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BABY GEOGRAPHY AND MORDELLICITY OF SURFACES

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Abstract. We prove strong Mordell for surfaces of general type and non-negative index over characteristic zero function fields by way of a, probably, more interesting lemma.

1 Introduction

There is a programme to establish the conjectured hyperbolicity of surfaces of general type, [McQ]. Their similarly conjectured Mordellicity over characteristic zero function fields is in principle amenable to the same strategy thanks to the technique introduced in the proof of Vojta's $1+\epsilon$ conjecture over function fields, [McQ13]. Nevertheless, the said technique which yields a derivative over the function field, and not just its base point, is delicate when there is bad reduction. As such, for the purpose of the present, we proceed classically by way of differentiating with respect to the base point, and make a start on the Mordell problem for surfaces as a function of their geography by way of:

Theorem 1.1. *Let $X\otimes K/K$ be a geometrically integral smooth minimal surface of general type and non-negative index, i.e. $c_1^2(X\otimes K)\geq 2c_2(X\otimes K)$ over a field of functions in characteristic zero. Denote by $Z\subset X\otimes K$ the proper Zariski closed subset whose geometrically irreducible components are rational or elliptic curves over $\overline{K}$, then there are constants $-\kappa(X\otimes K)<0$, and $\alpha(X\otimes K)\geq 0$ such that any algebraic point f of $X\otimes K\backslash Z$ satisfies:*

$$h_{\omega_{X\otimes K/K}}(f)\leq \kappa\mathrm{discr}(f)+\alpha. \tag{1}$$

In particular (cf. A) the K-rational points of $X\otimes K\backslash Z$ are contained in finitely many sub-varieties whose normalisations are iso-trivial.

The corresponding hyperbolicity theorem is due to Steven Lu, [Lu91], and is, in essence, Miyaoka's almost ampleness theorem, [Miy83]. By way of a standard variation, Section 1, on this theme we reduce, 1.4, along the lines that these references suggest, the theorem to the following, LEMMA 1.2. *Let $X \otimes K/K$ be a geometrically integral smooth surface over a field of functions in characteristic zero. Denote the base point by k, i.e. k is algebraically closed and K/k are the rational functions on a curve over k, and let $Y \subset \mathbb{P}(\Omega_{X\otimes K/k})$ be closed with every generic point finite over $X \otimes K$, then there are constants $-\kappa(X \otimes K, Y) < 0$, $\alpha(X \otimes K, Y) \geq 0$, and a proper closed subset $W \subset X \otimes K$, such that any algebraic point f on $X \otimes K \backslash W$ whose derivative f' over k belongs to Y satisfies:*

$$h_{\omega_{X\otimes K/K}}(f) \leq \kappa \mathrm{discr}(f) + \alpha. \tag{2}$$

Arguably, the lemma is more interesting than the theorem since it has no restrictions on chern numbers, and, so, has the potential to be applicable under less restrictive geographic constraints. In neither case are there restrictions on the degree $(K(f) : K)$, and in either case we work with normalised heights and discriminants. Strictly speaking the former depends on the choice of a model X/S for S/k smooth proper with function field K. Much of the time this is un-important, and the height notation is less cumbersome. Occasionally, however, there may be some hidden dependence, and when there is, notably 3.2–3.3, and the end of the proof, it is spelt out. Similarly, we use the dualising sheaf notation, $\omega_{X\otimes K/K}$, *etc.*, even though this is a bundle, so avoiding $K_{X\otimes K/K}$, and up to normalising by the reciprocal of $(K(f) : K)$, height along a bundle just means degree along the curve $f : T \to X$ in the model corresponding to the point, while discriminant means (geometric) genus of T, (14).

Finally let us give a guide to the proof of the lemma. After base change, one may suppose that Y is the union of its geometrically integral components. By hypothesis, each algebraic point in question lifts to at least one such by way of its derivative over k. Fixing our attention on such a component we may therefore replace X by it, and the condition $f' \in Y$ amounts to f is invariant by a foliation by curves $\mathscr{F}$. The key point is to bring the canonical bundle $K_{\mathscr{F}}$ of forms along this foliation into play, which is done by an optimal height estimate, 3.3, along $K_{\mathscr{F}}$, akin to those whether in the theorem, (1), or the lemma, (2), *i.e.* $\kappa = 1 + \epsilon$, such optimality being, incidentally, false, [ACLG12], at the level of the theorem. This statement needs generic resolution of foliation singularities, and is false otherwise, equivalently without resolution $K_{\mathscr{F}}$ should be written $\omega_{\mathscr{F}}$ and carries little information. Regardless for trivial reasons $K_{\mathscr{F}}$ is effective, and a little less trivially it has, Fact 3.6, a particularly elegant Zariski decomposition over the generic point, *i.e.* its nef. part is of the form $K_{\mathscr{F}_0}$ for $\mathscr{F}_0$ the induced foliation on some $X_0 \otimes K$ with quotient singularities obtained from contracting chains of invariant rational curves, $X \otimes K \to X_0 \otimes K$. By 3.3 we are therefore, 3.7, reduced to studying the situation where $K_{\mathscr{F}_0}$ has numerical Kodaira dimension 1. Since it is effective, much the same argument as employed in the classification theorem of foliated surfaces over a point establish that its Kodaira dimension is 1. Thus, while not logically necessary, it is very useful, and perhaps even desirable, to understand a priori how the proof goes of the related statement in the classification, [McQ08, IV.4.1–IV.5.2], of foliated surfaces over a point, *i.e.* the (foliated) minimal model of a foliated surface of general type with numerical (foliated) Kodaira dimension 1 and an effective canonical divisor is (either) of the natural foliations on a quotient of a product of curves by a finite group, since everything post the proof of 3.3, is a variant on this. Indeed, once the Kodaira fibration $X \otimes K \to B \otimes K$ of $K_{\mathscr{F}}$ is found it's even (76)–(77), a map of foliated varieties $(X \otimes K, \mathscr{F}) \to (B \otimes K, \mathscr{G})$, and by [Jou78], the latter must be a pencil of curves $B \to C$ over k. In terms of models, this forces the algebraic points to lie in the $\mathscr{F}$-invariant fibres $X \to C$, where the induced foliation is a quotient of a product of curves, and one concludes by adjunction (81)–(83).

Thus, relative to the geography, the proof is involved, which suggests that k rather than K differentiation is un-sustainable under less restrictive hypothesis.

2 Riemann-Roch Calculations

We have both an absolute base field which is algebraically closed of characteristic zero, say, $\mathbb{C}$ for convenience, rather than k, and the function field $K = \mathbb{C}(S)$ of a smooth projective curve S. Our interest is smooth surfaces over K, so, say, a proper family $X \to S$ of surfaces, which is generically smooth and geometrically integral, with, for convenience X non-singular. Over the generic fibre, we have a short exact sequence on $X \otimes K := X \times_S \operatorname{Spec}(K)$:

$$0 \to \Omega_{S/\mathbb{C}} \otimes K \xrightarrow{\sim} \mathscr{O}_{X\otimes K} \to \Omega_{X\otimes K/\mathbb{C}} \to \Omega_{X\otimes K/K} \to 0 \tag{3}$$

where, here, and elsewhere the superscript 1, will be omitted from the notation. In particular, (3) yields $s_t\,(\Omega^{\vee}_{X\otimes K/\mathbb{C}}) = s_t(\Omega^{\vee}_{X\otimes K/K})$, for s_t the Segre polynomial, so: $s_2\,(\Omega^{\vee}_{X\otimes K/\mathbb{C}}) > 0$, if $s_2\,(\Omega^{\vee}_{X\otimes K/K}) > 0$. At the level of the projective bundle,

$$P_{X\otimes K/\mathbb{C}} := \mathbb{P}(\Omega_{X\otimes K/\mathbb{C}}) = \mathbb{P}(\Omega_{X/\mathbb{C}}) \times_S \operatorname{Spec}(K) \tag{4}$$

with projection π to X and tautological bundle L, this says $L^4_{X\otimes K/\mathbb{C}} > 0$. Thus:

$$\chi\left(P_{X\otimes K/\mathbb{C}}, L^{\otimes n}_{X\otimes K/\mathbb{C}}\right) \sim \frac{n^4}{4!}\, L^4_{X\otimes K/\mathbb{C}} \gg n^4, \tag{5}$$

grows positively in n, for $c_1(X \otimes K)^2 > c_2(X \otimes K)$. We calculate the cohomology,

$$\mathrm{H}^i\left(P_{X\otimes K/\mathbb{C}}, L^{\otimes n}_{X\otimes K/\mathbb{C}}\right) \xrightarrow{\sim} \mathrm{H}^i(X \otimes K, \operatorname{Sym}^n \Omega_{X\otimes K/\mathbb{C}}), \tag{6}$$

where we profit from $R^j\pi_* L^{\otimes n}_{X\otimes K/\mathbb{C}} = 0$ for $j > 0$. To guarantee large h^0, we need only control H^2, which is isomorphic to $\mathrm{H}^0(X\otimes K, \operatorname{Sym}^n T_{X\otimes K/\mathbb{C}} \otimes \omega_{X\otimes K/\mathbb{C}})$. Whence we'll conclude that L is big provided that $T_{X\otimes K/\mathbb{C}}$ is not big. In fact,

Lemma 2.1. *Suppose that the generic fibre $X \otimes K$ is minimal of general type, and the family is not iso-trivial, then,* $\mathrm{h}^0\,(X \otimes K, \operatorname{Sym}^n T_{X\otimes K/\mathbb{C}}) = 0$, $\forall\, n \in \mathbb{Z}_{>0}$.

Proof. Suppose otherwise, and let C be a generic member of a sufficiently high multiple of $\omega_{X\otimes K/K}$. Then $\Omega_{X\otimes K/K}|_C$ is semi-stable, [Kob87, Theorem 8.3], and $\omega_{X\otimes K/K}\cdot C>0$ so in fact, $\Omega_{X\otimes K/K}|_C$ is ample, and whence

$$\mathrm{H}^0((X\otimes K,\mathrm{Sym}^n T_{X\otimes K/K})=0,\quad \forall n\in\mathbb{Z}_{>0}. \tag{7}$$

Now suppose there is a minimal $n\geq 1$ such that $\mathrm{h}^0\left(X\otimes K,\mathrm{Sym}^n T_{X\otimes K/\mathbb{C}}\right)\neq 0$, and consider the exact sequence afforded by (3),

$$0\to\mathrm{Sym}^{n-1}\Omega_{X\otimes K/\mathbb{C}}\to\mathrm{Sym}^n\Omega_{X\otimes K/\mathbb{C}}\to\mathrm{Sym}^n\Omega_{X\otimes K/K}\to 0 \tag{8}$$

then if $n\geq 2$ we obtain an element of $\mathrm{H}^0\left(X\otimes K,\mathrm{Sym}^n T_{X\otimes K/K}\right)$. By (7) this is nonsense, while again by (7) if $n=1$: $\Omega_{X\otimes K/\mathbb{C}}$ must be a split extension of $\Omega_{X\otimes K/K}$ by $\mathscr{O}_{X\otimes K}$, and so $X\otimes K/K$ is isotrivial. □

Consequently, if $X\otimes K/K$ is not iso-trivial,

$$\mathrm{h}^0(L^{\otimes n}_{X\otimes K/\mathbb{C}})\text{ grows like } n^4 \text{ for } s_2=c_1^2-c_2>0 \text{ on the generic fibre, so:} \tag{9}$$

FACT 2.2. *Let $X\otimes K/K$ be a surface of general type with $s_2(X\otimes K)>0$, then for H ample on X, there are constants $-\kappa(X)<0$, $\alpha(X)>0$, and a divisor $D\subset P_{X\otimes K/\mathbb{C}}$ such that any algebraic point f satisfies,*

$$\text{(a) } h_H(f)\leq\kappa\,\mathrm{discr}(f)+\alpha \text{ or (b) } f'\in D \tag{10}$$

wherein the heights and discriminants are understood to be normalised by the degree, cf. (14).

Proof. The iso-trivial case is in [Bog77], so we suppose otherwise, and write the algebraic point as $f:T\to X$, for $T\to S$ finite, $T/\mathbb{C}$ smooth. It has a derivative,

$$f':T\to\mathbb{P}(\Omega_{X/\mathbb{C}}) \tag{11}$$

and for L the tautological bundle on the latter:

$$L\cdot_{f'}T\leq(2g-2) \tag{12}$$

for g the genus of T. By (9) there is some $n\in\mathbb{Z}_{>0}$ such that $nL_{X\otimes K/\mathbb{C}}-\pi^*H$ is effective, say, D. Equivalently for $\overline{D}$ the closure in $\mathbb{P}(\Omega_{X/\mathbb{C}})$, there are divisors $F_+,F_-\geq 0$ supported in fibres of $X\to S$ such that,

$$nL=\pi^*H+\overline{D}+F_+-F_-. \tag{13}$$

The proposition is intended with normalised heights and discriminants, *i.e.*,

$$h_H(f)=\frac{H\cdot_f T}{(T:S)},\ \mathrm{discr}(f)=\frac{2g-2}{(T:S)}\text{ etc.} \tag{14}$$

so taking intersections with L we conclude. □

To improve this when the index is non-negative, requires

Lemma 2.3 (cf. [Miy83], [Lu91]). *Let $F/\mathbb{C}$ be a smooth minimal surface of general type and non-negative topological index with $Y \hookrightarrow \mathbb{P}(\Omega_{F/\mathbb{C}}) \xrightarrow{\pi} F$ a divisor each generic point of which dominates F then for L the tautological bundle on $\mathbb{P}(\Omega_{F/\mathbb{C}})$*

$$L^2 \cdot Y > 0 \tag{15}$$

unless the universal cover of F is a bi-disc.

Proof. Let $\rho : F \to F_0$ be the canonical model, then F_0 admits, [TY87], a Kähler-Einstein metric in the orbifold sense, *i.e.* on a smooth champ de Deligne-Mumford $p : \mathscr{F} \to F_0$ almost étale and bi-rational over its moduli F_0. Consequently, [Kob87, Theorem 8.3], $\Omega_{\mathscr{F}/\mathbb{C}}$ is K_{F_0} semi-stable, so by Hartogs' extension $\Omega_{F/\mathbb{C}}$ is $K_F = \rho^* K_{F_0}$ semi-stable. We have, however, the tautological sequence

$$0 \to N \to \pi^* \Omega_{F/\mathbb{C}} \to L|_Y \to 0 \tag{16}$$

so by hypothesis and K_F semi-stability

$$0 \le (Y : F)(\mathrm{c}_1^2 - 2\mathrm{c}_2) = (L|_Y)^2 + N^2 \le 2(L|_Y)^2. \tag{17}$$

As such we're done unless for every irreducible component Y' of Y:

$$0 = (Y' : F)(\mathrm{c}_1^2 - 2\mathrm{c}_2) = 2(L|_{Y'})^2 = 2(N|_{Y'})^2 \tag{18}$$

and without loss of generality Y is irreducible. It is defined by an injection

$$0 \to M \to \mathrm{Sym}^n \Omega_{F/\mathbb{C}}, \quad n \in \mathbb{Z}_{>0} \tag{19}$$

where M is a line bundle, and (16) is saturated, while by K_F semi-stability

$$K_F \cdot M \le \frac{n}{2} K_F^2 \tag{20}$$

so from

$$(L|_Y)^2 = n\mathrm{s}_2(Y) - K_F \cdot M, \tag{21}$$

we're equally finished unless we have equality in (20).

Now by Hartogs' extension (16) and (19) have analogues for $\Omega_{\mathscr{F}/\mathbb{C}}$ for line bundles $\mathscr{L}$, $\mathscr{N}$, $\mathscr{M}$ say, and a divisor $\mathscr{Y} \hookrightarrow \mathbb{P}(\Omega_{\mathscr{F}/\mathbb{C}})$ in the obvious notation whose pull-backs (as $\mathbb{Q}$-divisors after resolving any indeterminacy from Y to the moduli of $\mathscr{Y}$ if necessary) may well be different from their analogues over F, but their intersection numbers with $K_{\mathscr{F}} = p^* K_{F_0}$ are the same. In particular from (17)

$$K_{\mathscr{F}} \cdot \mathscr{L}|_{\mathscr{Y}} = K_F \cdot L|_Y = K_F \cdot N = K_{\mathscr{F}} \cdot \mathscr{N} > 0 \tag{22}$$

we must have $(\mathscr{L}|_{\mathscr{Y}})^2 = \mathscr{N}^2$. As such Bogomolov-vanishing, [Bog78]; (19) and the first equality in (17) combine to

$$0 \ge \mathscr{N}^2 = (\mathscr{L}|_{\mathscr{Y}})^2 = n\mathrm{s}_2(\mathscr{Y}) - K_{\mathscr{F}} \cdot \mathscr{M} = n(\mathrm{c}_2(F) - \mathrm{c}_2(\mathscr{F})). \tag{23}$$

If, however, ρ is non-trivial the rightmost term in (23) is strictly positive- each contraction of a connected component of the exceptional divisor reduces the topological Euler characteristic by at least 3/2- so we must have $\mathscr{F} = F_0 = F$; K_F is ample; (19) is an injection of line bundles by [Kob87, Theorem 8.3]; and $\pi : Y \to F$ is finite.

On the other hand we have a (non-zero since we're in characteristic 0) composition

$$\pi^* M \otimes L^{-n}|_Y = \mathscr{O}_Y(-Y) \to \Omega_{\mathbb{P}(\Omega_{F/\mathbb{C}})/\mathbb{C}} \to \Omega_{\mathbb{P}(\Omega_{F/\mathbb{C}})/F} = \pi^* K_F \otimes L^{-2} \qquad (24)$$

vanishing along some (possibly empty) divisor R which by (18) and equality in (20) has nil $\pi^* K_F$ degree, so, indeed R is empty since π is finite and K_F is ample. Consequently the co-kernel of the leftmost map, Ω_Y, in (24) is isomorphic to the kernel of the rightmost, $\pi^*\Omega_X$, while Y is l.c.i., and F is smooth whence π is also flat, so it's an étale cover. Finally, therefore, we can replace $\pi^*\Omega_F$ in (16) by the co-tangent bundle of the smooth surface Y, so, by (17) and [Kob87, Theorem 8.3], (16) is actually a split exact sequence, and the universal cover of Y, which is equally that of F, is a bi-disc. □

We can, therefore, improve Fact 2.2 to

Fact 2.4. *Again let $X \otimes K$ be a non-isotrivial minimal surface of general type, but now with non-negative topological index, τ, equivalently $c_1(X \otimes K)^2 \geq 2c_2(X \otimes K)$, then for any divisor, $D \subset P_{X\otimes K/\mathbb{C}}$, each generic point of which dominates $X \otimes K$ $L_{X\otimes K/\mathbb{C}}|_D$ is big, unless, after a base change in K, $X \otimes K$ admits an étale cover by a product of curves.*

Proof. We may suppose D is geometrically irreducible. There is an integer m and a line bundle M on $X \otimes K$ such that, $\mathscr{O}_P(D) \xrightarrow{\sim} L^{\otimes m}_{X\otimes K/\mathbb{C}} \otimes \pi^* M^\vee$. We wish to calculate the intersection number, $L^3_{X\otimes K/\mathbb{C}} \cdot D$ on the generic fibre. To this end, observe:

$$L^3_{X\otimes K/\mathbb{C}} \cdot D = L^3_{X\otimes K/\mathbb{C}} \cdot (m\, L_{X\otimes K/\mathbb{C}} - M) = m\, s_2\,(T_{X\otimes K/\mathbb{C}}) - \omega_{X\otimes K/\mathbb{C}} \cdot M. \qquad (25)$$

The map, $M \to L^{\otimes m}_{X\otimes K/\mathbb{C}}$, defined by D, may be pushed forward and we obtain a composition:

$$M \to \mathrm{Sym}^m \Omega_{X\otimes K/\mathbb{C}} \to \mathrm{Sym}^m \Omega_{X\otimes K/K}, \qquad (26)$$

while D irreducible implies the composite map $M \to \mathrm{Sym}^m\, \Omega_{X\otimes K/K}$ is non-zero, so we may saturate the composition (26) to obtain a divisor $Y \hookrightarrow \mathbb{P}(\Omega_{X\otimes K/K})$ each component of which dominates $X \otimes K$ and which (employing $L_\bullet$ for tautological bundles) is linearly equivalent to $mL_{X\otimes K/K} - \bar{M}$ for some line bundle $\bar{M} \geq M$ on $X \otimes K$. Plainly, by (3), $\omega_{X\otimes K/\mathbb{C}} = \omega_{X\otimes K/K}$, is nef so by (25)

$$L^3_{X\otimes K/\mathbb{C}} \cdot D - L^2_{X\otimes K/K} \cdot Y = \omega_{X\otimes K/K}(\bar{M} - M) \geq 0. \qquad (27)$$

On the other hand: by Weil rigidity, [Wei60], a non-isotrivial smooth bi-disc quotient is an étale quotient of a product of curves, so by hypothesis, and 2.3:

$$L^3_{X\otimes K/\mathbb{C}}\cdot D > \frac{3m}{2}\,\tau\,(X\otimes K)\geq 0. \tag{28}$$

To determine the cohomology $\mathrm{H}^i\,(D, L^{\otimes n}_{X\otimes K/\mathbb{C}})$, we use the exact sequence,

$$0\to L^{\otimes n}_{X\otimes K/\mathbb{C}}\,(-D)\to L^{\otimes n}_{X\otimes K/\mathbb{C}}\to L^{\otimes n}_{X\otimes K/\mathbb{C}}|_D\to 0 \tag{29}$$

so, on taking n sufficiently large, we are reduced to excluding the possibility that $\mathrm{H}^2\,(P_{X\otimes K/\mathbb{C}},\, L^{\otimes n}_{X\otimes K/\mathbb{C}})$ grows in dimension like n^3. Arguing as before 2.1 we must simply exclude that $\mathrm{H}^0\,(C, \mathrm{Sym}^n\,T_{X\otimes K/\mathbb{C}})$ grow like n^3, for C a generic member of a suitable multiple of $\omega_{X\otimes K/\mathbb{C}}$. Proceeding exactly as in the proof of 2.1 we see that we are done unless $\mathrm{H}^0\,(C, T_{X\otimes K/\mathbb{C}})\neq 0$. This of course forces $T_{X\otimes K/\mathbb{C}}|_C$ to split, and better still the semi-stability of $\Omega_{X\otimes K/K}|_C$ obliges,

$$0\to \Omega_{X\otimes K/K}|_C\to \Omega_{X\otimes K/\mathbb{C}}|_C\to \mathscr{O}_C\to 0$$

to be the Harder-Narismhan filtration for the bundle $\Omega_{X\otimes K/\mathbb{C}}$, and whence the tangent sheaf (over $\mathbb{C}$) of the orbifold covering of the canonical model of $X\otimes K$ encountered at the beginning of the proof of 2.3 splits, which again implies that $X\otimes K$ is isotrivial. □

We freely base change in K as necessary. The components of D appearing in (10)(b) which don't dominate $X\otimes K$, are the pull-backs of curves. If these curves aren't rational or elliptic, the height bound (10)(a), for a possibly different κ, is easy, *e.g.* [McQ13] is optimal, but [Voj91] is sufficient. Similarly lifting algebraic points to a covering étale over K changes the discriminant at most by a constant, so, again either of the above height bounds for curves over K gives the height bound (10)(a), and whence Fact 2.4 refines Fact 2.2 to:

FACT 2.5. *Let $X\otimes K/K$ be a surface of general type with $\tau(X\otimes K)\geq 0$, then for H ample, there are constants $-\kappa(X)<0$, $\alpha(X)>0$, and a sub-scheme $Y\subset P_{X\otimes K/\mathbb{C}}$, finite over $X\otimes K$ at each of its generic points such that any algebraic point f not belonging to the at worst finite set of rational or elliptic curves on $X\otimes K/K$ satisfies,*

$$\text{(a) } h_H(f)\leq \kappa\,\mathrm{discr}(f)+\alpha \text{ or (b) } f'\in Y. \tag{30}$$

3 Foliations by Curves

The goal is to prove that on surfaces of general type and non-negative index all algebraic points satisfy (30)(a). The obstruction is (30)(b), so (b) implies (a) will do. This no longer has anything to do with chern numbers, and we assert, Lemma 3.1. *Let $X \otimes K/K$ be a surface of general type and $Y \subset P_{X\otimes K/\mathbb{C}}$ a sub-scheme finite over $X \otimes K$ at each of its generic points, then for H ample there are constants $-\kappa(X,Y) < 0$, $\alpha(X,Y) > 0$ such that algebraic points f with $f' \in Y$ excluded from a proper closed subset in $X \otimes K/K$ satisfy,*

$$h_H(f) \leq \kappa \operatorname{discr}(f) + \alpha. \tag{31}$$

The proof will occupy the rest of the manuscript. Plainly we may suppose Y geometrically irreducible, and the points f are Zariski dense. The condition $f' \in Y$ is a first order O.D.E.. Since the points f also lift to Y by differentiation, we can replace X by Y without changing their discriminants, so without loss of generality the O.D.E. is linear, *i.e.* it's a foliation by curves $\mathscr{F}$. The condition $f' \in Y$ may thus be replaced by f is invariant by $\mathscr{F}$, which in turn is given by a short exact sequence,

$$0 \to \Omega_{X/\mathscr{F}} \to \Omega_{X\otimes K/\mathbb{C}} \to K_{\mathscr{F}} I_Z \to 0 \tag{32}$$

where the kernel is reflexive rank 2, $K_{\mathscr{F}}$ is the bundle of forms along the leaves, and Z the (generic) singular sub-scheme of $\mathscr{F}$. As such Y is in fact,

$$\operatorname{Proj}\left(\sum K_{\mathscr{F}.}^{\otimes n} I_Z^n\right). \tag{33}$$

The singularities are important, and we blow up to make them best possible. This means, functorially with respect to the ideas, canonical, [MP13]. By op. cit., canonical resolutions of foliations exist for 3-folds, but here we need much less, *i.e.* canonical over the generic point, so [SdS00] is perfectly adequate. Base changing as necessary, we may suppose that every singularity whose generic point is flat over S is K-rational. In the following the order of quantification is critical,

Fact 3.2. *Let X/S with H ample on X be a model of X_K such that the following hold,*

(a) *Y of 3.1 is a foliation by curves, $\mathscr{F}$, on X.*
(b) *Over some Zariski open $U \subset S$, every singularity of $\mathscr{F}|_U$ is canonical, and defines a section of $X|_U \to U$.*

Then understanding heights to be computed on this model, for every $\epsilon > 0$, there is a proper sub-variety $V_\epsilon \subset X \otimes K$ and $\alpha(\epsilon) > 0$ such that,

$$h_{K_{\mathscr{F}}}(f) \leq \operatorname{discr}(f) + \epsilon h_H(f) + \alpha(\epsilon), \quad f \notin V_\epsilon. \tag{34}$$

Proof. This is immediate from the substantially more general [McQ05] V.6.1. Since, it is critical to the proof, and the present case permits substantial simplification we give the details. Nevertheless some familiarity with Jordan decomposition, [Mar81], of vector fields is necessary, so, even though this is just linear algebra, 3.4, one may wish to cross reference op. cit. As in the proof of Fact 2.2 we identify f with a map from a (smooth) curve T finite over S. The derivative, (11), now factors through the sub-scheme, Y, defined by $\mathscr{F}$, which, (33), is isomorphic to the blow up in Z, so the tautological bundle L on the projectivised tangent space is isomorphic to $K_{\mathscr{F}}|_Y(-E)$ for E the exceptional divisor. We may suppose f does not factor through Z, whence we have a finer form of (12),

$$K_{\mathscr{F}} \cdot_f T = (2g-2) + \mathrm{s}_{f,Z} - \mathrm{Ram}_f \tag{35}$$

for g the genus of T, and $\mathrm{s}_{f,Z}$ the Segre class of f around Z. Any component of Z which is not flat over S

$$\text{can contribute at most a constant times } (T:S) \text{ to the Segre class,} \tag{36}$$

which we'll employ repeatedly in proving (the manifestly, (35), sufficient to imply Fact 3.2):

Claim 3.3. *Let everything be as in Fact 3.2, and for $U \subset S$ Zariski open let $\mathrm{s}_{f,Z\cap U}$ be the part of $\mathrm{s}_{f,Z}$ supported over U then up to an implied constant depending on U*

$$\mathrm{s}_{f,Z} - \mathrm{s}_{f,Z\cap U} \leq O_U((T:S)). \tag{37}$$

Moreover: there is a Zariski open U_ϵ, a constant $\alpha(\epsilon)$, and a proper closed sub-variety V_ϵ of $X|_{U_\epsilon}$ such that any invariant curve f not factoring through V_ϵ satisfies,

$$\mathrm{s}_{f,Z\cap U_\epsilon} - \mathrm{Ram}_f \leq \epsilon H \cdot_f T + \alpha(\epsilon). \tag{38}$$

Purely for notational convenience, we suppose in addition to Fact 3.2(a) and (b) that a blow up has been performed in every singular component flat over S. Whence the exceptional divisor is an algebraic invariant hypersurface through the singularity. Similarly we may suppose that over U at a given singularity, Z, there is a vector field ∂ generating the foliation. This descends to an $\mathscr{O}_U$-linear map,

$$\overline{\partial} \in \mathrm{End}(J_Z/J_Z^2) \tag{39}$$

for J_Z the reduced ideal supported on Z. Normalising so that $x=0$ is an equation for the exceptional divisor, should there be 2 such eigenvalues, then shrinking U as necessary we can write,

$$\partial = x\frac{\partial}{\partial x} + a\frac{\partial}{\partial y} + g\frac{\partial}{\partial z}, \quad a = \lambda y \mod J_Z^2 \tag{40}$$

where $(x,y) = J_Z \ni g$, and $\lambda : U \to \mathbb{A}^1$. Generically canonical is not equivalent to λ not identically positive rational, but, this latter condition can be achieved by blowing up, so, for notational convenience, we'll suppose it. Irrespectively to prove (37) in this case we can by (36) blow up in points as much as we wish, so, without loss of generality, every component of the singular locus of $\mathscr{F}$ flat over S is smooth at every $s \in S\backslash U$, and so, not just generically but in a sufficiently small neighbourhood of every closed point, such a component may be identified with the locus $x=y=0$. Consequently we can clear denominators in (40) by way of multiplication by a suitable function in order to obtain a local generator in a neighbourhood of any closed point of such a component of the form

$$\partial = hx\frac{\partial}{\partial x} + a\frac{\partial}{\partial y} + g\frac{\partial}{\partial z}, \quad h \neq 0 \bmod (x,y),\ a,g \in (x,y) \tag{41}$$

wherein h, a, g are regular functions, z may be identified with a coordinate at $s \in S$, and the origin with $s \in S \backslash U$. As such if $t \mapsto (x(t), y(t), z(t))$ is a branch of f cutting the singular locus in s then $x(t)z(t) \neq 0$ and (41) implies

$$\min\{\mathrm{ord}_t(x(t)), \mathrm{ord}_t(y(t))\} \leq \mathrm{ord}_t(g) = \mathrm{ord}_t(zh). \tag{42}$$

Consequently if we write $h = h_0(z) + h_1$ for some function $h_1 \in (x,y)$, then either $\mathrm{ord}_t(h) \leq \mathrm{ord}_t(h_0)$ and (42) implies (37) (for components which generically admit 2 eigenvalues) or

$$\mathrm{ord}_t(h_0) = \mathrm{ord}_t(h_1)\ (\geq \min\{\mathrm{ord}_t(x(t)), \mathrm{ord}_t(y(t))\}) \tag{43}$$

and (37) follows from combining (43) with $h_0 \neq 0$.

Now, to understand what invariant curves can meet Z over U one uses Jordan decomposition. As above we profit from (3.2)(b), to identify the given component of interest with S, and the Jordan decomposition takes place not in the completion in Z, but only in the completion in the maximal ideal in a point, to wit:

Revision 3.4. Let $s \in U \cap Z$ then in the completion, $\hat{\mathscr{O}}_s$, in $\mathfrak{m}(s)$ there is a Jordan decomposition of the generator ∂,

$$\partial = \partial_S + \partial_N,\ [\partial_S, \partial_N] = 0 \tag{44}$$

into semi-simple, ∂_S, and, topologically nilpotent, ∂_N, parts. Furthermore for some coordinates x, η_s, ζ_s with the latter formal functions in the completion at s and $\eta_s = y$, modulo $\mathfrak{m}(s)^2$,

$$\partial_S = x\frac{\partial}{\partial x} + \lambda(s)\eta_s\frac{\partial}{\partial \eta_s}. \tag{45}$$

In particular, if an invariant curve passes through the singularity, it's completion must be invariant by ∂_S, and ∂_N. Thus, supposing, as we may, that the curve is not the singular locus itself, we have the following cases,

(1) $\lambda(s) \notin \mathbb{Q}_+$, then any invariant curve must factor through $x = 0$, or $\eta_s = 0$, and the function ζ_s must vanish.
(2) $\lambda(s) = m/n \in \mathbb{Q}_+ \backslash \mathbb{Z}_{>0}$, then an invariant curve is exactly an element of the family $\zeta_s = 0$, $x^m = \rho\eta_s^n$, $\rho \in \mathbb{C}$.
(3) $\lambda(s) \in \mathbb{Z}_{>0}$, an invariant curve belongs to the family, $\zeta_s = 0$, $x^m = \rho\eta_s$, $\rho \in \mathbb{C}$, but exactly which elements of this family, if any, are invariant depends on ∂_N.

Jordan decomposition cannot be done uniformly along Z. In fact not even the hypersurface η need exist in the completion in Z over a Zariski open in S- as one might guess the problem here is when $\lambda(s) \in \mathbb{Z}_{>0}$. However (by linear algebra) for every $k \in \mathbb{Z}_{>0}$, to be chosen, we can achieve that there is an algebraic function y on a Zariski open such that for $s \in U_k$, a Zariski open depending on k,

$$\eta_s = y (\mathrm{mod}\, J_Z^k). \tag{46}$$

As such if I_k is the ideal $(x, y^k) \cap (y, x^k)$ and $f : \Delta \to X$ a germ of a formal invariant curve, pointed in p, crossing Z at some $s = f(p) \in U_k$ then,

$$\mathrm{ord}_p(f^* I_k) \geq \begin{cases} k & \text{Case (1) as above,} \\ \min\{H(\lambda(s)), k\} & \text{Cases (2) and (3)} \end{cases} \tag{47}$$

for $H(\lambda(s))$ the big height, *i.e.* $\max\{m, n\}$ in the above notation. There are only finitely many points in $\mathbb{Q}$ with big height less than k, so we throw these away to get a possibly different open U_k. Further every crossing of such a Z contributes exactly 1 to the difference between the ramification and the Segre class so by (47),

$$\mathrm{s}_{f, Z \cap U_k} - \mathrm{Ram}_f \leq \frac{1}{k} \cdot \mathrm{s}_{f|_{U_k}, I_k}. \tag{48}$$

Now for a section of H^d, $d \in \mathbb{Z}_{>0}$ over $X \otimes K$ to lie in $I_k \otimes K$ is at most $2k$ conditions, so for $d = 2\sqrt{k}$, or thereabouts, and k sufficiently large,

$$\Gamma(X \otimes K, H^{\otimes d} I_k) \neq 0. \tag{49}$$

Thus if f does not lie in the base locus, B_k of the linear system (49), and we shrink U_k so that the linear system is defined over the same, then:

$$\mathrm{s}_{f, Z \cap U_k} \leq \frac{2}{\sqrt{k}} H \cdot_f T \tag{50}$$

so by (48) we've done (38) in the 2-eigenvalue case.

In the 1-eigenvalue/saddle node case, we first do (37), so as in (41) we find a local generator of the foliation of the form

$$\partial = h x^{e+1} \frac{\partial}{\partial x} + a \frac{\partial}{\partial y} + g \frac{\partial}{\partial z}, \quad e \in \mathbb{Z}_{\geq 0}, \quad h \neq 0 \bmod (x, y), \ a, g \in (x, y) \tag{51}$$

where h, a, g are regular functions; $x = y = 0$ defines the reduced structure on the singular locus; and $x = 0$ is still the exceptional divisor. Now while $e = 0$ is not impossible, it's very rare so in principle the analogue of (42) for a germ of an invariant curve is much worse, to wit:

$$\mathrm{ord}_t(g) = \mathrm{ord}_t(zh) + e\,\mathrm{ord}_t(x) \tag{52}$$

Now $g \in (x, y)$, so blowing up in (x, y) as necessary (since any new components of the singular locus flat over S will, generically, have 2 eigenvalues) we may suppose that

$$g = x^m g_0,\ g_0 = z^n \text{ modulo } (x), \quad \text{for some } m \in \mathbb{Z}_{>0}\, n \in \mathbb{Z}_{\geq 0}, \tag{53}$$

and we distinguish the following cases: $m > e$ so (52) implies (37) by way of the same dichotomy encountered in (42)–(43); $m < e$ so without loss of generality $\mathrm{ord}_t(x) > n\mathrm{ord}_t(z)$ since- *cf.* (43)- we would otherwise have

$$\mathrm{s}_{Z,f}(s) \leq (e+1)\mathrm{ord}_t(x) \leq (e+1)n\mathrm{ord}_t(z) = (e+1)n(T:S) \tag{54}$$

and there's nothing to do, whence (52) becomes

$$n\mathrm{ord}_t(z) - \mathrm{ord}_t(zh) = (e-m)\,\mathrm{ord}_t(x) \tag{55}$$

and we're done again; otherwise $e = m$, $n = d+1$, $d \geq 0$ and, without loss of generality $\mathrm{ord}_t(x) > (d+1)\mathrm{ord}_t(z)$, while if we again write $h = h_0(z) + h_1$, $h_1 \in (x, y)$, then we can similarly suppose $\mathrm{ord}_t(h_1) > \mathrm{ord}_t(h_0)$, so

$$h_0(z) = \nu z^d + O(z^{d+1}), \quad \nu \neq 0. \tag{56}$$

On the other hand the identity (52) results from the differential identity

$$(x^e hz)\frac{x}{\dot{x}} = g\frac{z}{\dot{z}}, \quad (xz)(t) \neq 0 \tag{57}$$

afforded by (51), so from the leading term in t of (57), $\nu \in \mathbb{Q}_{>0}$, and

$$\mathrm{ord}_t(x) = \nu\mathrm{ord}_t(z) \tag{58}$$

which, *cf.* (54), completes the proof of (37) in the 1 eigenvalue case. Ironically (38) is easier here because there is a formal invariant hypersurface $\eta = 0$, the formal centre manifold, defined in the fibre over a Zariski open subset $U \subset S$ of the completion in Z such that the semi-simple part of the Jordan decomposition is uniformly of the form,

$$\eta\frac{\partial}{\partial\eta} \tag{59}$$

in some coordinate system ξ_s, η, ζ_s in $\hat{\mathscr{O}}_s$, with $\xi_s = 0$ the exceptional divisor. Thus invariant curves either factor through one of the exceptional divisor or $\eta = 0$. The former may be ignored, and in the latter case the induced foliation in $\eta = 0$ either leaves Z invariant, or it does not. In the former case there are no other invariant curves crossing Z over a sufficiently small Zariski open. In the latter case they can be taken of the form $\zeta =$ Constant, in some coordinate system x, η, ζ defined in the completion of Z over a Zariski open with $x = 0$ the exceptional divisor, and (η, x^k) algebraic for any k, which one then takes as I_k in (49) and argues as in (50). □

Now consider the composite of the natural maps,

$$\Omega_{K/\mathbb{C}} = \mathscr{O}_{X\otimes K} \to \Omega_{X\otimes K/\mathbb{C}} \to K_{\mathscr{F}}. \tag{60}$$

This is non-zero at every f, which are Zariski dense, so it defines a non-zero section Γ, which we confuse with the curve that it defines, of $K_{\mathscr{F}}$. To fix ideas: Γ has a Zariski decomposition, if the nef. part were big then there is a $\delta > 0$ such that, for f not factoring through a proper subvariety,

$$h_{K_{\mathscr{F}}}(f) \geq \delta h_H(f) \tag{61}$$

so (34) implies (31) and we're done. Consequently, what's required is some (foliated) minimal model theory in order to understand the non-generic case where this fails. We could appeal to the general theory of [McQ05], but we essentially only have a surface question, so things can be done by hand. Base changing in K we may suppose that every intersection of components of Γ is a section, and:

Fact 3.5. *Let $C \subset X \otimes K$ be a curve then:*

(a) *If C is not invariant by $\mathscr{F}$, then $(K_{\mathscr{F}} + C)\cdot C \geq 0$.*
(b) *If C is invariant and belongs to the support of Γ then it is smooth and satisfies: $K_{\mathscr{F}} \cdot C = -\chi_C + \mathrm{s}_Z(C)$.*
(c) *The invariant part of Γ is simple normal crossing, and the crossings in the same occur only at singularities of $\mathscr{F} \otimes K$.*

Proof. In (a), by definition, the composition:

$$\mathscr{O}_C(-C) \to \Omega_{X\otimes K/\mathbb{C}} \to K_{\mathscr{F}} \tag{62}$$

is non-zero. Similarly, if C is smooth, then $f^{-1}I_Z$ is a Cartier divisor, $\mathscr{O}_C(-E)$, say, with E of degree $\mathrm{s}_Z(C)$, and if, furthermore, C is invariant, then there is a surjection,

$$\Omega_{C/\mathbb{C}} \to K_{\mathscr{F}}|_C(-E) \to 0. \tag{63}$$

Should in addition C be in Γ, then $\Omega_{K/\mathbb{C}}$ maps to zero under the above, so there is even a surjection,

$$\omega_{C/K} \to K_{\mathscr{F}}|_C(-E) \to 0 \tag{64}$$

and both sides are line bundles, so this is an isomorphism. Whence we only require to prove the various smoothness and normal crossing assertions.

Now, in the proof of 3.3, we've already (implicitly) seen by way of the Jordan decomposition at a singularity, 3.4, that there are at most 2 formal invariant hypersurfaces containing the singularity which are flat over K, and that these are smooth with simple normal crossings. Consequently, we're reduced to proving that the invariant part of Γ is smooth outside the singularities. As such, let $\Sigma \hookrightarrow \Gamma\backslash Z$ be any (irreducible) multi-section of S, whence it's finite étale over some sufficiently small

Zariski open $U \subset S$. By the definition of Γ, Σ cannot be $\mathscr{F}$-invariant- otherwise the composite,

$$\Omega_{U/\mathbb{C}}|_{\Sigma} \xrightarrow{\sim} \Omega_{\Sigma/\mathbb{C}} \xrightarrow{\sim} K_{\mathscr{F}}|_{\Sigma} \tag{65}$$

would certainly not be null. Thus, shrinking U as necessary, we may suppose that $\mathscr{F}$ is everywhere transverse to Σ, and, better still, there is a function x vanishing on Σ and a generator ∂ of the foliation about the same such that $\partial(x) = 1$. At this point the Frobenius theorem is valid in the completion in Σ, more precisely for any function f on a formal neighbourhood:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n (\partial)^n (f) \tag{66}$$

converges in the completion, so there is a unique invariant hypersurface containing Σ, and it is smooth. Alternatively, since we're over $\mathbb{C}$, by the usual Frobenius theorem, take the germ of an analytic hypersurface which is Σ together with the unique germ of an invariant curve through each point. □

The first of these tells us that the conditions $K_{\mathscr{F}} \cdot C < 0$, and C contractible are incompatible, the second that if $C \subset \Gamma$ were to satisfy $K_{\mathscr{F}} \cdot C < 0$ then it must be a smooth invariant rational curve. The self intersection may well not be -1, so its contraction $X \to Y$ may lead to a quotient singularity. There is, however, a minimal smooth champ de Deligne-Mumford $\mathscr{Y} \to Y$, *the Vistoli covering champ*, whose moduli is Y, [Vis89, 2.8]. Both Fact 3.5, and its proof, are valid étale locally, so modulo understanding rational curve in the broad sense of a 1-dimensional champ with positive Euler characteristic, we may, *cf.* [McQ08, III.3-III.3.bis], continue contracting to obtain:

FACT 3.6. *There is a contraction $\rho : X \otimes K \to X_0 \otimes K$ to a surface with quotient singularities such that,*

- (a) $K_{\mathscr{F}_0} \xrightarrow{\sim} \mathscr{O}_{X_0 \otimes K}(\rho_* \Gamma)$, *and:* $K_{\mathscr{F}} = \rho^* K_{\mathscr{F}_0} + \sum_i a_i E_i$, $a_i \in \mathbb{Q}_+$, *for $E_i \subset \Gamma$ smooth invariant simple normal crossing rational curves.*
- (b) *The $\mathbb{Q}$-divisor $K_{\mathscr{F}_0}$ is nef., and the E_i are contracted by ρ.*
- (c) *The restriction, $p^* \Gamma_0$, to the Vistoli covering champ $p : \mathscr{X}_0 \otimes K \to X_0 \otimes K$ of $\rho_* \Gamma$ satisfies Fact 3.5(b) and (c).*

As it happens, [MP13, III.i.1] or [McQ05, I.6.11], and perhaps surprisingly, terminal foliation singularities are smooth points of a foliation on a Vistoli covering champ should this exist, more generally idem on the Gorenstein covering champ, *i.e.* smallest champ de Deligne-Mumford such that the canonical along the foliation is a bundle, so, in fact, cf. [McQ08, III.3.2], one can prove that the connected components of $X \to X_0$ are actually chains of rational curves- the hands own dual graph alternative in op. cit. also proves the same. However, we don't need to know this, and we observe that we have the following possibilities:

Alternatives 3.7. *Exactly one of the following occurs:*

(a) $K_{\mathscr{F}_0}$ *is big, and, as we've observed post* (61) *we're done by* (34).
(b) $K_{\mathscr{F}_0}$ *has numerical Kodaira dimension 0, so the map from* $\Omega_{K/\mathbb{C}}|_{X_0}$ *is an isomorphism. Whence* X_0 *is iso-trivial, and we're done again.*
(c) $K_{\mathscr{F}_0}$ *has numerical Kodaira dimension* 1.

We therefore exclusively concentrate on the alternative (c), and we assert:

Claim 3.8. *In the notation of Fact* 3.6*, not every curve in* Γ_0 *is invariant.*

Proof. We harmlessly confuse Γ_0 with its moduli, and suppose otherwise, then for $\rho^*\Gamma_0 = \sum a_j\, C_j$, $a_j \in \mathbb{Q}_+$ the given divisor expressed as a sum of irreducible components, the said components are smooth and normal crossing, Fact 3.5(c). The fact that the crossings only occur in foliation singularities implies,

$$(\omega_{X\otimes K/K} - K_{\mathscr{F}}).\, C_j \leq -\, C_j^2 - \sum_{k\neq j} C_k.\, C_j \tag{67}$$

for every j, while the fact that it is nef. of square 0 gives,

$$-\, a_j\, C_j^2 = \sum_{k\neq j} a_k\, C_k.\, C_j \tag{68}$$

again for every j. Whence multiplying (67) by a_j, summing, and combining with (68) gives

$$\omega_{X\otimes K/K}.\, p^*\,\Gamma_0 \leq K_{\mathscr{F}}.\, p^*\,\Gamma_0 = 0, \tag{69}$$

which is absurd since $\omega_{X\otimes K/K}$ is big. □

In addition we also make:

Further Claim 3.9. $\rho_*\Gamma$ *is supported on disjoint irreducible curves* C_i *such that the champs* $\mathscr{C}_i \to C_i$ *lying over any* C_i *in* $\mathscr{X}_0 \otimes K$ *are smooth and everywhere transverse to* $\mathscr{F}_0$.

Proof. Firstly at the level of $\mathbb{Q}$ divisors write $\Gamma_0 = \Gamma' + \Gamma''$ as a sum of its invariant and non-invariant part. By 3.8, $\Gamma' \neq 0$, and by Fact 3.5(a) it's nef, whence by the Hodge index theorem Γ' is parallel to Γ_0 in Néron-Severi, and so $\Gamma' \cdot \Gamma'' = 0$. This implies, however, that (68) also holds for $\rho^*\Gamma''$, so the proof of 3.8 actually proves $\Gamma'' = 0$.

Now let $\mathscr{C}$ be an irreducible component of Γ_0. As in (62) and (63) we have natural maps,

$$\mathscr{O}_{\mathscr{C}}(-\mathscr{C}) \to \Omega_{\mathscr{X}_0\otimes K/\mathbb{C}}|_{\mathscr{C}}, \quad \text{and,} \quad \Omega_{\mathscr{X}_0\otimes K/\mathbb{C}}|_{\mathscr{C}} \to K_{\mathscr{F}_0}|_{\mathscr{C}} \tag{70}$$

since $\mathscr{C} \subset \Gamma_0$ and not invariant this factors as a non-zero map,

$$\mathscr{O}_{\mathscr{C}}(-\mathscr{C}) \to \Omega_{\mathscr{X}_0\otimes K/K}|_{\mathscr{C}} \to K_{\mathscr{F}_0}|_{\mathscr{C}} \tag{71}$$

whence if either $\mathscr{C}$ were not smooth, nor everywhere transverse (so, inter alia, not containing a foliation singularity) this composition must vanish somewhere, and $\mathscr{C}^2 > 0$. This, however, implies $K_{\mathscr{F}_0}$ big, which is nonsense. □

Next we have to move the C_i, or better the champs $\mathscr{C}_i$ occurring in Further Claim 3.9. A useful simplification is that since $X \otimes K$ is of general type the $\mathscr{C}_i$ cannot have positive Euler characteristic, whence they admit étale covers by honest curves. Fix i, and let $C \to \mathscr{C}_i$ be such a cover, then as per [SGA-I, Exposé I, Théorème 8.3], this extends to an étale cover U with trace C of the completion of $\mathscr{X}_0$ in $\mathscr{C}_i$. Now we use the foliation to move C, to wit:

Final Claim 3.10. *Define $n \in \mathbb{Z}_{>0}$ by: $\Gamma_0|_U = nC$, and let U_m, $m \in \mathbb{Z}_{>0}$ be the mth thickening of C in U, then, for all $m \in \mathbb{Z}_{>0}$ $\mathscr{O}_{U_m}(C)$ is at worst $n+1$ torsion.*

Proof. By induction on m, with $m = 1$ following from the natural map (62) which affords Fact 3.5(a). For $m \in \mathbb{Z}_{>0}$, we have an exponential sequence,

$$\mathrm{H}^0(\mathscr{O}_{U_m}^{\times}) \xrightarrow{\delta} \mathrm{H}^1\left(C, I_C^m/I_C^{m+1}\right) \longrightarrow \mathrm{Pic}(U_{m+1}) \longrightarrow \mathrm{Pic}(U_m) \longrightarrow 0 \qquad (72)$$

so without loss of generality we have a $n+1$-torsion bundle L on U such that the isomorphism $L^{\vee}|_C \xrightarrow{\sim} \mathscr{O}_C(C)$ lifts to U_m. Now take a sufficiently fine open cover $\coprod U_\alpha \to U$, and denote by ∂_α, x_α a generator for the foliation, and a defining equation of C on a given U_α, then for $\zeta_{\alpha\beta}$ the locally constant transition functions for L, on combining with (72):

$$\mathrm{Pic}(U_{m+1}) \ni L + C = \left[\zeta_{\alpha\beta}\frac{x_\beta}{x_\alpha}\right], \quad \text{so } x_\alpha = \zeta_{\alpha\beta}(1 + h_{\alpha\beta})x_\beta \qquad (73)$$

for $h_{\alpha\beta}$ a 1 co-cycle in I_C^m/I_C^{m+1}. As such if $g_{\alpha\beta}$ are the transition functions for $K_{\mathscr{F}}^{\vee}$ then,

$$\partial_\alpha x_\alpha = \zeta_{\alpha\beta}(1 + (m+1)h_{\alpha\beta})g_{\alpha\beta}\partial_\beta x_\beta \pmod{I_C^{m+1}} \qquad (74)$$

which, since C is everywhere transverse, says that $L - (m+1)(C+L) - K_{\mathscr{F}}$ is trivial on U_{m+1}, or equivalently,

$$0 = (m+n+1)C + mL = (m+n+1)(L+C). \qquad (75)$$

The bundle $L + C$ is, by induction, in the image of the H^1 in (72), so we're done since both this group and $\mathrm{Im}(\delta)$ are K, so, in particular $\mathbb{Q}$, vector spaces. □

To profit from this, choose some C_i as in Further Claim 3.9, and apply Final Claim 3.10 to find a map $q : X_0 \otimes K \to B$ with connected fibres to a smooth curve containing C_i, whence, any other possible C_j as, at worst a multiple fibre. Base changing as appropriate we may suppose that $c_i = q(C_i)$ is K-rational, and denoting by n_i the multiplicity of the fibre supported on C_i define a champ $\mathscr{B} \to B$ over B with monodromy $\mathbb{Z}/n_i$ at each c_i, and trivial otherwise. By construction q

lifts to $q : \mathscr{X}_0 \to \mathscr{B}$, and $K_{\mathscr{F}_0} = q^* K_{\mathscr{G}}$ for some effective divisor $K_{\mathscr{G}}$ on $\mathscr{B}$. As the notation suggests this corresponds to a foliation on $\mathscr{B}$. Indeed, the composite:

$$q^*\Omega_{\mathscr{B}/\mathbb{C}} \to \Omega_{\mathscr{X}_0/\mathbb{C}} \to K_{\mathscr{F}_0} = q^* K_{\mathscr{G}} \tag{76}$$

comes from a map $\chi : \Omega_{\mathscr{B}/\mathbb{C}} \to K_{\mathscr{G}}$ since the fibres are proper and connected, while the natural diagram,

$$\begin{array}{ccc} \Omega_{\mathscr{X}_0/\mathbb{C}} & \longrightarrow & K_{\mathscr{F}} \\ \uparrow & & \| \\ q^*\Omega_{\mathscr{B}/\mathbb{C}} & \xrightarrow{q^*\chi} & q^*K_{\mathscr{G}} \end{array} \tag{77}$$

commutes. A priori, the image of χ may not be saturated, but if it were not so, around some fibre Q, say, then Q would have to belong to the support of Γ_0 and be $\mathscr{F}_0$-invariant, which is impossible by the same argument affording Further Claim 3.9. Whence, in the strongest possible sense, we have a map of foliated champs $(\mathscr{X}_0, \mathscr{F}_0) \to (\mathscr{B}, \mathscr{G})$, *i.e.* leaves go to leaves.

We can now use the classification theorem, [McQ08], to conclude. As noted, we're supposing that our algebraic points f are Zariski dense, so the same is true of their images in B. Whence, by [Jou78], $\mathscr{G}$ has a first integral. Necessarily this first integral cannot be defined over S, so we take models, with $r : B_S \to C$ the Stein factorisation of the integral over some smooth complex curve, so supposing an a priori base change in K to achieve that the fibres of q are geometrically integral, the composite $p : X \to C$ also has connected fibres. We need to compare the fibrations,

$$\begin{array}{ccccccc} p : X & \xrightarrow{\rho_S} & X_0 & \xrightarrow{q} & B_S & \xrightarrow{r} & C \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow & & \\ S & = & S & = & S & & \end{array} \tag{78}$$

where the subscript S indicates that we extend over S something that we have only previously defined at the generic point. We may suppose that X and B_S are regular models, *i.e.* smooth over $\mathbb{C}$ but according to the ad hoc way we're doing minimal model theory X_0 might be no better than normal and a bit of a mess over a proper closed subset of S. Actually, by [McQ05] we could avoid this difficulty, but it presents a limited problem. Let γ be the generic point of C. The foliation singularities of X_γ over $\mathbb{C}(C)$ have, so far, only been rendered canonical if they are also flat over S. Otherwise they belong to fibres. The resolution algorithm is by successive blow ups in closed geometric points, so it's Galois equivariant, whence further blowing up to obtain generically canonical over C changes nothing, and we assume that this has been done. By construction there is a (not necessarily effective) divisor F supported in fibres of π, and a model E_S of the $\mathbb{Q}$-divisor contracted by the ad-hoc minimal model procedure, Fact 3.6, such that,

$$K_{\mathscr{F}} = q^* K_{\mathscr{G}} + F + E_S \tag{79}$$

which equally restricts to the canonical bundle of the induced foliation in $p^{-1}(\gamma)$, because of the saturation of χ in (77). Thus for a possibly different divisor F_+ again supported in fibres, but now nef. and effective on B,

$$K_{\mathscr{F}} \leq q^*(K_{\mathscr{G}} + F_+) + E_S \tag{80}$$

and, again, similarly for the induced canonical bundle in the generic fibre of p. Thus the generic fibre of p is a foliated surface with numerical Kodaira dimension at most 1. Any foliated surface of numerical dimension at most 0 is covered by rational or elliptic curves or is dominated by an abelian surface. We can suppose S as hyperbolic as we like, so this cannot occur since the said group varieties would cover the fibres of π which we're supposing of general type. Similarly, the fibres of r dominate S, so the generic fibre of p has irregularity as high as we like (1 would do) so (foliated) abundance holds for the generic fibre of p. Again, by the hyperbolicity of S and the general type assumption on fibres of π, the generic fibre of p cannot be an elliptic fibration nor the suspension of a representation in PGL_2 or automorphisms of an elliptic curve, so it must be an isotrivial family of curves of genus $h \geq 2$, *i.e.* after a base change over the generic point of the generic fibre B_γ of r, the foliation in a generic fibre of q is the projection onto a curve Q_γ. The structure of such foliations $\mathscr{F}_\gamma$ on fibres X_γ of p is not particularly complicated, but merits a little attention. The minimal model algorithm is Galois equivariant, and there is a unique map to a unique minimal model $(M_\gamma, \mathscr{H}_\gamma)$, with M_γ a surface over $\mathbb{C}(C)$ with at worst isolated quotient singularities mapping to B_γ. Over the generic point of S, M_γ coincides with $X_0 \otimes K$. The Vistoli covering champ $\mathscr{M}_\gamma \to M_\gamma$ exists, is smooth, and maps smoothly to a champ $\mathscr{B}_\gamma$ over B_γ in such a way that for any $V \to \mathscr{B}_\gamma$ scheme like and étale, $\mathscr{M}_\gamma|_V \xrightarrow{\sim} V \times Q_\gamma$. All of which holds on replacing γ by a sufficiently small Zariski open $V \subset C$, and any algebraic point f may be supposed to map to some $c \in V$. As in the proofs of Facts 2.2 and 3.2 we identify f with $f : T \to X$ for T finite over S, so by adjunction,

$$\omega_X \cdot_f T = \omega_{X_c} \cdot_f T. \tag{81}$$

The surface X_c maps to the minimal model M_c, by ν say, and, the induced foliation on the champ $\mathscr{M}_c \to M_c$ is smooth, so:

$$\omega_{M_c} \cdot_{\nu f} T = K_{\mathscr{H}_c} \cdot_{\nu f} T \tag{82}$$

while any extension ω_{X_0} of $\omega_{X_0 \otimes K}$ to a bundle on X satisfies, $\omega_{X_0} = \nu^* \omega_M + F$ over a Zariski open in C, and, as ever F supported in fibres of π. Similarly over an open in C, $\nu^* K_{\mathscr{H}} \leq K_{\mathscr{F}}$ by the definition of canonical singularities, so for f outwith a proper sub-variety and some constant α depending only on the models:

$$\omega_{X_0} \cdot_f T \leq K_{\mathscr{F}} \cdot_f T + \alpha(T : S). \tag{83}$$

The bundle $\omega_{X_0 \otimes K}$ is certainly big, so we conclude to 3.1 from Fact 3.2.

A Isotriviality

By way of notation, let X/S be a flat family of normal projective curves or surfaces over a complex curve S with generic point $\mathrm{Spec}(K) \hookrightarrow S$ and denote by C a general fibre should the family be in curves, or a generic element $|nH|$ in a generic fibre, for some divisor H ample on the generic fibre, and n sufficiently large to be chosen.

FACT A.1. *Notation as above if $\Omega_{X/S} \otimes K|_C$ is ample, and $n \gg 0$, then $\Omega_{X/\mathbb{C}} \otimes K|_C$ is ample iff X/S is not isotrivial.*

Proof. Modulo the need for the normality hypothesis, the case of families of curves is trivial, while for surfaces we have an exact sequence,

$$\to \mathrm{Hom}_{X\otimes K}(\Omega_{X/\mathbb{C}}, \mathscr{O}_X) \to \mathrm{Hom}_C(\Omega_{X/\mathbb{C}}, \mathscr{O}_C) \to \mathrm{Ext}^1_{X\otimes K}(\Omega_{X/\mathbb{C}}, \mathscr{O}_X(-H)) \to \tag{84}$$

along with the local global spectral sequence,

$$\mathrm{H}^p(X \otimes K, \mathscr{E}xt^q(\Omega_{X/\mathbb{C}}, \mathscr{O}_X(-H)) \Rightarrow \mathrm{Ext}^{p+q}_{X\otimes K}(\Omega_{X/\mathbb{C}}, \mathscr{O}_X(-H)). \tag{85}$$

On the other hand, X is S_2 so by [SGA-II, Exposé VII, 1.2], there are no local Ext groups, so the above Ext^1 is just the $E_2^{(1,0)}$ term in the spectral sequence, which, [SGA-II, Exposé XII, 1.4], vanishes for $n \gg 0$. Consequently, there is a retract of the arrow,

$$0 \longrightarrow \Omega_{S/\mathbb{C}} \otimes K \longrightarrow \Omega_{X/\mathbb{C}} \otimes K \tag{86}$$

iff there is a retract on restriction to C, which, since $\Omega_{X/K}|_C$ is ample, is false iff every quotient of $\Omega_{X/\mathbb{C}}|_C$ has positive degree. □

FACT A.2. *If no model of the generic fibre of X/S admits a global vector field and it is dominated by an isotrivial variety, then some model of $X \otimes K$ is isotrivial.*

Proof. Again, the curve case is much easier, and we restrict to the surface case, so that by hypothesis we have a dominant rational map from some $V \times S$ to X, and without loss of generality $X/\mathbb{C}$ is smooth, albeit $\mathbb{Q}$-factorial would do. Cutting by hyperplanes as necessary, we can suppose that V is a surface, and we resolve the indeterminacy by way of,

$$V \times S \xleftarrow{p} \tilde{V} \xrightarrow{q} X. \tag{87}$$

Denote by A a very ample divisor on V, then over a generic $s \in S$, $q_* p^* A$ is a linear system with base locus of dimension at most zero, whence, in fact, empty base locus by Zariski-Fujita. Thus, for A sufficiently large, the above linear system determines a bi-rational map $r : X \otimes K \to Y$, with Y normal (since pull-back of ample by finite is ample and X is normal) such that, $B = r_* q_* p^* A$, or a large multiple, is as ample as we require. Now suppose that Y is not isotrivial. By hypothesis it is not ruled, so $\Omega_{Y/K}$ is semi-positive [Miy87], whence, up to replacing A by a multiple, it's ample on restriction to B provided Y does not admit a global vector field, and A.1 applies. If, however, we take a generic $s \in S$, so that p_s is defined on V_s with the co-dimension of V_s in V at least 2, then, contrary to hypothesis, the tangent space to $\mathrm{Hom}(V_s, Y)$ at p_s, *viz:* $\mathrm{Hom}_{V_s}((p_s)^* r^* \Omega_{Y/\mathbb{C}}, \mathscr{O}_{V_s})$ is necessarily zero since $(p_s)^* r^* \Omega_{Y/\mathbb{C}}|_A$ is ample. □

COROLLARY A.3. *If a normal curve or minimal surface over K of general type is dominated by an isotrivial variety, then it is isotrivial.*

Proof. The normal model of a curve or the minimal model of a surface of general type is uniquely unique, so if some model is isotrivial, A.2, the asserted model is too. □

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