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RESEARCH ARTICLE

Curves on surfaces of mixed characteristic

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In memoriam Torsten Ekedahl

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Abstract The classification of foliated surfaces (McQuillan in Pure Appl Math Q 4(3):877–1012, 2008) is applied to the study of curves on surfaces with big co-tangent bundle and varying moduli, be it purely in characteristic zero, or, more generally when the characteristic is mixed. Almost everything that one might naively imagine is true, but with one critical exception: rational curves on bi-disc quotients which aren't quotients of products of curves are Zariski dense in mixed characteristic. The logical repercussions in characteristic zero of this exception are not negligible.

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1 Introduction

For $X/\mathbb{C}$ a smooth algebraic surface with $\Omega^1_{X/\mathbb{C}}$ big, a theorem, [3], of Bogomolov asserts:

$$\textit{The set, } Z, \textit{ of rational or elliptic curves on } X \textit{ is finite,} \tag{1}$$

and, better still, for H an ample divisor on X:

$$\begin{aligned}&\textit{There is a constant } -\kappa(X,H)<0 \textit{ such that if } f: C\to X \textit{ is any map from}\\ &\textit{a smooth curve which doesn't factor through } Z, \textit{ then } H\cdot_f C \leqslant -\kappa\chi(C).\end{aligned} \tag{2}$$

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Wherein, the notation is constructed according to the mnemonic: if ω is a metric on X (or indeed any complex manifold) with (holomorphic) sectional curvature $-K < 0$, then (Gauss–Bonnet)

$$\int_C f^*\omega \leqslant -\frac{1}{K}\chi(C). \tag{3}$$

Nevertheless, one shouldn't, even if $Z = \varnothing$, imagine for a second that (2) is equivalent to negative sectional curvature, *e.g.* there are buckets of simply connected surfaces in Bogomolov's class with $Z = \varnothing$. Rather what (2) expresses is a weak, or (1) weaker still, algebraic vestige in complex dimension 2 ($\subset$ real dimension 4) of the implication that negative Ricci curvature implies negative sectional curvature in real dimension at most 3. Indeed, by [24], the canonical model of X admits (in the orbifold sense should this not coincide with the minimal model) a Kähler–Einstein metric of Ricci curvature -1, but, as we've remarked, will in all probability catastrophically fail to admit a uniformisation of the type encountered in real dimension 2 or 3. There are, of course, much better holomorphic vestiges of negative sectional curvature than (1)–(2), of which, the best to date is that Gromov's isoperimetric, [11], holds, *i.e.*:

$$\begin{gathered}\textit{There is a constant } -K(X) < 0 \textit{ such that}\\ \textit{if } f: \Delta \to X \textit{ is any holomorphic disc, then}\\ \mathrm{area}_\omega(f) = \int_\Delta f^*\omega \leqslant \frac{1}{K}\,\mathrm{length}_\omega(\partial_f) = \int_{\partial\Delta} |df|_\omega\end{gathered} \tag{4}$$

where for ease of exposition, we suppose $Z = \varnothing$, which gives us the right to take ω equal to the Kähler–Einstein metric in (4). In particular, in exceptional cases where one does have uniformisation, *e.g.* ball and bi-disc quotients, one sees that the right value of $-K = -\kappa(X, K_X)^{-1}$ whether in (2) or (4) is the holomorphic sectional curvature of the Kähler–Einstein metric, so $-2/3$ for balls, respectively $-1/2$ for the bi-disc, and, already [1], this turns out to be best possible in the restrictive algebraic context (2)–(3).

The much deeper inequality (4) while affording a revealing insight into the relation of (2) with the isoperimetric profile, is, however, only really our immediate interest in respect of the complexity, 1.9, of the algebraic surface part of its proof, [14], as opposed to the part, [12] or [6], valid on all almost complex manifolds in all dimensions, and such questions of complexity are in turn subordinate to the main question of how does Bogomolov's theorem vary in moduli. This already has its own interest in characteristic zero, so we set things up accordingly: S will be an irreducible affine scheme of finite type flat over a Noetherian integral domain of characteristic zero, and X/S a smooth family of S-projective surfaces, with say X and its generic fibre geometrically irreducible for convenience, then we have two subtly different theorems.

Theorem 1.1 *Suppose X/S above satisfies Bogomolov's condition $\Omega^1_{X/S}$ big, then exactly one of the following happens,*

(a) *There exists a constant $-\kappa < 0$, along with a closed nowhere dense sub-scheme $Z \subset X$, such that for every closed geometric point* $\mathrm{Spec}(k) \to S$ *and every sepa-*

rable map $f : C/k \to X$ from a smooth k-curve which doesn't factor through Z,

$$H \cdot_f C \leqslant -\kappa \chi_C. \tag{5}$$

(b) *The closed points $s \hookrightarrow S$ of positive residue characteristic are dense, and the generic fibre is dominated by a modification of a bi-disc quotient* (*including b.t.w. the case of a product of curves*).

Theorem 1.2 *Suppose X/S above satisfies Bogomolov's condition $\Omega^1_{X/S}$ big, then exactly one of the following happens,*

(a) *There is a closed nowhere dense sub-scheme $Z \subset X$ such that for every closed geometric point* $\mathrm{Spec}(k) \to S$, *every map $f : C/k \to X$ from a smooth rational or elliptic k-curve factors through Z.*
(b) *There exists a nowhere dense closed sub-scheme Z of X; a surjective map $S' \to U$ from an open dense affine sub-scheme $S' \subset S$ onto the spectrum of a sub-ring of $\mathbb{Q}$; finitely many bi-disc quotients Y_i/U which* (*over $\overline{\mathbb{Q}}$*) *don't admit an almost étale cover by a product of curves; and real quadratic number fields K_i* (*functorial in Y*) *such that infinitely many primes $p \in U$ are inert in each K_i; $X \times_S S'$ is dominated by an irreducible component of $Y_i \times_U S'$ for each i, and every rational or elliptic curve $C/k \to X$ which doesn't factor through Z is the image of some $f : \mathbb{P}^1_k \to Y_i$ of H-degree $O(p)$, for* $\mathrm{Spec}(k) \to S$ *a closed geometric point of characteristic p inert in some K_i, and f invariant by one of the natural foliations on Y_i/U arising from the bi-disc structure, yet missing the cusps should they exist.*

Before proceeding to discuss the theorems in mixed characteristic, let us note that the obvious combination of 1.1, Noetherian induction, the upper semi-continuity of h^0, and (to get the optimal Z as below) Riemann–Hurwitz gives

Corollary 1.3 *Let $X \to S$ be the universal smooth family of* (*minimal*) *surfaces of general type over some irreducible component of the moduli space $/\mathbb{C}$, then if the co-tangent bundle of the generic fibre is big, the set Z whose fibre over $s \in S$ is the rational and elliptic curves on X_s is a closed nowhere dense sub-scheme, and there exists a constant $-\kappa < 0$, such that for every closed geometric point $s \in S(\mathbb{C})$ and every map $f : C \to X_s$ from a smooth curve which doesn't factor through Z,*

$$H \cdot_f C \leqslant -\kappa \chi_C. \tag{6}$$

The case of 1.3 where the bigness of Ω^1 is guaranteed by Riemann–Roch, *i.e.* $c_1^2 > c_2$ is also a theorem of Miyaoka, [21], and, when the two overlap, Miyaoka's theorem is way better since it provides explicit (and very simple) functions of the Chern numbers for κ and the degree of Z, whereas 1.3 says such functions exist, but they could be anything.

Now, irrespectively of whether we're in mixed characteristic or not, it's well known, and easy, see the proof of 1.1 in Sect. 5, that Bogomolov's theorem reduces to studying curves invariant by a foliation $\mathcal{F}$, so 1.1 and 1.2 are really just corollaries of

Theorem 1.4 *Let $(X, \mathcal{F}) \to S$ be a family of foliations by curves on a family X/S of smooth projective surfaces of general type with X and its generic geometric fibre irreducible for convenience, then exactly one of the following happens,*

(a) *There exists a constant $-\kappa < 0$, along with a closed nowhere dense sub-scheme $Z \subset X$ such that for every closed geometric point $\mathrm{Spec}(k) \to S$ and every separable $\mathcal{F}\otimes k$ invariant map $f: C/k \to X$ from a smooth k-curve which does not factor through Z,*

$$H \cdot_f C \leqslant -\kappa \chi_C. \tag{7}$$

(b) *The closed points $s \hookrightarrow S$ of positive residue characteristic are dense, and the generic fibre is a modification of a bi-disc quotient (including b.t.w. the case of a product of curves) with $\mathcal{F}$ one of the natural foliations induced by the bi-disc structure.*

Theorem 1.5 *Hypothesis as in 1.4, then exactly one of the following happens,*

(a) *There is a closed nowhere dense sub-scheme $Z \subset X$ such that for every closed geometric point $\mathrm{Spec}(k) \to S$, every invariant map $f: C/k \to X$ from a smooth rational or elliptic k-curve factors through Z.*

(b) *There exists a nowhere dense closed sub-scheme Z of X; a surjective map $S' \to U$ from an open dense affine sub-scheme $S' \subset S$ onto the spectrum of a sub-ring of $\mathbb{Q}$; a bi-disc quotient Y/U which (over $\overline{\mathbb{Q}}$) does not admit an almost étale cover by a product; and a real quadratic number field K (functorial in Y) such that infinitely many primes $p \in U$ are inert in K; $X \times_S S'$ is a bi-rational modification of an irreducible component of $Y \times_U S'$ with $\mathcal{F}/S'$ the base change of one of the natural foliations on Y/U arising from the bi-disc structure, and every $\mathcal{F}$-invariant rational or elliptic curve not factoring through Z is $\mathcal{F}\otimes k$ invariant for $\mathrm{Spec}(k) \to S$ a closed geometric point of characteristic p inert in k. Moreover for each such k, with $p \gg 0$, there is a $\mathcal{F}\otimes k$ invariant generically embedded rational curve $f: \mathbb{P}^1_k \to Y$, with Y understood as a Deligne–Mumford champ if there are quotient singularities, missing the cusps E should these exist such that*

$$\frac{p}{2} \leqslant (K_Y + E) \cdot_f \mathbb{P}^1_k \leqslant (p-1)\, \mathrm{c}_{2,\log}(Y, E) \tag{8}$$

and, along which some non-classical modular form of weight $(2p, -2)$ vanishes.

At first sight 1.4.(b) may appear surprising, but it obviously happens. Specifically, say $f = f_1 \times f_2 : C \to X = X_1 \times X_2$ a curve on a product of hyperbolic curves over $\mathbb{F}_p$ to avoid pointless technicalities, then f separable is only equivalent to, say, f_1 separable and there's nothing to stop us composing f_2 with a huge multiple of Frobenius. As such *separable curves of (a fixed) genus g on a product of hyperbolic curves in characteristic p are not bounded in moduli once g is (depending on X) sufficiently large*, albeit, by Riemann–Hurwitz and separable/inseparable factorisations of the f_i, the above is the only way this can happen, while, plainly, there are no rational or elliptic curves. This brings us to the case of bi-disc quotients which (up to a finite group action) aren't products of curves, and as one sees there is a clear distinction between primes p which are split, as opposed to inert in the real quadratic field K of

1.2.(b). From the point of view of constructing counterexamples to boundedness in moduli of curves of genus g, the mechanism, 3.12, of the case of products of curves is equally valid, (48), when p is split, and, indeed, this is, 3.9, the only mechanism. Nevertheless, in the presence of cusps there is some subtlety, *i.e.* bounding rational curves at split p, see the proof of 1.5 in Sect. 5, has more to it than what one can say at a fixed prime.

This brings us to the amusing logical consequences of (8). More or less by definition, every theorem in algebraic geometry over $\mathbb{C}$ is an ACF_0 (first order theory of algebraically closed fields of characteristic zero) theorem. Unfortunately, and evidently, since there is quantification over a priori unbounded sets of curves in both (1) and (2) neither is a statement in ACF_0. One could, however, ask whether a theorem, $\mathcal{T}(f)$, about some type of curve, f on X, has a proof which is *essentially* ACF_0, *i.e.* does it have, for X fixed, the form

$$P_i \wedge T_j(f) \ \Rightarrow \ \mathcal{T}(f) \tag{9}$$

where P_i are finitely many ACF_0 theorems and $T_j(f)$ is a, possibly infinite, set of tautologies valid for all curves f that we wish to study. A priori, therefore, f could be arbitrary, fixed genus, or whatever, but in practice, and mathematical precision in 1.6, f is rational. For example any map $f: C/k \to X$ from a smooth curve to a geometric fibre has a factorisation $f = f_0 g$ into purely inseparable followed by separable, so, the existence of such a factorisation is such a tautology. Better still separable maps admit a derivative $f_1 := f_0'\colon C/k \to P_1 := \mathbb{P}(\Omega_{X/S}) \to P_0 := X$, and if L_1 is the tautological bundle then

$$L_1 \cdot_{f_1} C = -\chi(C) - \mathrm{Ram}_{f_0} \tag{10}$$

is another tautology, which can even be iterated to an infinite sequence of tautologies, *i.e.* for $P_m \hookrightarrow \mathbb{P}(\Omega_{P_{m-1}/S})$ the mth jet bundle, L_m its tautological bundle, and f_m the mth derivative of f_0,

$$L_m \cdot_{f_m} C = -\chi(C) - \mathrm{Ram}_{f_{m-1}}. \tag{11}$$

Now Bogomolov's condition is that for some a, b the ACF_0 statement

$$P_{a,b}(X)\colon\ \mathrm{H}^0(X, \mathrm{Sym}^a \Omega^1_X(-bH)) \neq 0 \tag{12}$$

is a theorem, so, choosing a, b appropriately and using just the tautology (10) the reduction of 1.1–1.2 to 1.4–1.5 is essentially ACF_0 in the sense of (9). On the other hand a theorem is ACF_0 iff it's ACF_p for $p \gg 0$, so a proof with the structure of (9) would imply a uniform bound in moduli for almost all p, and whence 1.5.(b) implies

Corollary 1.6 *For every bi-disc quotient X, possibly with cusps E and quotient singularities, in which case X is to be understood as a Deligne–Mumford champ, but not a finite quotient of a product of curves, and $\mathcal{F}$ either of the natural foliations the following* (*true*) *statement has no essentially* ACF_0 *proof;*

There are at most finitely many $\mathcal{F}$-invariant rational curves on $X_{\mathbb{C}} \backslash E$. (13)

Now while it's true that by 1.4.(b) we could construct for high genus curves on products of curves a related statement with no *essentially* ACF_0 proof, this is only because of an evident separability rather than "bi-separability" issue which we could have excluded via a more sophisticated variation of (9). We haven't bothered to do this, since it's not only irrelevant for rational curves, but, manifestly, the weaker the statement with no essentially ACF_0 proof is, the more interesting it becomes. For example, the tautologies (11) imply tautological inequalities,

$$L_m \cdot_{f_m} C < 0 \tag{14}$$

for rational curves, while for each m, our foliation, $\mathcal{F}$, defines a surface $X_m \hookrightarrow P_m$ in the mth jet space through which f_m must factor, so for all non-negative integers $n_1, \ldots, n_m$ with non-zero sum, n, all the ACF_0 statements

$$\mathrm{H}^0\big(X_m,\ L(\underline{n}) := L_1^{\otimes n_1} \otimes \cdots \otimes L_m^{\otimes n_m}\big) \neq 0 \tag{15}$$

are false on bi-disc quotients which aren't quotients of products of curves. Now, if there are cusps, the $L(\underline{n})$ of (15) aren't very interesting, *e.g.* not even pseudo effective [17, IV.5.7], but (13) is so weak that we have the right to do all this logarithmically, so the natural map $X_{m,\log} \to X$ is an isomorphism and the bundle in (15) (understood logarithmically if necessary) is just the nth power of the canonical of the foliation $K_{\mathcal{F}}^n$, so, we recover the well known fact, [23] or [17, IV.5.4], that $K_{\mathcal{F}}$, which is in fact nef. in characteristic zero, has Kodaira dimension $-\infty$, which as it happens, 3.3, is a lemma in the proof of 1.5.(b), but, in principle 1.6, or better, 1.5.(b) is a much stronger fact than $\kappa(\mathcal{F}) := \kappa(K_{\mathcal{F}}) = -\infty$, *c.f.* 1.9.(c) for another example.

As to the proof of the theorems themselves, they are largely an application of the classification theorem, [17], of foliated surfaces, the theorems of which (if not the proof) are statements in ACF_0, so by model completeness they're actually ACF_0 theorems, and can be taken to be the P_i in (9). The tautologies, T_j, of *op. cit.* while taking (10) as their starting point are much more sophisticated. In particular they are false if the fibre over the generic point of S does not have canonical, 2.3, foliation singularities, and require careful analysis, (83)–(97), of how invariant curves meet such singularities. The precise statement is the refined tautological inequality, 4.2, and it is delicate. For example in characteristic zero (14) for all m is equivalent to being rational, but not in characteristic p, 3.4, and one could reasonably call such curves *pseudo-rational*. Now the refined tautological inequality gives 1.4.(a) & 1.5.(a) more or less gratis when the foliation is of general type, *i.e.* $\kappa(\mathcal{F}) = 2$, but it does not imply that pseudo-rational curves invariant by such foliations aren't Zariski dense in mixed characteristic. Indeed it strongly exploits the difference between rational and pseudo rational—or genus g and pseudo genus g for that matter—and it may well be the case that there is a foliation of general type in mixed characteristic with Zariski dense pseudo-rational curves, for which, by the same reasoning as above, (15) would have to be false for all $\underline{n}$ but $K_{\mathcal{F}}$ would be big. In any case since we're also supposing that X has general type in the usual sense, the only other possibility is that $\mathcal{F}$ is one of the natural foliations on a bi-disc quotient. The case of which where $\kappa(\mathcal{F}) \geqslant 0$ is a product of curves, and we have the pleasing fact that foliated Mori theory hits the logical obstruction of 1.6 on the nose, *i.e.*

$$\mathcal{F} \text{ has only finitely many invariant rational curves} \tag{16}$$

is always a true statement for complex surfaces of general type, but

Corollary 1.7 *Let $\mathcal{F}$ be a foliation on a complex surface of general type, then T.F.A.E.*

(a) *The foliated Kodaira dimension satisfies $\kappa(\mathcal{F}) \geqslant 0$.*
(b) *Foliated abundance holds, i.e. $\nu(\mathcal{F}) = \kappa(\mathcal{F})$.*
(c) *The* (*true*) *statement* (16) *is an essentially* ACF_0 *theorem.*

As such it remains to indicate the proof of 1.5.(b), and the distinct behaviour of the invariant rational curves according to whether p is split or inert, which are, in fact, equivalent to the foliation being p-closed, 2.5, or not. In this context, it's relevant to observe that the way [14] would prove (13) is not Liouville's theorem but Baum–Bott residue theory, Appendix, which gives over $\mathbb{C}$ that the degree of the canonical, $K_{\mathcal{G}}$, of the other foliation along our curve is zero, and the rest is easy. Now a particular feature of the natural foliations on bi-disc quotients is that once they're p-closed, they're p-adically integrable, and at such p one can mimic, (50), Baum–Bott residue theory with values in a characteristic zero field. Consequently the proof of [14] may be pushed through. The case of 3.9 of rational or elliptic curves missing the cusps (whence a p-adic proof of (13) provided one chooses p correctly, *i.e.* split) is particularly easy, whereas the general case, see the proof of 1.5 in Sect. 5, is very much a variation on [14]. Otherwise p is inert and the locus of vanishing of the p-curvature is no longer the whole surface but a divisor cut out by the non-classical modular form of 1.5.(b), and if one looks at this divisor carefully enough, 3.6, one finds the rational curves (8). Complimentary to this is the behaviour at a fixed split prime. In characteristic zero, as the referee observed, 3.7, again by Baum–Bott, every invariant curve is in the resolution of the cusps. Similarly, at the split primes, there is a strong notion of invariance with analogous properties to that in characteristic zero, which, in terms of (a priori incompatible) local liftings equates to full p-adic invariance. The possibility of rational curves which might be invariant in this sense was already the reason for our variant, (50)–(51), of Baum–Bott, but it can be pushed to a strict analogue of the absence of any (non-trivial) invariant curves, to wit:

Compliment 1.8 (3.13) *Let* $(Y, \mathcal{F})$ *be the natural foliation on a bi-disc quotient of 1.5.*(b)*, and Y_p the reduction at a sufficiently large prime, p, split in the quadratic field of op. cit., then there are no curves other than those in* (*the possibly empty*) *resolution of the cusps satisfying the strong $\mathcal{F}$-invariance discussed above, i.e. that factorisation* (48) *occurs infinitely often.*

Amusingly, our variant of Baum–Bott immediately reduces this, (60), to a strictly weaker statement, 3.20, than the (foliated) Kodaira dimension being $-\infty$ modulo a split prime, but because there is, for p fixed, no relation between ACF_p and ACF_ℓ even for ℓ all other primes, one cannot, deduce this as above, (14)–(15) *et seq.*, from the existence of rational curve at the inert primes, and instead there is a lengthy diversion to push through a weak version of the non-existence of sections in characteristic zero as found in [17, IV.5.4].

Before concluding this introduction it is appropriate to make

Scholion 1.9 (*Complex hyperbolicity as an essentially* ACF$_0$*theorem*) We know this can't be true by 1.6, but remarkably, for surfaces in the Bogomolov class, (13) with entire instead of rational is almost the only problem. More precisely for pointed Riemann surfaces there is a tautology of the form (10) but for intersection numbers and Euler characteristics understood in the sense of Nevanlinna theory, leading to a refined tautological inequality (which was actually the motivation for 4.2) and one can ask for an *essentially* ACF$_0$ proof of (2), exactly as in (9) but with the implied tautologies, intersection numbers, *etc.* understood in the Nevanlinna sense. In particular, the original motivation of the classification theorem was (with a view to generalisation) to find an essentially ACF$_0$ proof of [14]. This is discussed in the introduction to [17] at length, so we won't go on about it here, but the basic irony of the proof of the classification theorem is (modulo soft theorems, Gromov's sense, of Brunella, [5], and Duval, [6]) is that it's essentially a corollary of what it was intended to give an essentially ACF$_0$ proof of, *i.e.* the main theorem on algebraic degeneracy of entire curves invariant by a foliation of [14]. As such, the proof of the classification is certainly not ACF$_0$. Nevertheless its statements are, so, as we've said, by model completeness they are ACF$_0$ theorems. Now argue as we did before, in the Nevanlinna setting the first step, *cf.* (12), is still essentially ACF$_0$, and we reduce to analytic curves invariant by foliations. Again, the $\mathcal{F}$ of general type case is gratis, while products of curves is easy, and again essentially ACF$_0$, so we're left with the class of surfaces in 1.6.

At this point *op. cit.* tells us that we're wasting our time, and we should look carefully at what is involved. To fix ideas, say our goal is no more than [14], *i.e.* there is no Zariski dense, $\mathcal{F}$ invariant entire curve, $f : \mathbb{C} \to X'$, for X' a smooth surface in the Bogomolov class which may be no better than a modification of a bi-disc quotient X. The cusps, E, are also $\mathcal{F}$ invariant, and no invariant curve can meet them, so we certainly have $f : \mathbb{C} \to X\backslash E$. There is, however, a possible issue with quotient singularities, *i.e.* we may only have an entire curve on the moduli of the Deligne–Mumford champ of (13) and not an entire curve in the orbifold/Deligne–Mumford sense in which the curves of (13) are rational. As such one cannot a priori use Liouville, and indeed there can be rational and elliptic curves on $X\backslash E$ if this is not understood in the orbifold sense. Nor, even supposing that it can be done when there are quotient singularities, is it necessarily appropriate to try and force a Liouville style argument, since if we want to understand the logical structure then less is best. Consequently, Baum–Bott residue theory still looks to be the way to go. Again this is absolutely trivial for bi-disc quotients in a way consistent with the intuition that (13) should be trivial, and one has that the Nevanlinna degree of the canonical, $K_{\mathcal{G}}$, of the other foliation is zero for free, with the subsequent steps being trivial given the refined tautological inequality which absorbs the problem of the quotient singularities. This line of reasoning has a number of further pleasing features, *viz*:

(a) A large chunk of Baum–Bott theory, *i.e.* with values in the base field, Appendix, is not only valid in the Zariski topology, but has perfect sense in ACF$_0$.

(b) Without further hypothesis, however, in characteristic p it, therefore, only calculates $\mathbb{F}_p$ valued intersection numbers of invariant bundles with invariant curves, and, as it happens, this is exactly how one gets the lower bound in (8).

(c) Under the p-adic integrability condition, not only does it work, 3.9, but the resulting proof, see the proof of 1.5 in Sect. 5, it affords of the boundedness of rational or elliptic curves at such primes is formally almost identical to what we've said above about the entire curve case over $\mathbb{C}$, and this is how it should be, *i.e.* a priori there's very little that one can say about a Zariski dense set of rational or elliptic curves in mixed characteristic, respectively a Zariski dense entire curve, other than tautologies of the form (10)–(11), respectively their Nevanlinna variants, and deductions of the form (9) by way of finitely many ACF_0 theorems. Baum–Bott with values in a characteristic zero field is, however, a statement which admits exactly this distinction, and it nails the logical problem posed by the rational curves of (8) on the nose, *i.e.* forget about everything one already knows, then, *in se*, the implication Baum–Bott with values in a characteristic zero field & (8) (in fact just any lower bound going to ∞ for rational or elliptic curves) $\Rightarrow \mathcal{F}$ is not p-adically integral for $p \gg 0$ holds.

(d) It's in excellent agreement with Miyaoka's proof, [21], of 1.3. In general terms, this consists of fixing the curve and making finitely many ACF_0 statements about it and the surface. Amongst these statements, the only one that doesn't have the form (9) is closure of certain global logarithmic 1-forms whose poles are allowed to depend on the curve. In the general case of 1.1.(a) & 1.2.(a) this strategy may well fail, but for $\mathcal{F}$-invariant curves on bi-disc quotients, it should go through in the presence of p-adic integrability.

Despite these highly attractive properties, it does not follow from 1.6 or even (8) that there is a logical necessity for Baum–Bott theory in proving the Green–Griffiths conjecture, [14], for surfaces in the Bogomolov class. It is, however, an absolute logical necessity that some non ACF_0 statements with very similar properties to (c) must intervene. The present scholion is, however, a best possible scenario wherein model completeness in ACF_0 gets us down to the bi-disc case. The practice is, currently, much worse, *i.e.* the classification theorem, [17], and whence all the results of the present article depend on the general Baum–Bott "residue estimate" of [14], or some appropriate variant thereof, [4]. This is much trickier than the (trivial over $\mathbb{C}$) bi-disc case, and nailing a similar "residue estimate", [18], for foliated 3-folds (whence, inter alia, Green–Griffiths for surfaces with $13c_1^2 > 9c_2$) can reasonably be described as difficult.

The original motivation for investigating the variation of (1)–(2) in mixed characteristic was an ingenious (unfortunately, its current status is we're stuck) idea of Fedor Bogomolov to reduce the study of the moduli of curves on surfaces of general type to the big co-tangent case, and I would certainly not have ventured into this area had he not prompted it. In questions of logic, I am always indebted to Ehud Hrushovski, and, in the particular, to an informative discussion—with an eye to getting more out of (8)—as to whether there was a better formulation of 1.6 than my *ad hoc* definition (9). At the risk of exposing my ignorance in such matters, with any misunderstanding being mine, his comments may usefully be noted: there is an elaboration of a language about the "generic curve" in course, but, currently, it couldn't be considered usable; secondly (my) incompetence in making logical definitions shouldn't distract from the limitations that (8) puts on any extension of the Lefschetz principle. This is, however, first and foremost a paper in mixed characteristic algebraic geometry, whose

presentation has been much improved by the carefully considered comments of the referee, *e.g.* 3.7, and, implicitly, the inclusion of the Appendix, on Baum–Bott theory. Finally the influences of Nick Shepherd-Barron and Torsten Ekedahl, both in the classification theorem and the present article are legion. I never met Ekedahl, but Nick has frequently communicated to me many of his insights. Amongst which, I asked Nick how one proved the "folk theorem" (I had heard a similar thing from Oort) 1.5.(b). He replied that it was Ekedahl who had told him, and Ekedahl had said it was just a matter of looking at where the p-curvature vanishes, and applying adjunction. As it happens, it took me a bit of thought to find the seemingly not so obvious adjunction, (32), but in terms of Ekedahl's exact sequence, (42), it's extremely natural, and to the author of the p-closed condition it would have been wholly obvious. Requiescat in pace.

2 Singularities

By a foliation (by curves) on an algebraic space, or indeed champ, X over a locally Noetherian base S we will mean a rank 1 quotient of $\Omega_{X/S}$, *i.e.* a short exact sequence,

$$0 \to \Omega_{X/\mathcal{F}} \to \Omega_{X/S} \to K_{\mathcal{F}}.I_Z \to 0. \tag{17}$$

This definition supposes a certain amount of regularity. If we were working with champs (which won't really be the case) then X should be Deligne–Mumford, otherwise $\Omega_{X/S}$ isn't defined, while to write the quotient as $K_{\mathcal{F}}.I_Z$, where $K_{\mathcal{F}}$ is a bundle and Z the singular locus supposed of co-dimension at least 2 amounts to supposing that the foliation is Gorenstein, *i.e.* given everywhere by a vector field non-vanishing in co-dimension 1.

If we further suppose that S is a field, k, say, of arbitrary characteristic and X is normal irreducible, then we can functorially extend the definitions of Mori theory. Details may be found in [17, I.1–I.2] for surface (which is largely our present interest), and [16, I.6–7], or [20, I.iii, III.i] in general. In particular it transpires that log-canonical and Gorenstein is equivalent to non-nilpotence (at closed points of Z of (17)) in the sense of the following

Revision 2.1 (cf. [13]) *We let A be a complete regular local ring containing a coefficient field, $\bar{k}$, supposed algebraically closed. Next let $\partial \in \mathfrak{m}\mathrm{Der}_{\bar{k}}(A)$. For every $n \in \mathbb{N}$ we have an exact sequence,*

$$0 \to \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \to \frac{A}{\mathfrak{m}^{n+1}} \to \frac{A}{\mathfrak{m}^n} \to 0.$$

We can consider ∂ as a $\bar{k}$-linear endomorphism, ∂_n, of $A_n = A/\mathfrak{m}^n$ for each n. Consequently ∂_n has a Jordan decomposition $\partial_{S,n} \oplus \partial_{N,n}$ into a semi-simple and nilpotent part. These are compatible with the restriction maps $A_{n+1} \to A_n$, and so on taking limits give a Jordan decomposition $\partial_S \oplus \partial_N$ of ∂.

Observe as an immediate consequence

Fact 2.2 *∂ is semi-simple iff there's a choice of generators $x_i \in \mathfrak{m}$, and $\lambda_i \in \overline{k}$ such that*

$$\partial = \sum_i \lambda_i x_i \frac{\partial}{\partial x_i} \in \mathrm{Der}_{\overline{k}}(A). \tag{18}$$

Where it's important to emphasise that semi-simplicity in 2.2 is to be understood in all of A, rather than just modulo $\mathfrak{m}^2$ since this is its sense in

Fact 2.3 ([20, III.i.3]) *A foliation in characteristic zero with log-canonical Gorenstein singularities has canonical singularities iff it has no semi-simple points at which, up to scaling, the λ_i of (18) are non-negative integers.*

Describing the nilpotent part is worth the trouble. Notice: $[\partial_S, \partial_N] = 0$, so given ∂_S as above we just compute a basis for fields which commute with it. Putting $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and $\Lambda \cdot -$, to be the usual inner product, albeit with values in $\overline{k}$, these are easily seen to be, cf. [13]

$$\begin{aligned} &\text{(a)} \quad x^Q x_i \frac{\partial}{\partial x_i}, \quad \Lambda \cdot Q = 0,\ \ Q = (q_1, \ldots, q_n),\ \ q_j \in \mathbb{N} \cup \{0\}, \\ &\text{(b)} \quad x^Q x_i \frac{\partial}{\partial x_i}, \quad \Lambda \cdot Q = 0,\ \ Q = (q_1, \ldots, q_n),\ q_j \in \mathbb{N} \cup \{0\} \text{ for } j \neq i, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q_i = -1 \end{aligned} \tag{19}$$

where of course $x^Q = x_1^{q_1} \ldots x_n^{q_n}$. As such, following [17, II.1.6]

Corollary 2.4 *Let ∂ be a non-singular derivation of a complete regular local ring A over an algebraically closed field $\overline{k}$ of characteristic $p > 0$ isomorphic to its residue field then there is a choice of coordinates $x, y_1, \ldots, y_n$ in the maximal ideal such that up to multiplication by a unit,*

$$\partial = \frac{\partial}{\partial x} + \sum_{i=1}^{n} x^{p-1} f_i(x^p, \underline{y}) \frac{\partial}{\partial y_i}.$$

Proof We can certainly multiply ∂ by a unit, in such a way that for some $x \in \mathfrak{m}$, $\partial x = 1$. Now consider $\widetilde{\partial} = x\partial$, and its Jordan decomposition $\partial_S \oplus \partial_N$. Trivially $\partial_S = x\partial/\partial x$, in some coordinate system $x, y_1, \ldots, y_n$. Observe that in our formulae for the nilpotent part we must have an exponent of x in the monomial $(x^Q x_i)$ at least 1, since $x | \widetilde{\partial}$, whence the claim. □

This is best possible, and so could reasonably be called the characteristic p Frobenius theorem. One can only do better if the following holds.

Definition 2.5 ([7]) The foliation is *p-closed* if for some, and in fact any, local generator ∂ of the foliation the fields ∂^p and ∂ are parallel.

The special coordinates of the divertimento can be used to 'compute' the p-curvature, to wit:

Fact 2.6 *The ideal where $\partial^p \wedge \partial$ vanishes is exactly the ideal cut out by the $f_i(x^p, \underline{y})$'s whence (albeit this doesn't require special coordinates) it is invariant.*

Proof We have $\partial^p(x) = 0$, while

$$\partial^p(y_i) = \partial^{(p-1)} x^{p-1} f_i = \sum_{a+b=p-1} \frac{x^b \partial^b}{a!(b!)^2}(f_i)$$

and for all i, $\partial(f_i)$, from which $\partial^b(f_i)$ for any b, belongs to $(f_1, \ldots, f_n)$. □

As such, the p-curvature can only vanish if all the $f_i = 0$, so

Fact 2.7 *The following are equivalent for a smoothly foliated irreducible algebraic space or indeed champ,*

(I) *The foliation is p-closed.*
(II) *There exists a closed point ξ such that in the complete local ring $\widehat{\mathcal{O}}_{X,\xi}$ there are coordinates $(x, y_1, \ldots, y_n)$, and the foliation has the form,*

$$(x, y_1, \ldots, y_n) \mapsto (x^p, y_1, \ldots, y_n). \tag{20}$$

(III) *For every point ξ there are coordinates in the complete local ring $\widehat{\mathcal{O}}_{X,\xi}$ such that the foliation has the form,*

$$(x, y_1, \ldots, y_n) \mapsto (x^p, y_1, \ldots, y_n). \tag{21}$$

Consequently, even if Z is empty, it's far from true that curves invariant by a foliation by curves have to be smooth, and this is the case even if the foliation isn't p-closed. For example,

$$\partial = \frac{\partial}{\partial x} + x^{p-1}(y^2 + x^p)\frac{\partial}{\partial y} \tag{22}$$

then the invariant curve $y^2 + x^p$ certainly isn't smooth, even though the foliation isn't p-closed, and, indeed, by 2.4 the singular locus of the set where the p-curvature vanishes is essentially arbitrary. It's therefore perhaps a little surprising that there are natural criteria whereby the locus of vanishing of the p-curvature is extremely well behaved at the singular points, *e.g.*

Fact 2.8 *Suppose A of 2.4 has dimension 2, and that ∂ has 2-invariant smooth transverse branches while being semi-simple modulo $\mathfrak{m}^2$ with the ratio, λ_0, of the eigenvalues not belonging to $\mathbb{F}_p$, then there is a choice of formal coordinates such that, up to rescaling by a constant,*

$$\partial = x\frac{\partial}{\partial x} + \lambda(x^p, y^p)\, y\frac{\partial}{\partial y}, \quad \lambda(0,0) = \lambda_0 \tag{23}$$

so, in particular, the ideal defined by $\partial^p \wedge \partial$ is just (xy).

Indeed, just apply (19)—by hypothesis we don't have to worry about (b).

3 Rational curves on bi-disc quotients

Now let us apply these considerations to find rational curves on bi-disc quotients, X, which are not (a precision that will subsequently be eschewed) finite quotients at a set of positive characteristics of positive (in fact about half) density. We begin with the smooth case, so in characteristic 0, Ω_X splits as

$$\Omega_X = K_{\mathcal{F}} \amalg K_{\mathcal{G}} \tag{24}$$

for 2-integrable foliations $\mathcal{F}$, and $\mathcal{G}$. By [9, 6.2.(iii)] there is a set of primes of positive density where $\mathcal{F}$ isn't p-closed (and, as it happens, at the same primes $\mathcal{G}$ isn't closed either, 3.8, but we don't need this for the moment). We work over the reduction modulo p at such a prime, with say base change to the algebraic closure k for convenience, and, of course $p \gg 0$ to guarantee not just good reduction, but also the splitting (24). The essential fact that we will use in analysing the locus where the p-curvature vanishes is

Revision 3.1 ([23] or [17, IV.5.4]) *In characteristic zero $K_{\mathcal{F}}$ and $K_{\mathcal{G}}$ are nef. line bundles of Kodaira dimension $-\infty$, so in particular*

$$K_{\mathcal{F}}^2 = K_{\mathcal{G}}^2 = 0, \quad 2K_{\mathcal{F}} \cdot K_{\mathcal{G}} = c_2 > 0, \quad K_X, \text{ is ample.} \tag{25}$$

So that before getting underway let's make

Warning 3.2 Over our current choice of k, $K_{\mathcal{F}} \otimes k$ and $K_{\mathcal{G}} \otimes k$ will be big, whence very far from nef., although (manifestly) (25) will continue to hold.

The first step is to write the locus of vanishing of the p-curvature (of $\mathcal{F}$) as a sum of irreducible divisors

$$P := pK_{\mathcal{F}} - K_{\mathcal{G}} = \sum_i n_i C_i \tag{26}$$

so that by 2.6 we know the Cartier divisors $\mathcal{O}_X(-n_i C_i)$ are $\mathcal{F}$-invariant. We wish, however, to know that the C_i themselves are $\mathcal{F}$-invariant. This will follow if we know that $p \not| n_i$, which follows a fortiori from

Fact 3.3 *Let N_p be the maximum of the n_i's in* (26) *then* $\limsup_p p^{-1} N_p = 0$.

Proof Suppose to the contrary that $\limsup_p p^{-1} N_p = \varepsilon > 0$, then for some infinite set of primes p, there is a curve D_p in characteristic p such that

$$K_{\mathcal{F}} - \varepsilon D_p \geqslant 0 \in \mathrm{NE}_1(X \otimes k) \otimes \mathbb{Q}. \tag{27}$$

Consequently, the degrees of the D_p are bounded independently of p, so they must belong to finitely many components of the Hilbert-scheme. Sub-sequencing as necessary, we can suppose that this component is the same for all p, and whence it's non-empty in characteristic zero. As such (27) may be supposed to hold for $D_p = D_0$ independent of p from which $K_{\mathcal{F}} - \varepsilon D_0$ is pseudo-effective in characteristic zero.

However, *cf.* [17, IV.5.7], by Zariski decomposition and Hodge-index in characteristic zero this forces $K_{\mathcal{F}}$ to be numerically equivalent to an effective divisor (whence effective since $q(X) = 0$) in characteristic zero contradicting 3.1. □

As such for $\nu_i : \widetilde{C}_i \to C_i$ the normalisations of the curves in (26); $-\chi_i = 2g_i - 2$ their geometric Euler characteristics; and r_i the ramification of the ν_i we obtain

$$K_{\mathcal{F}} \cdot C_i = -\chi_i - r_i. \tag{28}$$

Again, before proceeding, let us make another

Warning 3.4 It is immediate, from (25)–(26) that there are curves with $K_{\mathcal{F}} \cdot C_i < 0$ and a moments thought even shows that the tangent sheaf of C_i is even a bundle isomorphic to $K_{\mathcal{F}}^{\vee}|_{C_i}$. This does not, however, imply that the C_i are rational, since tangent vectors do not necessarily lift to the normalisation in positive characteristic. A perfectly good example is provided by (22), which has the further curious property that the derivative $\nu_i' : \widetilde{C}_i \to P_1 := \mathbb{P}(\Omega_X^1)$, and indeed all subsequent derivatives, $\nu_i^{(m)}$, to all higher jet spaces, P_m, has exactly the same ramification as ν_i. In particular if L_m were the tautological bundle on such spaces, then

$$L_m \cdot_{\nu_i^{(m)}} \widetilde{C}_i \text{ (here } = K_{\mathcal{F}} \cdot C_i) < 0, \quad \text{for all} \quad m \tag{29}$$

which for many proof theory purposes is just as interesting as C_i rational. Nevertheless, in characteristic p, the *pseudo rationality* condition (29) is not equivalent to rationality.

Complementary to (28) we have the tangency divisor of C_i with $\mathcal{G}$, *i.e.* the composition of

$$\mathcal{O}_{C_i}(-C_i) \to \Omega_X|_{C_i} \to K_{\mathcal{G}}|_{C_i} \tag{30}$$

which is necessarily non-zero since k is algebraically closed—albeit perfect, which is natural here, since everything is arithmetic would do. Consequently if t_i is its degree then

$$t_i = (K_{\mathcal{G}} + C_i) \cdot C_i \geqslant 0. \tag{31}$$

Putting (28) and (31) together, we therefore obtain

$$(pK_{\mathcal{F}} + K_{\mathcal{G}}) \cdot C_i = -p\chi_i - C_i^2 + (t_i - pr_i). \tag{32}$$

Now observe that by (25), $(pK_{\mathcal{F}} + K_{\mathcal{G}}) \cdot P = 0$, so there is at least one curve such that (32) is not positive, and we assert

Claim 3.5 *If, for C_i in the support of P, $(pK_{\mathcal{F}} + K_{\mathcal{G}}) \cdot C_i \leqslant 0$ then C_i is rational.*

Proof In the first place, we show $C_i^2 < 0$, since, otherwise $P \cdot C_i \geqslant 0$, and whence

$$K_{\mathcal{F}} \cdot C_i \geqslant (p+1)^{-1} K_X \cdot C_i > 0 \tag{33}$$

so that the left hand side of (32) is at least $2p(p+1)^{-1}K_X\cdot C_i > 0$, which is nonsense. Now suppose that C_i isn't rational, *i.e.* $\chi_i \leqslant 0$ then we must have

$$pr_i > t_i. \tag{34}$$

To exclude this follows a fortiori if it doesn't happen locally at any branch through a singular point of C_i, so to this end we take a generator, ∂, of $\mathcal{F}$ in the special coordinates, and notation, of 2.4 with A the completion of $\mathcal{O}_X$ in the said singularity, so that our branch is defined by a prime factor $g(x^p, y)$ of $f(x^p, y)$ being equal to zero, while the local contribution to the tangency divisor of the branch is the order of $g_y(x^p, y)$ along it. Now consider the normalisations, $\nu\colon \Delta \to \Delta^2$, $\nu'\colon \Delta \to \Delta^2$ of the irreducible plane curves $g(x^p, y) = 0$, and $g(z, y) = 0$, where $\Delta := \mathrm{Spf}\, k[[t]]$. These fit into a diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{t\mapsto(\bullet t^m,\, \bullet t^n)} & \Delta^2 \\ {\scriptstyle t\mapsto \bullet t^d}\downarrow & & \downarrow{\scriptstyle (x,y)\mapsto(x^p,y)} \\ \Delta & \xrightarrow{s\mapsto(\bullet s^{m'},\, \bullet s^{n'})} & \Delta^2 \end{array}$$

where (m, n), respectively (m', n') are relatively prime positive integers, and $\bullet$ denotes some unit. As such, we obtain

$$pm = dm', \quad n = dn'$$

and there are two cases to consider. In the first place suppose that $p|d$, then $m = m'$, $n = pn'$, and the ramification is at most $m - 1$. If, however, we write $g(x^p, y)$ as a power series in monomials $x^a y^b$, then we must have $am + bn \geqslant mn$, so the tangency divisor is at least

$$(m-1)n \geqslant p(m-1)$$

which contradicts (34). In the other case, $m' = pm$, $n = n'$, so the ramification is at most $n - 1$, while if we write $g(z, y)$ as a power series in monomials $z^a y^b$, then we must have $am' + bn' \geqslant m'n'$, and whence the tangency is at least

$$n(mp - 1).$$

From which (34) implies $n - 1 > n(m - 1/p)$, so $m = 1$, and the absurdity that the branch is unramified. □

It therefore remains to consider the general bi-disc quotient. In characteristic zero, after resolving the cusps, these have at worst isolated quotient singularities. In particular they are smooth champs in a particularly simple way, *i.e.* they admit a finite étale covering by a space, and since our interest is to find rational curves on such champs, there is absolutely no loss of generality in supposing that there are no quotient singularities. As such, we can work on a smooth surface X with simple normal crossing boundary

E, each connected component of which is a polygon of rational curves, and whence (24) generalises to a splitting

$$\Omega_X(\log E) = K_{\mathcal{F}} \amalg K_{\mathcal{G}}. \tag{35}$$

Remaining in characteristic zero for the moment, the foliation singularities, whether of $\mathcal{F}$ or $\mathcal{G}$, are identical with the singularities of E (which, b.t.w., is invariant by both foliations) and can be described [17, IV.2.2] in suitable formal coordinates by a generator

$$\partial = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y} \tag{36}$$

where λ is a real quadratic irrationality, and $xy = 0$ is the local equation for E. Consequently if we work modulo sufficiently large primes such that X, E have good reduction; (35) continues to hold; and $(\mathbb{F}_p(\lambda){:}\mathbb{F}_p) = 2$, so 2.8 applies. In particular, therefore, if we understand the locus of vanishing of the p-curvature logarithmically, then it does not contain any component of E, *i.e.* (26) holds with no C_i in the support of the cusps. On the other hand, [17, IV.2.2], $K_{\mathcal{F}} \cdot E' = K_{\mathcal{G}} \cdot E' = 0$ for every E' in the support of E, so P is disjoint from E, while $K_X + E$ descends to an ample divisor on the contraction of the cusps, whence everything is exactly as in the smooth case, and we deduce

Proposition 3.6 *Let (X, E) be a model over the integers of a smooth algebraic champ obtained by resolving the cusps (if any) of a bi-disc quotient no finite étale cover of which is a product of curves, then for $\mathcal{F}$ either of the natural foliations arising from the bi-disc structure there is a set, $\mathcal{P}$, of primes of positive density such that for k an algebraically closed field of characteristic $p \in \mathcal{P}$ there is a $\mathcal{F}$-invariant rational curve $f_p : \mathbb{P}^1_k \to X \backslash E$, and constants c, C, depending on X, H, where the latter is an ample divisor, such that*

$$pc \leqslant H \cdot_{f_p} \mathbb{P}^1_k \leqslant pC. \tag{37}$$

Proof Suppose first that $\liminf_{p\in\mathcal{P}} H \cdot_{f_p} \mathbb{P}^1_k$ is finite, then there is an irreducible component of the Hilbert scheme (say of a finite projective étale cover of X to avoid some technicalities) which contains a rational 1-cycle missing E for infinitely many primes, whence a non-trivial rational curve $f_{\mathbb{C}} : \mathbb{P}^1_{\mathbb{C}} \to X_{\mathbb{C}} \backslash E$, which is nonsense. On the other hand, the residue theory of Baum–Bott, [2], is, Appendix, valid in Hodge groups over k, and f misses E, so, (107), $K_{\mathcal{G}} \cdot_{f_p} \mathbb{P}^1_k$ vanishes modulo p, while $K_{\mathcal{F}} \cdot_{f_p} \mathbb{P}^1_k$ is at most -2. Consequently, since $K_{\mathcal{F}} + K_{\mathcal{G}}$ is ample on contracting the cusps, we must have $K_{\mathcal{G}} \cdot_{f_p} \mathbb{P}^1_k = pd$, $d \in \mathbb{N}$. Supposing, therefore, $p \gg 0$ to avoid bad reduction, we can take c in (37) anything such that $H - cK_{\mathcal{G}}$ is ample, while by (26), C is at worst $K_{\mathcal{F}} \cdot H$. □

As the referee observed this may usefully be contrasted with the situation in characteristic 0, or indeed, 3.13, sufficiently large primes where $\mathcal{F}$ is p-closed, to wit:

Remark 3.7 In characteristic 0 by the Frobenius theorem; the structure of canonical foliation singularities; and since everything is understood in the orbifold sense, the following are equivalent for a $\mathcal{F}$-invariant map $f: \Sigma \to X$ from a not necessarily compact Riemann surface

(a) $f^{-1}(E) \neq \varnothing$.
(b) $f^{-1}(\mathrm{sing}(\mathcal{F})) \neq \varnothing$.
(c) $f^{-1}(E) = \Sigma$.

If furthermore Σ is compact, and by a minor abuse of notation, we identify it with a smoothly embedded orbifold then if any of (a)–(c) were false, Σ would miss the singularities so by Baum–Bott theory, *cf.*, A.2, A.7 and A.6,

$$\Sigma^2 = K_{X/\mathcal{F}}(E)\cdot\Sigma = K_{\mathcal{G}}\cdot\Sigma = 0 \tag{38}$$

which, *cf.* proof of 3.3, is impossible by the index theorem. Consequently,

All invariant compact curves in characteristic zero must factor through E.

Or alternatively, apart from the cusps, one of the natural foliations on a bi-disc quotient has an invariant algebraic curve iff an étale cover (in the orbifold sense) is a product of curves.

Before proceeding to analyse the complimentary primes let us also make

Remark 3.8 If some étale cover of our bi-disc quotient were a product of curves, then, of course, the foliation is p-closed for all p and the divisor (26) never exists, nor, are there any rational curves. Similarly, in the hypothesis of 3.6, we can—[9, remark, p. 23] when there are no cusps, and [17, II.2.3] otherwise—take, for $p \gg 0$, the complement, $\mathcal{P}'$, to $\mathcal{P}$ to be primes where both $\mathcal{F}$ and $\mathcal{G}$ are p-closed. The decomposition $\mathcal{P}\amalg\mathcal{P}'$ of sufficiently large primes can be naturally identified with those, $\mathcal{P}$, which are inert, respectively, $\mathcal{P}'$, split in some real quadratic extension (depending naturally on X) $K/\mathbb{Q}$. It's both curious, and typical of this sort of problem, that the relation of K to the geometry is more obvious when there are singularities, *i.e.* in the notation of (36), it's just $\mathbb{Q}(\lambda)$.

As to what happens at $\mathcal{P}'$ the following is surprisingly, *cf.* 3.12, precise

Proposition 3.9 *Let (X, E) be as in 3.6 with (up to excluding a finite set of primes of bad reduction) $\mathcal{P}'$ the set of primes complimentary to $\mathcal{P}$ then for $f: C/k \to X$ any separable map from a smooth curve over k of characteristic $p \in \mathcal{P}'$ which doesn't factor through E there is a $m = m(f) \in \mathbb{N}\cup\{0\}$ such that*

$$(K_X+E)\cdot_f C \leqslant -(1+p^m)\,\chi(C\backslash f^{-1}(E)) \tag{39}$$

where for D a divisor on C, the Euler characteristic $\chi(C\backslash D)$ is $\chi(C)$ minus the number, D_{red} of points in D counted without multiplicity. In particular, therefore, for any such curve, $\chi(C\backslash f^{-1}(E)) < 0$, and over k, $X\backslash E$ is "algebraically hyperbolic".

Proof By hypothesis we have a non-trivial map

$$f^*\Omega_X(\log E) \to \omega_C(\log f^{-1}(E)) \tag{40}$$

and wild ramification is a help rather than a hindrance, *i.e.* the degree of the right hand side in (40) is at most $-\chi(C\backslash f^{-1}(E))$. Consequently if both the induced maps from $f^*K_{\mathcal{G}}$ and $f^*K_{\mathcal{F}}$ arising from combining (35) with (40) are non-trivial, then we get (39) with $m=0$, so without loss of generality f is $\mathcal{F}$, but not $\mathcal{G}$, invariant.

This much is characteristic independent, but by the definition of $\mathcal{P}'$, and [9, remark.(ii), p. 23] in the smooth case, respectively 2.8 should there be cusps, both our natural foliations are p-integrable. Even better, [9, remark.(iii), p. 23], they are actually p-adically integrable, so in particular $\mathcal{F}$ is a height m-foliation in the sense of [7] for all $m \in \mathbb{N}$, *i.e.* one can replace p by p^m in 2.7. This means that there are compatible factorisations,

$$X \xrightarrow{\rho_m} Y_m \xrightarrow{\sigma_m} X^{(m)} \tag{41}$$

of the mth powers, F_X^m, of the geometric Frobenius together with natural exact sequences

$$0 \to K_{\mathcal{F}}^{p^m} = \rho_m^* \sigma_m^* K_{\mathcal{F}}^{(m)} \to \rho_m^* \Omega_{Y_m^*}(\log E_m) \to \Omega_{X^*}(\log E) \to K_{\mathcal{F}}|_{X^*} \to 0 \tag{42}$$

where E_m is the divisor induced by the cusps on Y_m which will be singular if the cusps are non-empty, so, for the moment we only understand (42) in the complement, X^*, resp. Y_m^*, of the (foliation) singularities. Irrespectively, the height $(m+1)$-structure defines a section of the left hand side of (42) so, again, away from the singularities we still have a splitting

$$\Omega_{Y_m^*}(\log E_m) = \sigma_m^* K_{\mathcal{F}}^{(m)}|_{Y_m^*} \amalg K_{\mathcal{G}_m}|_{Y_m^*}, \quad \rho_m^* K_{\mathcal{G}_m} = K_{\mathcal{G}}. \tag{43}$$

In order to understand the singularities we require

Lemma 3.10 *Let everything be as above with W the Witt vectors of k, then in the completion, $\mathfrak{X}$, of a foliation singularity (in a local lifting over W) the foliation admits a semi-simple generator, i.e. coordinates x, y such that $T_{\mathcal{F}}|_{\mathfrak{X}}$ is generated by*

$$x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, \quad \lambda \in \mathbb{Z}_p. \tag{44}$$

Proof We prove, by induction, that the foliation has a sequence of compatible generators, ∂_m, and coordinates x_m, y_m such that (44) holds modulo p^m, with $x_m y_m = 0$ the cusps modulo p^m. In the case $m=1$, [17, II.1.3], the Jordan decomposition of 2.1 and p-closure combine to give what we need. Now consider going from m to $m+1$ with $x = x_m$, $y = y_m$ and a generator mod p^{m+1} normalised by $\partial(x) = x$, then linear algebra reveals that ∂ semi-simple modulo p^m implies that we still have a Jordan decomposition $\partial_S + \partial_N$ modulo p^{m+1}, with the nilpotent part exactly as in (19).(a), *i.e.*

$$\partial_N = \sum_{i+j\lambda=0\,(p^{m+1})} \left(\varepsilon_{ij} x^i y^j\right) y \frac{\partial}{\partial y}, \quad \varepsilon_{ij} = 0 \ \left(p^m\right). \tag{45}$$

Now to obtain a local generator, ∂_m, of the foliation at the corresponding singular point of the resulting foliation on Y_m of (41) from ∂_{m-1} one applies the difference operator,

$$\delta(\partial_{m-1}) := p^{-1}\left(\partial_{m-1}^{p} - \partial_{m-1}\right)$$

by first working modulo p^2, then reducing modulo p. Better still the complete local ring of Y_m is easily expressed in the Jordan coordinates, *i.e.*

$$k[[x^i y^j]], \quad i + j\lambda = 0 \ \ (p^m) \tag{46}$$

and we deduce that the Jordan decomposition of ∂_m is

$$\delta^m(\partial_S) + p^{-m}\partial_N.$$

Consequently, [17, II.1.3] applies again to deduce that $\partial_N = 0\,(p^{m+1})$, and whence (44). □

This may be applied to extend the definition of $K_{\mathcal{G}_m}$ across the singularities via

Fact/Definition 3.11 Say a meromorphic differential, ω, on an irreducible variety, Y, has *log-poles* along a Weil divisor, E, if everywhere locally there exist (possibly empty but finite) sets of functions a_i, z_i such that

$$\omega - \sum_i a_i \frac{dz_i}{z_i} \in \Omega^1_Y, \quad \text{support}(z_i) \subseteq E \ \text{ for all } i.$$

Then $K_{\mathcal{G}_m}$ extends over the singularities to a line bundle on Y_m with log-poles along E_m.

Proof In the coordinates of (44) the generator of $K_{\mathcal{G}}$ on X at a singularity is given by

$$\omega = \lambda \frac{dx}{x} - \frac{dy}{y} \tag{47}$$

which for $Z_{ij} = x^i y^j$ a function on Y_m as per (46) is equally

$$-j^{-1}\frac{dZ_{ij}}{Z_{ij}}, \ \text{provided } p \nmid j,$$

so ω of (47) is a closed differential on Y_m with log-poles along E_m. Now, over Y_m^*, sections of $K_{\mathcal{G}_m}$ are exactly the closed differentials in $K_{\mathcal{G}_{m-1}}$, and whence the claim follows by induction in m. □

Now suppose inductively (we already have the case $m = 1$) that $\rho_m f$ admits a factorisation

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle F_C^m}\downarrow & & \downarrow{\scriptstyle \rho_m} \\ C^{(m)} & \xrightarrow{f_m} & Y_m \end{array} \tag{48}$$

where F_C is Frobenius and f_m is separable. If, however, this were to fail for $\rho_{m+1} f$, then the composition (wherein wild ramification is again help rather than hinderance)

$$f_m^* K_{\mathcal{G}_m} \to f_m^* \Omega_{Y_m}(\log E_m) \to \omega_{C^{(m)}}(\log f_m^{-1}(E_m))$$

implied by (43), *cf.* (40), is non-trivial, so by (48), and 3.11,

$$K_{\mathcal{G}} \cdot_f C \leqslant -p^m \chi(C \backslash f^{-1}(E)) \tag{49}$$

which, by the definition of $\mathcal{F}$-invariance, already holds with $m = 0$ for $K_{\mathcal{F}}$ whence (39).

As such, there remains the possibility that (48) holds for all m. Here, if it were true, that f could be lifted to characteristic zero, then it would follow from A.7 that the degree of $K_{\mathcal{G}}$ along f would be zero. In characteristic p, however, the situation is in a sense simpler since by (43), (48), and 3.11

$$f^* K_{\mathcal{G}} = (F_C^m)^* K_{\mathcal{G}_m} \ \Rightarrow \ p^m \,|\, K_{\mathcal{G}} \cdot_f C \tag{50}$$

and since this is hypothesised to hold for all m,

$$K_{\mathcal{G}} \cdot_f C = 0 \ \text{ which implies: } (K_X + E) \cdot_f C = K_{\mathcal{F}} \cdot_f C \leqslant -\chi(C \backslash f^{-1}(E)) \tag{51}$$

and completes the proof since $K_X + E$ is ample on contracting E, so, $\chi(C \backslash f^{-1}(E)) < 0$ is nonsense. □

Having achieved our goal let us observe that 3.9 is optimal

Remark 3.12 Consider first the case where $X = X_1 \times X_2$ is a product of smooth geometrically irreducible hyperbolic curves, and everything is defined over $\mathbb{F}_p$ to avoid some technicalities. Then for any map $f: C/\mathbb{F}_p \to X$ from another smooth geometrically irreducible curve, if both projections, f_i, are separable, then by Riemann–Hurwitz we get (39) with $m = 0$. The condition of 3.9, however, is only that f is separable, so there's nothing to stop us replacing $f = f_1 \times f_2$ by $g_m := f_1 \times F^m f_2$ for as large a multiple, F^m, of Frobenius as we like. As such $K_X \cdot_{g_m} C$ grows linearly in p^m, while $\chi(C)$ is constant, so the appearance of p^m in (39) is unavoidable and *separable curves of bounded genus on X are not bounded in moduli*. Similarly, if one toys with the examples in [1], exactly the same mechanism should yield examples of products of curves where

$$K_X \cdot_{f_i} C_i \geqslant -(1 - \varepsilon)(1 + p^m)\, \chi(C_i) \tag{52}$$

for infinitely many separable curves $f_i: C_i \to X$. As such, if, for any bi-disc quotient, X, be it a product of curves or not, were we to define $\mathcal{P}_X$, $\mathcal{P}'_X$ as primes where the natural foliations aren't, respectively are, p-closed then 3.6, (which, 3.8, is void for products of curves) and 3.9 (just apply Riemann–Hurwitz and separable/inseparable factorisation) hold exactly as stated. In particular, (39) is in no way the result of non-classical behaviour of bi-disc quotients at $p \in \mathcal{P}'_X$—indeed the overwhelming evidence suggests that one ought to have p-adic uniformisation at such primes, while from the proof we see that the mechanism giving rise to the factor p^m in *op. cit.* is exactly the same as that for products of curves. Similarly, to see that the factor p^m is necessary in (39) one can exploit, at least when there are no cusps, that the canonical model of Y_m occurring in (41) behaves, [8], exactly as in characteristic zero.

Specifically, the 5-canonical map is very ample, while Y_m has exactly the same Chern numbers as X, so there is a very ample bundle $H = 5K_{Y_m}$ whose generic member is a smooth curve $C_m \hookrightarrow Y_m$ with $\chi(C_m)$ independent of m. The pull-back, $\rho_m^* C_m$, is, generically, an irreducible curve C' whose normalisation, $f: C \to C'$, satisfies (48). As such the degree of f grows like p^m, while $\chi(C)$ remains bounded. Again, however, in all likelihood, the examples in [1] can be tweaked to give the lower bound (52) on the nose, *i.e.* $\varepsilon = 0$ up to the addition of a constant $o(p^m)$.

Similarly, we can improve our understanding of the possibility that (48) occurs ad infinitum by establishing the characteristic p-analogue of the referee's observation, 3.7, *i.e.*

Compliment 3.13 *Let p be a sufficiently large prime as in 3.9, then irrespectively of whether there are cusps or not, any algebraic curve $f: C \to X$ (with f the normalisation of its image) admitting the factorisation (48) for all m belongs to the support of a cusp.*

This is a good bit trickier, *cf.* 3.20, than its characteristic 0 motivation 3.7, and will need some preliminaries, to wit:

Lemma 3.14 (*cf.* 3.7) *Let p and f be as in 3.13 with E the cusps then f is an embedding, and the following are equivalent*

(a) $f^{-1}(E) \neq \varnothing$.
(b) $f^{-1}(\mathrm{sing}(\mathcal{F})) \neq \varnothing$.
(c) $f^{-1}(E) = C$.

Proof We only do (b) implies (c) of the equivalence since everything else is not only similar but easier. Irrespectively, for some integers $a, b \in \mathbb{N}$ write the map f at the singular point as

$$f: t \mapsto (\xi t^a u, \eta t^b v), \quad u(0) = v(0) = 1, \quad \xi, \eta \in k$$

in the mod p^m Jordan coordinates of (46), then by hypothesis for every pair of integers (i, j) with $i + j\lambda = 0\,(p^m)$,

$$\xi^i \eta^j t^{ia+jb} f^*(u^i v^j) \tag{53}$$

is a function of t^{p^m}, and since without loss of generality $ab \neq 0$ we must therefore have

$$\frac{b}{a} = -\lambda \ \ (p^m). \tag{54}$$

The integers a, b are, however, the order of tangency between f and the branches of the cusps, so they're independent of m while λ is a quadratic irrationality, and whence (54) is nonsense for $m \gg 0$. □

Lemma 3.15 *If X_K is the fibre over the quotient field of the Witt vectors of k, then without loss of generality, the specialisation map*

$$\mathrm{Pic}(X_K) \to \mathrm{Pic}(X) \tag{55}$$

restricted to the torsion sub-groups is an isomorphism.

Proof The torsion in the left hand side of (55) is finite, and a given bundle is trivial in characteristic 0 iff it's trivial for $p \gg 0$, so (55) is certainly injective on torsion. As to surjectivity, whether over X or X_K a torsion bundle L is numerically trivial, so if A is ample $n(A+L)$ has, by Riemann–Roch, a section for $n \in \mathbb{N}$ independent of p. □

Lemma 3.16 *For p as in 3.13 the bundle $K_{\mathcal{F}}$ is nef.*

Proof Suppose to the contrary that there is a curve C with $K_{\mathcal{F}} \cdot C < 0$. As such in the notation of (41)–(43),

$$\rho_m^* K_{Y_m} \cdot C = (p^m K_{\mathcal{F}} + K_{\mathcal{G}}) \cdot C < 0, \quad \text{for} \quad m \gg 0.$$

Thus, following [22], bend and break applies (a priori there could be a problem with singularities, but on surfaces it's a non-issue, [8, 2.1], since the canonical can't go up on passing to a minimal resolution) so that for H an ample bundle on X and every geometric point c of C there is a rational curve $L_c^m \ni c$ such that

$$\sigma_m^* H \cdot L_c^m \leqslant -4 \cdot \frac{\sigma_m^* H \cdot (\rho_m)_* C}{K_{Y_m} \cdot (\rho_m)_* C} \xrightarrow{m \to \infty} -4 \cdot \frac{H \cdot C}{K_{\mathcal{F}} \cdot C}. \tag{56}$$

Now if M_c^m is the reduced pre-image of L_c^m in X then the left hand side of (56) is

$$p^{m-n} H \cdot M_c^m$$

where n is the largest integer such that the factorisation (48) (albeit with M_c^m rather than C) holds. Consequently, not only do the curves M_c^m belong to finitely many families, but the difference $m - n$ is bounded as $m \to \infty$. This is, however, exactly, 3.13, what we're trying to eliminate, but with the added hypothesis that the curve is rational, and this is impossible by (51), and 3.14. □

Lemma 3.17 *To prove 3.13 it is sufficient to exhibit whether for $K_{\mathcal{F}}$, or $K_{\mathcal{G}}$, two disjoint curves some rational multiples of which are numerically equivalent to $K_{\mathcal{F}}$, resp. $K_{\mathcal{G}}$.*

Proof We do $K_{\mathcal{G}}$ since this is the case we will use. Irrespectively, since $h^{1,0}$ is zero in characteristic zero and $p \gg 0$, the hypothesis is equivalent to,

$$X \to |nK_{\mathcal{G}}| \tag{57}$$

being a base point free linear system for some large n. Now let,

$$\beta \colon X \to B \tag{58}$$

be the Stein factorisation of (57), then the a priori problem we face is that β may not be generically smooth. This can only occur, however, if we have a factorisation

$$\beta\colon\ X \xrightarrow{F_X} X^{(1)} \xrightarrow{\beta^{(1)}} B$$

through Frobenius, and replacing X whether by $X^{(1)}$ or some other twist $X^{(m)}$, $m \gg 0$, we eventually conclude, in the obvious notation, that $\beta^{(m)}$ is generically smooth. Now, as it happens, under the hypothesis of 3.13, X is isomorphic to $X^{(m)}$, albeit it's better practice to replace B by $B^{(-m)}$ and twist back, but, in any case, we can, on ignoring its relation with (57), suppose without loss of generality that (58) is a generically smooth map whose fibres are numerically equivalent to rational multiples of $K_{\mathcal{G}}$. In particular, therefore,

$$\beta^*\colon \beta^* K_B \to \Omega^1_X$$

is non-zero, while its projection to $K_{\mathcal{F}}$ is zero by (25) and 3.16, so we must have an identity of $\mathbb{Q}$-divisors

$$K_{\mathcal{G}} = \beta^*(K_B + \Delta)$$

for some divisor $\mathbb{Q}$ divisor Δ on B, and since $K_{\mathcal{G}} \neq 0$ in Néron–Severi,

$$\text{degree}(K_B + \Delta) > 0. \tag{59}$$

On the other hand, Δ is of the form

$$\sum_i \left(1 - \frac{1}{e_i}\right) + \sum_j a_j$$

where $e_i \in \mathbb{N}$ prime to p arise from the tame ramification, and $a_j \geqslant 1$ from the wild ramification, so (wild helps, and just as in characteristic zero) the left hand side of (59) is at least $1/42$. Consequently a generic fibre of β has degree with respect to any ample bounded by 42 times that of $K_{\mathcal{G}}$, so for $p \gg 0$ it lifts to characteristic zero, which is absurd by 3.1. □

With the preliminaries in place we can give

Proof of 3.13 Modulo replacing Σ by C, (38) is valid as stated for essentially the same reasons—our variant, (50), of Baum–Bott, and 3.14, while $h^{1,0} = 0$ for $p \gg 0$, so (identifying C with its smooth image) there are integers $a, b \in \mathbb{N}$ such that

$$aK_{\mathcal{G}} = bC \in \text{Pic}(X). \tag{60}$$

As such if d is the g.c.d. of the pair a, b and $a = da'$, $b = db'$ there is a line bundle M such that

$$K_{\mathcal{G}} = b'M,\ C = a'M \in \text{Pic}(X) \ \text{ mod (torsion)}. \tag{61}$$

Appealing to 3.15 we can replace X by an étale cover (in the orbifold sense if we don't a priori simplify the problem by way of the residual finiteness of the orbifold fundamental group) independent of $p \gg 0$, so, without loss of generality, $a = a'$, $b = b'$ are relatively prime, and (61) holds exactly in $\mathrm{Pic}(X)$, rather than mod torsion.

In any case, since C cannot also be invariant by $\mathcal{G}$ we have a non-zero tangency divisor, (30), which by (38) is nowhere zero, so by (60) C is $a+b$ torsion. Better, by (48),

$$\begin{gathered}\mathcal{O}_C(C) \text{ (indeed any divisor that descends to every } Y_{m,F}) \\ \text{is } p^m\text{-divisible in } \mathrm{Pic}(C)(F)\end{gathered} \tag{62}$$

for F the finite field of definition of C, so it's actually ℓ-torsion, for ℓ the non-p part of $a+b$. On the other hand, lifting ℓ-torsion bundles is purely topological, so in the completion $\widehat{X}$ of X in C there is such a bundle, L, and for X_n the nth thickening in C, we attempt, by induction to show that $C|_{X_n}$ is isomorphic to $L|_{X_n}$. On a sufficiently fine étale cover $U \to \widehat{X}$, with $(s,t)\colon R = U \times_X U \rightrightarrows U$ the projections, we have, therefore, a generator x of $\mathcal{O}_U(-C)$, which is hypothesised to satisfy,

$$s^*x = l(1+fx^n)t^*x \mod (I_C^{n+2}), \quad n \geqslant 1 \tag{63}$$

where $l\colon R \to \mu_\ell$ is the transition function for $L^\vee$, and $f \in \mathcal{O}_U$. We have however, without loss of generality, a generator ∂ of $T_{\mathcal{G}}|_U$, with transition function $g\colon R \to \mathbb{G}_m$, and applying this to (63) we obtain

$$s^*(\partial(x)) = gl(1+(n+1)fx^n)t^*(\partial(x)) \mod (I_C^{n+1}) \tag{64}$$

which means (since $\partial(x)$ is invertible): if D is defined to be $C - L$ then

$$(K_{\mathcal{G}}+C)+nD = 0 \in \mathrm{Pic}(X_{n+1}). \tag{65}$$

Multiplying by a and combining with (60) we have, therefore:

$$(a+b)C + anD = (b+a(n+1))D = 0. \tag{66}$$

Now, this far, we're just following the characteristic zero proof of IV.5.1, case (d), in [17], in which one concludes from the exact sequence

$$\mathrm{H}^1(C, \mathcal{O}_C(-nC)) \xrightarrow{x\mapsto 1+x} \mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n) \to 0 \tag{67}$$

in which D belongs to the kernel, that it's zero iff it's torsion. In positive characteristic, however, we only know this is p-torsion, so we have a problem albeit since a, b are relatively prime:

we're done if $p|a$.

As such, we can suppose a, p relatively prime, and form the ath root $\pi\colon X_a \to X$ of C guaranteed by the torsion free version of (61), which from $(a,p)=1$ is an $\mathcal{F}$

equivariant operation, so $X_a \to [X_a/\mathcal{F}]$ is still a height m-foliation for all m, while by 3.14

$$\Omega_{X_a}(\log E_a) = K_{\mathcal{G}_a} \amalg \pi^* K_{\mathcal{F}}, \quad K_{\mathcal{G}_a} = \pi^* K_{\mathcal{G}} + (a-1)C_a,$$
$$\pi^* C = aC_a.$$

Consequently, we have the advantage of a global section, ω, of $K_{\mathcal{G}_a}$, where

$$K_{\mathcal{G}_a} = (b+a-1)C_a := cC_a \tag{68}$$

but with the disadvantage that X_a may no longer be liftable. Regardless we still have factorisations

$$F^m_{X_a}:\ X_a \xrightarrow{\rho_m} Y_{a,m} \xrightarrow{\sigma_m} X_a^{(m)}$$

wherein both C_a and $K_{\mathcal{G}_a}$ are invariant by ρ_m and we assert

Claim 3.18 *For every m there is a global section ω_m of $K_{\mathcal{G}_a}$ over $Y_{a,m}$ such that $\omega = \rho_m^* \omega_m$.*

Proof of claim By induction, wherein $\omega = \omega_0$, the assertion is equivalent to ω_m being closed. Observe that ω_m vanishes along $C_a^{(m)}$ (identified with its image under ρ_m) to order c by virtue of (68) and what we're trying to exclude, *viz*: 3.13, so $d\omega_m \neq 0$ implies

$$K_{\mathcal{G}_a} + \sigma_m^* K_{\mathcal{F}} = cC_a + \Delta = K_{\mathcal{G}_a} + \Delta, \quad \Delta \geqslant 0 \tag{69}$$

and whence $\sigma_m^* K_{\mathcal{F}} = \Delta$, where by 3.16 and the log-variant of 3.1, Δ cannot vanish on $C_a^{(m)}$. Alternatively, at the level of Picard groups, we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ker}(\rho^*_{m,m+1}) & \longrightarrow & \mathrm{Pic}(Y_{m+1,a}) & \xrightarrow{\rho^*_{m,m+1}} & \mathrm{Pic}(Y_{m,a}) \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}(C_a^{(m+1)})[p] & \longrightarrow & \mathrm{Pic}(C_a^{m+1}) & \xrightarrow{F^*_{C^{m+1}}} & \mathrm{Pic}(C_a^m)
\end{array} \tag{70}$$

wherein what we have to prove is that the class $K_{\mathcal{G}_a}(-cC_a^{(m+1)})$ in the middle group of the top row vanishes, given, that by indiction it belongs to the kernel. The obstruction to vanishing is, however, exactly the section of $\sigma_m^* K_{\mathcal{F}}$ defined by (69), *i.e.* we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ker}(\rho^*_{m,m+1}) & \xrightarrow{\mathrm{obs}} & \mathrm{H}^0(Y_{m,a}, \sigma_m^* K_{\mathcal{F}}) \\
\downarrow & & \downarrow \\
\mathrm{Pic}\big(C_a^{(m+1)}\big)[p] & \xrightarrow{\mathrm{obs}} & \mathrm{H}^0\big(C_a^{(m)}, K_{C_a^{(m)}}\big)
\end{array} \tag{71}$$

with injective horizontal arrows. The pull back of $K_{\mathcal{G}_a}(-cC_a^{(m+1)})$ along the middle arrow of (70) is, however, by (62), prime to p torsion, so its image in the left hand corner of (71) is zero, *i.e.* Δ of (70) vanishes on C_a which is nonsense. ■

Now the Cartier operator respects the foliations, so for every m, by 3.18 we have sections

$$\text{Cartier}\,(\omega_m) \in \mathrm{H}^0(Y_{m,a}^*, K_{\mathcal{G}}) = \mathrm{H}^0(Y_{m,a}, K_{\mathcal{G}}) \tag{72}$$

which by 3.17 must be parallel to ω_m, and wherein $Y_{m,a}^*$ is, once more, the complement in the foliation singularities. Should it be non-zero, then c of (68) satisfies

$$p\,|\,(c+1). \tag{73}$$

As such if we return to the induction (63)–(67), (where now $a = 1$ and $b = c$) with the additional information of (73), then the strategy of *op. cit.* can be changed to prove that C is $p\ell$ torsion on $\widehat{X}$. Specifically, one can, without reference to (73), already take f in (63) to be in the image of Frobenius, and one changes $l = l_{\text{old}}$ of *op. cit.* to,

$$l_{\text{new}} = l_{\text{old}} F^* h, \quad h\colon R \to \mu_p, \text{ a co-cycle,} \tag{74}$$

which is possible since in the flat topology,

$$0 \to \mathcal{O}_C(-nC) \xrightarrow{x \mapsto 1+x} \mu_{p,X_{n+1}} \to \mu_{p,X_n} \to 0 \tag{75}$$

is exact, and the leftmost sheaf in (75) has no H^2. Consequently either the coefficient in (66) is prime to p and D is zero, or we get that $p|n$, and we can continue by way of (74) thanks to (75).

Finally, therefore, we have the possibility that for every m the Cartier operators in (72) vanish. In particular, therefore, (73) cannot hold, and we further assert

Claim 3.19 *Under the above hypothesis there is a factorisation,*

$$\sigma_m : Y_{m,a} \to T \to X_a^{(m)} \tag{76}$$

wherein the former map is bi-rational, in fact it is an isomorphism off $C_a^{(m)}$ where, for x a local equation of the same, it has a cusp of the form

$$x \mapsto \left(x^{c+1}, x^{p^m}\right) \tag{77}$$

while the latter map in (76) is an irreducible α_{p^m}-torsor.

Proof First suppose $Y_{m,a}^* = Y_{m,a}$, *i.e.* there are no cusps. The case $m = 1$ is standard, and T is given by

$$\mathcal{O}_{X^{(1)}}[\xi] \subset \mathcal{O}_{Y_1}, \quad \omega = d\xi$$

wherein the latter equation is to be understood locally since the difference of two such solutions is a pth power. Consequently, by 3.18, we can certainly take the 2nd map in

(76) to be a tower of α_p torsors, but we need a little more. To this end, confusing push forward to $X^{(m)}$ of a structure sheaf with itself for notational convenience, observe that we have sheaves of rings,

$$\mathcal{O}_{Y_{m,a}} \supset \mathcal{O}_{Y^{(1)}_{m-1,a}} \supset \cdots \supset \mathcal{O}_{Y_1^{(m-1,a)}} \supset \mathcal{O}_{X_a^{(m)}}$$

and the nth inclusion may be defined locally as

$$\mathcal{O}_{Y_{m,a}} \ni f \quad \text{iff} \quad \left(\frac{D^{p^i}}{p^i!}\right)(f) = 0, \quad 0 \leqslant i < n,$$

for some divided symmetric powers of a local vector field D in the $\mathcal{G}_a$-direction. A priori such powers need not commute, but if we normalise them by

$$\left(\frac{D^{p^i}}{p^i!}\right)(\omega^{(i)}_{m-i}) = 1 \tag{78}$$

then for a local solution $d\xi = \omega$ off $C_a^{(m)}$ we have

$$\left(\frac{D^{p^i}}{p^i!}\right)(\xi^{p^i}) = 1$$

so, in fact, they commute, and the exponential

$$\sum_{n=0}^{p^m-1} \left(\frac{D^n}{n!}\right)$$

affords an α_{p^m} action on $Y_{m,a}$ wherever the normalisation (78) has sense, *i.e.* off $C_a^{(m)}$. Nevertheless, the divided symmetric powers defined by (78) extend meromorphically around $C_a^{(m)}$, and while this doesn't define an action on $Y_{m,a}$ it does define an action on T defined by (77). In the general case that $Y^*_{m,a} \neq Y_{m,a}$ all of the above is valid over the complement of the foliation singularities, so that by 3.14 the extension of the torsor over these is (locally) just $Y_{m,a} \to X^{(m)}$ itself. ■

To conclude the proof of 3.13, we have

$$\mathrm{H}^1(X_a, \alpha_{p^m}) = \mathrm{Hom}(\alpha_{p^m}, \mathrm{Pic}^0(X_a)) \qquad \text{for all } m,$$

and T in 3.19 is irreducible, so $\mathrm{Pic}^0(X_a)$ must be positive dimensional. On the other hand, $\Omega^1_{X_a}$ can have no section other than ω—otherwise we get either a section of $K_{\mathcal{F}}$ which for $p \gg 0$ lifts to characteristic zero, or a second section of $K_{\mathcal{G}_a}$ contradicting 3.17—so by Igusa's inequality, the Albanese,

$$\alpha : X \to E$$

is an elliptic curve, and our section ω is a pull-back along α, which, in turn, is generically smooth. Plausibly, however, α restricted to C_a is inseparable, but replacing X by Y_m for $m \gg 0$ we can suppose this doesn't happen, so, in fact, C_a is a (possibly multiple) fibre of α, and we reduce to 3.17. □

Given the length of the proof of 3.13, and its close relationship with the classification of foliations in characteristic zero it's worth concluding with

Remark 3.20 The characteristic zero version, 3.7, of 3.13 is not only simpler, but, at least implicitly, stronger, since it follows a fortiori from the characteristic zero lemma

$$\begin{array}{c}\textit{A foliation, } \mathcal{G}\textit{, admitting an everywhere transverse curve } C \textit{ with } C^2 = 0 \\ \textit{has Kodaira dimension } 1\textit{, i.e. the curve moves in a pencil.}\end{array} \tag{79}$$

which, as we've said, is [17, IV.5.1.(d)], and is necessarily reproduced here en passant, (61)–(67). Similarly, and with little change, [19, III.10], (79) is valid for families in characteristic zero. Equally, it may well be the case that (79) holds in characteristic p, but we certainly haven't proved it, or better we've only proved the very weak variant in which C is not only invariant by a foliation $\mathcal{F}$ transverse to $\mathcal{G}$, but infinitely so in the sense that (48) holds for all m. As such, and despite its similarity with (79), neither 3.13, nor its proof, establishes that at the split primes either of the natural foliations on a bi-disc quotient, which isn't a product of curves, have Kodaira dimension $-\infty$.

4 Refined tautology

Again, let X be an algebraic space or Deligne–Mumford champ over a locally Noetherian base S. For $\mathrm{Spec}(k) \hookrightarrow S$ a closed (of any characteristic) point, and C/k a proper smooth curve (or indeed proper 1 dimensional smooth Deligne–Mumford champ), we consider a separable map $f: C \to X$. As such if $P := \mathbb{P}(\Omega^1_{X/S})$, then by base change $\mathbb{P}(\Omega_{X/k}) = P \otimes_S k$, to which, by the definition of separability, there is a derived curve, $f': C \to \mathbb{P}(\Omega_{X/k})$, admitting the following

Tautology 4.1 *Let L be the tautological bundle on P, and $\chi_C = 2 - 2g(C)$ the geometric Euler-characteristic of the curve, then*

$$L \cdot_{f'} C = -\chi_C - \mathrm{Ram}_f \leqslant -\chi_C. \tag{80}$$

In the case that $f: C \to X$ is invariant by a foliation by curves $\mathcal{F}$, with X/S smooth to fix ideas, this can, in the notation of (17), be re-written as follows: outside of Z, the foliation defines a section of $\mathbb{P}(\Omega_{X/S}) \to X$, whose closure over Z is the blow up $\pi: \widetilde{X} \to X$ in Z, including any implied nilpotent structure. As such there is an exceptional divisor E, $L|_{\widetilde{X}} = \pi^* K_{\mathcal{F}}(-E)$, and, provided that f doesn't factor through Z, the intersection $E \cdot_{f'} C$ can be identified with the Segre class $\mathrm{s}_Z(f)$, so the first part of the tautology (80) becomes,

$$K_{\mathcal{F}} \cdot_f C = -\chi_C - \mathrm{Ram}_f + \mathrm{s}_Z(f). \tag{81}$$

We'd like to refine this by "removing" the Segre class term. This can be done for S of finite type over $\mathbb{Z}$, and $(X, \mathcal{F}) \to S$ a family of foliations by curves with canonical singularities over each generic point of S—otherwise it's false, but we don't need that. The precise statement is,

Fact 4.2 *Let $S/\mathbb{Z}$ be of finite type, with $(X, \mathcal{F}) \to S$ a family of foliations by curves on a proper (over S) Deligne–Mumford champ with canonical $\mathbb{Q}$-Gorenstein singularities over each generic point of S, then for every $\epsilon > 0$ there is a closed, nowhere dense, sub-champ Z_ϵ of X such that every curve $f : C/k \to X$ over every closed point* $\mathrm{Spec}(k) \hookrightarrow S$ *invariant by $\mathcal{F} \otimes_S k$ which doesn't factor through Z_ϵ satisfies,*

$$K_{\mathcal{F}} \cdot_f C \leqslant -\chi_C + \epsilon H \cdot_f C \tag{82}$$

where H is a line bundle on the moduli of X which is big, relative to S, over every generic point of the same.

This is stated in [16, V.6.1] for S a field of characteristic zero, wherein the situation even involves invariant discs, which don't have so much sense in mixed characteristic. The proof though, which is all of *op. cit.* Section V, continues to have perfect sense, and works in the generality stated. With respect to the immediate interests of this article, we only need the case of X/S a smooth geometrically connected family of projective surfaces, and since the theorem is stable under base change and throwing away nowhere dense closed sets (precisions which may well be subsequently eschewed) we can further suppose that each singularity is a section. Plainly the difficulty is to estimate $s_Z(f)$. This is, however, subordinate to a local question: how does a local invariant curve meet Z, so we can suppose everything complete in some connected component of Z. Now, as it happens, the pure characteristic 0 case isn't much easier than that of mixed characteristic, and, indeed, is strictly easier than the variation for algebraic points over characteristic zero function fields treated in [19, III.2]. As such we confine ourselves to mixed characteristic, while supposing, for ease of exposition, that S is an open subset of the spectrum of the ring of integers of a number field, K. In particular, after appropriate localisation and without loss of generality, we can suppose that the completion in Z is an affinoid $\mathrm{Spf}(B)$ with trace S on which the foliation is defined by an everywhere singular, but non-vanishing in co-dimension 1, vector field ∂.

Proof of 4.2 for mixed characteristic families of surfaces To fix ideas, let's begin with the easy case, *i.e.* a so called saddle node over K, equivalently the semi-simple part (over K) has only one non-zero eigenvalue. As such, over K, one can refine (19) to find formal coordinates x, y in $B_{\{K\}}$ (where $\{\}$ is formal localisation, which, [15], is not to be confused with localisation) such that

$$\partial = x\frac{\partial}{\partial x} + \frac{y^{r+1}}{1+\nu y^r}\frac{\partial}{\partial y}, \qquad \nu \in K, \quad r \in \mathbb{N}. \tag{83}$$

Now, while there is no comparably simple formula in all of B, observe that 4.2 is stable under blowing up in the singularities, which we can suppose done a priori so we have an exceptional divisor $E \supset Z$ which, without loss of generality, in (83) is the plane $y = 0$. Similarly, straightforward linear algebra shows that the plane $x = 0$ is a well defined $\mathcal{F}$-invariant formal sub-scheme in all of B, so over B we can suppose that our generator has the form

$$\partial = x\,\frac{\partial}{\partial x} + y^{r+1} b(y, x)\,\frac{\partial}{\partial y}, \qquad b(0, 0) = 1. \tag{84}$$

With this is mind, we next consider what happens at $s \in S$ of characteristic p, *i.e.* reduce modulo $k(s)$, and complete $Z \otimes k(s)$ to get a new local ring A_s in which 2.1 and (19) are valid. Consequently, for a change of coordinates of the form $(x, y) \mapsto (x, z = yu_s(x, y))$ where u_s is a unit in A_s, we can improve (84) to a Jordan decomposition

$$\partial = x\,\frac{\partial}{\partial x} + z^{r+1} b_s(z, x^p)\,\frac{\partial}{\partial z}, \qquad b_s(0, 0) = 1, \quad b_s \in A_s. \tag{85}$$

Plainly, this depends on s, but it has the manifest advantage that if $f = (x(t), z(t))$: $\Delta_s \to \mathrm{Spf}(A_s)$ is a $\mathcal{F}$-invariant map from the formal disc over $\overline{k}(s)$ then it must be invariant by both the semi-simple and nilpotent parts. As such if $x(t) \neq 0$ then, by invariance under the semi-simple part, $\dot{x} \neq 0$ and $z = z(t^p)$. Consequently for ord the order of vanishing under f^* at the closed point of Δ,

$$\mathrm{s}_Z(f)_{\mathrm{loc}} - \mathrm{Ram}_{f,\mathrm{loc}} \leqslant \mathrm{ord}(x) - \mathrm{ord}(\dot{x}) \leqslant 1 \leqslant \frac{1}{p}\,(E \cdot_f \Delta)_{\mathrm{loc}} \tag{86}$$

where the Segre class and intersection number are understood locally in the obvious way. In particular, for ε given as in 4.2 and $p \gg \varepsilon^{-1}$ we certainly have the kind of bound that we require to deduce (82) from (81). To deal with the alternative possibility that $x(t)$ is identically zero: observe that for r as in (84) and any $d \in \mathbb{N}$ this forces

$$\mathrm{s}_Z(f)_{\mathrm{loc}} \leqslant \frac{r+1}{d}\,\mathrm{s}_{Z_d}(f)_{\mathrm{loc}} \tag{87}$$

for Z_d the subscheme defined in all of $\mathrm{Spf}(B)$ by the ideal (x, y^d) of (84), *i.e.* $\mathrm{s}_{Z_d}(f)_{\mathrm{loc}}$ is equally the (local) intersection of f with the exceptional divisor of a weighted blow up with weights $(1, d)$. Now over the generic point Z_d has length d, while the number of sections of a multiple mH of an ample bundle grows like m^2, so for $d \ll m^2$

$$\Gamma(X_K, H^{\otimes m} I_{Z_d}) \neq 0, \quad m = O(\sqrt{d}). \tag{88}$$

Consequently, throwing away a nowhere dense (dependent on d) Zariski closed subset of X, we have the global inequalities

$$\mathrm{s}_{Z_d}(f) \ll \sqrt{d}\,H \cdot_f C \quad \Longrightarrow \quad \frac{r+1}{d}\,\mathrm{s}_{Z_d}(f) \ll \frac{1}{\sqrt{d}}\,H \cdot_f C \tag{89}$$

which is again an appropriate bound for $d \gg \varepsilon^{-2}$, *cf.* [19, (49)–(50)].

This reduces us to singularities, Z, whose semi-simple part over the generic point has two non-zero eigenvalues. To avoid some pointless technicalities we can (since it doesn't change $K_{\mathcal{F}}$) suppose that the singularities over the generic point are reduced in the sense of Seidenberg, *i.e.* either the previous case of nodes, or the ratio of the eigenvalues is not in $\mathbb{Q}_{>0}$. The Seidenberg property is stable under blowing up, so again, for convenience, we can suppose that there is an exceptional divisor $E \supset Z$, and whence the semi-simple part over K has the form

$$y\frac{\partial}{\partial y} + \lambda x\frac{\partial}{\partial x} \tag{90}$$

for $y = 0$ a local equation for E, and, after base change if necessary, $\lambda \in K\backslash\mathbb{Q}_{>0}$. The first of a couple of additional subtleties is that unlike (84) the invariant branch $x = 0$ of $\mathcal{F}$ which exists (just apply (19) and the Seidenberg condition) in the completion at the generic point, may fail to exist in the completion Spf (B) over all of S. More precisely, for $d \in \mathbb{N}$ some large integer to be chosen, linear algebra, *cf.* (19), reveals that the obstruction to finding a coordinate hypersurface $x = 0$ other than the exceptional divisor such that

$$\partial(x) \in (x) + I_Z^{d+1} \tag{91}$$

occurs at points $s \in S$ such that

$$\lambda \in \{1, \ldots, d\} \mod (\mathfrak{m}(s)). \tag{92}$$

Thanks to our Seidenberg hypothesis this only occurs at finitely many closed points—indeed their residue characteristics, p, must even satisfy $p \ll d$. Such quantification isn't so important however, since $d \gg \varepsilon^{-2}$ will be fixed once we have global sections of $H^{\otimes m}(x, y^d)$ over the generic point for m of size commensurable to $\sqrt{d}$, so we just discard fibres where (92) could happen.

Now the easy sub-case occurs at $s \in S$ where λ isn't in the prime sub-field of $k(s)$. By 2.8, $\mathcal{F}_s$ cannot be p-closed, while, conversely, the locus where $\partial^p \wedge \partial$ is zero contains all invariant curves, so our invariant curve must be one of the coordinate hypersurfaces in (23). One of these is the exceptional divisor, so we can ignore this possibility, while the other agrees with $x = 0$ of (91) to order d, and in either case the maximum possible value of the local contribution

$$\mathrm{s}_Z(f)_{\mathrm{loc}} - \mathrm{Ram}_{f,\mathrm{loc}} \tag{93}$$

is 1. As such, if we again put Z_d to be sub-scheme cut out by (x, y^d), but here with x as in (91), then

$$\mathrm{s}_Z(f)_{\mathrm{loc}} - \mathrm{Ram}_{f,\mathrm{loc}} \ll \frac{1}{d}\,\mathrm{s}_{Z_d}(f)_{\mathrm{loc}} \tag{94}$$

and, as in (89), we have what we need.

This leaves us with the possibility that λ belongs to the prime field $\mathbb{F}_p \hookrightarrow k(s)$, and we effect a change of coordinates $(x, y) \mapsto (x_s, y)$ to identify (on completion in $Z \otimes k(s)$) the semi-simple part

$$\partial_S(s) = y\frac{\partial}{\partial y} + \lambda_s x_s \frac{\partial}{\partial x_s}, \tag{95}$$

of a re-scaling of ∂ over $k(s)$, wherein the hypersurface $x_s = 0$ may be supposed in agreement with $x = 0$ to order $d + 1$. Writing our invariant curve as $f : t \mapsto (x_s(t), y(t))$, observe that the maximum possibility for (93) is still 1, and it would actually be negative if f were widely ramified. Plainly, this latter possibility can be ignored, but equally if $t \mapsto y(t)$ were wild then (86) would hold, while if $t \mapsto x_s(t)$ were wild then (94) holds as soon as $p \geqslant d$. Consequently, we can suppose that both the coordinate projections of f are tame. As such if m, n are the smallest positive integers whose ratio mod p is λ_s, then for ord again the order of vanishing on restriction to Δ, invariance by the semi-simple part (95) implies

$$\mathrm{ord}(x_s) \geqslant m, \qquad \mathrm{ord}(y) \geqslant n \tag{96}$$

or one of f^*x_s, f^*y vanishes identically. This latter possibility is exactly what we encountered in the previous discussion of $\lambda_s \notin \mathbb{F}_p$, whence, on treating it in the same way, we can suppose that (96) holds. On the other hand since $\lambda \in K\backslash\mathbb{Q}_{>0}$, any Arakelov intersection of $\lambda \in \mathbb{P}^1_S$ with m/n is, up to the irrelevance of a choice of metric on $\mathcal{O}(1)_{\mathbb{P}^1_{\mathbb{C}}}$, non-negative, so

$$\max\{m, n\} \gg p. \tag{97}$$

Finally, therefore, if $p \geqslant d$, then, either $n \gg p$ and we have the estimate (86) or $m \gg p$ and we appeal to (94). Having thus estimated all the possibilities for the local contribution to (93) we deduce the refinement (82) of the tautology (81). □

5 Surfaces of general type

Throughout this section S is an irreducible affine scheme of finite type over a Noetherian integral domain with generic point of characteristic zero, *e.g.* $\mathbb{Z}$, and X/S is a smooth family of S-projective surfaces of general type, with H a S-ample line bundle. The principle conclusions, 3.6 and 4.2 of Sect. 3, respectively Sect. 4, combine with the minimal model theory of [17] to yield

Proof of 1.4 By 3.6 and 3.12, (a) is hopelessly false should (b) occur, so it remains to prove (a) otherwise. By the classification theorem, [17], this amounts to the following distinct possibilities:

(i) The (foliation) canonical $K_{\mathcal{F}}$ is big on the generic fibre.
(ii) The generic fibre is as (b) but points of positive residue characteristic aren't dense.

The first case is immediate by 4.2 and a suitable choice of ε. In the second case, notation as in 3.9 albeit in characteristic zero and including the case of products of curves, we can either use 3.7, or, to keep the logical complexity down, but again, by classical Baum–Bott residue theory, or by (50), if one wants to be cute about it, conclude that (51) holds, while in characteristic zero, *cf.* 3.7, an invariant curve can't meet E, so the left hand side of (51) is $-\chi(C)$. □

Proof of the subtler theorem, 1.5 Plainly (a) holds for products of curves, and their quotients by finite group actions, while any other bi-disc quotient is defined over $\overline{\mathbb{Q}}$, and by 3.6 we have invariant rational curves satisfying (8) whenever p is inert. As such, by 1.4, it remains to deal with bi-disc quotients at split primes, and we can, of course, suppose that we're on a model with generically canonical singularities. Putting ourselves again in the notation of 3.9, this follows from (39) if there are no cusps, E. Otherwise, there is some subtlety, since invariant curves can meet E in positive characteristic. In the proof of 3.9, if (48) holds for all m, then we either appeal to 3.13, or, less demandingly, avoid using a sledge hammer on a nut, and use our variant, (50)–(51), of Baum–Bott, so we'd have

$$K_{\mathcal{G}} \cdot_f C = 0 \mod (p^m) \text{ for all } m \text{ which implies } (K_X + E) \cdot_f C \leqslant K_{\mathcal{F}} \cdot_f C \qquad (98)$$

and there's nothing to do by 4.2. If, however, we're in the infinitely more likely possibility that (48) holds for some m, but fails for some $m+1$, then, rather than the former equation in (98), we only have (49) which, since C is rational or elliptic, affords

$$(K_X + E) \cdot_f C \leqslant (1 + p^m) \, \text{cardinality} \, (f^{-1}(E)_{\text{red}}).$$

In the particular case that $m = 1$, we've already seen in (95)–(97) that

$$\text{cardinality} \, (f^{-1}(E)_{\text{red}}) \ll \frac{1}{p} E \cdot_f C$$

where the implied constant may be no better than the Arakelov height of λ, *i.e.* it's arbitrary. Nevertheless, this bounds the degree along $K_X + E$ by a constant multiple of the intersection with E, and this works in general—just do (95)–(97) modulo p^m instead of modulo p, which actually has some simplification since $xy = 0$ of *op. cit.* can be taken to be the equation of E, while by 3.10 we have full p-adic Jordan decomposition. As such we deduce ($p \gg 0$ to avoid any bad reduction issues) that any rational or elliptic curve $f : C/k \to X$ where the characteristic of k is a split prime, satisfies

$$(K_X + E) \cdot_f C \ll E \cdot_f C \qquad (99)$$

for an implied constant depending only on X.

Now suppose such rational or elliptic over $\overline{k}(s)$ are Zariski dense with $s \in \Sigma$ of residue characteristic, p, a split prime. By hypothesis, and the fact that we know, 1.4, that the theorem is true is in characteristic zero we can, *cf.* proof of 3.6, find for an infinite set of split p, a rational or elliptic curve, $f_p : C_p/\overline{k}(s) \to X$ such that for H ample

$$\liminf_{p \in \Sigma} H \cdot_{f_p} C_p = \infty. \qquad (100)$$

On the other hand, for $s \in \Sigma$ we have a specialisation map, [10, 20.3], on Néron–Severi

$$i_s : \text{NS}^1(X \otimes k(S))_{\mathbb{R}} \to \text{NS}^1(X \otimes k(s))_{\mathbb{R}}$$

whence, by duality, classes in $\mathrm{NS}_1(X\otimes k(S))_{\mathbb{R}}$ defined by

$$[\phi_p]\colon \mathrm{NS}^1(X\otimes k(S))_{\mathbb{R}} \to \mathbb{R}\colon \quad D \mapsto \frac{1}{H\cdot_{f_p} C_p}\, i_s(D)\cdot_{f_p} C_p$$

which for $p \gg 0$ certainly intersects any ample divisor, A, positively. On the other hand, $nH - A$ is ample for $n \gg 0$, and vice-versa, so for A fixed, $A\cdot[\phi_p]$ is bounded, and weak limits of the $[\phi_p]$ exist. Choose one, ϕ, say, and throw the rest of Σ, *i.e.* $[\phi_p]$ not converging to ϕ, away, then the underlying f_p are still Zariski dense by (100), so ϕ may be identified with a (non-zero) nef. class in $\mathrm{NS}^1(X\otimes k(S))_{\mathbb{R}}$. Now, up to introducing some harmless quotient singularities, there's no loss of generality in identifying $X\otimes k(S)$ with a resolution, π, of the cusps on the canonical model, [17, III.3], $(X_0, \mathcal{F}_0)$ of $\mathcal{F}$, and since ϕ is nef., then, sub-sequencing as necessary, by 4.2

$$\pi^* K_{\mathcal{F}_0}\cdot\phi = 0$$

while by (99), $E\cdot\phi > 0$. We have, however,

$$\phi = \pi^*(\pi_*\phi) + \frac{E\cdot\phi}{E^2}\, E$$

and $\phi^2 \geqslant 0$, so $\pi_*\phi$ is big and nef. in the Mumford intersection theory of X_0, which by Hodge affords the absurdity, [17, IV.5.5], that $K_{\mathcal{F}_0}$ is numerically trivial. □

Now we can apply this, following, [3], to curves on surfaces of general type in the usual way

Proof of 1.1, $\Omega^1_{X/S}$ big Let $\pi: P = \mathbb{P}(\Omega_{X/S}) \to X$ be the projective tangent space, and L its tautological bundle, then, by hypothesis, there is a $\delta > 0$ such that $L - \delta\pi^* H$ is effective. Again, by hypothesis, the derivative, $f'\colon C \to P$ of our curve exists, and satisfies 4.1. As such, we're done unless f' factors through a divisor $D \hookrightarrow P$ dominating X. Such a divisor defines a foliation, $\mathcal{F}$, by curves (on itself) given over the generic point by

$$\Omega^1_{D/S} \xrightarrow{\sim} \pi^*\Omega^1_{X/S} \to L|_D$$

so, without loss of generality, curves not satisfying (a) are invariant by a foliation on a surface dominating X, and we conclude by 1.4. □

Proof of the subtler theorem, 1.2 Exactly as above, but use 1.5 rather than 1.4. □

Appendix: Baum–Bott theory with values in the ground field

Let X/S be an algebraic, resp. complex, Deligne–Mumford champ over an algebraic, resp. complex, space. The Deligne–Mumford condition ensures that $\Omega^1_{X/S}$ is well defined, and whence for every $n \in \mathbb{N}$ there is, on identifying H^1 with Čech co-cycles, an Atiyah class,

$$\begin{aligned} \mathrm{H}^1(X, \mathrm{GL}_n) \ni E \mapsto \mathrm{at}(E) \in \mathrm{H}^1(X, \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathrm{End}(E)) : \\ G \mapsto d \log G := (dG) G^{-1} \end{aligned} \tag{101}$$

whose symmetric functions define for each $q \in \mathbb{N}$ Chern-classes of "Hodge type", *i.e.*

$$\mathrm{c}_q : \mathrm{H}^1(X, \mathrm{GL}_n) \to \mathrm{H}^q(X, \Omega^q_{X/S}). \tag{102}$$

As such the Atiyah class of a vector bundle, E, vanishes iff there is a connection

$$\nabla : E \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} E$$

and such vanishing implies, a fortiori, vanishing of the Chern classes of (102). Naturally, therefore, the Atiyah class is usually viewed as an obstruction, and, rather obviously, if one only asks that E admits a connection along certain directions then this obstruction is smaller, *i.e.* for any derivation $D : \mathcal{O}_X \to \mathcal{M}$ with values in an $\mathcal{O}_X$-module

Fact A.1 *The obstruction that a vector bundle E admits a connection along D, i.e. a map*

$$\nabla : E \to \mathcal{M} \otimes_{\mathcal{O}_X} E$$

satisfying the Leibniz rule, $\nabla(fs) = D(f)s + f\nabla(s)$, lies in

$$\mathrm{H}^1(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathrm{End}(E)). \tag{103}$$

Proof By definition, D corresponds to a map $\Omega^1_{X/S} \to \mathcal{M}$, so that the obstruction (103) is just the restriction of the Atiyah class (101). □

A generally valid example of such a bundle would be

Example A.2 If C is a family of invariant effective Cartier divisor, then by definition

$$\mathcal{O}_X(-C) \xrightarrow{D} \mathcal{M}(-C)$$

so we get (rather tautological) connections for $\mathcal{O}_X(-C)$, and $\mathcal{O}_X(C)$.

Consequently, irrespective of any integrability about the field of directions afforded by D

Corollary A.3 *If $f : Y \to X$ is a map such that we have a factorisation*

$$f^*\Omega^1_{X/S} \to f^*\mathcal{M} \to \Omega^1_{Y/S}$$

then for any bundle E admitting a connection along D the Hodge Chern classes

$$c_q(f^*E) \in H^q(Y, \Omega^q_{Y/S})$$

vanish.

Now the key point is that in the presence of integrability there is at least one rather interesting bundle admitting a connection along the leaves, *i.e.*

Fact A.4 *Suppose the sheaf of derivations, $T_{\mathcal{F}}$, vanishing along the kernel, $\Omega^1_{X/\mathcal{F}}$, of $\Omega^1_{X/S} \to \mathcal{M}$ is closed under bracket and $U \hookrightarrow X$ is the locus where $\Omega^1_{X/\mathcal{F}} \hookrightarrow \Omega^1_{X/S}$ defines a sub-bundle of $\Omega^1_{X/S}$, then $\Omega^1_{X/\mathcal{F}}|_U$ admits a connection along $\Omega^1_{X/S}|_U \to \Omega^1_{\mathcal{F}} := \mathrm{Hom}_{\mathcal{O}_X}(T_{\mathcal{F}}, \mathcal{O}_X)$.*

Proof Differentiation affords a map

$$\begin{aligned} d\colon \Omega^1_{X/\mathcal{F}} \to \Omega^2_{X/S} \text{ whose composition with} \\ \Omega^2_{X/S} \to \Lambda^2\Omega^1_{\mathcal{F}} \text{ is dual to the bracket,} \end{aligned} \tag{104}$$

so by hypothesis the former map in (104) factors through the kernel of the latter, which over the locus, U, is naturally the image of

$$\Omega^1_{X/\mathcal{F}} \otimes \Omega^1_{X/S} \to \Omega^2_{X/S}$$

and the quotient of this by the sub-image of $\Lambda^2\Omega^1_{X/\mathcal{F}}$ is, over, U: $\Omega^1_{X/\mathcal{F}} \otimes_{\mathcal{O}_X} \Omega^1_{\mathcal{F}}$. □

Typically one applies this in the form

Corollary A.5 *Suppose $\Omega^1_{X/\mathcal{F}}$, $\Omega^1_{\mathcal{F}}$ are bundles; U is of co-dimension 2; and X is S_2 then $\Omega^1_{X/\mathcal{F}}$ admits a connection,*

$$\nabla\colon \Omega^1_{X/\mathcal{F}} \to \Omega^1_{X/\mathcal{F}} \otimes_{\mathcal{O}_X} \Omega^1_{\mathcal{F}}. \tag{105}$$

Nevertheless,

Warning A.6 It does not follow from A.3 and A.5 that for $f\colon Y \to X$ factoring through a $\mathcal{F}$ invariant sub-variety that the Atiyah class, or even the Chern classes, whether of $f^*\Omega^1_{X/\mathcal{F}}$, or f^*C of A.2, vanishes. Indeed this only follows if (105) factors, around f, through

$$\Omega^1_{X/\mathcal{F}} \otimes_{\mathcal{O}_X} (\text{Image of } \Omega^1_{X/S} \to \Omega^1_{\mathcal{F}})$$

otherwise there is a residue/obstruction in

$$H^1\big(X, (\text{coker of } \Omega^1_{X/S} \to \Omega^1_{\mathcal{F}}) \otimes \mathrm{End}(\Omega^1_{X/\mathcal{F}})\big).$$

The resulting residues admit, [2] or [17, 1.3] for a characteristic free version, various formulae, of which the most relevant is:

Example A.7 Let $(X, \mathcal{F})$ be a foliated smooth surface with log-canonical singular locus Z over a field k for which every singularity is a reduced k-point and E an invariant simple normal crossing divisor such that at any singularity there are two branches of E then $\Omega^1_{X/F}(E)$ admits a connection without residues, *i.e.*

$$\nabla: \ \Omega^1_{X/\mathcal{F}}(E) \to \Omega^1_{X/\mathcal{F}}(E) \otimes_{\mathcal{O}_X} K_{\mathcal{F}} I_Z. \tag{106}$$

In particular, for p the characteristic of k and $f: \Sigma \to X$ any invariant curve

$$\Omega^1_{X/\mathcal{F}}(E) \cdot_f \Sigma = 0 \mod (p). \tag{107}$$

Proof The in particular follows from the general residue formula [17, 1.3.1] of which (106) is a minor variation. □

References

1. Autissier, P., Chambert-Loir, A., Gasbarri, C.: On the canonical degrees of curves in varieties of general type. Geom. Funct. Anal. **22**(5), 1051–1061 (2012)
2. Baum, P., Bott, R.: Singularities of holomorphic foliations. J. Differ. Geom. **7**(3–4), 279–342 (1972)
3. Bogomolov, F.A.: Families of curves on a surface of general type. Dokl. Akad. Nauk SSSR **236**(5), 1041–1044 (1977) (in Russian)
4. Brunella, M.: Courbes entières et feuilletages holomorphes. Enseign. Math. **45**(1–2), 195–216 (1999)
5. Brunella, M.: Plurisubharmonic variation of the leafwise Poincaré metric. Internat. J. Math. **14**(2), 139–151 (2003)
6. Duval, J.: Sur le lemme de Brody. Invent. Math. **173**(2), 305–314 (2008)
7. Ekedahl, T.: Foliations and inseparable morphisms. In: Bloch, S.J. (ed.) Algebraic Geometry. Bowdoin 1985. Proceedings of Symposia in Pure Mathematics, vol. 46.2, pp. 139–149. American Mathematical Society, Providence (1987)
8. Ekedahl, T.: Canonical models of surfaces of general type in positive characteristic. Inst. Hautes Études Sci. Publ. Math. **67**, 97–144 (1988)
9. Ekedahl, T., Shepherd-Barron, N., Taylor, R.: A conjecture on the existence of compact leaves of algebraic foliations. http://www.dpmms.cam.ac.uk/~nisb/
10. Fulton, W.: Intersection Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A, vol. 2, 2nd edn. Springer, Berlin (1998)
11. Gromov, M.: Metric Structures for Riemannian and Non-Riemannian Spaces. Modern Birkhäuser Classics. Birkhäuser, Boston (2007)
12. Kleiner, B.: Hyperbolicity using minimal surfaces (preprint)
13. Martinet, J.: Normalisation des champs de vecteurs holomorphes (d'après A.-D. Brjuno). In: Bourbaki Seminar, Vol. 1980/81. Lecture Notes in Mathematics, vol. 901, pp. 55–70. Springer, Berlin (1981)
14. McQuillan, M.: Diophantine approximations and foliations. Inst. Hautes Études Sci. Publ. Math. **87**, 121–174 (1998)
15. McQuillan, M.: Formal formal schemes. In: Berrick, A.J., Leung, M.C., Xu, X. (eds.) Topology and Geometry: Commemorating SISTAG. Contemporary Mathematics, vol. 314, pp. 187–198. American Mathematical Society, Providence (2002)
16. McQuillan, M.: Semi-stable reduction of foliations. IHES/M/05/02 (2005). preprints.ihes.fr/2005/M/M-05-02.pdf
17. McQuillan, M.: Canonical models of foliations. Pure Appl. Math. Q. **4**(3), 877–1012 (2008)
18. McQuillan, M.: Two Ext groups and a resdiue (2012). arXiv:1207.1155

19. McQuillan, M.: Baby geography and Mordellicity of surfaces. Geom. Funct. Anal. **26**(1), 337–357 (2016)
20. McQuillan, M., Panazzolo, D.: Almost étale resolution of foliations. J. Differential Geom. **95**(2), 279–319 (2013)
21. Miyaoka, Y.: The orbibundle Miyaoka–Yau–Sakai inequality and an effective Bogomolov–McQuillan theorem. Publ. Res. Inst. Math. Sci. **44**(2), 403–417 (2008)
22. Shepherd-Barron, N.I.: Miyaoka's theorems on the generic seminegativity of T_X and on the Kodaira dimension of minimal regular threefolds. In: Kollár, J. (ed.) Flips and Abundance for Algebraic Threefolds. Astérisque, vol. 211, pp. 103–114. Société Mathématique de France, Paris (1992)
23. Shepherd-Barron, N.I.: Infinite generation for rings of symmetric tensors. Math. Res. Lett. **2**(2), 125–128 (1995)
24. Tian, G., Yau, S.-T.: Existence of Kähler–Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. In: Yau, S.T. (ed.) Mathematical Aspects of String Theory. Advanced Series in Mathematical Physics, vol. 1, pp. 574–628. World Scientific, Singapore (1987)