Proof. Under the hypothesis $(X_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})$ has good reduction, failure of p-closure gives a global section of $K_{\mathcal{F}_{\mathfrak{m}}}^{p-1} \otimes K_{X_{\mathfrak{m}}}^{\vee}$. Now D_0 is nef., so we can find H_0 ample with $K_{\mathcal{F}_0} \cdot H_0 < 0$, and without loss of generality $H_{\mathfrak{m}}$ is still ample, yet,

$$H_{\mathfrak{m}} \cdot \{(p-1)\, K_{\mathcal{F}_{\mathfrak{m}}} - K_{X_{\mathfrak{m}}}\} = (p-1)\, H_0 \cdot K_{\mathcal{F}_0} - K_{X_0} \cdot H_0 < 0$$

for p sufficiently large, which is absurd. □

Now it would be very much preferable to have a singular variant of II.2.1. More precisely the quotient $\Omega_{X_0} \to K_{\mathcal{F}_0} \cdot I_Z \to 0$ defining the foliation gives a section of $\pi : \mathbb{P}(\Omega_{X_0}) \to X_0$ outside of the singularities which we can complete to a divisor $\widetilde{X}_0$ on $\mathbb{P}(\Omega_{X_0})$. Indeed $\widetilde{X}_0$ is just the blow up of X_0 in I_Z, so it may be singular if Z has nilpotent structure but regardless there is a well defined exceptional divisor E_0 and $L_0 = \pi^* K_{\mathcal{F}_0}(-E_0)$, where L_0 is the tautological bundle on $\mathbb{P}(\Omega_{X_0})$. An optimal singular variant of II.2.1 would be to replace $K_{\mathcal{F}_0}$ by L_0. As we'll see this is false, however it is so in a very subtle way. More precisely we'll content ourselves with looking for something slightly weaker. Specifically for C_0 a curve in X_0 we can take its proper transform $\widetilde{C}_0$ in $\widetilde{X}_0$ and define its Segre class or multiplicity along Z_0, $s_{Z_0}(C_0)$, to be $E_0 \cdot \widetilde{C}_0$. For convenience we'll suppose $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} \mathbb{Z}$ is finite over its image (i.e. everything is defined over a number field) and observe that the definition of Segre class makes sense locally where its just the multiplicity at $z \in Z$ if there is no nilpotent structure, then we have:

Lemma II.2.2. *Notations as above, and suppose $z \in \operatorname{sing}(\mathcal{F})$ is a canonical singularity without nilpotent structure and irrational eigenvalue λ, with $\varepsilon < \min\left\{\frac{1}{H(\lambda)}, 1\right\}$ where $H(\lambda)$ is the big height of λ in the Fubini-Study metric, then if D_0 is effective with its proper transform nef. and $K_{\mathcal{F}_0} \cdot D_0 < \varepsilon \operatorname{ord}_z(D_0)$, $(X_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})$ is p-closed for all maximal ideals $\mathfrak{m}$ of $\mathcal{O}$ such that p splits completely and is sufficiently large.*

Proof. What's at stake is to ask how large is the order of vanishing of the p-closure map $T_{\mathcal{F}}^p \to N_{\mathcal{F}} \cdot I_Z$ around p, where $N_{\mathcal{F}}$ is the normal bundle to the foliation. We make use of I.1.8, and let ∂ be a local generator of $T_{\mathcal{F}}$ about z with $\partial_S \oplus \partial_N$ a Jordan decomposition. Necessarily for p sufficiently large, we have $\partial_S = x \, \frac{\partial}{\partial x} + \lambda y \, \frac{\partial}{\partial y}$ for some suitable coordinates x, y in $\widehat{\mathcal{O}}_{X,z}$, and as ever,

$$\partial^p = \partial_S^p + \partial_N^p = \partial_S + \partial_N^p$$

so the order of vanishing of the p-closure map is the order of vanishing of, $\partial_S \wedge \partial_N^p + \partial_N \wedge \partial_S + \partial_N \wedge \partial_N^p$. Now $\partial_N^p \wedge \partial$ vanishes at least to order p unless $\lambda = 1$ in $k(\mathfrak{m})$ which given it's irrationality we might as well say doesn't happen by taking p sufficiently large.

Consequently we're reduced to looking at $\partial_N \wedge \partial_S$. The nilpotent part is given by the conditions of I.1.8 which are of the form $q_1 \, \lambda + q_2 = 0$ for $q_i \in \mathbb{N} \cup \{0\} \cup \{-1\}$, and the nilpotent part vanishes to order at least $q_1 + q_2 + 1$. Consequently if in standard coordinates $[S, T]$, say, on $\mathbb{P}^1$ we identify λ with $[\lambda, 1]$ and Q with the section $\gamma_Q = q_1 \, S + q_2 \, T$ of the tautological bundle then our condition becomes $\gamma_Q(\lambda) = 0$, and the order of vanishing is at least the big height of γ_Q in the Fubini-Study metric. Even better the pairing $(Q, \lambda) \mapsto \gamma_Q(\lambda)$ just corresponds to intersecting $\lambda \times Q$ with respect to the diagonal on $\mathbb{P}^1 \times \mathbb{P}^1$ in the Arakelov sense, and we have:

$$h(\lambda) + h(Q) = \frac{1}{(\mathcal{O}:Z)} \left\{ \sum_{\mathfrak{m}} \mathrm{ord}_{\mathfrak{m}}(\gamma_Q(\lambda)) \log(\# k(\mathfrak{m})) \right.$$
$$\left. - \sum_{k(\mathcal{O}) \underset{\sigma}{\hookrightarrow} \mathbb{C}} \frac{1}{2} \log \left(\frac{|q_1 \lambda^\sigma + q_2|^2}{(|q_1|^2 + |q_2|^2)(1 + |\lambda^\sigma|^2)} \right) \right\}.$$

Since λ is irrational, $\Lambda \cdot Q$ is not identically zero, whence if $\lambda q_1 + q_2 = 0$ and p splits completely we have

$$p \leq H(\lambda)\, H(Q)$$

which tells us that the nilpotent part vanishes to order at least $p/H(\lambda)$, so that in turn the p-curvature at z vanishes to order at least $p \min\left\{1, \frac{1}{H(\lambda)}\right\} > p\,\varepsilon$. Shrinking ε a little as necessary we may assume $\widetilde{D}_{\mathfrak{m}}$ is ample, and as Q divisors $p\,K_{\mathcal{F}} + N_{\mathcal{F}} - p \min\left\{1, \frac{1}{H(\lambda)}\right\} E_z$, E_z being the exceptional divisor over z, is effective which enjoys the absurdity of intersecting $\widetilde{D}_{\mathfrak{m}}$ negatively for p sufficiently large. □

Obviously the condition of p being totally split wasn't really necessary, the weaker condition $k(\mathfrak{m}) = \mathbb{F}_p$ would have worked equally well provided we took the worse bound of, $\min\left\{\frac{1}{(\mathcal{O}:Z)\, H(\lambda)}, 1\right\}$. What is essential in the proof is that λ is irrational. One might hope that rational is easier to treat, which is certainly true if the singularity were linearisable in characteristic zero, then we could actually get the lemma for $\varepsilon = 1 - \delta$, any $\delta > 0$, but if it isn't the p-closure map will only have a finite order at z independent of p, and the lemma appears hopeless. Nevertheless the lemma casts a lot of light on some examples discovered by Ekdhal, Shepherd-Baron, and Taylor, [ESB], of which the most interesting from our point of view is,

Example II.2.3. (Hilbert Modular Surfaces) Let $\mathcal{O}$ be the ring of integers in a real quadratic number field. The group $\Gamma = \mathrm{SL}_2(\mathcal{O})$ acts on two copies of the upper half plane $\mathfrak{H} \times \mathfrak{H}$ by way of,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) \mapsto \left(\frac{a\,z_1 + b}{c\,z_1 + d}, \frac{\overline{a}\,z_2 + \overline{b}}{\overline{c}\,z_2 + \overline{d}} \right)$$

where bar denotes Galois conjugation in $\mathcal{O}$. Modulo ± 1, the action is properly discontinuous and the resulting surface $\mathfrak{H} \times \mathfrak{H}/\Gamma$ is quasi-projective (Hilbert-Modular forms give the quasi-projective embedding) with at worst quotient singularities. Furthermore we can complete things to a projective surface S_0 by adding in finitely many cusps which are elliptic Gorenstien singularities resolved by elliptic polygons or a rational curve with a node. The surface S_0 is called the Baily-Borel compactification, and its minimal desingularisation $\rho : S \to S_0$ is what is usually termed a Hilbert-Modular surface. The foliations determined by the splitting descend to foliations $\mathcal{F}_0, \mathcal{G}_0$ on S_0. Neither is $\mathbb{Q}$-foliated Gorenstien in the presence of cusps, cf. IV.2.2, but they do have canonical singularities in the sense of I.2.10. There is a splitting $\mathcal{T}_{S_0} = \mathcal{T}_{\mathcal{F}_0} \oplus \mathcal{T}_{\mathcal{G}_0}$, and even better if D is the reduced simple normal crossing divisor contracted by ρ then it is invariant by the induced foliations $\mathcal{F}$ and $\mathcal{G}$, and both of these not only have canonical singularities but,

$$\Omega_S(\log D) = K_{\mathcal{F}} \oplus K_{\mathcal{G}} .$$

In particular $K_{\mathcal{F}}, K_{\mathcal{G}}$ are pseudo-effective divisors of numerical Kodaira dimension 1 and Kodaira dimension $-\infty$ (essentially because any section would force the foliation to be a fibration, cf. IV.5.1) so that for any $\varepsilon > 0$ there is an ample divisor H_ε on the blow up of X in any singularity z, say, of $\mathcal{F}$ such that $K_{\mathcal{F}} \cdot H_\varepsilon < \varepsilon \operatorname{ord}_z(H_\varepsilon)$ (otherwise $K_{\mathcal{F}}$ would have a section cf. IV.5.7). Now consider the case where z lies on a polygon. The singularities are exactly the crossings of the elliptic polygon, and within a polygon the eigenvalue at any one singularity can be calculated in terms of any other via the residue-formula I.3.1. Actually we're in the degenerate case of I.3.2 where the number of curves is exactly the number of singularities, and the eigenvalues may be computed as continued fractions in the intersection matrix, cf. IV.2.2. In any case the eigenvalues are thus in one to one correspondence with the non-principal ideal classes of $\mathcal{O}$, and are certainly irrational provided S is singular. Whence combining II.2.2 with II.1.3 we have for p sufficiently large,

$$(X, \mathcal{F}) \text{ is } p\text{-closed iff } p \text{ splits completely in } \mathcal{O} .$$

In particular, [EST], p closure for infinitely many p does not imply algebraicity of the leaves in characteristic zero. □

II.3. Adjunction

In the sequel to II.2.1 we introduced the Segre class of the foliation along the singular subscheme Z. Let's just note it more explicitly by way of a definition, viz:

Definition II.3.1. *Let $\Omega_X \to K_{\mathcal{F}} \cdot I_Z \to 0$ be a foliation by curves on a non-singular variety X, and $\widetilde{X}$ in $\pi : \mathbb{P}(\Omega_X) \to X$ the closure of the corresponding section over the smooth locus, then $\widetilde{X} \xrightarrow{\sim} \mathrm{Bl}_Z(X)$ (understood with nilpotent structure) and if E is the exceptional divisor and L the tautological bundle then $L = \pi^* K_{\mathcal{F}}(-E)$ and for C in X a curve we define the Segre class $s_Z(C)$ of Z along C to be $E \cdot \widetilde{C}$ where $\widetilde{C}$ is the proper transform.*

Now adjunction for $(X, \mathcal{F})$ a foliation by curves on a non-singular surface takes two forms either:

(a) C is a curve not invariant by the foliation and we have a natural map,

$$\begin{array}{ccc} \mathcal{O}_C(-C) & \longrightarrow & \Omega_X \otimes \mathcal{O}_C \\ & & \downarrow \\ & & K_{\mathcal{F}} \otimes \mathcal{O}_C \end{array}$$

so $(K_{\mathcal{F}} + C) \cdot C \geq 0$. In particular $K_{\mathcal{F}} \cdot C < 0$ implies $C^2 > 0$, so C is big and nef.

(b) C is invariant by the foliation, and $\nu : \widetilde{C} \to C$ is its normalisation then,

$$K_{\mathcal{F}} \cdot C = (2g(\widetilde{C}) - 2) + s_Z(C) - \mathrm{Ram}_{\widetilde{C}/C} \, .$$

On the other hand $s_Z(C) - \mathrm{Ram}_{\widetilde{C}/C} \geq 1$ if it's non-zero so if $K_{\mathcal{F}} \cdot C < 0$ then C is a rational curve (smooth if the singularities are canonical) with $s_Z(C) = 0$ or 1, giving $K_{\mathcal{F}} \cdot C = -2$ or -1 respectively. In the former case by I.3, the normal bundle $N_{\mathcal{F}}$ is flat along C, and the foliation is a (possibly singular) fibration in rational curves.

Tying things together, particularly in the context of II.2.1, we have:

Lemma II.3.2. (characteristic zero) *Let $(X, \mathcal{F})$ be a foliated surface with X non-singular and C a curve on which $K_{\mathcal{F}} \cdot C < 0$ then either,*

(a) *C is not invariant, and nef.*

(b) *C is an invariant rational curve with $K_{\mathcal{F}} \cdot C \in \{-1, -2\}$.*

II.4. Miyaoka's semi-positivity theorem

We now arrive at the main application of p-closure which by virtue of II.3 we'll render a little tidier, but in all essentials is due to Miyaoka, with the following proof being taken from Shepherd-Baron's simplification of the original arguement, cf. [S-B].

Proposition II.4.1. *Let $(X, \mathcal{F})$ be a foliation by curves enjoying canonical singularities on a non-singular surface in characteristic zero with M a nef. $\mathbb{R}$-divisor and C a curve with $K_{\mathcal{F}} \cdot C < 0$ then for every $x \in C$ there is a rational curve $L_x \ni x$ invariant by $\mathcal{F}$ with $K_{\mathcal{F}} \cdot L_x \geq -2$ and,*

$$M \cdot L_x \leq 4 \, \frac{M \cdot C}{-K_{\mathcal{F}} \cdot C} \, .$$

Proof. Firstly suppose C is invariant, then we take $L_x = C$, and we're done by II.3.1(b). So we're reduced by II.3.1(a) to C non-invariant and nef. We proceed in stages:

(a) H is an ample divisor and we look to get the slightly worse bound of $4+\varepsilon$ for any $\varepsilon > 0$. By II.2.1 we know that suitable specialisations of $(X, \mathcal{F})$ are p-closed for p sufficiently large. Abusing notation and continuing to use $(X, \mathcal{F})$ for the specialisation we know that the quotient $X/\mathcal{F}$ exists, and we let $\pi : Y \to X/\mathcal{F}$ be a minimal resolution of its singularities. Necessarily for some, possibly empty, effective contractible divisor E we have,

$$\pi^* K_{X/\mathcal{F}} = K_Y + E\,.$$

Now let D be the image in $X/\mathcal{F}$ of C, and $\widetilde{C}$ its proper transform in Y then we obtain,

$$K_Y \cdot \widetilde{D} \leq (p-1)\, K_{\mathcal{F}} \cdot D + K_X \cdot D$$

so that by [K-3], 4.13 and 3.2, there is a rational curve M_x through x such that,

$$G.\, M_x \leq \frac{4p\, H.\, C}{-(p-1)\, K_{\mathcal{F}} \cdot C - K_X \cdot C}$$

where G is the pull-back to $X/\mathcal{F}$ of the conjugate $H^{(1)}$ of H. Now let L_x be the pre-image of M_x on X, which is certainly rational, and suppose $\rho : X \to X/\mathcal{F}$ on restriction to L_x is separable then,

$$H.\, L_x = \frac{1}{p}\, G.\, M_x < 1\,, \text{ for } p \gg 0\,.$$

Consequently for $p \gg 0$ we obtain a rational integral sub-curve L_x of $\mathcal{F}$, i.e. $\rho\,|_{L_x}$ is inseparable, through x with,

$$H.\, L_x \leq (4+\varepsilon)\, \frac{H.\, C}{-K_{\mathcal{F}} \cdot C}$$

and the result follows by the representability of the Hilbert-scheme. Everything else is now formal as in [K1], pg. 146, i.e.

(b) Given (a) we know the proposition for any $4+\varepsilon$, $\varepsilon > 0$, and H an ample $\mathbb{Q}$-divisor. This immediately gives the proposition for $4+\varepsilon$, and M an ample $\mathbb{R}$-divisor. Now suppose M is simply nef., and H is ample, then for any $\delta > 0$, $M + \delta H$ is an ample $\mathbb{R}$-divisor and for $\varepsilon > 0$ we find $L_{x,\varepsilon,\delta}$ with,

$$(M+\delta H) \cdot L_{x,\varepsilon,\delta} \leq (4+\varepsilon)\, \frac{M \cdot C}{-K_{\mathcal{F}} \cdot C} + (4+\varepsilon)\, \delta\, \frac{H \cdot C}{-K_{\mathcal{F}} \cdot C}\,.$$

If for some $\varepsilon, \delta > 0$, $M \cdot L_{x,\varepsilon,\delta} \leq 4\, \frac{M \cdot C}{-K_{\mathcal{F}} \cdot C}$ then we're done. Otherwise,

$$H \cdot L_{x,\varepsilon,\delta} < \frac{\varepsilon}{\delta}\, \frac{M \cdot C}{-K_{\mathcal{F}} \cdot C} + (4+\varepsilon)\, \frac{H \cdot C}{-K_{\mathcal{F}} \cdot C}$$

for all $\varepsilon, \delta > 0$, in particular $\varepsilon = \delta = \frac{1}{n}$, $n \in \mathbb{N}$, say, with $L_{x,\varepsilon,\delta} = L_{x,n}$ of degree bounded independently of n. Whence the $L_{x,n}$ live in finitely many families, and for infinitely many n, $L_{x,n}$ is in exactly one of these, so its intersection with any $\mathbb{R}$-divisor is independent of n, and we obtain II.4.1. □

Proposition II.4.1 for arbitrary algebraic spaces will involve a more careful adjunction formula which we postpone until Chapter III. However we have proved,

Corollary II.4.2. *Let $(X, \mathcal{F})$ be a foliation on a 2-dimensional algebraic space with $K_{\mathcal{F}} \cdot D < 0$ for some nef. $\mathbb{R}$-divisor D. Then for any nef. $\mathbb{R}$-divisor M and $x \in X$ there is a rational integral sub-curve L_x through x, such that,*

$$M. L_x \leq 4 \frac{M. D}{-K_{\mathcal{F}} \cdot D}$$

where the above intersections are understood in the sense of Mumford.

Proof. For a general algebraic space take $p : (Y, \mathcal{G}) \to (X, \mathcal{F})$ a smooth projective resolution with canonical singularities then p^*D and p^*M are still nef. $\mathbb{R}$-divisors, and better still,

$$K_{\mathcal{G}} \cdot p^*D = K_{\mathcal{F}} \cdot D < 0$$

from which the corollary follows. □

The other important, and indeed formal corollary of II.4.1 is the cone theorem, i.e.

Corollary II.4.3. (Cone theorem) *Let $(X, \mathcal{F})$ be a foliation on a non-singular algebraic surface then there are finitely many invariant rational curves L_i, with $K_{\mathcal{F}} \cdot L_i \in \{-1, -2\}$ (the latter occuring only if the foliation is a pencil of rational curves) such that,*

$$NE_1(X)_{\mathbb{R}} = NE_{K_{\mathcal{F}} \geq 0}(X) + \sum_i \mathbb{R}_+ [L_i]$$

where $NE_1(X)_{\mathbb{R}}$ is the cone of effective curves, and $NE_{K_{\mathcal{F}} \geq 0}$ the upper half space on which $K_{\mathcal{F}}$ is positive. Even better every $K_{\mathcal{F}}$ negative extremal ray in NE is of the form $\mathbb{R}_+[L_i]$, for some i.

Proof. (Following [K1] pg. 160.) Consider the rational curves with $K_{\mathcal{F}} \cdot L_i < 0$. If there are infinitely many of these by a theorem of Jouanalou, [J], the foliation is a rational pencil. In this latter case there are only finitely many possibilities for a class $[L_i]$ in $NE_1(X)$ represented by an invariant rational curve. Whence the right hand side is a closed cone even if we take all invariant rational curves with $K_{\mathcal{F}} \cdot L_i < 0$. Now suppose the left and right hand sides are different then there is a nef. $\mathbb{R}$-divisor M strictly positive on the right hand side off 0 and $z \in NE_1(X)_{\mathbb{R}} \backslash \{0\}$ with $M \cdot z = 0$. Scaling we might as well say that $M \cdot \alpha \geq 1$ for α the class of any effective curve on the right, and $K_{\mathcal{F}} \cdot z < 0$. Now write z as a limit of effective cycles, $C_i = \sum a_{ij} C_{ij}$, $a_{ij} \in \mathbb{R}_+$, then for i sufficiently large there is a $j = j(i)$ such that $-K_{\mathcal{F}} \cdot C_{ij} > 0$ and,

$$\frac{M \cdot C_{ij}}{-K_{\mathcal{F}} \cdot C_{ij}} \leq \frac{M \cdot C_i}{-K_{\mathcal{F}} \cdot C_i}$$

now we can apply II.4.1 to find an invariant rational curve L_{x_i} with $K_{\mathcal{F}} \cdot L_{x_i} \geq -2$ and,

$$M \cdot L_{x_i} \leq 4 \, \frac{M \cdot C_i}{-K_{\mathcal{F}} \cdot C_i}$$

the limit as $i \to \infty$, is zero so we find a curve L_{x_i} with $M \cdot L_{x_i} \leq 1/2$, say, so we have the absurdity that L_{x_i} is not in the right hand side. $\square$

III. Minimal models

III.0. $K_{\mathcal{F}}$ vs. K_X

Consider the situation of the singularity in example I.2.1. This gives a foliated surface $(X, \mathcal{F})$ together with a modification $p : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ obtained by blowing up the singularity with exceptional divisor E, say, such that $K_{\widetilde{\mathcal{F}}} \cdot E = 1$. Hence as noted this singularity is K_X-terminal but in an adaptation of the usual terminology it is $K_{\mathcal{F}}$-log canonical. Evidently we have no desire to blow such curves down if we wish to construct a minimal model, and so throughout this section we make the following blanket assumption,

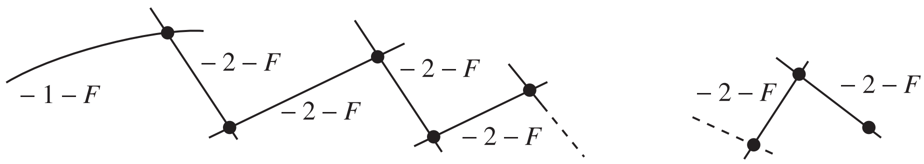
Hypothesis III.0.1. All foliated surfaces $(X, \mathcal{F})$ appearing in this section will be assumed to have canonical ($K_{\mathcal{F}}$) singularities. Any precisions on the singularities of X will be made according to the situation. For simplicity we will suppose that $(X, \mathcal{F})$ is not a fibration by rational curves.

Now we need a few definitions, viz,

Definitions III.0.2. *Let $(X, \mathcal{F})$ be a foliation on a 2-dimensional algebraic space and C a curve contained in the smooth locus of X, whence we may define à la* II.1 *the Segre class $s_{\mathrm{sing}(\mathcal{F})}(C)$ of C along any singularities of $\mathcal{F}$, including any possible nilpotent structure. Based on this we say,*

(a) *C is said to be a $-1\,F$-curve (respectively $-2\,F$-curve) if C is a smooth rational integral sub-curve of $\mathcal{F}$ with $s_{\mathrm{sing}(\mathcal{F})}(C) = 1$ (respectively 2).*

(b) *A -1-chain is a chain of smooth rational integral sub-curves of $\mathcal{F}$ contained in the smooth locus of X of the form,*



Following standard usage the $-1\,F$-curve at the beginning of the chain will be called the handle. The components will be labelled E_i, $i \geq 1$, in the natural way starting from the handle.

(c) *If there exists an integral sub-curve of $\mathcal{F}$ not contained in a -1-chain but which nevertheless meets the chain then it is called the tail. By* I.2.7 *if it exists then such a curve is unique.*

(d) *A -1-chain is said to be artificial if it may be contracted to a non-singular point of the underlying space X. If moreover this point is actually a non-singular point of the foliation then we say the chain is very artificial.*

Some remarks are necessarily in order. In light if II.4.1, C is a $-1\,F$-curve if and only if $K_{\mathcal{F}} \cdot C = -1$. Not surprisingly they are the correct analogue of $-1\,F$-curves in the usual sense. On the other hand $-2\,F$-curves are a little more subtle, certainly they satisfy $K_{\mathcal{F}} \cdot C = 0$, however, not only is the converse false (consider a foliation arising from a product of curves) but a $-2\,F$-curve need not be contractible (cf. for example IV.4.6 (ii)).

Finally we introduce a class of curves which do not properly have an analogue in the usual Enriques-Kodaira classification theory (although obviously they appear in a log-variant) i.e.

Definition III.0.3. *Let $(X, \mathcal{F})$ be an algebraic space, a possibly reducible but reduced curve Z contained in the smooth locus of X is said to be an elliptic Gorenstien leaf (e.g.l.) if it consists of a contractible cycle of $-2\,F$-curves or alternatively a contractible rational integral sub-curve with a simple node at a point of* $\mathrm{sing}(\mathcal{F})$ *without nilpotent structure. If it contains no curves of self-intersection -1, then we say additionally that Z is minimal.*

The terminology is easily justified since by I.2.6 a cycle of $-2\,F$-curves or a rational integral sub-curve with a simple node is certainly a leaf. On the other hand after a series of smooth contractions we can easily assume that our e.g.l. Z is minimal, and of course has euler characteristic zero. Consequently Z is the fundamental cycle of a contraction to an elliptic Gorenstien singularity. Now the unfortunate thing is (cf. IV.2.2) is that Z will not be contractible to a foliated Gorenstien singularity. The following is standard,

Fact III.0.4. Let $\pi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ denote the contraction in the category of algebraic spaces of a minimal e.g.l. Z to an elliptic Gorenstien singularity p then Y is $\mathbb{Q}$-foliated Gorenstien at p iff $K_{\mathcal{F}} \mid_Z$ is torsion.

Proof. We write $Z = \sum_i Z_i$, and observe $K_{\mathcal{F}} \cdot Z_i = 0 \; \forall\, i$. The only if part is clear, so all we need do is calculate $\pi_* \, K_{\mathcal{F}}^{\otimes n}$ for any $n \in \mathbb{N}$ with $\mathcal{O}_Z(n\, K_{\mathcal{F}}) \xrightarrow{\sim} \mathcal{O}_Z$. On the other hand the theorem of formal functions yields,

$$\pi_* \, K_{\mathcal{F}}^{\otimes n} = \varprojlim_k H^0(\mathcal{O}_{Z^{(k)}}, K_{\mathcal{F}}^{\otimes n})$$

where $Z^{(k)}$ is the k-th infinitesimal neighbourhood of Z. Equally we have of course, an exact sequence,

$$H^0(\mathcal{O}_{Z^{(k+1)}}, K_{\mathcal{F}}^{\otimes n}) \to H^0(\mathcal{O}_{Z^{(k)}}, K_{\mathcal{F}}^{\otimes n}) \to H^1(\mathcal{O}_Z\,(n\, K_{\mathcal{F}} - k\, Z))$$

where the final group is isomorphic to $H^0(\mathcal{O}_Z\,(k\, Z)) = 0$, and so we conclude $\pi_* \, K_{\mathcal{F}}^{\otimes n}$ has a nowhere vanishing section in a neighbourhood of p as required. □

III.1. Algebraic spaces and local theory

We work first of all in the category of algebraic spaces in order to compute what we must necessarily contract later in the algebraic category. We will freely use Mumford's intersection theory, and all identities involving "divisors" should therefore be understood modulo numerical equivalence in this sense.

Evidently a generalisation of the proof of II.4.1 will be required, and so we begin with,

Lemma III.1.1. *Let $p : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ be a map between foliated 2-dimensional normal algebraic spaces contracting an irreducible curve E, and let C in X be a curve with $\widetilde{C}$ its proper transform then,*

$$(K_{\mathcal{F}} + C).\, C = (K_{\widetilde{\mathcal{F}}} + \widetilde{C}).\, \widetilde{C} + \frac{(\widetilde{C}.\, E)}{(-E^2)} \{(\widetilde{C} + K_{\widetilde{\mathcal{F}}}).\, E\} .$$

Proof. Apply the definition of Mumford's intersection theory. □

Now let's get underway, and suppose $(X, \mathcal{F})$ is as in III.0.1 with the underlying space X non-singular, and let E be curve with $K_{\mathcal{F}}.\, E < 0$. As we have seen in II.4.1 it must be a $-1\, F$-curve, and better still by II.4.1 it is contractible. We wish to contract it and see what happens. The result may be best summarised by,

Proposition III.1.2. *Let $(X, \mathcal{F})$ be as above and $(Y, \mathcal{G})$ a foliation on a normal 2-dimensional algebraic space obtained from $(X, \mathcal{F})$ by contracting a -1-chain with C a curve in Y whose proper transform $\widetilde{C}$ meets the chain and $K_{\mathcal{G}}.\, C < 0$ then $\widetilde{C}$ is a $-2\, F$-curve which is in fact the tail of the given chain and as such fits into a new -1-chain which is itself contractible.*

Proof. We make use of the intermediary space obtained by contracting all but the last curve in the chain. Let's call this $(Z, \mathcal{H})$ and let $C^{\#}$ be the proper transform of C in Z, then by III.1.1 we have on denoting by E the push-forward to Z of the last curve in the chain,

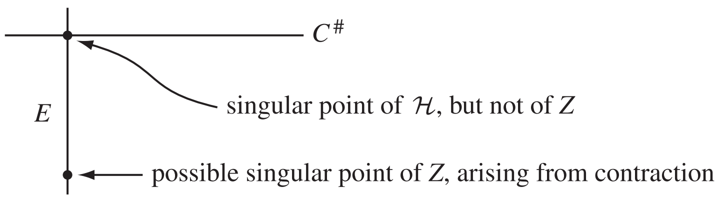
$$(K_{\mathcal{G}} + C).\, C = (K_{\mathcal{H}} + C^{\#}).\, C^{\#} + \frac{(C^{\#}.\, E)}{(-E^2)} \{(C^{\#} + K_{\mathcal{H}}).\, E\} .$$

Our aim is to show that if C is not an integral sub-curve of $\mathcal{G}$, then,

$$(K_{\mathcal{G}} + C).\, C \geq 0 .$$

This will easily follow by induction if we can show that the hypothesis $C^{\#}.\, E \neq 0$ implies $(C^{\#} + K_{\mathcal{H}}).\, E \geq 0$. This latter statement is itself however immediate by induction on the length of the chain. Consequenlty if C is not an integral sub-curve of $\mathcal{G}$ then its pull-back to X is a nef. $\mathbb{Q}$-Cartier divisor whose intersection with $K_{\mathcal{F}}$ is that of C with $K_{\mathcal{G}}$ which is negative, and so absurd by II.4.2.

Now given C is an integrable sub-curve of $\mathcal{G}$ meeting the chain, by I.2.6 this can only happen in the following way, i.e. in Z we have a picture,



Not surprisingly adjunction for $\mathcal{H}$ yields,

$$K_{\mathcal{H}} \cdot C^{\#} = K_{\mathcal{F}} \cdot \widetilde{C} = 2g(\widetilde{C}) - 2 + s_{\mathrm{sing}(\mathcal{H})}(C^{\#})$$

while on the other hand,

$$0 > K_{\mathcal{G}} \cdot C = K_{\mathcal{H}} \cdot C^{\#} + \frac{(K_{\mathcal{H}} \cdot E)}{(-E^2)}.$$

Better still repeated application of the latter formula shows that if A, say, is the intersection matrix of the chain used to obtain Z, then,

$$K_{\mathcal{H}} \cdot E = \frac{-1}{\det(-A)}$$

since $\det(-A)$ is a positive integer we must therefore have, $K_{\mathcal{F}} \cdot \widetilde{C} \leq 0$. Given that $\widetilde{C}$ meets $\mathrm{sing}(\mathcal{F})$ this can only happen if it is a -1 F-curve or a -2 F-curve. However the former possibility contradicts I.3.2, and II.4.2 and we're done. □

Remarks III.1.3. The proof actually establishes a number of additional useful facts. Namely let $(X, \mathcal{F})$, $(Y, \mathcal{G})$ be as given with $\widetilde{C}$ the tail of the -1-chain if it exists, C its push-forward to Y, and A the intersection matrix of the chain, then,

(a) $K_{\mathcal{G}} \cdot C = K_{\mathcal{F}} \cdot \widetilde{C} - \frac{1}{\det(-A)}$.

(b) If $\widetilde{C}$ is a -2 F-curve then $K_{\mathcal{G}} \cdot C < 0$ and we have a -1-chain obtained by adding $\widetilde{C}$ to the original chain which is itself contractible.

(c) Any two -1-chains of maximal length are disjoint, since otherwise we would have the situation eliminated in the proof, i.e. a tail which is a -1 F-curve.

(d) A -1-chain is contractible.

Finally let us note a fact which will ultimately prove rather useful, viz:

Proposition III.1.4. *A contraction of a -1-chain containing no exceptional curves of the 1^{st} kind inside a foliated surface $(X, \mathcal{F})$ with underlying space X non-singular cannot be foliated Gorenstien.*

Proof. Let $\pi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be the contraction of the chain supported on $E_1, \ldots, E_n$ say, and suppose $K_{\mathcal{G}}$ is foliated Gorenstien. A priori we know there are $a_i \in \mathbb{Q}_{>0}$ such that,

$$K_{\mathcal{F}} = \pi^* K_{\mathcal{G}} + \sum_i a_i E_i$$

as $\mathbb{Q}$-Cartier divisors. Now if $K_{\mathcal{G}}$ is Cartier then,

$$a_1 = K_{\mathcal{F}} \cdot \pi^* K_{\mathcal{G}} - K_{\mathcal{F}}^2 \in \mathbb{Z}$$

furthermore,

$$-1 = a_1\, E_1^2 + a_2$$

so in addition $a_2 \in \mathbb{Z}$, and one sees easily by induction that indeed all the $a_i \in \mathbb{Z}$. On the other hand if d is the discriminant of the negative of the intersection matrix of the chain then, $a_n = 1/d$, which is absurd. $\Box$

III.2. Contractions

We begin with $(X, \mathcal{F})$ as in III.0.1 with underlying space X non-singular. If $K_{\mathcal{F}}$ is nef. we do nothing. Otherwise there is a -1-chain. By the aforesaid result of Jouanalou, or indeed a usual Néron-Severi type finiteness argument, any -1-chain is contained in one of maximal length, and as observed these are all disjoint. Should there be a -1-chain which is artificial then we contract it, and so inductively obtain a foliated surface $(X', \mathcal{F}')$ obtained by contracting on $(X, \mathcal{F})$ with X' non-singular which contains no artificial -1-chains. Thus if $K_{\mathcal{F}'}$ is not nef. there is a finite disjoint union of maximal -1-chains none of which is artificial. A priori these can be contracted simultaneously to normal singularities on an algebraic space, however since they are chains of rational curves it is standard, cf. [A], that the unique space $(Y, \mathcal{G})$ obtained by this contraction satisfies,

(a) Y has rational singularities. Indeed they are log-terminal.

(b) Y is a $\mathbb{Q}$-factorial projective variety, and so all previous calculations in Mumford intersection theory remain valid in the Picard group with $\mathbb{Q}$-coefficients.

Evidently $(Y, \mathcal{G})$ has canonical singularities, and so we require to prove,

Proposition III.2.1. *With $(Y, \mathcal{G})$ as above, $K_{\mathcal{G}}$ is nef.*

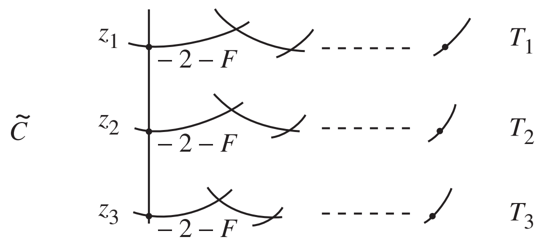
Proof. Without loss of generality, we may, in the above notations, replace $(X, \mathcal{F})$ by $(X', \mathcal{F}')$. We let C be a curve in Y with $K_{\mathcal{G}} \cdot C < 0$ and $\widetilde{C}$ its proper transform in X. A minor variation of the argument in III.1.2 reduces us to the case that C is an integral sub-curve of $\mathcal{G}$. Moreover if C is not the tail of any of the -1-chains that we have contracted then $(X, \mathcal{F})$ is isomorphic to $(Y, \mathcal{G})$ in a neighbourhood of C and so $K_{\mathcal{G}} \cdot C = K_{\mathcal{F}} \cdot \widetilde{C} \geq 0$ which is absurd. Whence let $T_1, \ldots, T_k$ be the maximal -1-chains which have $\widetilde{C}$ as their tail and $d_1, \ldots, d_k \in \mathbb{N}$ the determinants of the negative of the intersection matrices of the appropriate chains, then we have:

$$K_{\mathcal{G}} \cdot C = K_{\mathcal{F}} \cdot \widetilde{C} - \sum_{i=1}^{k} \frac{1}{d_i}\,.$$

Better still none of the chains are artificial, and so, $K_{\mathcal{G}} \cdot C \geq K_{\mathcal{F}} \cdot \widetilde{C} - k/2$. On the other hand we have by adjunction,

$$K_{\mathcal{F}} \cdot \widetilde{C} = (2g\,(\widetilde{C}) - 2) + s_{\mathrm{sing}(\mathcal{F})}\,(\widetilde{C}) \geq 2g\,(\widetilde{C}) - 2 + k\,.$$

Consequently C is a rational curve, and $1 \leq k \leq 3$. Should $k = 1$, then by III.1.2 $\widetilde{C}$ is a $-2\,F$-curve which fits into a new -1-chain so our given chain would not be maximal, contrary to assumption. Should $k = 2$, then in fact $\widetilde{C}$ is still a $-2\,F$-curve, and so we are reduced to considering the case $k = 3$. In this case $s_{\mathrm{sing}(\mathcal{F})}(\widetilde{C}) = 3$ and since maximal -1-chains are disjoint we must have the following picture,



Since the singularities Z_j, of the foliation, give local Segre classes of 1 for both $\widetilde{C}$ and the chain they cannot carry any nilpotent structure, so again by I.3.2, we must have $C^2 \geq 0$ contrary to II.4.2. □

Remark III.2.2. It is worth extending the above discussion to consider the possibility of an integrable sub-curve C whose proper transform meets several maximal non-artificial chains and such that $K_{\mathcal{G}} \cdot C = 0$. In the above notation, therefore, $k \leq 4$. The case $k = 1$ is impossible, while $k = 3$ or 4 imply by an identical argument to the above that $C^2 \geq 0$.

As hinted therefore the case $k = 2$ merits further discussion. In the notations of the proof this can only happen if $d_1 = d_2 = 2$. Consequently by systematically contracting rational curves of self-intersection -1 appearing in either chain, we may replace $(X, \mathcal{F})$ by a smaller model in which not only is the underlying space still non-singular but both the chains in question each consist of a single curve with self-intersection -2.

Two situations are possible. The easy one is if the intersection of $\widetilde{C}$ with the singular locus of $\mathcal{F}$, although 3, due to nilpotent structure on $\mathrm{sing}(\mathcal{F})$ is supported at only 2 points. In this case the union of the 3-curves T_1, $\widetilde{C}$, T_2 form a leaf of $\mathcal{F}$, and as such cannot meet any other integral sub-curve of the foliation. This leaf is contractible to a rational $\mathbb{Q}$-factorial singularity if and only if $C^2 < 0$. Let us note this with a name, viz,

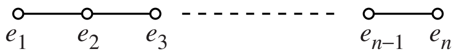
Definition III.2.3. (a) *A curve $\widetilde{C}$ of the form discussed above is said to be an easy tail.*

The next case is the more delicate, i.e. the singular locus of $\mathcal{F}$ meets $\widetilde{C}$ in 3-distinct points, and so there could be an integrable sub-curve meeting $\widetilde{C}$ in the point not determined by the -1-chains. The one saving grace is that $C^2 = \widetilde{C}^2 + 1$, and so is an integer. Not surprisingly we add to the above definition,

(b) *A curve $\widetilde{C}$ of the above form is said to be a bad tail.*

III.3. Proof of minimal model theorem

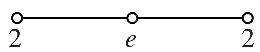
Keeping to the notations of the previous section we start with a foliated surface $(X, \mathcal{F})$ containing no artificial -1-chains, and contract all non-artificial -1-chains to obtain a foliated surface $(Y, \mathcal{G})$ with canonical singularities. By III.1.3 no $-2\,F$-curve can meet a non-artificial chain. Better still the set of $-2\,F$-curves is not only finite, but the union of its members is a disjoint union whether of chains, or cycles and indeed there may be rational integral sub-curves with a node. Regardless concentrating on the first fact we may freely contract $-2\,F$-curves with self-intersection -1. This step may be carried out in either X or Y without changing $K_{\mathcal{F}}$ or $K_{\mathcal{G}}$. The former is perhaps preferable, so let us add to our hypothesis that in X there are no $-2\,F$-curves with self-intersection -1. We proceed to represent the remaining chains or cycles of $-2\,F$-curves in the usual way by their dual graphs, for example:



putting weights on each node according to the negative of the self-intersection number of the corresponding rational curve. For such graphs, or indeed any connected sub-graph, we may form the associated quadratic form, which in light of the above conventions we call contractible if it is positive definite. Since the possibility of a node of weight one is excluded if C is a curve corresponding to a positive node then it is contained in a unique maximal contractible sub-chain or cycle.

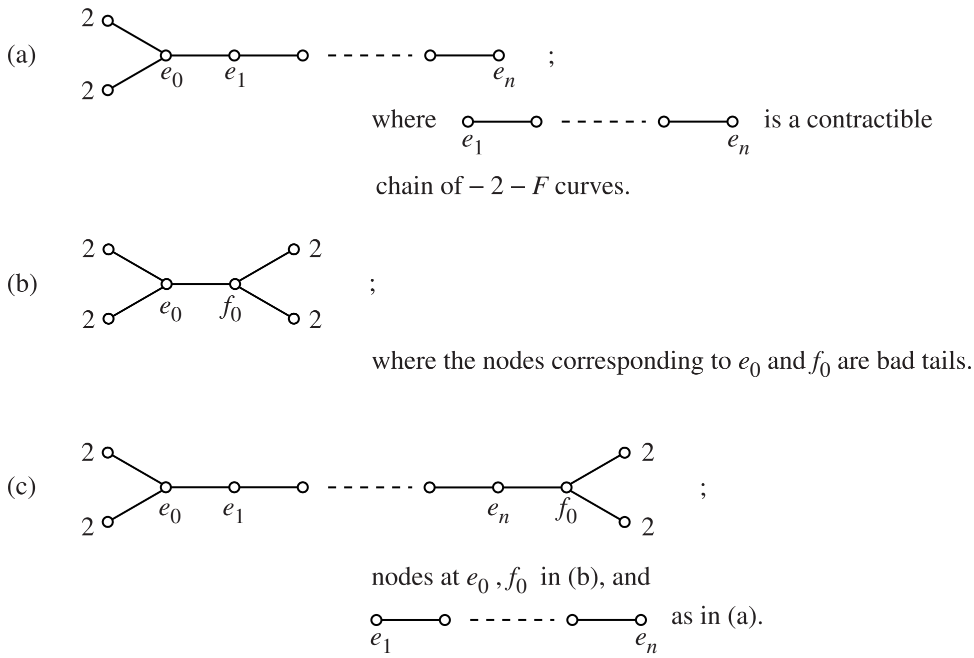
With this in mind we now contract once more all -1-chains in $(X, \mathcal{F})$ to get $(Y, \mathcal{G})$ with $K_{\mathcal{G}}$ nef. As ever the appropriate variation of III.1.1 guarantees the property that if C is not an integrable sub-curve and $K_{\mathcal{G}} \cdot C = 0$ then $C^2 \geq 0$, while if C is not a tail but nevertheless an integral sub-curve the property holds unless C is a $-2\,F$-curve or an e.g.l. supported on a rational curve with a node. We wish to contract these but the possibility of bad tails necessitates two cases.

We tackle the case of easy tails first. If an easy tail is contractible then we have a leaf in X of $\mathcal{F}$ whose dual graph is,



whith $e \geq 2$, and so this may as noted certainly be contracted to a rational $\mathbb{Q}$-factorial singularity through which no $-2\,F$-curves pass. Whence contracting all easy tails that we can, should there be no bad tails then all other tails satisfy the defining property that $K_{\mathcal{G}} \cdot C = 0 \Rightarrow C^2 \geq 0$. We thus concentrate on the $-2\,F$-curves. However all those not satisfying the defining property live in a finite number of disjoint contractible chains or e.g.l.'s not meeting any singularities of the underlying space that we may have so far introduced and so these may all be contracted to rational $\mathbb{Q}$-factorial singularities in the case of chains and elliptic Gorenstien singularities in the case of e.g.l.'s.

Now let us consider bad tails. Even if they exist we may argue as above unless either a bad tail intersects a contractible chain of $-2\,F$-curves, or indeed that bad tails intersect among themselves. Extending the previous conventions on dual graphs to this situation there are therefore three possibilities in X,



Now case (a) is simply either contractible or it is not, according to whether $e_0 \geq 2$ or not. In any case if it is contractible then it is contractible to a rational $\mathbb{Q}$ factorial singularity as in [A]. Evidently if it's not completely contractible then contracting the $-1\,F$ curves, and the $e_1, \ldots, e_n$ chain gives the defining property. Turning to cases (b) and (c) we know these cannot be contractible by I.3.2. Now if both $e_0, f_0 \leq 1$ then we just contract the $-1\,F$ curves, and the $e_1, \ldots, e_n$ chain to get the defining property. Otherwise say $e_0 \geq 2$, and we contract the subgraph of the form (a) and the remaining $-1\,F$ curves to obtain the conclusion.

We have thus shown that a foliated surface $(X, \mathcal{F})$ with underlying space non-singular admits a map to a minimal model with the given defining characteristics. Consequently it remains to extend this to a foliated algebraic space $(Z, \mathcal{H})$ with canonical singularities on which the singularities of the underlying space Z are normal but otherwise arbitrary. As noted in the introduction the former statement may be understood in Mumford's intersection theory. Whence let $\pi : (X, \mathcal{F}) \to (Z, \mathcal{H})$ be a relatively minimal projective desingularisation of the surface Z and Γ_k the fibre over a singular point z_k. Without loss of generality π is not the identity and necessarily, Γ_k is a connected curve, with underlying intersection matrix negative definite and indeed no curve component of any Γ_k is rational of self-intersection -1. There are two cases to consider, given $\Gamma = \Gamma_k$, say,

(a) No component of Γ is a $-1\,F$-curve. Thus since $(Z, \mathcal{H})$ has canonical singularities for all components E of Γ, $K_{\mathcal{F}} \cdot E = 0$. On the other hand if E is not an integral sub-curve of the foliation then, $E^2 \geq -K_{\mathcal{F}} \cdot E = 0$, which is absurd since every component is contractible. Thus every component is an integral sub-curve and if $K_{\mathcal{F}} \cdot E = 0$ then E is either a $-2\,F$-curve a rational integral sub-curve with a node or an elliptic curve contained in the smooth locus of $\mathcal{F}$. However in the latter case we again obtain the absurdity that $E^2 = 0$, and so Γ is a chain of $-2\,F$-curves, or an e.g.l.

(b) Γ contains $-1\,F$-curves. In this occasion let $T_1, \ldots, T_r$ be -1-chains within Γ of maximal length. By hypothesis none of them are artificial, and they are all disjoint. Contract them to obtain a new contractible curve Γ' inside a space $(Y, \mathcal{G})$, by III.2.1 suitably adapted, we obtain that for any curve E' in Γ', $K_{\mathcal{G}} \cdot E' \geq 0$, and so arguing as above we see that Γ must consist entirely of integral sub-curves, and so either Γ is contained in a -1-chain or any -1-chain which is maximal within Γ is actually maximal within X.

Now even though X may contain $-2\,F$-curves whose square is -1, we may still adapt our notations and discussion to this situation as follows. Either a given -1-chain in Γ has a tail in Γ or it does not. Since -1-chains are disjoint from each other in the latter case we have,

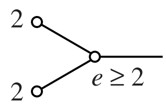
(b)(i) Γ is contained in a maximal -1-chain.

On the other hand contracting -1-chains which are in Γ must leave the tail contractible, but as in II.2.2 this is impossible unless the tail, in X, meets the singular locus of the foliation 3-times. If one of these points is of multiplicity 2 then we have,

(b)(ii) Γ is two -1-chains joined by their mutual tails, and the entire curve Γ is a leaf of $\mathcal{F}$, in particular it meets no other integral sub-curves.

The tricky case is of course 3-distinct points. Since Γ is connected, there are the 3 possibilities corresponding to the Dynkin diagrams (a), (b), (c) above, and indeed a fourth,

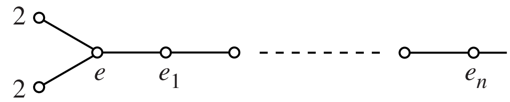
(b)(iii) Γ is two -1-chains (both of which are irreducible curves with self-intersection -2) joined by their mutual tails, but this combination is not a leaf. We represent such a Γ by the graph,

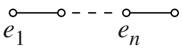


where the edge without a final node represents the singularity of the tail at which integral sub-curves may meet Γ.

In any case since Γ must be contractible the aforesaid Dynkin diagrams (b), (c) are excluded for the same reason as before, and so the final possibility is,

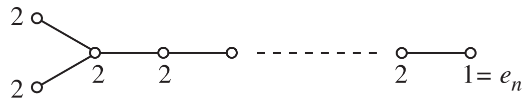
(b)(iv)



where 

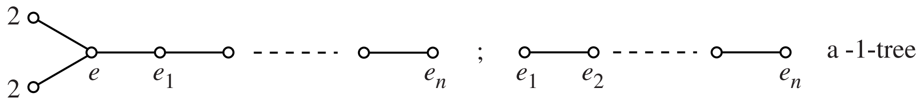
is a chain of $-2\,F$-curves with the nodeless edge employed as in (b)(iii).

At this point we may run our algorithm for constructing a minimal model from $(X, \mathcal{F})$ and see what happens. Firstly we contract chains of artificial -1-curves. Such chains cannot intersect Γ in cases (b)(i), (b)(ii), so do not affect its contractibility, unless of course in $b(i)$, Γ is contained in an artificial chain and we simply finish. On the other hand the process of contracting an artificial -1-curve is simply that of contracting a succession of rational curves with self-intersection -1, and as such either Γ remains contractible after this process or at some point during it we obtain the situation,



There being necessarily no extra edge left, contracting the non $-1\,F$ curves leaves two -1-chains intersecting, which is impossible, so after carrying out this step (b)(iii) and (b)(iv) remain contractible. In case (a) we only need to worry if Γ is a chain of $-2\,F$ curves. This will remain contractible unless both ends of the chain (respectively the single curve in Γ) become exceptional curves of the 1^{st} kind (respectively have square 0) which is certainly impossible for the foliation not a rational fibration.

Now consider the next step of systematically contracting $-2\,F$-curves of self-intersection -1. A priori there are no such curves in Γ, but after finitely many steps they map appear in either cases (a), (b)(i) or (b)(iv). Should such a curve appear in Γ we contract it, always giving preference to contraction of curves of square -1 which may subsequently appear as we continue to push Γ forward. Thus we arrive in a contraction $(X', \mathcal{F}')$ of $(X, \mathcal{F})$ with X' non-singular containing no artificial -1-chains or $-2\,F$-curves of self-intersection -1, with the push-forward Γ' of Γ continuing to be contractible and being of one of the above types. As ever we combine I.3.2 with II.4.2. In the first place exactly as per III.1.3 in case (a) Γ meets at most one -1 chain and the whole thing can be contracted. In (b)(i), Γ is immediately contracted by our algorithm as is (b)(ii). Cases (b)(iii) and (b)(iv) could give rise to (b) and (c) of our previous discussion, but there we have a choice and so we obviously make the one that contracts Γ. It could also happen that a -1 chain meets the free edge in (b)(iii) or (iv), say the former for simplicity. However since the chain is non-artificial contracting it leaves $e > 1$, and so we would have a contractible diagram of the form in X,



which we have already excluded in III.2.1; and so $(Z, \mathcal{H})$ must indeed map to a minimal model of the given form, and better still we have established that our minimal models are genuinely minimal for foliated algebraic spaces with canonical singularities.

Finally we turn to the question of uniqueness, and observe,

Lemma III.3.1. *If $(Y, \mathcal{G})$, $(Y', \mathcal{G}')$ are minimal models of foliated surfaces and $p : (Y', \mathcal{G}') \to (Y, \mathcal{G})$ a map between them which is defined everywhere then p is an isomorphism.*

Proof. Since Y, Y' are normal, it suffices to check that it is injective. Should it fail to be injective then some point $y \in Y$ has a one-dimensional connected fibre, F, say. However $(Y, \mathcal{G})$ is canonical, and $K_{\mathcal{G}'}$ is nef. so $K_{\mathcal{G}'} \cdot E = 0$ for all components E of F, which by the defining property contradicts the negative definite nature of the intersection form on F. □

Consequently given a canonical minimal model $(Y, \mathcal{G})$ and any resolution $p : (X, \mathcal{F}) \to (Y, \mathcal{G})$ should we construct a minimal model $(Y', \mathcal{G}')$ from $(X, \mathcal{F})$ according the proceeding algorithm then we have uniqueness if and only if we obtain a map from $(Y', \mathcal{G}')$ to $(Y, \mathcal{G})$. On the other hand if $q : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$ is the contraction of the -1 chains then this is unique since no two such can intersect. Evidently p and p', say, factor through q as $\pi \circ q$, $\pi' \circ q$, say, and we have uniqueness provided all curves with $K_{\mathcal{F}_0} \cdot C = 0$ are contracted by q and q'. The only way this cannot happen is if there is a curve C in $(Y, \mathcal{G})$ with $K_{\mathcal{G}} \cdot C = C^2 = 0$, so by the index theorem $K_{\mathcal{G}}^2 = 0$, whence we naturally subordinate uniqueness to classification. □

IV. Classification and abundance

IV.0. Definitions

Before getting underway, we take advantage of the existence of minimal models to make,

Definition IV.0.1. *Let $(X, \mathcal{F})$ be a model of a foliated surface with canonical singularities then we define the numerical Kodaira dimension $\nu(X, \mathcal{F})$ to be*

(i) $-\infty$ *if $K_{\mathcal{F}}$ is not pseudo-effective, i.e. $(X, \mathcal{F})$ is a fibration by rational curves.*

(ii) *Otherwise let $\pi : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be a map to a minimal model then if modulo numerical equivalence with $\mathbb{Q}$-coefficients we have,*
  (a) $K_{\mathcal{G}} = 0$, *then* $\nu(X, \mathcal{F}) = 0$
  (b) $K_{\mathcal{G}} \neq 0$, *but* $K_{\mathcal{G}}^2 = 0$, *then* $\nu(X, \mathcal{F}) = 1$
  (c) $K_{\mathcal{G}}^2 \neq 0$, *then* $\nu(X, \mathcal{F}) = 2$.

Despite a minor possibility of non-uniqueness it is an easy exercise in the use of the index theorem to prove that this is well-defined, and as such a birational invariant.

Similarly we have,

Definition IV.0.2. *Let $(X, \mathcal{F})$ be as above then we define the Kodaira dimension $\kappa(X, \mathcal{F})$ to be the Kodaira dimension of $K_{\mathcal{F}}$. By standard properties of Kodaira dimension this is independent of the choice of model provided that it has canonical singularities, and as such is a birational invariant.*

Finally we put these both together and say,

Definition IV.0.3. *Abundance holds for $(X, \mathcal{F})$, as above, if $\nu(X, \mathcal{F}) = \kappa(X, F) \geq 0$.*

IV.1. Rationality

The main aim of this section is simply to record (understanding surface to be a two dimensional algebraic space),

Theorem IV.1.1. *Let $(X, \mathcal{F})$ be a foliated surface with canonical singularities then $(X, \mathcal{F})$ is a fibration by rational curves iff there is a curve C with $K_{\mathcal{F}} \cdot C < -1$. The if direction remaining true if the underlying space X is non-singular.*

The necessity of the condition is obvious if X is non-singular, and whence for arbitrary canonical singularities by an immediate application of the definition. The sufficiency is an application of the cases considered in III which imply it could not be otherwise. Clearly the numerical Kodaira dimension and Kodaira dimension coincide.

Intuitively one expects that $\mathbb{P}^1$-bundles over smooth curves (with their natural fibration) ought to be the minimal models in this situation with uniqueness up to flipping fibers. We observe this is indeed so, viz,

Complement IV.1.2. Let $(X, \mathcal{F})$ be a foliated surface with canonical singularities which is a fibration by rational curves then there is a proper birational map $p : (X, \mathcal{F}) \to (X', \mathcal{F}')$ where the latter is just the natural fibration on a $\mathbb{P}^1$-bundle over a smooth curve.

Proof. We take $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ a resolution with underlying space Y regular, and let Γ be the connected curve over one of the finitely many possible singular points of X. After systematically contracting any artificial -1-chains in Γ, we observe that the arguments of III.1.2 and III.2.1, thanks to the contractibility of Γ, go through verbatim to show that contracting all the other -1-chains in Γ yields a map $q : (X, \mathcal{F}) \to (Z, \mathcal{H})$ such that $q(\Gamma)$ is contractible and $K_{\mathcal{H}}$ is non-negative on all integral sub-curves of $q(\Gamma)$ – note Γ cannot contain leaves since these are not contractible – so as ever by III.1.1, Γ consists of integral sub-curves. The quotient space $B = X/\mathcal{F}$, however, exists and is normal, whence a smooth projective curve. Consequently Γ is contained in a fibre of $\rho : X \to B$. Necessarily this fibre is singular. Let $F_1, \ldots, F_r$ be the singular fibres with $\sum_i F_i = \rho^* D$, say, then,

$$K_{\mathcal{F}} = K_X - \rho^* \, \omega_B(D) + \sum_{i=1}^{r} (F_i)_{\mathrm{red}}$$

whence if C is a general fibre, then $K_{\mathcal{F}} \cdot C = K_X \cdot C = -2$. Thus if F is any singular fibre then by adjunction (for K_X) it contains a rational integral sub-curve with $C^2 = -1$, so by a sequence of contractions of $-1\,F$- or $-2\,F$-curves we arrive at $(X', \mathcal{F}')$ provided we always give preference in contracting curves with $C^2 = -1$ to those that may appear in the push-forward of Γ. □

As a rather amusing corollary we obtain the following result which was already noted by M. Brunella, [Br1], in his classification of smooth foliations, viz:

Corollary IV.1.3. *Let $(X, \mathcal{F})$ be a smooth foliated surface (i.e. the underlying space X is non-singular and $\mathcal{F}$ has terminal singularities) then X is minimal unless $\mathcal{F}$ is the foliation obtained from blowing up the point of common intersection of a pencil of lines in $\mathbb{P}^2$.*

Proof. If the numerical Kodaira dimension of $(X, \mathcal{F})$ is $-\infty$, then this immediately follows by IV.1.2 since a smooth foliation is a minimal model. Otherwise let,

$$0 \to N \to \Omega_X \to K_{\mathcal{F}} \to 0$$

be the dévissage that defines the foliation, then again $(X, \mathcal{F})$ is a minimal model so in particular $K_{\mathcal{F}}$ is nef. Now suppose X is not a minimal surface and let E be an exceptional curve of the 1^{st} kind then $N \cdot E \le -1$, whence for any positive rational number ε we have,

$$(N - \varepsilon\, E)^2 \ge 2\,\varepsilon - \varepsilon^2 \,, \;\; K_{\mathcal{F}} \cdot (N - \varepsilon\, E) = c_2(X) - \varepsilon\, K_{\mathcal{F}} \cdot E\,.$$

Now suppose further that X is not birational to a $\mathbb{P}^1$-bundle over a curve of genus g, then since X is not minimal $c_2(X) > 0$ so that the above plus Riemann-Roch imply that for ε-sufficiently small $N - \varepsilon E$, and whence N is big, contrary to a famous estimate of Bogomolov, cf. [B1]. Whence X is a surface of the said type on which there are exceptional curves of the 1^{st} kind so,

$$c_1(X)^2 < 2\,c_2(X)\,.$$

However we have the basic estimate,

$$c_1(X^2) - 2\,c_2(X) = K_{\mathcal{F}}^2 \geq 0$$

which is absurd. □

IV.2. General type

Before progressing to the more delicate aspects of classification, it seems reasonable to note what the minimal model theorem gives in the case of foliations of general type, i.e. numerical Kodaira dimension 2. Clearly abundance holds in this situation but the minimal model theorem gives us much more, viz:

Theorem IV.2.1. *Let $(X, \mathcal{F})$ be the unique*[7] *minimal model of a foliation of general type and let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be the resolution of the cusps then there is an ample $\mathbb{Q}$-divisor H and an effective $\mathbb{Q}$-divisor Z (supported on the minimal) elliptic Gorenstien leaves such that,*

$$K_{\mathcal{G}} = \pi^*\,K_{\mathcal{F}} = H + Z\,.$$

Proof. This is basically standard cone theory à la [K1] but for want of a reference we give the details. In the first place observe that in Mumford's intersection theory $K_{\mathcal{F}}$ is strictly positive on the closed cone of effective curves, since if $\alpha \neq 0$ is in this cone and $K_{\mathcal{F}}.\,\alpha = 0$ then $\alpha^2 < 0$. Now let α_n be a sequence of effective divisors converging to α. Clearly α is not nef. so there is a curve C with $\alpha.\,C < 0$. Hence C is in the support of α_n for $n \gg 0$, say $\alpha_n = a_n\,C + \beta_n$ where again β_n is effective. Since $\alpha.\,C < 0$ there is a subsequence of the a_n which must converge to some positive real number a, and so $\alpha = a\,C + \beta$, where β is in the closed cone of effective curves. Consequently $K_{\mathcal{F}}.\,C = 0$, which forces $C^2 \geq 0$, which is absurd. Whence there is a rational $\varepsilon > 0$ such that $K_{\mathcal{F}}.\,C \geq \varepsilon\,\|C\|$ for some norm on $\mathrm{NS}_1^m(X)$ and C a curve. Equally if we take H to be ample on Y, and H_0 it's image in $\mathrm{NS}_1^m(X)$, then without loss of generality $K_{\mathcal{F}}.\,C \geq \varepsilon\,H_0.C$, while on the other hand for some positive rational cycle Z, supported on the elliptic gorenstein leaves, $\pi^*\,H_0 = H + Z$, so for any curve C on Y, $\pi^*\,K_{\mathcal{F}}(-\varepsilon Z).C \geq \varepsilon\,H.C$,

[7] Uniqueness in the general type situation is immediate from the discussion of III.3.

i.e. $\pi^* K_{\mathcal{F}} - \varepsilon Z$ is strictly positive on the closed cone of curves in Y, and Y is $\mathbb{Q}$-factorial, so this divisor is ample by Kleimann's criterion. □

Of course the presence of elliptic Gorenstien leaves in the above decomposition appears a priori rather ugly. Certainly one would have hoped for a base point freeness result for $K_{\mathcal{F}}$ on the minimal model. However eventual freeness of $K_{\mathcal{F}}$ whether on X or on Y, in the notations of the theorem, would imply that $K_{\mathcal{F}}$ is $\mathbb{Q}$-foliated Gorenstien on the minimal model, and this in general is impossible. Indeed we shall prove,

Theorem IV.2.2. *Let $\pi : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be the contraction of a minimal elliptic Gorenstien leaf on a foliated algebraic space with canonical singularities then the contraction is never $\mathbb{Q}$-foliated Gorenstien. Better still the germ of any such contraction is isomorphic to the germ of the natural foliation on the Baily-Borel compactification of a Hilbert modular surface.*

Proof. The question is completely local, so without loss of generality we may take Y to be a tubular neighbourhood of the elliptic Gorenstien leaf Z, say. We denote by $Z_1, \dots, Z_n$ the components of Z and proceed to calculate what it means for the contraction to be $\mathbb{Q}$-foliated Gorenstien.

Fix a component Z_i, and cover the smooth locus of $\mathcal{F}$ in a neighbourhood of Z_i by polydiscs Δ_α. Then the bundle N, say, in Ω^1_Y defining the foliation admits a description of the form, $N \mid_{\Delta_\alpha} = \mathcal{O}_{\Delta_\alpha} dz_\alpha$, for $z_\alpha \in \Gamma(\Delta_\alpha)$ which may in turn be taken as defining $\mathcal{O}_{\Delta_\alpha}(Z)$ locally. Consider the respective transition functions $g_{\alpha\beta}$, $h_{\alpha\beta}$ and observe the formulae,

$$dz_\alpha = g_{\alpha\beta}\, dz_\beta\,,\ z_\alpha = h_{\alpha\beta}\, z_\beta \Rightarrow dz_\alpha = h_{\alpha\beta}\, dz_\beta \quad \text{in} \quad \Omega^1_{\Delta_\alpha} \otimes \mathcal{O}_Z\,.$$

Consequently in $\mathcal{O}_Z \otimes \mathcal{O}_{\Delta_\alpha}$ we have $g_{\alpha\beta}\, h_{\alpha\beta} = 1$, and this represents the class in $\mathrm{Pic}(Z)$ of $\mathcal{O}_Z(N+Z)$. Now of course the intersection of Z with the smooth locus of $\mathcal{G}$ is just Z minus its singularities. So let p be such a singularity, let x_p, y_p be local equations for the components (in the strict Hensalisation) of Z through p, and ω_p a generator of N at p, then for a suitable polydisc Δ_p about p we have,

$$dz_\alpha = g_{\alpha p}\, \omega_p\,,\ z_\alpha = h_{\alpha p}(x_p\, y_p) \Rightarrow dz_\alpha = h_{\alpha p}\, d(x_p\, y_p) \quad \text{in} \quad \Omega^1_{\Delta_p} \otimes \mathcal{O}_Z\,.$$

In addition we note that ω_p is of the form, $x_p\, a_p\, dy_p + y_p\, b_p\, dx_p$, where $a_p(p)\, b_p(p) \neq 0$, so without loss of generality a_p, b_p are invertible over Δ_p.

Now by virtue of the fact that the contraction is elliptic Gorenstien $K_Y + Z$ may be assumed trivial. Whence $(X, \mathcal{F})$ is $\mathbb{Q}$-foliated Gorenstien by III.0.4 if and only if, $N+Z$ is torsion in $\mathrm{Pic}^0(Z)$. So let us suppose that indeed the singularity is $\mathbb{Q}$-foliated Gorenstien then for some $m \in \mathbb{N}$ should we put, $u_{\alpha\beta} = (g_{\alpha\beta}\, h_{\beta\alpha})^m$ and similarly for $\nu_{\alpha p}$ at the singularities then the corresponding cohomology class $\nu \in H'(Z, \mathcal{O}_Z^\times)$ must be trivial. For simplicity let us suppose that Z has more than one component then the pull-back of ν to $H^1(Z_i, \mathcal{O}_{Z_i}^\times)$, for any i, is also trivial, and we denote the two singularities by p and q. Further let's say that Z_i in a neighbourhood whether of p or q is given by x_p (resp. x_q) $= 0$, then $\nu_{\alpha p}$ (resp. $\nu_{\alpha q}$) is equal to b_p^{-m} (resp. b_q^{-m}). Whence if the cohomology class ν is trivial over Z_i there are functions f_α, f_p, f_q invertible on any Δ_α, Δ_p, Δ_q such that $\nu_{\alpha\beta} = f_\alpha/f_\beta$ etc., so without loss of generality $f_\alpha = 1$ for all α, and $f_p = b_p^m$ and $f_q = b_q^m$. Now consider the other curve Z_j through p then since the various f's may be assumed to come from Z we obtain in the same way $f_p = a_p^m$, and so the ratio $a_p/b_p(p)$ must be a root of unity. In the case of a rational curve with a node we obtain the same conclusion on considering the two branches through the singularity. At this point we choose a labelling of the singularities, of which there are n, and let $\lambda_1, \ldots, \lambda_n$ be the said ratios, or their inverses as appropriate, and apply the discussion of I.3 to obtain,

(a) $\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \mathrm{Re}\left(\lambda_i + \frac{1}{\lambda_{i+1}}\right)$, if $n \geq 2$ and $\lambda_{n+1} := \lambda_1$

or

(b) $-C^2 + 2 = \mathrm{Re}\,(\lambda_1 + 1/\lambda_1)$, if $n = 1$.

In either case the right hand side has modulus bounded by $2n$, which is strictly smaller than the modulus of the left hand side since Z is basic, which proves the first part.

Let's turn our hand now to the second part, and begin by observing that $\Omega_Y^1(\log Z)$ is an extension of $K_{\mathcal{F}}$ by $N(Z)$ and the obstruction to this splitting on a formal neighbourhood of Z is given by,

$$\varprojlim_m H^1(Z_n, N(Z) \otimes K_{\mathcal{F}}^\vee) = \varprojlim_m H^1(Z_n, K_{\mathcal{F}}^{-2})$$

where Z_m is the m^{th}-infinitesimal neighbourhood of Z. Since we know that $(X, \mathcal{F})$ is not foliated Gorenstien then $K_{\mathcal{F}}$ is not torsion so this obstruction certainly vanishes, i.e. on an infinitesimal neighbourhood of Z we have a splitting,

$$\Omega_Y(\log Z) = N(Z) \oplus K_{\mathcal{F}}\,.$$

Call the foliation defined by $N(Z)$, $\mathcal{H}$. The transverse structure will permit us to conclude that we have a Hilbert-Modular surface. The question is purely local, so we might as well say that Y is just its own completion in Z, or better an analytic neighbourhood since we want to consider its universal cover $\nu : \widetilde{Y} \to Y$. Necessarily $\pi_1(Y) = \pi_1(Z) = \mathbb{Z}$, and $\nu^{-1}(Z) \to Z$ the universal cover of Z is just an infinite chain of rational curves. Better still $\widetilde{Y}$ is still convex (or more precisely 1-pseudo convex) since at least one curve C in $\pi^{-1}(Z)$ has $C^2 \leq -3$. Consequently we can take the Stein factorisation, say $\pi : \widetilde{Y} \to \widetilde{X}$ to obtain an almost étale cover $\nu : \widetilde{X} \to X$, with exactly one point Z lying over our initial elliptic Gorenstien singularity. Moreover the induced foliations $\widetilde{\mathcal{F}}, \widetilde{\mathcal{H}}$ on $\widetilde{X}$ are foliated Gorenstien (n.b. the contractibility of Z forces $\mathcal{H}$ to have canonical singularities, c.f. the discussion concluding

IV.5.a) so certainly $\mathcal{T}_{\widetilde{X}}$ is locally free. This suggests that $\widetilde{X}$ in non-singular, and indeed this is true by known cases of the Zariski-Lipman conjecture, or alternatively it follows from IV.5.10(b) modulo a slightly more complicated arguement to take care of a possibly multiple tangent divisor between the foliations induced by $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{H}}$ on the minimal smooth resolution. In any case $\widetilde{X}$ is indeed non-singular, and the foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{H}}$ allow us to write it canonically as $\Delta \times \Delta \ni z$, with the splitting respecting the $\mathbb{Z}$ action and fixing z, so we're done. □

Now while as we will see in IV.5 that the natural foliations on Hilbert-Modular surfaces induced by the splitting of the upper half plane is of numerical Kodaira dimension 1, their germs are hardly likely to be unique to such things and so we have established that IV.2.1 is probably optimal.

IV.3. Vectorial foliations ($\nu(X, \mathcal{F}) = 0$)

In this chapter we investigate foliations of numerical Kodaira dimension 0. Ultimately we'll see that up to an almost étale covering they are defined by global vector fields whence the appellation. One might even suspect that a foliation of numerical Kodaira dimension 0 ought to be defined by a global vector field. Unfortunately this isn't quite true, indeed it is not even true modulo étale covers, but only modulo covers étale in codimension 1. Consequently before getting underway with the classification proper let us convince ourselves of this phenomenon by means of examples. The first example was brought to my attention by M. Brunella, the second by G. Mendes, cf. [Me2].

Example IV.3.0. (a) Consider the action of $\mathbb{Z}/3$ on $\mathbb{P}^2$ given in homogeneous coordinates by, $\sigma : [x, y, z] \mapsto [y, z, x]$, where σ is a generator of $\mathbb{Z}/3$. In which case there are 3-fixed points of this action, viz: $[\zeta, \zeta^2, 1]$ where ζ is a cube root of unity, and otherwise the action is free. Furthermore, under the standard embedding of $\mathbb{A}^2$ in $\mathbb{P}^2$ the foliation given by, $x\,\partial/\partial x - \zeta y\,\partial/\partial y$ is stable by this group action even though the vector field itself is not. Taking the quotient therefore yields a foliated surface $(X, \mathcal{F})$ with canonical foliation singularities, and underlying space X $\mathbb{Q}$-factorial, such that $K_{\mathcal{F}}^{\otimes 3} \xrightarrow{\sim} \mathcal{O}_X$ but $K_{\mathcal{F}}$ is not isomorphic to $\mathcal{O}_X$ so the foliation cannot be defined by a global vector field. Indeed if $p : (X', \mathcal{F}') \to (X, \mathcal{F})$ is the minimal smooth resolution then over each of the 3-singularities there are two exceptional curves E_{i1}, E_{i2}, $1 \leq i \leq 3$ each with square -2 forming a $-1\,F$-chain and, $K_{\mathcal{F}'} = 2/3(E_{11} + E_{21} + E_{31}) + 1/3(E_{12} + E_{22} + E_{32})$.

That this should occur as a quotient of a singular foliation is perhaps not so surprising. The second example, however, goes back to A. Lins Neto, [LN], and shows that this is possible even as a quotient of a smooth foliation, viz:

Example IV.3.0. (b) Consider in $\mathbb{P}^2$ the 9 lines defined in homogeneous coordinates by $(X^3 - Y^3)(Y^3 - Z^3)(Z^3 - X^3) = 0$. Denote by $L_1, \ldots, L_9$

these lines, then the union of the $L_i \cap L_j$, $i \neq j$, is 12 points $p_1, \ldots, p_{12}$ enjoying the following properties:

(i) Any p_k is contained in precisely 3-lines.

(ii) Any L_i contains precisely 4 of these points.

(iii) Any 3 points amongst the p_k not contained in a line are in general position.

As before with respect to the standard embedding of $\mathbb{A}^2$ in $\mathbb{P}^2$ for $t \in \mathbb{C}$, consider the foliations given by,

$$\mathcal{F}_t : (x^3-1)(x-ty^2)\frac{\partial}{\partial x} + (y^3-1)\,(y-tx^2)\,\frac{\partial}{\partial y}$$

with respect to all these foliations, the 9 lines form a configuration of invariant sub-curves. Consider the blow-up $\widetilde{X}$, say, of $\mathbb{P}^2$ in the 12-points with $\widetilde{L}_i$ the proper transforms of the 9 lines which are now all disjoint with $\widetilde{L}_i^2 = -3$. Moreover for a very general t we have,

(a) $\widetilde{\mathcal{F}}_t$ has canonical singularities.

(b) The $\widetilde{L}_i$ are the only invariant sub-curves of the foliation.

(c) $\widetilde{\mathcal{F}}_t$ has Kodaira dimension zero.

Consequently, $K_{\widetilde{\mathcal{F}}_t} = \frac{1}{3} \sum_{i=1}^{q} \widetilde{L}_i$ and blowing down all 9 lines gives a foliated surface $(X, \mathcal{F})$ with canonical foliation singularities, underlying space X $\mathbb{Q}$-factorial, and $K_{\mathcal{F}}^{\otimes 3} \xrightarrow{\sim} \mathcal{O}_X$. The difference however with the previous example is that there are no integrable complete sub-curves, and so as we will later $(X, \mathcal{F})$ admits a (3:1) cover by a Kronecker foliation on an abelian surface.

We thus begin the question of classification. As ever we take care of irregular surfaces first, viz:

Lemma IV.3.1. *Let $(X, \mathcal{F})$ be a model of a foliated surface with canonical singularities and $\nu(X, \mathcal{F}) = 0$, then abundance holds, and better still if X is a non-singular surface with non-zero irregularity then in fact $h^0(K_{\mathcal{F}}) \neq 0$ or the foliation is a fibration.*

Proof. The lemma is trivial for surfaces with zero irregularity, so suppose otherwise and without loss of generality let $(X, \mathcal{F})$ be the unique desingularisation of a minimal model, then we have an exact sequence,

$$0 \to N \to \Omega^1_X \to K_{\mathcal{F}} \cdot I_Z \to 0$$

giving rise to the foliation. Observe that by a classical lemma of Castelnuovo should $h^0(X, N) \geq 2$ then the foliation is a fibration, while if $q(X) = 1$ then $h^0(K_{\mathcal{F}}) \neq 0$ or the foliation is the Albanese fibration. Whence in either case the quotient $\pi : X \to X/\mathcal{F} := B$ exists and factors through the

Albanese. Better still the foliation is certainly an elliptic fibration, so by Kodaira's canonical bundle formula we obtain,

$$K_{\mathcal{F}} = \pi^* \left\{(R^1 \pi_* \mathcal{O}_X)^\vee\right\} \left(-\sum_i (D_i)\right)$$

the sum being taken over all fibres F_i such that the underlying space $(F_i)_{\text{red}}$ is singular and D_i is a proper effective Cartier divisor contained in F_i. To profit from this observation observe that by the minimal model theorem and the fact that π factors through the Albanese there is a $\mathbb{Q}$-Cartier divisor L of degree 0 on B, and $\mathbb{Q}$-Cartier divisors Z_{ij} supported on -1-chains inside F_i such that,

$$K_{\mathcal{F}} = \pi^* L + \sum_{ij} Z_{ij}$$

as $\mathbb{Q}$-Cartier divisors. Consequently if $p : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$ is a contraction of the -1-chains then, $p_*(K_{\mathcal{F}} \otimes \pi^* L^\vee)$ is torsion and the map π descends to X_0.

We may thus take a cover $q : (Y_0, \mathcal{G}_0) \to (X_0, \mathcal{F}_0)$ which is étale in codimension one and such that $K_{\mathcal{G}_0}$ is the pull-back of a Cartier divisor on B which is numerically equivalent to zero. Now putting $p : (Y, \mathcal{G}) \to (Y_0, \mathcal{G}_0)$ to be a minimal desingularisation, profiting from III.1.4, and essentially the arguments of III.3 (although see IV.3.3 et sequel) we see that $K_{\mathcal{G}}$ is numerically equivalent to zero as a Cartier divisor. Nevertheless the foliation is still an elliptic fibration so applying the previous formula for $K_{\mathcal{F}}$ but now to $K_{\mathcal{G}}$ we see that D_i must be a multiple of F_i by Zariski's lemma, so it is F_i and so continuing to denote the quotient by π we have, $\deg (R^1 \pi_* \mathcal{O}_Y)^\vee = 0$. Consequently the only multiple fibres of $\pi : Y \to Y/\mathcal{G}$ are non-singular elliptic curves so the foliation is smooth on Y (otherwise there would be a curve intersecting $K_{\mathcal{G}}$ positively). Appealing to [BPV] (III.18.3) we deduce that $K_{\mathcal{F}}$ is torsion and so conclude. □

Divertimento IV.3.1$^{1/2}$. Obviously what we've done in the course of the proof is under the hypothesis of positive irregularity with $K_{\mathcal{F}}$ not inheriting a global section from the space of 1-forms is to construct an almost étale cover $q : (Y, \mathcal{G}) \to (X_0, \mathcal{F}_0)$ with Y non-singular and $\pi : Y \to Y/\mathcal{G}$ an elliptic fibration with $R^1 \pi_* \mathcal{O}_Y$ torsion. Now if $Y/\mathcal{G}$ has positive genus then it's a theorem (cf. [R] E.5.1) that Y is a sesquielliptic surface, i.e. $Y = C \times E/G$ where G is an abelian group acting on an elliptic curve E by translations, on C any old how, and the diagonal action is free. The proof however has no need of this further hypothesis, so putting $Y/\mathcal{G} = B$ let's give the details since it's something that will arise often enough. The important point is that Kodaira's canonical bundle formula gives, $q = p_g + 1 \geq g(B) + 1$. Consequently if $\alpha : Y \to \mathrm{Alb}(Y)$ is the Albanese, then α cannot factor through B, and $\alpha \times \pi : Y \to \mathrm{Alb}(Y) \times B$ expresses Y as a finite cover of a family of elliptic curves in a fixed abelian variety. Such a family is necessarily constant, so in fact for some elliptic curve A which is a quotient of $\mathrm{Alb}(Y)$ every fibre Y_b admits a non-constant map to A, which in fact is Galois with abelian group, and the normalised fibre product $Y \times_B A$ is a product as required. Consequently we'll use the sesquielliptic terminology even for a base of genus 0, and when we come across this case again we'll explicitly note it in the classification.

We now change our notation slightly and denote by $(X', \mathcal{F}')$ the unique minimal desingularisation of the minimal model of a foliated surface $(X, \mathcal{F})$. In light of IV.3.1 and our supposition that $\nu(X, \mathcal{F}) = 0$ we deduce that $(X, \mathcal{F})$ is in fact $\mathbb{Q}$-foliated Gorenstien. Furthermore we may assume if $q(X') \neq 0$, then in fact $h^0(K_{\mathcal{F}'}) \neq 0$. To advance we need another little lemma, viz,

Lemma IV.3.2. *Let $(Y, \mathcal{G})$ be a foliated surface with $K_{\mathcal{G}}$ isomorphic to the structure sheaf in codimension 2, then in fact $(Y, \mathcal{G})$ is foliated Gorenstien and indeed $K_{\mathcal{G}}$ is not just the structure sheaf, but defines an element of $H^0(\mathcal{T}_Y)$.*

Proof. Recall that by definition $K_{\mathcal{G}}$ is the dual of the saturated subsheaf of $\mathcal{T}_Y$ which defines the foliation, so that this is just an easy exercise in local cohomology, cf. [M1], 2.3. □

Applying the lemma to the case of positive irregularity, we may therefore conclude that $K_{\mathcal{F}}$ is the structure sheaf, and the foliation is defined by a global vector field. Should the irregularity be zero then again there is a finite map, $p : (Y, \mathcal{G}) \to (X, \mathcal{F})$, from a variety Y which is normal, with p étale in codimension 1, and $K_{\mathcal{G}}$ again the structure sheaf, with the foliation once more being defined by a global vector field, cf. op.cit. We need to check that $(Y, \mathcal{G})$ is again canonical. More generally we prove,

Lemma IV.3.3. *Let $p : (Y, \mathcal{G}) \to (X, \mathcal{F})$ be étale in codimension one with $(X, \mathcal{F})$ a minimal model of a foliated surface with canonical singularities then $(Y, \mathcal{G})$ has canonical singularities, and Y is $\mathbb{Q}$-factorial outside of any e.g.l.'s.*

Proof. The $\mathbb{Q}$-factorial part of the proposition is well-known (e.g. [Ka], 3.2). Otherwise consider à la [CKM] a diagram of the following form,

$$\begin{tikzcd} F \arrow[r, "p"'] \arrow[d, hook] & E \arrow[d, hook] \\ (Y', \mathcal{G}') \arrow[r, "p"'] \arrow[d, "h"] & (X', \mathcal{F}') \arrow[d, "h"] \\ (Y, \mathcal{G}) \arrow[r, "p"'] & (Y, \mathcal{F}) \end{tikzcd}$$

where F, E are exceptional curves on some non-singular birational modifications corresponding to valuations with non-empty centres on X and Y, and e the ramification index of F over E. Given F such a diagram always exists for some E. We proceed to calculate the discrepancy $a(F, \mathcal{G})$ of $K_{\mathcal{G}}$ along F. We have, locally around F,

$$K_{\mathcal{G}'} = h^* K_{\mathcal{G}} + a(F, \mathcal{G}) F = p^* K_{\mathcal{F}'} - a(E, \mathcal{F}) p^* E + a(F, \mathcal{G}) F .$$

Now the key point is that the main theorem on minimal canonical models implies that E is an integral sub-curve of $\mathcal{F}'$. One therefore may check that locally on F, $K_{\mathcal{G}'} = p^* K_{\mathcal{F}'}$, and so: $a(F, \mathcal{G}) = (e+1)\, a(E, \mathcal{F}) \geq 0$, as required. □

Let us therefore consider the problem of a foliated surface $(Y, \mathcal{G})$ with canonical singularities which is defined by a global vector field with the additional property that $K_{\mathcal{G}} \xrightarrow{\sim} \mathcal{O}_Y$. Whence we put G to be the connected component of the identity in $\mathrm{Aut}(Y)$ and observe the identification,

Fact IV.3.4.

$$T_1 G = H^0(\mathcal{T}_Y) .$$

Better still for any point $y \in Y$, we have a map $f_y : G \to Y : g \mapsto y^g$, whose differential $df_y : T_1 G \to T_y Y$ is just the natural image by restriction under this identification of $H^0(\mathcal{T}_Y)$ in $T_y Y$.

Now let us turn to the singularities of the surface Y. Necessarily these are fixed points of G, so that if $p : (Y', \mathcal{G}') \to (Y, \mathcal{G})$ is a minimal resolution of singularities then Y' is G-equivariant. Moreover as in IV.3.1 essentially the same arguments as in III.3 shows that $K_{\mathcal{G}'} = \rho^* K_{\mathcal{G}} + E$, where E is a divisor supported on -1-chains which is not integral. However $(Y, \mathcal{G})$ is foliated Gorenstien so E must be zero by III.1.4. Now consider the problem of curves E in Y' with $E^2 = -1$. By adjunction these are invariant for any vector field, so they aren't just invariant for our foliation $\mathcal{G}$ but the whole group G. Since $K_{\mathcal{G}} \tilde{\to} \mathcal{O}_Y$, they can be blown down with discrepancy 0, and so we obtain a minimal surface S blown down from Y together with a foliation $\mathcal{H}$. The surface S is G-equivariant, $(S, \mathcal{H})$ has canonical singularities, and $K_{\mathcal{H}} \tilde{\to} \mathcal{O}_S$.

Firstly consider the case that G is not transitive, then the general integral sub-curve is rational or elliptic. By IV.1.1 the former is impossible. As ever the quotient space $B = S/\mathcal{H}$ exists and is a regular curve, and the quotient map $\pi : S \to B$ is an elliptic fibration. In addition as in IV.3.1,

$$\mathcal{O}_S \xleftarrow{\sim} K_{\mathcal{H}} \xrightarrow{\sim} (R^1 \pi_* \mathcal{O}_S)^\vee$$

and every fibre is thus an elliptic curve, so argueing as in op.cit., we deduce that $\mathrm{sing}(\mathcal{H}) = \varphi$, so $c_1^2(S) = c_2(S) = 0$, and $p_g(S) \geq g(B)$, the genus of B. Necessarily then, $q(S) \geq g(B) + 1$, so that an étale cover of S is a product $E \times B$ for some fixed elliptic curve B. Indeed more precisely we have,

Case IV.3.5 (a) $(Y, \mathcal{G})$ is the natural elliptic fibration on a sesquielliptic surface (cf. IV.3.$1^{1/2}$), so essentially just $E \times B$ for E an elliptic curve and B arbitrary, and the minimal model is unique.

Before continuing observe that this certainly follows from the above discussion, since $(S, \mathcal{H})$ regular and $K_{\mathcal{G}'} \xrightarrow{\sim} \mathcal{O}_{Y'}$ forces $S = Y'$. Further Y' contains no $-2\,F$-curves so by 3.3, $Y' = Y$.

So suppose that the action of G is transitive. The easiest case is if G is compact, then S is an abelian surface. Thus we either have a fibration by elliptic curves, and in particular IV.3.5 (a) above, or $(S, \mathcal{H})$ is a Kronecker foliation, i.e. defined by a global holomorphic one form ω which is necessarily some combination $\lambda\,\omega_0 + \mu\,\omega_1$ of a basis of $H^0(S, \Omega^1_S)$ in such a way that no leaf is compact. Similar considerations to the remarks following the above case, ensure that this case reduces to,

Case IV.3.5 (b) $(Y, \mathcal{G})$ is a Kronecker foliation on an abelian surface, and the minimal model is unique.

Otherwise G is an extension of an abelian variety A, say, by an affine linear group H, and we next consider the possibility that $A \neq 0$ but G is not compact. Choose a general point $s \in S$ and let N be the normaliser of the stabiliser of s in G then as in IV.3.4 we have a map, but now injective,

$$f_s : G/N \hookrightarrow S\,.$$

Since S is minimal, it is therefore the natural compactification of an extension of an elliptic curve E by $\mathbb{G}_a$ or $\mathbb{G}_m$, obtained by adding the section at infinity or sections at zero and infinity. Let $\pi : S \to E$ be the natural map. The foliation cannot be a fibration by rational curves and so the map,

$$\mathcal{O}_S \xleftarrow{\sim} \pi^*\,\omega_E \longrightarrow K_{\mathcal{H}}\,I_{\mathrm{sing}(\mathcal{H})} \xleftarrow{\sim} I_{\mathrm{sing}(\mathcal{H})}$$

is non-zero, whence necessarily the foliation is without singularities, and either $S = \mathbb{P}^1 \times E$ or the connected component of $\mathrm{Aut}(S)$ is the said extension. In either case we have a splitting,

$$T_S = T_{S/E} \oplus \pi^*\,T_E$$

and the foliation is everywhere transverse to the fibration over E. This excludes the possibility of rational integral sub-curves and forces the identification $(S, \mathcal{H}) = (Y, \mathcal{G})$. In addition even if S is split we still have natural sections at zero and infinity (respectively ∞) given by the zero locus of the $\mathbb{P}^1$-part of the vector field, and the structure group naturally reduces to $E \times \mathbb{G}_m$ (respectively $E \times \mathbb{G}_a$) and so we obtain,

Case IV.3.5 (c) The minimal model $(Y, \mathcal{G})$ is either,

(i) An *elliptic vector foliation*, i.e. Y is the natural equivariant compactification of an extension of E by $\mathbb{G}_a$, and the foliation is defined by a vector field everywhere transverse to the natural fibration by rational curves, with a unique compact leaf, the natural section at infinity.

(ii) A *semi-abelian foliation*, i.e. Y is the natural equivariant compactification of an extension of E by $\mathbb{G}_m$, the foliation is defined by a vector field everywhere transverse to the natural fibration by rational curves, and there are two compact leaves, the natural sections at zero and infinity.

In all cases the minimal model is unique.

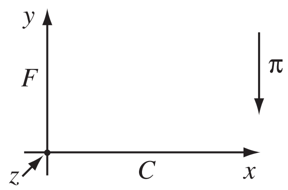
We are thus left with the possibility that the group G is affine and acting transitively on S, so S is rational. For the moment we ignore the possibility that S is $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ and consider only the cases that $\pi : S = \mathbb{P}(\mathbb{I} \oplus H^{\otimes n}) \to \mathbb{P}^1$, is a Hirzebruch surface H the tautological bundle on $\mathbb{P}^1$, $n \geq 2$. The necessary generality is however maintained by surreptitiously allowing the possibility that $n = 0$ or 1. In any case $c_2(S) = 4$, so if Z is the singular locus of $\mathcal{H}$ then $\deg Z$, counting nilpotents, is 4 (if $S = \mathbb{P}^2$, then $\deg Z = 3$, choose a canonical singularity and blow up in its reduced structure to get the case $n = 1$, and maintain $K_{\mathcal{H}}$ trivial). For any fibre F of π, we have $K_{\mathcal{H}} \cdot F = F^2 = 0$. However if F contains a singular point of $\mathcal{H}$ and F is not an integral sub-curve then $(K_{\mathcal{H}} + F) \cdot F > 0$, so a fibre containing a singularity is an integral sub-curve.

Consequently if we're on $\mathbb{P}^1 \times \mathbb{P}^1$ the situation is nice and clear. Either there are no saddle nodes and for each direction we have two invariant fibres, giving an invariant boundary ∂ consisting of 4 fibres and the foliation $\mathcal{H}$ defined by an injection of bundles,

$$\mathcal{O}_S \hookrightarrow T_S(\log \partial) \xrightarrow{\sim} \mathcal{O}_S \oplus \mathcal{O}_S$$

and since it's not a fibration by rational curves it must be a Kronecker type foliation (i.e. in standard coordinates given by $\frac{dx}{x} + \lambda \frac{dy}{y}$, $\lambda \notin \mathbb{Q}$) and everything is governed by the toric action, i.e. the group in question is $\mathbb{G}_m \times \mathbb{G}_m$. Otherwise there is a saddle node, and indeed by a trivial explicit calculation two saddle nodes, leading to two invariant fibres in one direction, each of which is the weak branch of the node, and one invariant fibre in the other direction which passes through both singular points. In particular we have an invariant boundary ∂ consisting of 3 invariant curves, in a suitable standard coordinate system the vector field in question is $x^2 \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$, $\lambda \in \mathbb{C}$, and the group in question is $\mathbb{G}_a \times \mathbb{G}_m$.

Evidently we wish to obtain the same conclusion in the general case $n \geq 1$. Here there is a canonical section C of π with $C^2 < 0$, which by adjunction is invariant. There is of course an invariant fibre F, so consider the following diagram which fixes our notations,



Now if the singular locus of $\mathcal{H}$ along F has Segre class 1 at z, then there must be another singularity w on F. We can blow up in w, and blow down F without changing any of our hypothesis except that n decreases by 1, and so if this situation persists reduce to the $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. $n = 0$, situation. ¿From which we deduce that we have at least 3 invariant curves, and a $\mathbb{G}_m \times \mathbb{G}_m$ or $\mathbb{G}_a \times \mathbb{G}_m$ action according to whether we have a 4^{th} or not. The tricky case is therefore if the Segre class of $\mathcal{H}$ along F is 2, i.e. F is a weak branch of a node. This of course implies that we have another singularity w on C, and without loss of generality we might as well say $\pi(z) = 0$, $\pi(w) = \infty$ in standard coordinates on $\mathbb{P}^1$. Now at z we know we can write our global vector field defining $\mathcal{H}$ as,

$$\partial = x\,\frac{\partial}{\partial x} + g(x)\left(y^2\,\frac{\partial}{\partial y}\right)$$

for y the natural coordinate for the contractible section of a Hirzebruch surface. If $\xi = 1/x$ then the natural transverse coordinate at ∞ is, $\eta = x^n y$, so that ∂ becomes,

$$\partial = -\xi\,\frac{\partial}{\partial \xi} + n\eta\,\frac{\partial}{\partial \eta} + \left(g\left(\frac{1}{\xi}\right)\,\xi^n\right)\,\eta^2\,\frac{\partial}{\partial \eta}\,.$$

So apart from the obvious fact that g has degree at most n, since $n \neq 0$ there must be a second singularity on the fibre at ∞, i.e. it cannot also be the weak branch of a node. Whence if there is only one singularity on F, we can find another invariant fibre F' on which there are two singularities, blow up in the one off C, and blow down to reduce n, thus arriving in the $\mathbb{P}^1 \times \mathbb{P}^1$ situation.

¿From which we can note the final two cases of the classification,

Case IV.3.5 (d) (i) $(Y, \mathcal{G})$ is a Kronecker foliation on an absolutely minimal toric variety. In this case the minimal model is not unique, and may be flipped in its birational equivalence class.

We justify this deduction as follows. Necessarily $(Y', \mathcal{G}')$ is obtained by sequentially blowing up $(S, \mathcal{H})$ in zero dimensional strata of the $\mathbb{G}_m \times \mathbb{G}_m$ action, and $(Y, \mathcal{G})$ is obtained by sequentially blowing down 1-dimensional strata. Consequently $(Y, \mathcal{G})$ is smooth outside of zero dimensional strata. Now it suffices for the argumentation that has lead us to deduce the nature of $(S, \mathcal{H})$ that $(X, \mathcal{F})$ contained no -1-chains, so suppose it is such a contraction of $(X', \mathcal{F}')$ rather than the actual minimal model, which in turn we denote by $(X_0, \mathcal{F}_0)$, and $(Y_0, \mathcal{G}_0)$ the corresponding cover, then any exceptional divisor whose centre lies over a singularity of $(X, \mathcal{F})$ has negative discrepancy. However any exceptional divisor over the pre-image of a singularity in $(Y, \mathcal{G})$ is such a divisor so that by IV.3.3 (or more precisely its proof) it must have negative discrepancy. Equally by virtue of our explicit description of $(Y, \mathcal{G})$ the discrepancy for the exceptional divisor over a zero dimensional stratum is zero, so the covering map is actually étale in a neighbourhood of the same. Moreover if an irreducible curve of the 1-dimensional stratum contained a point over a singular point of $(X, \mathcal{F})$ then the image of such a curve in $(X, \mathcal{F})$ meets the singular locus of the foliation in 2-points on the non-singular locus of X and also passes through one singular point of the underlying space, and so by the considerations of III.3 $K_{\mathcal{F}}$ has positive intersection with such a curve, which is absurd. Consequently if $T = \sum_i C_i$ is a chain of rational integrable sub-curves on X contracted by the minimal model procedure, then the cover is actually étale in a neighbourhood of T. Whence if d is the degree of the cover then the pre-image of any C_i is d disjoint rational curves, C_{ij} say, with $C_{ij}^2 = C_i$. Further for reasons of counting singularities the pre-image of T is thus a disjoint union of d-chains of contractible rational integrable sub-curves, and the cover $(Y_0, \mathcal{G}_0)$ of $(X_0, \mathcal{F}_0)$ is itself a minimal model as required.

Argueing similarly we obtain,

Case IV.3.5 (d) (ii) $(Y, \mathcal{G})$ is a *toric vector foliation*, i.e. a foliation of the above form on an absolutely minimal compactification of $\mathbb{G}_a \times \mathbb{G}_m$ (i.e. no component of the boundary may be contracted).

IV.4. Kodaira dimension 1

In this section we study foliations whose Kodaira dimension $\kappa(X, \mathcal{F})$, as opposed to its numerical Kodaira dimension, is equal to 1. We begin with a lemma, viz,

Lemma IV.4.1. *Let $(X, \mathcal{F})$ be a foliated surface with canonical singularities and underlying space non-singular such that $1 \geq \kappa(X, \mathcal{F}) \geq 0$ then any elliptic Gorenstien leaf admits a $\mathbb{Q}$-foliated Gorenstien contraction.*

Proof. By hypothesis some power of $K_{\mathcal{F}}$, say $K_{\mathcal{F}}^{\otimes m}$, has a global section ω, say, and there exists some elliptic Gorenstien leaf with fundamental cycle Z, say. Now we distinguish two cases:

(a) ω does not vanish on any component of Z.

In consequence if $\widehat{X}$ is the completion of X along Z then $K_{\mathcal{F}}^{\otimes m} \mid \widehat{X}$ has a nowhere vanishing section, so that certainly the space obtained by contracting Z is $\mathbb{Q}$-foliated Gorenstien.

(b) ω vanishes on some components of Z.

In this case we write $\mathrm{div}(\omega) = D + W$, where W is supported on Z and the curves in the support of D are distinct from Z. Moreover we can make this decomposition not just on $(X, \mathcal{F})$ but on $(Y, \mathcal{G})$, say, where $(Y, \mathcal{G})$ is obtained by contracting the rational singularities of A_n or D_n type so that $K_{\mathcal{G}}$ is nef. The Kodaira dimension is however less than 2 so we obtain, $K_{\mathcal{G}} \cdot D = 0$. Whence if C is a curve in the support of D then $C^2 \geq 0$. Bearing in mind once again that $K_{\mathcal{G}}$ is not big we obtain in addition that $C^2 = D^2 = D.\,C = 0$, and whence $W.\,D = 0$. Necessarily then $K_{\mathcal{G}}^2 = W^2 < 0$, which is absurd. □

As an immediate consequence of the lemma we obtain, on applying IV.2.2

Corollary IV.4.2. *If $(X, \mathcal{F})$ is a minimal model of a foliated surface with $\kappa(X, \mathcal{F}) = 1$, then X is $\mathbb{Q}$-factorial.*

Now let us concentrate on a minimal foliated surface $(X, \mathcal{F})$ of Kodaira dimension 1, and let $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ be the uniqueminimal resolution of the underlying space. In addition let two sections of $K_{\mathcal{F}}^{\otimes m}$ be given with underlying divisors $D = \sum_i a_i C_i$ and $D' = \sum_i a'_i C'_i$ then, $C_i \cdot C_j = C_i \cdot C'_j = C'_i \cdot C'_j = 0$, for all possible combinations of i and j. On the other hand, we can certainly find an effective divisor E and sub-divisors D_0, D'_0 of D and D' respectively such that,

$$m\,K_{\mathcal{F}} - E = D_0 = D'_0$$

and since $D_0 \cdot D'_0 = 0$ this implies that $|m\,K_{\mathcal{F}} - E|$ defines a base point free pencil. Naturally we let $\pi : X \to B$ be its Stein factorisation, then the above considerations imply that every fibre is an irreducible curve, and for any $b \in B$, $X_b \cdot E = 0$, so E itself is contained in a fibre, and whence there is an effective $\mathbb{Q}$-Cartier divisor K on B such that $K_{\mathcal{F}} = \pi^* K$. To proceed further we will require, to make a covering $p : (Y, \mathcal{G}) \to (X, \mathcal{F})$ such that $p^k K_{\mathcal{F}}$ is Cartier, and ultimately we'd like p to be almost étale. To this end we know we can choose some sum of smooth fibres X_b in the smooth locus of X where b is a very ample divisor on B such that for some positive integers $m, n, m\,K_{\mathcal{F}}$ is Cartier and equal to $n\,X_b$. Modulo an almost étale cover we can assume without loss of generality that m and n are relatively prime. Now over the smooth locus of X consider the covering $p' : (X', \mathcal{F}') \to (X, \mathcal{F})$ associated to the m^{th} root of $n\,X_b$. This is defined in the usual way, namely over a Zariski open U_α in the smooth locus $K_{\mathcal{F}}$ is Cartier with local generator $\partial_\alpha^\vee$, where of course ∂_α is a generator of $T_{\mathcal{F}}$, so that the section s associated to $n\,X_b$ may be written as $f_\alpha\,\partial_\alpha^{-m}$. Consequently in the algebra $\operatorname{Sym} T_{\mathcal{F}}$ over U_α we have an ideal, $\partial_\alpha^m - f_\alpha$, and these patch to define a coherent sheaf of ideals $\mathcal{I}$ on $\mathbb{V}(T_{\mathcal{F}})$, as ever over the smooth locus, whose zero locus is X'. We can of course, and do, complete over the singular locus by just taking the normalisation of X in the appropriate function field, which we continue to denote by X' which is smooth over the smooth points of X, ramified along X_b, and almost étale everywhere else. Better still since X_b is in the smooth locus, $((p')^* X_b)_{\text{red}} = D$, say, is Cartier, and $(p')^* K_{\mathcal{F}}^{\otimes m} = mD$, so after an almost étale cover of X', which by abuse of notation we'll continue to denote by X', $(p')^* K_{\mathcal{F}}$ is Cartier. Now let $p'' : (X'', \mathcal{F}'') \to (X, \mathcal{F})$ be a similarly associated ramified covering for a different bunch of smooth fibres X_c, and let K', K'' be the corresponding function fields. We put $K = K' \cap K''$ and take Y to be the normalisation of X in K. So suppose Y is not étale over the generic point of X_b, call the generic point of the pre-image of X_b, η (certainly this corresponds to an irreducible divisor) then we have an exact sequence,

$$\Omega_X \otimes \mathcal{O}_{Y,\eta} \to \Omega_Y \otimes \mathcal{O}_{Y,\eta} \to \Omega_{Y/X} \otimes \mathcal{O}_{Y,\eta} \to 0$$

with the quotient being non-zero. Since we're working generically at η, $X'' \to Y$ is flat around η, so take ξ a point in the pre-image of η, then we have a commutative diagram of exact sequences,

$$\begin{array}{ccccccc}
\Omega_X \otimes \mathcal{O}_{X'',\xi} & \longrightarrow & \Omega_Y \otimes \mathcal{O}_{X'',\xi} & \longrightarrow & \Omega_{Y/X} \otimes \mathcal{O}_{X'',\xi} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\Omega_X \otimes \mathcal{O}_{X'',\xi} & \longrightarrow & \Omega_{X''} \otimes \mathcal{O}_{X'',\xi} & \longrightarrow & \Omega_{X''/X} \otimes \mathcal{O}_{X'',\xi} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \Omega_{X''/Y} \otimes \mathcal{O}_{X'',\xi} & = & \Omega_{X''/Y} \otimes \mathcal{O}_{X'',\xi} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Now $\Omega_{X''/X} \otimes \mathcal{O}_{X'',\xi} = 0$ since $X'' \to X$ is almost étale, so $\Omega_{X''/Y} \otimes \mathcal{O}_{X'',\xi} = 0$. Further $\Omega_Y \otimes \mathcal{O}_{X'',\xi}$ is torsion free, whence the first map in the middle column is an isomorphism and $\Omega_{Y/X} \otimes \mathcal{O}_{X'',\xi} = 0$. However $\mathcal{O}_{Y,\eta} \to \mathcal{O}_{X'',\xi}$ is a flat map between discrete (rank 1) valuation rings so: $\Omega_{Y/X} \otimes \mathcal{O}_{Y,\eta} = 0$. So certainly $Y \to X$ is almost étale over the generic point of X_b, and by a rather easier arguement at the generic point of any other divisor, so indeed $p : Y \to X$ is almost étale, whence by [Ka] 3.2, Y is also $\mathbb{Q}$-factorial. Consequently let $\mathcal{G}$ be the induced foliation on Y, then $K_{\mathcal{G}} = p^* K_{\mathcal{F}}$, $(Y, \mathcal{G})$ has canonical singularities, and what remains to check is that $K_{\mathcal{G}}$ is Cartier. We'll do this under the hypothesis that the foliation does not coincide with the Kodaira fibration, whence to this end we let $\pi : Y \to \widetilde{B}$ be the Stein factorisation of the Kodaira fibration on Y. Over X we have a natural map,

$$\pi^* \omega_B \left(\sum_i (m_i - 1) \, C_i \right) \to K_{\mathcal{F}}$$

where $m_i \, C_i$ are the multiple fibres. Obviously this map may vanish on some fibres, but equally it can only vanish on fibres. So for D_i the reduced support of any fibre (whence D_i is irreducible) let a_i be the order of vanishing along D_i of the natural map $\pi^* \omega_B \to K_{\mathcal{F}}$ (so $a_i \geq m_i - 1$, if the fibre has multiplicity m_i) then $\pi^* \omega_B \left(\sum_i a_i D_i \right) \to K_{\mathcal{F}}$ is an isomorphism of reflexive sheaves, and what we require to prove is that $p^* \mathcal{O}_X(a_i D_i)$ is Cartier. Now let $\pi : X' \to B'$, and $\pi : X'' \to B''$ be the Stein factorisations of the Kodaira fibration on X' and X'' respectively. If we put m_i to be the multiplicity of the fibre containing D_i, and D'_i in X' any irreducible divisor over D_i then, if b_i, b'_i are local equations for $\pi(D_i)$ and $\pi(D'_i)$ respectively we have for some $e'_i \in \mathbb{N}$, $b_i = (b'_i)^{e'_i}$. Whence if $n'_i = \mathrm{ord}_{D'_i}(b'_i)$ then, $m_i = e'_i n'_i$ since p' is almost étale. Furthermore $K_{\mathcal{F}'}$ is Cartier, so if ∂ is a local generator of the foliation about D_i, supposed distinct from X_b, $n'_i \mid \mathrm{ord}_{D'_i}(\partial b'_i)$ so that $a_i = n'_i d'_i$ for d'_i a non-negative integer. Consequently, $(b'_i)^{d'_i} \in k(B')$ is a $\frac{m_i}{a_i}$-th root of b_i, i.e. $((b'_i)^{d'_i}_i)^{m_i} = b_i^{a_i}$, since this is equally true for $k(B'')$ and $\mathbb{C}$ contains all roots of unity, $b_i^{a_i/m_i} \in k(B') \cap k(B'') = k(\widetilde{B})$. Even better $b_i^{a_i/m_i}$ is an integral element so it's actually in $\mathcal{O}_{\widetilde{B}}$ and for $\widetilde{D}_i$ lying over D_i we have, $\mathrm{ord}_{\widetilde{D}_i}(b_i^{a_i/m_i}) = a_i$, so $p^* \mathcal{O}_X(a_i D_i)$ is Cartier as claimed. To summarise we have proved,

Lemma IV.4.3. *If $(X, \mathcal{F})$ is a minimal model with $\kappa(\mathcal{F}) = 1$, and $\pi : X \to B$ the Stein factorisation of the Kodaira fibration, then if π and $\mathcal{F}$ do not coincide, there is an almost étale cover $p : (Y, \mathcal{G}) \to (X, \mathcal{F})$ such that $K_{\mathcal{G}} = p^* K_{\mathcal{F}}$ is Cartier.*

Let us dispose immediately of the case where the lemma fails, i.e. the foliation and the fibration coincide. Manifestly the generic fibre X_b is thus a smooth invariant curve with $K_{\mathcal{F}} \cdot X_b$ elliptic. By Kodaira's canonical bundle formula any elliptic fibration has foliated Kodaira dimension at most 1, and by the classification of IV.3 at least one, unless after an almost étale cover the fibration is a product so we obtain,

Corollary IV.4.4(i) *The foliation is an elliptic fibration which up to almost étale covers is not the elliptic fibration on a product $E \times B$ over a smooth curve B, with E a fixed elliptic curve.*

Now it's worthwhile (particularly for understanding abundance) to tabulate the possibilities for the singularities of an elliptic fibration from the foliated perspective. Obviously, modulo some blowing up and down, this will just be the same as the list of Kodaira. We pursue, however, the question directly from the foliated point of view, introducing the question by way of,

Proposition IV.4.5. *With hypothesis as given the unique minimal desingularisation $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ of a minimal foliated surface of Kodaira dimension 1 is an isomorphism in a neighbourhood of an invariant fibre unless the fibration $\pi : X \to B$ is an elliptic fibration, or the fibre is the special invariant rational fibre described at the end of the proof.*

Proof. In light of the classification of the singularities of X there are only 3 possibilities for a singular fibre which meets an integral sub-curve, viz: -1-chains, a chain of $-2\,F$-curves, or D_n[8]. Let us for the moment call this information of types I, II and III respectively and let $\widetilde{C}$ in $\widetilde{X}$ denote the proper transform of a fibre C through which there is a singularity. Consequently if $T_1, \dots, T_k$ are the maximal -1-chains in $\widetilde{X}$ meeting $\widetilde{C}$ with discriminants $d_1, \dots, d_k$ of the corresponding intersection matrices normalised in a positive sense then as ever $s_{\mathrm{I}} \geq k$ and $d_i \geq 2$, $\forall\, 1 \leq i \leq k$. Furthermore we have the formula,

$$K_{\mathcal{F}} \cdot C = (2g-2) + s_0 + s_{\mathrm{I}} + s_{\mathrm{II}} + s_{\mathrm{III}} - \sum_{i=1}^{k} \frac{1}{d_i}$$

where s_{I}, s_{II}, s_{III} are the multiplicities of the Segre class of C along the singular locus of $\widetilde{\mathcal{F}}$ supported at the type I, II and III information respectively, s_0 the Segre class at any singularities of $\mathcal{F}$ outside of these, and of course g is the genus of C. Consequently if $k > 0$ then certainly $g = 0$, while if $k = 0$ then $g \leq 1$ and indeed $g = 1$ would imply that $s_{\mathrm{II}} + s_{\mathrm{III}} = 0$. This being absurd, we conclude that C is always of genus 0, and the above formula becomes,

$$2 + \sum_{i=1}^{k} \frac{1}{d_i} = s_0 + s_{\mathrm{I}} + s_{\mathrm{II}} + s_{\mathrm{III}}$$

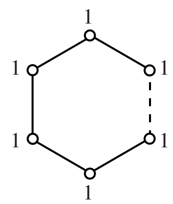
and we simply tabulate and consider all possibilities. Necessarily $k \leq 4$, $s_{\mathrm{I}} + s_{\mathrm{II}} + s_{\mathrm{III}} \geq 1$ and we have:

k	s_0	s_{I}	s_{II}	s_{III}
0	1	0	1	0
0	1	0	0	1
0	0	0	1	1
0	0	0	2	0
0	0	0	0	2
2	0	3	0	0
2	1	2	0	0
2	0	2	1	0
2	0	2	0	1
3	0	3	0	0
4	0	4	0	0

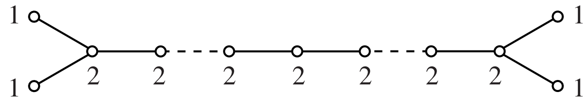
Naturally we begin with $k = 0$, so that $\widetilde{C}$ must be a $-2\,F$-curve, whence by the construction of the minimal model $\widetilde{C}^2 \leq -2$. On the other hand the intersection matrix of the curves in $p^{-1}(C)$ must be indefinite so we may apply the classification of integral quadratic forms to conclude that the said intersection matrix is one of the forms,

[8] We abuse standard convention, and for obvious reasons consider a "D_3" where the central node has valency 3, with an edge towards a possibly inexistent node, corresponding to a foliation singularity.

$\widetilde{A}_n$, $n+1$ vertices, $n \geq 1$



$\widetilde{D}_n$, $n+1$ vertices, $n \geq 6$

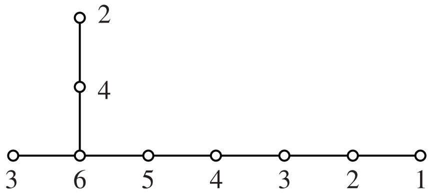


arising from the cases $s_{\mathrm{II}} = 2$, $s_{\mathrm{III}} = 0$ and $s_{\mathrm{II}} = 0$, $s_{\mathrm{III}} = 2$ respectively. In addition the fact that $C^2 = 0$ obliges all the nodes to correspond to curves with self-intersection -2 (i.e. this really is the usual Dynkin diagram rather than the dual graph of the fibre) and the corresponding fibre over $b = \pi(c)$ to be a multiple of the corresponding root system as indicated. Necessarily then the euler characteristic of the fibre $\widetilde{X}_b$ is zero as required.

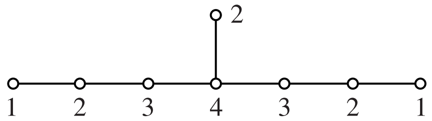
Now let us turn to the case $k = 2$. The sub-cases $s_{\mathrm{I}} = 3$ and $s_0 = 1$ are a little awkward, and we will return to them at the end. Regardless we must have $d_1 = d_2 = 2$, and so for $s_{\mathrm{I}} = 2$ the hypothesis that $C^2 = 0$ forces $\widetilde{C}^2 \geq -2$, and permits us to argue as above to conclude that the only possibility for the fibre $\widetilde{X}_b$ is $\widetilde{D}_n$, $n \geq 5$, corresponding to the case $s_{\mathrm{III}} = 1$, and that once again π is an elliptic fibration.

Next we take up the case $k = 3$, where we have the possible arrangements for the discriminants (d_1, d_2, d_3) being $(2,3,6)$, $(2,4,4)$ and $(3,3,3)$ respectively. Now if we consider the case $\widetilde{C}^2 \leqq -2$ then the various integrability conditions impose that the fibre must be a multiple of the root system in the respective Dynkin diagrams,

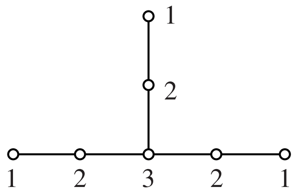
$\widetilde{E}_8$



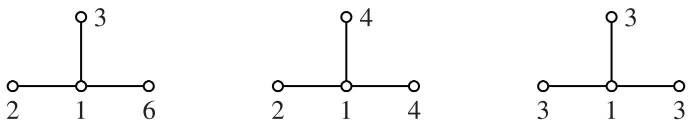
$\widetilde{E}_7$



$\widetilde{E}_6$



where in all cases the curve C corresponds to the unique node of valence 3 and once again the fibration is elliptic. Otherwise $\widetilde{C}^2 = -1$. We cannot however contract it to a canonical singularity of $(X, \mathcal{F})$ although we do observe that the fibre must arise as the intersection matrix of one of the following graphs, viz:

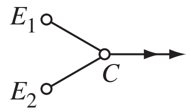


where now the weights denote the negative square of the self-intersection of the curve represented by the given node. In the first case contracting systematically and in order the curves with weights 1, 2 and 3 yields a rational curve with a cusp, while in the second case contracting the curves of weight 1 and 2 yields $\widetilde{A}_1$, i.e. two rational curves tangent in one point, while in the final case contracting the curve of weight 1 yields 3 lines meeting in a point. Regardless in all cases we conclude that π is an elliptic fibration.

So let us now take up the case of $k = 4$ then we must have that $d_1 = \cdots = d_4 = 2$, whence $\widetilde{C}^2 = -2$ and the fibre is necessarily the Dynkin diagram $\widetilde{D}_4$, and in particular forces π to be an elliptic fibration.

Finally let us return to the cases $k = 2$, $s_{\mathrm{I}} = 3$ or $s_0 = 1$ then $\widetilde{C}^2 = -1$ and the two -1-chains meeting $\widetilde{C}$ are in fact irreducible rational curves E_1, E_2 say of self-intersection -2.

Now if $s_{\mathrm{I}} = 3$, then at say the intersection of E_1 and C, E_1 is the strong branch of a node which by I.3.1 would force $E_1^2 = 0$. Since this is nonsense we have $s_0 = 1$, and $s_{\mathrm{I}} = 2$. Call the singularity not contained in E_1 or E_2, z. Again we can appeal to I.3.1 to see that the local contribution at z to C^2 in the residue formula is zero, and this can only happen if C is the strong branch of a node, i.e. in the Dynkin diagram notation we have,



where the free edge corresponds to the weak branch of the node. □

Ultimately however this explicit determination has little role when working on our almost étale covering $p : (Y, \mathcal{G}) \to (X, \mathcal{F})$ since the very fact that $K_{\mathcal{G}}$ is Cartier implies as ever by III that if $\rho : (Y', \mathcal{G}') \to (Y, \mathcal{G})$ is the unique minimal resolution then no -1 chains are contracted and $K_{\mathcal{G}'} = \rho^* K_{\mathcal{G}}$. Indeed, as in IV.3.5(d) we can even argue that $(Y, \mathcal{G})$ is a minimal. As in op.cit. use the subscript 0, e.g. $(X_0, \mathcal{F}_0)$ for the actual minimal model and $(X, \mathcal{F})$ for the contraction of the -1 chains, then we still get an almost étale cover $p : (Y, \mathcal{G}) \to (X, \mathcal{F})$. Now let C in Y be a $-2\,F$ curve, i.e. the push-forward of the same from the minimal smooth resolution $(Y', \mathcal{G}')$ then $K_{\mathcal{F}} \cdot p(C) = 0$. Further on C there are two singularities a and b, and the exceptional divisor of the blow up of either of these has discrepancy zero, so by IV.3.3, neither $p(a)$ nor $p(b)$ lie on the contraction of a -1 chain, with the discrepancy over both $p(a)$ and $p(b)$ understood with respect to the exceptional divisor on the blow up in the same still being zero. Appealing to $K_{\mathcal{F}} \cdot p(C) = 0$, we observe that the singular locus of X cannot meet C outside of either $p(a)$ or $p(b)$. So indeed if T in Y is a chain of $-2\,F$ curves then not only are the only possibilities for it to meet $\mathrm{sing}(Y)$ at the foliation singularities at the ends, but that the same is true for $p(T)$. However X was obtained by only contracting -1 chains, so infact X is smooth around $p(T)$, and since $Y \to X$ is almost étale, it's actually étale in a neighbourhood of T. Since everything is étale we obtain that a $-2\,F$ chain T in Y is contractible iff $p(T)$ is contractible. Consequently all $-2\,F$ chains in Y get contracted to the cover $p : (Y_0, \mathcal{G}_0) \to (X_0, \mathcal{F}_0)$ unless their images intersect. Whence suppose T, T' are maximal contractible chains in Y but $p(T) \cup p(T')$ is not contractible, thus as noted $p(T) \cap p(T') \neq \varphi$. Now either way $p(T) \cup p(T')$ must be a fibre of the Kodaira fibration $X \to B$, so it's pre-image in Y is disjoint union of fibres of $Y \to \widetilde{B}$ by an arguement of numerical connectedness and Zariski's lemma. Consequently if we add to our supposition, as we may, that T, T' are in distinct fibres of the Kodaira fibration we find there are curves C, C' not in T, T' such that some maximal contractible subchain whether of $C \cup T$ or $C' \cup T'$ has its image contracted by $X \to X_0$, so indeed Y_0 is minimal.

All of this in place and returning to the notation $p : (Y, \mathcal{G}) \to (X, \mathcal{F})$ for our cover of the minimal model the classification up to almost étale covers is now easy.

Case IV.4.4(ii) (Ricatti foliations) Here the Kodaira fibration $\pi : Y \to \widetilde{B}$ exhibits Y as a $\mathbb{P}^1$ bundle over $\widetilde{B}$. Indeed by the explicit tabulation of IV.4.5 given that $K_{\mathcal{G}}$ is Cartier, Y is non-singular, so this is true iff the generic fibre of X over B is $\mathbb{P}^1$. Necessarily, we have a natural isomorphism,

$$K_{\mathcal{G}} = \pi^* \, \omega_{\widetilde{B}} \left(\sum_i m_i \, b_i \right)$$

where the $m_i \in \mathbb{N}$ is the multiplicity along the fibre Y_{b_i} of the natural map from $\pi^* \, \omega_{\widetilde{B}} \to K_{\mathcal{G}}$, and $m_i \geq 1$ iff Y_{b_i} is an invariant fibre. Outside of the invariant fibres the foliation is everywhere transverse to the fibration by adjunction. This is a rather unusual thing for a curve to satisfy, and thus allows in the analytic topology a nice description. Specifically in a small tubular neighbourhood of a non-invariant fibre Y_b, projecting a leaf onto the point where it intersects Y_b gives a local first integral (cf. the graph construction of [BM]) and combining this with the projection π, gives over a sufficiently small neighbourhood Δ of b an isomorphism,

$$\begin{array}{ccc} Y_\Delta = \pi^{-1}(\Delta) = \mathbb{P}^1 \times \Delta & \xrightarrow[\nu]{} & \mathbb{P}^1 \\ {\scriptstyle \pi}\downarrow & & \\ \Delta & & \end{array}$$

such that locally over Δ the foliation is the fibration ν. Now let Δ_α be a cover by such discs of $Y \backslash \underset{i}{\cup} \; b_i$, and $[S_\alpha, T_\alpha]$ projective coordinates determined by this isomorphism. A priori the transition functions $g_{\alpha\beta}$ are in $\mathrm{PSL}_2(\Delta_{\alpha\beta})$, however these have to be locally constant along the leaves, so in fact must be in $\mathrm{PSL}_2(\mathbb{C})$, and the $[S_\alpha, T_\alpha]$ form a local projective system, i.e. the foliation is determined by a representation,

$$\rho : \pi_1(Y \backslash \underset{i}{\cup} \; b_i) \to \mathrm{PSL}_2(\mathbb{C}) \, .$$

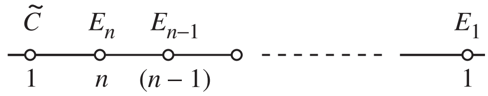
It's worth closing with a remark, or perhaps better a caveat, viz:

Z This is not the usual definition of Ricatti foliation. Indeed the main difference is that in the usual definition,

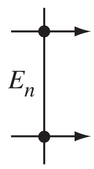
(a) the Kodaira dimension need not be 1, e.g. toric foliations are permitted.

(b) The a priori model on a $\mathbb{P}^1$-fibration is not taken to have canonical singularities.

Evidently Ricatti in the sense of IV.4.4(ii) is a priori weaker than this standard definition. However we have established that up to almost étale covers the two coincide. Indeed we've already seen an example in the form of the special rational fibre found in the proof of IV.4.5 (this corresponds to resolving a nilpotent singularity in the usual definition then blowing down the $-1\,F$ curves E_1, E_2 in the notations of op.cit.), which show that almost étale covers are wholly necessary. However the general situation requires a lot more than $(2:1)$ covers. Here is an example (as ever due to M. Brunella) consider over a disc Δ the foliation on $\mathbb{P}^1 \times \Delta$ given by, $n\,x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$ where y is a standard coordinate on $\mathbb{P}^1$ and x a coordinate on Δ, and $n \in \mathbb{N}$, $n \geq 2$. It's possible to fit these examples into global $\mathbb{P}^1$ bundles in such a way that the only algebraic curve in a neighbourhood of $x = 0$ is the fibre C itself. To resolve the singularity requires n-blow ups, these fit into a Dynkin diagram with the multiplicity as indicated, viz:



where the free edge denotes the singularity on the given curve, not contained in another curve of the diagram, and E_1 is the proper transform of the exceptional curve on the i^{th}-blow up. Now the key point is that E_n is not invariant and its contraction in the minimal model looks like,



the marked points being singularities, the horizontal arrows the possible directions of foliation branches at the same. In particular the Kodaira fibration has multiplicity n along E_n, and $n\,K_{\mathcal{F}}$ is the smallest multiple of $K_{\mathcal{F}}$ which is Cartier. Since n is arbitrary, it shows there is no hope of an effective generation result without almost étale covers.

Case IV.4.4(iii) (Turbulent foliations) Here the Kodaira fibration $\pi : Y \to \widetilde{B}$ simply expresses Y as a product $E \times B$ for E a fixed elliptic curve. Indeed a priori, by the tabulation of IV.4.5, the minimal resolution of singularities Y' could have singular fibres of type A_n over $\widetilde{B}$ if the generic fibre of $X \to B$ is elliptic. However we can pull the same projection onto fibres trick of IV.4.4(ii) to exhibit not just the foliation but also Y locally as a product $E \times \Delta$ in the analytic topology, which forces the modular invariant to be constant so there cannot be any A_n's and Y is non-singular. The fibration being elliptic without A_n's, once more gives $q(Y) = p_g(Y) + 1$, so Y is definitely sesquielliptic as per IV.3.1$^{1/2}$, and we might as well just take a further covering to make it a product for convenience. In particular the foliation is smooth, since a singularity in the fibre would give a singular fibre by IV.4.5. Just as in the Ricatti case we have a natural isomorphism,

$$K_{\mathcal{G}} = \pi^* \omega_{\widetilde{B}} \left(\sum_i m_i Y_{b_i} \right)$$

where m_i is the multiplicity of the natural map, $\omega_{\widetilde{B}} \to K_{\mathcal{G}}$ along Y_{b_i}, with $m_i \geq 1$ iff the fibre is invariant, and as we've already noted we can argue as above to see that the foliation is determined by a representation,

$$\rho : \pi_1(\widetilde{B} \setminus \bigcup_i b_i) \to \mathrm{Aut}(E)\,.$$

The final case is the easiest of all,

Case IV.4.4(iv) The Kodaira fibration exhibits $\pi : Y \to \widetilde{B}$ as an isotrivial family of curves of genus ≥ 2, so after an étale cover say $Y = C \times \widetilde{B}$ for C of genus ≥ 2, and the foliation is just the fibration $Y \to C$ and necessarily $g(C)$ and $g(\widetilde{B}) \geq 2$. Indeed as soon as $\pi : X \to B$ is generically of genus at least 2, the same is true of $Y \to \widetilde{B}$ and by IV.4.5, Y must be non-singular. Multiple fibres of π would be invariant, which is clearly absurd if the fibre were smooth and quite generally absurd in the singular case by the tabulation of IV.4.5, so in fact $\pi^* \omega_B \to K_{\mathcal{G}}$ is an isomorphism. Consequently π is an isotrivial family, and certainly $g(C)$, $g(\widetilde{B}) \geq 2$. Iso-trivial families of genus at least 2 can also occur in the previous case. Necessarily the Kodaira fibration has no invariant fibres, the base $\widetilde{B}$ has genus at least 2, and the representation ρ is torsion.

IV.5. Sporadic foliations

We begin with a lemma which shows that we have a stark choice i.e. if $\nu(X, \mathcal{F}) = 1$, where as ever $(X, \mathcal{F})$ is the minimal model of a foliated surface then either $\kappa(X, \mathcal{F}) = 1$ or $\kappa(X, \mathcal{F}) = -\infty$. Specifically we prove,

Proposition IV.5.1. *Hypothesis as above if $\kappa(X, \mathcal{F}) \geq 0$ then in fact $\kappa(X, \mathcal{F}) = 1$, so that in particular the classification is exactly as in the previous chapter.*

Proof. Suppose indeed $\kappa(X, \mathcal{F}) \geqq 0$, then there is an effective $\mathbb{Q}$-Cartier divisor, $\sum_i a_i C_i$, say, where C_i are irreducible curves which represents $K_{\mathcal{F}}$, and by IV.4.1 $(X, \mathcal{F})$ is $\mathbb{Q}$-foliated Gorenstien, and $C_i^2 = C_i \cdot C_j = 0 \ \forall\, i, j$. Now we consider various cases.

Firstly suppose some curve C from amongst the C_i is an integral sub-curve of $\mathcal{F}$. Then either C meets the singular locus of X or it does not. Should C not meet the locus where the underlying space X is singular then either C is an elliptic curve or a rational curve with a node. In either case, $K_X \cdot C = C^2 = 0$ and C is a divisor of elliptic fibre type. Alternatively C meets $\mathrm{sing}(X)$, and we may consider the unique minimal smooth resolution

$\rho : \widetilde{X} \to X$. The argument of IV.4.5 goes through verbatim to establish that $\rho^* C$ is again of elliptic fibre type, or the special rational fibre at the end of the proof. However this latter certainly moves in a 1-dimensional family. However if $q(X) > 0$ then the rational fibration determined by C is the Albanese, so certainly in this case $k(\mathcal{F}) = 1$. Now it is well known that a divisor of elliptic fibre type on a surface of positive, K_X, Kodaira dimension must arise from an elliptic fibration, and so in this case we may conclude. However, cf. [R], this is probably the least attractive part of ordinary classification, and in addition is not true for rational surfaces. Whence we will prove:

Divertimento IV.5.1 (bis) Let D on $\widetilde{X}$ be a divisor of elliptic fibre type such that D_{red} is a leaf of $\widetilde{\mathcal{F}}$ then $K_{\mathcal{F}}$ has Kodaira dimension 1, and the $K_{\mathcal{F}}$-Kodaira fibration is an elliptic fibration.

Proof. By IV.4.5 it suffices to show that the foliation has Kodaira dimension 1. We consider cases according to the irregularity. Observe that if the foliation is a fibration, then by Zariski's lemma it is an elliptic fibration and so we need only consider, $q(\widetilde{X}) = 2$ and $h^0(K_{\widetilde{\mathcal{F}}}) = 1$, $q(\widetilde{X}) = 1$ and $h^0(K_{\widetilde{\mathcal{F}}}) = 1$, or $q(\widetilde{X}) = 0$. In either of the cases $q(\widetilde{X}) > 0$ the divisor D must be a genuine elliptic curve, and the Albanese map must be to either an abelian surface or an elliptic curve. Consider the case $q(\widetilde{X}) = 1$, as we will see on returning to the proof of IV.5.1 we can suppose without loss of generality that D is in the support of the map from $\omega_{\mathrm{Alb}(\widetilde{X})} \to K_{\mathcal{F}}$. However we have a commutative diagram of natural maps,

$$\begin{tikzcd}
\omega_{\mathrm{Alb}(\tilde{X})}|_{D_{\mathrm{red}}} \arrow[r] \arrow[dr] & \omega_{D_{\mathrm{red}}} \arrow[d, "2"] \\
& k_{\mathcal{F}}|_{D_{\mathrm{red}}}
\end{tikzcd}$$

the composite diagonal map is zero, so the Albanese fibration is the desired fibration. Should $q(\widetilde{X}) = 2$, we may redure to the same situation after taking étale covers, on replacing the Albanese map by the map from Alb $(\widetilde{X})$ to the image of D in the same obtained via Poincaré decomposition.

Thus by Riemann-Roch, we are reduced to studying the case where $q(\widetilde{X}) = p_g(\widetilde{X}) = 0$. Call the image of D_{red} in $(X, \mathcal{F})$, C, then for some $m, n \in \mathbb{N}$ we must have,

$$m\, K_{\mathcal{F}} = n\, C\,.$$

Suppose that m, n have a common factor l, say, then $m = l\, m_0$, $n = l\, n_0$ with $m_0, n_0 \in \mathbb{N}$ and,

$$l\,(m_0\, K_{\mathcal{F}} - n_0\, C) = 0\,.$$

So we may pass to a cover which is étale in codimension 1, and apply IV.3.3 to conclude without loss of generality $(m, n) = 1$. Now consider the cover in $\mathbb{V}(K_{\mathcal{F}}^{-m})$ defined in codimension 2, corresponding to a m^{th}-root of

nC, and define a scheme Y by taking $\mathcal{O}_Y$ to be the normalisation of $\mathcal{O}_X$ in the function field of the same. This gives us a finite cover $\nu : Y \to X$ ramified over C, which locally in codimension 2 just corresponds to taking a m^{th}-root of a local equation for C. Whence there is a Weil divisor C' on Y such that, $\nu^* C = m\, C'$ as Weil divisors. Better still since C is an integrable sub-curve on X, $K_{\mathcal{G}} = \nu^* K_{\mathcal{F}}$, again a priori as Weil divisors, but a postiori as $\mathbb{Q}$-Cartier divisors since $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier on X. The proof of IV.3.3 only required that this identity be satisfied, so $(Y, \mathcal{G})$ has canonical singularities. Whence modulo passing to yet another cover, this time étale in codimension 1 to kill any torsion, and appealing to the minimal model theorem in its full generality for singular spaces we may assume without loss of generality that,

$$K_{\mathcal{F}} = n\, C\,.$$

We now distinguish various cases according to the type of C,

(a) C is an elliptic curve.

By adjunction, as applied in IV.4.5, C does not meet $\operatorname{sing}(X)$ and so $(X, \mathcal{F})$ is foliated Gorenstien. Whence by III.1.4, $K_{\widetilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$, and Riemann-Roch implies $H^1(\widetilde{X}, K_{\widetilde{\mathcal{F}}}) = 0$. However C is contained in the smooth locus of the foliation so its chern class is in,

$$\operatorname{Im}\{H^1(\widetilde{X}, K_{\widetilde{\mathcal{F}}})^\vee \to H^{1,1}(\widetilde{X})\}$$

which is absurd.

(b) $\rho^* C$ is a rational curve with a node or an elliptic polygon.

This is very similar to (a). Again $(X, \mathcal{F})$ is foliated Gorenstien, and $K_{\widetilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}} = nD$, where D is the said elliptic divisor, so that once more Riemann-Roch gives $H^1(\widetilde{X}, K_{\widetilde{\mathcal{F}}}) = 0$. Now in general there is an obstruction to an elliptic polygon being in,

$$\operatorname{Im}\{H^1(\widetilde{X}, K_{\widetilde{\mathcal{F}}})^\vee \to H^{1,1}\}$$

even when, as here, it passes through no other foliation singularity than the singularities of the curve itself. However in this particular case D is properly the Dynkin diagram $\widetilde{A}_n$, i.e. all components have self intersection -2, and an explicit calculation using I.3.1 shows that this can only happen if the eigenvalue at each singularity is -1. Consequently the obstruction posed by the residue map is zero, so we once more have the absurdity that D is zero in $H^{1,1}$.

(c) The elliptic divisor D corresponding to the appropriate multiple of $\rho^* C$ is one of the Dynkin diagrams $\widetilde{D}_n$, $n \geq 4$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$.

Since every component of D has self intersection -2 and $(X, \mathcal{F})$ is Gorenstien off D we have $K_{\widetilde{\mathcal{F}}} - D_{\text{red}} = \rho^*(K_{\mathcal{F}} - C)$, and $D_{\text{red}}^2 = -2$. Consequently

Riemann-Roch gives $\chi(\widetilde{X}, K_{\widetilde{\mathcal{F}}}) = 0$, and whence, $h^1(\widetilde{X}, K_{\widetilde{\mathcal{F}}}) = 1$. As ever consider the dévissage defining the foliation, i.e.

$$0 \to N \to \Omega_{\widetilde{X}} \to K_{\widetilde{\mathcal{F}}} \cdot I_Z \to 0$$

for Z the singular subscheme of $\widetilde{\mathcal{F}}$, then we get an exact sequence,

$$0 \to H^0(\widetilde{X}, K_{\widetilde{\mathcal{F}}} \cdot I_Z) \to H^1(\widetilde{X}, N) \to H^{1,1}\,.$$

Now D is certainly in the image of the last map whether by explicit calculation, or the fact that the number of curves exceeds the number of singularities. However $h^1(\widetilde{X}, N) = 1$, so there must be foliation singularities, say Z', other than those supported in D. The next thing we observe is that for $m \gg 0$, $H^0(\widetilde{X}, K_{\widetilde{\mathcal{F}}} + D_{\mathrm{red}}(-mD)) = 0$, so that Riemann-Roch gives $H^0(\widetilde{X}, N(-D_{\mathrm{red}} + mD)) \neq 0$. In particular for some effective divisor E, possibly zero, supported off D, $m_i \in \mathbb{Z}$, and C_i curves supported in D,

$$N = E + \sum_i m_i\, C_i\,.$$

However $\rho^* K_{\mathcal{F}} \cdot N = 0 = \rho^* K_{\mathcal{F}} \cdot C_i$, so $\rho^* K_{\mathcal{F}} \cdot E = 0$, and E must be supported on $-2\,F$ curves. Denoting by X' the contraction of the $-2\,F$ curves off D, we therefore have that $N\left(-\sum_i m_i\, C_i\right)$ is a torsion $\mathbb{Q}$-Cartier divisor on X', so modulo an almost étale cover we can suppose that $N\left(-\sum_i m_i\, C_i\right)$ is Cartier, which obliges and $-2\,F$ curve to have self intersection -2, and E to be zero. Consequently without loss of generality we have a global meromorphic 1 form ω in N with poles at most on D, so naturally $d\omega$ is a 2-form with poles of the said form and the issue is whether it's zero or not. Now there are definitely foliation singularities off D on $\widetilde{X}$, and by Zariski's lemma N is a negative multiple of D, and D comes from $H^1(X, N)$, so the residue of N at every singularity off D is zero. However, quite generally, the image of $c_1(N)$ under the residue map is to 1st order the trace of the linearisation of the vector field at the singularity, so all the eigenvalues off D must be -1, whence if $d\omega \neq 0$ there is a non-zero effective Cartier divisor E supported off D, and $m_i \in \mathbb{Z}$ with as before C_i curves supported in D such that,

$$K_X = E + \sum_i m_i\, C_i$$

and again $K_{\mathcal{F}} \cdot K_X = K_{\mathcal{F}} \cdot C_i$ forces E to be supported on $-2\,F$ curves, so once more modulo and almost étale cover $E = 0$, which is absurd unless the cover destroys the foliation singularities off D, which we might as well assume doesn't happen, so without loss of generality ω is closed. Given that ω has meromorphic poles, it's class in $H^1_{DR}(\widetilde{X}\backslash D)$ is non-zero, unless we have an elliptic fibration in which case we'd be done, so without loss of generality $(\widetilde{X}, D)$

has non-trivial logarithmic Albanese. However here we have a dévissage of the form,

$$0 \to N(D_{\mathrm{red}}) \to \Omega^1(\log D_{\mathrm{red}}) \to K_{\mathcal{F}} \cdot I' \to 0$$

where I' is a non trivial ideal with $\mathcal{O}_{\widetilde{X}}/I'$ supported off D, so in fact $H^0(\widetilde{X}, K_{\mathcal{F}} \cdot I') = 0$, and by the closure of global forms with logarithmic poles, the foliation must again be an elliptic fibration.

(d) Otherwise.

There are 3 remaining possibilities for the elliptic divisor D by the tabulation of IV.4.5. In all of these $\widetilde{C}$ is an exceptional curve of the 1^{st} kind and, $K_{\widetilde{\mathcal{F}}} = \rho^*(K_{\mathcal{F}} + C) - \widetilde{C}$ which again gives $\chi(\widetilde{X}, K_{\widetilde{\mathcal{F}}}) = 0$, and so (c) goes through verbatim.

Otherwise no curve C from amongst the C_i is an integral sub-curve, and indeed for any C in the support of $K_{\mathcal{F}}$ with $\rho : \widetilde{X} \to X$ as above we may appeal to the proof of II.4.1 and III.1.1 to conclude that in fact,

$$(K_{\widetilde{\mathcal{F}}} + \widetilde{C}).\, \widetilde{C} = (K_{\mathcal{F}} + C).\, C = 0$$

where $\widetilde{C}$ is the proper transform of C in $\widetilde{X}$. Better still,

(a) $\widetilde{\mathcal{F}}$ is smooth in a neighbourhood of $\widetilde{C}$.

(b) $\mathcal{O}_{\widetilde{C}}(-\widetilde{C})$ embeds as a line bundle in $\Omega^1_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{C}}$, so in fact $\widetilde{C}$ is smooth.

(c) $K_{\mathcal{F}} + C$ is Cartier, and is equivalent to a positive rational multiple of the $\mathbb{Q}$-Cartier divisor C.

In particular $(X, \mathcal{F})$ is foliated Gorenstien iff C is Cartier, which is in fact iff $\widetilde{X}$ is isomorphic to X around C, but all we'll really use is that the minimum integer n such that $\mathcal{O}_X(n\, K_{\mathcal{F}})$ or $\mathcal{O}_X(n\, C)$ is Cartier is the same for $K_{\mathcal{F}}$ and C. In any case $T_{\mathcal{F}}(-C)$ is also Cartier, so for a suitable open cover indexed by $\alpha \in A$, let ∂_α be a local generator for $T_{\mathcal{F}}(-C)$ and x_α the same for $\mathcal{O}_X(-n\, C)$. Call the respective transition functions $g_{\alpha\, \beta}$ and $f_{\alpha\, \beta}$ then,

$$\frac{\partial_\alpha\, x_\alpha}{x_\alpha} = \frac{\partial_\alpha\, f_{\alpha\beta}}{f_{\alpha\beta}} + \frac{\partial_\alpha\, x_\beta}{x_\beta} = g_{\alpha\, \beta}\, \frac{\partial_\beta\, x_\beta}{x_\beta} \pmod{\mathcal{I}_C}$$

and so $\partial_\alpha \log x_\alpha$ gives a global section of $\mathcal{O}_C(K_{\mathcal{F}} + C)$, which outside of the singularities is just n-times the usual section given by adjunction. Consequently it's non-zero, and gives an isomorphism $\mathcal{O}_C \xrightarrow{\sim} \mathcal{O}_C(K_{\mathcal{F}} + C)$ since $(K_{\mathcal{F}} + C) \cdot C = 0$ forces it to be nowhere vanishing. Now for $k \in \mathbb{N}$, let X_k be the thickening of C in X by the k^{th} symbolic power $\mathcal{I}_C^{(k)} = \mathcal{O}_C(-kC)$, and let's set about proving by induction that $\partial_\alpha \log x_\alpha$ also gives an isomorphism $\mathcal{O}_{X_k} \xrightarrow{\sim} \mathcal{O}_{X_k}(K_{\mathcal{F}} + C)$. Quite generally we have the exponential sequence,

$$0 \longrightarrow \mathcal{I}_C^{(k)}/\mathcal{I}_C^{(k+1)} \xrightarrow{\exp = 1+x} \mathcal{O}^*_{X_{k+1}} \longrightarrow \mathcal{O}^*_{X_k} \longrightarrow 0$$

giving rise to,

$$H^0(\mathcal{O}^*_{X_k}) \xrightarrow{\delta} H^1(C, \mathcal{I}_C^{(k)}/\mathcal{I}_C^{(k+1)}) \longrightarrow \mathrm{Pic}(X_{k+1}) \longrightarrow \mathrm{Pic}(X_k) \longrightarrow 0\,.$$

The basic fact to note for later reference is,

Fact. $\mathrm{Im}(\delta)$ is a subcomplex vector space of $H^1(C, \mathcal{I}_C^{(k)}/\mathcal{I}_C^{(k+1)})$.

Indeed for any $\rho \in \mathbb{C}$, and $h \in \mathcal{O}_{X_{k+1}}$ congruent to 1 modulo $\mathcal{I}_C$, we can define h^ρ by the usual Taylor expansion, so that indeed if g is another such function, $(hg)^\rho = h^\rho\, g^\rho$. On the other hand if $h \in H^0(\mathcal{O}^*_{X_k})$ then its restriction to C is a global unit, so a constant, say $\lambda \in \mathbb{C}^\times$ and $\delta(h) = \delta(h/\lambda) + \delta(\lambda) = \delta(h/\lambda)$, so to calculate $\delta(h)$ we can suppose without loss of generality that it's congruent to 1 modulo C. Lifting h with respect to our cover to $\mathcal{O}^*_{X_{k+1}}$ gives $p_\alpha \in \mathcal{I}_C$ with $h = (1+p_\alpha)$, and $(1+p_\alpha)(1+p_\beta)^{-1} = 1 + p_{\alpha\beta}$, for $p_{\alpha\beta} \in \mathcal{I}_C^{(k)}/\mathcal{I}_C^{(k+1)}$ a 1 cocycle representing $\delta(h)$. However $(1+p_\alpha)^\rho(1+p_\beta)^{-\rho} = (1+p_{\alpha\beta})^\rho = 1 + \rho\, p_{\alpha\beta}$, so $\delta(h^\rho) = \rho\, \delta(h)$, and we're done.

Now let's offer the precision that our cover is in the étale topology. For some positive integers p and q we have that $p(K_{\mathcal{F}} + C) = q(n\,C)$ on X, so our induction hypothesis gives that for a q-torsion valued cocycle $\zeta_{\alpha\beta}$ and a 1-cocycle $h_{\alpha\beta} \in \mathcal{I}_C^{(k)}/\mathcal{I}_C^{(k+1)}$ we have, $f_{\alpha\beta} = \zeta_{\alpha\beta}(1 + h_{\alpha\beta})$. Equally for some function $u_{\alpha\beta}$, $h^n_{\alpha\beta} = u_{\alpha\beta}\, x^k_\beta$ holds in $\mathcal{O}_X$, so that

$$\frac{\partial_\alpha\, x_\alpha}{x_\alpha} = \exp\left(\frac{k}{n}\, h_{\alpha\beta}\right) g_{\alpha\beta}\, \frac{\partial_\beta\, x_\beta}{x_\beta}\,.$$

Consequently if L is the bundle with transition functions $\exp\left(-\frac{h_{\alpha\beta}}{n}\right)$ then, $\partial_\alpha \log x_\alpha$, gives an isomorphism $\mathcal{O}_{X_{k+1}} \xrightarrow{\sim} \mathcal{O}_{X_{k+1}}(K_{\mathcal{F}} + C) \otimes L^k$. From which we obtain, $L^{kp} \xrightarrow{\sim} \mathcal{O}(-q\,n\,C) \xrightarrow{\sim} L^{-nq}$, so: $(nq + kp)L = 0$, and by our vector space fact we've finished our induction. Better still since C is $\mathbb{Q}$-Cartier the symbolic powers are certainly co-final in the $\mathcal{I}_C$-adelic topology, so if $\widehat{X}$ is the completion of X in C then $\mathrm{Pic}(\widehat{X}) = \varprojlim \mathrm{Pic}(X_k)$ and so in fact we establishes that the Hilbert scheme of X has positive dimension in a neighbourhood of $q\,n\,C$ (just consider the family $\mathbb{I}^q_{nC} = \mathrm{const}$). Whence there is a curve A, a family of curves $\nu : Y \to A$ with nC as a fibre and maps,

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\nu"] \arrow[dr, "\mu"] & X \times A \arrow[d] \\ A & X \end{tikzcd}$$

such that the projection $\mu : Y \to X$ is dominant, and $\mu^*\, K_{\mathcal{F}}$ is an effective $\mathbb{Q}$-Cartier divisor whose intersection with the generic fibre of A is zero. Whence if $Y^\#$ and $A^\#$ are the respective normalisations of Y and A then $K_{\mathcal{F}}\,|_{Y^\#}$ is the pull-back of a $\mathbb{Q}$-Cartier divisor on $A^\#$ so that $K_{\mathcal{F}}$ has Kodaira dimension 1 as required. $\square$

Next keeping to the hypothesis $\nu(X, \mathcal{F}) = 1$, let us deal as always with the case of irregular surfaces, i.e.

Lemma IV.5.2. *If $(X, \mathcal{F})$ is the minimal model of a foliated surface which is not a fibration by rational curves and $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ the unique smooth resolution, such that the underlying space $\widetilde{X}$ has $q(\widetilde{X}) \geq 1$, then $\kappa(X, \mathcal{F}) \geq 0$. Nor can the foliation be a fibration if $\kappa(X, \mathcal{F}) = -\infty$.*

Proof. We recall as in IV.3.1 that by restricting global holomorphic 1-forms to $K_{\widetilde{\mathcal{F}}}$ we see that the lemma is trivial, unless $h^0(\widetilde{X}, N) \geq 2$ if $q(\widetilde{X}) \geq 2$ or $h^0(X, N) = 1$ if $q(\widetilde{X}) = 1$, where as ever $0 \to N \to \Omega^1_{\widetilde{X}}$ is the saturated map defining the foliation. In the first case a well-known lemma of Castelnuovo, cf. [B], implies the foliation is a fibration while in the second case it must be the Albanese fibration. In either case the quotient space $\widetilde{X}/\widetilde{\mathcal{F}}$ exists by virtue of the singularities being canonical, and better still the quotient map $\pi : \widetilde{X} \to B := \widetilde{X}/\widetilde{\mathcal{F}}$ admits a factorisation through $\pi : X \to B$ where B is equally $X/\mathcal{F}$ and of course B is a smooth curve.

Now if π is an elliptic fibration then we may conclude by IV.3.1 and IV.4, otherwise the generic fibre has genus $g \geq 2$, and we're basically in the situation of a theorem of Arakelov, cf. [Sz], except that in op.cit. the fibration $X \to B$ is assumed to be semi-stable. This is, however, a limited problem. Indeed consider a semi-stable reduction of the fibration π, viz:

$$\begin{tikzcd} \widetilde{X} \arrow[dd, "\pi"'] & \widetilde{Y} \arrow[l, "\nu"] \arrow[dd, "\pi"'] \arrow[dr] & \\ & & Y \arrow[dl] \\ B & C \arrow[l, "\nu"] & \end{tikzcd}$$

with induced foliation/fibration $\widetilde{\mathcal{G}}$, $\mathcal{G}$ on $\widetilde{Y}$ and Y respectively. The map ν is ramified only in fibres and is chosen in such a way that blowing down exceptional curves of the 1^{st} kind in the fibres leads to a non-singular Y fibring over C with semi-stable fibres. From the foliation point of view Y is the minimal smooth resolution of the minimal model, and Arakelov's theorem asserts that $K_{\mathcal{G}}^2 > 0$ unless the fibration is isotrivial[9]. The map ν being ramified only in fibres implies $K_{\widetilde{\mathcal{G}}} = \nu^* K_{\mathcal{F}}$, and $(Y, \mathcal{G})$ has canonical singularities so we know that $K_{\widetilde{\mathcal{G}}}$ is big, if the fibration is not isotrivial. Better still bigness is equivalent to strict positivity on the closed cone of nef. $\mathbb{R}$-divisors, and $\nu^* : \mathrm{NS}^1(\widetilde{X})_{\mathbb{R}} \to \mathrm{NS}^1(\widetilde{Y})_{\mathbb{R}}$ gives an injection of the said cone on $\widetilde{X}$ into that of $\widetilde{Y}$, so if $K_{\widetilde{\mathcal{G}}}$ is big then $K_{\widetilde{\mathcal{F}}}$ is big. Consequently the fibration is isotrivial.

[9] The reference in question actually proves the theorem in positive characteristic, and so trivially implies it in characteristic zero. The positive characteristic arguement is not only substantially more elegant, but far simpler.

Since $K_{\mathcal{G}}$ is Cartier we may appeal to IV.4.4, to find curves D on $\widetilde{Y}$ with $K_{\widetilde{\mathcal{G}}} \cdot D = 0 = K_{\mathcal{F}} \cdot \nu_* D$, so from what we've classified already $X \to B$ isn't just isotrivial but is a product after an almost étale cover. $\square$

We are therefore left to investigate the possibility of $\nu(X, \mathcal{F}) = 1$, and $\kappa(X, \mathcal{F}) = -\infty$ on regular surfaces. As a warm up, and indeed in order to show that this can happen, let us consider:

Example IV.5.3. Let $(X, \mathcal{F})$ be a smooth foliated surface with $\nu(X, \mathcal{F}) = 1$ and $\kappa(X, \mathcal{F}) = -\infty$ then X is uniformised by the bi-disc, and indeed the natural foliation on any smooth surface X which is uniformised by the bi-disc is of this type unless it is isogeneous to a product of curves.

Proof. Suppose, $0 \to N \to \Omega^1_X$ defines the foliation. By IV.5.2 $q(X) = 0$, and so $c_2(X) > 0$. Thus a minor generalisation of IV.1.3 shows that N is nef., and $c_1^2(X) = 2c_2(X) > 0$. Whence X has general type, and is minimal as a surface. Better still if C is an exceptional curve of the second kind then $K_{\mathcal{F}} \cdot C = 0$, and since $(X, \mathcal{F})$ is minimal, $C^2 \geq 0$, which is absurd. Consequently K_X is ample and Ω_X admits an Einstein-Kahler metric. However we note that $K_X \cdot K_{\mathcal{F}} = K_X \cdot N = c_2(X)$, so Ω_X is not stable, and we have a splitting,

$$\Omega_X = N \oplus K_{\mathcal{F}} .$$

This then implies, cf. [Y], that X is uniformised by the bi-disc, and the foliation is just the natural foliation arising from this uniformisation[10]. Conversely given such a foliation $\nu(X, \mathcal{F}) = 1$, so by a combination of our various considerations in IV.5.1 and IV.4 we must have $\kappa(X, \mathcal{F}) = -\infty$ should X not be isogeneous to a product of curves.

The next question one may ask is whether such foliations can occur with singularities. As we have previously hinted this is indeed so. However before exploring a logarithmic analogue of the above let us prove a little lemma, viz:

Lemma IV.5.4. *Let $(X, \mathcal{F})$ be a foliated surface with underlying space X non-singular and Z a cyclic configuration of smooth rational integral sub-curves or a single rational integral sub-curve with a node such that $K_{\mathcal{F}} \cdot Z_i = 0$ for any component Z_i of Z then $(X, \mathcal{F})$ has canonical singularities in a neighbourhood of Z or $(X, \mathcal{F})$ is a pencil of rational curves. In particular in the former case Z is an elliptic Gorenstien leaf.*

Proof. Suppose for simplicity that Z has more than one component. The condition $K_{\mathcal{F}} \cdot Z_i = 0$ forces the support of $\mathrm{sing}(\mathcal{F}) = \mathrm{sing}(Z)$ in a neighbourhood of Z. Now a canonical resolution is obtained by blowing up in points of $\mathrm{sing}(\mathcal{F})(\mathbb{C})$. If we blow up in such a point then either the discrepancy is zero, and there is a new cycle consisting of the proper transform of Z and

[10] This is also in [Be] after some preparatory lemmas which are assumed within our general considerations of § II.

the exceptional curve which satisfies the same hypothesis, or the discrepancy is negative. If $(X, \mathcal{F})$ is not a rational pencil this discrepancy must be -1. We therefore obtain a chain of rational integral sub-curves all of which intersected with the canonical class give 0, and curves at either end intersecting in -1. Without loss of generality we may assume that the singularities at either end are canonical. Blowing up in any singular point not at the ends does not change this configuration if it has zero discrepancy. Otherwise as before the discrepancy must be -1, and we can take a sub-chain of the same type. However we must in doing this eventually arrive at the situation of canonical singularities and so we are done by III.1.2. $\square$

Now let us turn to our example,

Example IV.5.5. Let $(X, \mathcal{F})$ be a foliated surface with canonical singularities and underlying space non-singular on which every singularity is contained in an elliptic Gorenstien leaf then $\nu(X, \mathcal{F}) = 1$ and $\kappa(X, \mathcal{F}) = -\infty$ iff $(X, \mathcal{F})$ is an almost étale cover of one of the natural foliations on a Hilbert modular surface (which is not a fibration by algebraic curves).

Proof. Consider a Hilbert-Modular as a quotient of $\mathfrak{H}^2$ and consider the foliations arising from the splitting. Now consider the same on the Baily-Borel compactification, and call it $(Y_0, \mathcal{G}_0)$ with $\nu : (Y, \mathcal{G}) \to (Y_0, \mathcal{G}_0)$ an almost étale cover which kills the quotient singularities, and $\pi : (Y', \mathcal{G}') \to (Y, \mathcal{G})$ a resolution of the cusps, with Z the reduced divisor contracted by π, then $\Omega_{Y'}(\log Z) = K_{\mathcal{G}} \oplus M$ for some line bundle M, and if $(Y, \mathcal{G})$ is not a fibration then the lemma obliges the singularities of either foliation to be canonical, and so both $K_{\mathcal{G}}$ and M are nef. However global 1-forms with log-growth are still closed so the Bogomolov-Castelnuovo lemma can be generalised to oblige neither $K_{\mathcal{G}}$ nor M to be big. In particular therefore $K_{\mathcal{G}}^2 = M^2 = 0$. So the previous considerations of our classification imply that $\kappa(Y_0, \mathcal{G}_0) = -\infty$.

For the converse we put Z equal to the union of the e.g.l.'s then the hypothesis force the foliation to be log smooth, i.e. given by a short exact sequence of bundles,

$$0 \to M \to \Omega^1_X(\log Z) \to K_{\mathcal{F}} \to 0\,.$$

Further by a variation of IV.5.3, combined with I.3.1 we have $M^2 = 0$, and $K_{\mathcal{F}}$ is nef. Not surprisingly we wish to prove that M is nef. Observe that if for some curve C, $M \cdot C < 0$ then for a suitable $\varepsilon > 0$ we have,

$$(M - \varepsilon\, C)^2 > 0 \quad \text{and} \quad K_{\mathcal{F}} \cdot (M - \varepsilon\, C) = c_2(\Omega_X(\log Z)) - \varepsilon\, K_{\mathcal{F}} \cdot C\,.$$

This will certainly contradict the aforesaid Bogomolov-Castelnuovo estimates, if we can guarantee $c_2(\Omega_X(\log Z)) = K_{\mathcal{F}} \cdot (K_X + Z) = K_{\mathcal{F}} \cdot K_X$ is positive. Suppose otherwise, and observe that by IV.5.2 X is a regular surface, so we have an exact sequence:

$$0 \to H^0(X, K_X) \to H^0(X, \mathcal{O}_X(K_X + Z)) \to H^0(Z, \omega_Z) \to 0\,.$$

By hypothesis $H^0(Z,\omega_Z) \xrightarrow{\sim} H^0(Z,\mathcal{O}_Z)$ whence $K_X + Z$ is effective, and since $K_{\mathcal{F}} \neq 0$ but nef., we must have $\mathcal{O}_X(K_X + Z) \xrightarrow{\sim} \mathcal{O}_X$, so by Riemann-Roch, $\chi(X, K_{\mathcal{F}}) = \chi(X) > 0$, since X is regular, which is absurd. Thus $c_2(\Omega_X(\log Z)) > 0$, and M is nef. In particular therefore $K_X + Z$ is nef. and for any curve C, $K_X + Z.\, C \geq K_{\mathcal{F}}.\, C$. However if C is not in the support of Z then $K_{\mathcal{F}}.\, C = 0$ implies $C^2 \geq 0$, as $(X,\mathcal{F})$ is the minimal resolution of a minimal model, which would force $K_{\mathcal{F}}$ to be effective by the index theorem. Consequently we have,

(a) $(K_X + Z).\, C > 0$ for all curves C outside the support of Z.

Furthermore, $(K_X + Z)^2 = 2c_2(\Omega_X(\log Z)) > 0$ and as above $K_{\mathcal{F}}.\, (K_X + Z) > 0$, so as ever by Riemann-Roch we have,

(b) $K_X + Z$ is big.

Whence by [TY], $\Omega^1_X(\log Z)$ admits a Kahler-Einstein, which as before forces the decomposition $\Omega^1_X(\log Z) = M \oplus K_{\mathcal{F}}$ and the claimed isomorphism. □

We now proceed to show that the above examples are all essentially that may occur, including working in the full generality of Hilbert-Modular surfaces, where the minimal model has quotient singularities arising from contracting $-1\, F$ chains. As ever $p : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ is the unique desingularisation of a minimal model, with of course $\nu\,(X,\mathcal{F}) = 1$ and $\kappa(X,\mathcal{F}) = -\infty$. In particular $p^*\, K_{\mathcal{F}}$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor, and so by [De2], it admits a singular metric, such that if $p^*\, \overline{K}_{\mathcal{F}}$ denotes the said divisor with metric then,

$$c_1(p^*\, \overline{K}_{\mathcal{F}}) \geq 0$$

where the inequality is understood in the sense of currents[11]. Now $\nu(\mathcal{F}) = 1$, so $K^2_{\mathcal{F}} = 0$, and $c_1(p^*\, \overline{K}_{\mathcal{F}}) \neq 0$, so if it were defined $c_1(p^*\, \overline{K}_{\mathcal{F}})^2 = 0$. This suggests that we can define a foliation or lamination in some real sense as the kernel of $c_1(p^*\, \overline{K}_{\mathcal{F}})$, which if it were holomorphic would give a transverse holomorphic structure to $\mathcal{F}$. Indeed this is exactly what the Kodaira fibration does – except in the case of elliptic fibrations where they coincide – and it's exactly the interplay between the fibration and the foliation that leads to the classification. A priori, the suggestion that the kernel of a smooth positive, yet degenerate, $(1,1)$ form could lead to a holomorphic structure is somewhat outrageous. Such things go under the name of Monge-Ampère foliations, and there is a litany of counterexamples, cf. [Mo]. However if $\overline{\overline{K}}_{\mathcal{F}}$ were another metricisation, and everything made sense, then $c_1(p^*\, \overline{K}_{\mathcal{F}}) \cap c_1(p^*\, \overline{\overline{K}}_{\mathcal{F}}) = 0$, so such a kernel ought to be unique, whence morally at least continuous. Indeed we can even make a good moral arguement for holomorphicity. So let's do this setting up some notation for the future in the process, viz:

[11] This is only proved in op.cit. for Cartier divisors, however the extension to $\mathbb{Q}$-divisors is formal.

Definition IV.5.6. *Let $(X, \mathcal{F})$ be a foliation by curves on a non-singular variety X and $\overline{L} = (L, \| \ \|)$ a line bundle with metric then we define the Chern class of L along $\mathcal{F}$ to be,*

$$c_{\mathcal{F}}(\overline{L}) = \frac{-1}{2\pi\sqrt{-1}}\,\overline{\partial}(\partial_{\mathcal{F}} \log \|s\|^2) \otimes \partial_{\mathcal{F}}^{\vee}$$

where s is a local section of L, and $\partial_{\mathcal{F}}$ a local generator of $T_{\mathcal{F}}$. If the metric is smooth $c_{\mathcal{F}}$ takes values in $A_X^{0,1} \otimes K_{\mathcal{F}}$, or more generally $D_X^{0,1} \otimes K_{\mathcal{F}}$, i.e. distributions, or whatever according to the regularity class. Indeed restricting to Dolbeault cohomology $c_{\mathcal{F}}(\overline{L})$ is just the restriction to $H_{\overline{\partial}}^{0,1}(K_{\mathcal{F}})$ of $c_1(\overline{L})$ in $H_{\overline{\partial}}^{1,1}$.

Now consider a smooth point of the foliation on $(X, \mathcal{F})$ (let's say we're back to surfaces again for simplicity) then we can find coordinates x, y with $\partial_{\mathcal{F}} = \partial/\partial x$ a local generator for the foliation, and whence a non-canonical identification of $dx\, d\overline{x}\, dy\, d\overline{y}$ with $d\overline{x}\, dy\, d\overline{y} \otimes \partial_{\mathcal{F}}^{\vee}$ with respect to which,

$$c_1(\overline{L})^2 = 1/2\, c_1(\overline{L}) \wedge c_{\mathcal{F}}(\overline{L})$$

modulo the equation making sense, e.g. $c_1(\overline{L})$ having L_2-coefficients. One can make a more complicated formula at the singular points, but the important thing is that a singularity makes the right hand side smaller.

To fix ideas let's suppose momentarily that $(\widetilde{X}, \widetilde{\mathcal{F}}) = (X, \mathcal{F})$, and that $K_{\mathcal{F}}$ admits a smooth positive semi-definite metric then certainly, $c_1(\overline{K}_{\mathcal{F}}) \wedge c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) = 0$ everywhere, and the 1-form defining the hoped for transverse structure is the conjugate of $c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})$. We can use the Bochner technique to show holomorphicity. The $\partial\overline{\partial}$ formulation is convenient. Specifically if $e^{-2\varphi}$ is a local weight for the metric then since $c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})$ is $\overline{\partial}$-closed,

$$\begin{aligned}
\frac{\sqrt{-1}}{2\pi}\,\partial\overline{\partial}(e^{-2\varphi}\, c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge \overline{c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})}) &= \frac{-\sqrt{-1}}{2\pi}\,\partial\{e^{-2\varphi}\, c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})\overline{\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})}\} \\
&= \frac{-\sqrt{-1}}{2\pi}\, e^{-2\varphi}\, \nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge \overline{\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})} \\
&\quad + e^{-2\varphi}\, c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge \overline{c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})}\, c_1(\overline{K}_{\mathcal{F}})\,.
\end{aligned}$$

So if $\nu(\mathcal{F}) = 1$, the curvature term should vanish, the left hand side integrates out to zero and we obtain,

$$\int_X e^{-2\varphi}\, \nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge \overline{\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})} = 0\,.$$

On the other hand, $c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge c_1(\overline{K}_{\mathcal{F}}) = 0$, from which $\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge c_1(\overline{K}_{\mathcal{F}}) = 0$. Now suppose we're somewhere p, say, on X where $c_1(\overline{K}_{\mathcal{F}})$ doesn't vanish and write $c_1(\overline{K}_{\mathcal{F}})$ at p as $\frac{dx\, d\overline{x}}{-2\pi\sqrt{-1}}$, with $\frac{dx\, d\overline{x} + dy\, d\overline{y}}{-2\pi\sqrt{-1}}$ the Kahler form then,

(a) $\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge c_1(\overline{K}_{\mathcal{F}}) = 0$ implies that there is no $dy\, d\overline{y}$ term in $\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})$.

(b) Which in turn says, $e^{-2\varphi}\,\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})\wedge\overline{\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})} > 0$ at p, unless $\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})$ only has a $dx\, d\overline{x}$ term.

Putting everything together implies $c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge \nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) = 0$, and this is the holomorphicity equation. So what exactly are the problems. Unfortunately they are legion, namely:

(i) The Bochner formula appears to be unjustifiable without some sort of regularity for the local weight φ. At the very least it looks like we need that it's in the Sobolev space $W_{2,2}$ (here $W_{p,q}$ means p^{th}-derivatives being L_q) and maybe even $W_{4,2}$, whereas general nonsense only gives $W_{1,2-\varepsilon}$.

(ii) We need to understand where $c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})$ (equivalently $c_1(\overline{K}_{\mathcal{F}})$) vanishes. Indeed even in a smooth situation if $c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) = \overline{f}\, d\overline{x} + \overline{g}\, d\overline{y}$, say, in local coordinates x, y and $g \neq 0$ the holomorphicity equation gives $\overline{\partial}(f/g) = 0$. However if both f and g are zero, there's some sort of pole that needs to be controlled.

(iii) The holomorphicity equation, $c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge \nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) = 0$ is non-linear. Even worse it's invariant under bumping, so it's trivially easy to make sequences of $(1,0)$ forms ω_n converging to some distribution ω with $\omega_n \wedge \overline{\partial}\,\omega_n \to 0$ such that $\omega = 0$ does not define a field of holomorphic lines.

(iv) Even if all went well, and the kernel was a well defined holomorphic foliation, we have to exclude the possibility that it coincides with $\mathcal{F}$.

Leaving (iv) apart, all of which certainly implies that we're not going to get anywhere without some a priori regularity. Returning then to the general situation where $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ is the minimal smooth resolution of the minimal model and $\nu(\mathcal{F}) = 1$, but $\kappa(\mathcal{F}) = -\infty$, the following observation is fundamental,

Lemma IV.5.7. *Notations and hypothesis as above then $\forall\, x \in \widetilde{X}$, the Lelong number $\nu\,(x, c_1\,(p^*\,\overline{K}_{\mathcal{F}})) = 0$.*

Proof. Suppose otherwise. Then for some $\varepsilon > 0$ the set,

$$E_\varepsilon = \{x \in \widetilde{X} \mid \nu\,(x, c_1\,(p^*\,\overline{K}_{\mathcal{F}})) \geq \varepsilon\} \neq \varphi$$

and by the celebrated theorem of Siu, cf. [Si], it is in fact a closed analytic subset of $\widetilde{X}$. Whence there are two possibilities dim $E_\varepsilon = 1$ or dim $E_\varepsilon = 0$. Suppose the former then by Siu's decomposition theorem, cf. op.cit., there are real numbers $\nu_1, \ldots, \nu_k \geq \varepsilon$, curves $C_1, \ldots, C_k$ and a positive $(1,1)$ current T_ε such that,

$$c_1\,(p^*\,\overline{K}_{\mathcal{F}}) = \sum_{i=1}^{k} \nu_i\, C_i + T_\varepsilon$$

where C_i denotes in this formula the current of integration over the curve. Now use duality to project everything into $\mathrm{NS}_1(\widetilde{X})_{\mathbb{R}}$, lowering the ν_i a little to ν_i' to ensure that they are a rational we obtain a pseudo-effective $\mathbb{Q}$-divisor T, such that,

$$p^* K_{\mathcal{F}} = \sum_{i=1}^{k} \nu_i' C_i + T$$

at which point we may apply Zariski decomposition to T, and use the index theorem to deduce that $p^* K_{\mathcal{F}}$ is represented by a positive cycle. Since without loss of generality $\widetilde{X}$ is regular, this is absurd.

Otherwise E_ε must have dimension 0, and indeed this must hold $\forall\, \varepsilon > 0$, so the subset, $\{x \in \widetilde{X} \mid \nu(x, c_1(p^* \overline{K}_{\mathcal{F}})) > 0\}$ is countable. However by [De2], we must have in this situation,

$$c_1 (p^* \overline{K}_{\mathcal{F}})^2 \geq \sum_{x \in \widetilde{X}} \nu(x, c_1(p^* \overline{K}_{\mathcal{F}}))$$

which establishes the lemma. □

All be it that vanishing of Lelong numbers represents some sort of regularity it's unfortunately very far from forcing the local weight φ of the metric to enjoy anything near the kind of a priori regularity that would be required to force the Bochner type argument through. Consequently, following Brunella, [B], we look towards the obvious choice for metricising $K_{\mathcal{F}}$, namely the leafwise Poincaré metric. Over the locus $U \subset X$ where neither $\mathcal{F}$ nor X is singular, in terms of a local trivialisation ∂ of $T_{\mathcal{F}}$ this is defined at $x \in U$ by,

$$e^{\varphi(x)} = \|\partial\|(x) = \inf \left\{ \frac{1}{|R|} : f : (\Delta, 0) \to (U', x), f_* \left(\frac{\partial}{\partial z} \right) = R\partial \right\}$$

where U' is U together with the contraction of any $-1\,F$ chains (cf. V.1.2 for the raison d'être of this definition) and $f : \Delta \to U'$ is a $\mathcal{F}$-invariant map from the disc. For general nonsense reasons (i.e. everything which is Stein has a Stein neighbourhood) the function φ is upper semi-continuous. The definition extends across the singularities by the following key result of M. Brunella,

Fact. (Brunella cf. [Br4]) The local weight φ is pluri-subharmonic.

The singularities being in codimension 2, immediately means that φ extends across them to a psh function on the whole of X. The immediate concern is, of course, that φ may be identically $-\infty$, however, this is excluded since we know that $q(\widetilde{X}) = 0$, and so $K_{\widetilde{X}} \cdot K_{\mathcal{F}} > 0$ by Riemann-Roch, and any parabolic curves are algebraic, cf. V.1.2(a). Even better as will be shown in § V, with these hypothesis φ is actually continuous away from the singularities of X and $\mathcal{F}$, and the algebraic invariant curves passing through them. So we have a reasonable amount of regularity, and the following arguement of Brunella shows that for this metricisation the kernel of $c_1(\rho^* \overline{K}_{\mathcal{F}})$ has sense, viz:

Lemma IV.5.8. (Brunella cf. op.cit.) *Away from the singularities of $\widetilde{\mathcal{F}}$ and algebraic invariant curves passing through them $c_1(\rho^*\overline{K}_{\mathcal{F}})$ is absolutely continuous.*

Proof. By an approximation theorem of Demailly there is a sequence of smooth metrics with local weights φ_n say on $\rho^* K_{\mathcal{F}}$ such that, $\varphi_n \to \varphi_r$ monotone decreasing and $c_1(\rho^*\overline{K}^n_{\mathcal{F}}) \geq -\frac{1}{n}\,\omega$, where ω is a Kahler form. Now we work locally in some polydisc Δ, say, away from the said problem area. Here φ is continuous so there is a well defined measure on functions $f \in C^\infty_c(\Delta)$ defined by,

$$\frac{\sqrt{-1}}{\pi}\,\partial\overline{\partial}\,\varphi \wedge c_1(\rho^*\overline{K}_{\mathcal{F}})(f) := \frac{\sqrt{-1}}{\pi}\,c_1(\rho^*\overline{K}_{\mathcal{F}})(\varphi\,\partial\overline{\partial} f)\,.$$

Now φ_n tends to φ weakly, and,

$$\int_X c_1(\rho^*\overline{K}^n_{\mathcal{F}}) \wedge c_1(\rho^*\overline{K}_{\mathcal{F}}) = 0$$

with, $c_1(\rho^*\overline{K}^n_{\mathcal{F}})\wedge c_1(\rho^*\overline{K}_{\mathcal{F}}) \geq -\frac{1}{n}\,\omega \wedge c_1(\rho^*\overline{K}_{\mathcal{F}})$, so: $c_1(\rho^*\overline{K}^n_{\mathcal{F}})\wedge c_1(\rho^*\overline{K}_{\mathcal{F}}) \xrightarrow[w]{} 0$. Applying this ultimate observation locally to our test function f gives,

$$\lim_n c_1(\rho^*\overline{K}_{\mathcal{F}})\,(\varphi_n\,\partial\overline{\partial} f) = 0\,.$$

Since φ is locally bounded both the positive and negative parts of $\varphi\,\partial\overline{\partial} f$ are integrable with respect to the measure coefficients of $c_1(\rho^*\overline{K}_{\mathcal{F}})$ so the monotone convergence theorem gives,

$$\frac{\sqrt{-1}}{\pi}\,\partial\overline{\partial}\,\varphi \wedge c_1(\rho^*\overline{K}_{\mathcal{F}})\;|_{\Delta} = 0\,.$$

Now consider the decomposition of $c_1(\rho^*\overline{K}_{\mathcal{F}})$ into absolutely continuous and singular parts, say,

$$c_1(\rho^*\overline{K}_{\mathcal{F}}) = T_{abc} + T_{\mathrm{sing}}\,.$$

Since Δ doesn't contain any singularities let's take coordinates x, y with $y = 0$ defining the foliation. Observe that for continuous things distributional derivatives can be calculated naively so that by construction $\varphi_{x\overline{x}} = e^{2\varphi}$, whence it's certainly not singular, and since $T_{\mathrm{sing}} \geq 0$, we must have that $\varphi_{x\overline{y}}$ and $\varphi_{y\overline{x}}$ are L_1, with $T_{\mathrm{sing}} = \mu\, dy\, d\overline{y}$ for some measure μ. Locally on Δ we can use regularising kernels to make a monotone sequence of smooth psh approximations $\psi_k \to \varphi$ with, $\frac{\sqrt{-1}\,\partial^2\,\psi_k}{\partial x\,\partial\overline{x}} \geq e^{2\psi_k}$, and by a mixture of the monotone convergence theorem and the boundedness of φ,

$$\lim_k \sqrt{-1}\,\partial\overline{\partial}\,\psi_k\, c_1(\rho^*\overline{K}_{\mathcal{F}})(f) = \lim_k \sqrt{-1}\,c_1(\rho^*\overline{K}_{\mathcal{F}})(\psi_k\,\partial\overline{\partial} f) = 0$$

from which, $\mu(e^{2\psi_k}) \to \mu(e^{2\varphi}) = 0$. Since we're away from any algebraic curves through the singularities, we're a fortiori away from any curves where φ might be $-\infty$, so indeed $\mu = 0$, as claimed. ◻

Now we have a well defined kernel, and observe quite generally, in the notations of the proof that for any metricised line bundle $\overline{L}$, with $c_1(\overline{L})$ absolutely continuous and local weight ψ with $\psi_{x\overline{x}} \neq 0$,

$$c_1(\overline{L}) = \frac{\pi\sqrt{-1}}{\psi_{x\overline{x}}}\, \overline{c_{\mathcal{F}}(\overline{L})} \wedge c_{\mathcal{F}}(\overline{L}) + \frac{\sqrt{-1}}{\pi}\, \frac{\Delta(\psi)}{\psi_{x\overline{x}}}\, dy\, d\overline{y}$$

where $\Delta(\psi) = \psi_{x\overline{x}}\, \psi_{y\overline{y}} - |\psi_{x\overline{y}}|^2$, and the formula is understood as a pointwise identity of L_1-functions. Applying this to $\overline{K}_{\mathcal{F}}$ with the Poincaré metric gives, off the excluded algebraic set,

$$c_1(\overline{K}_{\mathcal{F}}) = \pi\sqrt{-1}\, e^{-2\varphi}\, \overline{c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})} \wedge c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})\, .$$

Consequently if it were defined the Bochner formula would give a pointwise identity; there being no need to integrate our the left hand side,

$$e^{-2\varphi}\, \nabla\, c_{\mathcal{F}}(\overline{K}_{\mathcal{F}}) \wedge \overline{\nabla c_{\mathcal{F}}(\overline{K}_{\mathcal{F}})} = 0\, .$$

In terms of local coordinates this amounts to, $\frac{\partial}{\partial \overline{x}}\left(\frac{\varphi_{y\overline{x}}}{\varphi_{x\overline{x}}}\right) = 0$, which in the presence of sufficient regularity is the standard holomorphicity equation for Monge-Ampère foliations. The transverse holomorphicity usually being trivial thanks to the identity $\frac{\partial}{\partial \overline{y}}\left(\frac{\varphi_{y\overline{x}}}{\varphi_{x\overline{x}}}\right) = \frac{\varphi_{x\overline{y}}}{\varphi_{x\overline{x}}}\, \frac{\partial}{\partial \overline{x}}\left(\frac{\varphi_{y\overline{x}}}{\varphi_{x\overline{x}}}\right)$ which follows from $\Delta(\varphi) = 0$. Our strategy then will simply be to push this analysis through by way of appropriate regularisation. Before getting under way, however, let's observe that if we succeed then we'll get our desired transverse foliation $\mathcal{G}$ on all of X. Indeed away from the singularities and the algebraic invariant curves through them $\varphi_{x\overline{x}}^{-1} = e^{-2\varphi}$ is still continuous, and $\mathcal{G}$ is defined by the 1-form,

$$\frac{\varphi_{y\overline{x}}}{\varphi_{x\overline{x}}}\, dy + dx = 0$$

so it's certainly transverse to $\mathcal{F}$ and not equal to it. Now consider a suitably small polydisc Δ meeting an algebraic invariant curve but not containing the singularities, then we certainly still have foliation coordinates x, y, with $y = \text{const.}$, the foliation, and on $\Delta^* = \Delta\backslash\{\text{said curve}\}$, $\mathcal{G}$ is still given by the same equation. Better still on Δ^*, $e^{-2\varphi}|\varphi_{y\overline{x}}|^2 = \varphi_{y\overline{y}}$ and φ is L_p for all $p < \infty$ by the vanishing of the Lelong numbers so if $d\mu$ is Lebesgue measure,

$$\begin{aligned}\left(\int_{\Delta^*} \left|\frac{\varphi_{y\overline{x}}}{\varphi_{x\overline{x}}}\right| d\mu\right)^2 &\leq \left(\int_{\Delta^*} |\varphi_{y\overline{x}}|\, e^{-2\varphi}\, d\mu\right)\left(\int_{\Delta^*} e^{-2\varphi}\, d\mu\right)\\ &\leq \varphi_{y\overline{y}}(\mathbb{I}_{\Delta}) \int_{\Delta^*} e^{-2\varphi}\, d\mu < \infty\, .\end{aligned}$$

Whence if $\frac{\varphi_{y\overline{x}}}{\varphi_{x\overline{x}}}$ is holomorphic on Δ^*, it's at worst meromorphic on Δ with a simple pole along the curve in question. Consequently $\mathcal{G}$ would be a well defined foliation on X in codimension 1, and since X is normal, well defined everywhere.

So let's justify the Bochner technique. Although everything is local it's convenient to keep the curvature notation $c_1, c_{\mathcal{F}}, \nabla$ etc., but this time we'll write $c_1(\psi)$, etc. for ψ a local weight, corresponding to a metric $e^{-2\psi}$ on $K_{\mathcal{F}}$. As in the proof of IV.5.8 f will be a positive compactly supported function on a polydisc Δ not containing the singularities nor meeting an algebraic invariant curve through them. Observe, moreover, that the proof of IV.5.8 gives quite generally that if $\varphi_k \to \varphi$ monotone, then, $c_1(\varphi_k) \wedge c_1(\varphi) \underset{w}{\longrightarrow} 0$. On the other hand we have,

$$c_1(\varphi_k) = \pi\sqrt{-1}\,\frac{\overline{c_{\mathcal{F}}(\varphi_k)} \wedge c_{\mathcal{F}}(\varphi_k)}{\varphi_{k,x\overline{x}}} + \frac{\sqrt{-1}}{\pi}\,\frac{\Delta(\varphi_k)}{\varphi_{k,x\overline{x}}}\,dy\,d\overline{y}\,.$$

Naturally we introduce $\lambda(\psi) = \frac{\psi_{x\overline{y}}}{\psi_{x\overline{x}}}$, for any ψ, so that our previous observation is equivalent to,

$$\varphi_{x\overline{x}}\,\varphi_{k,x\overline{x}}|\lambda(\varphi) - \lambda(\varphi_k)|^2 \underset{w}{\longrightarrow} 0 \quad \text{and} \quad \frac{\varphi_{x\overline{x}}}{\varphi_{k,x\overline{x}}}\,\Delta(\varphi_k) \underset{w}{\longrightarrow} 0\,.$$

In particular for $\varphi_k \to \varphi$ monotone obtained via regularising kernels in the usual way, $\varphi_{k,x\overline{x}} \geq e^{2\varphi_k} \geq e^{2\varphi}$ so $\lambda(\varphi_k) \underset{w}{\longrightarrow} \lambda(\varphi)$ in L_2. Now consider the connection ∇_k on $K_{\mathcal{F}}$ corresponding to the metric $\varphi^k_{x\overline{x}} = \left\|\frac{\partial}{\partial x}\right\|^2$. Then,

$$\frac{\sqrt{-1}}{2\pi}\int_\Delta f\partial\overline{\partial}\left\{\frac{c_{\mathcal{F}}(\varphi_k)\,\overline{c_{\mathcal{F}}(\varphi_k)}}{\varphi_{k,x\overline{x}}}\right\} = \frac{-\sqrt{-1}}{2\pi}\int_\Delta \frac{f}{\varphi_{k,x\overline{x}}}\,\nabla_k\,c_{\mathcal{F}}(\varphi_k)\,\overline{\nabla_k\,c_{\mathcal{F}}(\varphi_k)}$$
$$+\int_\Delta f\cdot c_1(\log\varphi_{k,x\overline{x}})\,\frac{c_{\mathcal{F}}(\varphi_k)\,\overline{c_{\mathcal{F}}(\varphi_k)}}{\varphi_{k,x\overline{x}}}\,.$$

Now $\varphi^{-1}_{k,x\overline{x}}\,c_{\mathcal{F}}(\varphi_k)\,\overline{c_{\mathcal{F}}(\varphi_k)} = \frac{1}{\pi^2}\,\varphi_{k,x\overline{x}}(dx\,d\overline{x} + \overline{\lambda(\varphi_k)}\,dy\,d\overline{x} + \lambda(\varphi_k)\,dx\,d\overline{y} + |\lambda(\varphi_k)|^2\,dy\,d\overline{y})$, and $\varphi_{k,x\overline{x}} \to \varphi_{x\overline{x}}$ in L_∞ while $\lambda(\varphi_k) \to \lambda$ in L_2, so the whole thing goes weakly to, $\varphi^{-1}_{x\overline{x}}\,c_{\mathcal{F}}(\varphi)\,\overline{c_{\mathcal{F}}(\varphi)}$ which is closed. So in fact,

$$\frac{1}{2\pi}\int_\Delta \frac{f}{\varphi_{k,x\overline{x}}}\left|\frac{\partial\lambda(\varphi_k)}{\partial x}\right|^2 d\mu = \lim_k \sqrt{-1}\int_\Delta f\,c_1(\log\varphi_{k,x\overline{x}})\frac{\overline{c_{\mathcal{F}}(\varphi_k)}\,c_{\mathcal{F}}(\varphi_k)}{\varphi_{k,x\overline{x}}}\,.$$

To estimate the integral on the right, observe that $\log\varphi_{k,x\overline{x}}$ is monotone decreasing to φ, so $c_1(\log\varphi_{k,x\overline{x}}) \wedge c_1(\varphi) \underset{w}{\longrightarrow} 0$. Defining a curvature function, $G(\psi) = e^{-2\psi}\,\frac{\partial^2\psi}{\partial x\,\partial\overline{x}}$ for any suitable ψ we can write this as,

$$e^{2\varphi}\,G(\log\varphi_{k,x\overline{x}})\,\varphi_{k,x\overline{x}}\,|\lambda(\varphi) - \lambda(\log\varphi_{k,x\overline{x}})|^2 \underset{w}{\longrightarrow} 0$$

and

$$\frac{e^{2\varphi}}{\varphi_{k,x\overline{x}}\,G(\log\varphi_{k,x\overline{x}})}\Delta(\log\varphi_{k,x\overline{x}}) \underset{w}{\longrightarrow} 0$$

while the integral we're seeking to estimate is,

$$\frac{1}{\pi}\int_\Delta f\,G(\log\varphi_{k,x\overline{x}})\,|\varphi_{k,x\overline{x}}|^2\,|\lambda(\varphi_k) - \lambda(\log\varphi_{k,x\overline{x}})|^2\,d\mu + \frac{1}{\pi}\int_\Delta f\,\frac{\Delta(\log\varphi_{k,x\overline{x}})}{G(\log\varphi_{k,x\overline{x}})}\,d\mu$$

the latter integral easily tends to zero by what we already have, and the former will also easily be so on obtaining an upper bound for the curvature. Denoting by $\Phi_k(x) = \sup_{x\in\Delta} \sup\{\varphi(y) \mid \rho_k(|x-y|) \neq 0\}$ where ρ_k is the symmetric bump function giving the convolution, we have,

$$|G(\log \varphi_{k,x\overline{x}})| \leq \frac{2\, e^{2\Phi_k}}{e^{4\varphi}} \, \|\varphi_x\|_{L_2} + \frac{e^{4\Phi_k}}{e^{4\varphi}}$$

since $|\varphi_{x\overline{y}}|^2 = \varphi_{x\overline{x}}\, \varphi_{y\overline{y}}$, the Garding inequality implies that φ_x is L_2 so indeed the curvature is bounded above, and whence,

$$\frac{\partial}{\partial x}\,(\lambda(\varphi))(f) = \lim_k \int \lambda(\varphi_k)\, f_x \, d\mu = \lim_k \int \frac{\partial}{\partial x}\,(\lambda(\varphi_k))\, f \, d\mu = 0\,.$$

For the transverse direction the basic fact is that $c_1(\varphi) = \pi\sqrt{-1}\ \overline{c_{\mathcal{F}}(\varphi)} \wedge c_{\mathcal{F}}(\varphi)\, e^{-2\varphi}$ is closed, which in terms of our test function f gives two sets of equations,

$$\int_\Delta f_y\, \lambda(\varphi)\, e^{2\varphi} - f_x |\lambda|^2\, e^{2\varphi}\, d\mu = 0 \quad \text{and} \quad \int_\Delta f_y\, e^{2\varphi} - e^{2\varphi}\, \overline{\lambda(\varphi)}\, f_x = 0\,.$$

Now for notational convenience put $\lambda = \lambda(\varphi)$ and take regularising convolutions $\varphi_k \to \varphi$, $\lambda_m \to \lambda$, $\overline{\lambda}_n \to \overline{\lambda}$ (n.b. $\lambda_m \neq \lambda(\varphi_m)$, it's just a convolution of λ) then possibly after subsequencing the 1^{st} of these converges weakly in L_∞, the other two weakly in L_2. Certainly $(\lambda_m)_x = 0$, and the following limits irrespective of the ordering in k, m, and n go to zero, viz,

$$\int (\lambda_m\, e^{2\varphi_k})_y\, f - (\lambda_m\, \overline{\lambda}_n\, e^{2\varphi_k})_x\, f\, d\mu \quad \text{and} \quad \int (e^{2\varphi_k})_y\, f - (e^{2\varphi_k}\, \overline{\lambda}_n)_x\, f\,.$$

However, $(\lambda_m\, e^{2\varphi_k})_y\, f - (\lambda_m\, \overline{\lambda}_n\, e^{2\varphi_k})_x\, f = (\lambda_m)_y\, e^{2\varphi_k}\, f + \lambda_m\, f(e^{2\varphi_k})_y - \lambda_m$ $f(\overline{\lambda}_n\, e^{2\varphi_k})_x$, so fixing m and replacing f by $\lambda_m\, f$ in the second integral gives,

$$\lim_{m\to\infty} \int_\Delta e^{2\varphi}\, f(\lambda_m)_y\, d\mu = 0\,.$$

Now since φ is continuous and psh, we may subsequence so that $e^{2\varphi_k}$ has a weak limit in $W_{1,2}$ say $e^{2\varphi}$ by a possible minor abuse of notation, and $e^{2\varphi_k} f \underset{w}{\longrightarrow} e^{2\varphi} f$ again in $W_{1,2}$ so,

$$\int_\Delta \lambda(e^{2\varphi} f)_y\, d\mu = 0\,.$$

Given that f is arbitrary, the space of functions $e^{2\varphi} f$ is dense in $W_{1,2}$ so indeed $\lambda_y = 0$, and $\varphi_{x\overline{x}}^{-1}\, \varphi_{y\overline{x}}$ is holomorphic on Δ.

At this point we have found our foliation $\mathcal{G}$, and we'll work on the minimal smooth resolution $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$. Denote by D the tangency divisor between $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ which we've shown to be $\widetilde{\mathcal{F}}$ invariant with simple poles, so it is the same for $\widetilde{\mathcal{G}}$ and $\Omega_{\widetilde{X}}(\log D)$ splits as $K_{\widetilde{\mathcal{F}}} \oplus K_{\widetilde{\mathcal{G}}}$. Now denote by $N_{\widetilde{\mathcal{F}}}$

and $N_{\widetilde{\mathcal{G}}}$ the respective logarithmic conormal bundles with $s_{\widetilde{\mathcal{F}}}(C)$, resp. $s_{\widetilde{\mathcal{G}}}(C)$ the Segre class along $\operatorname{sing}(\mathcal{F})$, resp. $\operatorname{sing}(\mathcal{G})$, of any curve while for $C \in D$, put $s(C, D) = D \cdot C - C^2$, i.e. the number of curves in D meeting C, then, for $C \subset |D|$, we have,

$$K_{\widetilde{\mathcal{F}}} \cdot C = -s_{\widetilde{\mathcal{G}}}(C) + s(C, D) \leq 0$$

and similarly for $K_{\widetilde{\mathcal{G}}} \cdot C$, irrespective of whether the latter has canonical singularities or not. To conclude that we really do have a Hilbert-Modular surface with the natural foliations there on, the first step is to show that $\widetilde{\mathcal{G}}$ has canonical singularities. Observe any singularity of $\widetilde{\mathcal{G}}$ is contained in a curve $C \subset |D|$, so the question is local about D. Furthermore $K_{\widetilde{\mathcal{F}}} \cdot C \leq 0$ for all $C \subset |D|$, so not only is D contracted by ρ, but it cannot be a D_n, so a connected component Γ can only be,

Cases IV.5.9.
(a) A (not necessarily maximal) $-1\,F$ chain for $\widetilde{\mathcal{F}}$.
(b) A (not necessarily maximal) chain of $-2\,F$ curves for $\widetilde{\mathcal{F}}$.
(c) A cycle of $-2\,F$ curves (or a rational curve with a node) for $\widetilde{\mathcal{F}}$.

Now as in the proof of the minimal model theorem the basic fact is that for any foliation with canonical singularities a chain of curves starting and ending in a $-1\,F$ curve which are joined by $-2\,F$ curves cannot be contractible, or even easier an invariant rational curve intersecting the canonical in -2 isn't contractible either. So consider firstly case (c), for any $C \subset \Gamma$, $s(C, D) = s_{\widetilde{\mathcal{G}}}(C) = 2$, so $K_{\widetilde{\mathcal{G}}} \cdot C = 0$, and the singularities of $\widetilde{\mathcal{G}}$ coincide with those of Γ. Suppose, therefore, that one of these is non-canonical. Blow up in it and call the exceptional divisor E, then either it's discrepant or not. If it's not $K_{\widetilde{\mathcal{G}}} \cdot E = 0$ and $E^2 = -1$, so E is invariant, with $s_{\widetilde{\mathcal{G}}}(E) = \widetilde{\Gamma} \cdot E = 2$, where $\widetilde{\Gamma}$ is the proper transform of Γ, and we still have a splitting, i.e. we've changed nothing, so we might as well say that we've a priori blown up sufficiently in order to guarantee negative discrepancy. In which case on blowing up $\widetilde{\Gamma}$ is two $-1\,F$ curves for $\widetilde{\mathcal{G}}$ joined by a (possibly empty) chain of $-2\,F$ curves for $\widetilde{\mathcal{G}}$ or an invariant rational curve intersecting the canonical in -2. Of course we still may not have canonical singularities, but the singularities of the induced foliation, which we'll still call $\widetilde{\mathcal{G}}$ to avoid complicating the notation, still coincide with those of $\widetilde{\Gamma}$. Blowing up in one of those we either keep a chain with the same properties in the non-discrepant case, or split into two disjoint chains with the same property. In any case we eventually contradict the contractibility of D. Case (b) automatically reduces to the latter part of case (c), while for case (a) the $-1\,F$ chain for $\widetilde{\mathcal{F}}$ is also a $-1\,F$ chain for $\widetilde{\mathcal{G}}$ but in the opposite sense, i.e. in the following Dynkin diagram for Γ, the $K_{\widetilde{\mathcal{F}}}$ intersection numbers are on the bottom, those for $\widetilde{\mathcal{G}}$ on the top,

$$\begin{array}{cccccccccccc}
0 & & 0 & & 0 & & & & 0 & & 0 & & -1 \\
\circ & \text{---} & \circ & \text{---} & \circ & & \text{- - - - - -} & & \circ & \text{---} & \circ & \text{---} & \circ \\
-1 & & 0 & & 0 & & & & 0 & & 0 & & 0
\end{array}$$

Here the $\widetilde{\mathcal{G}}$ and Γ singularities may not a priori coincide on the curve on the left, say by virtue of a singularity $z \in \operatorname{sing}(\widetilde{\mathcal{G}})\backslash\Gamma$. However blowing up in z in the non-discrepant case only adds a node to the left, the pattern of 0's and -1's being the same for the induced foliation, so we can assume that we're only one blow up away, and we again conclude by the latter discussion for (c).

Now supposing that $\widetilde{\mathcal{G}}$ is not a rational fibration, the next step is to consider what happens to curves Γ contracted by a minimal model map $\sigma : (\widetilde{X}, \widetilde{\mathcal{G}}) \to (X', \mathcal{G}')$ for $\widetilde{\mathcal{G}}$. As ever we proceed by cases,

Cases IV.5.10. (a) Γ is a maximal $-1\,F$ chain for $\widetilde{\mathcal{G}}$. Ordering the curves in the chain in the usual way call them $C_1, \ldots, C_n$. Then we have the sub-cases,

(i) $C_1 \subset |D|$, and let $C_1, \ldots, C_k$ be the maximal sub-chain of T in D connected to C_1. Suppose $k < n$, so in particular $n \geq 2$. Now suppose further $k+1 < n$, then $C_{k+2} \subset |D|$, and $K_{\widetilde{\mathcal{F}}} \cdot C_{k+2} = s(C_{k+2}, D) - 2$. Further there is a maximal sub-chain of T in D connected to C_{k+2}, say $C_{k+2}, \ldots, C_{k+m}$, $m \geq 2$. If $m = 2$, $s(C_{k+2}, D) = 0$, and C_{k+2} isn't contractible or $m \geq 3$ implying $K_{\widetilde{\mathcal{F}}} \cdot C_{k+2} = K_{\widetilde{\mathcal{F}}} \cdot C_{k+m} = -1$ and $C_{k+2}, \ldots, C_{k+m}$ non-contractible, both of which are absurd. Consequently $n = k+1$, and $(K_{\widetilde{\mathcal{F}}} + C_n) \cdot C_n = -s_{\mathcal{G}}(C_n) + D \cdot C_n = -1$, so by adjunction C_n is $\widetilde{\mathcal{F}}$ invariant, so is tangent to $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$, i.e. $C_n \subset |D|$, contrary to hypothesis, implying thus that T is a component of D.

(ii) C_1 is not in D. Consequently $n \geq 2$, and $C_2 \subset D$, with $K_{\widetilde{\mathcal{F}}} \cdot C_2 = -2 + s(C_2, D)$. By the contractibility of C_2, $s(C_2, D) \geq 1$, i.e. $n \geq 3$, and we have a maximal sub-chain, $C_2, \ldots, C_k$ in D, with $k \geq 3$. As ever $K_{\widetilde{\mathcal{F}}} \cdot C_k = -1$, and we have a contradiction.

(b) Γ is a maximal $-2\,F$ chain, not meeting any $-1\,F$ chain. This cannot happen since a maximal sub-chain in D has both ends enjoying intersection -1 with $K_{\widetilde{\mathcal{F}}}$, or -2 if the length is 1.

(c) The dynkin diagram D_n, as described in III.3, including the degenerate case D_3. Here if $n \geq 4$, there is a maximal sub-chain in D of the part of the diagram consisting of the $-2\,F$ chain, which as in (b) cannot exist. So we're reduced to the degenerate case, and if C is the curve with $K_{\widetilde{\mathcal{G}}} \cdot C = 1$, then since $C^2 \leq -2$, $D \cdot C = 2$, so $(K_{\widetilde{\mathcal{F}}} + C) \cdot C = -1$. Whence again C would have to be $\widetilde{\mathcal{F}}$ invariant, and find itself in the support of D, which cannot happen.

(d) An elliptic Gorenstien leaf. If the whole leaf isn't in D, then there is a non-empty maximal chain of $-2\,F$ curves in both T and D, which as above cannot exist, so indeed the whole leaf is also elliptic Gorenstien for $\widetilde{\mathcal{F}}$.

(e) A rational curve with a node is necessarily invariant for $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$.

Whence in summary σ can only contract $-1\,F$ chains or e.g.l.'s, these are connected components of D and everything contracted by σ is contracted by ρ. This was of course under the hypothesis that $(\widetilde{X}, \widetilde{\mathcal{G}})$ wasn't a rational fibration, however to prove that T was in D used no property of $\widetilde{\mathcal{F}}$, so conversely all

$-1\,F$ chains contracted by ρ are $\widetilde{\mathcal{G}}$ invariant. Consequently $\widetilde{\mathcal{G}}$ is never a fibration since $\pi : \widetilde{X} \to B = \widetilde{X}/\widetilde{\mathcal{G}}$ exists in this case by the canonicity of the singularities and we can find fibres $\widetilde{X}_b$ disjoint from D, so everywhere transverse to $\widetilde{\mathcal{F}}$ from which the absurdity $K_{\widetilde{\mathcal{F}}} \cdot \widetilde{X}_b = \widetilde{X}_b^2 = 0$. A priori σ is not unique since $\widetilde{\mathcal{G}}$ may have Kodaira dimension positive. Certainly we have $\nu(\widetilde{\mathcal{G}}) \geq 0$, and $K(\widetilde{\mathcal{G}}) = 1$ is excluded by Bogomolov's generalisation of the Castelnuovo trick, i.e. if $h^0(N_{\widetilde{\mathcal{F}}}(D)^{\otimes m}) \geq 2$, $m \in \mathbb{N}$, we can take roots to show that $\widetilde{\mathcal{F}}$ is a fibration. Thus it remains to exclude, $\nu(\widetilde{\mathcal{G}}) = \kappa(\widetilde{\mathcal{G}}) = 0$. Here we can contract the -1 chains for $\widetilde{\mathcal{G}}$, which are likewise for $\widetilde{\mathcal{F}}$ with the inverse ordering by, say, $\sigma^{\#} : (\widetilde{X}, \widetilde{\mathcal{G}}) \to (X^{\#}, \mathcal{G}^{\#})$ so that $K_{\mathcal{G}^{\#}} = N_{\mathcal{F}^{\#}}(D^{\#})$, $D^{\#} = \sigma_{\#}(D)$, has canonical singularities for $\mathcal{F}^{\#}$ and $\mathcal{G}^{\#}$ and pass via an almost étale cover to obtain positive irregularity, which is absurd. Thus also for $\mathcal{G}$, $\kappa(\mathcal{G}) = -\infty$, and $\nu(\mathcal{G}) = 1$, and whence $\rho = \sigma$ by the mutually unique contraction property with respect to either foliation. Whence finally we've got enough to apply [TY] or [KoN], where there is a more detailed treatment of the quotient singularities, i.e. for $\sigma^{\#}$ as above, and $D^{\#} = \sigma_{\#}(D)$ the elliptic Gorenstien leaves, necessarily in the non-singular locus of $X^{\#}$,

(a) $K_{X^{\#}} + D^{\#} = K_{\mathcal{F}^{\#}} + K_{\mathcal{G}^{\#}}$ is nef, and $(K_{X^{\#}} + D^{\#}) \cdot C = 0$ for C any curve iff $C \subset D^{\#}$.

(b) $(K_{X^{\#}} + D^{\#})^2 = 2\,K_{\mathcal{F}^{\#}} \cdot K_{\mathcal{G}^{\#}} = 2\,\rho^*\,K_{\mathcal{F}} \cdot K_X > 0$, and $\rho^*\,K_{\mathcal{F}} \cdot (K_{X^{\#}} + D^{\#}) = \rho^*\,K_{\mathcal{F}} \cdot K_{\widetilde{X}} > 0$ so $K_{X^{\#}} + D^{\#}$ is big.

Consequently $(\widetilde{X}, \widetilde{\mathcal{F}})$ is one of the naturally occuring foliations on a surface uniformised by the bi-disc, and the uniformisation map itself gives an almost étale cover $\nu : (S, \mathcal{H}) \to (X, \mathcal{F})$ such that the unique minimal desingularisation $(\widetilde{S}, \widetilde{\mathcal{H}}) \to (S, \mathcal{H})$ is obtained purely by resolving cusps.

V. Hyperbolicity

V.1. What we know and what we don't

The crudest possible understanding of the word hyperbolic is that it is not parabolic. For example for X an algebraic variety one can ask whether or not there are any non-trivial maps $f : \mathbb{C} \to X$. Surprisingly by a theorem of Brody, his reparameterisation lemma, this turns out to be a rather good definition if X is projective. When X is quasi-projective, however, reparameterisation seems to give no useful information (unless the boundary is of a rather special form), a 1^{st} approximation therefore to a correct definition is to ask that there are no maps $f : \mathbb{C} \to \overline{X}$ with Zariski dense image. One can of course pose exactly the same question in a foliation context, which indeed we'll do by means of a definition,

Definition V.1.1. *A foliated variety $(X, \mathcal{F})$ is said to be weakly hyperbolic modulo ∂ (∂ being a priori a closed subvariety, but in practice as the notation suggests a simple normal crossing divisor) if every non-trivial invariant map $f : \mathbb{C} \to X$ factors through ∂.*

Not surprisingly we'll be dealing with foliations on surfaces, so let $(X_0, \mathcal{F}_0)$ be a minimal model of a foliated surface with canonical singularities and, $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$ its unique minimal desingularisation. Evidently, irrespective of minimality, if $(X, \mathcal{F})$ is weakly hyperbolic modulo ∂ then we can take ∂ to be the finite set of rational and elliptic invariant curves. This set is in any case always finite regardless of whether $(X, \mathcal{F})$ is weakly hyperbolic or not, unless of course the foliation is a rational or elliptic pencil. For the moment then let's fix ∂ to be so, and note:

Fact V.1.2. (a) If $\rho^* K_{\mathcal{F}_0} \cdot K_X > 0$ then $(X, \mathcal{F})$ is weakly hyperbolic modulo ∂.

Indeed this is the main theorem of [M1]. To be precise the said theorem is only stated for X of general type, however, this is only used to guarantee $\rho^* K_{\mathcal{F}_0} \cdot K_X > 0$. A far more immediate demonstration, without going through the details of op.cit., is provided by Brunella's residue theorem for transverse invariant measures, [Br2], but more of this later. In addition, the refined tautological inequality of [M1], gives:

Fact V.1.2. (b) If $(X, \mathcal{F})$ is of general type then it is hyperbolic modulo ∂.

Tying this up with the classification, everything with $\nu(\mathcal{F}) \leq 0$ has all invariant curves parabolic, and if $\nu(\mathcal{F}) = 1$ and $\kappa(\mathcal{F}) = -\infty$ then indeed $\rho^* K_{\mathcal{F}_0} \cdot K_X > 0$ by Riemann-Roch so we can appeal to V.1.2 (a). Other $\nu(\mathcal{F}) = 1$ cases have $\kappa(\mathcal{F}) = 1$, and two are trivially easy: evidently elliptic fibrations have all invariant curves parabolic, while it is never so for isotrivial families of curves of genus at least 2. Amongst Ricatti and turbulent cases there is a distinguished sub-case:

Fact V.1.2. (c) If $\pi : (X_0, \mathcal{F}_0) \to B$ is the Kodaira fibration of a turbulent or Ricatti foliation which is everywhere transverse to π then it's weakly hyperbolic modulo ∂.

Again the key is the tautological inequality. This easily shows that if T is a transverse invariant measure associated to f, then T is homologous to a fibre. Whence if $\nu : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X_0, \mathcal{F}_0)$ is an almost étale covering à la IV.4.3 such that $\widetilde{X}$ is non-singular with $\widetilde{\mathcal{F}}$ everywhere transverse to the induced fibration $\widetilde{\pi} : \widetilde{X} \to \widetilde{B}$, then $\mathrm{discr}_{Y/\mathbb{C}}(r)(\overline{H}_{\widetilde{f}} Y(r))^{-1} \to 0$ as $r \to \infty$ cf. V.3 where $\widetilde{f} : Y \to \widetilde{X}$ is a lifting of f. In particular if $\widetilde{T}$ is the associated transverse invariant measure, the tautological inequality still forces $\widetilde{T}$ to be homologous to a fibre, and we'll observe in completing the proof of V.5.1(a) that this is impossible for cohomological reasons.

The general Ricatti and turbulent situations are quite delicate. Certainly to have a transcendental invariant map $f : \mathbb{C} \to X$ requires that the base B of the Kodaira fibration has genus zero or 1. Indeed argueing as above we know that there must be at least one invariant fibre so this excludes the possibility of turbulent over an elliptic base, whence:

Fact V.1.2. (d) If $\pi : (X, \mathcal{F}) \to B$ is neither a rational nor elliptic fibration nor given by a global vector field (up to almost étale covers) and $f : \mathbb{C} \to X$ is an invariant map with Zariski dense image then X is either Ricatti or turbulent over a rational base or Ricatti over an elliptic base.

An exact determination depends on the singularities of the minimal model $(X_0, \mathcal{F}_0)$ since apart from a couple of obvious cases to be discussed presently the curve f must pass through $\mathrm{sing}(X_0) \cup \mathrm{sing}(\mathcal{F}_0)$, so in principle an explicit description should be possible. An easier, and directly relevant question, is to discuss the leaves of $(X_0, \mathcal{F}_0)$ outside the singularities, i.e. is there an invariant map $f : \mathbb{C} \to X_0 \backslash Z$, where Z is any singularity of X_0 or $\mathcal{F}_0$, which is not the contraction of a $-1\,F$ chain, and which has order over the contraction of any -1 chain in its image divisible by the determinant of the intersection matrix of the same. The reason for treating the contraction of the $-1\,F$ chains almost as if they were smooth is because our usual discrepancy arguement of IV.3.3 means that locally everything is just a finite quotient of the smooth situation, so that the holonomy groupoid is still defined, all be it after an analogue of semi-stable reduction to take account of the finite holonomy about the contractions, and whence the notion of leaf on the terminal locus $X_0 \backslash Z$ makes sense. Better still if $\nu : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X_0, \mathcal{F}_0)$ is an almost étale cover with $\widetilde{X}$ non-singular then $\widetilde{Z} = \nu^{-1}(Z)$ is precisely the singular locus of $\widetilde{\mathcal{F}}$, as ever by IV.3.3, and leaves on $\widetilde{X} \backslash \widetilde{Z}$ go to leaves on $X \backslash Z$ (indeed this preservation property under Gorenstien coverings could have been used to define Z in the first place). Now suppose we have such a map $f : \mathbb{C} \to X_0 \backslash Z$ in the Ricatti over an elliptic base situation. We know that there is at least one invariant fibre, which by the very definition of leaf cannot meet f (something confirmed incidentally by IV.4.5) so this is also impossible, and the only options are Ricatti or Turbulent, with B rational and one or two invariant fibres. In order to write these down nicely, we need to show that $\widetilde{B}$ is still rational after the almost étale covering ν. Evidently this will follow if we know that the lifting $\widetilde{f}$ is still a map from $\mathbb{C}$, but this is immediate from the prescription on the ramification over the contractions of $-1F$ chains, and a local identity between the index of Gorenstien covers and order of holonomy. Now we have the full force of our monodromy description on $\widetilde{B}$, where there are necessarily one or two invariant fibres containing all the singularities, so in fact every leaf is uniformised by $\mathbb{C}$. Consequently, we'll use the slightly abusive notational shorthand of writing, $(\widetilde{X},\widetilde{\mathcal{F}}) \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{G}_a\,,\ \mathbb{P}^1 \times \mathbb{G}_m\,,\ E \times \mathbb{G}_a\,,\ E \times \mathbb{G}_m\,,$ for E an elliptic curve, to denote whether $(\widetilde{X},\widetilde{\mathcal{F}})$ is either Ricatti or turbulent with one or two invariant fibres. In any case all these examples occur, and are easily written down, for example if $b(x)$ is any polynomial of degree n with $b(0) \neq 0$ then $x^n \frac{\partial}{\partial x} + b(x)\, y\, \frac{\partial}{\partial y}$ gives an example isomorphic to $\mathbb{P}^1 \times \mathbb{G}_a$. In summary we have,

Fact V.1.2. (e) Either every leaf of $X_0\backslash Z$ is parabolic or a parabolic leaf is algebraic. The former case occurs iff the foliation is a rational or elliptic fibration or (up to almost étale covers) generated by a global vector field or a Ricatti or turbulent foliation of the form $\mathbb{P}^1 \times \mathbb{G}_m\,,\ \mathbb{P}^1 \times \mathbb{G}_a\,,\ E \times \mathbb{G}_m\,,\ E \times \mathbb{G}_a\,.$ This latter class we might as well give a name to such as almost vectorial, after all after flipping fibres to $\mathbb{P}^1 \times \mathbb{P}^1$ or working on $\mathbb{P}^1 \times E$ they can all be explicitly written down.

Now this is a very nice qualitative picture, but from a practical point of view wholly useless. When indeed there are no rational or elliptic invariant curves (or more generally when the totality of these is contractible) one can appeal to Brody's reparameterisation lemma to get a meaningfull quantitative statement. In general, however, there seems to be no way to relate this picture to quantitative statements like the size of invariant discs or the curvature as it varies from leaf to leaf. The minimal meaningful way to pose the curvature question appears to be,

Definition V.1.3. *Let $(X_0, \mathcal{F}_0)$ be a minimal model of a foliated surface with canonical singularities and $Z = \mathrm{sing}(X_0) \cup \mathrm{sing}(\mathcal{F}_0)$, excepting the contraction of $-1\,F$ chains, with x in the smooth locus of X_0, and ∂ a local generator of $\mathcal{F}_0$ at x then we define the small invariant Kobayashi metric,*

$$\|\partial\| = \inf\left\{ \frac{1}{R} : f : (\Delta, 0) \to (X_0\backslash Z, x) \text{ is invariant, } f_*\left(\frac{\partial}{\partial z}\right) = R\partial, \right.$$
$$f \text{ has order over the contraction of any } -1F \text{ chain}$$
$$\left. \text{divisible by the determinant of the same} \right\}$$

where Δ is the unit disc.

The small invariant Kobayashi metric is of course nothing other than the Poincaré metric along the leaves, and is thus naturally understood in the orbifold sense around the contraction of a $-1F$ chain. We could of course, and in many ways it would have been preferable, have defined it in terms of maps into all of X_0. The trouble is that this latter definition is not upper semi-continuous on the total space $T_{\mathcal{F}_0}$, whereas V.1.3 trivially is (any compact subset of a leaf in the smooth locus can be deformed). Certainly V.1.2(e) guarantees that for $(X_0, \mathcal{F}_0)$ not one of our finite list of exceptions the metric is non-zero outside of rational and elliptic curves. Unfortunately it fails to give any other basic information, e.g. is the Poincaré metric continuous. This difficulty is resolved by a proper definition of hyperbolicity as dictated by the principle of A. Bloch, "nihil est in infinito quod non fuerit prius in finito", cf. [Bl], namely that weak hyperbolicity, quod est infinito, ought to be a manifestation of a deeper finiteness property such as the convergence of discs. Unfortunately the literature on this subject is rather confused, so we'll have to devote the next couple of subsections to sorting out what this means. However, once we get the definitions sorted out the aim of this chapter will be to turn what we do know about hyperbolicity in infinito, into what for the moment we don't in finito, and to calculate explicit bounds for the degeneration of the Poincaré metric along ∂ in the process.

V.2. Preliminaries

Foliations are unimportant to the preliminaries so we'll work simply with a projective variety X and maps of discs to it, which we'll denote $\mathrm{Hom}(\Delta, X)$. The basic operation is a function $T(r)$ of the radius $0 \leq r < 1$ which is analogous to the pushforward to a point in ordinary intersection theory, i.e.

Definition V.2.1. *Let τ be a $(1,1)$ form on Δ then,*

$$T(r)\,(\tau) := \int_0^r \frac{dt}{t} \int_{\Delta_t} \tau$$

where Δ_t is the disc of radius t.

A priori this seems a rather bizaire definition, however, it's the absolutely correct one from an intersection theory point of view. Specifically let $\overline{L}$ be a metricised Cartier divisor on X, and $c_1(\overline{L})$ it's Chern class in $A^{1,1}(X)$ then we can define $\overline{L}._f\,\Delta(r)$ as $T(r)(f^* c_1(\overline{L}))$, better still if s is a meromorphic section of L with f^*s neither identically zero nor infinity. Then,

$$\overline{L}._f\,\Delta(r) = \sum_{0<|z|<r} \mathrm{ord}_z(f^*s) \log \frac{1}{|z|} - \int_{|z|=r} \log \|f^*s\| \frac{d\theta}{2\pi} + \log \|f^*s\|\,(0)$$

where for convenience we suppose 0 is neither a zero nor pole to avoid a more complicated formula. Consequently if s is actually in $H^0(X, L)$, $\|f^*s\|$ is bounded by a constant $C_{\overline{L}}$ independent of f, and,

$$\overline{L}._f\,\Delta(r) \geq C_L + \log \|f^*s\|\,(0)$$

i.e. the intersection with effective divisors is, basically, positive. When dealing with $f(0)$ fixed independently of f the error is negligible, for f varying, however, there is need for a good deal of care. Now the formula for $\overline{L}._f\,\Delta(r)$ could, by means of a moving lemma, have been taken for the definition of the intersection product. The advantage of the function $T(r)$ is that it allows us to extend the definition beyond divisors, i.e.

Definition V.2.2. *For $f \in \mathrm{Hom}(\Delta, X)$ and $r \in (0,1)$ we have a current $T_f(r)$ defined via,*

$$T_f(r) : A^{1,1}(X) \to \mathbb{C} : \eta \mapsto T(r)\,(f^*\eta)\,.$$

Observe in the first instance that the degree controls the derivative, i.e.

Lemma V.2.3. *Let $f : \Delta \to \mathbb{P}^n$ be a holomorphic map to projective space, and fix $0 < r < 1$. Then for the Fubini-Study metric adapted to a particular choice of coordinates depending only on a fixed sufficiently small neighbourhood of $f(0)$,*

$$|df(z)|^2_{\mathrm{euc}} \leq c_1\, R^{-1}\, \mathcal{O}_{\mathbb{P}^n}\overline{(1)}._f\,\Delta(r) + c_2$$

for $|z| < \frac{1}{2}R$, where $R = \min\{r/n\exp(-\mathcal{O}_{\mathbb{P}^n}\overline{(1)}._f\,\Delta(r)), r\}$, and the implied constants depend only on the said neighbourhood of $f(0)$ with degrees being measured in the said metric.

Proof. A disc is Stein, so fixing a trivialisation of $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ allows us to write $f = [f_0, \ldots, f_n]$ in projective coordinates. Everything is homogeneous under the unitary group so we may as well say $f(0) = [1, \ldots, 1]$. In particular if H_i is the i^{th}-coordinate hyperplane then,

$$\mathcal{O}_{\mathbb{P}^n}\overline{(1)}._f\,\Delta(r) = -\int_{|z|=r} \log\frac{\|f^*\mathbb{1}_{H_i}\|}{\|f^*\mathbb{1}_{H_i}\|(0)} + \sum_{0<|z|<r} \mathrm{ord}_z(f^*H_i)\log r/|z|\,.$$

Now for convenience put, $T_f(r) = \mathcal{O}_{\mathbb{P}^n}\overline{(1)}._f\,\Delta(r)$, and observe that if $f(z) \in H_i$ with $|z| < r$, then:

$$|z| \geq \frac{r}{n}\exp(-T_f(r))\,.$$

So put $R = \min\left\{r, \frac{r}{n}\exp(-T_f(r))\right\}$ then for $|z| < R$, $f_i(z) \neq 0$. In particular if we identify affine n-space $\mathbb{A}^n$ with $\mathbb{P}^n \backslash H_0$ then Δ_R maps to $\mathbb{A}^n$ under f with standard coordinates $g_i = f_i/f_0$ and each g_i is a unit on Δ_R.

Consequently Jensen's formula gives,

$$\frac{g_i'(z)}{g_i(z)} = \int_0^{2\pi} \log|g_i\,(\mathrm{R}e^{i\theta})|\,\frac{2\,\mathrm{R}e^{i\theta}}{(\mathrm{R}e^{i\theta} - z)^2}\,\frac{d\theta}{2\pi}\,.$$

We require to control the modulus of the logarithm in the integrand, so to this end observe that inversion on $\mathbb{G}_m^n = \{x_1 \ldots x_n \neq 0\} \subset \mathbb{A}^n$ is well defined on a modification $\rho : W \to \mathbb{P}^n$ which is an isomorphism outside crossings of hyperplanes. In particular if $i : W \to \mathbb{P}^n$ is the extension of inversion then,

$$i^* \mathcal{O}_{\mathbb{P}^n}(1) = \rho^* \mathcal{O}_{\mathbb{P}^n}(n) - E$$

for E an effective divisor on W contracted by ρ to a sub-scheme of the coordinate hyperplanes. Over $f(\Delta_R)$ we know how to invert, so,

$$\frac{1}{2} \int_{|z|=R} \log \left(1 + \sum_{i=1}^{n} \frac{1}{|g_i|^2} \right) \frac{d\theta}{2\pi} \leq n\, T_f(R) + C_n$$

where C_n is an absolute constant depending only on n. A similar bound for $\log \left(1 + \sum |g_i|^2\right)$ is even easier, and so we obtain, for $|z| < R/2$,

$$\left| \frac{g_i'}{g_i} \right| \leq 8R^{-1}(n+1)\, T_f(R) + C_n$$

with C_n as before up to its definition. Now consider the Fubini-Study metric for the particular choice of coordinates, with ω the corresponding $(1,1)$ form then,

$$\omega \leq \frac{\sqrt{-1}}{\pi} \sum_i \frac{dx_i \, d\overline{x}_i}{(1 + |x_i|^2)}$$

so that plugging in our estimates for g_i' gives the lemma. □

Now certainly the lemma depends on the whereabouts of $f(0)$, but given a sequence of maps f_n we can always subsequence so that the $f_n(0)$ converge. At this point the choice of a metric adapted to $f(0)$ is irrelevant, given that the boundedness or otherwise of $\overline{H}\cdot_{f_n} \Delta(r)$, for H ample is wholly independent of the metric in the limit.

This is of course just another application of Jensen's formula which in the context of the degree $\overline{H}\cdot_{f_n} \Delta(r)$, r fixed, $n \to \infty$ or n fixed $r \to 1$ manifests itself by way of $\lim(\overline{H}\cdot_{f_n} \Delta(r))^{-1}\, T_{f_n}(r)$ having a subsequence that converges to a positive harmonic current. Certainly therefore dependence on the metric is sufficiently minimal that it will often be dropped from the notation. However, harmonic is not really good enough for applications. Especially in the presence of foliations where the Baum-Bott residue philosophy takes place at the level of Dolbeault cohomology, and not at a Hodge type level of forms modulo $\partial\overline{\partial}$ boundaries as is the case with metrics on bundles. The following lemma is therefore the main technical innovation of this section, and is designed to overcome the problems inherent in reparameterisation, viz:

Lemma V.2.4. *Let X be a projective variety, and $\overline{H}$ an ample divisor with strictly positive metric and $f_n : \Delta \to X$ a sequence of maps such that for some $0 < r < 1$, $\overline{H}\cdot_{f_n} \Delta(r) \to \infty$, then after subsequencing we can find $R \in (r, 1)$ such that,*

$$\frac{1}{\overline{H}\cdot_{f_n} \Delta(R)}\, T_{f_n}(R) \underset{w}{\longrightarrow} T$$

where $T \neq 0$ is a closed positive current.

Proof. Any sequence f_n has a subsequence such that $\frac{1}{\overline{H}._{f_n}\Delta(R)} T_{f_n}(R) \underset{w}{\longrightarrow} T$ and, $T(c_1(\overline{H})) = 1$, so $T \neq 0$. The difficulty is to prove closure. Consequently for α a smooth $(1,0)$ form consider,

$$|T_f(R)\,(\overline{\partial}\alpha)| = \frac{1}{2}\left|\int_{\Delta_R} \overline{\partial}\log|z|^2\,\alpha\right| \leq \|\alpha\|_{\sup}\int_{\Delta_R}|df|_{\mathrm{euc}}\,\frac{dt\,d\theta}{2\pi}$$

for a suitable determination of the sup-norm independent of f. Now observe,

$$\overline{H}._f\,\Delta(R) = \int_0^R \frac{dt}{t}\int_0^t I_2(s)\,sds = \int_0^R I_2(s)\,sds\int_s^R\frac{dt}{t} = \int_0^R I_2(s)\,s\log R/s\,ds$$

where $I_2(s) = \int_{|z|=s}|df|^2_{\mathrm{euc}}\,\frac{d\theta}{2\pi}$. Rearranging we have,

$$I_f(R) := \int_0^R I_2(t)\,t\,|\log t|\,dt = T_f(R)\,(c_1(\overline{H})) + |\log R|R\,\frac{d}{dR}\,T_f(R)\,(c_1(\overline{H}))\,.$$

Consider for a suitable $\varepsilon > 0$ to be chosen,

$$\begin{aligned}\int_{R\geq|z|\geq\varepsilon}|df|_{\mathrm{euc}}\,\frac{dt\,d\theta}{2\pi} &\leq \int_\varepsilon^R I_2(t)^{1/2}\,dt\\ &\leq \left(\int_\varepsilon^R t\,|\log t|\,I_2(t)\;dt\right)^{1/2}\left(\int_\varepsilon^R\frac{dt}{t\,|\log t|}\right)^{1/2}\\ &\leq I_f^{1/2}(R)\log^{1/2}\,\frac{|\log\varepsilon|}{|\log R|}\,.\end{aligned}$$

Now let's use the $\ll$ notation for inequality up to a constant that depends on what we can fix, i.e. K, r, X, H etc. but definitely not f. Then for any $\varepsilon \ll \min\{r\exp(-T_f(r)\,(c_1(\overline{H})), r\}$ we may apply V.1.7 to conclude,

$$\begin{aligned}\int_{|z|\leq R}|df|_{\mathrm{euc}}\,\frac{dt\,d\theta}{2\pi} &\ll I_f^{1/2}(R)\log^{1/2}\,\frac{|\log\varepsilon|}{|\log R|} + \varepsilon^{1/2}\,T_f^{1/2}(r)\,(c_1(\overline{H}))\\ &\ll I_f^{1/2}(R)\log^{1/2}\left(\frac{T_f(R)}{|\log R|}\,(c_1(\overline{H}))\right)\end{aligned}$$

for a suitably optimal ε. It therefore remains to show that $I_f(R)$ is not that much bigger than $T_f(R)$ for a suitable R independent of f. This is easily done by a standard piece of measure theory,

Fact V.2.5. Let $\delta > 0$ and S_n a sequence of increasing differentiable functions on $[0,1)$ with $S_n(r_0) \geq n^{2/\delta}$ for some $r_0 \in (0,1)$ then the set,

$$\{1 > r \geq r_0 : S_n'(r) \geq S_n^{1+\delta}(r)\,(1-r^2)^{-1},\ \text{for some } n\}$$

has finite measure in the induced hyperbolic measure on the unit interval, i.e. $(1-t^2)^{-1}\,dt$.

Proof. The set in question has measure bounded by,

$$\sum_n \int_{r_0}^1 \frac{S_n'(r)}{S_n^{1+\delta}(r)}\, dr \le \sum_n \int_{S_n(r_0)}^\infty \frac{dx}{x^{1+\delta}} = \frac{1}{\delta}\sum_n \frac{1}{S_n(r_0)}\,\delta = \frac{\pi^2}{6\delta} < \infty\,.$$

Fixing $\delta > 0$ we can certainly subsequence to obtain, $T_{f_n}(r)\,(c_1(\overline{H})) \ge n^{2/\delta}$, consequently we can find $R \in (r,1)$ with,

$$I_{f_n}(R) \le \frac{|\log R|\, T_{f_n}^{1+\delta}(R)(\overline{c_1(H)})}{(1-R^2)}$$

for all n. Since $T_{f_n}(R) \ge T_{f_n}(r)$ we obtain a closed current as claimed after suitable subsequencing. □

Let's close this section by way of some paranthetical remarks which will save time later. In the first place the technique of V.2.5 is extremely robust, and easily extends to a situation involving both sequences of varieties and functions, so in practice a sequence of increasing differentiable functions S_{nk} then we have,

Variation V.2.6. Notations as above for $r \in (0,1)$ outside a set of finite hyperbolic measure,

$$S_{nk}'(r) \le \{S_{nk}(r) + 2e\} \log^{n^2+k^2+1}\{S_{nk}(r) + 2e\}\,(1-r^2)^{-1}\,.$$

Proof. Indeed writing $F_{nk}(r) = S_{nk}(r) + 2e$, we have $\log F_{nk}(0) > 1$, and the set in question has measure bounded by,

$$\sum_{n,k} \int_{F_{nk}(0)}^\infty \frac{dx}{x \log^{n^2+k^2+1} x} = \sum_{n,k} \frac{1}{(n^2+k^2)(\log F_{nk}(0))^{n^2+k^2}} < \infty\,.$$

In which context it is also useful to note the functoriality properties of this construction. Quite generally we can extend our intersection product to maps $f : Y \to X$ where $p : Y \to \Delta$ is a ramified cover of the disc – note, and not withstanding the uniformisation theorem, this is not just generalisation for the fun of it, since the critical point is how the curve is exhausted with such curves naturally arising by way of covering constructions,

$$\begin{array}{ccc} Y & \xrightarrow[\tilde{f}]{} & Z \\ \downarrow p & & \downarrow \rho \\ \Delta & \xrightarrow[f]{} & X \end{array}$$

with $\rho : Z \to X$ a map of algebraic varieties. Naturally the intersection product on Y is defined by pushing forward forms to Δ, then applying our previously defined $T(r)$. Even better for sequences of maps arising from covering constructions everything goes through verbatim (the wholly general situation requires minor changes, and due to its academic nature is ignored), i.e.

Variation V.2.7. Notations as above, and suppose further that our triples $f_n : Y_n/\Delta \to X$ arise from lifting maps $g_n : \Delta \to W$, i.e. we have for each n commutative diagrams,

$$\begin{array}{ccc} Y_n & \underset{f_n}{\longrightarrow} & X \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p} \\ \Delta & \underset{g_n}{\longrightarrow} & W \end{array}$$

with $g_n(0)$ bounded away from the ramification, for n sufficiently large, of the supposed generically finite map p, then for $R \in [r, 1)$ outside a set of finite hyperbolic measure $T(R)$ is closed, where $T(R)$ is a weak limit of

$$T_{f_n}(r) : A^{1,1}(X) \to \mathbb{R} : \eta \mapsto \deg(r)\,(f_n^* \eta)$$

normalised by the degree, $(\overline{H} \cdot_{f_n} Y_n(r))^{-1}$.

The final thing to bear in mind is that T enjoys reasonable functoriality properties. Specifically consider a commutative diagram with the vertical map on the right dominant,

$$\begin{array}{ccc} \widetilde{Y}_n & \underset{\widetilde{f}_n}{\longrightarrow} & Z \\ \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho} \\ Y_n & \underset{f_n}{\longrightarrow} & X \\ \downarrow{\scriptstyle p_n} & & \\ \Delta & & \end{array}$$

Evidently we consider $\widetilde{p}_n : \widetilde{Y}_n \to \Delta$ as the composition $\rho_n^* p_n$, and, for any $r \in (0, 1)$, $\rho_* T_{\widetilde{f}_n}(r) = (\widetilde{Y}_n : Y)\, T_{f_n}(r)$, whence,

Fact V.2.8. Let $\widetilde{T}, T$ be the associated currents then up to suitable normalisation, $\rho_* \widetilde{T} = T$.

Here we're deliberately vague. Obviously $(\widetilde{Y}_n : Y)$ constant in n is a without loss of generality hypothesis. What's more delicate is what ample bundle we use. The ample degree on Z will always dominate that on X, and when the converse holds the proposition is up to an unimportant choice of constant unambigous. Typical situations where this will be true are ρ generically finite and $f_n(0)$ bounded away from a suitable closed subvariety or Z the projective/log projective tangent space, and f_n some subsequence of divergent discs. In any case the worst that could happen is that normalising by the Z degree, $\rho_* \widetilde{T} = 0$, which with one exception typically implies trivially what we need to know, so that generally the issue of normalisation will be passed over without comment.

V.3. Compactness

Our starting point is,

Lemma V.3.1. *Let $f_n \in \mathrm{Hom}(\Delta, \mathbb{P}^N)$ be a sequence of maps with continuous extensions to the closed unit disc $\overline{\Delta}$ with $H._{f_n} \overline{\Delta}$ bounded uniformly in n then for any $r < 1$ we can subsequence in such a way that the f_n may be written in suitable homogeneous coordinates as $[f_{n0}, \dots, f_{nN}]$ where the f_{ni} converge for all i.*

Proof. Without loss of generality we may suppose the f_n accumulate somewhere, so say $1 = [1, \dots, 1]$ in an appropriate projective coordinate system with respect to which we may metricise H by way of the Fubini-Study metric. Consequently if $X_i \in \Gamma(\mathbb{P}^N, H)$ is the equation of a coordinate hyperplane then modulo further subsequencing and enlarging r a little as necessary we can suppose that for each i, the $f_n^{-1}(X_i) \cap \Delta_r$, counted with multiplicity are a convergent sequence of 0 cycles in $\overline{\Delta}_r$ which are actually contained in $\overline{\Delta}_s$ for some $s < r$, which we'll write as,

$$f_n^{-1}(X_i) = \sum_{z \in \overline{\Delta}_s} a_{ni}(z)[z] \to \sum_{z \in \overline{\Delta}_s} a_i(z)[z]$$

where necessarily $\exists t > 0$, $a_{ni}(z) = 0$, $|z| \leq t$, and all i independently of n. The disc being Stein we can write $f_n |_{\Delta_r}$, for an appropriate $1 > R > r$ as,

$$f_n(z) = \left[\prod_{w \in \overline{\Delta}_s} (z-w)^{a_{n0}(w)}, \prod_{w \in \overline{\Delta}_s} (z-w)^{a_{n1}(w)} u_1, \dots, \prod_{w \in \overline{\Delta}_s} (z-w)^{a_{nN}(w)} u_N \right]$$

where the $u_1, \dots, u_N$ are units. Now consider the meromorphic functions g_{ni} defined as $f_n^* X_i / f_n^* X_0$ and apply Jensen's formula, i.e. for $z \in \Delta_r$,

$$\log |g_{ni}(z)| \cdot |P_{ni}(z)| = \int_{|w|=R} \log |g_{ni}(w)| \ \mathrm{Re} \left\{ \frac{w+z}{w-z} \right\} \frac{dw}{2\pi\sqrt{-1}\, w}$$

where $P_{ni}(z)$ is the Blaschke product,

$$P_{ni}(z) = \prod_{w \in \overline{\Delta}_s} \left\{ \frac{R^2 - \overline{w}\, z}{R(z-w)} \right\}^{a_{ni}(w) - a_{n0}(w)} .$$

Consequently employing the technique of V.2.3, or more correctly its proof, to bound the integrals over $|w| = R$ of $\log(1 + |g_{ni}|)$ and $\log\left(1 + \frac{1}{|g_{ni}|}\right)$ independently of n we conclude that for some constant C independent of n, $z \in \Delta_r$,

$$|u_{ni}|(z) \leq C \cdot \frac{(R^2 - rs)^a}{R^b}$$

for non-negative integers a, b independent of n determined by the signs of $a_i(w) - a_0(w)$, so that the u_{ni} converge, and we're done. $\square$

Despite appearances, as Brunella pointed out to me, this doesn't guarantee the convergence of f_n in the compact open topology. The classic example is,

$$f_n : \Delta \to \mathbb{P}^1 : z \to \left[z - \zeta, z - \zeta + \frac{1}{n} \right]$$

for any ζ in the disc punctured at the origin. Evidently this problem presents itself quite generally at any point in the common zero locus of the limit $f_0, \ldots, f_n$. On the other hand the maps f_n manifestly "converge", and so the only real question is in what space. The answer is, of course, provided by the Bishop-Gromov compactness theorem which we summarise according to our needs. In the first place we have,

Definition V.3.2. *A disc with bubbles Δ^b is a connected 1-dimensional complex space with singularities at worst nodes exactly one of whose components is the unit disc Δ, and every connected component R_z, $z \in R(\Delta^b) := \Delta \cap \mathrm{sing}(\Delta^b)$, of the complement of Δ is a tree of smooth rational curves.*

Certainly for $f : \Delta^b \to X$ without a bubble at the origin, and $\overline{L}$ a metricised line bundle we can easily extend the intersection product by way of,

$$\overline{L}\cdot_f \Delta^b(r) := \overline{L}\cdot_f \Delta(r) + \sum_{\substack{z \in R(\Delta^b) \\ |z| < r}} \log \frac{r}{|z|} \int_{R_z} f^* c_1(\overline{L})$$

so that necessarily we have similar positivity properties for intersections with effective divisors to the intersection product over discs. In addition we can define a graph Γ_f as,

$$\Gamma_f = (\mathrm{id} \times f)(\Delta) \bigcup_{z \in R(\Delta^b)} z \times f(R_z) \subset \Delta \times X .$$

Moreover the compact open topology on $\mathrm{Hom}(\Delta, X)$ is equivalent to the Hausdorff topology on the graphs of maps, so that quite naturally we can consider $f : \Delta^b \to X$ as a boundary point if its graph is a limit of graphs Γ_{f_n}, i.e.

Definition V.3.3. *Say that $f_n \in \mathrm{Hom}(\Delta, X)$ converges to $f : \Delta^b \to X$ if for $r \in (0,1) \backslash \{|z| : z \in R(\Delta^b)\}$ the graphs $\Gamma_{f_n} \times_\Delta \overline{\Delta}_r$ converge to $\Gamma_f \times_\Delta \overline{\Delta}_r$ in the Hausdorff metric.*

We can certainly add such boundary points to our Hom sets, to get a new space $\overline{\mathrm{Hom}}(\Delta, X)$ topologised by embedding it in the space of subvarieties of $\Delta \times X$ in the Hausdorff metric over compact subsets of Δ, and with this in mind a convenient formulation of Gromov's compactness theorem in light of V.3.1 and a standard diagonalisation arguement is,

Fact V.3.4. ([Gr]) Let $C : (0,1) \to \mathbb{R}_+$ be any function and $K \subset \mathrm{Aut}(\Delta)$ compact then,

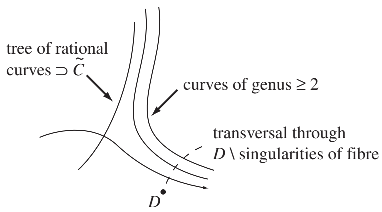
$$\{f \in \overline{\mathrm{Hom}}(\Delta, X) : \overline{H}_{\cdot\alpha^* f}\, \Delta(r) \leq C(r), \alpha \in K\}$$

is compact. In particular $f_n \in \overline{\mathrm{Hom}}(\Delta, X)$ has a convergent subsequence iff for some subsequence f_k, radii $r_m \to 1$ and $\alpha_k \in \mathrm{Aut}(\Delta)$ convergent, $\overline{H}_{\cdot\alpha_k^* f}\, \Delta(r_m)$ is bounded independently of k for each m.

As we've defined it as a subset of the set of subvarieties of $\Delta \times X$ in the Hausdorff topology, $\overline{\mathrm{Hom}}(\Delta, X)$ is a minimal compactification of what may be termed bounded subsets of $\mathrm{Hom}(\Delta, X)$. Now certainly $\overline{\mathrm{Hom}}(\Delta, X)$ $= \mathrm{Hom}(\Delta, X)$ if X doesn't contain rational curves, but conversely we have,

Fact V.3.5. $\mathrm{Hom}(\Delta, X)$ is a propersubset of $\overline{\mathrm{Hom}}(\Delta, X)$ if the smooth locus of X contains rational curves.

Proof. Taking hyperplane sections, it's clearly sufficient to treat the case of surfaces – actually even curves would do but we'd like to make examples of sequences of discs not wholly contained in the curve. Equally provided we make appropriately generic examples we can always blow them down so we might as well say C is a smooth rational curve in a smooth surface X. Choose a point $\infty \in C$, then we can find an affine neighbourhood U of $C \backslash \infty$, and whence $\mathcal{I}_C \otimes \mathcal{O}_U$ is generated by a single equation, x, say, which defines a pencil $x =$ constant. Passing to a modification $\rho : \widetilde{X} \to X$ we obtain a Lefschetz pencil. Denote by $\widetilde{C}$ the proper transform of C, then for a sufficiently generic choice of pencil, the generic element is a hyperbolic curve, and some component D of the fibre containing $\widetilde{C}$ minus the singularities is hyperbolic and does not blow down to a point under ρ. Around a chain of rational curves in the fibre containing $\widetilde{C}$ as a component which without loss of generality we can suppose meets D our fibration looks like,



(V.3.6)

Consequently if we uniformise the curves of genus ≥ 2 indicated in the figure by the disc with the origin centered on the indicated transversal then we obtain a sequence of discs that converges to a disc with bubbles on $\widetilde{X}$, and by the particular choice of D this blows down to the same on X. $\square$

Now suppose more generally that we're in a quasi-projective situation. The appropriate level of generality is to consider $X \backslash D$, for a pair (X, D) with log-canonical singularities and the components of D, $\mathbb{Q}$-Cartier divisors, although regardless of the singularities if we have a map from $\mathbb{P}^1$ or even just $\mathbb{A}^1$ to the smooth locus of $X\backslash D$ then 2.2.1 goes through verbatim to show that $\mathrm{Hom}(\Delta, S)$ is a strict subset of $\overline{\mathrm{Hom}}(\Delta, X)$, while in the converse direction we may appeal to,

Lemma V.3.7. [G] *Let $f_n : Y \to X\backslash D$ be a sequence of maps converging uniformly on compact subsets of a Riemann surface to $f : Y \to X$, then either, f maps to $X\backslash D$ or D.*

Now necessarily if a sequence $f_n : \Delta \to X$ with degrees uniformly bounded on a sequence of radii $r_k \to 1$ converges to a disc with bubbles, then for some $0 < |\zeta| < 1$, there are maps $\varphi_n : \Delta_{R_n} \to \{z \mid |z-\zeta| < \varepsilon_n\}$ with $R_n \to \infty$ and $\varepsilon_n \to 0$ such that $f_n \circ \varphi_n$ converges uniformly on compact subsets of $\mathbb{C}$ to a rational map $f : \mathbb{A}^1 \to X$ (cf. [Sik]). Whence if in fact the f_n are maps to $X\backslash D$ then either $X\backslash D$ or D must contain an affine line. Better still if we write $D = \Sigma D_i$ in terms of its components then the line isn't just in D but $D_i\backslash \bigcup_{j\neq i} D_j$ for some i, and should this happen we have,

Fact V.3.8. Notations as above, if $X\backslash D$ and $D_i\backslash \bigcup_{j\neq i} D_j$ do not contain any affine lines then $\overline{\mathrm{Hom}}(\Delta, X\backslash D) \subset \mathrm{Hom}(\Delta, X)$.

There are plenty of natural situations where the hypothesis that $D_i\backslash \bigcup_{j\neq i} D_j$ contains no lines is guaranteed, e.g. X a toric variety and $X\backslash D$ a torus, while for surfaces we have the wholly general,

Fact/Definition V.3.9. (Log minimal model theorem) Suppose $X\backslash D$ is not fibred by lines then there is a contraction $\rho : (X, D) \to (X', D')$ within our given category of singularities with $X\backslash D = X'\backslash D'$ such that $D'_i\backslash \bigcup_{j\neq i} D'_j$ does not contain affine lines. Even better if (X, D) has general type then we may even insist that each $D'_i\backslash \bigcup_{j\neq i} D'_j$ is hyperbolic, in which case we say that the boundary D is *hyperbolic.*

Evidently this is very close to the situation that we're likely to encounter in the study of the invariant Poincaré metric à la V.1.3, i.e. all the invariant affine lines have been contracted. This is, however, not quite true due to the presence of saddle nodes, nevertheless we have,

Fact V.3.10. (Brunella's no bubbling lemma, [Br4]) Notations as per V.1.3, with the obvious extension of $\overline{\mathrm{Hom}}$ to the invariant context and the need for maps to have order the discriminant of any contracted chain they may pass through then,

$$\overline{\mathrm{Hom}}_{\mathcal{F}}(\Delta, X_0\backslash Z) = \widetilde{\mathrm{Hom}}_{\mathcal{F}}(\Delta, X_0\backslash Z)$$

where $\widetilde{\ }$ is used to clarify that we're in the Brunella subspace of $\mathrm{Hom}_{\mathcal{F}}(\Delta, X\backslash Z)$ prescribed by the above ramification condition.

With these considerations in mind the proper reflection, "in finito", of weak hyperbolicity for projective varieties is,

Definition V.3.11. (a) *For ∂ a proper subvariety of X we say X is hyperbolic modulo ∂ if a sequence of maps $f_n : \Delta \to X$ of which a subsequence doesn't converge in $\overline{\mathrm{Hom}}(\Delta, X)$ is, in the compact open sense, contained in arbitrarily small neighbourhoods of ∂. Idem in the foliated case with* Hom *replaced by the set of invariant maps* $\mathrm{Hom}_{\mathcal{F}}$.

Observe that by V.3.5 the above definition is best possible. Indeed to fix ideas, say X is a surface then since everything which is Stein has a Stein neighbourhood at the very least Z must contain all the rational and elliptic curves, so in fact there must be a finite set of such. If there are only elliptic curves then we could replace $\overline{\mathrm{Hom}}$ by Hom, but this is probably a fairly special situation. In the quasi-projective situation the most generally valid definition is,

Definition V.3.11. (b) *Let (X, D) be a projective variety with boundary and ∂ a proper subvariety of $X\backslash D$ then we say (X, D) is hyperbolic modulo ∂ if a sequence of maps $f_n : \Delta \to X\backslash D$ of which a subsequence doesn't converge in $\overline{\mathrm{Hom}}(\Delta, X)$ is, in the compact open sense, contained in arbitrarily small neighbourhoods of $\partial \cup D$.*

Again as we've seen this is best possible. On the other hand given that hyperbolicity necessitates general type at least for varieties, rather than foliations, we can propose a definition excluding the possibility of arbitrary proximity to D, provided D is hyperbolic in the sense of V.3.9, i.e.

Definition V.3.11. (c) *Everything as per V.3.11(b), but suppose additionally that D is hyperbolic (V.3.9) then if f_n not enjoying a convergent subsequence obliges it to be arbitrarily close to $\bar{\partial}$, we say that (X, D) is complete hyperbolic modulo ∂.*

One should bear in mind however that although such a compactification of $X\backslash D$ exists for surfaces, higher dimension seems anything but obvious. In any case the hyperbolicity of X – respectively (X, D)-modulo ∂ leads to a natural metric as per V.1.3, at least in the smooth case – respectively log-smooth – on T_X, respectively $T_X(-\log D)$, which we'll give in the more general log case, i.e.

Defintion V.3.12. (Kobayashi-Royden) *Let (X, D) be a smooth projective variety, with simple normal crossing boundary, which is hyperbolic modulo ∂ then for $x \in X\backslash D\backslash\partial$, and $\partial \in T_X(-\log D) \otimes \mathbb{C}(x)$,*

$$\|\partial\|_{\mathrm{Kob}} = \inf\left\{\frac{1}{R} \mid f : (\Delta, 0) \to (X\backslash D, x) : f_*\left(\frac{\partial}{\partial z}\right) = R\partial\right\}.$$

Now while the metric is certainly non-degenerate off ∂ under the hyperbolicity hypothesis, which motivates the same as Kobayashi's definition of hyperbolicity, hyperbolicity in the sense of V.3.11 implies more. Specifically,

Fact V.3.13. Let (X, D) be hyperbolic modulo ∂, then $\| \;\|_{\mathrm{Kob}}$ is continuous on the total space of $T_X(-\log D)$ off $D \cup \partial$.

Proof. Upper semi-continuity is as ever just the property that everything Stein has a Stein neighbourhood. In the other direction, let $x \in X \backslash D \cup \partial$, $x_n \to x$, and $(x_n, \partial_n) \to (y, \partial)$ in $T_X(-\log D)$, with $(f_n)_* \left(\frac{\partial}{\partial z}\right) = R_n \partial_n$. Now f_n converges to a disc with bubbles, $f : \Delta^b \to X$, which cannot bubble at the origin since $\partial \cup D$ necessarily contains the bubbling locus, and $x \notin \partial \cup D$, whence there is a fixed $\varepsilon > 0$ such that the discs $f_n \mid_{\Delta_\varepsilon}$ converge to $f \mid_{\Delta_\varepsilon}$, thus if $f_* \left(\frac{\partial}{\partial z}\right) = R\partial$, then $\varepsilon R_n \to \varepsilon R$, and we're done. □

In the more specific foliated case, the hyperbolicity of $(X, \mathcal{F})$ modulo ∂ comfortably implies the continuity of the invariant Poincaré metric of V.1.3, by a minor variant of the above to take care of the ramification at the $-1F$ chains by way of the algebraic nature of the Gorenstien covering. On the other hand, with our notations as per op.cit., and given Brunella's no bubbling lemma an even stronger result is on the horizon for invariant maps missing the singularities for which we prepare for by way of,

Definition V.3.14. *Let H be a subset of $\overline{\mathrm{Hom}}(\Delta, X)$, then we say that it is compact modulo ∂ if it is closed, and every sequence in H which doesn't enjoy a convergent subsequence is arbitrarily close to ∂*

Note in particular that if H is a Hom space, as opposed to a $\overline{\mathrm{Hom}}$ space, then the convergence is necessarily taking place without bubbles. We will need one final preliminary, which for want of a better place we'll put here,

Lemma V.3.15. *Suppose a sequence $\{f_n\} \in \mathrm{Hom}(\Delta, X)$ is given which is not eventually arbitrarily close to a Zariski closed subset ∂ of X, and that no subsequence of $\{f_n\}$ converges then after subsequencing there is a convergent sequence $\alpha_n \in \mathrm{Aut}(\Delta)$ such that $\mathrm{dist}(\alpha_n^* f_n(0), \partial)$ and $\left\|(\alpha_n)_* (f_n)_* \left(\frac{\partial}{\partial z}\right)\right\|(0)$ are bounded away from zero independently of n.*

Proof. By hypothesis there is a disc of radius $r < 1$, with $f_n(\overline{\Delta}_r) \not\subset U$ for U some open neighbourhood of ∂, and $|df_n|_{\mathrm{euc}}$ large somewhere on $\overline{\Delta}_r$. Provided we simply take automorphisms that move the origin to somewhere in $\overline{\Delta}_r$ these will converge after subsequencing, so we might as well say that $|df_n|_{\mathrm{euc}}(0)$ is as large as we like. So consider a connected component U'_n of points in $\overline{\Delta}_r$ which are a suitable distance ε away from ∂, which contains an open subset of points U''_n of points a distance 2ε away. Now take a small square in U''_n and extend it inwards as a rectangle until it touches U'_n (e.g. if the square is centered on the positive imaginary axis, the rectangle is given by taking the y coordinate as small as possible, remembering without loss of generality that $0 \notin U''_n$). This gives us a closed rectangle $\overline{R}_n$, say, in $\overline{\Delta}_r$ containing points u_n, v_n with $\mathrm{dist}(f_n(u_n), \partial) \geq 2\varepsilon$ and $\mathrm{dist}(f_n(v_n), \partial) = \varepsilon$. Equally we're done unless the derivative on $\overline{R}_n$ is very small, in particular bounded, so the same is true for the composite g_n of f_n with the obvious map $\overline{S} \to \overline{R}_n$ from the closed square of length 1. Denoting inverse image of this latter map by $\sim$ we can subsequence so that $g_n \to g$, $\widetilde{u}_n \to u$, and $\widetilde{v}_n \to v$, with of course

$g_n(\widetilde{u}_n) \to g(u)$ and $g(\widetilde{v}_n) \to g(v)$, so g is non-constant. Take any point where dg is non-zero, and denote by z_n its image in R_n then $|d f_n(z_n)|_{\mathrm{euc}}$ is eventually bounded away from zero as required. □

V.4. Comparison of conformal structures

It's an easy consequence of hyperbolicity in the sense of V.3.11 that if X is hyperbolic and $f : C \to X$ a map from an algebraic curve, there is a constant $\alpha(X)$ such that,

$$H._f C \leq \alpha(X)(2g-2)$$

where H is ample on X, and g the genus of C. Indeed what is hoped is that if we extend the notion of curve sufficiently generally then such inequalities might even be equivalent to hyperbolicity. Here the notion of curve is somewhat inverse to the uniformisation theorem. All be it that most algebraic curves are hyperbolic, they are nevertheless distinguished amongst all Riemann-surfaces by the fact that they admit a finite covering of $\mathbb{P}^1$, in terms of which the genus is easily expressed by way of the Riemann-Hurwitz formula. In this respect there is another distinguished class of Riemann-surfaces, which are those normally found in Nevanlinna theory, namely finite ramified covers $p : Y \to \mathbb{C}$ of the line. Here the analogue of genus is a function of $r > 0$,

$$\mathrm{discr}_{Y/\mathbb{C}}(r) = \sum_{0<|p(y)|<r} \mathrm{ord}_y(\mathrm{Ram}_p) \log \left|\frac{r}{p(y)}\right| + \sum_{y \in p^{-1}(0)} \mathrm{ord}_y(\mathrm{Ram}_p) \log r$$

this is obviously reasonable in light of the nature of the intersection product, which as we've said one defines in the obvious way via V.2.1. Even though any other Riemann-surface is uniformised by the disc it will be important to work in the generality of finite ramified covers $p : Y \to \Delta$. Naturally the discriminant should take account of the negative curvature of the disc, and so is defined as,

$$\begin{aligned}\mathrm{discr}_{Y/\Delta}(r) = (Y : \Delta) \log(1-r^2)^{-1} &+ \sum_{0<|p(y)|<r} \mathrm{ord}_y(\mathrm{Ram}_p) \log \left|\frac{r}{p(y)}\right| \\ &+ \sum_{y \in p^{-1}(0)} \mathrm{ord}_y(\mathrm{Ram}_p) \log r\end{aligned}$$

the $\log(1-r^2)^{-1}$ is also wholly reasonable being $T(r)$ (Poincaré metric). In any case the discriminant should be considered as the natural measure of size of the conformal structure on Y induced from $T_Y \to p^* T_\Delta$. For a map $f : Y \to X$ we wish to compare this with the conformal structure arising from $T_Y \to f^* T_X$. This is done via the comparison of conformal structures map F, which we'll require in the greater generality of projective varieties X with simple normal crossing boundary D, and is defined as,

$$F : Y \to P(T_X(-\log D) \oplus \mathbb{1}) : y \mapsto \left[f_* \left(\frac{\partial}{\partial y} \right), p_* \left(\frac{\partial}{\partial y} \right) \right] .$$

Being a priori well defined except at points where f and p are ramified, and being well defined even there since Y is normal. In local coordinates say $x_1, \ldots, x_n$ with the divisor being given by $x_1 \ldots x_k = 0$ then of course,

$$F(y) = \left[\frac{x_1'(y)}{x_1}, \ldots, \frac{x_k'(y)}{x_k}, x_{k+1}'(y), \ldots, x_n'(y), p'(y) \right] .$$

Now let $\overline{M}$ be a metricisation of the tautological bundle of $P(T_X(-\log D) \oplus \mathbb{1})$ arising from a given fixed metric on $T_X(-\log D)$ and the trivial metric on the trivial bundle, then the comparison of conformal structures theorem morally says,

$$\overline{M}._F \, Y(r) \leq \mathrm{discr}_{Y/\Delta}(r) + \mathrm{rad}_{f,D}(r) + (\text{small error})$$

with the intersection product is extended to Y via p in the obvious way, and,

$$\begin{aligned} \mathrm{rad}_{f,D}(r) = & \sum_{0<|p(y)|<r} \min\{1, \mathrm{ord}_y(f^* D)\} \log \left| \frac{r}{p(y)} \right| \\ & + \sum_{y \in p^{-1}(0)} \min\{1, \mathrm{ord}_y(f^* D)\} \log r \, . \end{aligned}$$

The tricky thing is that in order to work with sequences of maps of discs, we should be very careful in computing the error. Consequently the precise formulation is,

Proposition V.4.1. *Notations as above there is a constant $C > 0$ depending on (X, D), the metrics etc. but wholly independent of f and an increasing C^2-function $S_f(r)$ such that for $f(y) \notin D$ if $p(y) = 0$,*

$$\begin{aligned} \overline{M}._F \, Y(r) + \frac{1}{2} \log \left(1 + \frac{\left\| f_* \left(\frac{\partial}{\partial y} \right) \right\|^2}{\left| p_* \left(\frac{\partial}{\partial y} \right) \right|^2} \right) (0) \leq & \frac{(Y:\Delta)}{2} \log \frac{(r\, S_f'(r))'}{r} + \mathrm{rad}_{f,D}(r) \\ & + \mathrm{discr}^{\#}_{Y/\Delta}(r) + \log^+ \overline{H}._f \, Y(r) \\ & + \log^+ |\log \|\mathbb{1}_D\|| \, (0) + C \end{aligned}$$

and $S_f(r) \leq C(\overline{H}._f \Delta(r) + \log \log^2 \|\mathbb{1}_D\| + C)$, where as ever $\overline{H}$ is a positively metricised ample bundle, $\mathrm{discr}^{\#}$ is the finite part of the discriminant (i.e. $\mathrm{discr} + (Y:\Delta) \log(1-r^2)$), and the short hand $\cdot (0)$ – which will be used both here and elsewhere – denotes evaluating the said function by summing over its values in $p^{-1}(0)$.

Proof. The hyperplane at ∞, $P(T_X(-\log D))$ naturally represents M as a divisor. We use it to calculate the intersection. Specifically if $\| \ \|_{\log}$ denotes the norm on $T_X(-\log D)$ and $\frac{\partial}{\partial y}$ is a fixed trivialisation of T_Y then,

$$\overline{M}_{\cdot F}\, Y(r) = -\int_0^r \frac{dt}{t}\int_{Y(t)} dd^c \log \frac{\left|p_*\left(\frac{\partial}{\partial y}\right)\right|^2}{\left|p_*\left(\frac{\partial}{\partial y}\right)\right|^2 + \left\|f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}}$$

$$= \mathrm{rad}_{f,D}(r) + \mathrm{discr}^{\#}_{Y/\Delta}(r) - \frac{1}{2}\log\left(1 + \frac{\left\|f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}}{\left|p_*\left(\frac{\partial}{\partial y}\right)\right|^2}\right)(0)$$

$$+ \frac{1}{2}\int_{\partial Y(r)} \log\left(1 + \frac{\left\|f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}}{\left|p_*\left(\frac{\partial}{\partial y}\right)\right|^2}\right) p^* \frac{d\theta}{2\pi}$$

where of course $Y(r)$ is the pre-image under p of the disc of radius r. Now using the symbol $\ll$ to denote an inequality up to multiplication by a constant depending only on X, D the various bundles, metrics etc., i.e. anything except f and Y, we have,

$$\frac{\left\|f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}}{\left|p_*\left(\frac{\partial}{\partial y}\right)\right|^2}\prod_i \frac{1}{\log^2\|1\!\!\mathrm{I}_{D_i}\|}\, p^*\left(\frac{dz\, d\overline{z}}{-\sqrt{-1}\pi}\right) \ll c_1(\overline{H}) - \sum_i dd^c \log\log^2\|1\!\!\mathrm{I}_{D_i}\|$$

where D_i are the components of D and since the proposition is independent of the metric on D we may without loss of generality suppose $\sup\limits_X \|1\!\!\mathrm{I}_D\| \leq 1/10$. Now put,

$$S_f(r) = \int_0^r \frac{dt}{t}\int_{\Delta_t}\left(1 + \frac{\left\|f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}}{\left|p_*\left(\frac{\partial}{\partial y}\right)\right|^2}\right)\prod_i \log^{-2}\|1\!\!\mathrm{I}_{D_i}\|\,\frac{d\theta}{2\pi}$$

then by the concavity of the logarithm,

$$\frac{1}{2}\int_{|z|=r}\log\left(1 + \frac{\left\|f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}}{\left|p_*\left(\frac{\partial}{\partial y}\right)\right|^2}\right)\frac{d\theta}{2\pi} \leq \frac{(Y:\Delta)}{2}\log\frac{(r\, S_f'(r))'}{r}$$

$$+ \log\int_{\partial Y(r)} |\log 1\!\!\mathrm{I}_D|\, p^*\left(\frac{d\theta}{2\pi}\right)$$

and the logarithm of latter integral is bounded by $\log^+ \overline{H}_{\cdot f} Y(r) + \log^+ \big|\log \|1\!\!\mathrm{I}_D\|\big|(0)$, while,

$$S_f(r) \ll \overline{H}_{\cdot f}\, \Delta(r) + \log\log^2\|1\!\!\mathrm{I}_D\|\,.$$

$\square$

Of course what arises in applications is not $P(T_X(-\log D)\oplus 1)$ but $P(T_X(-\log D))$. Whence let $\overline{L}$ be the metricised tautological bundle on the latter, with metric arising from a choice of metric on $T_X(-\log D)$ and observe,

Corollary V.4.2. *Notations as above with particular reference to V.4.1 we have,*

$$\overline{L}_{\cdot f'}\, Y(r) + \frac{1}{2}\log\left\| f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}(0) + N_{f'}(r) \leq (\textit{same bound as in V.4.1})$$

where, $N_{f'}(r) = \sum_{\substack{0<|p(y)|<r \\ z\notin f^{-1}(D)}} \log\frac{r}{|p(y)|}\ \mathrm{ord}_z(\mathrm{Ram}_f)$.

Proof. Observe that the projection $\rho : P(T_X(-\log D)\oplus \mathbb{1}) - - \to P(T_X)$ is well defined after blowing up the zero section $[0] = P(\mathbb{1})$. Call the said modification $\nu : \widetilde{P} \to P(T_X(-\log D)\oplus\mathbb{1})$ and the extension of the projection ρ, then for the natural induced metrics we have an isomorphism of metricised divisions,

$$\nu^*\,\overline{M} = \rho^*\,\overline{L} + \overline{\widetilde{[0]}}\,.$$

Further with this metricisation,

$$f^*\,c_1(\overline{\widetilde{[0]}}) = -dd^c\log\frac{\left\|f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}}{\left|p_*\left(\frac{\partial}{\partial y}\right)\right|^2 + \left\|f_*\left(\frac{\partial}{\partial y}\right)\right\|^2_{\log}}\,.$$

Consequently a direct calculation of $\overline{\widetilde{[0]}}_{\cdot f}\,\Delta(r)$ yields the result. □

Consider, now, the case where the map $f:\Delta\to X$ is a leaf of a foliation by curves then on restricting to f', L is very nearly the canonical bundle $K_{\mathcal{F}}$. On the other hand $K_{\mathcal{F}}$ is bigger than L by roughly the singularities of the foliation. Our goal is to refine V.4.2 so as to get a similar inequality for $K_{\mathcal{F}}$. Since we are intending to work with families of maps this will require some care, over and above that of [M1]. In particular we must work in a logarithmic context, so to start with $(X,\mathcal{F})$ will be a foliation by curves on a non-singular surface with canonical singularities, and the boundary D will be the algebraic invariant curves passing through the singularities. Necessarily this is simple normal crossing. Now consider the effet of blowing up in the singularities. If the eigenvalue of the singularity is non-zero this just has the obvious sense. Should it be zero, i.e. a so called saddle node, this means the minimal resolution of the blow up in the singular subscheme of the singularity with nilpotent structures. We may describe this in a formal neighbourhood with coordinates x, y, say, as follows. The foliation is given by a field $\partial = x(1+\nu y^{e-1})\,\frac{\partial}{\partial x} + y^e\,\frac{\partial}{\partial y}$ for some $e\geq 2$ and $\nu\in\mathbb{C}$. The so called weak branch given by $x=0$ determines a formal germ at the origin, and after blowing up once one should blow up again in the intersection of the exceptional divisor with the proper transform of the weak branch, repeating this process e times to obtain the minimal resolution. Now given $(X,D,\mathcal{F}) = (X_0,D_0,\mathcal{F}_0)$ we blow up in the singularity if D does not contain all the branches. This gives us a modification X_1 of X with induced foliation $\mathcal{F}_1$ and a new boundary D_1, which is the proper transform of D together with the exceptional curves. Now just iterate the whole procedure to get a sequence of modifications $(X_k, D_k, \mathcal{F}_k)$, denote by π_{mk} the projection from the m^{th}-modification to the k^{th}, for $m > k$. Over a singularity z of X_{k-1}, let E_{kzj} be the basis of exceptional curves between X_k and X_{k-1} contracted to z such that $E_{kzi} \cdot E_{kzj} = -\delta_{ij}$. Evidently if z has no nilpotent structure there is just one such curve, otherwise there are $e(z)$ where e is the degree of the node, and we order these according to the natural way that they appear in the resolution process described above. It is convenient to put $E_{kz} = \sum_{j=1}^{e(z)} E_{kzj}$. Furthermore on any X_k we have the logarithmic tautological bundle, L_k, say, for $\Omega_{X_k}(\log D_k)$ together with the formula, for $k \geq 2$,

$$\pi_{k0}^* K_{\mathcal{F}} = L_{k-1} + \sum_z \sum_{j=1}^{e(z)-1} E_{kzj}$$

where the sum is taken over saddle nodes on X_{k-1} through which the weak branch is not in the boundary D_{k-1}. In particular $L_k = L_{k-1} + F_k$ where F_k is an effective Cartier divisor on X_{k+1} contracted by $\pi_{k+1,k-1}$. Now put metrics on everything, and use C_k to denote a constant depending only on the various spaces, bundles, and metrics together with k. Obviously $C_k = C_k'$ means modulo the definition of C_k, rather than some identity of numbers. In any case let $f : \Delta \to X = X_0$ be an invariant map not factoring through D with $f(0)$ not contained in the singularities then, for $k \geq 2$,

$$\begin{aligned}
\overline{K}_{\mathcal{F}} \cdot_f \Delta(r) &= \overline{L}_{k-1} \cdot_f \Delta(r) + \sum_z \sum_{j=1}^{e(z)-1} \overline{E}_{kzj} \cdot_f \Delta(r) \\
&\leq \overline{L}_k \cdot_f \Delta(r) + \sum_z \sum_{j=1}^{e(z)-1} \overline{E}_{kzj} \cdot_f \Delta(r) - C_k \sum_{z \in \operatorname{sing}(\mathcal{F}_0)} \log f^* \| \mathbb{1}_{E_{1z}} \| (0) \, .
\end{aligned}$$

Observe that the lifting of f to X_k can only intersect the boundary at a singularity where there are not two algebraic branches and $k \geq 2$ call these singularities bad, then we also have

$$\operatorname{rad}_{f,D_k}(r) \leq \sum_{z \in \operatorname{sing}(\mathcal{F}_{k-1})_{\text{bad}}} \overline{E}_{kze(z)} \cdot_f \Delta(r) - C_k \sum_{z \in \operatorname{sing}(\mathcal{F}_0)} \log f^* \| \mathbb{1}_{E_{1z}} \| (0) + C_k'$$

where of course $e(z) = 1$ for a non-node which is bad. Consequently if we put, $\operatorname{dist}(\operatorname{sing}(\mathcal{F}), f(z)) := \left(\prod_{z \in \operatorname{sing}(\mathcal{F}_0)} \| \mathbb{1}_{E_{1z}} \| \, (f(z)) \right)$ and combine with V.4.1 we have,

$$\overline{K}_{\mathcal{F}.f}\,\Delta(r) + \frac{1}{2}\log\left\| f_*\left(\frac{\partial}{\partial z}\right)\right\|^2(0) \le \sum_{z\in \mathrm{sing}(\mathcal{F}_{k-1})_{\mathrm{bad}}} \overline{E}_{kz\cdot f}\,\Delta(r)$$
$$- C_k \log \mathrm{dist}(\mathrm{sing}(\mathcal{F}), f(0))$$
$$+ C_k' + \log^+ |\log \|\mathbb{I}_{D_k}\|| (0)$$
$$+ \frac{1}{2}\log\frac{(r\,S'_{f_k}(r))'}{r}$$

where f_k is the lifting of f to X_k, and S_{f_k} is as per V.4.2. Whence to obtain a bound for the $K_{\mathcal{F}}$ degree we have to control the intersection with the bad singularities. Observing that these are in one to one correspondence with formal branches through the singularities of $\mathcal{F}_0$ which are not contained in the boundary, this is done by means of:

Lemma V.4.3. *Let $(Y, \mathcal{F})$ be a foliated non-singular surface with canonical singularities and b a formal branch of $\mathcal{F}$ through a canonical singularity z through which there is not an algebraic invariant curve. Let Y_1 be obtained by blowing up $Y_0 = Y$ in z, with b_1 the proper transform of the branch $b_0 = b$, Y_z from Y_1 by blowing up the crossing point of b_1 with the exceptional divisor etc., and E_k the exceptional curve on Y_k blown down by $\rho_{k,k-1}$ (ρ_{km} from Y_k to Y_m, $m < k$ being the projection) then for H ample there is a positive rational constant α such that for all k, every sufficiently large and divisible multiple of $\alpha\,[\sqrt{k}]^{-1}\rho_{k0}^*\,H - E_k$ is generated by its global sections outside the exceptional curves other than E_k.*

Proof. For any integer $m > 0$, let us consider the bundle $F_{k,m}$ on Y_k defined by,

$$F_{k,m} = m\,\rho_{k0}^*\,H - \sum_{i=1}^{k}\rho_{ki}^*\,E_i$$

then $F_{k,m}^2 = m^2\,H^2 - k > 0$ for $m \ge \sqrt{k}$. Further $\rho_{k0}^*\,H\cdot F_{k,m} = m\,H^2 > 0$, so $F_{k,m}$ is certainly big. To get appropriate global generation requires an understanding of the intersection of $F_{k,m}$ with various curves. We firstly consider the exceptional ones, so let $\widetilde{E}_{ki}$ be the proper transform of E_i on Y_k, $i \le k$, and note:

$$-E_i\cdot(\rho_{ki})_*\,(\widetilde{E}_{kj}) = \begin{cases} 1, & j = i \\ -1, & j = i-1 \\ 0 & \text{otherwise}\end{cases}$$

Consequently, $F_{k,m}\cdot\widetilde{E}_{kj} = 0$ for $j < k$, and $F_{k,m}\cdot E_k = 1$. Next let $\widetilde{C}$ be the proper transform of a curve C on Y then,

$$\widetilde{C} = \rho_{k0}^*\,C - \sum_i a_i\,\rho_{ki}^*\,E_i$$

where, $a_i = \widetilde{C} \cdot \rho_{ki}^* E_i = E_i \cdot (\rho_{ki})_* \widetilde{C} \in \mathbb{N} \cup \{0\}$. Now suppose C is not invariant by the foliation. The canonical bundle of $\mathcal{F}$ on Y_k is precisely that of $\mathcal{F}$ on Y_0, as ever say $K_{\mathcal{F}}$, so:

$$(\rho_{k0}^* K_{\mathcal{F}} + \widetilde{C}) \cdot \widetilde{C} = K_{\mathcal{F}} \cdot C + C^2 - \sum_i a_i^2 \geq 0 \,.$$

While we also have,

$$F_{k,m} \cdot \widetilde{C} = m\, H \cdot C - \sum_i a_i \,.$$

Not surprisingly we control $\sum\limits_i a_i$ by way of the usual ℓ_1 versus ℓ_2 inequality, and the index theorem. Specifically:

$$\sum a_i \leq \sqrt{k} \left(\sum a_i^2\right)^{1/2} \leq \sqrt{k} \left(K_{\mathcal{F}} \cdot C + \frac{(H \cdot C)^2}{H^2}\right)^{1/2} \leq \alpha' \sqrt{k}\, H \cdot C$$

for a suitable constant α'. So that, $F_{k,m} \cdot \widetilde{C} \geq (m - \alpha' \sqrt{k})\, H \cdot C$. Finally consider the possibility that $\widetilde{C}$ is invariant. By our hypothesis there is at most one such and it is transverse to the given branch so,

$$F_{k,m} \cdot \widetilde{C} = m\, H \cdot C - 1 \geq (m-1)\, H \cdot C \,.$$

Consequently for an appropriate $\alpha > \alpha'$ and $\widetilde{C}$ a non-exceptional curve, we have for $m = \alpha\, [\sqrt{k}]$, $F_{k,m} \cdot \widetilde{C} > 0$, with $F_{k,m} \cdot E = 0$ if and only if E is one of the exceptional curves $\widetilde{E}_{ki}$ for $i \neq k$. Evidently the curves $\underset{i<k}{\cup} \widetilde{E}_{ki}$ may be contracted to a $\mathbb{Q}$-factorial singularity so sufficiently large divisible multiples of $F_{k,m}$ are generated by global sections, while for a possibly slightly larger α,

$$\alpha\, [\sqrt{k}]^{-1} H - E_k = \frac{1}{k} F_{k,m} + \sum_{i=1}^{k-1} \frac{i}{k} \widetilde{E}_{ki} \,.$$

$\square$

Putting everything together we obtain,

Theorem V.4.4. *There are constants α depending only on the fixed data on X, and C_k, C_k' depending on both the same and $k \in \mathbb{N}$ such that for $f : \Delta \to X$ an invariant map with $f(0)$ not contained in the singularities and f not factoring through an algebraic curve passing through the singularities we have,*

$$\begin{aligned} \overline{K}_{\mathcal{F}\cdot f}\, \Delta(r) + \log \left\| f_* \left(\frac{\partial}{\partial z}\right) \right\| (0) \leq\; & \frac{1}{2} \log \frac{(r\, S'_{f_k}(r))'}{r} + \frac{\alpha}{\sqrt{k}}\, \overline{H}_{\cdot f}\, \Delta(r) \\ & + \log^+ |\log \| \mathbb{1}_\partial \| | \,(0) \\ & - C_k \log \operatorname{dist}(\operatorname{sing}(\mathcal{F}), f(0)) + C_k' \end{aligned}$$

where ∂ is the set of algebraic invariant curves through the singularities.

Given the rather involved nature of what has gone before, let's clarify the essence of what's going on, by noting:

Remark V.4.5. Morally V.2.7 says for any $\varepsilon > 0$, $\overline{K}_{\mathcal{F}}.f\,\Delta(r) \leq \mathrm{discr}(r) + \varepsilon\,\overline{H}.f\,\Delta(r)$. As ever we don't write this because of the issues of controlling sets of finite measure in bounding the various derivatives of S_{f_k}, and the error term at the origin. It's this latter feature that requires the full force of V.4.3, since for a fixed origin the trivial remark that $F_{k,m}$ is big would be wholly sufficient. For foliations by curves in arbitrary dimension a corresponding bigness statement, and whence a bound on the degree of $K_{\mathcal{F}}$ for curves with fixed origin is wholly tractable, the global generation is however more subtle.

V.5. Applications

As we've seen in V.4 the natural boundary which intervenes in discussing the hyperbolicity of a foliated surface $(X, \mathcal{F})$ is the algebraic invariant curves through the singularities. Indeed this is hardly surprising since it's even the same for fibrations. Consequently our notations are as follows, $(X_0, \mathcal{F}_0)$ is a minimal model of a foliated variety, $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$ its minimal resolution of singularities and ∂_0 the set of invariant algebraic curves through $\mathrm{sing}(\mathcal{F}_0) \cup \mathrm{sing}(X_0)$. Evidently hyperbolicity is only meaningfull if $\nu(\mathcal{F}_0) \geq 1$. With this in mind we can rectify our purely qualitative knowledge of V.1 by means of,

Theorem V.5.1(a). *Notations as above, and $\nu(\mathcal{F}_0) \geq 1$, then if $(X_0, \mathcal{F}_0)$ is neither Ricatti nor turbulent with invariant fibres then it is hyperbolic modulo ∂_0.*

Proof. We can clean up using V.3.4 and V.3.15 to suppose in the event of the failure of the theorem that there is a sequence of maps $f_n \in \mathrm{Hom}_{\mathcal{F}}(\Delta, X)$ with $f_n(0) \in K$ a compact subset of X such that $\mathrm{dist}(K, \partial \cup \mathrm{sing}(\mathcal{F}))$ and $\left\| (f_n)_* \left(\frac{\partial}{\partial z}\right) \right\| (0)$ are bounded away from zero independently of n, and $\overline{H} \cdot_{f_n} \Delta(r)$ goes to infinity, where ∂ is now $\rho^{-1}(\partial_0)$ together with any curves contracted by ρ, and as ever $\overline{H}$ is a strictly positive metricisation of some ample.

We take $\pi_k : (X_k, \mathcal{F}_k) \to (X, \mathcal{F})$ to be the modifications of V.4 (if we're consistent about numbering the minimal model is $(X_{-1}, \mathcal{F}_{-1})$) and let f_{kn} be the lifting of f_n to X_k. Our goal is to find a radius where we can estimate the error of V.4.1 independently of n. However this is precisely what our minor variation of V.2.6 is designed for, so that after subsequencing to ensure that for some $1 > r_0 > 0$, $\overline{H} \cdot_{f_n} \Delta(r_0)$ grows sufficiently rapidly in n, we obtain for r outside a set of finite hyperbolic measure, and for all k,

$$\frac{1}{2} \log \frac{(r\, S'_{f_{kn}}(r))'}{r} \leq \mathrm{discr}(r)(1 + \circ_k(\overline{H} \cdot_{f_n} \Delta(r)))$$

where the $\circ_k$ denotes that the error goes to zero for k fixed and $n \to \infty$. Since V.2.6 is also satisfied we can subsequence as in V.2.3 to obtain a closed positive non-zero invariant current T (modulo normalisation by a suitable ample) such that thanks to V.4.4 and the derivative being bounded away from zero at the origin,

$$K_{\mathcal{F}} \cdot T \leq \alpha / \sqrt{k}$$

for any k, so indeed $K_{\mathcal{F}} \cdot T = 0$. Observe, further, that since $f_n(0)$ is bounded away from all algebraic curves through the singularities, and the singularities themselves, it is a fortiori so at any curve E contracted by the minimal model map $\rho : (X, \mathcal{F}) \to (X_0, \mathcal{F}_0)$ so indeed $E \cdot T \geq 0$. Whence, $\rho^* K_{\mathcal{F}_0} \cdot T = E \cdot T = 0$. The only way this can happen is if the projection $P(T)$ of T to $\mathrm{NS}_1(X)_{\mathbb{R}}$ via duality is parallel to $\rho^* K_{\mathcal{F}_0}$. This is certainly rubbish for $\mathcal{F}$ of general type. In the case $\rho^* K_{\mathcal{F}_0} \cdot K_X > 0$ or equivalently $\kappa(\mathcal{F}) = -\infty$ we have a choice between the methodology of [M1] to extract a contradiction, or Brunella's residue theorem. In the latter case the non-general type case has $\kappa(\mathcal{F}) = -\infty$ so by, IV.5.7, the current T is diffuse, and Brunella's residue theorem immediately gives $K_{\mathcal{F}_0} \cdot K_X \leq 0$. In the former case, we have a philosophically preferable emphasis on the numerical properties of $K_{\mathcal{F}}$. Profiting from the language of currents to rewrite the original argument more cleanly the theorem of Demailly in op.cit. implies that if T has positive Lelong number anywhere then it's big so $K_{\mathcal{F}_0} \cdot T > 0$. Since this is absurd we must have zero Lelong number everywhere, which then implies $(K_X - K_{\mathcal{F}}). T = 0$, by a substantially more trivial use of the Baum-Bott philosophy. It remains to discuss Ricatti and turbulent cases without invarinat fibres. To this end take an almost étale cover $\nu : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X_0, \mathcal{F}_0)$ such that the Kodaira fibration $\pi : \widetilde{X} \to \widetilde{B}$ is relatively smooth. Argueing as in V.1.2(c) and above we can associate a current $\widetilde{T}$ to the possibly ramified liftings $\widetilde{f}_n$ of f_n, which again must be parallel to $K_{\widetilde{\mathcal{F}}}$. However $\widetilde{T}$ is invariant so its image under restriction from $H^{1,1}$ to $H^1(\widetilde{X}, K_{\widetilde{\mathcal{F}}})$ is zero, while having no invariant fibres implies that $c_1(K_{\widetilde{\mathcal{F}}})$ is non-zero in $H^1(\widetilde{X}, K_{\widetilde{\mathcal{F}}})$, which is absurd. □

Due to the highly quantitative nature of V.4.4 we can give bounds, indeed best possible, for the degeneration of the invariant Kobayashi metric of $(X_0, \mathcal{F}_0)$, i.e. for $x \notin \mathrm{sing}(X_0) \cup \mathrm{sing}(\mathcal{F}_0)$ and ∂ a local generator for $T_{\mathcal{F}_0}$,

$$\|\partial\|_{\mathrm{Kob}} = \inf\{\frac{1}{R} : f : (\Delta, 0) \to (X_0, x), f_* \left(\frac{\partial}{\partial z}\right)(0) = R\partial, f \text{ invariant}\}$$

this is of course, contrary to the notation, smaller than the small Kobayashi/ Poincaré metric as defined in V.1.3 in the presence of singularities. In any case let Z be the singularities then we have,

Theorem V.5.1(b). *Everything as in part (a) and notations as above. There is constant $\alpha > 0$ such for all $\varepsilon > 0$ there exists $\beta(\varepsilon) > 0$ for which,*

$$\|\partial\|_{\mathrm{Kob}}(x) \geq \frac{\beta(\varepsilon)\,\mathrm{dist}(x, Z)^{\alpha}}{|\log \|\mathrm{II}_{\partial_0}\|\,(x)|^{1+\varepsilon}}\,.$$

Proof. For the everywhere transverse case of Ricatti and Turbulent foliations there is nothing to prove, and the only other case with $\nu(\mathcal{F}_0) = 1$ is that of Hilbert-Modular surfaces, which follows by direct calculation, so we're reduced to general type. Consequently by taking k sufficiently large we find $\delta > 0$ such that for any $f \in \mathrm{Hom}_{\mathcal{F}}(\Delta, X)$,

$$\begin{aligned}\varepsilon\, \overline{H}_{\cdot f}\, \Delta(r) + \log \left\| f_* \left(\frac{\partial}{\partial z}\right) \right\| (0) \leq & -\alpha \log \mathrm{dist}(\rho\, f(0), Z) \\ & + \log^+ |\log f^* \|\mathrm{II}_{\partial}\|| (0) \\ & + \frac{1}{2} \log \frac{(r\, S'_{f_{kn}}(r))'}{r} + C\end{aligned}$$

for some constant C. If the theorem is false there is a sequence of maps f_n with derivatives at the origin going to infinity and by part (a) these must be in arbitrarily small neighbourhoods of ∂. In any case $\overline{H}._{f_n}\,\Delta(r)\to\infty$ for some $r\in(0,1)$ and we can proceed as before (bearing in mind a $\log\log^+|\log f^*\|\mathbb{1}_\partial\||(0)$ will enter into the error for S) to derive a contradiction from,

$$\varepsilon\,\overline{H}._{f_n}\,\Delta(r)\ll\log\overline{H}._{f_n}\,\Delta(r)\,.$$

□

So a combination of general nonsense reasons as per V.3, and the no bubbling lemma gives the continuity of the Poincaré metric, should it not be identically zero, outside of ∂_0 and the singularities. In addition, and obviously, the lower bound for the degeneration of the invariant Kobayashi metric implies the same for the Poincaré metric, which since the latter is p.s.h. implies that it has no Lelong numbers except maybe at the singularities for $(X_0,\mathcal{F}_0)$ not Ricatti or Turbulent with invariant fibres. However, as we've seen in the course of the discussion of Sporadic foliations, invariant multiple fibres, i.e. the natural map from $\omega_B\to K_{\mathcal{F}_0}$ being multiple must force the Poincaré metric in this latter case to have Lelong numbers. Whether without multiple fibres a bound of the form V.4.1(b) can be given for the degeneration of the Poincaré metric seems a little unclear. Regardless for what concerns the compactness of the leaf space we can go much further thanks to Brunella's no bubbling lemma. Indeed let us take up the notation of V.1.3 and define an algebraic invariant curve $C\subset X_0$ to be parabolic if $C\backslash Z$ is (orbi)-uniformised by $\mathbb{C}$, then unsurprisingly,

Complement V.5.2. If all the leaves of $(X_0,\mathcal{F}_0)$ are not parabolic, i.e. we don't have a foliation in the exceptional list V.1.2(e), then the Brunella space, $\widetilde{\mathrm{Hom}}_{\mathcal{F}}(\Delta,X_0\backslash Z)$, is compact modulo (the necessarily algebraic and finite in number) parabolic leaves.

Proof. As ever let $\rho:(X,\mathcal{F})\to(X_0,\mathcal{F}_0)$ be the minimal smooth resolution. Whence suppose the contrary, i.e. that there is a sequence $f_n\in\widetilde{\mathrm{Hom}}_{\mathcal{F}}(\Delta,X_0)$ not arbitrarily close to the parabolic leaves, without a convergent subsequence. We already know that the liftings $\widetilde{f}_n$ to X are arbitrarily close to the set of algebraic leaves through the singularities, so modulo subsequencing and composing with convergent sequences of automorphisms of Δ, and appealing to the no bubbling lemma it will suffice to bound the degrees of the $\widetilde{f}_n$ at appropriate radii under the additional hypothesis that the $\widetilde{f}_n(0)$ are a bounded distance independent of n, away from both the contracted curves and the proper transforms of the parabolic ones while accumulating at a smooth point of some curve C such that $\rho(C)$ is hyperbolic. Consequently we proceed much as before, but with a different strategy for singularities $z\in\mathrm{sing}(\mathcal{F})\cap C$. Specifically if there is an algebraic curve through z other than C we proceed as before, i.e. do nothing, while if there is not we blow up once.

Observe that in the first case $\widetilde{f}_n$ may be supposed not to pass through z, since otherwise it would factor through C, nor does it in the second case by virtue of the definition of the Brunella space $\widetilde{\mathrm{Hom}}$. With these preliminaries we apply the log-tautological inequality exactly as before except that we omit C, including instead any exceptional curves as prescribed above. Associating an appropriate current T with the data we may write it as $T' + \lambda C$, where $\lambda \geq 0$ is it's Lelong number along C, and find that it satifies,

$$K_{\mathcal{F}}.\, T' + \lambda\{2g(C) - 2 + (\mathrm{sing}(\mathcal{F}) \cap C)_{\mathrm{red}}\} \leq 0$$

where naturally $\#\,(\mathrm{sing}(\mathcal{F}) \cap C)_{\mathrm{red}}$ is the number of singularities of $\mathcal{F}$ along C counted without multiplicity. Now for every curve E_i contracted by a $-1F$ chain,

$$E_i.\, (T' + \lambda C) \geq 0$$

so if $\sum_i a_i\, E_i$ is a chain contracted to a point $z \in X_0$ with its discrepancies, then,

$$\left(\sum_i a_i\, E_i\right) \cdot T' \geq \frac{-\lambda}{e(z)}$$

where $e(z)$ is the determinant of the intersection matrix of the chain. Putting this together we have,

$$K_{\mathcal{F}_0} \cdot \rho_* T' + \lambda \left\{ (2g-2) + \#\,(Z \cap C)_{\mathrm{red}} + \sum_z \left(1 - \frac{1}{e(z)}\right) \right\} \leq 0$$

where Z is the singularities of $\mathcal{F}_0$ except the contraction of -1 chains, and the sum is taken over the images of contractions meeting C. The factor multiplying λ is, therefore, the negative of the orbifold euler characterictic, which by hypothesis is positive so $\lambda = 0$, and $\rho_* T = 0$, which as ever is nonsense by virtue of the positivity of intersection of T with the contracted curves. □

V.6. A surface complement

Similar methodology can be applied to the study of the hyperbolicity of surfaces themselves, although at this point Brunella's residue theorem is not just convenient, but indispensible. The applicability stems from the following observation of Bogomolov,

Lemma V.6.1. ([B2]) *Let (X, D) be a smooth surface with simple normal crossing boundary of logarithmic general type and $s_2(X, D)$ $(= c_1^2(X, D) - c_2(X, D)) > 0$ then the tautological bundle L on $\pi : \mathbb{P}(\Omega_X(\log D)) \to X$ is big.*

Now let's try and apply this in combination with V.4.2 and V.2.6 to the study of $\mathrm{Hom}(\Delta, X)$. Taking $\overline{H}$ to be a positively metricised ample divisor then the lemma implies (indeed it is equivalent) to the existence of some $b \in \mathbb{N}$

and a naturally metricised effective divisor $\overline{F}$ such that $\overline{L}^{\otimes b}$, metricised as in V.4.2, is equal to $\pi^*\overline{H}(\overline{F})$. So that for $f \in \mathrm{Hom}(\Delta, X)$ with $f(0) \notin F$,

$$b\,\overline{L}._{f'}\,\Delta(r) \geq \overline{H}._{f}\,\Delta(r) + \log\|f'^*\,\mathbb{I}_F\|(0) + \mathrm{const}$$

where the constant is just a lower bound for $-\log\|\mathbb{I}_F\|$. Now suppose further that $f_n \in \mathrm{Hom}(\Delta, X)$ don't converge in $\overline{\mathrm{Hom}}(\Delta, X)$ then we can subsequence to guarantee $\overline{H}._{f_n}\,\Delta(r_0)$, for some fixed r_0, as large a function of n as we like, so that applying V.4.2 and V.2.6 we obtain for $r \in (0,1)$ outside a set of finite hyperbolic measure,

$$\begin{aligned} \log\|df_n\|(0) + \overline{L}._{f'_n}\,\Delta(r) \quad &\leq \quad (1 + \circ(\overline{H}._{f_n}\,\Delta(r))\log(1-r^2)^{-1} \\ &+ \quad \mathrm{rad}_{f_n,D}(r) + \log|\log f_n^*\,\mathbb{I}_D|(0)\,. \end{aligned}$$

Certainly we can restrict to the case $\mathrm{rad}_{f_n,D} = 0$, e.g. $f_n : \Delta \to X\backslash D$ or D empty, but the real problem concerns what happens at zero. Equally it makes little difference whether we prove that the f_n converge in $\overline{\mathrm{Hom}}$ or that the $\alpha_n^* f_n$ converge, where α_n are a convergent sequence of automorphisms, so provided that the f'_n are not contained in arbitrarily small neighbourhoods of $F \cup \pi^{-1}(D)$, composing with a convergent sequence of automorphisms implies by V.3.15 that,

$$\overline{H}._{f_n}\,\Delta(r) \leq C(r)$$

and whence a contradiction. The real difficulty arises, therefore, when the sequence f_n becomes arbitrarily close to some divisor, which by the way contains as a strict, and relatively straightforward subcase, the possibility that the f_n are actually in the divisor, which modulo a simple reduction is wholly covered by the results of V.5.

The immediate difficulty is that the divisor F is potentially a complete mess. Concentrating on its horizontal components, these correspond to saturated maps,

$$0 \to M \to \mathrm{Sym}^d\,\Omega_X(\log D)$$

for $d \in \mathbb{N}$ and M a line bundle, which don't factor through a lower symmetric power. Abusing notation slightly, and denoting the component equally by F, with U the universal sub-bundle over the same we have a factorisation,

$$\pi^* M \otimes \mathcal{O}_F \to U \otimes \pi^*\,\mathrm{Sym}^{d-1}\,\Omega_X(\log D) \otimes \mathcal{O}_F \to \pi^*\,\mathrm{Sym}^d\,\Omega_X(\log D) \otimes \mathcal{O}_F$$

So passing to a ramified cover $\rho : S \to X$ we have a fibre square,

$$\begin{array}{ccc} \mathbb{P}(\rho^*\Omega_X(\log D)) & \xrightarrow[\rho]{} & \mathbb{P}(\Omega_X(\log D)) \\ \downarrow & & \downarrow \\ S & \xrightarrow[\rho]{} & X \end{array}$$

with all the horizontal components of $\rho^* F$ enjoying degree 1. Further passing to a resolution of singularities not just of S but of $\rho^{-1}(D)$, which again somewhat abusively we'll denote $\rho : (S, |D|) \to (X, D)$, we have the natural map,

$$\rho^* \Omega_X(\log D) \to \Omega_S(\log |D|)$$

and a birational map, $\mathbb{P}(\Omega_S(\log |D|) \dashrightarrow \mathbb{P}(\rho^* \Omega_X(\log D))$ which is an isomorphism over the generic point of S. Resolve this latter map by, $\psi : P \to \mathbb{P}(\Omega_S(\log D))$, and consider what's become of the maps f_n. In the first place these do not necessarily lift to S, but we do have a sequence of liftings to ramified coverings,

$$\begin{array}{ccc} Y_n & \xrightarrow[\widetilde{f_n}]{} & S \\ \downarrow p_n & & \downarrow \\ \Delta & \xrightarrow[f_n]{} & X \end{array}$$

with the derivatives $\widetilde{f_n'} : Y_n \to \mathbb{P}(\Omega_S(\log |D|))$ of the proper transforms $\widetilde{f_n}$ lifting $f_n' : \Delta \to \mathbb{P}(\Omega_X(\log D))$ to P. Whence if the f_n are eventually arbitrarily close to some divisor on $\mathbb{P}(\Omega_X(\log D))$ then the $\widetilde{f_n'}$ are the same for its pull-back to P. Consequently since P is isomorphic to $\mathbb{P}(\Omega_S(\log |D|))$ over the generic point the $\widetilde{f_n'}$, defined on Y_n, are arbitrarily close to a divisor each horizontal component of which has degree 1.

This basically completes the reduction of the hyperbolicity of $X \backslash D$ to a foliation problem. Specifically we are led to consider sequence of triples,

$$\begin{array}{ccc} Y_n & \xrightarrow[f_n]{} & S \\ \downarrow p_n & & \\ \Delta & & \end{array}$$

of maps of finite covers of the disc whose ramification is not worse than the degree such that for some sufficiently large simple normal crossing divisor D on S and foliations $\mathcal{F}^i$, $1 \leq i \leq r$, the derivatives f_n' are arbitrarily close, in the compact open sense to the divisor in $\mathbb{P}(\Omega_S)$ cut out by D and the foliations. Naturally for some ample divisor H on S we suppose the degrees $H \cdot_{f_n} Y_n(s)$ unbounded for some $0 < s < 1$, and that the f_n – or more precisely, simply the $f_n(0)$ – are not arbitrarily close to some Zariski closed subset V to be specified. With this in mind the following lemma will do nicely,

Lemma V.6.2. (Notations as above modulo replacing S by X) *For any $\varepsilon > 0$ and any sufficiently large, depending on ε, proper birational modification ρ : $(\widetilde{X}, \widetilde{D}) \to (X, D)$ with $f_n' : Y_n \to \mathbb{P}(\Omega_{\widetilde{X}}(\log \widetilde{D}))$ the logarithmic derivative we* *have for $r \in (0,1)$ outside of a set of finite hyperbolic measure, and $n = n(\varepsilon)$ sufficiently large,*

$$(K_{\widetilde{X}} + \widetilde{D})\cdot_{f_n} Y_n(r) \leq L\cdot_{f_n} Y_n(r) + \varepsilon\, H\cdot_{f_n} Y_n(r)$$

where L is the tautological bundle on $\mathbb{P}(\Omega_{\widetilde{X}}(\log \widetilde{D}))$, and momentarily, contrary to standard usage $\widetilde{D}$ is permitted to be bigger than $\rho^{-1}(D)$.

In order to conclude to a hyperbolicity statement from the lemma, one first notes the minor fact that if the f_n are not arbitrarily close to some divisor ∂ in X then the same is true for the $\widetilde{f_n}$ at the S level with respect to $\rho^{-1}(\partial)$. Thus by a priori composing with convergent sequences of automorphisms of the disc we can guarantee,

$$E\cdot_{\widetilde{f_n}} Y_n(r) \geq \text{const} + \log \|\widetilde{f}_n^* \, \mathbb{1}_E\| \, (0) \geq \text{const}$$

for any E in the support of $\rho^{-1}(\partial)$, and so guarantee the conditions announced prior to the lemma. In any case, given the lemma, we have for $\varepsilon > 0$,

$$\begin{array}{rcl} (K_S + |D|)\cdot_{f_n} Y_n(r) + \log \|df_n\|(0) & \leq & \operatorname{rad}_{f_n,D}(r) + \operatorname{discr}^{\#}_{Y_n/\Delta}(r) \\ & + & C(r) + \varepsilon\, H\cdot_{f_n} \Delta(r) \end{array}$$

and by including the ramification in our excluded set, we can estimate $\operatorname{discr}^{\#}$ in the obvious way by the ramification divisor, so that restricting to one of the natural cases where the radical is zero, we'll obtain,

$$(K_X + D)\cdot_{f_n} \Delta(r) \leq C(r) + \varepsilon\, H\cdot_{f_n} \Delta(r)$$

from which the hyperbolicity of $X \backslash D$ modulo some effectively computable, if rather unclear, proper Zariski closed subset.

The important things to draw from this discussion are in the first place that we need the full force of the main lemma, i.e. it would be wholly insufficient to simply bound the degree of the maps $f_n : Y_n \to S$ in terms of the ramification, since this would not a priori imply a similar bound for the maps $f_n : \Delta \to X$. Secondly, although in the case of $c_1^2 > c_2$ where the specific problem is that our maps may intersect some horizontal divisor negatively, we might hope to play some numerical game between X and S (all be it, that it's not clear what) to reduce to some easier problem involving less components, this isn't the point. More precisely if we hope to conclude to results on the hyperbolicity of algebraic surfaces, by use of analogues of V.6.1 for higher jet bundles, then we must be able to exclude systematically, by way of appropriate analogues of V.6.2, the degeneration at the level of lower order jets as the next example illustrates.

Example V.6.3. The Bloch Theorem

The Bloch Theorem concerns the hyperbolicity of $(\mathbb{P}^2, D)$, where D is a divisor consisting of 4 lines in general position. The arguements of this section work quite generally for any number of lines, and leads to specific quantification of the value distribution of discs about the same. However the case of at least 5 lines (all be it with worse quantification) follows from the above combination of V.6.1 with V.6.2,, and things are combinatorially a bit involved so we'll stick to the case of 4.

To begin with denote by P, respectively $P_{\log}$, the bundles $\mu : \mathbb{P}(\Omega_{\mathbb{P}^2}) \to \mathbb{P}^2$, and $\nu : \mathbb{P}(\Omega_{\mathbb{P}^2}(\log D)) \to \mathbb{P}^2$. The bundle P is equally smoothly fibred by λ say, over $(\mathbb{P}^2)^\vee$, which we think of as a smooth connection on $\mathbb{P}^2$, or equivalently a smooth foliation $\mathcal{F}$ on P arising from a splitting of the bundle of 2-jets, J, i.e. the quotient of Ω_P by the universal sub-bundle. Equally if $J_{\log}$ is the quotient of $\Omega_{P_{\log}}(\log \nu^* D)$ by the corresponding universal sub-bundle we have an induced connection corresponding to a singular foliation $\mathcal{F}_{\log}$, with a splitting of the natural exact sequence for $J_{\log}$, i.e.

$$0 \to L_{\log} \to J_{\log} \underset{\longrightarrow}{\overset{\text{split}}{\longleftarrow}} \nu^* K_{\mathbb{P}^2}(D) \otimes L_{\log}^{-2} \to 0$$

where L, and $L_{\log}$ are the respective tautological bundles on P and $P_{\log}$. Now argueing as above this time with respect to the second derivatives $f_n'' : \Delta \to \mathbb{P}(J_{\log})$, for $f_n : \Delta \to \mathbb{P}^2 \backslash D$ a sequence of divergent maps which are not eventually arbitrarily proximate to D, it's immediate that we're done unless the f_n'' are eventually arbitrarily close to the divisor in $\mathbb{P}(J_{\log})$ defined by the connection. In order to reduce to V.6.2 we have to relate $P_{\log}$ to P. Observe that over a line C in D, both these bundles posses canonical sections, i.e.

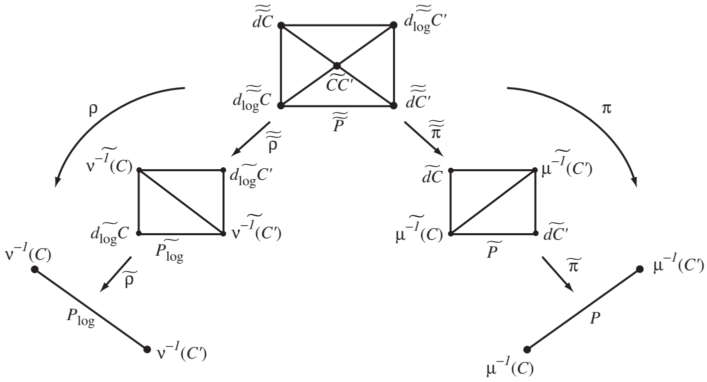
(i) The derivative, dC,

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-C) \to \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_C \to \omega_C \to 0$$

(ii) The residue, $d\log C$,

$$0 \to \mathcal{O}_C(K_X + D) \to \Omega_{\mathbb{P}^2}(\log D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to 0\,.$$

The possibilities for resolving the birational map $P_{\log} \dashrightarrow P$ are not unique. The correct one is to first blow up in all the $d\log C$, which by the way are disjoint, then blow in the crossings of the proper transforms of the $\nu^{-1}(C)$, $C \subset |D|$. The final exceptional divisors so obtained are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$'s, so blow down in the other direction, or alternatively they're the blow up in the crossings of the proper transforms of the $\mu^{-1}(C)$ after blowing up in the dC. Note also that the proper transforms of the $\nu^{-1}(C)$, respectively $\mu^{-1}(C)$ blow down in the other direction. Let's summarise all this in terms of some maps, and simplicial complexes defined by the intersection of various divisors, which in turn fixes our notation, viz,



The canonical bundles of the various foliations are related by way of the formula,

$$\rho^* K_{\mathcal{F}_{\log}} - \sum_C d\,\widetilde{\widetilde{\log}\, C} = K_{\widetilde{\widetilde{\mathcal{F}}}} = \pi^* K_{\mathcal{F}} + \sum_{CC} \widetilde{CC'}$$

where the negative discrepancy on the left reflects the fact that the $d\widetilde{\log C}$ are not invariant, and of course the positive discrepancy on the right that the $\widetilde{CC'}$ are so. Consequently the natural divisor to work with, at least initially, is,

$$\widetilde{\widetilde{D}} := \sum_C \widetilde{\widetilde{dC}} + \sum_{CC'} \widetilde{CC'}\,.$$

However we also have other interesting divisors arising from the pencils/foliations defined by any pair of lines. In $(\mathbb{P}^2)^\vee$ these correspond to six lines, which we'll denote by F's, meeting in the 4 points, [C], corresponding to our given curves. This information incorporates itself naturally into our data since $\widetilde{P}$ arises as the fibre product,

$$\begin{array}{ccccc}
\widetilde{P} & \xrightarrow{\widetilde{\lambda}} & (\widetilde{\mathbb{P}^2})^\vee & \longleftarrow & \widetilde{\lambda}(\widetilde{dC}) \\
{\scriptstyle\widetilde{\pi}}\downarrow & \square & \downarrow{\scriptstyle\widetilde{\pi}} & \square & \downarrow \\
P & \xrightarrow[\lambda]{} & (\mathbb{P}^2)^\vee & \hookleftarrow & [C]
\end{array}$$

where the middle vertical arrow is the blow up in the 4, aforesaid points, and we denote by $\widetilde{F}$ the pull-back to $\widetilde{P}$ of the proper transform of the various F's, with $\widetilde{\widetilde{F}}$ its proper transform on $\widetilde{\widetilde{P}}$. At the $\widetilde{\widetilde{P}}$ level all divisors appearing so far are simple normal crossing, with the $\widetilde{\widetilde{\mathcal{F}}}$ invariant ones being $\widetilde{D}$ together with the $\widetilde{\widetilde{F}}$'s. Moreover the pull-back to $\widetilde{\widetilde{P}}$ of the universal subbundle $U_{\log}$ on $P_{\log}$ is still a subbundle of $\Omega_{\widetilde{\widetilde{P}}}(\log \widetilde{D} + \widetilde{\widetilde{F}})$ (although having a bundle is notationally more important than anything else) so we get an induced jet bundle $\widetilde{\widetilde{J}}$ with dévissage induced by $\widetilde{\widetilde{\mathcal{F}}}$,

$$0 \to \rho^* \nu^* K_{\mathbb{P}^2}(D) \otimes \rho^* L_{\log}^{-2} \left(\sum_C d\widetilde{\widetilde{\log}}\, C + \sum_F \widetilde{\widetilde{F}} \right) \to \widetilde{\widetilde{J}} \to K_{\widetilde{\widetilde{\mathcal{F}}}} \to 0\,.$$

Now suppose that the derivatives of our maps $f_n' : \Delta \to P_{\log}$ are not eventually arbitrary close to $\bigcup\limits_F \rho(\widetilde{\widetilde{F}})$ which by the way contain the $d\log C$, then after composition with convergent sequences of automorphisms of the disc the radical of $\widetilde{\widetilde{F}}$ for the induced maps $\widetilde{\widetilde{f_n}} : \Delta \to \widetilde{\widetilde{P}}$ may comfortably be estimated by,

$$\mathrm{rad}_{\widetilde{\widetilde{f_n}}, \widetilde{\widetilde{F}}}(r) \leq \widetilde{\widetilde{F}} \cdot_{\widetilde{\widetilde{f_n}}} \Delta(r) + \mathrm{const}$$

where the implied constant is independent of n. Consequently applying the tautological inequality for $(\widetilde{\widetilde{P}}, \widetilde{\widetilde{D}} + \widetilde{\widetilde{F}})$ with respect to the induced maps $\widetilde{\widetilde{f_n}}$ and their derivatives, and argueing as in the applicability of V.6.1 we're certainly going to bound the degree of the $\widetilde{f_n}$, unless the derivatives are eventually arbitrarily proximate to the divisor in $\mathbb{P}(\widetilde{\widetilde{J}})$ defined by the above dévissage.

At this point let's work in terms of the induced currents in the sense of V.2.4. Call an appropriate such on $\mathbb{P}(\widetilde{\widetilde{J}}), T$. We may devide T into two parts, according to the singularities of $\widetilde{\widetilde{\mathcal{F}}}$ which occur at the intersections $\widetilde{\widetilde{F}} \cap E_{CC'}$. Denote the said singular curves S, and use the Skoda-El Mir theorem to write,

$$T = \mathbb{1}_S T + T_{\mathcal{F}}$$

where the closed current $T_{\mathcal{F}}$ is invariant by $\widetilde{\widetilde{\mathcal{F}}}$ in the strongest possible sense, i.e. $\widetilde{\lambda}_* \, \widetilde{\widetilde{\pi}}_* \, T_{\mathcal{F}} = 0$. Furthermore the full foliation on $\widetilde{\widetilde{P}}$ is given by,

$$0 \to \widetilde{\widetilde{\pi}}^* \, \widetilde{\lambda}^* \, \Omega_{(\widetilde{\mathbb{P}}^2)^\vee}(\log \widetilde{F} + \widetilde{dC}) \to \Omega_{\widetilde{\widetilde{P}}}(\log \widetilde{\widetilde{D}} + \widetilde{\widetilde{F}}) \to K_{\widetilde{\widetilde{\mathcal{F}}}} \to 0$$

and so,

$$K_{\widetilde{\widetilde{\mathcal{F}}}} \cdot T_{\mathcal{F}} = (K_{\widetilde{\widetilde{P}}} + \widetilde{\widetilde{D}} + \widetilde{\widetilde{F}})\, T_{\mathcal{F}}$$

while by accident more than design,

$$K_{\widetilde{\widetilde{\mathcal{F}}}} \cdot \mathbb{1}_S T = \sum_{CC'} (\lambda(CC') + E_{CC'} \cdot \mathbb{1}_S T) = (K_{\widetilde{\widetilde{P}}} + \widetilde{\widetilde{D}} + \widetilde{\widetilde{F}})\, \mathbb{1}_S T$$

where $\lambda(CC')$ is the Lelong number of T along the component of S determined by C and C'. Consequently on combining with the original tautological inequalities for $\mathbb{P}(J_{\log})$ and $\mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))$ we obtain,

$$0 \geq (K_{P_{\log}} + D) . \rho_* T + \left(\sum_C d\log C + \widetilde{\widetilde{F}} \right) T \geq (K_{P_{\log}} + D) . \rho_* T\,.$$

From which we deduce that the Bloch-Theorem reduces to V.6.2 in the specific form,

Fact V.6.3bis. Bearing in mind V.3.8, if a sequence $f_n \in \mathrm{Hom}(\Delta, \mathbb{P}^2 \backslash D)$ which is not eventually arbitrarily close to D doesn't have a subsequence convergent in $\mathrm{Hom}(\Delta, \mathbb{P}^2)$, the derivatives $f_n' : \Delta \to \mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))$ are eventually arbitrarily close to the divisors defined by the pencils of lines defined by any two pairs of lines in D, together with the divisor D itself.

In order to fully profit from the example let's offer some remarks,

Remarks V.6.4. (i) The analysis here is very much inverse to Bloch's. The connection corresponds to the Wronskian, the divisors F to various sub-Wronskians. Bloch begins by excluding the possibility that the f_n could be arbitrarily proximate to some appropriate divisors, then he does the some for the sub-Wronskian, and eventually the Wronskian. This reverse ordering has some technical advantage, and corresponds to working with the modifications involved in V.6.2 as well as those which we've just seen. On the other hand what Bloch regards as his main technical innovation makes no appearance, although it may be related to V.6.5.

(ii) V.6.2 in this particular case is an easy exercise, but only because it's straightforward to identify appropriate bundles flat along the various pencils after a modicum of blowing up. However the algebraic nature of the foliations in question, doesn't of itself simplify things that much, except that harmonic rather than closed currents would suffice.

Having hopefully illustrated its importance let us turn to the proof of V.6.2.

Proof. Obviously we blow up until we reach the point where all of our foliations have canonical singularities. Thus we implicitly use the fact, specific to surfaces, that modifications of foliations with canonical singularities still enjoy the same. The same principle even allows us to assume the marginally stronger class of what are classically termed reduced singularities, and whence the technical convenience that any of our foliations has exactly two branches at a given singularity. Regardless we are now in a situation where any exceptional curve on a further blow up must be invariant by all the foliations, so that after further modifications we might as well suppose that we begin from a point where a curve C in the support of D with respect to a given foliation $\mathcal{F}^i$ is either invariant by it or nowhere tangent to it, and in addition that at any crossing of D there passes at least one curve invariant by all the foliations, while at any singularity of any foliation there is at least one curve in the support of D invariant by it.

Evidently the meaning of sufficiently large is in terms of modifications as employed to conclude V.4.4, which we flesh out in this multi-foliated and quasi-projective situation. We also insist that our divisor is sufficiently large, i.e. that D contains any curve which is invariant and passes through at least one singularity of some $\mathcal{F}^i$, and call this initial modification (X_0, D_0). Now suppose our foliation $\mathcal{F}^i$ is given by a dévissage,

$$0 \to N_0^i \to \Omega_{X_0} \to K_{\mathcal{F}_0^i} \cdot I_{Z_0^i} \to 0$$

We can divide our divisor D_0 as $\Delta_0^i + \partial_0^i$ where Δ_0^i is the part invariant by $\mathcal{F}^i$, and consider the corresponding log dévissage,

$$0 \to N_0^i(\Delta_0^i) \to \Omega_{X_0}(\log D_0) \to K_{\mathcal{F}_0^i}(\partial_0^i) \cdot I_{W_0^i} \to 0\,.$$

Given that there is at least one $\mathcal{F}^i$ invariant curve through any crossing of D_0, and D_0 is everywhere transverse to $\mathcal{F}^i$ where it isn't invariant, $W_0^i \subset Z_0^i$. Better still since at every singular point there is at least one $\mathcal{F}^i$ invariant curve C in D_0, if Z_0^i has degree 1 at this point, W_0^i is empty. The structure of W_0^i therefore comes from the saddle nodes, and W_0^i is empty or not according to whether or not, the weak branch is in D_0.

Now certainly amongst the foliations singularities may coincide, but we might as well insist that we've blown up sufficiently that this will be the case iff the branches coincide. In particular if a point is a singularity of two foliations $\mathcal{F}^i$ and $\mathcal{F}^j$ then a given branch is algebraic (whence in the support of D_0) for one iff it's algebraic for the other. Whence define Z_0 to be the set of points in X_0 which are saddle nodes for some foliation $\mathcal{F}^i$ whose weak branch at the point is not algebraic. Define X_1 to be the blow up of X_0 in Z_0, D_1 to be the pull-back of D_0 (by construction this is automatically reduced and contains the exceptional divisor), Z_1 to be the induced saddle nodes with non-algebraic branches, and continue inductively to get a sequence of modifications,

$$(X_n, D_n) \to (X_{n-1}, D_{n-1}) \to \cdots \to (X_1, D_1) \to (X_0, D_0)$$

with $\rho_{pq} : (X_q, D_q) \to (X_p, D_p)$ the projection for $q > p$, and for $z \in Z_0$, E_{zq} the exceptional curve on (X_q, D_q) over z blown down by $\rho_{q,q-1}$. Observe that our foliations $\mathcal{F}^i$ transform according to one of the following possibilities in a neighbourhood of $z \in Z_0$,

(i) $\mathcal{F}^i$ doesn't have a saddle node at z, then all induced singularities on X_q have degree 1. Whence if $\mathcal{F}_q^i$ is the induced foliation and $D_q = \Delta_q^i + \partial_q^i$ the decomposition into invariant and non-invariant parts, $K_{\mathcal{F}_q^i}(\partial_q^i)$ is a quotient of $\Omega_{X_q}(\log D_q)$.

(ii) $\mathcal{F}^i$ has a saddle node at z, with non-algebraic weak branch b, then for a subcheme W_q^i supported at the crossing of $\widetilde{b}$ with the exceptional divisor, say z_q, $K_{\mathcal{F}_q^i}(\partial_q^i)\, I_{W_q^i}$ is a quotient of $\Omega_{X_q}(\log D_q)$.

Now let us turn to our maps $f_n : Y_n \to X$, and their various proper transforms, also denoted f_n, to any X_q. Certainly we have various logarithmic derivatives, $f_n' : Y_n \to \mathbb{P}(\Omega_{X_q}(\log D_q))$ which thanks to the log-tautological inequality, and the general considerations of V.2 give for all q and $r \in (0,1)$ outside a set of finite hyperbolic measure positive currents,

$$T'_{nq}(r) : A^{1,1}(\mathbb{P}(\Omega_{X_q}(\log D_q)) \to \mathbb{C} : \eta \mapsto (H._{f_n} Y(r))^{-1} \int_0^r \frac{dt}{t} \int_{Y(t)} f_n^* \eta$$

converging weakly, after suitable subsequencing, to closed positive currents T'_q, provided of course that we excluded a sufficiently large Zariski subset in order to guarantee the degree on X_0 measured with respect to H, dominates the degree with respect to an ample divisor on X_q. Within $\mathbb{P}(\Omega_{X_q}(\log D_q))$, which we may as well denote P_q, with π to X_q the projection, we have two kinds of divisors that interest us, one being $\pi^* C$ for C a component of D, the other the closure of the section in codimension 2 defined by the foliation $\mathcal{F}^i$, which we'll denote X^i_q. Necessarily X^i_q is isomorphic to the blow up of X_q in W^i_q, so it's a section away from saddle nodes z, but in general for a node of degree $d_i(z)$ at least 3, it will be a blow up in a non-reduced scheme, so it's singular, and its minimal smooth resolution $\widetilde{X}^i_q$ is around the node isomorphic to $X_{q+d_i(z)}$. By hypothesis T'_q is supported only on these divisors. Before breaking T'_q up according to its support, observe that for any C in D, $(K_X + D).\, C \leq 0$ iff C is rational and meets $D \backslash C$ in at most 2 points. Call such curves parabolic, and observe further that it is a harmless hypothesis to suppose that if C in D is not parabolic and $C' \neq C$ in D meets C then C' is parabolic, i.e. just blow up in the crossings of D, which we might as well have done a priori so that this holds for all D_q, $q \geq 0$. Now we can devide T'_q up into a sum of closed positive currents according to the rule,

$$T'_q = \sum_{C \subset D_q^{n\text{par}}} \pi^* \, \mathbb{1}_C \, T'_q + \sum_{C \subset D_q^{\text{par}}} \pi^* \, \mathbb{1}_{C \backslash D_{np}} \, T'_q + \sum_i \mathbb{1}_{X^i_q \backslash D_q} \, T'_q$$

where $D_q = D_q^{\text{par}} + D_q^{n\text{par}}$ is the decomposition of D_q into its parabolic and non-parabolic components.

Given the nature of our preliminaries we take care of $\mathbb{1}_{X^i_q \backslash D_q} T'_q$ first. For simplicity let's write this as $\mathbb{1}_{\mathcal{F}^i_q} T_q$, by construction $\pi_* \, \mathbb{1}_{\mathcal{F}^i_q} T_q$ is a transverse invariant measure for $\mathcal{F}^i_q$, which can have atomic components on at most invariant curves wholly contained in the smooth locus. Locally over the points z, if any, where X^i_q isn't a section we have an exceptional Cartier divisor $E^i_q(z)$ whose pull-back to the minimal smooth resolution $X_{q+d(z)}$ is $\sum_{k=1}^{d_i(z)} \rho^*_{q+d_i,k} \, E_k(z)$. In any case if L_q is the tautological bundle on P_1, then,

$$L_q \otimes \mathcal{O}_{X^i_q} = \mathcal{O}_{X^i_q} \left(K_{\mathcal{F}^i_q} + \partial^i_q - \sum_z E^i_q(z) \right) .$$

Furthermore by the logarithmic variant of Brunella's residue theorem, for T on X_q a transverse $\mathcal{F}^i_q$ invariant measure, without atomic support $(N^i_q + \Delta^i_q).\, T \leq 0$. However this inequality equally holds trivially for any invariant curve wholly contained in the smooth locus of $\mathcal{F}^i_q$, so it holds for $\pi_* \, 1\!\!1_{\mathcal{F}^i_q} T_q$, which implies,

$$L_q \cdot 1\!\!1_{\mathcal{F}^i_q} T_q \geq \pi^*(K_{X_q} + D_q) \, 1\!\!1_{\mathcal{F}^i_q} T_q - \sum_z E^i_q(z) \cdot 1\!\!1_{\mathcal{F}^i_q} T_q \, .$$

On bearing in mind that $\pi_* \, 1\!\!1_{\mathcal{F}^i_q} T_q$ is nef. for any q, we can therefore apply V.4.3 to obtain,

$$\sum_z E^i_q(z) \cdot 1\!\!1_{\mathcal{F}^i_q} T_q \leq (\mathrm{const}) \, [\sqrt{q}]^{-1}$$

for some absolute constant depending only on X, so that for some q sufficiently large, $L_q \, 1\!\!1_{\mathcal{F}^i_q} T_q \geq \pi^*(K_{X_q} + D_q).1\!\!1_{\mathcal{F}^i_q} T_q - \varepsilon$, and we're reduced to obtain a similar inequality for the part of T'_q supported over D.

To this end consider for C in D the canonical section, defined by the short exact sequence,

$$0 \to \mathcal{O}_C(K_X + D) \to \Omega_X(\log D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to 0$$

where the last map is the residue map. For simplicity of notation consider q as fixed, and let P_C be the fibre of P over C with S the said section. Certainly we have,

$$\mathcal{O}_{P_C}(S) = \mathcal{O}_{P_C}(L_q - \pi^*(K_X + D))$$

so an estimate of the form $S \cdot 1\!\!1_{P_C} T' \geq 0$ will serve us nicely. Better still for C parabolic, $S^2 \geq 0$, so such an estimate trivially holds for any current supported in P_C, in particular for $1\!\!1_{P_C \setminus \pi^* D^{\mathrm{npar}}} T$, and we're reduced to study C non-parabolic.

At this point it becomes clear why we wished to simplify the notations. Specifically we consider a sequence of modifications,

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 = P$$

with P_1 the blow up of P_0 in S, so that the proper transform P^1_C of P_C is isomorphic to P_C whence in this sense we blow up P_1 in S and continue ad infinitum. Here we'll use π_{mn} for the projection between P_n and P_m, and F_n the exceptional divisor blown down by $\pi_{n-1,n}$.

To set the scene on P_1, P^1_C can be blown down, and the resulting variety is $\mathbb{P}(\Omega_{X_q}(\log D'_q))$ where $D_q = D - C$, with P^1_C blowing down to the derivative of C. Now since any derived type current with non-trivial push-forward in $\mathbb{P}(\Omega_{X_q}(\log D'_q) \otimes \mathcal{O}_C)$ is supported on C this shows that there are no non-exceptional components of the induced current supported on F_q. However this holds much more generally, but let's first set up the notation and clarify the above remarks.

Our sequence $f'_k : Y_k \to P$, has a lifting to any P_n which leads to an induced current T_n (strictly speaking T_{nq}, but q will play no further role). We're interested in the decomposition,

$$T_n = 1\!\!1_{P^n_C} T_n + 1\!\!1_{\pi^*_n C \setminus P^n_C} T_n + 1\!\!1_{P_n \setminus \pi^*_n C} T_n$$

and we've already remarked that $(\pi_{10})_*(\mathbb{1}_{\pi_{01}^* C\setminus P_C^1} T_1) = 0$, however suppose this held more generally, i.e., we have,

Claim. $(\pi_{n0})_* \, (\mathbb{1}_{\pi_n^* C\setminus P_C^n} T_n) = 0$.

Then we may conclude as follows. In the first place this forces $\mathbb{1}_{P_C^n} T_n = \mathbb{1}_{P_C} T_0$, so that $(F_1+\cdots+F_n)\cdot \mathbb{1}_{P_C^n} T_n = n\, S\cdot \mathbb{1}_{P_C} T_0$. Furthermore $\mathbb{1}_{\pi_{0n}^* C\setminus P_C^n}$ T_n is homologous to a positive sum of curves contracted by π_{n0}, and $F_1 + \cdots + F_n$ is non-positive on such so,

$$0 \le (F_1+\cdots+F_n)\cdot T_n \le n\, S\cdot \mathbb{1}_{P_C} T_0 + \sum_{i=1}^{n} F_i \cdot \mathbb{1}_{P_n\setminus \pi_{0n}^* C} T_n$$

where the first inequality holds since $f_n(0)$ is supposed bounded away uniformly from C. Finally note that P_C^n intersects positively any current supported off it, so:

$$\sum_{i=1}^{n} F_i \cdot \mathbb{1}_{P_n\setminus \pi_{0n}^* C} T_n \le P_C \cdot \mathbb{1}_{P_0\setminus \pi_{01}^* C} T_0$$

since the latter is fixed independently of n, our claim indeed implies that $S\cdot \mathbb{1}_{P_C} T_0 \ge 0$.

It remains therefore to justify the claim. To this end suppose locally C is given by an equation $x=0$, and y is the other coordinate. In a neighbourhood of S we have a coordinate $t = \frac{dy}{dx/x}$ (or $\frac{dy/y}{dx/x}$ if we're close to a crossing of D) with respect to which $\pi_{0n} : P_n \to P$ is the minimal smooth resolution of the ideal $I = (t^n, x)$, i.e. $\bigotimes_{i=1}^{n} \pi_{in}^* \, \mathcal{O}_{P_i}(-F_i) = \pi_{0n}^{-1} I$, so to be an open neighbourhood a distance ε off P_C^n amounts to the condition,

$$\frac{|x|}{|x|+|t|^n} \ge \varepsilon \Leftrightarrow |dy| \text{ (resp. } \left|\frac{dy}{y}\right| \text{ near a crossing)} \le \left(\frac{1-\varepsilon}{\varepsilon}\right)^{\frac{1}{n}} \; \frac{|dx|}{|x|^{1-1/n}}\,.$$

Consequently the claim will follow from an appropriate estimate for the measure, $T\left(\mathbb{1}_{\delta,\varepsilon}\, \frac{dx\, d\overline{x}}{|x|^{2-2/n}}\right)$ for $\mathbb{1}_{\delta,\varepsilon}$ a small neighbourhood a distance δ around C, but ε off P_C^n. More canonically in terms of the norm $\mathbb{1}_C$, we may observe for δ sufficiently small, determined by n,

$$\frac{1}{n^2}\,\frac{dx\, d\overline{x}}{|x|^{2-2/n}} \gg\ll 4d\, \|\mathbb{1}_C\|^{1/n}\, d^c \|\mathbb{1}_C\|^{1/n} + |x|^{\frac{1}{n}}\, \omega$$

for ω_n a smooth form depending on n. Equally $4d\, \|\mathbb{1}_C\|^{1/n}\, d^c\|\mathbb{1}_C\|^{1/n}$ is just $dd^c\|\mathbb{1}_C\|^{2/n}$ this time up to a factor of $\|\mathbb{1}_C\|^{2/n}\times$ smooth, so the following

Jensen formula, with $\mathbb{1}_\delta$ denoting points with $\|\mathbb{1}_C\| \leq \delta$, will do nicely,

$$\begin{aligned} T_{f_k,Y_k}(r)\left(\mathbb{1}_\delta \frac{dd^c\|\mathbb{1}_C\|^{2/n}}{\delta^{2/n}}\right) &= \int_{\partial Y_k(r)} \mathbb{1}_\delta \left(\frac{\|\mathbb{1}_C\|^{2/n}}{\delta^{2/n}} - 1\right) p_k^*\left(\frac{d\theta}{2\pi}\right) \\ &- \frac{1}{n}\int_{\partial Y_k(r)} \mathbb{1}_\delta \log\frac{\|\mathbb{1}_C\|}{\delta}\, p_k^*\left(\frac{d\theta}{2\pi}\right) \\ &- \mathbb{1}_\delta \left(\frac{\|\mathbb{1}_C\|^{2/n}}{\delta^{2/n}} - 1\right)(0) \\ &+ \frac{1}{n}\mathbb{1}_\delta \log\frac{\|\mathbb{1}_C\|}{\delta}(0) \\ &+ \frac{1}{n}\sum_{0<|p_k(y)|<r} \mathrm{ord}_y(f_k^*D)\log\left|\frac{r}{p_k(y)}\right| \\ &- T_{f_k,Y_k}(r)\left(\mathbb{1}_\delta\, c_1(\overline{C})\right). \end{aligned}$$

From which we see that the measure $T\left(\mathbb{1}_\delta \frac{dx\,d\overline{x}}{|x|^{2-2/n}}\right)$ goes to zero like $\delta^{2/n}$. Whence if we take $\delta = \varepsilon^2$, we've comfortably established our claim.

□

Manifestly therefore whenever the condition that a sequence of maps $f_n : \Delta \to X$ or more generally $X\backslash D$ enjoys the property that their derivatives are eventually arbitrarily close to some divisor on $P(T_X)$ but not eventually arbitrarily close to a divisor in X, the main lemma and the tautological inequality apply to bound the K_X-degree. Consequently provided X has general type and the divisor is taken to include the curves contracted to the canonical (not just minimal) model we obtain convergence of a subsequence in $\overline{\mathrm{Hom}}(\Delta, X)$, while evidently in the quasi-projective case we work with the log-canonical model.

The question arises therefore of what becomes of a sequence of maps $f_k : \Delta \to X$ which are arbitrarily close to some divisor ∂ in X. Apart from the need to assume simple normal crossings, the proof of V.6.2 points us to a wholly general estimate.

Under these more restrictive hypothesis, and in the notations of the proof it will be more than sufficient to prove $S.\,\mathbb{1}_{P_C}\,T_0 \geq 0$ for S the residual section associated to a non-parabolic curve $C \subset \partial$. The key to doing this is to work with the tautological bundle L on $\mathbb{P}(\Omega_X(\log\partial))$ in a metric which is complete for $X\backslash\partial$, whence to make this difference clear we'll use L^c for L metricised in such a way, and L^{sm} for a smooth metric. Provided we work with L^c and evaluate derivatives of maps at the origin in the complete metric we have an arguably more satisfactory inequality with much the same sort of error but with the $\log\log^2\|\mathbb{1}_\partial\|$ at the origin terms replaced by log of the degree of $-dd^c\log\log^2\|\mathbb{1}_\partial\|$. As such V.4.2's marginally non-tautological content is the estimate for the difference between degrees calculated in L^{sm} rather than L^c in terms of the $\log\log^2\|\mathbb{1}_\partial\|(0)$ term which is best possible. Consequently if $T_k \to T$ is a sequence of normalised currents associated to the degree function for maps from various discs it is not, in general, true that $L^c \cdot T_k \to L^c \cdot T$ when the latter is calculated in the sense of currents, since it would necessarily be $L^{\mathrm{sm}} \cdot T$. The fact that this is so is why the L^c formulation has not been employed as it would be unsuitable for foliations where L must be compared with $K_{\mathcal{F}}$ by way of smooth metrics. For maps in a tubular neighbourhood of curves without other restrictions, however, it is ideal provided we bear in mind that we're working with $\lim_k L^c \cdot T_k$ everything goes through, and the key points are:

Facts V.6.5. (a) (Notations as above) Let $f_k : \Delta \to X$ be a sequence of maps of unbounded degree at some radius $r_0 \in (0,1)$ with D a divisor on X then, after subsequencing, for $r \in (0,1)$ outside a set of finite hyperbolic measure we have,

$$T_{f_k}(r)(-dd^c \log\log^2 \|\mathbb{1}_D\|) \leq \log(1-r^2)^{-1} + 0(\log \overline{H}\cdot_{f_k}\Delta(r) + \log N_{f_k,D}(r))$$

where, as ever, $\overline{H}$ is positive ample, and $N_{f_k,D}$ is the counting function.

(b) In particular if $f'_k : \Delta \to \mathbb{P}(\Omega_X(\log \partial))$ are the logarithmic derivatives and S the residual section associated to a divisor omitted by the f_k with $\|\mathbb{1}_S\|$ calculated in the complete metric then after subsequencing we can find a convergent sequence of automorphisms $\alpha_k \in \mathrm{Aut}(\Delta)$ such that $\alpha_k^*(f'_k)^* \|\mathbb{1}_S\|(0)$ is uniformly bounded away from 0 in k.

Proof. (a) This is just an application of Jensen's formula used to yield integral representations of $\log f_k^*\|\mathbb{1}_D\|$. The model is the proof of the logarithmic derivative lemma in [L2].

(b) In the coordinates x, y, t of the proof of V.6.2, and away from a crossing of ∂ (otherwise even easier) the condition $\|\mathbb{1}_S\| \leq \varepsilon$ amounts to,

$$\frac{|t|}{|t| + \frac{1}{|\log x|}} \leq \varepsilon \Leftrightarrow |t| \leq \frac{\varepsilon}{(1-\varepsilon)}\,|\log x|^{-1}$$

and the statement follows from the corresponding Jensen formula for $dd^c \log\log^2 \|\mathbb{1}_C\|$.

□

Applying this to our situation, or more correctly a minor variant to cope with several residual sections at the same time, and denoting by F_1^c the metricisation of the exceptional divisor over the blow up of S in the complete metric with $T_k \to T$ the limiting sequence of degree functions after appropriate composition with bounded automorphisms, we obtain $\lim_k F_1^c \cdot T_k \geq 0$ regardless of the proximity of the maps f_k to C at the origin. Better still letting ρ_ε be a sequence of bump functions in a neighbourhood of C and unravelling the Skoda-El Mir theorem we have,

$$\lim_{\varepsilon \to 0} \lim_{k \to \infty} T_k\{(1-\rho_\varepsilon)\, c_1(F_1^c)\} = 0$$

since $\|\mathbb{I}_S\|$ blows no worse than $|\log\|\mathbb{I}_C\||$ around C, and whence,

Fact V.6.6. (Notations as above) Suppose for $r_0 \in (0,1)$, the degrees $\overline{H}_{\cdot f_k}\,\Delta_0(r)$ with respect to a positive metricised ample are unbounded in k and the f_k are arbitrarily close to ∂, then after subsequencing, and composing with convergent automorphisms, we have for $n \in \mathbb{N}$ and $r \in (0,1)$ outside a set of finite measure,

$$L^c_{\cdot f'_k}\,\Delta(r) \geq (K_X + \partial)_{\cdot f_k}\,\Delta(r) - \frac{a}{n}\,H_{\cdot f_k}\,\Delta(r) + \frac{1}{n}\log\|f_k^*\,\mathbb{I}_{\partial_0}\|\,(0)\,,$$

provided $k = k(n) \gg 0$, with some constant independent of the variables, and ∂_0 the part of the boundary not omitted by the maps.

Proof. The fact that the error term at the origin only involves ∂_0 is the above discussion. Everything else is immediate from V.6.2 up to being careful about how the current is normalised. On observing however, and quite generally, that on any modification $\psi : \widetilde{V} \to V$ with exceptional divisors E_i, and H ample on the base, some $\psi^* H\left(-\sum_i \delta_i\,E_i\right)$ for δ_i as small as we like is still ample we have our required normalisation with any extra error absorbed into the inequality itself.

Now let's turn this into an estimate for the degeneration of the Kobayashi metric on a quasi-projective surface (X,D) and indeed the complete hyperbolicity modulo ∂_0. We necessarily suppose not only that (X,D) has log general type, but that (X,D) is the log-canonical model, or equivalently $K_X + D$ is ample. The unfortunate thing is that the set ∂_0 on X could be highly singular and for $D \neq \varphi$, meet the boundary in an extremely messy way. Consequently to apply V.6.5 we must pass to a resolution $\rho : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth, and the total pull-backs of D and ∂_0 have simple normal crossings. Our boundary ∂ then will be the union of $\widetilde{D}$, the reduced total pull-back of D, and $\widetilde{\partial}_0$, any component of the reduced pull-back that isn't in $\widetilde{D}$. An appropriate measure of the singularity of our initial situation is provided by the discrepancy, i.e.

$$K_{\widetilde{X}} + \widetilde{D} = \rho^*(K_X + D) + \sum_i a_i\,E_i$$

where the E_i are the curves contracted by ρ, and $a_i \in \mathbb{Q}_{\geq 0}$. Obviously to keep the situation as natural as possible ρ should be minimal, i.e. first pass to the unique minimal smooth resolution (X', D') of (X,D), then take a minimal resolution for the divisor $D' + \partial'$, where the latter is the proper transform of ∂. In any case denote by $\|\ \|^c_{\widetilde{\log}}$ a complete norm on $T_{\widetilde{X}}(-\log\partial)$ then we have,

Corollary V.6.7. *Notations as above (X,D) is complete hyperbolic modulo Z, and for all $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that for any $f \in$* $\mathrm{Hom}(\Delta, X\backslash D)$,

$$\prod_i \|\mathbb{I}_{E_i}\|^{a_i+\varepsilon}(0)\,\|d\widetilde{f}\|^c_{\widetilde{\log}}(0) \leq \frac{C(\varepsilon)}{\|\mathbb{I}_{\widetilde{\partial}_0}\|^{1+\varepsilon}}$$

where $\widetilde{f} : \Delta \to \widetilde{X}$ is the proper transform.

Apart from bearing in mind that this and V.6.6 are the unique situations where the degree on X may not dominate that on $\widetilde{X}$, and so far an ample bundle we must use $\rho^*(K_X + D)\left(-\sum_i \delta_i E_i\right)$ for appropriately small δ_i, and whence the ε of the announcement, this is just an application of V.6.5 as per the methodology of V.5.1(b), and V.5.2, with the various inequalities for the $K_{\mathcal{F}}$ degree replaced by the said bound for the $K_{\widetilde{p}} + \widetilde{D} + \widetilde{\partial}_0$ degree.

V.7. Variations and Reflections

As we remarked in V.4, questions of value distribution are naturally opposite to the uniformisation theorem in the sense that if B is one of the simply connected Riemann-surfaces $\mathbb{P}^1$, $\mathbb{C}$ or Δ then the value distribution of maps to a variety X should be thought of in terms of triples,

$$\begin{array}{ccc} Y & \xrightarrow[f]{} & X \\ \downarrow p & & \\ B & & \end{array}$$

even in the case of $B = \Delta$, where we've seen the importance of the choice of exhaustion in deducing V.6.1 from V.6.2. In a number theory spirit such triples can be thought of as points in $X(\overline{B})$ for $\overline{B}$ an algebraic closure, although in the disc case one should be careful that the ramification isn't too close to the origin in the sense of V.2.7, so as to guarantee the rest of V.2. Obviously this is okay in all natural examples where the p-ramification will arise by lifting maps to ramified covers of some algebraic variety.

In any case two important points emerge,

(a) Provided it applies, Brody's reparameterisation trick for deducing hyperbolicity statements from lack of maps from the complex line is radically easier than working directly with sequences of discs.

(b) What is the precise relation between the canonical degree and the discriminant, or better the tautological degree, since the ratio of the two is very much a sectional curvature viewed from a global rather than infinitesimal perspective.

As an example of (a) we have,

Example V.7.1. *Let X be a surface such that for some $m \in \mathbb{N}$, $h^0(X, S^m\,\Omega^1_X) \geq 2$. Then, X is hyperbolic iff X contains no rational or elliptic curves or is covered by an abelian surface.*

Proof. The only if direction is trivial, while contrary wise the hypothesis force X to be of general type. Regardless reparametrisation implies that hyperbolicity follows if we can show that there is no non-trivial holomorphic map $f : \mathbb{C} \to X$. Whence without loss of generality we may reparametrise it in such a way that it has bounded derivative, cf. [L2]. Now consider a holomorphic symmetric m-form ω on X then, $f^* \omega = \lambda\, dz^{\otimes m}$ where λ is a bounded holomorphic function on the complex line, i.e. a constant. If λ is zero then certainly f will be an integrable sub-curve of a foliation on a m-fold cover of X, so by V.1.2(a) we have 2-holomorphic m-forms ω_1, ω_2 say on X and non-zero complex numbers λ_1, λ_2 with $f^* \omega_i = \lambda_i\, dz^{\otimes m} \neq 0$. Consequently $f^*(\lambda_2\, \omega_1 - \lambda_1\, \omega_2) = 0$ and reasoning as above we conclude. □

While, at least for foliations, we have an extremely satisfactory inequality V.4.4, which is wholly optimal for leaves. One should not however be misled whether by V.4.4 or V.6.2 into believing that this sort of sectional curvature of the canonical bundle is always -1, which is very much the suggestion of Vojta's canonical height inequality conjectures, i.e.

Example V.7.2. (Ball Quotients) Let X be an algebraic variety which is a quotient of the n-ball, B_n, and $f : \Delta \to X$ a geodesic for the natural holomorphic connection then for r outside a set of finite hyperbolic measure,

$$K_{X \cdot f}\, \Delta(r) = \frac{(n+1)}{2}\ \mathrm{discr}(r) + \mathrm{o}(K_{X \cdot f}\, \Delta(r))\,.$$

Proof. This of course can be done by direct infinitesimal calculation, but let's review it from our global perpective, i.e. consider the second jet space, Q, of X. This is a $\mathbb{P}^{n-1}$ bundle over the projective tangent space $P = \mathbb{P}(\Omega_X)$ corresponding to the bundle J on P defined as Ω_P modulo the universal sub-bundle. In particular it is an extension,

$$0 \to L \to J \to \Omega_{P/X} \to 0$$

where L is the tautological quotient bundle on P, and the connection corresponds to a splitting of J. In particular it gives a foliation by curves $(P, \mathcal{F})$ on P with canonical bundle $K_{\mathcal{F}} \xrightarrow{\sim} L$. Now the situation in the normal direction is nice and easy. Specifically everything is $SU(n,1)$ invariant, so by just working over the ball we find a transverse invariant measure which allows us to do harmonic theory along the leaves. The important consequence of which is that the Chern class of the normal bundle which is only a priori a $\overline{\partial}$ boundary along the leaves is actually $\partial\overline{\partial}$ along the leaves so its degree along any geodesic is bounded by a constant. Whence,

$$K_{P \cdot f}\, \Delta(r) = L_{\cdot f}\, \Delta(r) + 0(1)\,.$$

On the other hand, $K_P = 2K_X - nL$, so appealing to the tautological inequality we establish our claim. □

This does not however violate the basic suggestion of Lang-Vojta that value distribution of maps is governed in a quantitative way à la V.5.4 by the canonical class – all be it that there isn't much evidence beyond dimension 2. Staying in dimension 2, let us give some elaboration on the remarks of the introduction. To begin with we recall the construction of the jet spaces of Arrondo, Sols, and Speiser, [ASS], or Demailly, [De1], associated to a non-singular surface X with simple normal crossing boundary D.

V.7.3. Construction of jet spaces

(a) Put $(X_0, D_0) = (X, D)$, $X_1 = \mathbb{P}(\Omega_X(\log D))$, π the projection to X and $D_1 = \pi^{-1}(D)$.

(b) Suppose for $n \in \mathbb{N}$ we have constructed log-jet spaces (X_n, D_n) such that $X_n = \mathbb{P}(J_n)$ and J_n is a rank 2 locally free quotient of $\Omega_{X_{n-1}}(\log D_{n-1})$ (or rank dim X locally free quotient in general) with projection π to X_{n-1} respecting the boundaries. Denote by i the natural embedding of X_n in $\mathbb{P}(\Omega_{X_{n-1}}(\log D_{n-1}))$, and consider the diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i^*U_{n-1} & \longrightarrow & \pi^*\,\Omega_{X_{n-1}}(\log D_{n-1}) & \longrightarrow & L_n & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i^*U_{n-1} & \longrightarrow & \Omega_{X_n}(\log D_n) & \longrightarrow & J_{n+1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_{X_n/X_{n-1}} & = & \Lambda^2 J_n \otimes L_n^{-2} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where the first row is the universal exact sequence on $\mathbb{P}(\Omega_{X_{n-1}}(\log D_{n-1}))$ pulled back to X_n, and L_n is the universal quotient bundle on X_n.

(c) Define J_{n+1} to be the extension of $\Lambda^2 J_n \otimes L_n^{-2}$ by L_n in the far column of the diagram. It certainly satisfies the induction hypothesis above, and so the definition makes sense.

We make the following easy observations. They are essentially due to Green and Griffiths, [GG], but their jet spaces were rather bigger. As ever we confine ourselves to X a surface,

V.7.4. Facts about jet spaces

(a) If $j : \Delta \to X$ is the germ of a map from a smooth curve (algebraic, holomorphic, formal or whatever one prefers) there is a natural mapping $j^{(n)} : \Delta \to X_n$ obtained by differentiating n-times.

(b) $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ is a tower of $\mathbb{P}^1$-bundles. The extra coordinate on $X_i \to X_{i-1}$ is $\frac{d^i y}{dx^i}$ in terms of a local coordinate system x, y on X. Consequently any non-degenerate ordinary holomorphic differential equation on X of order n say may be interpreted as a divisor W, say, in X_n dominating X_{n-1}. In turn the said O.D.E. induces a foliation by curves $\mathcal{F}$ (thus it is automatically integrable) on W and there is a one to one correspondence between solutions of the O.D.E. on X and integrable sub-curves of the foliation via n^{th}-order differentiation and pushing down respectively.

(c) If (X, D) has log-general type then Riemann-Roch ought to yield positive integers n, a_i such that $\sum_{i=1}^{n} a_i L_i \mid_{X_n}$ is big. For example L_1 is big if $c_1^2 > c_2$, and $2L_1 + L_2$ is big if L_1 isn't big and $c_1^2 > \frac{9}{13} c_2$. Unfortunately the general Riemann-Roch calculation is a bit fastidious, although probably nothing that a computer can't handle, and is in any case fairly irrelevant since the corresponding calculation is easy for Green and Griffiths' original jet spaces, and by the tautological inequality we'll therefore know all about the value distribution of maps provided they're not in a divisor on the Green-Griffiths jet space in the compact or parabolic theories, respectively not close to such a divisor in the disc theory. On the other hand the jets of V.7.3 are just the Green-Griffiths jets modulo parameterisations, so by looking at all parameterisation of maps, we conclude that we can prove some suitable analogue of V.6.1 and its subsequent application, provided the maps in question aren't solutions of ODE's in the sense of (b) for the compact and parabolic theories, repectively not close to solutions in the discs theory.

At this point something like the Green-Griffiths conjecture fits into a wholly general programme,

V.7.5. Foliated Mori programme

(a) Given an O.D.E. with holomorphic coefficients on an algebraic surface X identify it with the pair $(W, \mathcal{F})$ say, where W is a desingularisation of the corresponding divisor in whatever jet space and $\mathcal{F}$ the induced foliation. Define the canonical bundle of the O.D.E. to be $K_{\mathcal{F}}$.

(b) Prove the existence of a canonical resolution of $(W, \mathcal{F})$. There are many partial results in this direction, but essentially the question is wide open.

(c) Try to run the minimal model programme. In particular prove a cone theorem for $K_{\mathcal{F}}$, and ultimately construct a minimal model.

(d) Classify the non-general type case, and analyse questions of hyperbolicity and value distribution in this case directly.

(e) Prove a value distribution inequality of second main theorem type for $K_{\mathcal{F}}$ à la V.4.4, and deduce the same for $K_X + D$ in the general type case, and employ ordinary classification otherwise. Modulo a technical point of cone theory, the strategy of [M1] recalled in V.5.1 ought always to permit this provided a canonical, rather than simply minimal, model exists.

Points (a)–(e) trivially imply the Green-Griffiths conjecture, if not quite hyperbolicity in the sense of V.1.4. However, what's more important is that they give structure to the conjecture, so that the conjecture itself becomes irrelevant. Specifically the ODE's of general type will not have transcendental parabolic solutions so the only possibility for a counterexample becomes something of small Kodaira dimension, which one can reasonably hope to classify. Now these things of small foliated Kodaira dimension considered as ODE's on surfaces of general type either admit transcendental parabolic solutions or they don't. Which of the two possibilities is irrelevant. In the latter case the conjecture holds, and that's nice, in the second there is a clear algebraic structure explaining why it fails.

Finally let's play the devil's advocate a bit, namely:

V.7.6. Objections to the programme

(i) Point (b) has been open a long time. Quite true, but it wasn't even appreciated what resolution of foliation singularities should mean in higher dimension, where as canonical singularities is not only an optimal definition but it even naturally brings a whole gambit of valuation theoretic machinery into play. Nevertheless this will require work, since it's canonical resolution that reduces any analysis, so far, to elementary calculus.

(ii) Point (c) is hardly a joke in the ordinary K_X case, so how can one expect to prove a foliated version of it. This is not a problem since despite living on higher dimensional varieties, all the foliations in question are by curves, and share a remarkable number of parallels with 1-dimensional classification. Not only has the foliated cone theorem already been proved, [B-M], but the minimal model theorem is coming along very nicely. Indeed we have systematically already seen that the $K_{\mathcal{F}}$ and K_X theories are almost disjoint. The only place we used K_X in a serious way was in IV.1.2 to construct the minimal model in the case that $\nu(\mathcal{F}) = -\infty$, i.e. a pencil of rational curves, which from a hyperbolicity point of view isn't exactly a priority.

(iii) A priori knowledge of the hyperbolicity of the solutions was used to finish the classification.

This is true, but not wholly relevant since the ought to part of V.7.5(e) is quite definitely will provided the curves in the face of the cone defined by $K_{\mathcal{F}\cdot -} = 0$ exhibit the elliptic behaviour that we've observed in dimension 2, as such this objection is of a more philosophical than practical nature. Moreover the continuity of the Poincaré metric is definitely an easier question than the Bloch hyperbolicity of invariant maps, given which, and quite generally, Brunella's p.s.h. variation theorem will in the non-general type case, give a transverse structure determined by a degenerate Monge-Ampère equation. What's more relevant is the non-algebraic nature of the solution to an algebraic problem. Unfortunately an algebraic way to proceed is not so obvious.

The only obvious stratagem is by way of positive characteristic where II.2.2 gives us a start. Specifically we can use what we have of classification prior to resolving the case of $\kappa(\mathcal{F}) = -\infty$ to show without loss of generality that something with this property is defined over a number field k. Then using Tchebetarov's theorem we can appeal to op.cit. to prove that at every prime p totally split in k the foliation is p-closed, provided there is an irrational eigenvalue. Even granted an irrational eigenvalue, respectively p-closure for all split primes, it's really not obvious how to finish, but certainly it's a start. The more serious difficulty is that if all the eigenvalues were rational, apart from being highly non-generic, there doesn't seem to be any extra information that can be gained from positive characteristic. There are of course through the singularities, in the non p-closed case, more or less unique invariant curves which one could hope to lift, and so force log smoothness in characteristic zero, but some appeal here has to be made to the special condition of $\kappa(\mathcal{F}) = -\infty$, since in general this certainly cannot be done.

VI. Rational points

VI.1. Hyperbolicity and Mordellicity

As we've defined it hyperbolicity is a compactness property of maps, while Mordellicity is a compactness property of points. Considered from a schematic point of view, however, the points are infact the generic points of appropriate 1-dimensional objects, so that the points are actually sections of fibrations. More generally we can look at algebraic points, and over function fields this translates into commutative diagrams of the form,

$$
\begin{array}{ccc}
\widetilde{Y} & \xrightarrow{f} & X \\
 & \searrow & \downarrow \pi \\
 & & B
\end{array}
$$

where B is a curve, $Y \to B$ a finite cover, and $\pi : X \to B$ not just proper, but also relatively projective. Evidently π is fixed, and is a model of $X \otimes k(B) \to k(B)$, the original variety over our field of functions. Not surprisingly we'll work in the quasi-projective setting, so this means assuming X non-singular and introducing a reduced simple normal crossing divisor D, each of whose components is flat over B. We'll also assume that the reduction of the singular fibres of π are simple normal crossing, and that together with D they form a simple normal crossing divisor.

The degree function in this setting is, thefore, nothing other than the standard degree function on X, and the analogue of Gromov's compactness theorem is Grothendieck's existence and properness of Hilbert Schemes. In particular if a sequence of algebraic points of degree bounded with respect to some ample bundle on X is Zariski dense, then one of finitely many (depending on the said degree bound) families of maps from B-schemes of bounded degree cover X, and X is isotrivial, i.e., after base change, X is generically a product with a variety X_0 defined over k, and the study reduces to that of curves on X_0 where appropriate compactness properties be it of algebraic curves or more generally discs are implicitly assumed, since otherwise we wouldn't be addressing the problem in the first place. As an introduction to how this relates to hyperbolicty, we may consider the original Mordell conjecture, i.e. relative dimesnion 1, over function fields, where it is immediately deducible from,

Fact VI.1.0 Suppose $(X, D)/B$ has relative dimension 1, and is both relatively logarithmically minimal and not isotrivial - so we're supposing the singularities are bit worse than we said we would, but this is temporary- then if $b_1, ..., b_k$ are the fibres of bad reduction, and ∂ is the reduction of D together with the said fibres, then (X, ∂) is complete hyperbolic iff the generic fibre is.

proof Suppose the contrary, i.e. $\mathrm{Hom}(\Delta, X\backslash\partial)$ is not relatively compact in $\mathrm{Hom}(\Delta, X)$ then by Brody's reparametrisation lemma we have a map $f : \mathbb{C} \to X$ which is the uniform limits of maps from big discs, $f_n : \Delta_{R_n} \to X\backslash\partial$, $R_n \to \infty$. This is, however, excluded by the hypothesis, which even force $B\backslash\{b_1, ..., b_k\}$ to be hyperbolic, in combination with V.3.7. □

Indeed there's a lot more information here than just the compactness of sections defined on the whole of B, and what we're actually getting is compactness for sections defined on any subset of B. However, just as the inequality V.4.2 for curves, where the canonical and tautoligical bundles coincide is more delicate than Bloch hyperbolicity, a more precise quantification is undoubtedly desirable. Consequently, and in slightly greater generality, we will always be concerned with sequences of maps $f_n : Y_n/B \to X$, such that for H ample on X the normalised degrees $\frac{1}{(Y_n:B)} H._{f_n} Y_n$ are unbounded, and enjoy Zariski dense image to avoid reducing to a smaller dimensional problem, again implicitly supposed soluble.

To warm up consider the data of such a sequence with the said reductions. To each rational point, $f_n : Y_n \to X$ we may associate a class in $\mathrm{NS}_1(X_k)$ as follows. Consider the open immersion $j : X_k \hookrightarrow X$, then we have an exact sequence,

$$0 \longrightarrow N \longrightarrow \mathrm{NS}_1(X) \underset{j^*}{\longrightarrow} \mathrm{NS}_1(X_k) \longrightarrow 0$$

where the kernel N is supported on fibres. Choose a section s, say, of j, and define $[Y_n]_s \in \mathrm{NS}_1(X_k)$ by duality via the formula,

$$D.[Y_n]_s := s(D)._{f_n} Y_n \,.$$

A priori this isn't terribly functorial, but in the "limit" it is not too bad. Observe that if H is ample on X and B is a big divisor on the generic fibre then for some $\varepsilon > 0$ we obtain,

$$s(B) = \varepsilon H + D + F \quad \text{and} \quad H = \varepsilon\, s(B) + D' + F'$$

where D, D' are effective and F, F' are $\mathbb{Q}$-divisors supported on fibres, so that:

Claim VI.1.1. *Notations as above,* $\frac{[Y_n]_s}{B.[Y_n]_s}$ *has a bounded subsequence in* $\mathrm{NS}^1(X_k)_{\mathbb{R}}$*, and as such has an accumulation point.*

Proof. By the first of the two formulae we may freely replace B by H since the set $\{f_n(Y_n)\}$ is supposed dense. By the second we obtain that for A an ample divisor on the generic fibre, $\frac{A.[Y_n]_s}{B.[Y_n]_s}$ is bounded. The Néron-Severi group is generated by finitely many ample divisors, and so we conclude. □

In particular, therefore, up to sub-sequencing, and choice of normalisation this construction is functorial, and enjoys functoriality properties similar to V.2.8.. Such considerations are valid in all characteristics, but a substantially more striking result holds in characteristic zero, which can be described in terms of currents or the Leray spectral sequence for the De-Rham co-homology of a fibration. The former keeps us closer to what has gone before, so let's explain things from that point of view. To begin with let X_{b_i} be the singular fibres, then we have a saturated map,

$$\pi^* \omega_B \left(\sum_i X_{b_i} - (X_{b_i})_{\mathrm{red}} \right) \to \Omega_X(\log D)$$

and by a quite substantial abuse of notation, we make,

Definition VI.1.2 $\Omega_{X/B}(\log D)$ *is the quotient of the above map*

This is non-standard, e.g. if D is empty, the usual definition is the quotient of the possibly non-saturated map from ω_B. Evidently $\mathbb{P}(\Omega_{X/B}(\log D)) \hookrightarrow \mathbb{P}(\Omega_X(\log D))$ is the irreducible divisor defined by the said (saturated) map, and over a singularity of a singular fibre, the reduced fibres of the latter and the former coincide. Furthermore we may consider the derivatives $f'_n : Y_n \to \mathbb{P}(\Omega_X(\log D))$, and the associated currents of integration T'_n, which as ever we may normalise by the degree to obtain a closed positive current T on $\mathbb{P}(\Omega_X(\log D))$ as the weak limit of $(H \cdot_{f_n} Y_n)^{-1} T_n$ for which the basic observation is,

Fact VI.1.3 Notations as above, with $U = \mathbb{P}(\Omega_X(\log D)) \backslash \mathbb{P}(\Omega_{X/B}(\log D))$ and ρ the projection to X, then $\rho_*(\mathrm{1\!I}_u T) = 0$.

Proof. The question is local, and vacuous over the singularities of fibres, so we may as well say b is a coordinate on B, with $\pi^* b = z^m$, for z a coordinate on X. Further let $x_1, \ldots, x_N$ be a coordinate system on X with $x_1 \ldots x_d = 0$ defining D. Consider a polydisc V_ε in U a distance ε away from $\mathbb{P}(\Omega_{X/B}(\log D))$, which amounts to $\|dz\| > \varepsilon$, where,

$$\|dz\|^2 = \frac{|dz|^2}{\sum_{i=1}^{d} \left| \frac{dx_i}{x_i} \right|^2 + \sum_{i=d+1}^{N} |dx_i|^2}$$

Consequently for any $f_n : Y_n/B \to X$, over the open set $(f'_n)^{-1}(V_\varepsilon)$ we have the estimate,

$$\frac{1}{\varepsilon^2} f_n^*(dz\, d\overline{z}) > \sum_{i=1}^{N} f_k^*(dx_i\, d\overline{x}_i) \,.$$

On the other hand, $f_n^*(dz\, d\overline{z}) = p_k^* \left(\frac{db\, d\overline{b}}{m^2 |b|^{2(1-1/m)}} \right)$, so for ω some Kähler form on X we have,

$$\int_{(f'_n)^{-1}(V_\varepsilon)} \omega \le \frac{C}{\varepsilon^2} (Y_n : B)$$

where the constant, C, is independent of everything except X and B, so given our supposition that $\frac{1}{(Y_n:B)} H \cdot_{f_n} Y_n$ is unbounded, we're done. □

Observe, of course, that for any $f : Y/B \to X/B$ with L the tautological bundle in $\mathbb{P}(\Omega_X(\log D))$ we don't just have a tautological inequality, but a tautological identity,

$$L._{f'} Y = (2g(Y) - 2) + \mathrm{rad}_{f,D} - \sum_{y \notin f^{-1}(D)} \mathrm{ord}_y(\mathrm{Ram}_f)\,.$$

and so a substantially simpler variation of V.6.2 immediately yields,

Corollary VI.1.4 *Suppose additionally that X is a surface, i.e. X/B is a family of curves, then $\forall \varepsilon > 0$, there is a constant $C(\varepsilon)$ such that if $f : Y/B \to X/B$ is a map of B-schemes then,*

$$(\omega_{X/B} + D)._f Y \le (1 + \varepsilon)(2g(Y) - 2 + \mathrm{rad}_{f,D}) + C(\varepsilon)(Y : B)\,.$$

One should note that VI.1.4 is, modulo the question of quantification in ε, wholly optimal, and infact wholly new in the non-isotrivial case, a particularly well known isotrivial example being $\mathbb{P}^1 \backslash \{0, 1, \infty\}$, i.e. Mason's 'abc' for polynomials. From which two important things emerge,

(a) It is important not just to prove compactness, but some actual degree bound à la VI.1.4. In the case of discs this is akin to the difference between a compactness statement, and an isoperimetric inequality, which we achieved satisfactorly for foliations, i.e. V.4.4, or the obvious quasi-projective variant thereof, but didn't quite manage for more general maps to surfaces, albeit that the explicit quantification on the Kobayashi metric, V.6.6, ought to be a good approximation.
(b) Morally, VI.1.3, reduces all Mordellicity problems to hyperbolicity problems, and 'curvature calculations' in the sense of V.6.2. There appears to be, however, a serious problem with bad-reduction -which, incidentally, is very hard to accept given VI.1.1- an illustration of which is that the only aspect of VI.1.4 which is not absolutely trivial is the use of the strategy of the last step of the proof of V.6.2 to control the bad reduction.

To illustrate this, let us consider Bogomolov's lemma, V.6.1, in this context, albeit that we first establish a little notation, i.e. k is the field of rational functions on B, the base is $\mathbb{C}$, generic fibres will be denoted by a subscript k, and irrespective of our current context we have the following blanket suppositions on the B-schemes, X/B whose Mordellicity we may wish to pursue, viz:

Assumptions VI.1.5. Notations as above then we suppose further that,

(a) X_k/k is smooth and geometrically integrable.

(b) X_k/k has general type

(c) X_k/k is not isotrivial.

In particular for Bogomolov's lemma, supposing the boundary D empty for purely notational simplicity, our set up is as follows. A non-singular 3-fold X is given together with a fibration $\pi : X \to B$ satisfying VI.1.5 with the additional hypothesis that $c_1(X_k)^2 > c_2(X_k)$, and X_k is minimal. We let L be the tautological bundle on $\mathbb{P}(\Omega^1_X)$, and observe that over the generic fibre we have an exact sequence,

$$0 \to \mathcal{O}_{X_k} \to \Omega_X \otimes_{\mathbb{C}} k \to \Omega_{X_k} \to 0\,. \tag{VI.1.6}$$

So that in particular, $s_t\,(\Omega^\vee_X \otimes_{\mathbb{C}} k) = s_t\,(\Omega^\vee_{X_k})$, where s_t is the Segre polynomial, and whence: $s_2\,(\Omega^\vee_X \otimes_{\mathbb{C}} k) = s_2\,(\Omega^\vee_{X_k}) > 0$. On the other hand from the point of view of the projective bundle, $\lambda : P = \mathbb{P}\,(\Omega_X \otimes k) \to X_k$ this says $L^4 > 0$, and so as ever by Riemann-Roch, $\chi\,(P, L^{\otimes n}) \sim \frac{n^4\,L^4}{4!}$, grows positively in n. We calculate the cohomology.

Firstly, note that, $H^i(P, L^{\otimes n}) \xrightarrow{\sim} H^i(X_k, \mathrm{Sym}^n\ \Omega_X \otimes k)$, since $R^j\,\lambda_*\,\mathcal{O}_P = 0$ for $j > 0$. So we need only concern ourselves with the H^2, which is isomorphic to $H^0(X_k, \mathrm{Sym}^n\ T_X \otimes K_{X_k} \otimes k)$. Whence we may conclude that L is big provided that T_X is not big. In fact we will prove,

Lemma VI.1.7. *Notations and assumptions as above then,* $h^0\,(X_k, \mathrm{Sym}^n\ T_X) = 0$, $\forall\, n \in \mathbb{N}$.

Proof. Suppose otherwise, and let C be a generic member of a sufficiently high multiple of K_{X_k}. Then $\Omega_{X_k}\mid_C$ is semi-stable, and $K_{X_k}.\,C > 0$ so in fact, $\Omega_{X_k}\mid_C$ is ample. Now choose the smallest $n \geq 1$ such that $h^0\,(X_k, \mathrm{Sym}^n\ T_X) \neq 0$, and consider the exact sequence,

$$0 \to \mathrm{Sym}^{n-1}\ \Omega_X \otimes k \to \mathrm{Sym}^n\ \Omega_X \otimes k \to \mathrm{Sym}^n\ \Omega_{X_k} \to 0$$

then if $n \geq 2$ we obtain an element of $H^0\,(X_k, \mathrm{Sym}^n\ T_{X_k})$, while if $n = 1$, given that $H^0\,(X_k, T_{X_k}) = 0$ the exact sequence VI.1.2 must split, and so X_k/k is isotrivial. Consequently, for a suitably generic C we obtain a map, from $S^n\,\Omega_{X_k}\mid_C$ to $\mathcal{O}_C$ which contradicts the aforesaid ampleness. $\square$

Now let us apply this observation. We know there is a positive integer n such that, $n\,L\,(-\lambda^*\,K_{X_k})$ is effective on the generic fibre, so for some divisor F, possibly not effective, but with support in the fibers of π we may find an effective divisor D in $\mathbb{P}\,(\Omega^1_X)$ such that,

$$n\,L = \lambda^*\,K_X + D + F$$

we may further note that $Y._{f'}\,F$ is bounded in modulus by $0\,((Y : B))$ where the implied constant depends only on X. Consequently if $f : Y/B \to X$ is a map of B-schemes corresponding to an algebraic point then if $f'\,(Y) \not\subset D$ the tautological identity yields,

$$K_{X\cdot f}\,Y \leq n\,(2g\,(Y) - 2) + 0\,((Y : B)) \tag{VI.1.8}$$

where of course $g\,(Y)$ is the genus of Y, and we're done by the compactness of the Hilbert-Scheme. Seemingly better still, we can treat the degenerate case $f'\,(Y) \subset D$ in light of VI.1.3, so that following a generic covering construction as per the immediate sequel to V.6.1, we ought to have a closed current T with exactly the same formal properties as that whose intersection with L we were able to bound below by way of the curvature calculation VI.6.2. Remarkably, this doesn't quite work. What we've said is absolutely true for suitable specialisations of the currents appearing in VI.1.3 at very general fibres, and indeed since all but finitely many exceptions to these contribute next to nothing to the numerical properties we may even content ourselves to say generically. However, at what may be termed the places of bad reduction, not just of the fibration but also of the current, we know nothing, and from the point of view of specialising currents, unlike co-homology classes, there's no reason that these bad fibres don't support all the intersection. Consequently, while accepting that this is a stupid, and undoubtedly resolvable problem, let us turn to a more classically algebro-geometric approach to Mordellicity, and see how the classification of foliations yields fruit, albeit that we now make the substantially weaker hypothesis of positive (infact non-negative would be sufficient in light of the classification theorem) topological index on the generic fibre, i.e. $c_1(X_k)^2 > 2\,c_2(X_k)$.

Before proceeding to the simplifications which this additional hypothesis permits, observe that for studying rational and algebraic points it wholly suffices for to obtain a bound akin to (VI.1.8) outside a closed subset Z of X_k- and indeed such a restriction is even necessary in the presence of generic rational or elliptic curves- whence without loss of generality we may assume that D dominates X. Necessarily there is an integer m and a line bundle M on X such that, $\mathcal{O}_P\,(D) \xrightarrow{\sim} L^{\otimes m} \otimes \lambda^*\, M^{\vee}$. We wish to calculate the intersection number, $L^3_{\cdot}\, D_k$ on the generic fibre. To this end we observe,

$$L^3_{\cdot}\, D_k = L^3_{\cdot}\,(m\,L - M) = m\,s_2\,(T_X \otimes k) - K_{X_k}.\,M\,.$$

On the other hand, D defines a map, $0 \to \lambda^*\,M \to L^{\otimes m}$, which we may push forward to obtain maps,

$$\begin{tikzcd}
\mathcal{O} \arrow[r] & M \arrow[r] \arrow[dr] & \mathrm{Sym}^m\,\Omega_X \otimes k \arrow[d] \\
 & & \mathrm{Sym}^m\,\Omega_{X_k} \arrow[d] \\
 & & 0
\end{tikzcd}$$

For D_k irreducible, the composite map $M \to \mathrm{Sym}^m\;\Omega_{X_k}$ is non-zero, and so we may apply K_{X_k} semi-stability of the cotangent bundle to obtain,

$$M \cdot K_{X_k} \leq \frac{m}{2}\, c_1\left(X_k\right)^2$$

and whence,

$$L^3 \cdot D_k \geq \frac{3m}{2}\, \tau\left(X_k\right) > 0\,.$$

Let us therefore examine the cohomology $H^i\left(D_k, L^{\otimes n}\right)$ for $D_k \hookrightarrow P_k$ irreducible. Profiting from the exact sequence,

$$0 \to L^{\otimes n}\left(-D_k\right) \to L^{\otimes n} \to L^{\otimes n} \mid_{D_k} \to 0$$

and taking n sufficiently large, we are reduced to excluding the possibility that $H^2\left(P_k, L^{\otimes n}\right)$ grows in dimension like n^3. Argueing as before we must simply exclude that $H^0\left(C, \operatorname{Sym}^n T_X\right)$ admits a similar estimate, for C a generic member of a suitable multiple of K_{X_k}. Proceeding exactly as in VI.1.7 we see that we are done unless $H^0\left(C, T_X\right) \neq 0$. This of course forces $T_X \mid_C$ to split, and better still the semi-stability of $\Omega_{X_k} \mid_C$ obliges,

$$0 \to \Omega_{X_k} \mid_C \to \Omega_X \otimes k \mid_C \to \mathcal{O}_C \to 0$$

to be the Harder-Narismhan filtration for the bundle $\Omega_X \otimes k$, and whence the tangent sheaf of the canonical model of X_k splits, which again implies that X_k is isotrivial. Thus we have deduced that for any irreducible divisor D_k in P_k dominating X_k, $L \mid_{D_k}$ is big. Thus should $f'(Y)$ be contained in such a divisor then again we obtain (VI.1.8) unless $f'(Y)$ is contained in a divisor within this divisor. Being prepared as ever to throw away closed subsets of X_k we thus obtain a subvariety Z_k of P_k, generically finite over X_k such that either (VI.1.8) holds or $f'(Y) \subset Z_k$. Taking a component S_k of Z_k and desingularising it we have therefore proved (VI.1.8) unless a dense set of sections (or sections on ramified covers to be precise) of $\pi : S \to B$ are integrable sub-curves of a possibly singular foliation by curves on the model S.

Now since all our maps $f : Y \to X$ whose derivatives find themselves in D lift to S without changing Y, we might as well say that $X = S$ provided we forget anything about Chern class inequalities and minimality. A foliation $\mathcal{F}$ by curves X has as ever a canonical bundle $K_{\mathcal{F}}$, and generically a singular sub-scheme Z supported in codimension 2 with ideal I_Z giving rise to a commutative diagram,

$$\begin{tikzcd}
\mathcal{O} \arrow[r] & \mathcal{O}_{X_k} \arrow[r] \arrow[dr] & \Omega_X \otimes k \arrow[r] \arrow[d] & \Omega_{X_k} \arrow[r] & 0 \\
 & & K_{\mathcal{F}} \otimes k\ I_Z \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

and we've established that our algebraic points considered as maps of curves $f : Y/B \to X/B$ are invariant by such a foliation. The advantage here is that the induced map, $\mathcal{O}_{X_k} \to K_{\mathcal{F}} \otimes k$ is non-zero, so $K_{\mathcal{F}} \otimes k$ is effective.

VI.2. An inequality for $K_{\mathcal{F}}$

Our set up is precisely the same as at the end of the last chapter, except that we profit from I.2.8 in order to choose a model $(X, \mathcal{F})$ which has generically canonical foliation singularities over the generic fibre, and to re-iterate, without any loss of generality (otherwise our goal would be trivial) we may assume that given an ample divisor H on X the ratios $\frac{1}{(Y_n:B)} H._{f_n} Y_n$, are unbounded for our set of rational points. Our goal will be in the spirit of V.4.4, i.e. to obtain a bound for the intersection of $K_{\mathcal{F}}$ along Y_n in terms of the genus and degree. Given such data, we may appeal to VI.1.1, and find for some subsequence $\mathcal{N}$ of our index set an accumulation point $\Phi_{\mathcal{N}}$ in the Néron- Severi group of the generic fibre, the calculation of whose degree precisely reflects the same for the Y_n, up to a small error. To proceed further, and following [M1], we introduce,

Definition VI.2.1. *Given a foliation by curves $(X, \mathcal{F})$ define its graph, $X(\mathcal{F})$ to be the associated divisor in $\mathbb{P}(\Omega^1_X)$, i.e.* $\mathrm{Proj}\left(\sum\limits_{n=0}^{\infty} K_{\mathcal{F}}.\, I_Z^n\right)$.

Now the graph is certainly birational to what we started with, but it may have underlying space singular so call, $\widetilde{X}(\mathcal{F})$ a resolution, which in our specific case we may insist to be minimal on the generic fibre. In this way we obtain for a singular foliation an infinite sequence of foliated 3-folds,

$$(X, \mathcal{F}) = (X_0, \mathcal{F}_0) \leftarrow (X_1, \mathcal{F}_1) \leftarrow (X_2, \mathcal{F}_2) \leftarrow \dots \qquad \text{(VI.2.2)}$$

such that $(X_i, \mathcal{F}_i)$ is obtained as the induced foliation on $\widetilde{X}_{i-1}(\mathcal{F}_{i-1})$ and if ρ_i from X_i to X_0 is the projection then, $\rho_i^* K_{\mathcal{F}} = K_{\mathcal{F}_i}$ on the generic fibre. Furthermore given a set of rational points on $(X, \mathcal{F})$ with the habitual hypothesis these lift to rational points on each $(X_i, \mathcal{F}_i)$ to which, modulo taking successively finer subsequences, we can associate homology classes Φ_0, Φ_1, Φ_2, etc. such that if ρ_{ij}, $i > j$, is the map from X_i to X_j then $(\rho_{ij})_* \Phi_i = \Phi_j$.

We apply these observations to the proof of

Theorem VI.2.3. *Let $(X, \mathcal{F})$ be a foliation by curves on a surface X/B with generically canonical foliation singularities then there is a proper closed subset Z of X such that for all $\varepsilon > 0$ there is a constant C_ε such that if $f : Y/B \to X/B$ is a map of B-schemes not factoring through Z which are integrable sub-curves of $\mathcal{F}$, then the following inequality holds,*

$$\frac{1}{\nu+1} K_{\mathcal{F}}\cdot_f Y \leq_{\mathrm{exc.}} (2g(Y) - 2) + \varepsilon\, H._f Y + C_\varepsilon (Y : B)$$

where H is ample, the subscript exc. denotes that the points are excluded from a finite set, and ν is the degree of the singularity with the most nilpotent structure.

Proof. By [J] and [V3] we may certainly assume that for any $\varepsilon > 0$ the exceptions to the inequality are dense.

Now suppose the contrary and consider the chain VI.3.3, then each X_i admits a map $\mathbb{P}(\Omega_{X_{i-1}})$ and as such is equipped with a tautological bundle L_i via pull-back. For each map f we trivially have,

$$L_{i\cdot_f} Y = (2g-2).$$

In addition for some divisor E_i contracted by $\rho_{i,i-1}$ we have that, $\rho_i^* K_{\mathcal{F}} = L_i(-E_i)$ on the generic fibre, and so for a constant C_i,

$$K_{\mathcal{F}\cdot_f} Y \leq_{\text{exc.}} (2g-2) + E_{i\cdot_f} Y + C_i (Y : B).$$

So that if $\varepsilon > 0$ is given, $E_{i\cdot_f} Y \geq_{\text{exc.}} \varepsilon\, \rho_i^* H_{\cdot_f} Y + \mathcal{O}_{\varepsilon,i}((Y:B))$. We can thus find homology classes Φ_i, which are in fact nef., such that $E_{i\cdot} \Phi_i \geq \varepsilon + \Delta\nu$, for all i, and which push forward one to the other, where Δ is the limit over the points of $2g-2/H_{\cdot_f} Y$, after suitable subsequencing.

Now there is no harm in a priori taking k sufficiently large to ensure that every generic singularity has as it's reduced structure a k-point. So for suitable coordinates x, y the completion in the singularity at the generic point has ring of functions $k[[x,y]]$, and for an appropriate function z on B, x, y, z is a coordinate system. Then I.2.8, or more precisely its proof with notations as in op.cit., gives the following possibilities for the singularities:

(a) $\partial = \partial_k + c\,\frac{\partial}{\partial z}$, where ∂_k is canonical for the k-foliation and without nilpotent structure and we identify $\mathrm{Der}_{\mathbb{C}}(k)$ with the module generated by $\partial/\partial z$. Then the exceptional curve just corresponds to a blow up in a smooth point, and this curve is in the support of the canonical map, $\mathbb{I} \to K_{\mathcal{F}}$.

(b) $\partial = \partial_k + c\,\frac{\partial}{\partial z}$, where ∂_k is canonical for the k-foliation but nilpotent, and the reduction of the divisor of c has simple normal crossings, with support in the branches of ∂_k. Here the exceptional divisor is simply a chain of rational curves obtained by blowing up in the reduced structure at the foliation singularities. The length of the chain is at most the degree of nilpotency of ∂_k and each element of the chain is again in the support of the canonical map from $\mathbb{I} \to K_{\mathcal{F}}$.

(c) $\partial = x\,\frac{\partial}{\partial x} + y^e\,\frac{\partial}{\partial z}$ in a suitable coordinate system. Again we obtain a chain of rational curves, e of them, obtained by blow ups in foliation singularities and of course the chain is contained in the support of the map from $\mathbb{I} \to K_{\mathcal{F}}$.

It follows from this description and the fact that the exact sequence (VI.1.2) splits on restriction to any algebraic point that if $\widetilde{E}_{i+1,k}$ is the proper transform in X_{i+1} of the k^{th}-exceptional divisor then,

$$\sum_k \widetilde{E}_{i+1,k\cdot_f} Y \leqq 0_i((Y:B)) + (2g(Y)-2)$$

for all algebraic points, which in terms of our homology classes becomes,

$$\sum_{\kappa} \widetilde{E}_{i+1,k}.\, \Phi_{i+1} \leq \Delta\,.$$

Now consider the difference between Φ_i and Φ_{i-1}, if we choose an orthogonal basis F_{ij} for the exceptional curves in the natural way, i.e. just pull-back to X_i the divisor corresponding to an exceptional curve which arises during any of the blowing ups from X_{i-1} to X_i, then for some positive real numbers α_{ij} we have,

$$\Phi_i = \rho^*_{i,i-1}\, \Phi_{i-1} - \sum_j \alpha_{ij}\, F_{ij}\,.$$

Better still with respect to this basis, $E_i = \sum_j F_{ij}$. Call E_{ij} good if its centre on X_{i-1} is contained in the proper transform of E_k in X_{i-1} for any $k \leq i-2$ and bad otherwise, then we have,

$$\sum_{j\,\text{good}} \alpha_{ij} \leq \nu\, \Delta$$

and so in particular,

$$\sum_{j\,\text{bad}} \alpha_{ij} \geq \varepsilon\,.$$

Now, not only is this true for all i, but the number of bad E_{ij} is bounded independently of i by twice the degree of the initial singular locus on X_0. Call this number d, then:

$$\sum_{j\,\text{bad}} \alpha_{ij}^2 \geq \frac{\varepsilon^2}{d}\,.$$

However since each Φ_i is nef., we conclude $\Phi_0^2 \geq \frac{i\varepsilon^2}{d}$ for any i, which is absurd. $\square$

VI.3. End of demonstration

Evidently we have now proved VI.1.8 in the case that the foliation is of general type, and so what remains is to carry out an analogue of our previous analysis should it be otherwise. We may certainly suppose after doing a little more blowing up that the divisor D, given via the natural map from $\mathbb{I} \to K_{\mathcal{F}}$ has simple normal crossings and of course $(X, \mathcal{F})$ has generically canonical foliation singularities. Observe further that one of our key formulas continues to hold, viz:

Formulas VI.3.1. (a) If the closure of a curve C in X_k in X is not an integrable hypersurface then,

$$(K_{\mathcal{F}} + C).\, C \geq 0\,.$$

Observe further that a weak analogue of adjunction also holds, viz:

(b) If a curve C in the support of D is an integrable hypersurface then,

$$K_{\mathcal{F}} \cdot C = (2g(C) - 2) + s_{\mathrm{sing}(\mathcal{F})}(C)$$

where the Segre class along the singular locus is defined as before. The formula (b) is less obvious and so we discuss it a little. Consider the restriction to the graph $\widetilde{X}$ of $(X, \mathcal{F})$ of the universal exact sequence in $\mathbb{P}(\Omega_X)$, i.e.

$$0 \to \widetilde{N} \mid_{\widetilde{X}} \to \Omega_X \mid_{\widetilde{X}} \to K_{\mathcal{F}}(-E) \to 0$$

where E is the exceptional Cartier divisor corresponding to blowing up in $Z = \mathrm{sing}(\mathcal{F})$. Since by hypothesis C is smooth the lifting $\widetilde{C}$ to X is isomorphic to C on the generic fibre and we have maps,

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \widetilde{N}|_{\widetilde{X}} \arrow[d] & & \\
\mathcal{O} \arrow[r] & \mathcal{O}_C(-C) \arrow[r] \arrow[ur] & \Omega_X|_{\widetilde{C}} \arrow[r] & \Omega_C|_{\widetilde{C}} \arrow[r] & 0
\end{tikzcd}$$

from which we certainly deduce a surjective map,

$$\Omega_C \mid_{\widetilde{C}} \to K_{\mathcal{F}}(-E) \mid_{\widetilde{C}} \to 0$$

and the hypothesis that C is in D forces the formula. Better still we obtain,

(c) If a curve C in the support of D is an integrable hypersurface, and $N \hookrightarrow \Omega_X$ is the rank 2 bundle defining the foliation then the natural map from $N \to \Omega_{C/B} \otimes k$ is zero.

This final observation yields the important

Lemma VI.3.2. *Two integrable hypersurfaces C, C' in the support of D can only meet generically in a foliation singularity.*

Proof. Suppose there is no singular point in the intersection then there is a neighbourhood U of C containing $C \cap C'$ which does not contain any foliation singularities which are flat over B. Whence if U_0 is the part of U over a suitable complement of points in B, we have,

$$c_1(C' \mid_{U_0}) \in \mathrm{Im}\{H^1(U_0, N) \to H^1(U_0, \Omega_X)\}.$$

Now U_0 contains $C \otimes k$ and so, $c_1(C' \mid_{C \otimes k})$ is in the image of the natural map from $H^1(U_0, N)$ to $H^1(C \otimes k, \Omega_{C \otimes k/k})$ which is zero, and this is absurd. Otherwise there are n singular points $z_1, \ldots, z_n$, say, in the intersection, which by construction is simple so if we blow up in these points to obtain $\widetilde{C}$ and $\widetilde{C'}$ say then by the above $\widetilde{C} \cdot \widetilde{C'} = 0$, and so $C \cdot C' = n$ as required. □

It is equally true that a self evident variant of this lemma would imply that if a rational integral $\mathcal{F}$-hypersurface C were contained in D, it must meet the singular locus to prevent X being fibered by $\mathbb{P}^1$'s. Given all of this the analysis of III carries over verbatim, – the obstruction group for doing the residue theory is just $H^1_Z(X \otimes k, K_{\mathcal{F}} \cdot I_Z)$. The only important thing to observe is that the obstruction group is one dimensional when it should be, i.e.

Lemma VI.3.3. *If two integrable hypersurfaces C, C' in the support of D meet in a foliation singularity, then that singularity is without nilpotent structure provided C and C' have local Segre class* 1 *along the singular locus of the foliation.*

Proof. We may take, in appropriate local coordinates, C, C', to be $x = 0$, and $y = 0$ respectively, so that if z is a coordinate on the base then the hypothesis oblige that we can write the derivation defining the foliation as,

$$\alpha\, x \, \frac{\partial}{\partial x} + y\, \beta \, \frac{\partial}{\partial y} + x\, y\, \gamma \, \frac{\partial}{\partial z}$$

since the respective Segre classes are 1, this forces α and β to be units. □

The corollary of the lemma is therefore that our residue calculations continue to work, viz:

Corollary VI.3.4. *Suppose $\{C_i\}$ is a collection of integrable hypersurfaces contained in the support of D such that for any C_i the local Segre class of the foliation singularities is everywhere generically* 1 *and the number of C_i exceeds the number of singularities they contain then the intersection matrix $\{C_i \cdot C_j\}$ cannot be negative definite.*

Proof. As per I.3.2 denote by W the set of singularities contained in $\bigcup\limits_i C_i$ then for each $w \in W$ there is a map Res_w from the vector space V in $H^1\left(X_k, \Omega^1\right)$ generated by the C_i to the local obstruction group, which by the lemma is isomorphic to k. The total obstruction group in a neighbourhood of the C_i has dimension at most $\#\, W$ so there are $\lambda_i \in k$, not all zero, such that, $\lambda_1\, C_1 + \ldots + \lambda_{k_n}\, C_{k,n} \in \mathrm{Im}\left\{H^1\left(U_0, N\right) \to H^1\left(U_0, \Omega_X\right)\right\}$ for a suitable neighbourhood $U_0 \supset \bigcup\limits_i C_i \otimes k$ as in IV.4.2. Thus argueing as therein, for any j,

$$\sum_i \lambda_i \left(C_i \cdot C_j\right) = 0$$

since the λ_i are not all zero, $\{C_i \cdot C_j\}$ cannot be negative definite.

Now we can conclude to the theorem if the foliation has general type. We certainly know that it cannot have Kodaira dimension 0, since à la IV.3

some cover of the minimal model would be isotrivial. Whence we are reduced to consider the possibility of numerical Kodaira dimension 1. To this end let $(S, \mathcal{G})$ be a model over B of the minimal model then the argument of IV.4.1 forces $(S_k, \mathcal{G}_k)$ to be $\mathbb{Q}$-foliated Gorenstien, and indeed if $p : X \to S$ is the projection then,

$$p_* \left(D_k\right) = K_{\mathcal{G}_k} = \sum_i a_i \, D_i \otimes k$$

where D_i are irreducible $\mathbb{Q}$-Cartier, and $D_i \cdot D_j = 0$ for all possible combinations of i and j. Since we are supposing that the generic fibre of X has general type the situation is marginally easier than before. In particular suppose one of the D_i were an integrable hypersurface then generically $p^* \, D_i = \sum_j b_j \, E_j$, say, is a normal crossing divisor with the crossings occuring only in the foliation singularities. Moreover it is nef. of square 0 so we have the formulae,

$$-\, b_j \, E_j^2 = \sum_{k \neq j} b_k \, E_k \cdot E_j$$

$$(K_X - K_{\mathcal{F}}). \, E_j \leq -\, E_j^2 - \sum_{k \neq j} E_k \cdot E_j$$

for all j. So that multiplying the latter by b_j and combining we obtain, $K_X \cdot p^* \, D_i \leq K_{\mathcal{F}} \cdot p^* \, D_i = 0$, which is absurd since K_X is big. Consequently without loss of generality no D_i is an integrable hypersurface, and argueing as in the non-invariant case of IV.5.1 bis (all we need to do is work in the relative Picard group over the generic fibre) we may deduce that the actual Kodaira dimension is 1. In particular we obtain, possibly after modifying some fibres over B, a map $q : S \to C$ where the generic fibre of C over B is a smooth curve, $K_{\mathcal{F}} \otimes k$ is the pull-back of a $\mathbb{Q}$-Cartier divisor on C, the fibres are of course connected so that any $D_i \otimes k$ is in one of the fibres. At this point we can make an almost étale cover $\nu : (\widetilde{S}, \widetilde{\mathcal{G}}) \to (S, \mathcal{G})$ as in IV.4.3 such that the covering is generically foliated Gorenstien. Since the curves f_n may meet the singularities of S this may lead to ramification. This is, however, bounded by a constant times the multiplicity at the singular point. Even better all the singularities lies in D, so if in the notations of VI.2, $\widetilde{\Phi}$ lifts Φ and $\widetilde{\Delta}$ is the discriminant and $K_{\mathcal{F}} \cdot \Phi = \Phi$ so $\widetilde{\Delta} \leq 2\Delta$, whence moduo leaving the $-2\,F$ curves alone we might as well say $(X, \mathcal{F}) = (S, \mathcal{G})$. On the other hand the singular locus of $\mathcal{F}$ is contained in D, so by VI.4.1 (a) it is generically empty, and of course $K_{\mathcal{F}}$ is the pull-back of a Cartier divisor on C over the generic point. Consider therefore the natural map,

$$q^* \, \Omega_C \otimes k \to q^* \, K_{\mathcal{F}} \otimes k$$

then since no component of D is an integrable hypersurface, this map is actually surjective, and indeed is the pull-back of a surjective map $\Omega_C \otimes k \to K_{\mathcal{F}} \otimes k$ on C. Let us call the kernel M then $q^* \, M$ maps to N under pull-back of differentials. Necessarily then the image of any $f : Y/B \to X/B$ in C is an integrable sub-curve of the foliation by curves so defined in C. Since without loss of generality these images are dense by a result of Jounalou, [J], the foliation on C must be a pencil. Now if we were only interested in rational points, then C would have to be iso-trivial, so without loss of generality $C \xrightarrow{\sim} C_1 \times B$, for some curve C_1, whence $K_{\mathcal{F}} \otimes k$ would be trivial forcing X iso-trivial. In general however, especially as our covering construction may have introduced ramification, we let $\nu : C \dashrightarrow C_1$ be the Stein factorisation of the corresponding pencil, and note that the image of an algebraic point of interest is constant in C_1, and whence for some $c \in C_1$, corresponds to a map, $f : Y \to X_c$. Observe that by abundance and base point freeness for 3-folds together with [Mi1] we may assume without loss of generality that the total space X has general type, so modulo excluding a suitable closed subset we may suppose X_c is a smooth surface of general type. Furthermore $K_X \mid_{X_c}$ is numerically equivalent to X_c, so consider a minimal model X_0 of X_c, then by [Mi1],

$$K_{X_0} \cdot_f Y \leq a\,(c_1\,(X_0)^2, c_2\,(X_0)) + b\,(2g\,(Y) - 2)$$

where a is a function of $c_1\,(X_0)^2$, $c_2\,(X_0)$ and b is a constant. Now observe that the number of monoidal transformation required to obtain X_c from X_0 is at most $c_2\,(X_c)$ and that the 5-canonical map is very ample on the canonical model of X_c, so that for some functions a', b' of $c_1\,(X_c)^2$ and $c_2\,(X_c)$ we obtain,

$$K_X \cdot_f Y \leq a' + b'\,(2g\,(Y) - 2)$$

the chern numbers are of course constant in a family, and so we may easily conclude to VI.1.8. In the specific case of rational points the usual logarithmic Miyaoka-Yau inequality works equally well since the singularities that may occur in lifting to the almost étale cover are bounded in complexity as ever by the proof of VI.4.

An inequality of the form VI.1.8 easily excludes that the rational points are dense, since it forces them to be contained in finitely many families, one of which must dominate X should it be otherwise, and whence X would be isotrivial.

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