Rational Criteria for Hyperbolicity

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PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD

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Prooemium

Abstract

This is only a book in the abstract sense. Its reality is a collection of 4 papers. The papers, however, have a common goal of understanding the categorical framework of curvature in the context of quasi-projective, varieties. The present is, therefore, a survey rather than a preface. For ease of exposition a choice of square root of -1 will be supposed, but avoiding this leads to better understanding. Illustrative examples of the theory are provided by xi.1 & 2, which one may wish to consult before a logical reading.

i. Mori theory and curvature

Mori theory is the study of holomorphic volume. In contrast to real volume on an oriented manifold, the bundle of holomorphic volume forms on a complex manifold is invariably highly non-trivial. It is, for example, ample whenever the Ricci curvature is negative, and conversely, if it's ample, Yau's solution of the Calabi conjecture even gives a metric with Ricci curvature -1. Negative Ricci curvature is not, however, a property of length, but of volume. Combinatorially, it manifests itself by way of an isoperimetric inequality,

$$\mathrm{vol}(B) \ll \mathrm{area}(\partial B)$$

for all balls, B, open in the variety, and implied constant independent of the ball.

Plainly smoothness is not a functorial property of this discussion. An algebraic variety X may perfectly well have well defined holomorphic volume without being smooth, *i.e.*, it is, by definition, Gorenstein. In these circumstances, a local chart will come equipped not only with a holomorphic volume element dV, but an anti holomorphic one, $d\overline{V}$, too. It would thus appear that we have a natural way, at least locally, to measure real volume by way of the measure $dV \wedge d\overline{V}$. Unfortunately, it is perfectly feasible that this is infinite, when it is finite the singularity is said to be canonical (or more generally klt if it were only $\mathbb{Q}$-Gorenstein). This terminology is somewhat regrettable, but in fairness has good algebraic motivation in the form of,

i.1 Conjecture *Let X be smooth, and of general type, i.e. K_X is big, then the canonical ring,*

$$\bigoplus_{m \in \mathbb{N} \cup \{0\}} H^0(X, K_X^{\otimes m})$$

is finitely generated.

In particular if the conjecture is true, which it is if X has dimension at most 3, then each variety of general type has what should rightly be regarded as a 'canonical model', X_0, obtained by taking the projective variety associated to the said ring, with canonical singularities being precisely those which appear on such models.

Of itself this wholly serves to clarify the meaning of Ricci curvature, which, as a question about metrics is ambiguous in this generality, but in terms of volume perfectly clear. Indeed a real volume form may be identified with a metric on K_X, of which we can compute the curvature, which we require to be positive. Consequently its top power is a new volume form, and Ricci curvature -1 is equivalent to this being the same as what we started with. Whence,

i.2 Further conjecture *A projective variety X is a canonical model iff it admits a metric of Ricci curvature -1.*

Strictly speaking this isn't quite correct, since a real volume form cannot distinguish the possibility that the complex volume is only defined up to some finite monodromy around the singularities. There is, however, a smallest object, the Gorenstein covering stack $\mathcal{X} \to X$ on which the ambiguity is removed, and at this point the if direction is true even if the metric on $K_{\mathcal{X}}$ were only continuous. As such, geometrisable singularities might have been a better name.

From the combinatorial perspective of an isoperimetric inequality, we have already seen that for the volume to be finite and compatible with the complex structure necessitates canonical singularities. A more delicate combinatorial instrument arises in exploiting the natural extra operator d^c by way of an extra integration. For X of dimension n this involves pointed balls, or more generally pointed Stein spaces (B, b) together with an exhaustion function ϕ satisfying,

$$(dd^c\phi)^n = \delta_b$$

while taking the value 0 on the boundary, so that for $r < 0$ we may replace the volume by a Nevanlinna type integral,

$$\fint_{\phi<r} \omega := \int_{-\infty}^{r} dt \int_{\phi<t} \omega (dd^c\phi)^{n-1}$$

for, say, ω a Kähler metric on our variety (restricted from Fubini-Study on a projective embedding is fine, since the definition is rather robust). This expression always has sense. Nevertheless, for the corresponding isoperimetric inequality,

$$\log |J|(b) + \fint_{\phi<r} \omega \leq \log \int_{\phi=r} |J| d^c\phi (dd^c\phi)^{n-1}$$

with J a suitable Jacobian type function to hold for all balls requires a canonical model. There is nothing mysterious about this. The double integration procedure is intrinsic to exploiting the complex structure, cf. [M3], while quite generally on fixing a real volume form, equivalently metricisation $\bar{K}_{\mathcal{X}}$ of the canonical bundle, we have,

i.3 Tautological inequality *The Gorenstein covering stack $\mathcal{X} \to X$ has canonical singularities iff for every ball mapping to it*

$$\log |J|(b) + \fint_{\phi<r} c_1(\bar{K}_{\mathcal{X}}) \leq \log \int_{\phi=r} |J| d^c\phi (dd^c\phi)^{n-1}$$

Here, for convenience, one can even suppose that the map from the ball is unramified at its origin b, and all we've done is integrate by parts, and exploit (bearing in mind [K1] VI.1.4.3) the algebraic definition, which itself simply asks that the Jacobian of an algebraic map to a canonical singularity should always be holomorphic. Ampleness of $K_{\mathcal{X}}$ is, therefore, what turns this tautology into a genuine isoperimetric inequality, and indeed conversely should we have,

i.4 Final Conjecture *All projective varieties that don't have general type enjoy a model which can be built up from,*

(i) *Canonical models in the sense of* i.1.

(ii) *Calabi-Yau's, i.e.* $K_X = 0$ *in rational co-homology.*

(iii) *Fano varieties, i.e.* $-K_X$ *ample.*

Again, this is known in dimension at most 3. Needless to say, we allow exactly the same possibilities for singularities in (ii) & (iii) that we allow in (i). As such, the existence of a Ricci flat metric in (ii) is, in general, an open question. In the third case, there is no particular reason to believe that Ricci curvature conveys a meaningful idea. After all, there are already proper parabolic 1 dimensional orbifolds, which are functorially speaking Fano, but do not admit a metric of strictly positive Ricci curvature. The correct description, therefore, would appear to be in terms of their rational chain connectedness, [K&], and, since the number of deformation types of such examples is expected to be finite in each dimension, explicit description of their anti-canonical ring, [CR].

One should also recognise the obstruction to negative Ricci curvature posed by rational curves not only globally by (iii), but also locally in the form of Mori's bend and break technique, [K1] III. Regrettably, the proof has restrictive conditions on singularities, and whence it has been displaced from its formerly central role. Nevertheless, conceptually speaking, it is a critical indication that the most important obstruction to negative curvature is posed by the presence of rational curves.

ii. Relativisation

A fundamental observation of Grothendieck is the importance of working relatively. Classically this meant always considering families of schemes $X \to S$, over an arbitrary base scheme S, rather than just the case where S is a closed point of characteristic zero, *i.e.* a variety over the complex numbers. The many startling consequences that Grothendieck managed to deduce from this philosophy clearly indicate that this is anything but generalisation for generalisations sake. Nor is the importance of this confined to the rather pathological category of schemes. It is a wholly general observation applicable to almost all mathematics.

In light of [A4] the relativisation of n-dimensional Mori theory over a base scheme, or better algebraic space, is easily understood as a problem in $n+1$-dimensional Mori theory. Complete functoriality, however, is only restored by working with algebraic (which will always mean Deligne-Mumford) stacks. Unsurprisingly, families of curves are well understood. Here the singularities are relatively canonical iff the singularities of any curve in the family are at worst nodes, and the existence of the moduli stacks $\mathcal{M}_{g,r}$ of stable r-pointed curves of genus g implies that every such family has a minimal, in fact canonical model. This latter statement is functorially true, notwithstanding that i.1 is false, [K], for the relative canonical bundle of $\mathcal{M}_{g,1} \to \mathcal{M}_g$, for $g \geq 3$, since, no matter how you understand it, the relative canonical bundle is relatively ample. The situation for families of surfaces, cf. [KS], is similar, but for the moment not quite so clean.

As to the metric theory of $\mathcal{M}_{g,1} \to \mathcal{M}_g$, the relative Kắhler-Einstein metric is the Weil-Petersson metric for $\mathcal{M}_{g,1}$, $g \geq 2$. Alternatively, it is defined at a relatively smooth point of $\mathcal{M}_{g,1}$ by appealing to the uniformisation theorem for the unique hyperbolic curve through the point obtained by deleting the other curves which are in the same fibre. This defines a metric on the relative canonical bundle, with mild singularities at the (relative) singularities, which has relative curvature -1, and chern form, positive outwith the singularities, equal to Weil-Petersson. The relative Ricci curvature problem for families of surfaces is more challenging.

There is, however, no reason to limit this highly satisfactory discussion to a base which is a scheme. Indeed, functorially speaking, the definition of scheme, or indeed space with structure of any kind be it analytic, differentiable, or whatever is close to nonsensical. Again, the correct observation is due to Grothendieck, *i.e.* what's really at stake is not a 'space' with local structure and rules for glueing it together, but the local structure with glueing per se. This leads to the notion of champs, which is unquestionably preferable to its deliberate mis-translation, stack. Our own particular case of interest will be that of the classifying stack $[X/\mathcal{F}]$ of a foliation by curves $\mathcal{F}$ on a projective variety $X/\mathbb{C}$, [II] A.1. In truth there are technical issues of definition about the singularities, but in supposing X normal we will always eschew these by way of the S_2 property. In particular, there is a well defined sheaf of relative volumes, and we call the foliation Gorenstein if these form a bundle $K_{\mathcal{F}}$. The descent datum for the structure sheaf is encoded in the natural differentiation operation, $\partial_{\mathcal{F}} : \mathcal{O}_X \longrightarrow K_{\mathcal{F}}$, and a bundle E on X comes with a descent datum away from the singularities, *i.e.* is a bundle on $[X/\mathcal{F}]$ when the latter is well defined, iff there is, over X, a leafwise holomorphic connection,

$$\nabla : E \longrightarrow E \otimes K_{\mathcal{F}}$$

satisfying, for s a section and f a function, $\nabla(fs) = \partial_{\mathcal{F}}(f)s + f\nabla(s)$.

Such is the conceptual/functorial basis of our set up. As such, we are not addressing the structure of the classifying stack $[X/\mathcal{F}]$ per se, *i.e.* doing dynamics, but rather the properties of the relative volume, which, for reasons of dimension is now a length. Nevertheless there are dynamical issues which will have to be addressed, while the ultimate theorems on relative hyperbolicity are very close to giving the essential information necessary for a study of the classifying stack in the spirit of [KM], *i.e.* valuative separatedness. Our basic conclusion will be that the perception of a 'schematic observer' on $[X/\mathcal{F}]$ of X differs from that of an observes perception of $\mathcal{M}_{g,1}$ from $\mathcal{M}_g$ neither more nor less than the difference between a given compact and a given non-compact Riemann-surface. Better still, this study encompasses not just the isoperimetric inequality for balls, but even for discs, whenever the latter can be reduced to a foliation problem, which is indeed the case on algebraic surfaces. One should though be warned that the said reduction to a foliation problem is not in dimension 2, but, for c_1^2 large, on foliated varieties of dimension as high as 11, and much higher still if c_1^2 is very small.

iii. The pro-eminence of hyperbolicity

On a curve C, there is only one foliation, $C \to \mathrm{pt}$, so curves foliated by curves are curves, and there's nothing here to be said that hasn't been said already. A foliated surface $S \to [S/\mathcal{F}]$, however, is a different matter. With probability 1 there will be singularities, and the (Gorenstein) log-canonical ones can be described as follows: If $\mathfrak{m}$ is the maximal ideal of a singularity, then a derivation ∂ defining the foliation linearises as an element,

$$\partial \in \mathrm{End}\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right)$$

which may well be zero. The singularity is log-canonical (functorially with respect to the ideas, and in fact in all dimensions, [II] I.6) iff ∂ is non-nilpotent. Almost invariably a singularity which is log-canonical is also canonical. The exception for surfaces occurs when, after multiplication by a unit, we can find coordinates x, y in the complete local ring $\hat{\mathcal{O}}$ such that,

$$\partial = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} \in \mathrm{Der}(\hat{\mathcal{O}}), \quad p, q \in \mathbb{N}$$

A general canonical/log-canonical singularity is the quotient of such, or of a smooth point, by a finite group. This much is true for the definitions which we have given so far, since these suppose that things are at least $\mathbb{Q}$-Gorenstein. On a surface, Mumford's intersection theory allows one to define canonical without necessarily supposing this, and it leads to another possibility which can be modeled on a cusp at infinity for either of the natural foliations on a bi-disc quotient.

The upshot of this is that i.1 fails for an arbitrary $\mathcal{F}$ of general type, *i.e.* $K_{\mathcal{F}}$ big on a smooth S with the foliation singularities being canonical, and unlike the $\mathcal{M}_g$ example the canonical model need not necessarily have projective moduli or for that matter be a scheme, albeit that the resolution of any cusps will restore projectivity, and the resolution in question would at least be minimal, *i.e.* enjoy $K_{\mathcal{F}}$ nef. As such the correct description of general type is necessarily metrical, or better, given the uniformisation theorem, in terms of the space of invariant maps from the disc, $\mathrm{Hom}_{\mathcal{F}}(\Delta, S)$. Again this looks to an a priori tautology describing canonical singularities,

iii.1 Refined tautological inequality *A foliated surface $S \to [S/\mathcal{F}]$ has canonical singularities iff for every map invariant map $f : \Delta \to S$ not factoring through the (necessarily simple normal crossing) divisor Z consisting of invariant curves through the singularities we have,*

$$\log \|f'(0)\| + \fint_{\Delta(r)} f^* c_1(K_{\mathcal{F}}) \leq \log \int_{\partial\Delta(r)} \|f'\| d^c \log |z|^2 + \text{small error}$$

Here norms of derivatives are understood in a complete metric for $S \backslash Z$ and again, for convenience, we suppose f relatively unramified at the origin. Its strict analogy with i.3 notwithstanding, this is considerably less trivial, [III] III.1.2. Not all of Z is needed if we're prepared to make a slightly more complicated statement, but for weak algebraic branches of saddle nodes (*i.e.* the aforesaid linear endomorphism is singular, which, by the way, never occurs in algebraically integrable examples) it is unavoidable.

The discussion of nodes, and their generalisations, is postponed to viii. & x. In our immediate context, we see that general type makes iii.1 into a genuine area vs. length isoperimetric inequality, and we conclude that $S \to [S/\mathcal{F}]$ is a continuously varying family of Riemann surfaces. In particular the uniformisation theorem defines a continuous metric on $K_{\mathcal{F}}$ which certainly has relative curvature -1, and by [Br4] a non-negative curvature form. The said curvature form imitates the Weil-Petersson metric, and although it's transverse degeneration is tricky to specify, it is certainly not everywhere transversely degenerate. Consequently the family $S \to [S/\mathcal{F}]$ varies in moduli, and this modular variation distinguishes general type from any other $S \to [S/\mathcal{F}]$ which is relatively hyperbolic.

This much certainly takes care of analogues of i.1 & i.2 for foliated surfaces, and leaves us to investigate i.4. This is the content of [M2], which it is necessary to review. Starting from a model $S \to [S/\mathcal{F}]$ with canonical singularities, one seeks to contract some curves to obtain the fundamental, already implicit in the general type discussion, dichotomy of Mori theory,

iii.2 Lemma *Either $K_{\mathcal{F}}$ is nef. or (remember we're working relatively) $S \to [S/\mathcal{F}]$ is a fibration in rational curves.*

In op. cit. this was achieved as an application of Miyaoka's partial extension of Mori's bend and break technique to foliations, which on surfaces already gave the dichotomy that either $K_{\mathcal{F}}$ was pseudo effective or $\mathcal{F}$ was a pencil of rational curves, so the extra work in iii.2 revolved around allowing mild singularities on the ambient space S, and pushing the thing to an actual canonical model, *i.e.* making an already nef $K_{\mathcal{F}}$ as positive as possible, so: for C a curve, $K_{\mathcal{F}}.C = 0 \Rightarrow C^2 \geq 0$.

Given the lemma, one sets off in the usual way. Indeed if $K_{\mathcal{F}}$ is zero in rational co-homology then the foliation is given by a global vector field on an almost étale cover of the canonical model, while otherwise if some multiple of $K_{\mathcal{F}}$ has a section then there is a pencil of sections, and, up to finite group actions, the foliation is either an isotrivial fibration in curves of genus at least 2, or the suspension of a representation of the fundamental group Γ of a quasi-projective curve, in either the automorphism group of $\mathbb{P}^1$, Ricatti, or that of an elliptic curve, E, turbulent, so, better: $[X/\mathcal{F}]$ is a $\mathbb{P}^1$ bundle or a genus one curve in the étale topus of $B\Gamma$.

This does not, however, exclude that we find ourselves in the situation where $K_{\mathcal{F}}$ is non-zero in rational co-homology, yet $K_{\mathcal{F}}^2 = 0$, and there are no global sections of any power. In other words, the numerical Kodaira dimension $\nu(\mathcal{F}) = 1$, but the actual Kodaira dimension $\kappa(\mathcal{F}) = -\infty$. 'Worse' still, on a smooth bi-disc quotient in either of it's natural foliations, this actually happens. At this point we see that the basic philosophy underlying i.4, abundance, is false. One might respond, however, that if one is already prepared to consider $S \to [S/\mathcal{F}]$ as a fibration, one shouldn't expect that the 'Kodaira fibration' will necessarily be algebraic, but rather takes the form $S \to [S/\mathcal{G}]$, for $\mathcal{G}$ a foliation by curves, everywhere transverse to $\mathcal{F}$, which at the price of allowing cusps is exactly what happens, and whence outwith the general type case, the one exception not already listed is a singular bi-disc quotient in either of it's natural foliations.

Even though there is a sense in which this rescues i.4, *i.e.* things of small Kodaira dimension have a simple structure, one must also recognise the manifest ways in which i.4 is wrong in the relative setting. Indeed the basic principle of i.4, which there is no reason to doubt in the absolute case, is that the isoperimetric inequality fails because the volume expands in the direction of the Kodaira fibration. For an isotrivial family $S \to [S/\mathcal{F}]$ of genus $g \geq 2$, however, even though abundance holds, the relative isoperimetric inequality is not degenerate, and the situation is relatively hyperbolic. Abundance, however, is a red herring, and a conceptual unity can only be restored by a complete rejection of the building block picture of i.4, both algebraically and analytically, in favour of,

iii.3 Guiding principle *What is required is an a priori understanding of the relative curvature of $S \to [S/\mathcal{F}]$ in terms of the numerical properties of $K_{\mathcal{F}}$. A postiori this will give a simple list of things which are not of general type. Such listing, however, is no longer an integral part of the theory, but an application.*

Already, the need for iii.3 should be clear from the structure which we've described for foliated surfaces $S \to [S/\mathcal{F}]$, and its incompatibility with i.4. Any doubt should be dispelled by reviewing how one arrives to the said list in dimension 2. To begin with, an obvious strategy for approaching $\nu(\mathcal{F}) = 1$ is to say that there should be a semi-positive metricisation $\bar{K}_{\mathcal{F}}$ of $K_{\mathcal{F}}$ with $c_1(\bar{K}_{\mathcal{F}}) \geq 0$, and the desired $\mathcal{G}$ should be the kernel. Supposing, say even a smooth metricisation, such so called Monge-Ampère foliations are not usually holomorphic. Moderately surprisingly, the existence of $\mathcal{F}$ says this won't be a problem by way of a Bochner type technique, [M2] IV.5.6, which actually works in all dimensions. Unfortunately, this technique requires some minimal regularity for the metric- at the very least $c_1(\bar{K}_{\mathcal{F}})$ ℓ_2- whereas general nonsense need not give anything better than measure coefficients. As such the problem is one of sufficient a priori regularity, which transpires to be a question of a priori hyperbolicity. Indeed, as one would suspect, *i.e.* powers of $K_{\mathcal{F}}$ have big H^1, Riemann-Roch allows one to suppose that $K_S.K_{\mathcal{F}} > 0$, and under this condition the main theorem of [M1] had proved that any invariant $f : \mathbb{C} \to S$ factored through one of finitely many rational or elliptic curves. Whence the right thing to do, [Br4], is to use this and the uniformisation theorem to get a natural metric on $K_{\mathcal{F}}$, so, in fact a relative metric with relative curvature -1. A priori, however, the chern form of this natural metricisation of $K_{\mathcal{F}}$ may not be non-negative. That, in fact, it is non-negative, is the main theorem of [Br4], which indeed holds quite generally, [Br5], since in any dimension it may be reduced to a surface question. Consequently, the leaf wise regularity is excellent, but one isn't out of the woods since we only really know about the curvature of fibres $S \to [S/\mathcal{F}]$ over closed points, and not how they vary, *i.e.* transversally this metricisation may be a complete mess. What is, therefore, required and, indeed sufficient on account of the symmetries of this metricisation which cause the initial Bochner formulation to localise in any dimension, is to strengthen [M1] from a description of the fibre over a closed point to (analytic) scheme points, *i.e.* a compactness theorem for the space $\mathrm{Hom}_{\mathcal{F}}(\Delta, S)$ of invariant maps from the disc.

This certainly establishes that iii.3 will lead to a complete description of the relative holomorphic volume $X \to [X/\mathcal{F}]$, for X of any dimension. There is, however, a legitimate question as to whether a more algebraic perspective in the spirit of i.4 has been overlooked. In this respect one should note that even where abundance holds, it is not true that there exists some universal n such that $K_{\mathcal{F}}^{\otimes n}$ always has a section. Whence, it is not clear that the classification of foliations on surfaces is a 1st order statement, so it's already an open question if a purely algebraic proof exits. The only realistic alternative would appear to be via the action of Frobenius in positive characteristic. Even here, though, there must be doubt since rational curves invariant by either of the natural foliations of a Shimura surface are Zariski dense over $\mathbb{Z}$, whence, even though this and cuspidal variants are the only other examples where this occurs, the curvature perspective is, at least naively, no longer valid in mixed characteristic. This phenomenon on Shimura surfaces occurs exactly for the primes p where $[S/\mathcal{F}]$ is meaningless. Conversely iii.3 is appealing to the existence of $[S/\mathcal{F}]$ as an actual honest champs analytique.

iv. The key theorem for surfaces

The irony of iii.3 should not be lost. The original conception which motivated [M2] was to make [M1] as simple as its analogue for volumes on surfaces by way of an appropriate variant of i.4. Given that the key point was a better version of [M1], the introduction to [M2] is confused. Let us therefore consider some different perspectives on the main theorem of [M1], beginning with how it first appears here, [III] III.1.3, in strengthened form,

iv.1 Theorem *Let $S \to [S/\mathcal{F}]$ be a minimal model of a foliation by curves with canonical singularities on a surface such that one of $K_{\mathcal{F}}^2$, $K_S.K_{\mathcal{F}}$ is strictly positive then there is a proper Zariski subset Z (actually the algebraic curves through the singularities) such that any sequence $f_n : \Delta \to S$ of invariant maps not arbitrarily close to Z has a subsequence which converges to at worst an invariant disc with bubbles, i.e. it is (relatively) hyperbolic modulo Z.*

Now it is a quite general fact, [III] I.2.5, that a sequence of divergent discs define, via the Nevanlinna area function, and after subsequencing a closed positive current. Better still if the discs aren't arbitrarily close in the compact open sense to some divisor, this current is nef. Moreover, since the discs are invariant, closure corresponds to a descent datum for this current to $[S/\mathcal{F}]$, *i.e.* it defines a transverse invariant measure $d\mu_{S/\mathcal{F}}$. The refined tautological inequality, iii.1, for the Nevanlinna area of $K_{\mathcal{F}}$ along discs may then be applied to yield,

iv.2 Proper Context *Suppose $S \to [S/\mathcal{F}]$ fails to be relatively hyperbolic in the strong sense of disc convergence, equivalently our guiding principle iii.3 fails, then there is a nef. transverse invariant measure $d\mu_{S/\mathcal{F}}$ satisfying $K_{\mathcal{F}}.d\mu_{S/\mathcal{F}} \leq 0$.*

Observe, why this is the proper context. In the first pace we have so far used nothing beyond the supposition that the singularities of $S \to [S/\mathcal{F}]$ are canonical. We can, though, appeal to the basic dichotomy iii.2 to suppose that $K_{\mathcal{F}}$ is nef. The cases $\nu(\mathcal{F})$ being 0 or 2 are easily eliminated, so in fact $K_{\mathcal{F}}$ and $d\mu_{S/\mathcal{F}}$ are non-zero and parallel in real homology by the index theorem. Consequently: although we know there are examples where we will never be able to represent $K_{\mathcal{F}}$ by a curve as the abundance perspective i.4 inevitably looks for, the failure of iii.3, produces a special representative, which isn't any old measure, but one coming from $[S/\mathcal{F}]$.

To see what's so special about $d\mu_{S/\mathcal{F}}$, let's put ourselves in the case $\nu(\mathcal{F}) = 1$. Here, $K_S.d\mu_{S/\mathcal{F}} = \omega_{[S/\mathcal{F}]}.d\mu_{S/\mathcal{F}}$. The bundle $\omega_{[S/\mathcal{F}]}$ lives on $[S/\mathcal{F}]$, where the measure $d\mu_{S/\mathcal{F}}$ has top weight, so, for dimension reasons, their intersection must be zero, whence $K_S.K_{\mathcal{F}} = 0$, and we've proved iv.1. The only fallacy in this argument results from the loose way in which we've been discussing $[S/\mathcal{F}]$ at the singularities, which means that $\omega_{[S/\mathcal{F}]}.d\mu_{S/\mathcal{F}}$ is not necessarily zero, but rather calculated by a residue around $\mathrm{sing}(\mathcal{F})$. This is equally true of any invariant bundle, and poses the question as to whether this can be controlled by the winding number of the measure around the singularities. Functorially with respect to the ideas this means the Lelong number of $d\mu_{S/\mathcal{F}}$, and indeed we have,

iv.3 Lemma *If the Lelong number λ_z of some invariant measure $d\mu$ at $z \in \mathrm{sing}(\mathcal{F})$ is zero then the residue at z along $d\mu$ associated to any invariant line bundle is zero.*

The proof of this in [M1] works for any isolated log-canonical singularity in any dimension, and in this generality is given in [IV] IV.3.1., albeit that higher dimension requires much more. Regardless the proof of iv.1 can now be completed, [III] III.1.3, by appealing to some standard results about Lelong numbers, [S] & [D1], and extensive use of the index theorem to prove that $d\mu_{S/\mathcal{F}}$ is actually an invariant nef. $\mathbb{R}$ divisor F. In fact, this is even true if $K_S.K_\mathcal{F} \neq 0$, so what gives iv.1 is the elliptic fibring lemma,

iv.4 Lemma *If $S \to [S/\mathcal{F}]$ has canonical singularities, then for any invariant effective $\mathbb{R}$-divisor F,*

$$(\omega_{[S/\mathcal{F}]} + D).F \leq 0$$

where D is the necessarily integral divisor on which F is supported.

On the canonical model, every connected component of F must be irreducible, so,

iv.5 Variant *Let $S \to [S/\mathcal{F}]$ be the canonical model of a foliated surface with $K_S.K_\mathcal{F} \neq 0$, then at least one of the following occurs,*

(i) *$S \to [S/\mathcal{F}]$ is relatively hyperbolic modulo the necessarily proper algebraic set defined by invariant curves through the singularities.*

(ii) *There is an invariant rational curve C which moves and $K_\mathcal{F}.C = 0$.*

Notice both possibilities can occur simultaenously, since 'canonical model' implicitly supposes $K_\mathcal{F}$ nef., so the latter case is the suspension of a representation of the fundamental group of a quasi-projective curve in $\mathrm{Aut}(\mathbb{P}^1)$. Whence, both at once is the rule, and if it's further prescribed that the discs don't meet $\mathrm{sing}(\mathcal{F})$, then the latter only happens without the former if the representation is either trivial or cyclic, in which cases the leaves are all $\mathbb{C}$'s or $\mathbb{C}^\times$'s respectively.

Regardless, what's important to recognise in the above variant is that if invariant discs were to get bigger and bigger away from a prescribed divisor, then the associated limit current of iv.2 is in fact a movable, invariant, rational curve.

v. Minimal models

The key theorem for surfaces is therefore achieved by a delicate interplay between the refined tautological inequality, iii.1, a modicum of transverse dynamic, iv.3, and the fundamental Mori dichotomy iii.2, *i.e.* the minimal model lemma. The importance of the latter, however, could easily be overlooked. It is, in fact, being used repeatedly to localise the tautological inequality, and whence pin down the obstructions to iii.3. Indeed the tautological inequality is a statement about parameterised discs. A parameterised disc is, however, from the perspective of the Nevanlinna area, a 'compact' object in an Arakelovian sense. Consequently, since the right hand side of the tautological inequality is the negative of a conformal Euler characteristic, it makes no more sense to try to localise it around pre-images of analytic neighbourhoods of sub-varieties than it would to claim that every open subset of $\mathbb{C}$ was simply connected. Whence, the localisability of the tautological inequality, around subvarieties, whenever $K_{\mathcal{F}}$ is nef., constitutes the essential relation between iii.2 and iii.3, and underlines the importance of proving iii.2 in all dimensions.

This project is carried out for any projective variety, X, of any dimension, n, in the relative setting of a foliation by curves $X \to [X/\mathcal{F}]$ with log-canonical singularities in [I] and [II]. There is a sense in which the minimal model lemma can be treated as a black box, and each paper has its own introduction, so our description will be brief. The overlap between the two is small, and most of that is justified by the fact that the overlap which does occur provides a more leisurely, and classical, introduction to the more technical character of the latter which takes place wholly in the 2-category of algebraic stacks. The former is also concerned with foliations of any rank - including the trivial foliation $X \to \mathrm{pt}$- and it's main theorem confirming the essential role of rational curves is self explanatory. It also displays the power of considering a foliation as a groupoid, even if its formulation is a little rudimentary in comparison with the more refined presentation of [II] II.1, where its ideas are extended to give an optimal bend & break estimate for foliations by curves. As such the key points in [II] are the analysis of extremal rays, and the extremely concrete flip theorem. Both of these take place within the perspective of [G3] that one should first identify the natural soft (2-)category that one is working with, so here Deligne-Mumford stacks or even just formal stacks, then tame its worst excesses, which, functorially with respect to the idea, means appeal to the projectivity of the moduli, or, for formal schemes, a projective embedding. Up to an extra operation, flaps, [II] also conracts and flips conic pencils into fibrations.

One should not, however, confuse a minimal model with a canonical model. A unified definition which takes account of what we've seen on surfaces and the $\mathcal{M}_{g,1} \to \mathcal{M}_g$ example is as follows: A minimal model $X \to [X/\mathcal{F}]$ is canonical if every curve $C \subset X$ with $K_{\mathcal{F}}.C = 0$ and nil natural map $\Omega^1_{[X/\mathcal{F}]} \to (\Omega^1_C)^{\vee\vee}$ intersects every effective Cartier divisor non-negatively. The condition is almost self explanatory, *i.e.* we wish to optimise the relative positivity of $K_{\mathcal{F}}$, but a little subtle at the singularities. There are, for example, in the algebraicly integrable case no curves in $\mathrm{sing}(\mathcal{F})$ with this property, cf. ix & x. In particular, the existence of a canonical model is considerably harder than a minimal one, and this harder theorem, although highly relevant, is not proved. Canonical models constitute the proper parallel of the standard theory of minimal models, and it's here, on for example 3-folds, that one finds a Mori type list, [Mo], that requires the blowing down of quadrics, whereas minimal foliated theory can be down with quotient singularities.

vi. Nihil est in infinito quod non fuerit prius in finito

This is Bloch's maxim. The a priori understanding of how invariant discs meet $K_{\mathcal{F}}$ is critical to the entire enterprise. As such, the most important theorem of [III] is the extension of the refined tautological inequality iii.1 to all dimensions. The statement [III] II.3.1 is optimal. For convenience, let us consider a special case,

vi.1 Example *Let $X \to [X/\mathcal{F}]$ be a projective variety foliated by curves with canonical $\mathbb{Q}$ Gorenstein singularities, and H ample, then for every $\epsilon > 0$ there is a proper subvariety Z_ϵ such that if $f : C \to X$ is an invariant map from a curve of genus g not factoring through Z_ϵ then,*

$$K_{\mathcal{F}}\cdot_f C \leq (2g-2) + \epsilon H\cdot_f C$$

The statement for discs has the same general form, following the model of iii.1, which itself is deduced as a corollary of the general statement. The reason why the refined tautological inequality holds is explained in the introduction to [III]. Plainly there may not be a sufficient number of proper invariant curves to test that the singularities are canonical using vi.1, but when extended to discs it is, again, wholly equivalent to the singularities being canonical.

Our emphasis, however, is on discs on which [III] contains a number of essential preliminaries. Of which the most important are the foliation free tautological inequality I.4, and extracting a closed current from divergent discs I.2.5. There is also a clear need for I.3 since the literature on disc convergence in complex function theory appears to be very confused. A good introduction to this material is [M3].

The final section III of [III] contains some illustrative surface examples,

- The proof of the key theorem iv.1 for surfaces, III.1.
- The Bloch theorem on the complete hyperbolicity of $\mathbb{P}^2\backslash\{4 \text{ lines}\}$, which is in fact a rather simple example involving foliated 3 folds, *i.e* a modification of $P(T_{\mathbb{P}^2}) \to (\mathbb{P}^2)^\vee$ to accommodate the log-structure, III.3.
- A strong hyperbolicity theorem in the spirit of Bloch for (log) surfaces with $c_1^2 > c_2$, or better the conclusion that discs which diverge on an algebraic surface S must lead to a derived current T' on $P(T_S)$ which is Zariski dense, i.e. for every Zariski open $U \subset P(T_S)$, $1\!\!1_U T' \neq 0$, III.4.

All of these examples are wholly within the context of the previous discussion, with the exception of the last which uses Brunella's extension of the elliptic fibring lemma iv.4 to invariant measures on foliated surfaces, [Br1], in an essential way. Before commenting on the intrinsic interest of this theorem, its principle relevance to the overall discussion is that it introduces a foliation free technique, cf. [III] III.4.1, for studying currents associated to supposed bigger and bigger discs on a surface, provided the support of this current is a, possibly irreducible, curve. This will replace the elliptic fibring lemma iv.4, when we eventually turn our attention to how the theory applies to the hyperbolicity of algebraic surfaces in general.

It is in the context of algebraic surfaces that the strong hyperbolicity theorem for surfaces with $c_1^2 > c_2$ is particularly noteworthy. The wholly general difficulty is that the study of the isoperimetric inequality for discs on an algebraic surface, unlike that for parabolic Riemann surfaces, can not quite be reduced to a question of the relative hyperbolicity of a foliated higher dimensional variety $X \to [X/\mathcal{F}]$, since the word variety supposes irreducible, and for arbitrary discs one only gets connected. Consequently, when strong hyperbolicity statements on surfaces are deduced from the theory of foliated varieties it is because [Bro] and [Kl] are applicable. The immediate case being discussed here, however, not only avoids this, but constitutes the unique known theorem on disc convergence where one cannot do better than convergence to a disc with bubbles. It is important to note that the proof does not fall into the general pattern of localising a tautological inequality via minimal model theory, but rather seeks lower bounds for the tautological bundle L on $P(T_S)$ intersected with various currents. Understanding anything in the proof beyond III.4.1 is therefore not per se necessary to the overall structure of the theory, but if one does understand it, one will be profoundly grateful as to how the minimal model theorem simplifies the study.

vii. Geometric measure theory

Given the emphasis on closed positive currents, a brief intermission on their general properties is appropriate. On surfaces the theory of such currents is particularly satisfactory because of Siu's theorem, [S], on the upper semi-continuity of Lelong numbers in the Zariski topology, and the resulting decomposition theorem. Although this theorem is valid in all dimensions it lacks the consequences in arbitrary dimension that it does in dimension 2. Now the Lelong number measures the concentration of a closed positive current at a point, and it's not hard to make local examples which show that a similar jolly cannot exist for measuring concentration at, say, a closed surface in a 3-ball. However, what underscores the definition of Lelong number is that a point doesn't have a boundary. The upshot of this observation, [IV], III.3.2, which needs some Kắhlerianity if the type of the current is other than dimension $(1,1)$ is that closed positive currents of dimension (d,d) on a projective variety have a vast number of parallels with algebraic cycles of dimension d. Whence, it seems reasonable to describe such a current as a geometric measure.

An immediate consequence of this observation is that it gives a satisfactory decomposition for free. Indeed one has the Skoda-El Mir theorem to the effect that if $d\mu$ is a geometric measure and $Z \subset X$ Zariski closed then one can localise $d\mu$ as,

$$d\mu = \mathbb{1}_{X\setminus Z} d\mu + \mathbb{1}_Z d\mu$$

with both parts of the decomposition geometric measures on X. For example if $d\mu$ was a sum of cycles, then the former is the sum of those cycles whose generic point doesn't lie in Z. What, though, is missing to turn this into the makings of a genuine decomposition theorem is to know that $\mathbb{1}_Z d\mu$ is a geometric measure on Z. This is, however, a trivial consequence of the properness of Z, [IV] III.3.3., which equally liberates the study of geometric measures from the un-healthy, and un-functorial, notion of smoothness. Going further, one, therefore, simply inducts on dimension of subvarieties, [IV] VI.3, to decompose a geometric measure into a countable sum of push-forwards of diffuse measures on subvarieties, the <u>components of the measure</u>, including possibly X itself, where diffuseness is defined by the condition that $\mathbb{1}_Z d\mu = 0$, for all proper subvarieties.

This is, however, just the tip of an iceberg, since all of the essentials of Fulton's algebraic intersection theory, [F], now work for geometric measures. The most immediately pertinent are deformation to the normal cone, [IV] III.5, and the segre class, $s_Z(d\mu)$ of a measure $d\mu$ along a subvariety Z, or more accurately of $\mathbb{1}_{X\setminus Z} d\mu$ along Z. Needless to say this coincides with Fulton's segre class when $d\mu$ is a cycle, and the Lelong number if Z is a point. In the case that $d\mu$ has dimension $(1,1)$, it is functorially with respect to the ideas, the winding number of $d\mu$ around Z. Should $d\mu$ arises from a sequence of discs this ought not be confused with the multiplicity, which is both functorially and actually different, even for points on surfaces. The segre class measures the size of discs near Z, while the multiplicity indicates how singular they are. The former always dominates the latter. The converse is rare.

This combines rather nicely with the notion of diffuseness, which as its name suggests is not wholly conducive to the measure concentrating at a subvariety. *e.g.*
vii.1 Fact *Let $d\mu$ be a $(1,1)$ diffuse geometric measure on a projective variety, then for Z a subvariety of dimension* d *at most one of the following can occur,*

(i) There is a nef. line bundle, L, with $L^{d+1} \neq 0$, and $L.d\mu = 0$.

(ii) The segre class $s_Z(d\mu) \neq 0$.

As the assumptions suggests, this is a counting argument, cf. [IV] V.1.4 & V.2.2, with sections of line bundles. In the foliated situation L will be $K_{\mathcal{F}}$, so that we have yet another example of the pervasiveness of the minimal model lemma.

viii. Transverse invariant measures

If we combine everything that we have so far with a view to applying 3.2, and investigating a priori estimates for the curvature $X \to [X/\mathcal{F}]$ of a foliated variety, we see that we can suppose $K_{\mathcal{F}}$ nef., and in the event of relative hyperbolicity failing produce a transverse invariant measure $d\mu_{X/\mathcal{F}}$ arising from bigger and bigger discs which satisfies $K_{\mathcal{F}}.d\mu_{X/\mathcal{F}} \leq 0$ thanks to the refined tautological inequality, whence by the minimal model theorem $K_{\mathcal{F}}.d\mu_{X/\mathcal{F}} = 0$. It is also the basic consequence of Nevanlinna theory, as opposed to using un-parameterised discs, that we may, without loss of generality, suppose $d\mu_{X/\mathcal{F}}$ is nef., *i.e.* meets every effective Cartier divisor non-negatively. To set the scene one notes a generalisation, [IV] II.2.2, of the index theorem on surfaces to n dimensional varieties:
viii.1 Fact (over any field) *Suppose a nef. divisor L has the property that $L^n = 0$, but $L^{n-1} \neq 0$, then the only class in Néron Severi which meets every divisor non-negatively and is in the plane $L = 0$ is L^{n-1} itself.*

In our foliated context, we certainly have the picture that we're looking for if our foliation is of general type, so that in all dimensions iv.2 takes the form,
viii.2 Proper Context *Suppose a minimal model $X \to [X/\mathcal{F}]$ fails to be relatively hyperbolic modulo a proper sub-scheme then there is a nef. transverse invariant measure $d\mu_{X/\mathcal{F}}$ with $K_{\mathcal{F}}.d\mu_{X/\mathcal{F}} = 0$. In particular $\nu(\mathcal{F}) \leq n-1$, and, viewing $d\mu_{X/\mathcal{F}}$ as an operator on divisors, $K_{\mathcal{F}}^{n-1}$ is parallel to it in Néron-Severi.*

So, certainly we have a curious representative for the class of $K_{\mathcal{F}}^{n-1}$, and vii.1 suggests why it is special. Again, Riemann-Roch points to the condition $K_X.K_{\mathcal{F}}^{n-1} > 0$ for $K_{\mathcal{F}}^n = 0$, so that viii.2 becomes, $\omega_{X/\mathcal{F}}.d\mu_{X/\mathcal{F}} > 0$. As ever the line bundle $\omega_{X/\mathcal{F}}$ on X comes with a descent datum for $X \to [X/\mathcal{F}]$, so its intersection with any transverse invariant measure $d\nu$ with $\mathbb{I}_{\mathrm{sing}(\mathcal{F})} d\nu = 0$ is given by a residue around $\mathrm{sing}(\mathcal{F})$, which is subordinate to the general residue symbol, [BB], [IV] IV.1.4, $Res(d\nu)$ governing the intersection of invariant bundles with measures. The question arises, therefore, as to whether we can relate this to the segre class of $d\nu$ around $\mathrm{sing}(\mathcal{F})$ in the spirit of iv.3.

Already for saddle nodes on surfaces, which is the unique difficulty in iv.3, this requires a little imagination. The technique employed also covers isolated log-canonical singularities, but, otherwise, it isn't at all clear that the residue is well defined except as a principle value which makes essential use of the properness of $\mathrm{sing}(\mathcal{F})$, *i.e.* there is no a priori evidence that Res can be calculated locally. Nor is it a question of commutative algebra - sheaves of distributions are injective modules, with even the surface case of iv.3 requiring the $\ell_\infty^\vee$ regularity of measures.

These difficulties require us to reflect on the nature of the residue symbol. Certainly we could make theory till the cows come home about residues, and relative Riemann-Roch/index theorems for $X \to [X/\mathcal{F}]$, but what is required is to relate the a priori highly analytic residue symbol, Res, to meromorphic functions on X, and in particular the residual measure around the singularities. The archetypal example of how to calculate a residue symbol is Grothendieck's algebraic De-Rham theorem. This is not really our situation, since functorially speaking a relative algebraic De-Rham theorem would involve computing residues around divisors transverse to the foliation, and, indeed if the divisor were everywhere transverse then for a foliation of any dimension Grothendieck's argument would go throw mutatis mutandis. Nevertheless, the tangency of our residue to the foliation notwithstanding, a paradigm with the algebraic De-Rham theorem emerges, *i.e.* the statement of the algebraic De-Rham theorem doesn't require resolution of singularities, but its proof does, and in the same way, our own residue symbol is under control if it is so on a resolution of singularities, [IV] IV.2.4. Certainly, there is a mute, and possibly unintelligible, logical point as to whether the algebraic De-Rham theorem would have failed without resolution of singularities, but in our current context, at least on the practical level, the intervention of a foliated version of resolution of singularities, LCR (log-canonical resolution), [IV] I.6.1 should be unsurprising.

Postponing for the moment some commentary on the hypothetical nature of LCR, it, nevertheless immediately brings to light that our residues are controlled by a $\ell_\infty^\vee$, or better measure, type regularity. Indeed just as a linear map comes equipped with a sub-space on which its semi-simple part is nil, so, at least formally, log-canonical singularities come with invariant formal sub-schemes uniquely defined by the loss of the non-nilpotence on restriction of a local generator of the foliation. Extending, the notation of saddles on surfaces we term these weak branches. Their existence in higher dimension is tied to particular irreducible, not connected components, of $\mathrm{sing}(\mathcal{F})$, so that even formally their behaviour can differ radically according to the sub-scheme in which one completes. Indeed on 3-folds, one encounters beats, [IV] I.5.1, which completed in a point look almost trivial, but which, for a fixed isomorphism class at a point, admit an infinite dimensional space of possibilities when completed in certain singular curves. Needless to say the analytic theory is more complicated still, but pretending that formal and analytic neighbourhoods are the same for the moment, one notes that the relations amongst one forms provided by a foliation by curves allows one to define residual measures around weak branches provided that these are defined in a small neighbourhood of a proper connected component of $\mathrm{sing}(\mathcal{F})$, [IV] III.6.3, and that such measures are zero iff the symbol Res is zero, op. cit. IV.1.

This certainly moves the discussion forward, but leaves open the possibility that such residual measures have nothing to do with meromorphic functions. Indeed though these measures are supported on $\mathrm{sing}(\mathcal{F})$, they a priori dominate every segre class associated to every possible scheme structure on $\mathrm{sing}(\mathcal{F})$. The ubiquity of integration by parts is not, however, confined to plagiaristic extension of Lelong's study of residual measures at points to the case of proper sub-varieties, and a trick emerges. Specifically, when one has many meromorphic functions, one has many positive forms, which, naively speaking, have the form $dd^c(\phi)$, for some potential ϕ. This is, of course, strictly speaking, false since there are residues, but it vastly widens the scope of things that we can hope to compute by integration by parts, and, in fact, [IV] IV.6.6, residual measures around weak branches fall into this trap. Better still, the double integration by parts in question clears up many of the problems associated with formal vs. analytic properties of weak branches, and shows that these weak residual measures will vanish provided that not only the segre class at the singularities of our invariant measure is nil, but that the logarithms of sufficiently many meromorphic functions are ℓ_1 after we cap with a Kǎhler form.

Consequently, what remains to be addressed is the wholly non-trivial condition that we can find enough meromorphic functions which are absolutely integrable against our invariant measure. Indeed, in principle, there could be no such. The situation is curious, since for f a meromorphic function,

$$dd^c \left(\log|f| d\mu_{\mathcal{X}/\mathcal{F}}\right) = d\sigma_0 - d\sigma_\infty$$

where the latter are the residual measures around the zeroes and poles. Consequently, if we were to form a relative Laplacian, Δ along the foliation against our measure, then $\Delta(f)$ is itself a measure, and one might suspect that we can extend elliptic regularity to deduce that $\log|f|$ is ℓ_1. This is wholly false, even in favourable circumstances where other considerations might allow us to suppose that the residual measures in question are nil. The root cause of the difficulty is that elliptic regularity, even locally, uses that the product of compacts is compact. This may occur in the guise of a sobolev lemma, or it may form an entire theory, [Gr1], it, is nevertheless, an unsustainable hypothesis in the context of relative products over $[X/\mathcal{F}]$. Even worse, as the example of $\mathcal{M}_{g,1} \to \mathcal{M}_g$ shows even universal closedness is insufficient for a relative Laplacian to exhibit transverse regularity, unless our measure was actually a δ-function at a closed point of $\mathcal{M}_g$.

This leads to an important digression on the functorial aspects of measure theory, [II] A.3, *i.e.* what properties are preserved by descent, or more accurately, co-descent, [SGA]I.VI. Specifically, to endow $[X/\mathcal{F}]$ with any sort of covering, we must be prepared to consider an étale map of infinite degree as an open. Such a map though allows us to push forward functions with compact support to functions with compact support, and in general maps with this property lead to a natural site, $\underline{E}QF$, the essentially quasi-finite site, op. cit. A.3.3, which respects (co-variantly) the functorial properties of standard measure theory. Critically to our current considerations there is a Krein-Milman type theorem for extremal measures on any $\underline{E}QF$-stack, leading to an honest space structure on the set of extremal measures, or 'points' which permits any invariant measure to be expressed as a Radon measure over points, A.3.14. In the context of $[X/\mathcal{F}]$, if the foliation were a fibration a point is, of course, a δ-functions on the base, and in general they are invariant measures whose space of invariant functions are just the constants, so for X-compact, harmonic functions constant. This is almost a parody of elliptic regularity, but nevertheless a wholly non-trivial property that can be exploited. The key to this exploitation is the relative integral from a compact space to a $\underline{E}QF$-stack. This is well defined irrespectively of whether the map is proper or not, and is a particular example of co-descent, which also leads to a Fubini theorem, A.3.16/17. In the case of $X \to [X/\mathcal{F}]$, these considerations combine to the principle (which is probably a theorem but is used in an ad hoc manner, [IV] IV.6) that the ℓ_1 norm of a function against an extremal measure is controlled by it's relative ℓ_1 derivatives. Better than this one cannot do, op. cit. IV.5.5, but it's enough to deduce,

viii.3 Theorem [IV] IV.7.3 *Let* $X \to [X/\mathcal{F}]$ *be a* $\mathbb{Q}$ *gorenstein foliation by curves on a projective variety, and suppose* LCR$^+$, *then for* $d\nu$ *a transverse invariant measure with* $\mathbb{I}_{\mathrm{sing}(\mathcal{F})} d\nu = 0$ *the residue* $d\nu$ *around* $\mathrm{sing}(\mathcal{F})$ *associated to any invariant line bundle is zero if* $s_{\mathrm{sing}(\mathcal{F})}(d\nu) = 0$.

Here LCR$^+$, [IV] IV.4.3, is a version of LCR with plausible convergence assumptions, since despite all that's been said, the purely formal nature of weak branches isn't quite enough to push things through, and one needs just a little convergence. As such, the definition of plausible is compatible with actual theorems on surfaces, and 3-folds. This rather lengthy dynamical digression completed, we can turn to its application to hyperbolicity, and observe that vii.1, viii.1, 2 & 3 instantly combine to show that we're on the right track,

viii.4 Central corollary *Let things be as per viii.2, but with* $K_X.K_{\mathcal{F}}^{n-1} \neq 0$ *then* $d\mu_{X/\mathcal{F}}$ *has no diffuse component, i.e. there is a countable union* $Z = \cup_i Z_i$ *of algebraic subvarieties such that* $d\mu_{X/\mathcal{F}} = \mathbb{I}_Z d\mu_{X/\mathcal{F}}$.

ix. The key theorem for 3-folds

The key theorem for surfaces is immediate from viii.4, and, conversely viii.4 is the extension to arbitrary dimension of the key theorem for surfaces. It is not, however, the required theorem in all dimensions. The reason why viii.4 gives the right theorem for surfaces is that linear algebra limits the number of invariant branches through a canonical surface singularity to two. Something broadly similar, [IV] I.3.2, is true for invariant surfaces in 3-folds, so the real a priori difficulty that viii.4 is inviting on 3-folds is that there is a Zariski dense set of invariant curves, $\{C_i\}$ with $K_{\mathcal{F}}.C_i = 0$.

Certainly such curves are rational, and they even meet the locus where the foliation is smooth in $\mathbb{G}_m$'s, so it's already a non-trivial fact that we've gone from the possibility of some highly arbitrary sequence of discs to find many rational curves. This is not, however, what our guiding principle iii.3 of finding numerical conditions for a priori hyperbolicity is about, and we require to control such curves.

Plainly the very problem of such curves on the minimal model of a foliated 3-fold can, after subsequencing, be encoded in an easy example of an invariant measure,

$$d\mu := \lim_{H.C_i \to \infty} \frac{1}{H.C_i} \int_{C_i}$$

where H is ample, and we require to study the atomic part, *i.e.* where one has point masses on $[X/\mathcal{F}]$, which is again a possibly countable sum of such curves.

Needless to say, one analyses such curves, [IV] II.3-8, in the spirit of the analysis of extremal rays in [II] II.5-8. The question though, is more difficult, since now the holonomy over the smooth locus can be $\mathbb{Z}$, when before it was finite. Obviously curves which move are good, and we perform some étale surgeries, to get all but finitely many of the remaining ones as nice as possible, *i.e.* not meeting any others. Better still what remains can be flopped, or technically more accurately flapped, *i.e.* (weighted) blown up then (weighted) blown down in the other direction. Nevertheless, by now we've lost all projectivity, and even if we hadn't there's no reason to believe that this weak sense of flopping would necessarily preserve it. Thus we need a reason to exclude, for example, that one can perform canonical contractions in the category of algebraic spaces ad infinitum. The situation is tamed by the invariant measure itself, and indeed a property particular to it that it has a well defined proper transform for a modification in any subvariety - a general measure only admits this if the subvariety has nil measure. Combining this with all our other knowledge, [IV], V.2.3, we see that a parabolic measure, *i.e.* a nef. transverse invariant measure with this extra critical property, and of course meeting $K_{\mathcal{F}}$ in zero, can only have finitely many curves in its support that do not move.

Despite the slightly ad hoc character of the above analysis which keeps the ambient singularities no worse than quotient, it still clearly indicates that if we proceeded functorially (which would imply allowing ambient LCIQ singularities) we could for $K_X.K_{\mathcal{F}}^2 > 0$ perform contractions and flops to arrive to the possibility that $d\mu_{X/\mathcal{F}}$ is wholly supported in $\mathrm{sing}(\mathcal{F})$, while it's already true that if $K_X.K_{\mathcal{F}}^2 < 0$ then for $d\mu_{X/\mathcal{F}}$ without support in $\mathrm{sing}(\mathcal{F})$ we find an invariant rational curve which moves. Whence, in this latter 'Ricatti' case, there is a map to a foliated algebraic surface $S \to [S/\mathcal{G}]$, such that $[X/\mathcal{F}]$ is a $\mathbb{P}^1$ bundle in the étale topus of $[S/\mathcal{G}]$.

As such, the critical new problem that we're presented with is that some part of $d\mu_{X/\mathcal{F}}$ could be supported in $\mathrm{sing}(\mathcal{F})$. Here the precise definition of transverse invariant measure is important, and we optimise our descent datum by insisting, as we may, in the spirit of the definition of canonical models that, $\Omega^1_{[X/\mathcal{F}]} \wedge d\mu_{X/\mathcal{F}} = 0$. Whence the only way that this problem can occur on a 3-fold is for curves in $\mathrm{sing}(\mathcal{F})$ where the foliation has a generator at a generic point of such of the form,

$$\partial = x\frac{\partial}{\partial x} + z^{p+1}\frac{\partial}{\partial y},\ p \in \mathbb{N},$$

so that in the $x = 0$ plane, $\mathrm{sing}(\mathcal{F})$ is an invariant curve for the induced foliation.

The plane $x = 0$ is the weak branch, and defines a formal subscheme, $\mathfrak{W}$, with trace the components of $\mathrm{sing}(\mathcal{F})$ where there is exactly one eigenvalue. It need not, however, extend to a formal subscheme with trace all of $\mathrm{sing}(\mathcal{F})$ as the example of the beast shows, [IV] I.5. Nevertheless, there is an induced foliation, $\mathfrak{W} \to [\mathfrak{W}/\mathcal{G}]$, and we can modify everything by invariant blowing up to insist that $\mathfrak{W}$ is smooth, with the singularities of $\mathcal{G}$ canonical. The canonical bundles are related by,

$$K_{\mathcal{G}} = K_{\mathcal{F}}|_{\mathfrak{W}}(-Z)$$

where Z is an effective divisor in $\mathfrak{W}$, supported on $\mathfrak{W} \cap \mathrm{sing}(\mathcal{F})$. Concomitantly, if $S := \mathbb{I}_{\mathrm{sing}(\mathcal{F})} d\mu_{X/\mathcal{F}}$, then S is certainly a cycle of curves invariant by $\mathcal{G}$, but we only know that $K_{\mathcal{F}}.S = 0$, whereas to deduce that even one of the curves in the support of S were rational or elliptic would require, $K_{\mathcal{G}}.S \leq 0$.

The right object of study, however, isn't S but the part of $d\mu_{X/\mathcal{F}}$ in $\mathfrak{W}$. Given that $\mathfrak{W}$ may be devoid of a strict analytic existence this is a priori defined in terms of components of the measure whose supporting variety V meets the trace of $\mathfrak{W}$, and whose completion in this intersection factors through $\mathfrak{W}$. In our current 3-fold context we can, without loss of generality, suppose that V is a curve, and we augment S to a (finite) cycle T containing the extra curves in the support of $d\mu_{X/\mathcal{F}}$ which 'factor through' $\mathfrak{W}$, which, in turn, extends around T. Remarkably,

ix.1 Fact [IV] V.4.1 *T is a nef. cycle on $\mathfrak{W}$, i.e. for any effective divisor D in the trace of $\mathfrak{W}$, $D.T \geq 0$.*

Better still it's easy to make a canonical contraction, $\rho : \mathfrak{W} \to \mathfrak{W}_0$, of components of the trace such that the induced foliation $\mathcal{G}_0$ has $K_{\mathcal{G}_0}$ nef., which implies, supposing $\mathfrak{W}$ non-algebraic, that $\rho_* T$ is a divisor of elliptic fibre type. Whence,

ix.2 Finiteness in 3-D *Let $d\mu_{X/\mathcal{F}}$ be a parabolic invariant measure on a minimal foliated 3-fold $X \to [X/\mathcal{F}]$ not of general type with $K_X.K_{\mathcal{F}}^2 \neq 0$ then,*

$$d\mu_{X/\mathcal{F}} = \sum_i d\mu_i + \sum_j L_j + \sum_k E_k$$

where, $d\mu_i$ are measures on the finitely many invariant surfaces where the foliation is in conics, the L_j are positive multiples of finitely many invariant rational curves, and the E_k positive multiples of the above, likely inexistent, elliptic divisors in $\mathfrak{W}$.

This is the essential of the extension to 3-folds of the key theorem on surfaces. To get from here to strict analogues of iv.1 or iv.5 requires some contractions and flops to either kill ($K_X.K_{\mathcal{F}}^2 > 0$) or move ($K_X.K_{\mathcal{F}}^2 < 0$) the part outwith $\mathfrak{W}$. This isn't too difficult, and only requires repeating what we've already done in the slightly more general situation of not just quotient, but LCIQ singularities on X. A full key theorem, however, would require moving the elliptic curve, which doesn't quite follow from the analogous result for surfaces, [M2] IV.5.2, since $\mathfrak{W}$ is formal. In applications to algebraic surfaces, however, ix.2 is overkill.

x. Weak Branching

The structure of the part of a parabolic measure within, or more accurately, infinitely tangent to, a weak branch on a 3-fold is particularly simple, and it is critical to have a sound understanding of why ix.1 is true, and the essentially formal nature of the minimal model theorem which it then exploits. The latter is the essence of the approach to bend & break in [I], but the only precedent for the former is [III] III.1.4 during the proof of the strong hyperbolicity theorem for $c_1^2 > c_2$. Nevertheless, it is a general argument which, in any dimension, allows us to work with the part of a measure inside a formal weak branching scheme, $\mathfrak{W}$, in much the same way that we might consider the component of a measure inside an honest sub-variety. Again, though its existence is little better than formal, we define a component of a measure as 'factoring through' a weak branch, whenever the completion of its supporting variety in the relevant components of $\mathrm{sing}(\mathcal{F})$ does. The key to 'seeing inside' $\mathfrak{W}$ is ix.1, and its generalisations. These profit from the fact that the measures which we are studying always admit proper transforms. Whence, irrespective of the fact that $d\mu_{X/\mathcal{F}}$ may have support in $\mathfrak{W}$, we can skew a weighted blow up of any component, Z, of the trace in favour of $\mathfrak{W}$. Alternatively one can make a chain of blow ups à la [III] III.4.1, but using stacks and weighted blow ups is more conceptual. Regardless given such a weighted blow up with exceptional divisor E, and proper transform $\tilde{d\mu}$, one knows a priori that $E.\tilde{d\mu} \geq 0$, so there is hope that if we skew the weight sufficiently in favour of $\mathfrak{W}$, we can deduce that $Z.T \geq 0$. As one might expect, any part of $\tilde{d\mu}$ on E is actually helping this plan, and what makes it succeed is the essentially repellent nature of the dynamics around $\mathfrak{W}$, which leads to a nil contribution from everything else as the weight goes to ∞. The proof, even on 3-folds, is best understood with the aid of the heuristic, (*i.e.* supposing a little analytic convergence), considerations preceding the the general case, [IV] VI.5.3.

The plan then is to extend the induction implied by viii.4 from projective varieties to weak branching formal schemes. Unfortunately, by the very definition of a weak branch, $\mathfrak{W}$, we can infer nothing about the singularities of the induced foliation from the ambient ones, consequently, in order to continue our study by a decreasing induction on the dimension of weak branches, we must have LCR. This ultimate intervention of LCR is logically clearer than in viii.3, where its necessity is mute. Certainly this is the unique class of singularities where minimal models exist, and the refined tautological inequality is even equivalent to canonical singularities. In appealing to this machinery, the study of the part of a measure within a weak branch is not, however, simply trying to maximise the amount of minimal model theory in the discussion. It, infact, indicates a highly intrinsic relation between LCR and hyperbolicity. Thanks to [BM], LCR is a local problem of local uniformisation type, while the presence of (strong) hyperbolicity serves to identify valuations of ever decreasing discrepancy with analytic and even formal invariant sub-schemes. This is, however, the essence of finding a contact structure which permits one to conclude resolution. In so much as one can, without loss of generality, assume that the Zariski closure of any weak branch is the whole variety, it isn't just true that the presence of foliated hyperbolicity for a given foliated variety indicates that it should have a log-canonical resolution, but so should all of its weak branches.

Appealing, once again, to the LCR hypothesis we can therefore proceed to the creation of the necessary machinery for a key theorem in all dimensions, inducting by decreasing dimension of both sub-varieties and formal (or better, formal formal, [M4]) sub-schemes: the minimal model programme can be extends to this generality, [IV] VI.6, and the slightly nasty induction occasioned by weak branches inside weak branches can be carried out, [IV] VI.7. This latter phenomenon first occurs in dimension 4, which is wholly representative of the general case.

xi. Algebraic Surfaces

The original motivation of this work was the reduction by Green & Griffiths, [GG], of the study of entire mappings $f : \mathbb{C} \to S$ to algebraic surfaces of general type to the study of entire mappings invariant by a foliation by curves on higher dimensional varieties. They further conjectured that such a map should factor trough a rational or elliptic curve, and, whence, at least qualitatively, suggested a certain equivalence between negative Ricci, and negative sectional curvature on surfaces, or, closer to our setting, an equivalence between the isoperimetric inequality for balls and that for discs. The exact role of foliations by curves in a quantative version of this equivalence is satisfactory whenever the surface contains no rational or elliptic curves by [Bro] & [Kl]. Otherwise there are issues as the difference between the proof of quantative disc convergence if $c_1^2 > c_2$, [III] III.2, and the comparatively simpler [M1] suggest. If, however, we confine our attention to the isoperimetric inequality for parabolic Riemann-surfaces, then a quantative relationship remains a problem about foliations on higher dimensional (irreducible) varieties.

This is summarised in [IV] VI.1-2, and given its motivating nature it seems reasonable to explain the inductive character of the machinery by way of a solution to this conjecture in the presence of LCR, or, more accurately LCR$^+$. In fact, as we've suggested, the conjecture may well imply LCR for ODE's on surfaces, and whence, irrespective of the truth of LCR, an identification of hyperbolicity of surfaces with algebraic conditions (*i.e.* those of Green & Griffiths plus LCR). In any case, the only extra thing that we have to observe is that the natural inductive structure is not quite by dimension of invariant subvarieties of a foliated variety, but by the order of the ODE. The relation between these is a consequence not just of the minimal model theorem, but also the corollary that a rational (not just bi-rational) map from a foliated variety to a minimal one can be resolved by invariant blowing up, [IV] VI.3. This leads to a rather simpler induction than a key theorem in all dimensions would demand, and whence,

xi.1 Theorem *Let S be a surface with canonical (*i.1 *sense) singularities, and no rational or elliptic curves, then supposing LCR$^+$ the following are equivalent,*

(i) S does not admit an almost étale cover by an abelian surface.
(ii) S is a canonical model of an algebraic surface in the sense of i.1.
(iii) S satisfies the isoperimetric inequality for balls.
(iv) S satisfies the isoperimetric inequality for discs.

This is equally true more generally for any 2-dimensional algebraic stack or indeed log-stacks with stable curve without connected elliptic components as boundary, and volume etc. calculated in a complete metric, and, needless to say, minor modifications of the definitions of rational, elliptic, and abelian. In the presence, however, of rational curves there is need for extra work of the type that we did on 3-folds to guarantee that the support of our measure was finite. Consequently, appealing to [III] III.3.2 we can only guarantee that the number of these is finite when either op. cit. or the 3-fold theorem apply, *e.g.* $c_1^2 > \frac{2}{3}c_2$, which is related to,

xi.2 Theorem *Let S be a surface of general type such that any rational or elliptic curve is contained in a divisor Z, then supposing LCR$^+$ there is a constant $\kappa > 0$, depending only on S and a given ample divisor H, such that for any holomorphic map $f : Y \to S$ from a proper cover $p : Y \to \mathbb{C}$ of the line not factoring via Z,*

$$\fint_{Y(r)} f^* c_1(H) \leq_{\text{exc}} -\kappa\, \chi_Y(r)$$

where $Y(r)$ is the pre-image of the disc of radius r, exc *denotes that r is excluded from a set of finite Lebesgue measure, and χ is the Nevanlinna Euler characteristic.*

In particular, with the theorem's hypothesis, curves of genus g are bounded in moduli, and any entire map $f : \mathbb{C} \to S$ factors through Z. Equally, the theorem is valid for stacks and log-stacks of dimension 2.

Opus Primum

Rational curves on foliated varieties, with F.A. Bogomolov

INTRODUCTION

In a series of papers, notably [B1] and [B2], the first author showed, amongst other things, that a surprising interplay between the classical Frobenius theorem on the integrability of vector fields closed under Lie bracket and various algebro-geometric considerations gave rise to some rather strong restrictions on the 'size' of subbundles of the cotangent bundle, with a particular corollary being inequalities for Chern numbers on algebraic surfaces. In refining this circle of ideas Y. Miyaoka, [Mi1], established that any quotient of the cotangent bundle of a surface of general type and positive index was big. Rather more remarkably, Miyaoka, [Mi2], subsequently considered the problem of subbundles of the tangent bundle with positive slope along a generic complete intersection of ample divisors, and by extending Mori's bend and break technique to 'deformations along a foliation' showed that these hypothesis implied the existence of covering families of rationally chain connected varieties in the direction of the foliation. The surface case of this result is particularly clean, since it asserts that either the bundle of top weight forms $K_{\mathcal{F}}$ along the leaves is pseudo-effective, or the foliation is a fibration by rational curves. Continuing in these directions the second author in extending the results of the first author on boundedness of moduli of curves of a given genus on surfaces to curves with boundary employed Miyaoka's semi-positivity theorem in an essential way, cf. [M1]. The use of the said theorem therein was to force, under the hypothesis of a dense parabolic leaf, the existence of a global vector field defining the foliation on a crude version of what might be considered it's minimal model, i.e. a normal algebraic space on which the positive part of the Zariski decomposition of $K_{\mathcal{F}}$ coincides with the push forward of the latter. It was natural however to examine this question more carefully, not just in terms of a more delicate structure of the minimal model but to introduce the study of the birational geometry of foliations per se, cf. [M2], [Br2], [Br3]. As ever a prerequisite for such a study is the understanding of subvarieties on which the cotangent bundle of our foliation is negative.

To fix ideas let us consider a foliation by curves, $\mathcal{F}$, on a smooth variety X. This is equivalent to giving a rank 1 torsion free quotient of Ω_X, whose Chern class we denote by $K_{\mathcal{F}}$. Miyaoka's theorem then asserts that if C is a 'sufficiently movable' curve then either $K_{\mathcal{F}}.C \geq 0$ or $\mathcal{F}$ is a (possibly singular) fibration by rational curves. The main problem here is the hypothesis, 'sufficiently movable' which more precisely means that C moves in a family $\{C_t \mid t \in T\}$ covering X such that generically C_t does not meet the singularities of $\mathcal{F}$, whereas one really wishes to understand the implications of the hypothesis $K_{\mathcal{F}}. C < 0$ for any curve C. The difficulties in extending Miyaoka's method, or its refinement by Shepherd-Barron, [SB], to this situation are formidable. Firstly one must establish that the hypothesis imply that the divided symmetric power algebra of $K_{\mathcal{F}}$ extends across a finitely generated extension of $\mathbb{Z}$, [Mi2], or rather more straightforwardly the foliation is defined by an inseparable scheme quotient in positive characteristic, [SB]. Even then there is the added complication that C may pass through the foliation singularities and so Mori's bend and break technique may not apply. The key to resolving this problem § 2 lies in finding a $\mathcal{F}$-invariant surface S containing C [1], and so reduce the study to something more tractable. In finding our surface, however, we necessarily show that the leaves through C are algebraic curves, and whence reprove Miyaoka's theorem in this case without any appeal to reduction in positive characteristic, thanks to a theorem of Arakelov, [A], all be it that the most satisfying proof of Arakelov's theorem is to proceed via positive characteristic, cf. [S2].

With this example in mind let us consider a more general situation. We denote by $(X, \mathcal{F})$ any variety equipped with an integrable (i.e. closed under Lie-bracket) foliation $\mathcal{F}$. The singularities of X are not our interest, but rather those of $\mathcal{F}$. What this latter should mean is a measure of how far away the foliation is from being given everywhere locally by a relatively smooth fibration. To understand this it is convenient to introduce an ambient smooth space M. The foliation is given by a subsheaf $\mathcal{T}_{\mathcal{F}}$ of $\mathcal{T}_X$ of rank r, say, and for any $x \in X$ we have the residue map,

$$\mathcal{T}_{\mathcal{F}} \otimes k(x) \longrightarrow T_M \otimes k(x) \,.$$

Should this map be an injection onto a subspace of dimension r, then Nakayama's lemma forces $\mathcal{F}$ to be a bundle in a neighbourhood of x, and better still the Frobenius theorem goes through verbatim to force $\mathcal{F}$ to be given locally by a relatively smooth fibration. Naturally then we introduce the notion of weak regularity, 1.1, which requires $\mathcal{T}_{\mathcal{F}}$ to be a bundle, and put:

$$\operatorname{sing}(\mathcal{F}) = \{x \in X \mid \dim(\operatorname{Im}\{\mathcal{T}_{\mathcal{F}} \otimes k(x) \longrightarrow T_M \otimes k(x)\}) < r\} \,.$$

This variety stratifies naturally according to the rank of the image map, and for C any curve in X, we denote by $r(C)$ the generic rank. We may now state,

Main Theorem (a) *Let $(X, \mathcal{F})$ be a weakly regular (integrable) foliated variety and C a curve in X with $T_{\mathcal{F}}|_C$ ample then for all $x \in C$ there is a $\mathcal{F}$-invariant rationally connected subvariety $V_x \ni x$ of dimension $r(C)$.*

Here we use reduction modulo p, but only to resolve the problem for a foliation such that $X/\mathcal{F}$ exists as a scheme quotient over $\mathbb{C}$, so in fact,

(b) *Notations and hypothesis as above then the minimal degree of the rational curve connecting any two points in V_x is effectively computable. In particular there is a rational curve $L_x \ni x$ tangent to $\mathcal{F}$ such that for any nef. $\mathbb{R}$-divisor H,*

$$H.\, L_x \leq 2(r(C)+1)\, \frac{H.\, C}{-K_{\mathcal{F}^{\cdot}_{r(C)}}\, C} \,.$$

[1] Added in preparation: J.-B. Bost, [Bo], used what may be considered an arithmetic version of this trick which independently led him to discover the geometric variant, and its higher dimensional generalisations à la § II.1.

In this notation, $\mathcal{F}_{r(C)}$ is the induced foliation on the subscheme of X where the generic rank is $r(C)$. It is not immediately clear that intersecting with the canonical class has any sense for $r(C) < r$, but well definedness will emerge in the course of the proof. One should also note that there is no need to suppose $T_{\mathcal{F}}$ is saturated in $\mathcal{T}_X$ provided there is closure under Lie-bracket. Without closure under bracket one can of course find a foliation $\mathcal{G}$ corresponding to the minimal subsheaf $\mathcal{T}_{\mathcal{G}}$ of $\mathcal{T}_X$ closed under the same. It may happen, however, that this is not a bundle in a neighbourhood of our curve C, although it will certainly be as 'ample' as one needs. This poses serious technical problems akin to the difficulty of doing deformation theory on singular varieties, and so one only obtains (a) and (b) under the weaker hypothesis that $\mathcal{G}$ is weakly regular in a neighbourhood of C. This is however perfectly sufficient to recover Miyaoka's semi-positivity theorem, where X is supposed normal and C moves in a large base point free family.

Our study however has so far revealed nothing if $r(C) = 0$. For foliations by curves this is equivalent to our curve being wholly contained in the singular locus of the foliation. To address this question we have recourse to the notion of foliated canonical singularities. This class of singularities may be understood as the maximal one for which birational geometry of foliations makes sense, equally the precise definition à la Kawamata-Mori et al. is given in I.1. To explain its implications observe that for a curve C with generic rank $r(C)$ contained in a component Y of the locus where the rank is the same there is an exact sequence of sheaves,

$$0 \longrightarrow \mathcal{N}_{r(C)} \longrightarrow T_{\mathcal{F}} \otimes \mathcal{O}_Y \longrightarrow T_{\mathcal{F}_{r(C)}}$$

where we take this as the definition of $\mathcal{N}$, and we have,

(c) *Let $(X, \mathcal{F})$ be a weakly regular (but not necessarily integrable) foliated variety with canonical singularities and C a curve with $r(C)$ the generic rank of the foliation along C then $\mathcal{N}_{r(C)} \otimes \mathcal{O}_C$ has a non-positive rank 1 quotient.*

Thus we have established that the rationally connected subvarieties guaranteed by (a) exhaust the ampleness of $T_{\mathcal{F}}|_C$, and we deduce,

Corollary (IV.2.1) *Let $(X, \mathcal{F})$ be a variety foliated by curves with foliated Gorenstein and canonical singularities (I.1.1) then there are countably many $\mathcal{F}$-invariant rational curves with $K_{\mathcal{F}}.\, L_i < 0$ such that if $\overline{\mathrm{NE}}(X)$ is the closed cone of effective curves and $\overline{\mathrm{NE}}(X)_{K_{\mathcal{F}} \geq 0}$ the subcone on which $K_{\mathcal{F}}$ is positive then,*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_{\mathcal{F}} \geq 0} + \sum_i \mathbb{R}_+ [L_i] \,.$$

Better still,

(a) *The rays $\mathbb{R}_+ [L_i]$ are locally discrete in the upper half space $\mathrm{NS}(X)_{K_{\mathcal{F}} < 0}$.*

(b) *If $(X, \mathcal{F})$ is not a ruling by rational curves $K_{\mathcal{F}}.\, L_i = -1$, $\forall i$, and $L_i \cap \mathrm{sing}(\mathcal{F}) \neq \varphi$. Otherwise, $K_{\mathcal{F}}.\, L_i \in \{-1, -2\}$.*

(c) *Every extremal ray in the half space $\mathrm{NS}(X)_{K_{\mathcal{F}} < 0}$ is of the form $\mathbb{R}_+ [L_i]$.*

Whence a smooth foliation either has $K_{\mathcal{F}}$ nef or it's a fibring by rational curves, and quite generally there are supporting Cartier divisors for extremal rays. Whence, we have the first step in a minimal model programme for foliations by curves.

It remains to thank M. Spivakovsky for his expertise (and patience) in explaining the nature of algorithmic desingularisation, but without Cécile the contents of this article would have lingered indefinitely in the realm of 'in preparation'.

I. SINGULARITIES

I.1 Definitions

Our objects of study are foliated varieties, i.e. a normal variety X equipped with a foliation $\mathcal{F}$, which we denote by $(X, \mathcal{F})$. On the other hand we do not wish to think of this as two separate objects, but rather as a unified whole. As such the singularities which we wish to study have a priori nothing to do with the space X, although as suggested above we will make the technically convenient, and rather mild assumption that X is normal. To proceed further let us note that the precise definition of a foliation is simply a saturated subsheaf of the tangent sheaf, i.e.

$$\mathcal{T}_{\mathcal{F}} \hookrightarrow \mathcal{T}_X \ .$$

If in addition the subsheaf $\mathcal{T}_{\mathcal{F}}$ is closed under Lie bracket, then we say that $(X, \mathcal{F})$ is integrable. One has of course the classical theorem of Frobenius over the complex numbers, which asserts that if X is smooth and $\mathcal{T}_{\mathcal{F}}$ is a subbundle of X at some point, then closure under Lie bracket is equivalent to the foliation being given locally as a fibration, whence the appellation.

Now in the study of singularities of a variety per se, the main protagonists are the cotangent sheaf, and the canonical bundle. The former is the more classical and its relation with local algebra rather well understood. The latter is rather more recent, but its study is the essential prerequisite for birational geometry. Since our results will also encompass the case of the trivial foliation, i.e. simply the study of curves on varieties, it is not surprising that we will encounter a similar phenomenon. However rather than the cotangent sheaf of the foliation, we will work with the tangent sheaf, via which we introduce our first definition, viz:

I.1.1. Definition *A foliated variety $(X, \mathcal{F})$ is said to be weakly regular if it is given by a subbundle $T_{\mathcal{F}}$ sitting as a saturated subsheaf of the tangent sheaf $\mathcal{T}_X$.*

This definition is immediately deserving of comment. In the case of the trivial foliation the definition asserts that the tangent sheaf of the variety is a bundle. This is a priori strictly weaker than the assertion that the variety is regular, i.e. that the cotangent sheaf is a bundle. It is however a conjecture of Zariski and Lipmann that the two are equivalent. At the other extreme if $(X, \mathcal{F})$ is a foliation by curves, given that X is normal, weak regularity is equivalent to being foliated Gorenstien as introduced in [M2], and amounts to the foliation being given everywhere by a vector field. Given that the said vector field is itself allowed to vanish in codimension 2, every foliation by curves on a non-singular variety is weakly regular. On the other hand for a singular variety X, the condition is highly non-trivial, and can give a great deal of information about the foliation, cf. op. cit.

The role of normality of the underlying space in the above is fairly unimportant, where it really makes its appearance is in the consideration of the canonical sheaf of the foliation, i.e. the dual of the top exterior power of the tangent sheaf $\mathcal{T}_{\mathcal{F}}$. Naturally we denote this by $K_{\mathcal{F}}$, and follow Kawamata, Mori et al., by introducing a discrepancy function to measure the singularities. Specifically suppose $K_{\mathcal{F}}$ is "$\mathbb{Q}$-Cartier". For the moment let us be deliberately vague about what this may mean, and consider any proper birational map $p : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ from a foliated variety to our original variety then for some divisors E_i contracted by p, and rational numbers a_i, we must have:

$$K_{\widetilde{\mathcal{F}}} \equiv p^* K_{\mathcal{F}} + \sum_i a_i E_i$$

where "$\equiv$" is some suitable equivalence relation of divisors, such as rational or numerical. Given that $\widetilde{X}$ is necessarily normal, whence non-singular in codimension 1, the numbers a_i depend only on E_i considered as prime valuations of the field of functions of X, and so we may define a map:

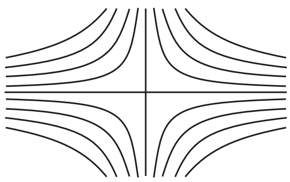
$$a(\ , X, \mathcal{F}) : \{\text{prime valuations } E \text{ of } k(X)^\times\} \longrightarrow \mathbb{Q}$$

where of course we implicitly assume that the valuations have non-empty centre, so that $a(\ , X, \mathcal{F})$ is even defined at the level of germs. Finally we are in a position to introduce the discrepancy of a foliated space, and to make another definition, i.e.

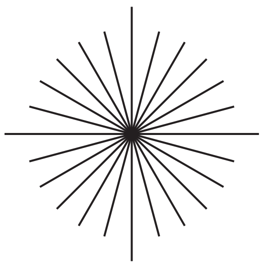
$$\operatorname{discrep}(X, \mathcal{F}) := \inf_E a(E, X, \mathcal{F})$$

I.1.2. Definition *A foliated space $(X, \mathcal{F})$ is said to have canonical singularities if* $\operatorname{discrep}(X, \mathcal{F}) \geq 0$*, and terminal singularities should this inequality be strict.*

Necessarily in the case of the trivial foliation, $K_{\mathcal{F}}$ and K_X coincide, so that the definitions of canonical and terminal do like wise. However when the rank of the foliation differs from the dimension of X this is absolutely not so. For example consider the vector field $\partial = x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y}$ in a neighbourhood of the origin in $\mathbb{C}^2$. One may easily draw this, viz:



The underlying space X is certainly smooth, and as such has terminal singularities in the usual sense. However the pair $(X, \mathcal{F})$ is our object of study and this has discrepancy zero, i.e. it is properly canonical without being terminal. One can go further, and consider the vector field $\partial = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$. Again this is easily drawn,



The underlying space is as before, but now the discrepancy is actually -1, for the foliated space. Actually these two examples are fairly representative of what can happen for an underlying smooth space X of dimension 2 together with a foliation by curves. Canonical singularities here are a slight generalisation, or more accurately the functorial correction, of so called reduced singularities or those of Poincaré-Dulac type, cf. [M2], and every foliation by curves on a surface may be resolved to one with canonical singularities. Furthermore, as a moment's reflection on our initial example shows, this is best possible, i.e. unlike the trivial case one cannot resolve to a birational model with terminal singularities. Ultimately, however, one gains a rather better feeling for the definition and its interplay with the singularities of the underlying space by considering blow ups in foliation equivariant centres. This of course means that we specify a subvariety Y of X, together with its sheaf of ideals $\mathcal{I}_Y$ and over an affine open subset U of X ask that for all derivations $\partial \in \Gamma(U, \mathcal{T}_{\mathcal{F}})$, $\partial(\mathcal{I}_Y) \subset \mathcal{I}_Y$. This definition behaves well with respect to localisation, so it easily globalises, and we have:

I.1.3. Lemma *Let $p : \widetilde{X} = \mathrm{Bl}_Y(X) \to X$ be a blow up of a foliated variety $(X, \mathcal{F})$ in a $\mathcal{F}$-equivariant centre Y then we have a natural map,*

$$p^* \mathcal{T}_{\mathcal{F}} \longrightarrow \mathcal{T}_{\widetilde{\mathcal{F}}} .$$

proof. The question is local, so we may assume that X is affine, say $\mathrm{Spec}A$, and $\mathcal{I}_Y$ is just the sheafication of an ideal I of A. The assertion is then simply that if $\partial \in \mathrm{Der}(A)$ lies in the tangent sheaf of the foliation then the a priori meromorphic vector field $p^* \partial$ is in fact holomorphic. This is easily verified, since locally a function f on $\widetilde{X}$ is of the form $\frac{g}{h^d}$ where d is a non-negative integer, $g \in I^d$, and h being in I, where naturally we think of ourselves as looking at functions on the $h \neq 0$ part, then of course:

$$\partial(f) = \partial\left(\frac{g}{h^d}\right) = \frac{h\,\partial g - dg\,\partial h}{h^{d+1}}$$

and since I^d is also $\mathcal{F}$ equivariant, $\partial(f)$ is a function as required. $\square$

As a simple illustration of the lemma consider a foliation by curves $(X, \mathcal{F})$ with underlying space X non-singular. Locally the foliation is given by a vector field ∂, and where ∂ vanishes is the singular locus of $\mathcal{F}$. Any point x in the singular locus is of course $\mathcal{F}$ equivariant, so in a neighbourhood of x the discrep is always less than or equal to zero. Consequently under such hypothesis if a 'singularity' is 'terminal' then the vector field is non-zero, i.e. the foliation is locally a smooth fibration. The essence of this conclusion extends, as we shall see, to arbitrary weakly regular integrable foliations. However it is not a classification of arbitrary terminal singularities, which will be postponed until [II]. In any case to justify the above remark regarding weakly regular foliations we will need a further lemma, viz:

I.1.4. Lemma *Let $p : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ be the blow up of a foliated variety (in characteristic zero) in a $\mathcal{F}$-equivariant centre Y as in* I.1.3, *and let $\rho : (X^{\#}, \mathcal{F}^{\#}) \to (\widetilde{X}, \widetilde{\mathcal{F}})$ be its normalisation then in fact ρ is $\mathcal{F}$-equivariant, i.e. we have a natural map,*

$$\rho^* p^* \mathcal{T}_{\mathcal{F}} \longrightarrow \mathcal{T}_{\mathcal{F}^{\#}} .$$

proof. The assertion precisely regards the locus where $X^{\#}$ is not isomorphic to $\widetilde{X}$. Better still since $X^{\#}$ is S_2 we only require to show the existence of the map in codimension 1, so let $D^{\#}$ in $X^{\#}$ be an irreducible divisor with generic point $\eta^{\#}$. If $\rho|_{D^{\#}}$ is an isomorphism generically then for any derivation ∂ in $\mathcal{T}_{\mathcal{F}}$, $\rho^* p^* \partial$ is a derivation of the local ring $\mathcal{O}_{X^{\#},D^{\#}}$ by the previous lemma, and there is nothing to do. Otherwise let D in $\widetilde{X}$ be the image, and η a generic point of D. Since X is normal, D is necessarily a component of the exceptional divisor on $\widetilde{X}$, and as such is $p^* \mathcal{T}_{\mathcal{F}}$ invariant. Whence, let us consider the blow up of $\widetilde{X}$ in D around a neighbourhood of η. Using the same letter for the space and the local neighbourhood we have maps,

$$\begin{array}{ccccc}
\widetilde{X} & \longrightarrow & \mathrm{Bl}_D X & \longrightarrow & X^{\#} \\
{\scriptstyle\pi}\Big\uparrow & & {\scriptstyle\pi}\Big\uparrow & & {\scriptstyle\pi}\Big\uparrow \\
D & \longrightarrow & \cup_i D_i & \longrightarrow & D^{\#}
\end{array}$$

where the first square is a fibre square. The fibre over D being generically finite to one, but definitely not an isomorphism. Now if D_j, say, is the image of $D^{\#}$ then either $D^{\#}$ is isomorphic to D_j, and applying lemma I.1.3 forces $\rho^* p^* \partial$ to be a derivation of $\mathcal{O}_{X^{\#},D^{\#}}$ for any ∂ in $\mathcal{T}_{\mathcal{F}}$. Otherwise it is not, and we once more make a local invariant blow up, this time in D_j. On the other hand since there is always a non-trivial fibre over the generic point of the centre of such a blow up, and all such blow ups are locally dominated by $X^{\#}$. This cannot continue indefinitely, from which we conclude. □

Unlike the previous lemma, characteristic zero is essential here in order to deduce that the centres of the local blowing ups are equivariant under the foliation – should the exceptional divisor have multiplicity p along some component in $\mathrm{char} = p > 0$ we'd certainly be in trouble. Applying the lemma to arbitrary Gorenstien foliations by curves implies that either the foliation is locally integrable by the usual Frobenius type procedure or it has at best a canonical singularity. The difficulty in extending such considerations to arbitrary foliations relies on identifying suitable invariant centres. We will come back to this, and a more precise discussion of the above remarks on integrability in I.2. For the moment let us complete this introduction to singularities by way of some remarks on the condition $K_{\mathcal{F}}$ is "$\mathbb{Q}$-Cartier".

The most obvious sense of this is of course to consider the open embedding $j : (X_{sm}, \mathcal{F}_{sm}) \hookrightarrow (X, \mathcal{F})$ of the locus where say both X and $\mathcal{F}$ are smooth, and to demand that there is a positive integer m such that $j_* K_{\mathcal{F}_{sm}}^{\otimes m}$ is a Cartier divisor. This is rather strong, and so we term it $\mathbb{Q}$-foliated Gorenstien, and of course foliated Gorenstien if we can take $m = 1$. Equally for 2-dimensional normal algebraic spaces, the definition of "$\mathbb{Q}$-Cartier" can be understood in a linguistically abusive, though not mathematically abusive, sense via Mumford's intersection theory. These remarks are, however, all rather paranthetical since we will be almost exclusively concerned with foliations which satisfy the Gorenstien condition.

I.2 Towards an ideal situation

We now wish to concentrate on how to ameliorate the singularities of a foliated variety in a neighbourhood of a curve. In this section we will concentrate on foliations by curves. This not only provides some calculations essential to the general case, but illustrates the key features of what we are after without the technical complications that arise in the higher rank case. The ideal of course would be to find a neighbourhood of the curve, birationally, where the foliation is everywhere integrable. In absolute generality this is impossible, indeed it is even so on smooth surfaces. Nevertheless the impossibility only occurs for curves invariant by the foliation. Since we intend to allow arbitrary singularities on the underlying space X we will consider an embedding $X \hookrightarrow M$ of our variety into a smooth variety. In addition everything will be arbitrarily local, so let's just work in the analytic topology. Observe that for a vector field ∂ on X and $x \in X$ it is completely unclear from the definition whether $\partial \neq 0$ in $\mathcal{T}_X \otimes k(x)$ implies $\partial \neq 0$ in $T_M \otimes k(x)$. Indeed this difficulty is at the root of the Zariski-Lipmann conjecture. Whence although it may depend on the embedding let's call ∂ non-singular at x if ∂ is non-zero in $T_M \otimes k(x)$, and observe the following straightforward generalisation of the Frobenius theorem,

I.2.1. Lemma *Let x be a non-singular point of a Gorenstien foliation by curves $(X, \mathcal{F})$ then there is a map $\pi : X \to Y$ of analytic spaces around x such that $\pi^* \Omega_Y$ is the conormal bundle of $\mathcal{F}$ in X, and π is relatively smooth.*

proof. We proceed in the obvious way. Namely if I is the ideal of X in M we have the usual short exact sequence,

$$I/I^2 \longrightarrow \Omega_M|_X \longrightarrow \Omega_X \longrightarrow 0\,.$$

By the non-singularity hypothesis there is a vector field ∂ on M, non-vanishing at x, with $\partial(I) \subset I$ which induces our given foliation on X, where of course we permit as much localisation as we need. Now let $z_1, \ldots, z_n$ be coordinate functions on M, then without loss of generality $\partial(z_1) = 1$, as usual we may put,

$$y_i = \sum_{n=0}^{\infty} \frac{(-1)^n\, z_1^n\, \partial^n\, z_i}{n!}\,, \qquad 2 \leq i \leq n$$

and one easily checks $\partial y_i = 0$. Consequently we just put Y to be the image of X in $\mathbb{C}^{n-1}$ under the map, $(y_2, \ldots, y_n) : M \to \mathbb{C}^{n-1}$. $\square$

Now we would like to obtain this situation around any point of a suitable curve C in X. The definition of suitable here is that C is not invariant by the foliation, so in particular the singular locus of the foliation meets C in a bunch of points. Let us concentrate on one of them, i.e. denote by X a sufficiently small neighbourhood of the point. It may of course happen that C is singular at such a point. The point is $\mathcal{F}$ equivariant so we are happy to blow up in it, so that whether self evidently or by a minor adaptation of lemma I.1.4 we note:

I.2.2. Fact *Notations as above, there is a sequence of varieties,*

$$X = X_0 \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_n = \widetilde{X}$$

where $X_i \to X_{i-1}$ is obtained by blowing up in a foliation equivariant centre, or by normalisation, such that the proper transform of C in $\widetilde{X}$ is non-singular at the induced foliation singularities.

Of course we like such sequences since they are "unramified" in the foliation direction by virtue of lemmas I.1.3 and I.1.4, so now we want to use a similar sequence, starting with C smooth at $\mathrm{sing}(\mathcal{F})$, to get into the local integrability situation of I.2.1.

Consequently let $x_1, \dots, x_n$ be coordinates on our ambient smooth space M, and suppose C is given by $x_2 = \cdots = x_n = 0$. We of course consider a vector field ∂ on M which leaves the ideal I of X invariant, and restricts to our given foliation on X which is supposed Gorenstien. Necessarily we are supposing that ∂ is singular at the origin, we write $\partial = a_i \frac{\partial}{\partial x_i}$, index notation, and consider how ∂ transforms around C under the blow up $p : \widetilde{M} \to M$ in the origin. We have a new coordinate system $x_1 = \xi_1$, $x_i = \xi_1 \, \xi_i$, with C transforming to $\xi_2 = \cdots = \xi_n = 0$. Putting ν_i to be the multiplicity of a_i at the origin, $a_i = \widetilde{a}_i \, x_1^{\nu_i}$, and understanding this notation in the natural way when some $a_i = 0$, we obtain:

$$p^* \partial = \widetilde{a}_1 \, \xi_1^{\nu_1} \, \frac{\partial}{\partial \xi_1} + \sum_{i=2}^{n} (\widetilde{a}_i \, \xi_1^{\nu_i - 1} - \widetilde{a}_i \, \xi_1^{\nu_1 - 1} \, \xi_i) \, \frac{\partial}{\partial \xi_i} \, .$$

Better still, we also have,

$$\mathrm{mult}_0(\widetilde{a}_i|_C) = \mathrm{mult}_0(a_i|_C) - \nu_i \, .$$

Now we distinguish two cases. In the first $a_1 \neq 0$, then on replacing p by a sequence of blow ups invariant in the pull-back of ∂ (so even invariant by the foliation in the case of canonical singularities) we have without loss of generality,

$$p^* \partial = \xi_1^{\nu_1} \, \frac{\partial}{\partial \xi_1} + \sum_{i=2}^{n} b_i \, \frac{\partial}{\partial \xi_i} \, .$$

Furthermore some b_i is not identically zero on C, so that possibly after some more blow ups of the same form, the pull-back of ∂ is a regular, and non-singular derivation along the proper transform of C. In the case that a_1 is identically zero, the same conclusion is even more immediate. Now let us summarise these reflections by way,

I.2.3. Definition *Call a foliation $\mathcal{F}$ on a variety X smoothly integrable at a point $x \in X$ if it arises as in lemma* I.2.1.

Whence,

I.2.4. Proposition *Let C be a non-invariant curve in a Gorenstien foliated variety $(X, \mathcal{F})$ then there is a neighbourhood $(X_0, \mathcal{F}_0)$ of C together with a proper birational map $p : (\widetilde{X}_0, \widetilde{\mathcal{F}}_0) \to (X_0, \mathcal{F}_0)$ such that $\widetilde{\mathcal{F}}_0$ is smoothly integrable around every point of the proper transform of C, and better still there is a natural map by pulling back, $p^* \, T_{\mathcal{F}_0} \to T_{\widetilde{\mathcal{F}}_0}$.*

I.3 General case

We now wish to study arbitrary integrable and weakly regular foliations around curves. Our only restriction will be that the curve is not contained in the singular locus of the foliation. The study again being local this statement is to be understood in terms of the foliation not having full rank in the tangent space of some smooth variety M into which X is embedded. The tricky thing here is that a singular point x may no longer be foliation invariant. Indeed by definition they are invariant precisely when the map at the residue field level, $T_{\mathcal{F}} \otimes k(x) \to T_M \otimes k(x)$ is zero, so we have to do a little more work to identify invariant centres. To this end, and much as before, let $x_1, \ldots, x_n$ be coordinates on M and write,

$$\partial_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}, \qquad 1 \le i \le r, \quad 1 \le j \le n$$

where r is the rank of the foliation. It goes without saying that the ∂_i leave the ideal I_X of X in M invariant, and induce our given weakly regular foliations $\mathcal{F}$. Not surprisingly the matrix $A = [a_{ij}]$ of functions on X will play a key role, let us denote by s the dimension of the image of $T_{\mathcal{F}} \otimes k(x)$ in $T_M \otimes k(x)$ around our point of study x. Observe that by row and column reduction of matrices we may find $s \times (n-s)$ and $(r-s) \times (n-s)$ matrices B, D of functions in the maximal ideal at x such that without loss of generality A has the form,

$$\begin{bmatrix} I & B \\ 0 & D \end{bmatrix}$$

Equally, for the same reason, there is a $r \times (n-r)$ matrix of meromorphic functions A_0, such that,

$$A = \left[I \vdots A_0 \right].$$

Now with these notations let us pause to consider the case of $r = s$ and the Frobenius theorem in this context. This time let's start with the definition, viz:

I.3.1. Definition *A foliated variety $(X, \mathcal{F})$ is said to be smoothly integrable at $x \in X$, if there is a relatively smooth map $\pi : X \to Y$ in a neighbourhood of x such that $\pi^* \Omega_Y$ generates the conormal bundle of $\mathcal{F}$.*

Then of course we have,

I.3.2. Lemma *Notations as above if $s = r$ at $x \in X$, then $(X, \mathcal{F})$ is smoothly integrable at x.*

proof. Proceeding as in lemma I.2.1 we can actually choose our coordinate functions on M, such that A_0 is not only a matrix of functions around x, but in fact the first row is identically zero. Now let us consider invariance under Lie bracket. By the invariance of I under the lifting of our various vector fields, this may simply be calculated in M and restricted to X. Consequently understanding A_0 as a $(r-1) \times (n-r)$ matrix of functions, we obtain a $(r-1) \times (r-1)$ matrix Λ of functions on X around x such that,

$$\left[0 \vdots \frac{\partial A_0}{\partial x_1} \right] = \Lambda \left[I \vdots A_0 \right]$$

which of course forces $\frac{\partial}{\partial x_1} A_0$ to be identically zero on X. Consequently if we let X_1 be the image of X in $\mathbb{C}^{n-1}$ under the map $(x_2, \ldots, x_n) : M \to \mathbb{C}^{n-1}$, and π the induced map then π is relatively smooth and there is a foliation $\mathcal{F}_1$ on X_1, closed under Lie-bracket of rank $(r-1)$ which induces our given foliation, so that we may conclude by induction. □

We can now turn to the situation of $s < r$ and see how far this discussion can be pushed. Allowing a meromorphic A_0, shows that at the first stage of the induction procedure (assuming of course $s \neq 0$) that there are $(s-1)$ meromorphic vector fields on X_1, which together with $\partial/\partial x_1$ generate our given foliation. Proceeding by induction gives a prefered coordinate system such that,

$$\partial_i = \frac{\partial}{\partial x_i}\ ,\ 1 \leq i \leq s \quad \partial_i = \sum_{j>s} a_{ij} \frac{\partial}{\partial x_j}\ ,\ i > s$$

with the a_{ij} holomorphic functions of $x_{s+1}, \ldots, x_n$ which all vanish at the origin. We have thus identified, locally, a suitable foliation invariant centre, viz: $x_{s+1} = \cdots = x_n = 0$. By way of our curve, C, passing through our singular point, we first consider its image under $(x_{s+1}, \ldots, x_n) : M \to \mathbb{C}^{n-s}$, and ask whether the curve itself is singular there or not. Argueing exactly as in I.2.2 we may via a sequence of equivariant blow ups pass without loss of generality to the situation where the image of C is given by $x_{s+2} = \cdots = x_n = 0$.

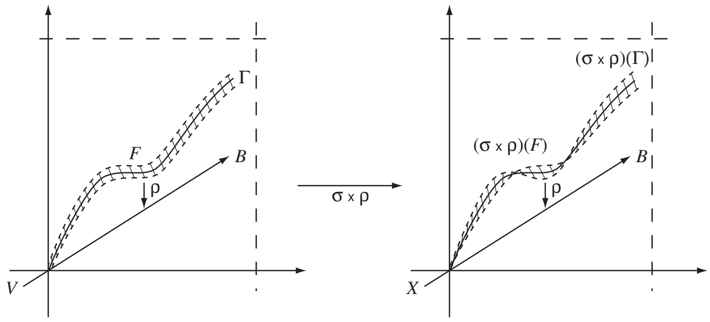
Now for a curve C whose tangent space does not generically factor through that of the foliation, we may proceed more or less as in the case of foliations by curves, namely, blow up in centres of the type we have identified until such times as the rank increases. Once the rank increases change coordinates in the obvious way, resolve any singularities on the new projection of our curve, then blow up in some foliation equivariant centres until we increase the rank again. Consequently we obtain,

I.3.3 Proposition *Let C be a curve in a weakly regular foliated variety $(X, \mathcal{F})$ whose tangent space does not generically factor through $\mathcal{F}$ and which is not contained in the singular locus of $\mathcal{F}$ then there is a neighbourhood $(X_0, \mathcal{F}_0)$ of C together with a proper birational map $p : (\widetilde{X}_0, \widetilde{\mathcal{F}}_0) \to (X_0, \mathcal{F}_0)$ such that $\widetilde{\mathcal{F}}_0$ is smoothly integrable at every point in the proper transform of C, and pull-back of derivations yields a natural map $p^* T_{\mathcal{F}_0} \to T_{\widetilde{\mathcal{F}}_0}$.*

II. ALGEBRISATION

II.1 The Graphic neighbourhood

Let us concentrate our attention in this section on a curve C inside a foliated variety $(X, \mathcal{F})$ with the foliation of rank r, integrable, and with weakly regular singularities where additionally we will suppose that C is neither contained in the singularities nor does its tangent space factor through $T_{\mathcal{F}}$. Now by the considerations of § 1, we may find an open neighbourhood X_0 of C (either formally or in the analytic topology) together with a proper birational map (in the category of analytic spaces) $p : \widetilde{X}_0 \to X_0$ such that the induced foliation $\widetilde{\mathcal{F}}_0$ is smoothly integrable in a neighbourhood V of the proper transform $\widetilde{C}$ of C. At this point we wish to consider the induced foliation $\mathcal{F}^{\#}$ on $V \times B$, where B is the normalisation of C, in a neighbourhood of the graph Γ of the natural map from B to $\widetilde{C}$. The tangent bundle of $\mathcal{F}^{\#}$ is simply the pull-back of that of $\widetilde{\mathcal{F}}_0$, and is of course smoothly integrable around Γ. Whence let us take a union of Δ_α, $\alpha \in A$, of small open analytic sets covering Γ, and denote by $\pi_\alpha : \Delta_\alpha \to Z_\alpha$ the relatively smooth map of analytic spaces which yields $\mathcal{F}^{\#}|_{\Delta_\alpha}$. Further for each α, we have an ideal I_α of functions on Δ_α, generated by functions on Z_α vanishing on Γ. Necessarily the I_α patch and define a smooth analytic subvariety F of $\cup\Delta_\alpha$ of dimension $r+1$, such that the normal bundle of Γ in F, $N_{\Gamma|F}$, is isomorphic to $T_{\widetilde{\mathcal{F}}}|_\Gamma$. Rather more intuitively what we have done is create an analytic space F by adding to each point of B the germ of the locally smooth integrable subvariety through each point of $\widetilde{C}$ guaranteed by I.3.4, while equipping F with a map ρ to B, and σ to X, i.e.



With this in mind we can deliver the coup de grâce to the transcendental nature of our problem by way of,

II.1.1 Fact *If $T_{\mathcal{F}}|_C$ is ample then the Zariski closure of $\sigma \times \rho\,(F)$ is of dimension $(r+1)$, and as such every $\mathcal{F}$-integrable subvariety through a point of $(\sigma \times \rho)(\Gamma)$ is algebraic.*

proof. Since F comes equipped with a projection to B which pushes forward an integrable subvariety to a point, it is wholly sufficient to prove the claim on the Zariski closure. Equally both X and B are algebraic, so all we need show is that for any line bundle L on F there is a constant $C(L)$ such that $h^0(F, L^{\otimes n}) \leq Cn^{r+1}$, for all positive integers. Better still if $\widehat{F}$ is the completion of F along C, we have an injection $H^0(F, L^{\otimes n}) \hookrightarrow H^0(\widehat{F}, L^{\otimes n})$, so we might as well just consider $\widehat{F}$, and this is essentially a trivial exercise. Specifically for $m \in N$, let F_m be the m^{th} infinitesimal thickening, and observe by construction that there is a map $T_{\mathcal{F}}|_\Gamma \to N_{\Gamma|F}$ which is generically an isomorphism, so $N_{\Gamma|F}$ is ample. On the other hand we have the usual exact sequence,

$$0 \longrightarrow H^0(\Gamma, \operatorname{Sym}^m N^\vee_{\Gamma|F} \otimes L^{\otimes n}) \longrightarrow H^0(F_{m+1}, L^{\otimes n}) \longrightarrow H^0(F_m, L^{\otimes n})\,.$$

Necessarily the first group vanishes for $m \leq C(L)n$, where the constant $C(L)$ is of the form $0\,(|\deg_\Gamma\, L|)$, and whence,

$$h^0(\widehat{F}, L^{\otimes n}) \leq \sum_{k=0}^{Cn} h^0(\Gamma, \operatorname{Sym}^k N^\vee_{\Gamma|F} \otimes L^{\otimes n}) \leq Cn^{r+1}$$

where the last inequality may involve a slightly different constant, but nevertheless only depends on L as required. □

Lest there be any confusion let us make,

II.1.2 Remark Evidently the role of the analytic topology is only for convenience of exposition, since the above is really a proposition about formal schemes.

II.2 Cleaning up

We will continue to concentrate on the example of the previous section. Thanks to II.1.1, we have obtained an algebraic variety W of dimension $r+1$, fibered over B by ρ, together with a section s of ρ, such that every fibre of W over B projects to a $\mathcal{F}$-invariant subvariety of X through the corresponding point of C. Even better our bundle of derivations $T_{\mathcal{F}}$ on X, lifts naturally to a bundle of $\mathcal{O}_B$-derivations, which we will continue to denote by $T_{\mathcal{F}}$, on W since after all dualising coherent sheafs is compatible with flat pull-back. Now intuitively we'd like to think of W/B as relatively smooth in a neighbourhood of $s(B)$ with relative tangent bundle $T_{\mathcal{F}}$. However, for the reasons detailed in I.2, this is potentially rather false around points where $\mathcal{F}$ is not smoothly integrable. Whence we seek to resolve W to $\widetilde{W}$ in such a way that $T_{\mathcal{F}}$ will admit a map to $T_{\widetilde{W}/B}$ around the section, and such that in a neighbourhood of the section the fibration will be smooth. In light of lemma I.1.3, what is therefore required is equivariant desingularisation with respect to $T_{\mathcal{F}}$ viewed as a bundle of derivations. Considered with respect to a fixed smooth embedding $W \hookrightarrow M$ we have as ever a stratification of the singularities by closed subschemes W_s defined as,

$$W_s = \{w \in W \mid \dim\left(\operatorname{Im}\{T_{\mathcal{F}} \otimes k(w) \longrightarrow T_M \otimes k(w)\}\right) \leq s\}.$$

Now let $Y \hookrightarrow W$ be a smooth centre in which we may wish to blow up in order to carry out the desingularisation algorithm of [BM]. For $y \in Y$, there exists a unique s such that $y \in W_s \backslash W_{s-1}$, where by convention W_{-1} is empty. We may apply the discussion pre-I.3.3, to find a coordinate system $x_1, \ldots, x_n$ on M in the analytic topology with respect to which a basis of $T_{\mathcal{F}}$ around y is given by,

$$\partial_i = \frac{\partial}{\partial x_i}\,,\ 1 \leq i \leq s, \quad \partial_i = \sum_{j=s+1}^{n} a_{ij}\,\frac{\partial}{\partial x_j}\,,\ s < i \leq r$$

where $a_{ij} = a_{ij}(x_{s+1}, \ldots, x_n)$. Furthermore we have a locally smooth map, $\pi : W \to Z$ around y, given by $(x_{s+1}, \ldots, x_n)$. By construction the algorithmic desingularisation procedure respects π, so we may find functions $f_1, \ldots, f_m$ of $x_{s+1}, \ldots, x_n$ which generate $I_Y \otimes \mathcal{O}_{W,y}$, where the local ring is understood formally or analytically. With these preliminaries in mind for any derivation ∂ of $T_{\mathcal{F}}$ over an open subset U containing y, our explicit choice of coordinates imply,

$$\partial\,(I_Y) \subset \mathfrak{m}\,(y)\,.$$

Since this holds for all ∂ and all $y \in Y$, we see that Y is in fact $\mathcal{F}$ equivariant. Blowing up in Y therefore leaves $T_{\mathcal{F}}$ as a bundle of derivations on the blow up, and so we may continue with the algorithmic desingularisation procedure to obtain a $\mathcal{F}$-equivariant resolution $\widetilde{W}$ of W. Our section s necessarily lifts to a section $\widetilde{s}$ of $\widetilde{W}$ which immediately forces $\widetilde{W}$ to be relatively smooth in a neighbourhood of $\widetilde{s}(B)$. Changing notations slightly, we have therefore established,

II.2.1 Better Fact *Let $(X, \mathcal{F})$ be a weakly regular foliated variety of rank r and $f : B \to X$ a map from a smooth curve such that $f^* T_{\mathcal{F}}$ is ample, and $f(B)$ is not contained in the singularities of $\mathcal{F}$ nor generically tangent to $\mathcal{F}$, then there is a smooth algebraic variety W of dimension $(r+1)$, equipped with projections σ, ρ*

to X and B respectively together with a section s of ρ such that if W_b denotes the smooth subvariety passing through $s(b)$ then $\sigma(W_b)$ is an $\mathcal{F}$-invariant subvariety through $f(b)$. In addition there is a natural map, $\sigma^* T_{\mathcal{F}} \to T_{W/B}$ in a neighbourhood of the section.

II.3 Complements

Firstly we consider the case where our curve C is not contained in the singularities but may be generically tangent to $\mathcal{F}$. Again let B be the normalisation of C, and continue to denote by $\mathcal{F}$ the induced foliation on the product $X \times B$, with Γ the graph of B. We observe that Γ is now a curve which is NOT generically tangent to $\mathcal{F}$, but $T_{\mathcal{F}}|_{\Gamma}$ is ample if it were already so for $T_{\mathcal{F}}|_{C}$. Passing to a modification $\widetilde{X \times B}$ of our product around Γ, as in I.3.4 we obtain a neighbourhood V of the proper transform $\widetilde{\Gamma}$ on which the induced foliation $\widetilde{\mathcal{F}}$ is smoothly integrable. Whence by II.1.1 there is a $\mathcal{F}$-invariant subvariety W_x through every point x of Γ of dimension the rank of $\mathcal{F}$. Projecting forward to X gives a $\mathcal{F}$-invariant subvariety $W \supset C$ of the appropriate dimension. The equivariant desingularisation arguements of II.2 go through verbatim to yield a resolution $\widetilde{W}$ such that $T_{\widetilde{W}}$ is ample on the proper transform $\widetilde{C}$ of C, and indeed we may even take the later isomorphic to B should we wish.

Let us finally consider the case where our curve C is contained in the singular locus, and let s be the generic rank of $\mathcal{F}$ along C. Denote therefore by Y a component of X_s containing C. The coordinate system pre- I.3.3 imply that Y is $\mathcal{F}$-invariant so let $\mathcal{G}$ be the induced foliation of rank s. The immediate thing to note is that $\mathcal{G}$ is not necessarily weakly regular. However at a point y of C where the rank drops to $t < s$ we are guaranteed a space of derivations of Y of dimension $s - t$ if C is not tangent to $\mathcal{G}$, respectively $s - t - 1$ otherwise, which are not identically zero. The resolution procedure of I.3 therefore goes through verbatim, and we make neighbourhoods of the normalisation B of C exactly as before which are equivariant under $T_{\mathcal{F}}$. Since Y is also $T_{\mathcal{F}}$-equivariant, and the new foliation is generically a quotient of $T_{\mathcal{F}}$, we therefore have:

II.3.1 Final Fact *Let s be the generic rank of $\mathcal{F}$ along C, then for every point $x \in C$ there is a smooth algebraic variety W_x of dimension s together with a map $\sigma : W_x \to X$ such that $\sigma(W_x) \ni x$ is a $\mathcal{F}$-equivariant subvariety. Better still if $f : B \to C$ is the normalisation then either,*

(a) *$T_{\mathcal{F}_s}$ is not generically tangent to C, and the W_x are the fibres of a smooth variety $\rho : W \to B$ through a section s, and there is a natural map, $\sigma^* T_{\mathcal{F}} \to T_{W/B}$, generically an isomorphism around B.*

(b) *$T_{\mathcal{F}_s}$ is generically tangent to C, W_x is independent of x and contains a copy of B such that there is a map of pairs $(W, B) \to (X, C)$ and a natural map $\sigma^* T_{\mathcal{F}} \to T_W$, generically an isomorphism around B.*

III. DEFORMATION THEORY

III.1 The Set up

We have thus reduced our initial highly transcendental problem to an algebraic one, either to produce rational curves in a smooth variety in which there is a curve on which the ambient tangent space is ample, or to find rational curves in a family of varieties over a curve in which there is a section on which the relative tangent space is ample. The latter is less standard and implies the former by the graph construction à la II.3 so we will concentrate upon it. The former case is more or less in the literature, modulo one little trick which can easily be extracted from our discussion. We will of course be following Mori's method of reduction modulo p, so in this section we make our set up with the necessary precision, and summarise the appropriate minor variations that we require from [K1], specifically § I.2 and § II.3.

To this end let B be a smooth projective curve over a field k of arbitrary characteristic, let $q : C \to B$ be a finite morphism from another curve, $\pi : X \to B$ a projective flat family and $f : C/B \to X/B$ a morphism of B-schemes (with respect to the structure maps q and π) where X/B is supposed relatively smooth in a neighbourhood of $f(C)$. We wish to study the scheme of B-morphisms, $\mathrm{Hom}_B(C, X)$ in a neighbourhood of f. The next proposition contains all that we will need,

III.1.1 Proposition *Notations as above then,*

(a) *The tangent space of* $\mathrm{Hom}_B(C, X)$ *at* f *is isomorphic to,* $H^0(C, f^*\, T_{X/B})$.

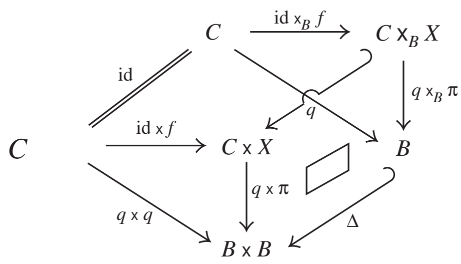
(b) *The dimension of every irreducible component of* $\mathrm{Hom}_B(C, X)$ *at* f *is at least,*

$$h^0(C, f^*\, T_{X/B}) - h^1(C, f^*\, T_{X/B})\,.$$

(c) *The deformations of* f *are unobstructed if* $H^1(C, f^*\, T_{X/B}) = 0$. *In particular should this occur* $\mathrm{Hom}_B(C, X)$ *is smooth at* f.

(d) *If* $Z \subset X$ *has codimension at least* 2, *and* $H^1(C, f^*\, T_{X/B}(-c)) = 0\ \forall$ *geometric points* c *of* C *then a generic* $g \in \mathrm{Hom}_B(C, X)$ *has image disjoint from* Z.

proof. For part (a), we wish to consider the graph Γ of f in the relative product $C \times_B X$ and calculate its conormal bundle. To this end consider the commutative diagram,



where by definition the vertical square is Cartesian, and Γ is the image of $\mathrm{id} \times_B f$.

This in turn leads to a commutative diagram of exact sequences of sheaves on Γ, viz:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & \frac{I_{\Delta,B\times B}}{I^2_{\Delta,B\times B}}\otimes\mathcal{O}_\Gamma \arrow[r, equal] \arrow[d] & \frac{I_{\Delta,B\times B}}{I^2_{\Delta,B\times B}}\otimes\mathcal{O}_\Gamma \arrow[d] & & \\
0 \arrow[r] & \frac{I_{\Gamma,C\times X}}{I^2_{\Gamma,C\times X}} \arrow[r] \arrow[d] & \Omega_{C\times X}\otimes\mathcal{O}_\Gamma \arrow[r] \arrow[d] & \Omega_\Gamma \arrow[r] \arrow[d, equal] & 0 \\
 & \frac{I_{\Gamma,C\times_B X}}{I^2_{\Gamma,C\times_B X}} \arrow[r] \arrow[d] & \Omega_{C\times_B X}\otimes\mathcal{O}_\Gamma \arrow[r] \arrow[d] & \Omega_\Gamma \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where the notation $I_{\cdot,\cdot}$ means the ideal of whatever subscheme inside the other. Now the middle row is exact because of the natural splitting of $\Omega_{C\times X}$, while the middle column is exact for general nonsense. Specifically X is by hypothesis smooth around Γ, so $\Omega_{C\times X}$ is a bundle, the conormal sheaf $I_{\Delta,B\times B}/I^2_{\Delta,B\times B} \xrightarrow{\sim} \Omega_B$ is also a bundle, and the map from Ω_B to $\Omega_{C\times X}$ is not zero, so it must be an injection of sheaves. Whence the left hand column is in fact exact, and so also is the bottom row by the 9 lemma, and/or a trivial diagram chase. From which we conclude an isomorphism via the splitting of the middle row,

$$I_{\Gamma,C\times_B X}/I^2_{\Gamma,C\times_B X} \xrightarrow{\sim} \Omega_\Gamma\backslash\Omega_{C\times_B X}\otimes\mathcal{O}_\Gamma \xrightarrow{\sim} \Omega_{X/B}\otimes\mathcal{O}_\Gamma$$

which proves (a). Better still $C\times_B X$ is non-singular around Γ by hypothesis, Γ is itself smooth, so the deformations are generically unobstructed which proves (b) and (c). To prove (d), we start by taking a smooth open neighbourhood U of f in $\mathrm{Hom}_B(C,X)$. For any geometric point $c\in C$, and for g in a possibly smaller U, we have by hypothesis, $H^1(C, g^*\,T_{X/B}(-c)) = 0$. Consequently if $\mathrm{Hom}_B(C,X,g|_c)$ is the space of morphisms taking c to $g(c)$, then a minor variation of the previous arguement forces $U\cap\mathrm{Hom}_B(C,X,g|_c)$ to be smooth around g of dimension $h^0(C,g^*\,T_{X/B}) - \dim(X/B)$. On the other hand,

$$\{g\in U \mid g(C)\cap Z\neq\varphi\} = \bigcup_{c\in C}\bigcup_{x\in Z}\{g\in U \mid g(x)=z\}\,.$$

So that this space has dimension at most,

$$h^0(C,g^*\,T_{X/B}) - \dim(X/B) + \dim C\times_B Z \leq \dim U - 1$$

which completes the proof of (d). □

III.2 Bend and Break

We begin with a simple lemma,

III.2.1 Lemma *Let E be an ample vector bundle on a smooth curve C over a field of positive characteristic, then there is a sufficiently high power $F_n : C \to C$, of the absolute Frobenius F such that for all geometric points c of C,*

$$H^1(C, F_n^* E(-c)) = 0 .$$

proof. Fix an ample divisor D on C, and using F_n to denote as large a power of Frobenius as we require, we obtain by Riemann-Roch global sections of $F_n^* E$ $(-D)$. Consequently for E of rank more than 1 we obtain a dévissage of bundles,

$$0 \longrightarrow E' \longrightarrow F_n^* E \longrightarrow E'' \longrightarrow 0$$

where E' is ample of rank 1. Moreover E'' is ample, and the proposition is clear for line bundles via Serre-duality, so for some high power of Frobenius F_m and $c \in C$ any geometric point we obtain a short exact sequence,

$$0 \longrightarrow F_m^* E'(-c) \longrightarrow F_{m+n}^* E(-c) \longrightarrow F_m^* E''(-c) \longrightarrow 0$$

where by induction the kernel and cokernel are acyclic. □

Now let us return to the notation of the previous section, viz: B, C smooth projective curves, $q : C \to B$, $\pi : X \to B$ the structure maps, with the latter family flat, and $f : C/B \to X/B$ a map of B-schemes such that X is smooth in a neighbourhood of $f(C)$. Moreover let everything be over a field of positive characteristic p, $f^* \Omega_{X/B}$ ample, and f not contained in a fibre of B. The map π trivially yields an inseparable equivalence relation π on X, with quotient $Y = X/\Pi$ factoring the geometric Frobenius on X (i.e. if locally b is a coordinate on B, and $x_1, \ldots, x_n$ the fibre coordinates, $X \to Y : (b, x_1, \ldots, x_n) \mapsto (b, x_1^p, \ldots, x_n^p)$). Denote the corresponding factorisation as below,

$$\begin{array}{ccc} X & & \\ \rho \downarrow & \searrow F_X & \\ Y = X/\Pi & \longrightarrow & X^{(1)} \end{array}$$

then Y is a normal $\mathbb{Q}$-factorial variety with $\rho^* K_Y = p\, \omega_{X/B} + \pi^* \omega_B$, where we define $\omega_{X/B} = K_X - \pi^* \omega_B$. Better still since Y is normal there is a subset Z of X of codimension at least 2 such that Y is smooth outside $\rho(Z)$. Consequently for a generic point $x \in X$, proposition III.1.1 and lemma III.2.1 guarantee the existence of a curve $C' \ni x$, not contained in a fibre of B, algebraically equivalent to a large multiple of the geometric Frobenius composed with f, so at the numerical level, $C' \approx p^n\, C$, say.

Now let H be any nef. $\mathbb{R}$-divisor on X, and suppose $\rho^* K_Y \cdot C < 0$ then by [K1] there is a rational curve M_x in Y containing x such that,

$$\begin{aligned} \sigma^* H^{(1)} \cdot M_x &\le 2 \dim X \{ \frac{\sigma^* H^{(1)} \cdot \rho(C')}{-K_Y \cdot \rho(C')} \} \\ &= 2 \dim X \{ \frac{pH \cdot C}{-(p\, \omega_{X/B} \cdot C + \pi^* \omega_B \cdot C)} \} \end{aligned}$$

Consequently if L_x is the rational curve in the pre-image of M_x under ρ, then either,

(a) $\rho|_{L_x}$ is inseparable, and:

$$H \cdot L_x \leq 2 \dim X \, \frac{pH \cdot C}{-(p\,\omega_{X/B} \cdot C + \pi^*\,\omega_B \cdot C)}$$

or

(b) $\rho|_{L_x}$ is birational, and:

$$H \cdot L_x \leq 2 \dim X \, \frac{H \cdot C}{-(p\,\omega_{X/B} \cdot C + \pi^*\,\omega_B \cdot C)} \,.$$

This is all we'll need of the bend and break technique so let's denote it by a number III.2.2, and conclude this section.

III.3 Finding rationally connected varieties

We now consider our data of maps $q : C \to B$, $\pi : X \to B$ and $f : C/B \to X/B$ with all the previous hypothesis on smoothness ampleness etc., and proceed to show that the fibres of X/B are rationally chain connected. None of our hypothesis are changed by supposing X non-singular, so let's throw that in for good measure. As usual we take our given data over $\mathbb{C}$ and find an integral affine $\mathbb{Z}$-scheme, S, of finite type over which everything is defined and our initial characteristic zero situation corresponds to the generic fibre. So in fact we'll actually use C, B, X, q, π, f, etc. for schemes and maps thereof, and rather abusively denote the generic fibre by the subscript $\mathbb{C}$. In any case given $\varepsilon > 0$ for a closed point s of S with residue field of sufficiently large positive characteristic, we may apply the considerations of the previous section to conclude the existence of a component W_ε of the Chow scheme parametrising 1-cycles with rational components of degree at most $2 \dim X \, \frac{H \cdot C}{-\omega_{X/B} \cdot C} + \varepsilon$ such that the map from the universal family $\mathcal{C}_\varepsilon$ over W_ε to X_s is dominant. Since this holds for all s in an open set, there is in fact such a component over the generic fibre. Better still since this must be true for all $\varepsilon > 0$ and there are at most finitely many such components so there is a component W of the Chow scheme parametrising 1-cycles with rational components of degree at most $2 \dim X \, \frac{H \cdot C}{-\omega_{X/B} \cdot C}$ such that the universal family $\mathcal{C}$ dominates X. Moreover the universal family is proper over X, and by the inequality III.2.2(b), these rational 1-cycles are all contained in fibres of π.

Now, we are in characteristic zero, so we can find a map well defined in codimension 1, $\nu : X_{\mathbb{C}} \dashrightarrow X_{1,\mathbb{C}}$, where $X_{1,\mathbb{C}}$ is smooth, with generic fibre a smooth rational curve in the family parametrised by $W_{\mathbb{C}}$. These fit into a diagram of $B_{\mathbb{C}}$-schemes, Forgetting for a moment the subscript $\mathbb{C}$ we require to calculate $\omega_{X_1/B} := K_{X_1} - \pi^*\,\omega_B$. Note that since $\pi^*\,\omega_B$ is saturated in Ω_X over an open set which surjects onto B, the same is true for $\pi^*\,\omega_B$ in Ω_{X_1}, so that outside of a closed set T which is at worst the union of something of codimension at most 2 and components of some fibres which do not meet the said open neighbourhood we have a map of bundles, which is injective as a map of sheaves,

$$\nu^*\,\Omega_{X_1/B} \longrightarrow \Omega_{X/B} \,.$$

Now without loss of generality we may assume that everything is defined over the same $\mathbb{Z}$-scheme S of finite type, and consider a generic deformation $g : C_m \to X$ of a sufficiently large power of the geometric Frobenius $C_m \to C$ of our given curve composed with f. Specifically we wish to calculate the degree of $g^* \nu^* \Omega_{X_1/B}$ via our knowledge of $f^* \Omega_{X/B}$. So in the first place let,

$$0 = T_0 \subset T_1 \subset \cdots \subset T_\iota = g^* T_{X/B}$$

be the Harder-Narasimhan filtration of $g^* T_{X/B}$. By definition we have,

$$\mu_{\max}(g^* T_{X/B}) = \mu(T_1) > \cdots > \mu(T_{\iota-1} \backslash T_\iota) = \mu_{\min}(g^* T_{X/B}).$$

In addition by lower semi-continuity in our family of deformations, and given we take g generic,

$$\mu_{\min}(g^* T_{X/B}) \geq \mu_{\min}(F_m^* f^* T_{X/B})$$

where $F_m : C_m \to C$ is the appropriate power of the geometric Frobenius. However by the ampleness of $T_{X/B}$ and the commutativity of Proj with base change, there is an $\varepsilon > 0$ such that $\mu(Q) \geq \varepsilon p^m$ for any line quotient Q of $F_m^* f^* T_{X/B}$. On the other hand if g is the genus of C then by Riemann-Roch we can find such a quotient with $\mu(Q) \leq \mu_{\min}(F_m^* f^* T_{X/B}) + g$, so taking $p \gg 0$ we may as well say, $\mu_{\min}(F_m^* f^* T_{X/B}) \geq \varepsilon p^m$ for a possibly smaller ε, and whence: $C_{m \cdot g}\, \omega_{X_1/B} \leq -\varepsilon p^m$. Turning now to the inseparable scheme quotient $\rho : X_1 \to Y_1$, defined via $\pi : X_1 \to B$ as before, through a generic point x of X_1 we can find for $p \gg 0$ a deformation missing the set T and any subset of codimension 2 we may introduce by passing to the quotient and so find $L_x \subset X_1$ a rational curve containing x on which ρ is inseparable, and,

$$H \cdot L_x \leq 2 \dim X \, \frac{H \cdot C}{\left(\varepsilon - \frac{\pi^* \omega_B \cdot C}{p^m}\right)}.$$

Lifting everything to characteristic zero as previously we get a ruling on X_1 by rational curves, which maps trivially to B, and a quotient $\nu : X_1 \dashrightarrow X_2$ by this ruling, well defined in codimension 2. We may thus continue by induction to establish that the fibres of X/B are rationally chain connected, with an appropriate bound on the degree of the connecting rational curves, by way of Tsen's theorem.

The entire discussion evidently goes through quasi-verbatim on replacing B by a point. The extra little idea in this case which yields a result not already in the literature is to use the deformations of f in X guaranteed a priori in positive characteristic, rather than trying to establish sufficient regularity of the quotient map $X \to X_1$ around C in characteristic zero.

IV. PRINCIPAL RESULTS AND COROLLARIES

IV.1 The main theorem

Consider a weakly regular integrable foliated variety $(X, \mathcal{F})$. The discussion of the previous chapters shows that if C is a curve in X, and $T_{\mathcal{F}}|_C$ is ample then through every point $x \in C$ there is a rationally chain connected variety whose dimension is the generic rank along C and whence by [K&] rationally connected, $\mathcal{F}$-invariant subvariety V_x with appropriate bounds on the degree of the connecting curves. It therefore remains to discuss the case of curves contained in the singular loss of $\mathcal{F}$ in more detail which is where the hypothesis of canonical singularities will intervene. To illustrate the principle idea let's suppose for the moment X is non-singular, and $\mathcal{F}$ is a foliation by curves. Denote by Z any subscheme contained in the singular locus of $\mathcal{F}$ considered scheme theoretically (i.e. if locally $\partial = a_i \frac{\partial}{\partial x_i}$, then the ideal of the singular locus is that generated by the a_i). Observe that we have maps,

$$\begin{tikzcd}
I_Z \arrow[dd] & \subset \mathcal{O}_X \arrow[d, "d"] & & & \\
& \Omega_X \arrow[r] \arrow[drrr, "D"'] & K_{\mathcal{F}} \otimes I_{\mathrm{sing}(F)} & \subset & K_{\mathcal{F}} \otimes I_Z \arrow[d, "\otimes \mathcal{O}_Z"] \\
I_Z/I_Z^2 \arrow[rrrr, dashed] & & & & K_{\mathcal{F}} \otimes I_Z/I_Z^2
\end{tikzcd}$$

We claim that the total composite of the maps on the right factors through I_Z/I_Z^2, and better still defines an $\mathcal{O}_Z$-linear map, which we will denote by D. The linearity is automatic by the Leibniz rule, and this in turn automatically forces the said factorisation. Let's observe some simple facts about this map. Firstly suppose $W \subset Z$ then we get maps for either subscheme, and a natural commutative diagram,

Now suppose we're considering a singular point x, with coordinate functions $x_1, \ldots, x_n$, and say ∂ a derivation defining the foliation at the point, then of course, $\partial = \sum_i a_i \frac{\partial}{\partial x_i}$, but $a_i - a_{ij}\, x_j \in \mathfrak{m}^2$, the square of the ideal at the point, with a_{ij} constants. In this case D is just the linear transform of the residue of the cotangent bundle given by,

$$D : \Omega_X \otimes k(x) \longrightarrow \Omega_X \otimes k(x) : dx_i \mapsto a_{ij}\, dx_j \,.$$

This map may of course be zero, but a moments thought shows that this cannot happen if the singularities are canonical. Indeed for general Z, we can just use the maps in the middle row together with the natural map from I_Z/I_Z^2 to the cotangent bundle to define a map, again denoted D, from $\Omega_X \otimes \mathcal{O}_Z$ to $\Omega_X \otimes \mathcal{O}_Z(K_{\mathcal{F}})$ which at every point is as above. Whence in turn for each $1 \leq n \leq \dim X$ we obtain via symmetric functions a global section, $S_n \in \Gamma(Z, \mathcal{O}_Z(nK_{\mathcal{F}}))$. The issue is therefore whether S_n is zero or not. If not this contradicts the ampleness of $T_{\mathcal{F}}$ on taking Z to be our curve, so what we'll show is that if S_n is zero for all n, the singularity is not canonical. It is wholly sufficient to prove this at a singular point, so say notations as above with x the origin. If all the symmetric functions vanish then the matrix $[a_{ij}]$ is nilpotent. Linear changes of coordinates conjugate the matrix, so we can suppose:

$$\partial = \sum_{i=k}^{n-1} x_i \frac{\partial}{\partial x_{i+1}} + \delta$$

where $\delta \in \mathfrak{m}^2 T_{X,x}$. We blow up in the origin, and look on the $x_n \neq 0$ patch, i.e. change coordinates to $x_n = \xi_n$, $x_i = \xi_i \xi_n$, $i < n$. Denote the blow up by $p : \widetilde{X} \to X$, then we have:

$$p^* \partial = \sum_{i=k}^{n-2} \xi_i \frac{\partial}{\partial \xi_{i+1}} + p^* \delta (\operatorname{mod} \mathfrak{m}^2 T_{X,0})$$

where $\mathfrak{m}$ continues to denote the maximal ideal at the origin. Superficially it may appear that we have reduced the complexity of the matrix, this may not however be the case since we can only guarantee,

$$p^* \delta = \sum_{i=1}^{n-1} a_i \xi_n \frac{\partial}{\partial \xi_i} \; (\operatorname{mod} \mathfrak{m}^2 T_{X,0})$$

and so the dimension of the eigenspace may not increase. Now writing things rather more invariantly we have a new linear map,

$$\widetilde{D} : \Omega_{\widetilde{X}} \otimes k(0) \longrightarrow \Omega_{\widetilde{X}} \otimes k(0)$$

with $\widetilde{D}\, d\xi_n = 0$, and all the terms, $\widetilde{\delta}$ say, that we have lumped under the appellation $\operatorname{mod} \mathfrak{m}^2$ enjoy the additional property of being divisible by ξ_n, with the exception of a term of the form $-\sum_{i \neq n} \xi_{n-1} \xi_i \frac{\partial}{\partial \xi_i}$. So let's blow up in the origin again, but this time look at the $\xi_{n-1} \neq 0$ patch on the blow up, i.e. put $\xi_{n-1} = \zeta_{n-1}$, $\xi_i = \zeta_i \zeta_{n-1}$, so that denoting the blow up morphism again by p, and the maximal ideal of the new origin still by $\mathfrak{m}$, we have,

$$p^* \partial = \sum_{i=k}^{n-3} \zeta_i \frac{\partial}{\partial \zeta_{i+1}} + \sum_{i=1}^{n-2} a_i \zeta_n \frac{\partial}{\partial \zeta_i} \; (\operatorname{mod} \mathfrak{m}^2 T_{X,0})$$

where the respective sums are understood to be zero if $k < n-3$ or $1 < n-2$, respectively. The new linear map $\approx{D}$, say, still has characteristic polynomial zero, but has at least one more eigenvector than D, so by induction, we conclude that our singularity could not have been canonical. Evidently to conclude the main theorem in this case it is sufficient that the singularities are canonical in dimension 1, i.e. canonical outside of a bunch of points. Furthermore by embedding in a smooth manifold, and using lemma I.1.4 to control any non-normality that the blow up procedure may introduce, we see that the hypothesis that X is smooth is not really essential, and whence arrive to our theorem in the case of foliations by curves.

The general situation is rather more delicate. We start as ever with a weakly regular foliated variety $(X, \mathcal{F})$ of rank r, and a curve C in X with s the generic rank of $\mathcal{F}$ along C. We have for Y a component of X_s containing C an exact sequence of the form,

$$0 \longrightarrow \mathcal{N} \longrightarrow T_{\mathcal{F}} \otimes \mathcal{O}_Y \longrightarrow \mathcal{T}_Y$$

where the image has rank s, and all the maps are generically of the same rank after tensoring with $\mathcal{O}_C$. Consequently if $f : B \to C$ is the normalisation, then there is a rank $r - s$ subbundle N of $f^* T_{\mathcal{F}}$ saturating the map on the left. This bundle admits a rather clean geometric description as follows, viz: we can find a neighbourhood V of the proper transform $\widetilde{C}$ of C in some $\mathcal{F}$-equivariant modification of Y such that the induced foliation $\mathcal{G}$ is smoothly integrable with tangent bundle $T_{\mathcal{G}}$, say, and so N must be the kernel of the map of bundles $\underset{\sim}{f^*} T_{\mathcal{F}} \twoheadrightarrow f^* T_{\mathcal{G}}$. Now let $\mathcal{I}$ be the ideal sheaf of $\widetilde{C}$ in the ambient modification $\widetilde{X}$ of X, then we have an $\mathcal{O}_{\widetilde{C}}$ linear map,

$$D : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{I}/\mathcal{I}^2 \otimes \mathrm{Ker}\,\{T_{\mathcal{F}} \to T_{\mathcal{G}}\}^{\vee}\,.$$

This map is not in general extendable to Ω_V, but no matter, since $T_{\mathcal{F}} \to T_{\mathcal{G}}$ is a surjection of bundles around $\widetilde{C}$, should E be the double dual of $f^*\,\mathcal{I}/\mathcal{I}^2$ we in fact have an induced map of bundles,

$$D : E \longrightarrow E \otimes N^{\vee}\,.$$

Finally letting L be the tautological bundle on $\mathbb{P}(N^{\vee})$ we have a map, $D : E \to E \otimes L$, and global sections given by symmetric functions $S_d \in H^0(\mathbb{P}(N^{\vee}), L^d)$. On the other hand these functions are non-zero if and only if, in our usual coordinate system, the matrix a_{ij} of functions in $x_{s+1}, \ldots, x_n$ is non-nilpotent for each $s < i \leq r$. The identical analysis to before shows this is not possible for canonical singularities, and whence we get our desired global section over B of $\mathrm{Sym}^d\, N^{\vee}$ for some d.

IV.2 Foliations by curves

In the case of foliations by curves there are some particularly beautiful corollaries of the main theorem, since the canonical bundle and cotangent bundle of the foliation now coincide. In particular let $(X, \mathcal{F})$ be a variety foliated by curves, then à la Mori we introduce the closed cone $\overline{\mathrm{NE}}(X)$ inside $\mathrm{NS}_1(X)_{\mathbb{R}}$ generated by effective curves and consider the subcone,

$$\overline{\mathrm{NE}}(X)_{K_{\mathcal{F}} \geq 0} := \{\alpha \in \overline{\mathrm{NE}}(X) \mid K_{\mathcal{F}} \cdot \alpha \geq 0\}\,.$$

Then we obtain,

IV.2.1 Theorem *Let $(X, \mathcal{F})$ be a variety foliated by curves with foliated Gorenstien and foliated canonical singularities then there are countably many $\mathcal{F}$ invariant rational curves L_i with, $K_{\mathcal{F}} \cdot L_i < 0$ such that,*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_{\mathcal{F}} \geq 0} + \sum_i \mathbb{R}_+[L_i]\,.$$

Better still,

(a) *The rays $\mathbb{R}_+[L_i]$ are locally discrete in the upper half space,* $\mathrm{NS}_1(X)_{K_{\mathcal{F}} < 0}$.

(b) *If $(X, \mathcal{F})$ is not a ruling by rational curves $K_{\mathcal{F}} \cdot L_i = -1$, $\forall i$.*

(c) *Otherwise, $-2 \leq K_{\mathcal{F}} \cdot L_i \leq -1$.*

(d) *Every extremal ray R of $\overline{\mathrm{NE}}(X)$ (i.e. $\alpha + \beta \in R$, $\alpha, \beta \in \overline{\mathrm{NE}}(X) \Rightarrow \alpha, \beta \in R$). Lying in the half space $\mathrm{NS}_1(X)_{K_{\mathcal{F}} < 0}$ is of the form $\mathbb{R}_+[L_i]$.*

proof. Everything in the corollary except (b) and (c) follows verbatim for the corresponding theorem for K_X, as found in [K1], Theorem III.1.2. To prove (b) and (c) consider an embedding of X in a smooth ambient manifold M. Observe firstly that the foliation defines a quasi-section (i.e. a section in codimension 2) of

$\mathbb{P}(\Omega^1_X) \to X$. Let $\widetilde{X}$ be the closure of this section, and M the tautological restricted to it. Further let L be the normalisation of any one of our L_i, then L maps to $\widetilde{X}$, by $\widetilde{f}$ say, and:

$$M \cdot_{\tilde{f}} L = -2 - \mathrm{Ram}_f$$

where $f : L \to L_i \hookrightarrow X$ is the corresponding map. Moreover we put,

$$s_{\mathrm{sing}(\mathcal{F})}(L) = K_{\mathcal{F}} \cdot_f L - M \cdot_{\tilde{f}} L \geq 0$$

which is precisely the contribution of the locus where $\mathcal{F}$ is singular in the sense of I.2. Now if L_i is singular at $x \in X$, then in fact x is a singular point of f, so by a sequence of $\mathcal{F}$-equivariant blow ups we can assume that $L_i = L$ (remember $K_{\mathcal{F}}$ cannot change). Whence,

$$-1 \geq K_{\mathcal{F}} \cdot_f L = s_{\mathrm{sing}(\mathcal{F})}(L) - 2 \geq -2$$

and -2 is obtained only if L does not pass through the singularities of $\mathcal{F}$. Consequently $\mathcal{F}$ must be smoothly integrable in a neighbourhood of L, and we can apply the algorithmic decomposition procedure once more to obtain a desingularisation $X^{\#}$ of X which is $\mathcal{F}$-equivariant around L. However the classical Frobenius theorem now forces L to have flat, whence trivial normal bundle in $X^{\#}$. Consequently L moves in a $\dim X - 1$ family which covers $X^{\#}$. Should C be any curve in this family, $K_{\mathcal{F}} \cdot C = K_{\mathcal{F}} \cdot L < 0$, so there is an $\mathcal{F}$-invariant rational subcurve of bounded degree through the generic point of X as required. □

We may also observe from the proof that to have an extremal ray on a foliated variety which is not a fibration in rational curves requires singularities, and so we even have,

IV.2.2 Corollary *Let $(X, \mathcal{F})$ be an everywhere smooth foliation by curves which is not a fibration in rational curves then $K_{\mathcal{F}}$ is nef.*

There is yet another case where usual Mori theory considerations, cf. [K1], yield a theorem of some interest. Call $H_R \in \mathrm{NS}^1(X)$ a supporting function of an extremal ray R if H_R is nef., and $H_R \cdot \alpha = 0$ iff $\alpha \in R$, then:

IV.2.3 Theorem *Hypothesis as in $IV.2.1$, and $R \subset \overline{\mathrm{NE}}(X)$ an extremal ray in the half space $\mathrm{NS}_1(X)_{K_{\mathcal{F}}} < 0$ then there is a $\mathbb{Q}$-Cartier divisor H_R which is a supporting function for R, and moreover $n\, H_R - K_{\mathcal{F}}$ is ample for $n \in \mathbb{N}$ sufficiently large.*

Opus Secundum

Semi-Stable Reduction of Foliations

INTRODUCTION

The object of this paper is to prove a minimal model theorem for foliations by curves. The bi-rational geometer needs little raîson d'être for such an action, but otherwise one might first wish to read the introduction to [IV] to see what this means in less algebraic terms. Nevertheless, minimal model lemma (length of proof not withstanding) is a substantially more accurate description, since even in the case of standard Mori theory it is hard to view the minimal model theorem as anything other than a means to the end of,

- Proving abundance, and exhibiting a non-general type variety as a Mori fibre space.
- 'Geometricising' a variety of general type by way of a Kăhler-Einstein metric on its canonical model.

So even in dimension 3 there's still work to be done on this latter question.

As we've said, however, the set up here is different: functorially with respect to the ideas we're after a relative minimal model theorem for $X \to [X/\mathcal{F}]$, where the latter is the classifying stack of a foliation by curves $\mathcal{F}$ on a projective variety X, or at least it would be if the latter had sense around the singularities, cf. the appendix. Fortunately, however, for X normal, the singularities are in co-dimension 2, so indeed there is a well defined reflexive rank 1 sheaf, $K_{\mathcal{F}}$, which is generically the relative canonical bundle. Unsurprisingly, we'll never allow the singularities of X to be any worse than $\mathbb{Q}$-factorial, so, a fortiori, the foliation is $\mathbb{Q}$-gorenstein, *i.e.* for some $m \in \mathbb{N}$, the reflexive hull of $K_{\mathcal{F}}^{\otimes m}$ is a bundle. We'll also need a sensible point of departure, *i.e.* the very concept of the birational geometry of $(X, \mathcal{F})$ must make sense, or what amounts to the same thing the singularities of $(X, \mathcal{F})$ are log-canonical.

In the happy event that the leaves of $\mathcal{F}$ were all algebraic, then Deligne & Mumford, [DM], have already done most of the work since $\mathcal{M}_{g,1} \to \mathcal{M}_g$ from the moduli stack of stable pointed curves of genus g to the moduli stack of curves has a nef. relative dualising sheaf for $g \geq 1$. There are, however, several important points to be drawn from this example. In the first place the model isn't just minimal but in some sense canonical. Indeed, although for $g \geq 3$ the relative dualising sheaf is big, no power of it is generated by sections, [K]. One might, therefore, say that in the relative setting canonical models don't exist. On the other hand the relative bundle is strictly positive on every curve contained in a fibre, and although it's zero on curves in the (relative) singular locus these aren't truly invariant, cf. [IV] VI.4. As such $\mathcal{M}_{g,1}$ is very much 'as good as one can do' for the universal foliation by curves of genus g, and in fact, one could easily do much worse since if $\tilde{\mathcal{M}}_{g,1} \to \mathcal{M}_{g,1}$ is a blow up in the singular locus, its relative canonical bundle over $\mathcal{M}_g$ is unchanged, so, certainly, still nef.. It will, however, vanish on some sub-curves of those fibres which become semi-stable rather than stable. There is, of course, a final issue that an abstract appeal to [DM] doesn't solve, namely a precise form for the rational map from an appropriate X with algebraically integrable foliation to $\mathcal{M}_{g,1}$, and if one bears this and the previous reflections on $\mathcal{M}_g$ in mind, one will have reasonable feeling for what the minimal model theory of foliations by curves consists of, albeit that the lack of invariant subvarieties excludes any hope that we can wholly reduce to a 'surface question'.

Now to the nitty-gritty of why this can be done in all dimensions without too much fuss as opposed to it's notoriously difficult K_X counterpart. To start with we're going to work throughout with algebraic (i.e. Deligne- Mumford) stacks, and needless to say characteristic zero. Unfortunately, there's an educational issue here, but Keel & Mori, [KM], have proved (modulo an old lemma of Vistoli, [V]) that these are a harmonious co-existence between groups and rings in the étale topology, and indeed, minor tweaking of their proof gives the same for any separated *fpqf* stack, I.1.3, not to mention a Zariski topology variant, I.3.2, provided we're prepared to allow extension of finite groups by abelian varieties, and finite group schemes respectively. Apart from some other minor generalities on semi-stable reduction of 1-dimensional stacks, I.4, the almost mandatory commentary on intersection theory on stacks I.8 (and, by the way, though we don't do it, much of the technical complication of [V] can be avoided by appeal to the jet groupoid II.1.1), and weighted projective structures as stacks I.9, what's really key in the preliminaries is the discussion of the Gorenstein covering stack I.5, and it's relation to singularities I.6-7. In traditional language this amounts to:

• The singularities of the index 1 cover faithfully reflect those we started with.

which is already false for actual Mori theory on a surface, *e.g.* a quotient of Du-Val may only be klt. As such, the importance of the Gorenstein covering stack should not be underestimated, especially as it serves to give the necessary extension of [I] to $\mathbb{Q}$-gorenstein varieties, II.3.3, and from there the cone theorem in optimal generality II.4.1. This compares sharply with the K_X case where the bend and break method isn't known to work without restrictive hypothesis such as LCI, and for that matter the un-necessarily restrictive results even for LCIQ, cf. [K2] 4.12, occasioned by not using stacks. The other critical difference between $K_{\mathcal{F}}$ and K_X theory is that in the former the normal cone of an extremal ray actually does determine, and accurately reflects, the entire structure of a formal neighbourhood, II.6.2. This rather surprising fact results from computing the formal holonomy, *i.e.* the obstruction to formally integrating the foliation in the neighbourhood of an extremal ray, II.5-6. Here we make some appeal to elementary homotopy, but this is really just for convenience since what's at stake is simply some patching conditions which, for a little extra work, could have been done wholly algebraically. This strongly suggests that we'll be able to do everything without leaving the 2-category of smooth stacks, and indeed, I.7, even under the further hypothesis that the non-scheme like points are (foliated) terminal, albeit that there is a caveat here due to Felipe Cano, [C2], that some singularities can be gorenstein and canonical only on permitting non-scheme like structure at the singularities themselves. Fortunately, the essentially local nature of flipping means this isn't an issue IV.10.

Given these pre-liminaries, we describe the locus of an extremal ray in §III, with the essential conclusion, III.5.1, being that up to at worst some extra $\mathbb{Z}/2$ monodromy this is a bundle of radially foliated (i.e. lines through the origin) weighted projective stacks. Here, and elsewhere, the lack of anyone having taken the trouble to prove the existence of a Hilbert-Scheme for stacks is a bit of a nuisance, but we invariably muddle through by either comparing stacks with the bug eyed covers of [K2] or appealing to the Chow Scheme of the moduli. From here till the end is a relatively standard application of the Artin's various convergence theorems, [A1], [A2], [A3], but not [A4], with projectivity issues being covered by an appropriate extension, IV.1.4, of Castelnuovo's contraction criterion in the spirit of [K1] III.1.8.1. Consequently we conclude to our main lemma the minimal model theorem for foliations not in conics, IV.7.4 & IV.10.2, along with an important 'uniqueness' property, whose strong form is that any rational (not just bi-rational) map to a minimal model can be resolved by invariant blowing up IV.11.3. We also don't forget the conic case, and in fact show that by a succession of flips and contractions, and, when necessary, an additional operation of flap, we can obtain a model which is either a bundle of radially foliated weighted projective stacks in the Zariski topology or a relatively smooth $\mathbb{P}^1$ bundle in the étale topology, IV.8.8.

Given the highly algebraic nature of the main lemma, the appendices on foliations and groupoids in analytic spaces may seem out of place. Their goal, however, is a c_0 version of our initial paragraph complementing [KM]. The main result, A.3.14, is, infact, a rather coarse c_0 variant of a GIT moduli space, but good enough to conclude that the c_0 theory of the classifying stack of a foliation is not, perhaps, quite as bad as one might fear.

I am indebted to Abramovich, Bogomolov, Brunella, Cano, Corti, and McKernan for many valuable suggestions, and comments, but, as ever, nothing would ever get done without Cécile.

I. PRELIMINARIES ON FOLIATED STACKS

I.1. Moduli Problems

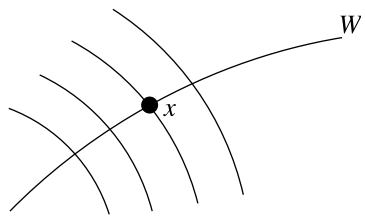
The general moduli problem for groupoids in algebraic spaces is basically solved by [KM]. Nevertheless there are a couple of outstanding issues which are worth clearing up by way of a closer look at op. cit. To this end let us take up verbatim the problem together with the notations from the same, i.e.

$$R \ \underset{s}{\overset{t}{\rightrightarrows}} \ X$$

is a groupoid in, say, finitely presented algebraic spaces over some base, or just finite type should the base be locally Noetherian, such that the source, s, and sink, t, are flat. Manifestly we require the map $j = s \times t$ to be separated, to have any hope, however, of forming a GC quotient X/R we must also suppose that the stabiliser $j : S := j^{-1}(\Delta_x) \to X$ is proper. Let us, therefore, suppose this and see if following [KM] we can solve the GC quotient problem. The question is, of course, local on X, and the key point of op. cit. is that one can replace X by an appropriate transversal (slice in [KM]) to the orbits. Specifically, the slightly technical, though cunning, definition of transversal is,

I.1.1 Definition. (cf. [KM] III.3, step 2) For $x \in X$ a closed geometric point call $W \ni x$ a C-M transversal if around x it is defined by $\dim_x \overline{\mathrm{orb}(x)} := \overline{t(s^{-1}(x))}$ equations and its intersection with $\overline{\mathrm{orb}(x)}$ is zero dimensional at x.

The picture is, of course, clear, albeit illuminating, i.e.



as is the existence of such transversals at generic points of $\mathrm{orb}(x)$, i.e. we can freely localise X, and so eventually just lift parameters in the local ring of $\overline{\mathrm{orb}(x)}$ at x to X. What is less clear, however, is,

I.1.2 Fact. ([KM], 3.3(i)) *Suppose simply that $j = s \times t$ is separated, then for any given geometric point $x \in X$ there is an isomorphic geometric point $w \in X$ together with a C-M transversal $g : W \ni w \hookrightarrow X$ such that,*

$$p : W \times_{(g,t)} R \underset{s}{\longrightarrow} X$$

is flat. In particular, [KM] 3.1, *the groupoid,*

$$R|_W \ \underset{s}{\overset{t}{\rightrightarrows}} \ W$$

defines around w, the same G-C quotient problem as the groupoid $R \rightrightarrows X$ around x.

Proof. This is precisely the proof of [KM] 3.3, however, there is a little confusion there about j being quasi-finite, so let's check that j separated suffices. In the first place, being Cohen-Macauly is an open condition, so we may choose a geometric point $w \in \mathrm{orb}(x)$, such that, $\mathrm{orb}(x)$ is C-M at w. Now consider, $t : s^{-1}(w) \to X$. This is a principle homogeneous space under $\mathrm{Hom}(w, w)$, so in particular its flat over $\mathrm{orb}(x)$, and $s^{-1}(w)$ is C-M along $t^{-1}(w) \cap s^{-1}(w) = \mathrm{Hom}(w, w)$, which is precisely the condition that we need to guarantee the asserted flatness of p. $\square$

Now once we have this, we can throw in the condition of proper stabiliser, and see what happens. Shrinking as necessary we may as well suppose that $\mathrm{orb}(w)$ contains exactly one geometric point, and that any other orbit, $\mathrm{orb}(w')$, $w' \in W$, is finite. Consequently, $j : R \to W \times W$ is quasi-proper with a (quasi) Stein type factorisation $R \underset{j_0}{\longrightarrow} R' \underset{j'}{\longrightarrow} W \times W$, where j_0 is actually proper, and j' quasi-finite. Better still, since the orbits are finite and without loss of generality W is affine both s and t are also quasi-proper, with quasi-Stein factorisations $s = s' \circ s_0$, $t = t' \circ t_0$ through the same R', and all of this is independent of flat base change. As such, the GC quotient problem for $R \rightrightarrows W$, will follow from the main theorem of [KM] provided that $R' \rightrightarrows W$ is a flat groupoid. This is however a well known theorem about abelian schemes, i.e. $s' : R \to R'$ is a family of equidimensional abelian varieties with a section so by [Mu1] 6.1.4, it is an abelian scheme, so in particular s_0 is flat whence s' is flat. This is in fact best possible provided the desired object of a moduli problem is a separated space, so let's summarise by way of,

I.1.3 Complement to [KM]. *Let $R \rightrightarrows X$ be a flat groupoid with proper stabiliser, then X/R has a G-C quotient as a possibly non-separated algebraic space. The said quotient is separated if $R \to X \times X$ is proper.*

I.2. Stacks versus Orbifolds

Let us immediately apply our previous moduli considerations to separated algebraic stacks for the $fppf$ topology. Indeed let $\mathcal{X}$ be such a stack, and $V \to \mathcal{X}$ a $fppf$ atlas then we can form the flat groupoid,

$$R := V \times_{\mathcal{X}} V \ \underset{s}{\overset{t}{\rightrightarrows}}\ V$$

where the maps are simply the necessarily flat projections. We therefore have,

I.2.1 Variant. *Let $\mathcal{X}$ be a $fppf$ algebraic stack, and suppose to be on the safe side the base is locally Noetherian, then the following are equivalent,*

(I) *$\mathcal{X}$ is separated.*

(II) *There is a proper moduli map $\pi : \mathcal{X} \to X$, with X a separated algebraic space.*

(III) *For some (whence all) $fppf$ atlases $V \to \mathcal{X}$, $s \times t : R \to V \times V$ is proper.*

Proof. (II) $\Rightarrow$ (I) is immediate, (I) $\Rightarrow$ (III) is basically [LM] 7.8, along with the general results of § 10 of the same. This leaves (III) $\Rightarrow$ (II), and we already have a separated moduli space $X := R/V$ by I.1.3, so it remains to check that π is proper, which follows by op. cit. 7.3 and the universal submersivity of π. $\Box$

Notice also that in the course of constructing X, we constructed an intermediate $fpqf$ stack $\mathcal{X} \underset{\rho}{\longrightarrow} \mathcal{X}' \underset{\sigma}{\longrightarrow} X$, where now σ isn't just proper but finite. Observe further that if $S \underset{j}{\longrightarrow} V$ is the stabiliser group scheme of our atlas $V \to \mathcal{X}$, then we've already establised in the sequel to I.1.2 that the geometrically reduced connected component S^0 of the identity is an abelian scheme over V. We have moreover a $fppf$ groupoid,

$$R^0 = \{\alpha \times g \times \beta\,;\ \beta \circ g = g \circ \alpha\} \subset S^0 \times_s R_t \times S^0 \rightrightarrows S^0$$

together with a flat map of groupoids $[R^0 \rightrightarrows S^0] \to [R \rightrightarrows V]$, while the properness of $R \to V \times V$ implies the same for $R^0 \to S^0 \times S^0$. Consequently we may apply I.1.2, to conclude the solution of the GC quotient problem for $[R^0 \rightrightarrows S^0]$ in the form of an abelian "scheme" (inverted commas seem preferable to writing relative abelian algebraic space) $\alpha : A \to X$. Equally we have a map, $S^0 \to A \times_X V$, which is an isomorphism on geometric points, and whence by rigidity an isomorphism stricta dictum. In consequence, and with the aid of [LM] 10.7, we may deduce,

I.2.2 Yet another Variant. *Things as in* I.2.1, *then yet another equivalent condition is,*

(IV) *There is a proper map $\rho : \mathcal{X} \to \mathcal{X}'$ to a separated algebraic $fpqf$ stack, suth that ρ is a bundle of classifying spaces for A in the flat topology (or indeed the étale topology should it exist) of $\mathcal{X}'$, i.e. for $W \to \mathcal{X}'$ a small open, $\mathcal{X} \times_{\mathcal{X}'} W = B_{\text{ét}}\, A \times_X W$.*

Plainly this more or less reduces the study of separated $fppf$ stacks to that of separated $fpqf$ stacks, so let's concentrate on the latter, although we'll still have a few further comments on how to reduce the general case even more. Regardless we now wish to show, following [V], that a separated $fpqf$ stack is an orbifold (in fact provided that one allows non-separated algebraic spaces, separation isn't important). The question is of a local nature, be it by I.2.1(II) or equivalently [KM] § 4, i.e. we wish to show that a fpF (where F here, and elsewhere means finite) groupoid $R \rightrightarrows V$, with V the spectrum of a strictly Henselian local ring, is equivalent to a classifying stack $[W/G]$ for G some group scheme, and $W \to V$ finite flat. In the first place R is a finite disjoint union, say, $\coprod_g V_g$, of spectra of local Henselian rings. Moreover base change by a finite flat strictly Henselian $\mathcal{O}_V$ algebra neither changes the GC quotient problem, nor does it increase the number of connected components of R, so without loss of generality we may suppose that each $s : V_g \to V$ has a section f_g. Now let G be an index set for the set of connected components, with distinguished element 1 corresponding to the connected component of the identity, and observe:

I.2.3 Facts. (a) *The groupoid structure on $R \rightrightarrows V$, induces a group structure on G, according to $V_{g\ t} \times_s V_h \to V_{hg}$.*

(b) *Each V_g is isomorphic to V_1, by way of,*

$$V_1 \to V_g : \alpha \mapsto f_g(t(\alpha)) \circ \alpha \qquad V_g \to V_1 : \beta \mapsto f_{g-1}(t\beta) \circ \beta\,.$$

Indeed (a) is clear, and while it's not a priori wholly transparent that the maps in (b) are actually inverse, it's never the less the case since over the closed point it's true, so it follows everywhere by Nakayama's lemma and the flatness of R over V.

As such in characteristic zero, we're actually done, since quite generally $V_1 \rightrightarrows V$ is a groupoid, and without loss of generality the orbit of the closed point is itself, so restricting to the closed point we find an affine group scheme with 1 point, from which, [KM] 6.5, the classifying stack $[V_1 \rightrightarrows V]$ is an honest space if the characteristic is neither positive nor mixed. In general, however, things are a bit more complicated. In the first place, one notes that if $\tilde{V} \subset V_1$ is the first order thickening of the identity then $\tilde{V} \rightrightarrows V$ is still a flat groupoid, and by virtue of working to 1^{st}-order, there is an isomorphism $\tilde{V}_s \times_s \tilde{V} \xrightarrow{\sim} \tilde{V}_t \times_s \tilde{V}$ which gives $\tilde{V} \underset{s}{\longrightarrow} V$ the structure of a group scheme over V acting on V. One then descends this to the moduli, and goes through the rigmorale of [SGA]3 VII to find a flat group scheme on the same which acts on V. Nevertheless such a refinement isn't of immediate necessity, so we'll omit its pursuit, and make,

I.2.4 Hypothesis. The implicit base of this discussion has characteristic zero, or more generally the characteristic of every closed geometric point is prime to the order of the stabiliser groups of $\mathcal{X}'$.

In which case we've established,

I.2.5 Fact. *Suppose* I.2.4, *and let $\mathcal{X}'$ be a $fpqf$ stack, then in fact it's a Deligne-Mumford stack, and,* cf. [V] 2.8, *it's even an orbifold, i.e. there exists an étale cover $\coprod_\alpha U_\alpha \to X$ of the moduli, such that $\mathcal{X}' \times_X U_\alpha = [V_\alpha/G_\alpha]$, where $\coprod_\alpha V_\alpha \to \mathcal{X}'$ is an atlas, and G_α a finite group acting on V_α.*

Basically I.2.5, is the stack version of [KM] 6.5, i.e. algebraic $fpqf$ spaces are algebraic spaces. In any case, the usual definition of orbifold pre-supposes that G_α is a subgroup of automorphisms of V_α (it also supposes V_α smooth, which clearly we won't) so let's try and get to grips with this. As a result, let us further make,

I.2.6 Hypothesis. $\mathcal{X}'$, equivalently X, has only one generic point.

What this amounts to at the level of the V_α's is that provided U_α is irreducible then each one has finitely many components $V_{\alpha i}$, all of which are isomorphically permuted by G_α. As a result if H is the automorphism group of the generic geometric point, then we have an exact sequence,

$$0 \to G'_\alpha \to G_\alpha \twoheadrightarrow G''_\alpha \hookrightarrow \mathrm{Aut}(V_\alpha)$$

whith H isomorphic to G'_α. More generally, even if U_α isn't irreducible, then on restricting to an appropriate open subset, one sees that H is still the kernel of $G_\alpha \to \mathrm{Aut}(V_\alpha)$, and we obtain a stack $\mathcal{X}'/H$ with moduli X, which is locally the orbifold $[V_\alpha/G''_\alpha]$. Notice in addition,

I.2.7 Fact. *Suppose* I.2.4 *and* I.2.6, *then for any test scheme T we have a fibre* *square,*

$$\begin{tikzcd}
\mathcal{X}' \arrow[d] & {[T/\mathbb{H}]} \arrow[l] \arrow[d] \\
\mathcal{X}'/H & T \arrow[l]
\end{tikzcd}$$

where $\mathbb{H}/T$ is a group scheme over T, isomorphic to H at geometric points.

Proof. Locally on T this is clear. In general, the patching data for the G'_α's, when restricted to T gives exactly the patching data of a group scheme $\mathbb{H}/T$, which is manifestly H over the geometric points. □

A useful corollary of which is,

I.2.8 Corollary. *Let $\mathcal{C}/\mathbb{C}$ be a smooth one dimensional Deligne-Mumford stack, with H its generic stabiliser, then if $\mathcal{C}/H$ is not a so called bad orbifold there is an étale map $C \to \mathcal{C}$, for some honest curve C, and even in the bad case there is a map, albeit ramified, $\mathbb{P}^1 \to \mathcal{C}$.*

This also seems an appropriate, for want of a better, place to make,

I.2.9 Warning. Given a Deligne-Mumford stack $\mathcal{X}$ with moduli X, and $Y \hookrightarrow X$ a closed integral subspace the closed integral sub-stack associated to Y, $\mathcal{Y}$, say, in $\mathcal{X}$ is given locally, in terms of our $\pi_\alpha : V_\alpha \to U_\alpha$ by taking $\pi_\alpha^{-1}(Y)_{\mathrm{red}}$, i.e. the analogue of irreducible subvariety, at least if U_α is irreducible, is a reduced variety whose irreducible components are permuted by a group action.

Finally let's put all of this together for a separated $fppf$ stack $\mathcal{X}$ under the hypothesis I.2.4 and I.2.6. Specifically we can find an étale cover $\coprod_\alpha U_\alpha \to X$ of the moduli, together with an étale atlas $\coprod_\alpha V_\alpha \to \mathcal{X}$ such that for some abelian scheme A/X we have finite groups $G_\alpha \in \mathrm{Ext}(H_\alpha, H)$, with H_α a subgroup of $\mathrm{Aut}(V_\alpha)$ such that, $\mathcal{X} \times_X U_\alpha \xrightarrow{\sim} [V_\alpha/G_\alpha \times A_{U_\alpha}]$. Denoting by $\mathcal{X}/A$ our old $\mathcal{X}'$, and $\mathcal{X}/A/H$ our old $\mathcal{X}'/H$, it's definitely not true that $\mathcal{X}/A$ can be realised as a product of $\mathcal{X}/A/H$ with the classifying stack BH, nor is it true that we can relate $\mathcal{X}$ to $\mathcal{X}/A$ in this sort of way. Nevertheless the obstruction to the latter is quite small, i.e. $\mathrm{Ext}^1_{\mathcal{X}/A}(\mathcal{I}, \pi^*A)$, where $\mathcal{I}$ is the inertia group stack of $\mathcal{X}/A$, and is as near zero as makes no difference, i.e.

I.2.10 Fact. *Let $T \to \mathcal{X}/A/H$ be a test scheme, then for $\mathbb{H}$ as in* I.2.7, *and some étale cover $T' \to T$ we have a fibre square,*

$$\begin{tikzcd}
\mathcal{X} \arrow[d] & BA \times_X [T'/\mathbb{H}] \arrow[l] \arrow[d] \\
\mathcal{X}/A/H & T' \arrow[l]
\end{tikzcd}$$

Proof. Indeed the obstruction to splitting $\mathcal{X}_T$ lies in, $H^1_{\text{ét}}(T, \mathrm{Hom}(\mathbb{H}, A))$ so this is clear. □

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Regardless the important point is that there obviously isn't much to an arbitrary separated $fppf$ stack beyond what one finds in the Deligne-Mumford set up, and so we'll make,

I.2.11 Convention. From now on, unless stated to the contrary, stack will mean separated Deligne-Mumford stack of finite type over some base (ultimately $\mathbb{C}$, but for the moment let's permit otherwise).

I.3. Topologies

Let's now give a letter, say, S, to our implicit base, supposed locally Noetherian to be on the safe side, and consider the category $\mathcal{C}$ of projective schemes over S. On $\mathcal{C}$ we have 4-topologies of interest, i.e. ét F, ét, fpF, $fpqf$, with respect to which we may form either spaces (i.e. quotients by an equivalence relation for the given topology) or stacks. As such consider, the following diagram of 2-categories, and categories, where the maps are the obvious ones, i.e.

$$\begin{tikzcd}
\textit{fpF}\text{ stacks} \arrow[rr] \arrow[dr, "\sim"', "\text{if I.2.4 holds}"] \arrow[dd] & & \textit{fpqf}\text{-stacks} \arrow[dr, "\sim", "\text{if I.2.4 holds}"'] \arrow[dd] & \\
& \text{ét } F \text{ stacks} \arrow[rr] \arrow[dd] & & \text{ét stacks} \arrow[dd] \\
\textit{fpF}\text{ spaces} \arrow[rr] \arrow[dr, "\sim"] \arrow[dd, "\wr"] & & \textit{fpqf}\text{ spaces} \arrow[dr, "\sim"] \arrow[dd, "\wr"] & \\
& \text{ét } F \text{ spaces} \arrow[rr] \arrow[dl, "\sim"'] & & \text{ét spaces} \arrow[dl, "\sim"'] \\
\begin{array}{c}\text{Schemes of}\\ \text{finite type}/S\end{array} \arrow[rr] & & \begin{array}{c}\text{Algebraic spaces of}\\ \text{finite type}/S\end{array} &
\end{tikzcd} \tag{I.3.1}$$

Now the basic point is that the vast majority of this diagram is redundant. The isomorphisms in the bottom triangles are [SGA]3 V.7, and [KM] 4.2 and 6.5 respectively. In fact we could even replace F (= finite) by proper, and qf (= quasi-finite) by proper, and quasi-proper, all of which is just as well since it's about as easy to describe a local ring in the flat topology as it is to define the set of all sets. We've also seen that under what amounts to a near enough best possible hypothesis, I.2.4, that the vertical arrows in the upper square are isomorphisms. It's also true that a Deligne-Mumford stack is proper over its moduli, so the only two things that really make a difference are,

(a) The bottom arrow from schemes to algebraic spaces.

(b) The difference between the quotient V/G of an affine by a group (or more generally group scheme), and the classifying stack $[V/G]$.

Indeed this distinction is wholly sharp, since we assert,

I.3.2 Claim. *The left hand face of the diagram is the fibre over schemes of the right hand face, i.e. a Deligne-Mumford stack has projective moduli iff it is an ét F stack.*

Proof. The if direction is again [SGA]3 V.7, at least if the moduli is separated, but even if it's not, it's not separated for the wholly scheme like reason of having too many points, and whence coincides with the algebraic space quotient of [KM]. Conversely let, $\pi : \mathcal{X} \to X$ be the supposed scheme like moduli of a Deligne-Mumford stack. The question is local on X, so we can suppose that X is the spectrum of a local ring, and let $q : U \to X$ be its strict Henselisation. By [V] 2.8 (and/or I.2.5), there is a strictly Henselian local ring V and a finite group G, such that, $\mathcal{X} \times_X U \xrightarrow{\sim} [V/G]$. As ever denote by p_i, $i = 1$ or 2, the projections from $U \times_X U$ to U, then since $\mathcal{X}$ is defined over X, we have a descent datum of classifying stacks,

$$\phi : p_1^*[V/G] = [p_1^* V/G] \xrightarrow{\sim} p_2^*[V/G] = [p_2^* V/G] \,.$$

Now V is strictly Henselian, and without loss of generality G is the stabiliser of the closed point, so in fact this data amounts to two different descent data, i.e.

$$\varphi : p_1^* V \xrightarrow{\sim} p_2^* V \,, \qquad \Phi : G \xrightarrow{\sim} G$$

such that $\varphi(v^g) = \varphi(v)^{\Phi(g)}$, and of course compatibility with the p_{ij}'s from $U \times_X U \times_X U$ to $U \times_X U$. Consequently V descends to a local affine Y, finite over X, and G to a finite flat group scheme $\mathbb{G}/X$ acting on Y. In particular, over geometric points $\mathbb{G}$ is G, and $\mathcal{X} = [Y/\mathbb{G}]$. □

Notice that the argument of the proof applies to other contexts, for example,

I.3.3 Sub-Fact. ("$GAGA$ for stacks") *Let $\mathcal{X}_{\rm an}$ be an analytic Deligne-Mumford stack whose moduli (which necessarily exists by an easy variant of* [KM]*) is a compact Moishezon manifold $X_{\rm an}$, then $\mathcal{X}_{\rm an}$ is algebraic, i.e. the analytic stack associated to a Deligne-Mumford stack.*

Proof. The compactness of $X_{\rm an}$ implies, and in fact this is all we need it for, that the ramification of $\mathcal{X}_{\rm an} \to X_{\rm an}$ is algebraic. As such the question is again local, so we can proceed as above on replacing X by a strictly Henselian ring, and U by its analytification, or indeed, completion. □

Taking all of this into account, we therefore have several different topologies for a stack $\pi : \mathcal{X} \to X$, together with the obvious functors,

$$\begin{array}{ccccc}
\mathcal{X}_{\rm fl} & \longrightarrow & \mathcal{X}_{\rm \acute{e}t} & \longrightarrow & \mathcal{X}_{\rm Zar} \\
{\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\
X_{\rm fl} & \longrightarrow & X_{\rm \acute{e}t} & \longrightarrow & X_{\rm Zar}
\end{array}$$

where we avoid discussing the right most of these if X is not a scheme. The important thing (as Abramovich pointed out to me) is,

I.3.4 Further Fact. (cf. [AV]) *Regardless of whether we're is the flat, étale, or Zariski topology, or for that matter analytic if things are over $\mathbb{C}$, π_* is acyclic on coherent sheaves on supposing* I.2.4.

Proof. As per op. cit., for flat, étale or analytic, this just amounts to the vanishing of $H^i(G, M)$, for M a G-module, with $\frac{1}{|G|} \in \mathrm{Aut}(M)$, which is exactly what I.2.4 guarantees. In the Zariski case, we're looking at $[Y/\mathbb{G}]$ in the notations of the proof of I.3.3, so for a suitable étale cover $\tilde{X} \to X$, and M a coherent $\mathbb{G}$-module on Y, we have a spectral sequence,

$$\check{H}^i(\tilde{X}/X, H^j(G, M \otimes_{\mathcal{O}_X} \mathcal{O}_{\tilde{X}})) \Rightarrow H^{i+j}(\mathbb{G}, M)$$

so again we're done, since all the rings in question are local. □

Consequently for $\mathcal{M}$ a coherent sheaf on a stack, the symbol $H^i(\mathcal{X}, \mathcal{M})$ is not just wholly unambiguous, but equal to the equally unambiguous symbol, $H^i(X, \pi_* \mathcal{M})$. In particular if X is projective, then $\mathcal{X}$ is cohomologically projective, i.e.

I.3.5 Sub-Fact. *Let H be ample on X and suppose* I.2.4, *then for $\mathcal{M}$ on $\mathcal{X}$ coherent, there is an integer $n_0 = n_0(\mathcal{M})$, such that for $n \geq n_0$, $H^i(\mathcal{X}, \pi^* H^{\otimes n} \otimes \mathcal{M}) = 0$, $i \geq 1$.*

We should of course also check,

I.3.6 Hilbert 90. Let $\mathcal{X}$ be an algebraic stack, then,

$$H^1_{\text{fl}}(\mathcal{X}, \mathrm{GL}_n) = H^1_{\text{ét}}(\mathcal{X}, \mathrm{GL}_n) = H^1_{\mathrm{Zar}}(\mathcal{X}, \mathrm{GL}_n)$$

where the last group is only understood to have sense if the moduli is a scheme.

Proof. This is just the usual Hilbert 90, together with a G-action that respects descent. □

At which point it's pretty clear that a stack with projective moduli is every bit as good as projective variety, and so it seems appropriate to make,

I.3.7 Parenthesis. Indeed the only difference between the stack and its moduli is that the former has too many tangent vectors (whence I.3.5 does not imply, as it couldn't, that a stack with a cohomologically ample bundle is embeddable in projective space), as such a separated stack is not separated in the classical sense and is more akin to a non-separated algebraic space with the right number of points, but the wrong number of tangents (cf. [K2]). Unlike the non-separated space, however, which locally thinks itself wholly separated, the stack knows that it has too many tangent vectors, and if the moduli is projective it is in many many ways a better object than a non-projective scheme (e.g. I.3.5, Kodaira vanishing holds too, etc.). The conclusion then, following Gromov, [G1], would be to consider stacks (at least separated ones) as a natural soft category for the ideas of algebraic geometry, i.e. just as integrability isn't that important for complex structure, neither is separation, in its classical manifestation, for things which are polynomial.

I.4. Resolution and Reduction

At this point we confine ourselves to characteristic zero, and of course finite type. In the first place recall the definition of algorithmic resolution of singularities according to [BM], i.e.

I.4.1 Definition. Given a reduced scheme V (or even algebraic space, or even formal scheme or formal algebraic space) there is a sequence of modifications,

$$V = V_0 \leftarrow V_1 \leftarrow \cdots \leftarrow V_n = \tilde{V}$$

such that $V_{i+1} \to V_i$ is a blow up in a smooth centre $Z(V_i)$, $\tilde{V}$ is non-singular, and the process is invariant with respect to smooth maps, i.e. if $\rho : W \to V$ is smooth then the algorithmic resolution,

$$W = W_0 \leftarrow W_1 \leftarrow \cdots \leftarrow W_n = \tilde{W}$$

is the cartesian product of that of V by ρ, or if one prefers, $Z(W_i) = \rho^{-1} Z(V_i)$.

The applicability of this to stack resolution is clear. Indeed even in the case of Artin stacks which we momentarily will permit. Specifically let, $\coprod_\alpha X_\alpha = X_0 \to \mathcal{X}$ be a smooth presentation of an integral Artin stack, by integral schemes and consider the subschemes $Z_{\alpha 0}$ of X_α corresponding to the smooth centres of the 1^{st}-stage of algorithmic resolution. Necessarily we have a fibre square of smooth maps,

$$\begin{tikzcd} X_\alpha & X_\alpha \times_{\mathcal{X}} X_\beta \arrow[l, "p_\beta"'] \arrow[d, "p_\alpha"] \\ \mathcal{X} \arrow[u, phantom] \arrow[from=u, "p_\alpha"'] & X_\beta \arrow[l, "p_\beta"'] \end{tikzcd}$$

Although perhaps not integral, $X_\alpha \times_{\mathcal{X}} X_\beta$ is reduced, with a well defined algorithmic centre $Z_{\alpha\beta}$, and of course $p_\beta^{-1}(Z_\alpha) = p_\alpha^{-1}(Z_\beta)$. Consequently the $Z_{\alpha 0}$ patch to a smooth sub-stack $\mathcal{Z}_0$ of $\mathcal{X}$ and almost by definition we have a diagram,

$$\begin{tikzcd} \coprod_\alpha X_\alpha \arrow[d, "\rho_\alpha"'] & \coprod_\alpha \mathrm{Bl}_{Z_{\alpha_0}}(X_\alpha) \arrow[l] \arrow[d, "\tilde{p}_\alpha"] \\ \mathcal{X} & \mathrm{Bl}_{\mathcal{Z}_0}(\mathcal{X}) \arrow[l] \end{tikzcd}$$

with $\tilde{p}_\alpha$ a smooth presentation, and whence,

I.4.2 Fact. (characteristic 0) *Any integral Artin stack $\mathcal{X}$ admits a resolution $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ by a sequence of modifications in smooth sub-stacks. Better still the process is even algorithmic with respect to smooth maps of stacks $\mathcal{Y} \to \mathcal{X}$, and enjoys an embedded variant for closed substacks $\mathcal{Z}$ of $\mathcal{X}$.*

Equally however, one is frequently presented with a smooth stack $\mathcal{X}$, whose moduli space X is singular. Here algorithmic resolution doesn't a priori help, since it leaves $\mathcal{X}$ unmodified, and it would certainly be desirable to have a sequence of modifications which didn't just smooth $\mathcal{X}$ but also its moduli. This is however, and again Abramovich pointed it out, impossible by [O], and so let's turn to that which is directly relevant, i.e. semi-stable reduction. Specifically let $\rho : \mathcal{X} \to \mathcal{C}$ be a proper family of 1-dimensional stacks over a normal 1-dimensional stack with $\mathcal{X}$ smooth. We already know that we can find a map $C \to \mathcal{C}$ from a smooth curve, and since our interest is reduction by base change, we may as well suppose that $\mathcal{C}$ is a honest curve. Consequently the fibres $\mathcal{X}_c$, for $c \in C$, are well defined substacks to which we may apply embedded resolution and so arrive to the situation that $\rho : \mathcal{X} \to C$ is a family of 1-dimensional stacks, such that the singularities of $\mathcal{X}_c$ are no worse than nodes. Notice also that since $\mathcal{X}$ has a moduli space, the components of $\mathcal{X}_c$ are well defined and so as ever we can write,

$$\mathcal{X}_c = \sum_{i=1}^{t} n_i(c)\,\mathcal{X}_i$$

for $n_i \in \mathbb{N}$ the multiplicities. For each singular point c, choose $n(c)$ such that each $n_i(c)$ divides it, let B be a map from a curve to C with the orbifold/stack structure defined by the $n(c)$, and $\mathcal{X}_B \to B$ the base change, with $\tilde{\mathcal{X}}_B$ its algorithmic resolution. For any $U \to \mathcal{X}_B$ étale, the corresponding resolution $\tilde{U}$ of U is, by virtue of being algorithmic, the unique minimal resolution of U and so thanks to the standard argument for stable reduction we have concluded,

I.4.3 Fact. (characteristic 0) *Any proper map $\rho : \mathcal{X} \to \mathcal{C}$ from a smooth 2-dimensional stack to a smooth 1-dimensional stack with fibres at worst nodes admits a semi-stable reduction, i.e. there is a map from an honest smooth curve $B \to \mathcal{C}$ such that if $c\tilde{X}_B$ is the algorithmic resolution of the base change then the fibres are still no worse than nodes, and are even reduced.*

Notice additionally that we can combine all of this with the uniformisation theorem of Corollary I.2.8 to obtain,

I.4.4 Lemma. *Let $\rho : \mathcal{X} \to \mathcal{C}$ be as in the previous fact and suppose further that modulo its generic stabiliser the generic point of $\mathcal{X}$ over $\mathcal{C}$ is not a bad orbifold then there is a family $\tilde{\rho} : \mathcal{Y} \to B$ of 1-dimensional semi-stable stacks (i.e. with reduced fibres) over a curve B, which is scheme like in codimension 2, together with a commutative diagram,*

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\rho"'] & \mathcal{Y} \arrow[l] \arrow[d, "\tilde{\rho}"] \\ \mathcal{C} & B \arrow[l] \end{tikzcd}$$

such that the ramification of $\mathcal{Y}/\mathcal{X}$ is supported only in the fibres of $\tilde{\rho}$.

Proof. We may as well apply I.2.8 in order to say that $\mathcal{C}$ is actually a curve C and $\mathcal{X}/C$ is semi-stable. Denote by K the function field of C, and apply I.2.8 to $\mathcal{X}_{\overline{K}}$. Consequently for some curve $C' \to C$, and $Y \to C'$ a semi-stable family we have a diagram,

$$\begin{tikzcd} Y \arrow[r, dashed, "\varphi"'] \arrow[d] & \mathcal{X} \arrow[d] \\ C' \arrow[r] & C \end{tikzcd}$$

with the dotted arrow φ rational, but étale over the generic fibre. Now form the graph of φ, which may develop non-scheme like points, and apply semi-stable reduction to obtain a family $\mathcal{Y} \to B$ for an appropriate B, satisfying all of the lemma except perhaps that it may have non-scheme like points in codimension 1.

However if this were to happen, then in local coordinates, x on the base, and y around a component of the fibre we would have a finite group G of automorhisms of the bi-disc leaving x fixed with generic stabiliser non-zero on $x = 0$, which in turn would force $\mathcal{Y}$ to be non-scheme like generically, so we're done. □

I.5. Foliations and Gorenstien Stacks

Suppose now that $\mathcal{X}$ is a normal stack, then since it is the quotient of an étale groupoid, albeit in the 2-category of stacks, there is no difficulty in defining its cotangent sheaf $\Omega_{\mathcal{X}}$, and whence the notion of foliated stack, i.e.

I.5.1 Definition. A foliation $\mathcal{F}$ on $\mathcal{X}$ is a rank 1 (coherent) quotient of $\Omega_{\mathcal{X}}$.

Arguably this is a less than intelligent point of view, nevertheless to avoid getting into technical issues about 3-categories we'll postpone a more sensible discussion till later, and confine ourselves to the directly relevant question of the well definedness of the canonical bundle, viz,

I.5.2 Definition. If the 1$^{\text{st}}$ Chern class of $\mathcal{F}$ exists as a line bundle on $\mathcal{X}$, e.g. $\mathcal{X}$ non-singular, then we say that $(\mathcal{X}, \mathcal{F})$ is foliated Gorenstien, and write $K_{\mathcal{F}}$ for the corresponding bundle. In the case that $K_{\mathcal{F}}$ is only a $\mathbb{Q}$-bundle, then we say that $(\mathcal{X}, \mathcal{F})$ is $\mathbb{Q}$-foliated Gorenstien. Note also the notation $T_{\mathcal{F}}$ for the dual bundle, or more generally $\mathbb{Q}$-bundle.

An important, indeed the only necessary, example of a foliated Gorenstien stack is that associated to a $\mathbb{Q}$-foliated Gorenstien normal scheme, or algebraic space, X. In order to convey the extra structure of the scheme case let us concentrate upon it. Here we can take an open covering U_α by sufficiently small affines, together with n_α the smallest integer such that $\mathcal{O}_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha}(n_\alpha K_{\mathcal{F}})$ by way of some local section s_α, and subsequently take the corresponding root V_α, i.e.

I.5.3 Construction. (characteristic 0, algebraically closed base) Let $U'_\alpha \subset U_\alpha$ be the open set where $\mathcal{O}_{U_\alpha}(K_{\mathcal{F}})$ is a bundle (necessarily $\operatorname{codim}(U_\alpha \backslash U'_\alpha) \geq 2$, since X is normal). Cover U'_α by small affines U_{α_i}, and write $s_\alpha = f_{\alpha_i} \partial_{\alpha_i}^{-n_\alpha}$ where ∂_{α_i} is a local generator of $\mathcal{O}_{U_{\alpha_i}}(T_{\mathcal{F}})$, and consider the sheaf of ideals $\mathcal{I}_\alpha$ in $\mathbb{V}(\mathcal{O}_{U'_\alpha}(T_{\mathcal{F}}))$ defined locally by $\partial_{\alpha_i}^{n_\alpha} - f_{\alpha_i}$, with V'_α the corresponding subschme, necessarily non-singular where U'_α is, so in particular normal. Naturally we put V_α to be the integral closure of U_α in the function field of V'_α, and observe that the extension of function fields $k(V_\alpha)/k(U_\alpha)$ is a cyclic Kummer extension with Galois group $G_\alpha = \mathbb{Z}/n_\alpha$.

The final observation results, of course, from the triviality of cohomology with values in any locally constant sheaf in the Zariski topology, and of course we obtain an action of G_α on V_α. We wish, now, to consider how these constructions patch, so consider what happens over the overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$. To this end let $d_{\alpha\beta}$ be the g.c.d. of n_α and n_β, $m_\beta = n_\alpha/d_{\alpha\beta}$, $m_\alpha = n_\beta/d_{\alpha\beta}$, and $n_{\alpha\beta} = n_\alpha n_\beta/d_{\alpha\beta}$ then over $U_{\alpha\beta}$ there is a unit $h_{\alpha\beta}$ such that,

$$s_\alpha^{\otimes m_\alpha} = h_{\alpha\beta}\, s_\beta^{\otimes m_\beta} \in \mathcal{O}_{U_{\alpha\beta}}(n_{\alpha\beta} K_{\mathcal{F}})\,.$$

Over the Gorenstien locus, everything is intrinsic so consider an affine $U_{\alpha\beta i} \subset U'_{\alpha\beta} = U'_\alpha \cap U'_\beta$, where $K_{\mathcal{F}}$ admits a trivialisation with respect to which s_α, s_β correspond to non-vanishing functions f_α and f_β respectively then the coordinate ring of $V_\alpha \times_{U'_{\alpha\beta i}} V_\beta$ has the form,

$$\frac{\mathcal{O}_{U'_{\alpha\beta i}}[S,T]}{(S^{n_\alpha} - f_\alpha,\, T^{n_\beta} - f_\beta)}$$

for some indeterminates S and T. We can consider this as some disjoint union of components of,

$$S^{n_\alpha} - f_\alpha = \prod_{\zeta \in \mu_{m_\beta}} (T^{n_\beta} - \zeta f_\beta) = 0 \subset U'_{\alpha\beta i} \times \mathbb{A}^2$$

where of course the above product is just $T^{n_{\alpha\beta}} - h_{\beta\alpha} S^{n_{\alpha\beta}}$ so that if $\tilde{U}_{\alpha\beta}$ is $U_{\alpha\beta}(h_{\alpha\beta}^{1/n_{\alpha\beta}})$ then $V_\alpha \times_{U'_{\alpha\beta}} V_\beta$ is open in $V_\alpha \times_{U'_{\alpha\beta}} \tilde{U}_{\alpha\beta}$ which in turn is open in $V_\alpha \times_{U_{\alpha\beta}} \tilde{U}_{\alpha\beta}$, whence the normalisation $V_{\alpha\beta}$ of $V_\alpha \times_{U_{\alpha\beta}} V_\beta$ is naturally contained in $V_\alpha \times_{U_{\alpha\beta}} \tilde{U}_{\alpha\beta}$ and thus étale over V_α, and indeed by symmetry over V_β.

The data $(V_\alpha, G_\alpha, U_\alpha)$ therefore determines a Q-variety, which in turn defines a stack $\mathcal{X}$ by way of the groupoid,

$$\coprod_{\alpha,\beta} V_{\alpha\beta} \rightrightarrows \coprod_{\alpha} V_\alpha$$

which enjoys the particularly simple property of being covered by Zariski open substacks of the form $[V_\alpha/G_\alpha]$. Manifestly its moduli space is X, and the induced foliation on $\mathcal{X}$ is Gorenstien, by virtue of $\mathcal{X}$ being S_2.

Let us note this more explicitly by way of a definition, and partial converse, viz:

I.5.4 Definition. Let $(X, \mathcal{F})$ be a $\mathbb{Q}$-Gorenstien foliated scheme then the above constructed stack $\mathcal{X}$ along with the induced foliation $\mathcal{F}$ will be called the associated foliated Gorenstien covering stack. Conversely if $(\mathcal{X}, \mathcal{F})$ is a foliated Gorenstien stack then its moduli space $(X, \mathcal{F})$ is a $\mathbb{Q}$-foliated-Gorenstien algebraic space. The constructions are not however inverse to each other as the stack may posses a generic stabiliser.

In any case the extent to which $(\mathcal{X}, \mathcal{F})$ has non-scheme like points is easily determined. Indeed let us introduce,

I.5.5 Definition. For $x \in X$, the index of $\mathcal{F}$ at x is the smallest integer $n(x)$ such that $\mathcal{O}_{X,x}(n(x)K_\mathcal{F})$ is a bundle.

Then unsurprisingly,

I.5.6 Fact. *The index is the same irrespective of whether we consider the local ring $\mathcal{O}_{X,x}$ in the Zariski or étale topology or even in its completion at the maximal ideal, while for $\xi \in \mathcal{X}$ any geometric point over a geometric point $x \in X$, $\mathrm{Aut}(\xi) = \mathbb{Z}/n(x)\mathbb{Z}$.*

Proof. To begin with let V be an étale neighbourhood of x, lying over the Zariski neighbourhood U, with $\mathcal{O}_V(mK_\mathcal{F})$ a bundle. Plainly we can suppose V irreducible, $m \leq n$, and consider the associated Gorenstien loci V' over U'. Denoting as ever $p_{ij} : V \times_U V \times_U V \to V \times_U V$, $p_i : V \times_U V \to V$ the various projections then over U' we have the standard isomorphism, $\varphi : p_1^* \mathcal{O}_{V'}(K_\mathcal{F}) \xrightarrow{\sim} p_2^* \mathcal{O}_{V'}(K_\mathcal{F})$, which extends to an isomorphism,

$$\varphi^m : p_1^* \mathcal{O}_V(mK_\mathcal{F}) \xrightarrow{\sim} p_2^* \mathcal{O}_V(mK_\mathcal{F})$$

by virtue of the S_2 condition. On the other hand φ satisfies the cocycle condition, $p_{31}^*\varphi = p_{32}^*\varphi\, p_{21}^*\varphi$ over V', so the same is true of φ^m over V, whence, $\mathcal{O}_U(mK_{\mathcal{F}})$ is a bundle, and the étale and Zariski indices coincide. Better still if formally we have an isomorphism, $s : \hat{\mathcal{O}} \xrightarrow{\sim} \hat{\mathcal{O}}(mK_{\mathcal{F}})$, $m \in \mathbb{N}$, with $\hat{\mathcal{O}}$ the corresponding complete local ring, then by Artin's implicit function theorem, [A1], together with Nakayama's lemma we have the same on an étale neighbourhood, from which our initial assertion.

For the second assertion, we start off with a Galois covering $V \to U$ with group $G = \mathbb{Z}/n(x)\mathbb{Z}$ defined over some Zariski open neighbourhood $U \ni x$ by way of I.5.3. Now let $y \in V$ be a geometric point over x, and denote by H the stabiliser of y in G. Consider the corresponding quotient $W = V/H \to U$, then G/H acts freely on the pre-image of x, so shrinking U as necessary we may as well say that $W \to U$ is étale. Now cover the Gorenstien locus U' of U by affines U_i with, $\mathcal{O}_{U_i}(T_{\mathcal{F}}) = \mathcal{O}_{U_i}\,\partial_i$, and $f_i\,\partial_i^{-n(x)}$ the corresponding section defining V, then the Galois extension $k(W)/k(U)$ contains $f_i^{1/d}$ for $d = \#(G/H)$, which yield a nowhere vanishing section of $\mathcal{O}_{W'}\left(\frac{n(x)}{d}\,K_{\mathcal{F}}\right)$. On the other hand W is certainly normal, so we even have a nowhere vanishing section of $\mathcal{O}_W\left(\frac{n(x)}{d}\,K_{\mathcal{F}}\right)$, with $W \to U$ étale, from which $\frac{n(x)}{d} \geq n(x)$, and thus the lemma. $\Box$

We may also note in consequence that the index is upper semi-continuous, so the non-scheme like locus is closed. In any case the non-scheme like nature of points on $\mathcal{X}$ also admits, under certain circumstances, an interpretation in terms of holonomy. Indeed consider the rather particular case where the non-scheme locus $\mathcal{B}$ is such that $\mathcal{X}$ and the foliation are smooth around $\mathcal{B}$, and everywhere transverse to each other. For a geometric point $b \in \mathcal{B}$ we may therefore apply the Frobenius theorem to find a pointed analytic disc $f : (\Delta, 0) \to (\mathcal{X}, b)$ invariant by $\mathcal{F}$. Projecting to X, we have that locally around $\pi(b)$ both X and $\mathcal{F}$ are smooth away from $B := \pi(\mathcal{B})$, as such if $j : L = \pi \circ f(\Delta^\times) \xrightarrow{\sim} \Delta^\times := \Delta\backslash\{0\} \hookrightarrow X$ is our local leaf we have a locally constant sheaf of algebras $j^{-1}\mathcal{O}_{\mathcal{F}} := \{z \in \mathcal{O}_{X_{\mathrm{an}}} : \partial(z) = 0\}$ where $-\mathrm{an}$ denotes analytic functions and ∂ is understood to generate the foliation on an appropriate open. Consequently if X has dimension $(n+1)$ we can consider the holonomy representation,

$$\rho : \mathbb{Z} = \pi_1(\Delta^\times) \to \mathrm{Aut}(\mathbb{C}\,[z_1, \ldots, z_m])\,.$$

We assert,

I.5.7 Claim. *If n is the index at $\pi(b)$, then* $\mathrm{Im}(\rho) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$.

Proof. Provided our original disc $f : \Delta \to \mathcal{X}$ is sufficiently small, then manifestly the holonomy is trivialised on the covering $f(\Delta^\times)$ of L, so, $\#\,\mathrm{Im}(\rho) \leq \mathbb{Z}/m\mathbb{Z}$. Now suppose conversely that the holonomy is actually smaller than this. It will suffice to consider its linearisation, i.e. the local system on $\Delta^\times$ whose corresponding bundle is $f^*\pi^*N_{L/X}|_{\Delta^\times}$, which we already know has constant part $dz_1, \ldots, dz_m$ with respect to the appropriate connection, for some functions $z_1, \ldots, z_m$ in an étale neighbourhood of b defining the foliation. To assert that the holonomy is smaller then $\mathrm{Aut}(b)$ amounts to claiming that we can find a non-trivial subgroup G of $\mathrm{Aut}(b)$, and $dw_1, \ldots, dw_m \in \Gamma(\Delta^\times, f^*\pi^*N_{L/X})$ invariant by G yet giving the same constant subsheaf as that defined by the dz_i. Consequently since the dw_i's are related to the dz_j's by a constant matrix in $\mathrm{GL}_m(\mathbb{C})$ over $\Delta^\times$, we see that G operates trivially on the dz_i's. On the other hand for any $\bar{d}$, G is a commutative group of automorphisms of the finite dimensional vector space, $\frac{\mathcal{O}_{\mathcal{X},b}}{m(b)^{\bar{d}}}$, so in the completion $\hat{\mathcal{O}}_{\mathcal{X},b}$ we certainly have a linearisable action which on the tangent space at b, acts non-trivially in the foliation direction and trivially normal to it. Whence if $V \ni b$ is the corresponding étale neighbourhood then V/G is non-singular so the index is strictly less than itself, which is nonsense. □

I.6. Intermission on Singularities

The discussion will be local and purely scheme like, so let $X = \mathrm{Spec}\,\mathcal{O}_{X,Y}$ be the germ of a normal scheme of finite type localised at a subvariety Y of codimension ≥ 2. We not only wish to consider a foliated germ $(X, \mathcal{F})$ on the same, but also a boundary divisor B. The foliation will, of course, be supposed $\mathbb{Q}$-Gorenstien as will the possibly empty set of prime components B_i be supposed $\mathbb{Q}$-Cartier with $B_i \supset Y$. Along the boundary there will be a symbol for invariance, i.e. $\varepsilon_i = 0$ or 1, according to whether B_i is invariant by $\mathcal{F}$ or not, together with weights $e_i \in \mathbb{N} \cup \{\infty\}$. The weights, however, will not be in the standard Mori theory sense, cf. [K3], but rather given by,

I.6.1 Warning/Definition. If B_i are the prime components of B, and $e_i \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ the weights, then $B = \sum_i \left(1 - \frac{1}{e_i}\right) B_i$. In any case, we have well defined multiplicity and discrepancy functions,

$$\nu_i, a_{\mathcal{F}} : \{\text{prime valuations of } \mathbb{C}(X) \text{ centered on } Y\} \to \mathbb{Q}$$

which we think of in the usual way, i.e.

I.6.2 Comment/Definition/Further Warning. To such a valuation we can associate a proper bi-rational modification $\rho : \tilde{X} \to X$, indeed even by a sequence of blow ups in smooth centres if we wish, and an exceptional divisor E so that the valuation ring is $\mathcal{O}_{\tilde{X},E}$. An important invariant of this situation is,

$$\varepsilon(E) := \begin{cases} 0 & \text{if } E \text{ is invariant by the induced foliation } (\tilde{X}, \tilde{\mathcal{F}}) \\ 1 & \text{if } E \text{ is not invariant by the induced foliation } (\tilde{X}, \tilde{\mathcal{F}}) \end{cases}$$

In particular the Chern class of a log-foliation, say, $(\tilde{X}, \rho^{-1}(Y), \tilde{\mathcal{F}})$ is not calculated as in the K_X case but in terms of ε. Specifically the log-canonical class of $(X, B, \mathcal{F})$ is $K_{\mathcal{F}} + \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i\, B_i$ and we define the germ $(X, B, \mathcal{F})$ to be,

(I) Terminal if $a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i\, \nu_i(E) > 0$

(II) Canonical if $a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i\, \nu_i(E) \geq 0$

(III) Log-terminal if $a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i\, \nu_i(E) > -\varepsilon(E)$

(IV) Log-canonical if $a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \varepsilon_i \, \nu_i(E) \geq -\varepsilon(E)$

where the various inequalities are understood to hold for all E.
Manifestly boundary components B_i which are invariant by $\mathcal{F}$ in no way effect the definitions, i.e. they may be freely added or removed without changing the nature of the singularity. Whence without loss of generality we will subsequently suppose $\varepsilon_i = 0$, and introduce, $\nu(E) = \sum_i \left(1 - \frac{1}{e_i}\right) \nu_i(E)$.

Now basically the problem with these definitions is that they're absolutely impossible to calculate locally unless everything is Gorenstien, and the boundary components are Cartier. Equally evidently we can reduce to this case by covering constructions, therefore we will now make,

I.6.3 Hypothesis. For purposes of classifying singularities of types (I)–(IV) we will suppose for the rest of this section $(X, \mathcal{F})$ Gorenstien and each component B_i Cartier.

It is also a pain in the neck to work with non-smooth things, so consider:

I.6.4 More Notation. Fix an ambient smooth embedding $X \hookrightarrow M$ of dimension the embedding dimension of X at M. Denote by ∂ a local generator for $\mathcal{F}$ around Y, which we may, without loss of generality, suppose restricted from M in such a way that the lifted foliation $(M, \mathcal{F})$ leaves X invariant. Now fix a quasi coefficient field $K \subset \mathcal{O}_{M,Y}$ of $\mathbb{C}(Y)$, then we can write (non-canonically),

$$\partial = \partial_K \oplus \delta$$

where $\partial_K \in \mathrm{Der}_K(\mathcal{O}_{M,Y})$, $\delta \in \mathcal{O}_{M,Y} \otimes_K \mathrm{Der}_{\mathbb{C}}(K)$. Nevertheless if $\mathcal{F}$ is singular at Y, ∂_K may be considered functorially to 1^{st} order, i.e. modulo $\mathfrak{m}^2_{M,Y}$, be it in,

$$\mathrm{End}\left(\frac{\mathfrak{m}_{X,Y}}{\mathfrak{m}^2_{X,Y}}\right) \quad \text{or} \quad \mathrm{End}\left(\frac{\mathfrak{m}_{M,Y}}{\mathfrak{m}^2_{M,Y}}\right)$$

which are in fact the same thing, since we took M to have the embedding dimension of X. Consequently we may talk about ∂_K being nilpotent or not, according to whether the said linearisation is, or not. We can also do all of this at the level of completions in Y (so $K = \mathbb{C}(Y)$), where we'll use the same notations up to adding a $\wedge$. Furthermore we will employ,

I.6.5 (FCP). (Formal checking principle) Completion is faithfully flat, so realising valuations by blow ups in smooth centres, we can simply calculate the discrepancy by way of the completion. We can also make converse type statements, provided we're careful not to blow up in non-algebraic centres.

We begin with a lemma, viz:

I.6.6 Lemma. *Hypothesis as per* I.6.3, *and suppose further that B has precisely one component, transverse to $\mathcal{F}$, and of multiplicity 1 at Y then for any finite weight $(X, B, \mathcal{F})$ is terminal, and canonical if the weight is infinite.*

Before proceeding with the proof, observe:

I.6.7 Sub-Lemma. *Everything as above, with E a rank 1-discrete valuation centered on Y then $\varepsilon(E) = 0$.*

Proof. Let $\tilde{\partial}$ in $\mathcal{O}_{\tilde{X},E}$ be a local generator of the induced foliation. Passing to completions we can find an element $x \in \mathfrak{m}_{X,Y}$ such that $\tilde{\partial} x = 0$. Consequently if

μ is the valuation of x, π a uniformising parameter, then for some $\tilde{x}$ of valuation zero, we have:

$$0 = \mu\,\pi^{\mu-1}\,\tilde{x}\,\tilde{\partial}\,\pi + \pi^{\mu}\,\tilde{\partial}\,\tilde{x}$$

so indeed $\tilde{\partial}\,\pi \in \mathfrak{m}_{\tilde{X},E}$ as required.

Observe in passing a little fact implicit in the sub-lemma,

I.6.8 Fact. *Suppose there is a* $x_1 \in \mathfrak{m}_{M,Y}$ *with* $\partial x_1 \neq 0$, *so without loss of generality* 1, *then we can find embedding coordinates* $x_1, \dots, x_d \in \mathfrak{m}_{M,Y}$ *such that not only do we have,* $\mathcal{O}_{\hat{M}} = \mathbb{C}(Y)[[x_1, \dots, x_d]]$, *but* $\hat{\partial} = \partial/\partial x_1$.

Proof. Apply the formal Frobenius theorem (or more accurately its proof, cf. II.1), and profit from the fact that $\mathbb{C}(Y)$ is finite over some $\mathbb{C}(y_1, \dots, y_{\dim(Y)})$.

In any case back at the lemma, we may now proceed to,

Proof. (of I.6.6) Let e_1 be the weight of the said component, B_1, with x_1 a local equation. Moreover take $\tilde{\partial}$, π, $\mathcal{O}_{\tilde{X},E}$ etc. as per the proof of I.6.7/I.6.8, and write, $x_1 = \pi^{\nu_1}\,\tilde{x}_1$, with $\tilde{x}_1 \in \mathcal{O}^*_{\tilde{X},E}$, $\partial = \pi^{-a}\,\tilde{\partial}$, then,

$$\mathcal{O}^*_{X,Y} \ni \partial x_1 = \pi^{\nu_1 - a}\,\tilde{\partial}\,\tilde{x}_1 + \nu\,\pi^{\nu_1 - a - 1}\,\tilde{x}_1\,\tilde{\partial}\,\pi$$

whence $a \geq \nu_1$, so if e_1 is the weight, $a - \left(1 - \frac{1}{e_1}\right)\nu_1 \geq \frac{\nu_1}{e_1}$, as required.

Note that in particular,

I.6.9 Corollary. $(X, \mathcal{F})$ *is terminal iff* $\mathcal{F}$ *is smooth at* Y, *and transverse to it.*

Proof. In the if direction, by hypothesis we can find a B satisfying I.6.6, so $(X, B, \mathcal{F})$ and a fortiori $(X, \mathcal{F})$ is terminal. Conversely if $\mathcal{F}$ is either singular at Y or leaves it invariant, then blowing up in Y followed by an invariant/algorithmic resolution yields valuations with $a_{\mathcal{F}} = 0$.

In fact we've even proved,

I.6.10 Sub-Corollary. $(X, \mathcal{F})$ *is terminal iff it's log-terminal.*

Proof. Terminal implies log-terminal is trivial. Conversely everything is Gorenstien, so log-terminal implies canonical. However if $\mathcal{F}$ is either singular at Y or not transverse to it, then, the blow ups employed in I.6.9 yield valuations with $\varepsilon(E) = 0$, and discrepancy 0.

This is of course useful in establishing,

I.6.11 Principle Fact. *The following are equivalent,*

(I) $(X, B, \mathcal{F})$ *is terminal.*

(II) $(X, B, \mathcal{F})$ *is log-terminal.*

(III) *Either* B *consists of a single component of multiplicity* 1, *transverse to* $\mathcal{F}$ *and of finite weight, or* B *is empty, and in both cases* $(X, \mathcal{F})$ *is terminal.*

Proof. (I) $\Rightarrow$ (II) is trivial, so suppose $(X, B, \mathcal{F})$ is log-terminal, then $(X, \mathcal{F})$ is too, so by I.6.10 $(X, \mathcal{F})$ is terminal. Now consider the sequence of modifications

$$X = X_0 \underset{\alpha}{\longleftarrow} X_1 \underset{\beta}{\longleftarrow} X_2 \underset{\gamma}{\longleftarrow} \tilde{X}$$

where X_1 is the blow up of X in Y, with E_1 the exceptional divisor, X_2 is the blow up of X_1 in the singular locus of the induced foliation, E_2 the exceptional divisor, and $\tilde{X}$ an invariant/algorithmic resolution of X_2. Observe also by I.6.8, that the induced singular locus in X_1 is a section, S, say, over Y so irrespective of whether X_1 is normal or not $\alpha^* T_{\mathcal{F}}(-E_1)$ is already saturated in $\mathcal{J}_{X_1}$, as such $K_{\mathcal{F}_1} = \alpha^* K_{\mathcal{F}} + E_1$ is an honest canonical class of the induced foliation $\mathcal{F}_1$. Better still since γ and β are equivariant blow ups,

$$\alpha^* \beta^* \gamma^* K_{\mathcal{F}} + \gamma^* \beta^* E_1 = \gamma^* \beta^* K_{\mathcal{F}_1} \geq K_{\tilde{\mathcal{F}}}$$

so that if E is a valuation on $\tilde{X}$ with centre on X_1 contained in E_1 and e the multiplicity of E then, $a_{\mathcal{F}}(E) \leq e$. However if $\nu_{i,Y}$ is the multiplicity of B_i at Y then, $\nu_i(E) \geq e\, \nu_i(E)$ so if $\varepsilon(E) = 0$ we must have,

$$0 < a_{\mathcal{F}}(E) - \sum_i \left(1 - \frac{1}{e_i}\right) \nu_i(E) \leq e \left\{1 - \sum_i \left(1 - \frac{1}{e_i}\right) \nu_i \right\}$$

while if $\varepsilon(E) = 1$, $a_{\mathcal{F}}(E) \leq e - 1$, so in either case, profiting from $e_i \geq 2$,

$$\sum_i \nu_{i,Y} < 2$$

which certainly establishes that we have at most one component of multiplicity 1, and manifestly of finite weight.

Now suppose that B_1 is not transverse to $\mathcal{F}$, then its proper transform in X_1 must contain S with multiplicity μ say. Consequently if the centre of E on X_1 is actually S (and of course such an E exists, and we can even take its centre on X_2 to be any component of E_2) with e' the multiplicity of E_2 along E, then, arguing as before,

$$0 < e' \left\{1 - \left(1 - \frac{1}{e_1}\right) \nu_{1,Y} - \left(1 - \frac{1}{e_1}\right) \mu \right\}$$

which is rubbish since $\mu, \nu_{1,Y} \in \mathbb{N}$. As such $(X, B, \mathcal{F})$ necessarily satisfies I.6.6, which in turn imply terminality. $\square$

As to canonical and log-canonical singularities the important cases are,

I.6.12 Further Fact. *Suppose $B = \varphi$ then the germ $(X, \mathcal{F})$ is canonical iff $\mathcal{F}$ is smooth at Y or the field ∂_K does not have nilpotent linearisation nor does it satisfy the equivalent conditions,*

(1) *The field ∂_K is a semi-simple $\widehat{K}$ derivation of $\widehat{\mathcal{O}}_{X,Z}$ with, after multiplication by a unit, positive integer eigenvalues.*

(2) *D is not nilpotent and Z is the centre of a valuation with negative $(X, \mathcal{F})$ discrepancy.*

(3) *Z is the centre of a valuation with discrepancy -1 such that the maximal ideal in $k(X)$ of the said valuation is not invariant for the induced foliation.*

Proof. That nilpotent fields aren't canonical follows from [I] IV.1. Everything else will follow from the equivalence of the said conditions, so consider,

(1) $\Rightarrow$ (2) It suffices to exhibit the centre via a chain of blow ups, and of course we can just work in the completion provided we blow up in algebraic centres. Observe that with the hypothesis, Z is in fact the singular locus of the foliation, and it's

easy to construct a sequence of blow ups in the singular locus of the foliation, determined by the Euclidean algorithm for the eigenvalues, which eventually lead to a valuation of negative discrepancy centred on Z.

(2) $\Rightarrow$ (3) Suppose we have a discrete rank 1 valuation ring $R \supset \mathcal{O}_{X,Z}$ dominantly, and an induced derivation $\widetilde{\partial}$ of R with $\partial = \pi^e \widetilde{\partial}$, $e \in \mathbb{N}$, π a unformising parameter. We can certainly extend all this to the completion, where we let F^n in $\widehat{\mathcal{O}}_{X,Z}$ be $\mathfrak{m}_R^n \cap \widehat{\mathcal{O}}_{X,Z}$. In particular, $\partial(F^n) \subset F^{n+e-1}$, and e is certainly at least 1. If, however, $e > 1$, then $\partial^k(f) \to 0$ as $k \to \infty$ for any f in $\widehat{\mathcal{O}}_{X,Z}$. On the other hand by a theorem of Chevalley the F^n induce the $\mathfrak{m}_{X,Z}$-adic topology, so ∂ is nilpotent.

(3) $\Rightarrow$ (1) We might as well just start with a discrete rank 1 valuation ring $R \supset \widehat{\mathcal{O}}_{X,Z}$, with uniformiser π, where by hypothesis $\partial = \pi\, \widetilde{\partial}$, and $\widetilde{\partial}\, \pi$ is a unit. We can certainly choose π and a quasi coefficient field L of R containing $\widehat{K}$ such that if we write $\widehat{\partial} = \partial_K \oplus \delta$, with $\delta \in \widehat{\mathcal{O}}_{X,Z} \otimes \mathrm{Der}_k(\widehat{K})$, then $\partial_K = \pi^a\, \widetilde{\partial}_K$, $\delta = \pi^b\, \widetilde{\delta}$, where $a, b \in \mathbb{Z}$ and $\widetilde{\delta} \in R \otimes \mathrm{Der}_k(L)$. On the other hand,

$$\widetilde{\partial} = \pi^{a-1}\, \widetilde{\partial}_K + \pi^{b-1}\, \widetilde{\delta}$$

and $\widetilde{\partial}_K, \widetilde{\delta}$ are independent, so $a, b \geq 1$. However $\widetilde{\delta}\, \pi = 0$, so $a = 1$, and $\widetilde{\partial}_K\, \pi$ is a unit. Furthermore we have an injection,

$$F^n/F^{n+1} \hookrightarrow \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$$

the left hand side is a finite $\widehat{K}$ vector space, the right hand finite over the residue field ℓ, say, of R. In either case ∂_K is a linear map for the appropriate field, and a quick calculation shows that it's just multiplication by $n\, \widetilde{\partial}\, \pi$ on the right hand side. Consequently if $F^n/F^{n+1} \neq 0$ then all the eigenvalues of ∂_K are the same. Even better, since the characteristic is zero, taking the trace shows that they actually take values in $\widehat{K}$, so after multiplying $\widetilde{\partial}$ by a unit in $\widehat{\mathcal{O}}_{X,Z}$ we might as well say $\widetilde{\partial}\, \pi = 1$. Now consider the sequence of integers n_k defined by n_{k+1} is the minimum integer with $F^{n_{k+1}} \neq F^{n_k}$, $n_1 = 1$, and put $\mathcal{O}_k = \widehat{\mathcal{O}}_{X,Z}/F^{n_k}$. Further we have an exact sequence,

$$0 \longrightarrow F^{n_k}/F^{n_{k+1}} \longrightarrow \mathcal{O}_{k+1} \longrightarrow \mathcal{O}_k \longrightarrow 0$$

and $F^{n_k}/F^{n_{k+1}} \hookrightarrow \mathfrak{m}^{n_{k+1}-1}/\mathfrak{m}^{n_{k+1}}$, so ∂_K acts semi-simply on it with eigenvalue, $n_{k+1} - 1$, where: $n_{k+1} > n_{k+1} - 1 \geq n_k$. Whence, by induction, we conclude that ∂_K is semi-simple on $\mathcal{O}_{k+1}$ with positive integer eigenvalues $< n_{k+1}$. Finally for any $d \in \mathbb{N}$, $\widehat{\mathcal{O}}_{X,Z}/\mathfrak{m}_{X,Z}^d$ is a quotient of a sufficiently large $\mathcal{O}_k$, so that (1) holds. $\square$

In a sense therefore, log-canonical is cleaner, and in many ways preferable since we have,

I.6.13 Further Fact bis. *Again $B = \varphi$ then a singular germ $(X, \mathcal{F})$ is log-canonical iff the linearisation of ∂_K is non-nilpotent.*

Proof. For X smooth, one can just blow up in the obvious way using the Euclidean algorithm for the λ_i, and reduce to I.6.12. This is surprisingly tedious to justify if X is normal, however [M2] I.1.1(b) actually shows for a valuation E,

(i) If the discrepancy is ≤ -2 then ∂_K is nilpotent.

(ii) If the discrepancy is -1, and $\varepsilon(E) = 0$ then ∂_K is nilpotent.

Consequently if ∂_K is not nilpotent, we always have $a_{\mathcal{F}}(E) \geq -\varepsilon(E)$, which a postiori even proves (by uniqueness of the valuation with discrepancy -1) that the aforesaid obvious resolution $\rho : (\tilde{X}, \rho^{-1}(Y), \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ is log-terminal. $\square$

Although we'll have no need of it, this seems an appropriate place to list what else can occur, albeit without proof by virtue of the aforesaid utilitarian considerations, i.e.

I.6.14 Fact. *Let $(X, B, \mathcal{F})$ be a germ of a foliation singularity, with $D \neq \varphi$, which is log-canonical then in fact it's canonical, and should it not be terminal the possibilities are,*

1) *$(X, \mathcal{F})$ is terminal and the weight is infinite.*

2) *$(X, \mathcal{F})$ is terminal, the weight is 2, B has multiplicity 1 at Y, and there is a simple tangency between B and $\mathcal{F}$, i.e. if f, ∂ are a local equation, and a local generator respectively then, $\partial^2 f \neq 0$.*

3) *$(X, \mathcal{F})$ is terminal, the weight is 2, B is a possibly non-irreducible, i.e. there may be two components, tacnode enjoying a simple tangency with $\mathcal{F}$.*

Notice that despite lack of immediate relevance, these are not just curiosities. Obviously (1) is essential for a sensible theory in the infinite weight case, while the importance of (2) rests on the fact that although a Bertini type theorem, i.e. for H generic ample $(X, H, \mathcal{F})$ has log-canonical singularities if $(X, \mathcal{F})$ does, is impossible for infinite weight, it is perfectly possible for weight 2. Furthermore, it is the case, cf. IV.9, that the entire log-minimal model programme for foliations follows from the weight 2 case, so that ultimately a perfectly general definition would have been given had all the weights been taken to be 2.

I.7. Smooth Singular Stacks

Intermission over let's return to stacks, i.e. $(\mathcal{X}, \mathcal{F})$ will be a normal foliated integral stack with moduli $(X, \mathcal{F})$, projective, albeit that this isn't necessary for the present discussion, and of course $\pi : \mathcal{X} \to X$ the moduli map. For any geometric point ξ of $\mathcal{X}$ we can perfectly well make,

I.7.1 Definition. $\mathcal{F}$ has terminal, canonical, log-terminal, log-canonical singularities at ξ iff for some étale (and in fact any) étale neighbourhood $V \ni \xi$ with the image of ξ a closed subvariety Y, the germ $(\operatorname{Spec} \mathcal{O}_{V,Y}, \mathcal{F})$ is terminal, canonical, etc.

Indeed the definitions are independent of étale coverings, so there's no ambiguity in what we've said, and we can talk about $\mathcal{O}_{\mathcal{X},\xi}$ being terminal, canonical etc. Our goal is to relate the singularities of $\mathcal{F}$ or $\mathcal{X}$ to those of X. To begin with we assert,

I.7.2 Claim. *Let $(X, \mathcal{F})$ be $\mathbb{Q}$-Gorenstien, with $\pi : (\mathcal{X}, \mathcal{F}) \to (X, \mathcal{F})$ its Gorenstien covering stack, then the singularities of $(X, \mathcal{F})$ are terminal, respectively canonical, respectively etc. iff those of $(\mathcal{X}, \mathcal{F})$ are.*

Proof. By definition $\mathcal{X}$ and X have the same geometric points, so the question is purely local, whence let $Y \subset X$ be a subvariety with $\mathcal{Y} \subset \mathcal{X}$ a closed substack above it, then of course we have,

$$\pi : (\operatorname{Spec} \mathcal{O}_{V,Z}, \mathcal{F}) \to (\operatorname{Spec} \mathcal{O}_{X,Y}, \mathcal{F})$$

for an appropriate $V \to \mathcal{X}$ étale, with π finite and étale in codimension 2. A priori for $\mathcal{E}$ a discrete rank 1 valuation of $\mathcal{O}_{V,Z}$ lying over a valuation E of $\mathcal{O}_{X,Y}$ the almost étale nature of π yields the formula,

$$a_{\mathcal{F}}(\mathcal{E}) = a_{\mathcal{F}}(E)e + \varepsilon(E)(e-1)$$

where e is the order of ramification at $\mathcal{E}$. So one way round this is clear, i.e. whatever we have on $(X, \mathcal{F})$ we certainly obtain on $(\mathcal{X}, \mathcal{F})$. Conversely we have to be careful about ε. By construction $(\mathcal{X}, \mathcal{F})$ is Gorenstien so if the singularity is terminal (equivalently log-terminal) then by I.6.7, $\varepsilon(E) = 0$, so the same holds for $(X, \mathcal{F})$. In a similar vein we can consider the canonical case. Indeed if R is the valuation ring of $\mathcal{E}$ then the precise local uniformisation procedure whereby $\mathcal{E}$ is obtained from $\operatorname{Spec} \mathcal{O}_{V,Z}$ by a way of a sequence of blow ups is rather easy, i.e.

$$\operatorname{Spec} \mathcal{O}_{V,Z} = V_0 \leftarrow V_1 \leftarrow \cdots \leftarrow V_n = \tilde{V}$$

where V_{i_n} is the blow up of V_i, in the centre Z_i, say, of the valuation on V_i. As such we have two possibilities,

(a) All the centres, $Z_0 = Z, \ldots, Z_{n-1}$ are invariant by a local generator ∂ of $\mathcal{F}$ around Z.

(b) There exists $0 \leq p < n-1$, such that $Z_0, \ldots, Z_{p-1}$ are invariant by ∂, but Z_p is not.

Now if $\rho_p : V_p \to V_0$ is the projection, then in the latter case we may apply the Frobenius theorem à la I.6.8 to $\rho_p^* \partial$ around Z_p since this doesn't require normality, to conclude that $\varepsilon(\mathcal{E}) = 0$. In the former case, $\rho_n^* \partial$ is still an honest derivation, so the discrepancy is non-positive, but everything is canonical so in fact $\rho_n^* \partial$ generates the induced foliation around $\mathcal{E}$, and direct calculation shows that again $\varepsilon(\mathcal{E}) = 0$. Exactly the same line of reasoning works for (b) in the log-canonical case, so in fact the only thing that could go wrong is that $\mathcal{E}$ is the valuation of discrepancy -1, in which case our discrepancy formula (more by luck than design) gives $a_{\mathcal{F}}(E) = -1$, so $(X, \mathcal{F})$ is log canonical too. □

Now let's go a bit further. Specifically let $(\mathcal{X}, \tilde{\mathcal{F}})$ be any smooth foliated stack, and $(X, \mathcal{F})$ its necessarily normal, and supposed projective moduli. Appealing to I.5.3, we know that things must factor, i.e. there is a diagram,

$$\begin{tikzcd} (\mathcal{X}, \tilde{\mathcal{F}}) \arrow[r, "\sigma"] \arrow[dr, "\pi"'] & (\mathcal{X}', \mathcal{F}) \arrow[d, "\rho"] \\ & (X, \mathcal{F}) \end{tikzcd}$$

with $(\mathcal{X}', \mathcal{F})$ the Gorenstien covering stack. Associated to this data we have a ramification locus with prime components B_i in X, covered by some integral substacks $\mathcal{D}'_i$, and $\mathcal{D}_i$ of $\mathcal{X}$ and $\mathcal{X}'$ respectively. We're interested in the properties of $K_{\mathcal{F}}$ so quotienting out by the generic stabiliser of $\mathcal{X}$ is harmless, whence suppose indeed that the generic stabiliser is zero, then the stabiliser of a generic geometric point of $\mathcal{D}_i$ is of the form $\mathbb{Z}/e_i$ for $e_i \in \mathbb{N}_{\geq 2}$, and indeed,

$$K_{\tilde{\mathcal{F}}} = \pi^* K_{\mathcal{F}} + \sum_i \varepsilon(B_i)\, \pi^* \left\{ \left(1 - \frac{1}{e_i}\right) B_i \right\}$$

while of course $K_{\mathcal{F}}$ is equally the canonical on $\mathcal{X}'$. In order to keep the notation manageable, let's abusively talk about valuations $\mathcal{E}$ of $\mathcal{X}$ centered on geometric points ξ, since as we've observed this is well defined up to étale covers, which themselves don't effect the singularities. With this in mind the formula for the discrepancy of a valuation $\mathcal{E}$ lying over E in X with multiplicity e becomes,

$$\frac{1}{e}\, a_{\tilde{\mathcal{F}}}(\mathcal{E}) = a_{\mathcal{F}}(E) - \sum_i \varepsilon_i(E) \left(1 - \frac{1}{e_i}\right) \nu_i(E) + \varepsilon(E) \left(1 - \frac{1}{e}\right) E\,.$$

Thus if we do the obvious thing and consider the boundary $B = \sum_i \left(1 - \frac{1}{e_i}\right) B_i$ on X, then,

$$(\mathcal{X}, \tilde{\mathcal{F}}) \text{ terminal at } \xi \Rightarrow (X, B, \mathcal{F}) \text{ terminal at } \pi(\xi)$$

since as we've seen, in this case $\varepsilon(E) = \varepsilon(\mathcal{E}) = 0$. For the same reason, canonical at ξ gives canonical at $\pi(\xi)$, and as per the proof of I.7.2, accident rather than design, gives log canonical at ξ implies $(X, B, \mathcal{F})$ log canonical at $\pi(\xi)$. The converse implications are of course trivial.

Consequently the key, and rather simple, proposition that keeps the singularities under control while running the minimal model programme may now be stated, viz:

I.7.3 Fact. *The following are equivalent,*

(I) *There is a smooth foliated stack $(\mathcal{X}, \tilde{\mathcal{F}})$ with moduli $(X, \mathcal{F})$ which at the non-scheme like points is terminal.*

(II) *The Gorenstien covering stack, $(\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$ is terminal at its non-scheme like points, and $\mathcal{X}'$ is everywhere smooth.*

(III) *$(X, \mathcal{F})$ is a normal $\mathbb{Q}$-foliated Gorenstien scheme, which is terminal at every point of* $\mathrm{sing}\,(X)$.

Proof. (II) $\Rightarrow$ (I) is trivial, (I) $\Rightarrow$ (III) is the above discussion, which leaves (III) $\Rightarrow$ (II), so drop the $'$ in the notation for the Gorenstien cover. In any case $\mathcal{X}$ is certainly smooth, indeed isomorphic to X, outside of $\mathrm{sing}\,(X)$, and again, by the above terminal at its non-scheme like points. Now suppose $\mathcal{X}$ wasn't smooth then algorithmic resolution I.4.1 guarantees the existence of valuation $\mathcal{E}$ with centre contained in $\mathrm{sing}\,(\mathcal{X})$ such that (employing the aforesaid minor abuse of notation) every vector field on $\mathcal{X}$ lifts without poles to $\mathcal{O}_{\mathcal{X},\mathcal{E}}$. However this would mean that the centre in question was no better than canonical, which is nonsense. □

Notice also that we can even employ these ideas to classify an appropriate class of smooth foliated stacks on which we will work,

I.7.4 Further Fact. *There is a one to one correspondence between the following sets,*

(A) *Smooth foliated stacks* $(\mathcal{X}, \tilde{\mathcal{F}})$ *with log-canonical singularities, terminal at their non-scheme like points, so in particular* $\tilde{\mathcal{F}}$ *transverse to the divisional ramification of* $\mathcal{X}$ *over* X.

(B) *Foliated* $\mathbb{Q}$*-Gorenstien, normal log-varieties* $(X, B, \mathcal{F})$ *with log-canonical singularities, no boundary component invariant, and terminal singularities at every point of* $\mathrm{sing}(X) \cup |D|$, *where* $|\ |$ *denotes support.*

The precise way to go between (A) and (B) will be explained in the course of the proof. However rather plainly B will be the boundary divisor associated to the ramification as previously introduced.

Proof. Indeed the said previous discussion, has already associated such a data (B) to an object of (A). To go from (B) to (A), we first take the Gorenstien covering stack $\pi : (\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$. By [V] 2.5, this is a smooth stack, and to each prime component B_i of the boundary, we have a corresponding integral divisorial stack $\mathcal{D}'_i$, from which a boundary stack $\mathcal{D}'$ with the obvious weights. Better still, $\mathcal{X}' \to X$ is almost étale, so the easy part of I.7.2 (although the less trivial converse is still true) gives,

(i) $\mathcal{X}'$ is terminal at its non-scheme like points

(ii) $(\mathcal{X}', \mathcal{D}', \mathcal{F})$ is terminal at $|\mathcal{D}'|$.

As such we may apply I.6.11, to conclude that $|\mathcal{D}'|$ is smooth, i.e. the $\mathcal{D}'_i$ are smooth and disjoint, and the weights e_i are finite. Now let $\coprod_\alpha U_\alpha \to X$ be a covering by appropriately small affines, such that $\mathcal{X}'$ is covered by classifying stacks of the form $[V_\alpha/G_\alpha]$, for $\pi_\alpha : V_\alpha \to U_\alpha$ an index 1-cover. By virtue of (ii) and I.6.11, any $\mathcal{D}'_i$ pulled back to V_α is a smooth (though perhaps not connected) divisor $B'_{i\alpha}$ invariant by G_α equal to the reduction of the pull-back of $B_i \cap U_\alpha$, with the total divisor $\sum_i B'_{i\alpha}$ smooth, albeit very definitely not connected if the boundary has more than 1 component. Now all the patches are affine, and without loss of generality sufficiently small to ensure that each $\mathcal{O}_{V_\alpha}(-B'_{i\alpha})$ is generated by a single equation $f_{i\alpha}$, of which we may simple extract the e_i^{th}-root for every i without in any way affecting smoothness since all the components are disjoint. Consequently we obtain coverings, $\tilde{V}_\alpha \to V_\alpha$ ramified over $\sum_i B'_{i\alpha}$, and in fact there would be no loss of generality in a priori insisting that for each α, there is at most one boundary component meeting U_α, say $i(\alpha)$, so that $U_\alpha = \tilde{V}_\alpha/\tilde{G}_\alpha$ where $\tilde{G}_\alpha$ is an extension of the form,

$$0 \to \mathbb{Z}/\mathbb{Z}e_{i(\alpha)} \to \tilde{G}_\alpha \to G_\alpha \to 0\,.$$

Regardless the construction clearly respects the patching data on $\mathcal{X}'$, i.e. we have a commutative diagram with the horizontal arrows étale,

$$\begin{tikzcd}
\tilde{V}_\alpha \arrow[d] & (\tilde{V}_\alpha \times_X \tilde{V}_\beta)^\# \arrow[l] \arrow[r] \arrow[d] & \tilde{V}_\beta \arrow[d] \\
V_\alpha & (V_\alpha \times_X V_\beta)^\# \arrow[l] \arrow[r] & V_\beta
\end{tikzcd}$$

where $\#$ denotes normalisation. As such the desired stack $\mathcal{X}$ is, therefore, the classifying stack of the groupoid,

$$\coprod_{\alpha,\beta}(\tilde{V}_\alpha \times_X \tilde{V}_\beta)^\# \rightrightarrows \coprod_\alpha \tilde{V}_\alpha \, .$$

$\square$

Of course none of this requires algebraicity of the moduli, and is wholly valid for arbitrary foliated (separated) stacks with appropriate singularities. However since the index is the same for the Zariski and étale topologies, we even have,

I.7.5 Remark. If the moduli of a foliated stack $(\mathcal{X}, \mathcal{F})$ of type (A) is a scheme then in fact $\mathcal{X}$ is a Zariski stack in the sense of being locally in the Zariski topology an orbifold of the form $[V/G]$, for G not just a group scheme, but a group.

While we may equally note,

I.7.6 Summary/Conclusion. Let $(X, B, \mathcal{F})$ be a normal $\mathbb{Q}$-Gorenstien foliated log-variety with finite weights on the non-invariant components of the boundary, enjoying terminal singularities on both the same as well as $\mathrm{sing}(X)$, along with log-canonical singularities elsewhere then,

There is a particularly nice class of smooth stacks $(\mathcal{X}, \mathcal{F})$, as described by (A), whose minimal model theory accurately reflects that of $(X, B, \mathcal{F})$.

I.8. Remarks on Intersection Theory

What follows are a few essentially obvious remarks on the structure of the Picard group of a stack over an algebraically closed field of characteristic zero, so let's say $\mathbb{C}$, along with the calculation of intersection numbers, amongst which the most important example will be the adjunction formula for invariant sub-stacks in the presence of a foliation. Nevertheless there seems to be a little bit of confusion in the literature as to the simplicity of what's involved, e.g. the emphasis in [Mu2] on Cohen Macauly coverings, while we wish to emphasise bundles as opposed to cycles as found in [V].

Regardless let's begin with the purely local case, i.e. U and V the spectra of strictly Henselian local rings, and G a finite group such that $\pi : [V/G] \to U$ is the moduli map of the classifying stack of some G-action on V. In this case the Serre-Hochschild spectral sequence yields an exact sequence,

$$0 \to H^1(G, \mathcal{O}_V^\times) \to H^1([V/G], \mathbb{G}_m) \to H^1(V, \mathbb{G}_m)^G \, .$$

The last group is of course 0, so the calculation of the local Picard group reduces to considering the exact sequence,

$$H^0(G, \mathcal{O}_V^\times/\mathbb{C}^\times) \to \mathrm{Hom}(G, \mathbb{C}^\times) \to H^1(G, \mathcal{O}_V^\times) \to H^1(G, \mathcal{O}_V^\times/\mathbb{C}^\times)$$

On the other hand $\mathcal{O}_V$ is a local ring, so $\mathcal{O}_V^\times/\mathbb{C}^\times$ is torsion free. Better still $\mathcal{O}_U$ is Henselian and an invariant element $\mathcal{O}_V^\times/\mathbb{C}^\times$ lifts to an element of $\mathcal{O}_V^\times$ satisfying a monic polynomial over $\mathcal{O}_U$, so

$$\mathbb{C}^\times \backslash \mathcal{O}_U^\times \to H^0(G, \mathcal{O}_V^\times/\mathbb{C}^\times)$$

is surjective. Equally, and wholly generally, torsion free divisible G-modules are acyclic, from which we conclude,

$$\mathrm{Hom}(G, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Pic}([V/G]) \, .$$

In particular if $\pi : \mathcal{X} \to X$ is the moduli, projective or not, and n the highest common multiple of the stabilisers of geometric points, then in the étale topology,

$$R^1\pi_*\,\mathbb{G}_m \otimes \mathbb{Z}\left[\frac{1}{n}\right] = 0\,.$$

Continuing to work in the étale topology, the Leray spectral sequence yields a short exact sequence,

$$0 \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^1_{\text{ét}}(\mathcal{X}, \mathbb{G}_m) \to H^0_{\text{ét}}(X, R^1\pi_*\,\mathbb{G}_m)$$

and so we certainly deduce,

$$\pi^* : \mathrm{Pic}(X) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \xrightarrow{\sim} \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right]\,.$$

Let us make explicit the inverse isomorphism. To this end let L be a line bundle on $\mathcal{X}$, and $\coprod_\alpha U_\alpha \to X$ a sufficiently fine étale covering with, $\coprod_\alpha [V_\alpha/G_\alpha] \to \mathcal{X}$ the corresponding covering of $\mathcal{X}$. Refining the cover as necessary we may suppose that we have isomorphisms,

$$s_\alpha : \mathcal{O}_{V_\alpha} \xrightarrow{\sim} L_{V_\alpha}$$

which in turn yield a G_α cocycle according to the rule,

$$G_\alpha \to \mathcal{O}^\times_{V_\alpha} : g \mapsto s^g_\alpha\, s^{-1}_\alpha$$

and without loss of generality we may suppose that this is zero modulo $\mathbb{C}^\times$, and so, $s^g_\alpha = \chi(g)\, s_\alpha$, for χ a character of G_α. Consequently,

$$Ns_\alpha := \prod_g s^g_\alpha : \mathcal{O}_{V_\alpha} \xrightarrow{\sim} L^{\otimes|G_\alpha|}_{V_\alpha}$$

is a G_α-invariant isomorphism, and $g_{\alpha\beta} = (Ns_\alpha)^{n/|G_\alpha|}\,(Ns_\beta)^{-n/|G_\beta|} \in \mathcal{O}^\times_{U_{\alpha\beta}}$ yields the desired cocycle on X.

In any case if we subsequently suppose that X is a scheme then we can make,

I.8.1 Provisional Definition. Denoting by π_* the inverse of the above isomorphism, put:

$$c_1 : \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \to \mathrm{CH}^1(X) \otimes \mathbb{Z}\left[\frac{1}{n}\right] : L \mapsto c_1(\pi_* L)\,.$$

The use of the word provisional is, of course, a result of the related notion of Cartier divisor on $\mathcal{X}$, i.e. the group,

$$\mathrm{Div}(\mathcal{X}) := \Gamma(\mathcal{X}, \mathcal{K}^\times_{\mathcal{X}}/\mathcal{O}^\times_{\mathcal{X}})\,.$$

As such to give a Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ amounts, for a sufficiently fine cover as above, to the giving of rational functions f_α on V_α, together with transition functions $g_{\alpha\beta} \in \mathcal{O}^*_{V_{\alpha\beta}}$. Consequently by sending a divisor to its transition functions, we obtain an exact sequence,

$$0 \to k(X)^\times \to \mathrm{Div}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})$$

where the last map is surjective if the moduli is a scheme without generic stabilisers, by way of

$$H^0(\mathcal{X}, \pi^* \mathcal{K}_X \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D})) = H^0(X, \mathcal{K}_X \otimes \pi_* \mathcal{O}_{\mathcal{X}}(D)),$$

at least supposing I.2.4, and even otherwise there's always a map in the right direction. Furthermore we have, following [V], a Chow group $\mathrm{CH}^1(\mathcal{X})$ of integral divisorial substacks modulo rational equivalence together with a cycle class map,

$$c_1 : \mathrm{Div}(\mathcal{X}) \to \mathrm{CH}^1(\mathcal{X})$$

defined in the usual way, i.e. send the local equations of the Cartier divisor f_α to the corresponding integral substacks with multiplicity. As ever this c_1 is injective modulo $k(X)^\times$, respectively surjective, according to whether $\mathcal{X}$ is normal, respectively factorial. Also continuing to follow op.cit. we have a map,

$$\pi_* : \mathrm{CH}^1(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \to \mathrm{CH}^1(X) \otimes \mathbb{Z}\left[\frac{1}{n}\right]$$

defined in the usual way, i.e. an integral substack $\mathcal{Y}$ of $\mathcal{X}$ with image Y in X, gets sent to $\deg(\mathcal{Y}/Y)Y$, where $\deg(\mathcal{Y}/Y) \in \mathbb{Z}\left[\frac{1}{n}\right]$ is as per op. cit. 1.15. In order, then, to turn our provisional definition I.8.1 into an actual definition we wish to check,

I.8.2 Fact. *The following diagram commutes,*

$$\begin{array}{ccc}
\mathrm{Div}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] & \longrightarrow & \mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] \\
\downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_1} \\
\mathrm{CH}(\mathcal{X}) \otimes \mathbb{Z}\left[\frac{1}{n}\right] & \xrightarrow[\pi_*]{} & \mathrm{CH}(X) \otimes \mathbb{Z}\left[\frac{1}{n}\right]
\end{array}$$

Proof. For a sufficiently fine cover as above let f_α be local equations for the Cartier divisor $\mathcal{D}$. By a minor variation of our previous considerations, we may, without loss of generality suppose $f_\alpha^g = \chi_\alpha(g)\, f_\alpha$, for $g \in G_\alpha$, and χ_α a character. As such $(Nf_\alpha)^{n/|G_\alpha|}$ are local equations for an étale Cartier divisor on X. On the other hand X is a scheme, so modulo refining the cover Hilbert 90 implies that the $(Nf_\alpha)^{n/|G_\alpha|}$ are actually rational functions on X, so that the image of the horizontal followed by the vertical is simply $\frac{1}{|G_\alpha|}\,\mathrm{div}(Nf_\alpha)$, independently of α.

The other direction is just an exercise in unravelling the definitions of [V], and indeed is purely local about the generic point of the image D in X of a component of the support of $\mathcal{D}$. In addition by op. cit. 3.7, or more correctly the proof, we can suppose both $\mathcal{X}$ and X are normal and irreducible. As such if H is the generic stabiliser of $\mathcal{X}$, then $\mathcal{X}/H$ is a classifying stack of the form $[\tilde{X}/\mathbb{Z}/n]$ where $\rho : \tilde{X} \to X$ is the extraction of a n^{th}-root of D. Consequently if $\mathcal{D}$ has multiplicity m at the said root, then the said definition (op. cit. 3.6) gives,

$$\pi_*\, c_1(\mathcal{D}) = \frac{m}{n(\#\, H)}\, D$$

which is indeed $\frac{1}{|G|}\,\mathrm{div}(f)$, for f a local equation of $\mathcal{D}$ on the étale neighbourhood $\tilde{X}$, and G the group affording the stack structure. $\Box$

Putting all of this together, we can certainly declare an end to the provisional character of I.8.1, and furthermore make,

I.8.3 Definition. Let $\mathcal{C}$ be a 1-dimensional stack with $\pi : \mathcal{C} \to C$ the necessarily projective moduli, then we have a map,

$$\deg : \mathrm{Pic}(\mathcal{C}) \to \mathbb{Q} : L \mapsto \deg(c_1(\pi_* L)) .$$

So that if $f : \mathcal{C} \to \mathcal{X}$ is a map to a stack. Then we have,

$$\mathrm{Pic}(\mathcal{X}) \underset{f^*}{\longrightarrow} \mathrm{Pic}(\mathcal{C}) \underset{\deg}{\longrightarrow} \mathbb{Q} : L \mapsto L \cdot_f \mathcal{C} := \deg(c_1(\pi_* f^* L)) .$$

Notice in particular that if we consider the square of maps,

$$\begin{array}{ccc} \mathcal{X} & \underset{f}{\longleftarrow} & \mathcal{C} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \underset{f}{\longleftarrow} & C \end{array}$$

and view L as a $\mathbb{Q}$-divisor on X, then provided every generic point has trivial stabiliser the degree $L \cdot_f C$ coincides with that defined in the usual way (i.e. $L^{\otimes n}$ is Cartier, so $\frac{1}{n} \deg(f^* L^{\otimes n})$ is well defined) by virtue of the identity $f^* \pi^* = \pi^* f^*$ at the level of Pic. Nevertheless there is an assumption involved here, so let us make,

I.8.4 Warning. There will arise situations in which $\pi : \mathcal{C} \to C$ does not have a trivial stabiliser at each of its generic points, so that for D a $\mathbb{Q}$-Cartier divisor on X which is Cartier on $\mathcal{X}$ it will not be the case that,

$$D \cdot_f \mathcal{C} = D \cdot_f C .$$

On the other hand if C is irreducible, and G the generic stabiliser, then:

$$D \cdot_f \mathcal{C} = \frac{1}{|G|} D \cdot_f C$$

so that the important property of positivity/negativity of the intersection number is independent of whether we employ I.8.3 or the above more classical considerations.

As an example to illustrate the correctness of the above degree conventions, we may consider,

I.8.5 Example. (Riemann-Hurwitz) Let $\mathcal{C}$ be a smooth irreducible 1-dimensional stack with generic stabiliser G and moduli $\pi : \mathcal{C} \to C$. Furthermore consider $\mathcal{C}/G$ as an orbifold in the usual way, i.e. with signatures $e_i \in \mathbb{N}\backslash\{1\}$ at points $c_i \in C$, then,

$$\deg(\omega_{\mathcal{C}}) = \frac{1}{|G|} \left\{ \deg(\omega_C) + \sum_i \left(1 - \frac{1}{e_i} \right) \right\} .$$

In addition if $f : \mathcal{C}' \to \mathcal{C}$ is a map between such stacks, then,

$$\deg(\omega_{\mathcal{C}'}) = (\mathcal{C}' : \mathcal{C}) \deg(\omega_{\mathcal{C}}) + \deg(\mathrm{Ram}_f) .$$

Proof. In fact we have a map, $\pi^* \omega_C \to \omega_{\mathcal{C}}$, so $\omega_{\mathcal{C}}$ is actually a Cartier divisor, and if we consider c_i as points on $\mathcal{C}$ then, at the level of $\mathrm{CH}^1(\mathcal{X})$,

$$c_1(\omega_{\mathcal{C}} \otimes \pi^* \omega_C^\vee) = \sum_i (e_i - 1)\, c_i$$

and of course each c_i has degree $\frac{1}{|G|\,e_i}$, so this establishes the first part. The second is immediate by the very definition of Ram_f, although what should be born in mind is that,

$$(\mathcal{C}' : \mathcal{C}) = \frac{\# G'}{\# G}\,(C' : C)$$

where G', C' etc. have the obvious meaning. $\square$

In any case let us progress from these essentially obvious considerations, to the equally obvious but rather more pertinent adjunction formula. The set up is, of course, that $(\mathcal{X}, \mathcal{F})$ is a foliated-Gorenstien stack, and $f : \mathcal{C} \to \mathcal{X}$ a smooth invariant 1-dimensional stack not wholly contained in the singular locus $\mathcal{Z}$. We have a map,

$$\Omega_{\mathcal{X}} \to K_{\mathcal{F}} \cdot \mathcal{I}_{\mathcal{Z}} \to 0$$

and whence an embedding,

$$i : \mathrm{Bl}_{\mathcal{Z}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Proj}\left(\bigoplus_{n \geq 0} K_{\mathcal{F}}^n \cdot \mathcal{I}_{\mathcal{Z}}^n\right) \hookrightarrow \mathbb{P}(\Omega_{\mathcal{X}})\,.$$

In particular if ρ denotes the projection of either of these stacks to $\mathcal{X}$, with L the tautological bundle on the latter and $\mathcal{E}$ the exceptional divisor on the former, then:

$$i^* L = \rho^* K_{\mathcal{F}}(-\mathcal{E})\,.$$

Furthermore the derivative $f' : \mathcal{C} \to \mathbb{P}(\Omega_{\mathcal{X}})$ factors generically through the blow up, and since everything is irreducible this is true everywhere, so in fact,

$$f^* K_{\mathcal{F}} = (f')^* L + \tilde{f}^* \mathcal{E}$$

where of course $\tilde{f}$ is the lifting of f to the blow up. As such we introduce,

I.8.6 Definition. Let things be as above then the Segre class of f at $\mathcal{Z}$ is defined to be,

$$s_{\mathcal{Z}}(f) = \deg(\tilde{f}^* \mathcal{E})\,.$$

This is of course wholly independent of any foliation hypothesis, and is an easily computed invariant of a purely local character, e.g. if $f^{-1}(\mathcal{Z})$ is a single smooth scheme like point c whose image in $\mathcal{X}$ is also scheme like, with $a_1, \ldots, a_m$ local equations for $\mathcal{Z}$ about $f(c)$ then,

$$s_{\mathcal{Z}}(f) = \min_i \mathrm{ord}_c(f^* a_i)\,.$$

In addition $(f')^* L$ is the image of the natural map,

$$f^* \Omega_{\mathcal{X}} \twoheadrightarrow (f')^* L \hookrightarrow \omega_{\mathcal{C}}$$

so by the very definition of ramification, we obtain,

I.8.7 Fact. (Adjunction Formula) *Everything as above, including conventions on degrees, then:*

$$K_{\mathcal{F}} \cdot_f \mathcal{C} = \deg(\omega_{\mathcal{C}}) + s_{\mathcal{Z}}(f) - \deg(\mathrm{Ram}_f)$$

Now there is a critical, and wholly specific to characteristic to zero, adjunction inequality. To state it properly observe that every closed geometric point of $\mathcal{C}$ is a Cartier divisor, so that $f^{-1}(\mathcal{Z})_{\mathrm{red}}$ is a well defined Cartier divisor on $\mathcal{C}$, so that with this in mind we have,

I.8.8 Sub-Fact. (Adjunction Inequality) *Again things as above then,*

$$K_{\mathcal{F}} \cdot_f \mathcal{C} \geq \deg(\omega_{\mathcal{C}}) + \deg f^{-1}(\mathcal{Z})_{\mathrm{red}} \, .$$

Proof. By the Frobenius theorem (cf. II.1 for the proper generality) $f(\mathcal{C})$ is smooth away from $\mathcal{Z}$, so identifying $\tilde{f}^*\mathcal{E}$ with $f^{-1}(\mathcal{Z})$ it will suffice to prove the inequality of Cartier divisors,

$$f^{-1}(\mathcal{Z}) \geq \mathrm{Ram}_f + f^{-1}(\mathcal{Z})_{\mathrm{red}} \, .$$

This is a purely local question about closed points $c \in \mathcal{C}$, with $f(c) \in \mathcal{Z}$, so we may suppose that everything is scheme like. As such let y be a coordinate around c, $x_1, \ldots, x_n$, embedding coordinates around $f(c)$, with $a_1, \ldots, a_m$ local equations for $\mathcal{Z}$. By construction, $(a_1, \ldots, a_m) \subset (x_1, \ldots, x_n)$, so that,

$$f^{-1}(\mathcal{Z}) \geq (\min_i \mathrm{ord}_c(f^* x_i))[c] \, .$$

On the other hand,

$$\min_i \mathrm{ord}_c(f^* x_i) = \mathrm{ord}_c(\mathrm{Ram}_f) + 1$$

so we're done. □

As such we have,

I.8.9 Corollary/Remarks. *Certainly then if* $K_{\mathcal{F}} \cdot_f \mathcal{C} < 0$ *then* $\deg(\omega_{\mathcal{C}}) < 0$*, and* $\mathcal{C}$ *is a rational stack. Manifestly we could list in terms of various signatures the possibilities for such curves in terms of the non-scheme like points, order of ramification, etc. The cleanest case, however, is when* $f^{-1}(\mathcal{Z})$ *is scheme like, so that* $\deg f^{-1}(\mathcal{Z})_{\mathrm{red}} \geq 1$*, and the number of possibilities is dramatically reduced.*

I.9. Weighted Projective Stacks

The normal definition of a weighted projective space, $\mathbb{P}(a_0, \ldots, a_r)$ is the quotient of $\mathbb{P}^r$ by the natural coordinate action of $\mu_{a_0} \times \cdots \times \mu_{a_r}$. This might, perhaps, lead to the belief that weighted projective stack simply means the classifying stack of such a quotient. Such a supposition would, however, be wholly false. The actual objects in question have a couple of different flavours, of which the more important one is more properly to be considered the higher dimensional generalisation of the so called bad orbifold which has moduli $\mathbb{P}^1$ and non-trivial automorphism group at precisely one point. Let us, therefore, proceed from this latter standpoint by way of,

I.9.1 Definition. Let $a_1, \ldots, a_r \in \mathbb{N}$ be given, and denote by $\mu_{\underline{a}}$ the product of multiplicative groups $\mu_{a_1} \times \cdots \times \mu_{a_r}$. We first of all form the weighted projective space, $\mathbb{P}(1, a_1, \ldots, a_r)$ by way of the standard coordinate action,

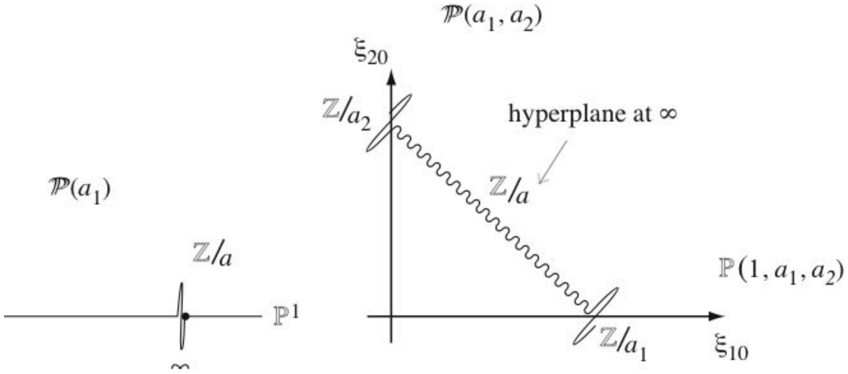
$$\mu_{\underline{a}} \times \mathbb{P}^r \to \mathbb{P}^r : (\zeta_1, \ldots, \zeta_r) \times [X_0, \ldots, X_r] \mapsto [X_0, \zeta_1 X_1, \ldots, \zeta_r X_r] \, .$$

Over $\mathbb{P}(1, a_1, \ldots, a_r)$ there is a unique stack, $\pi : \boldsymbol{\mathcal{P}} \to \mathbb{P}(1, a_1, \ldots, a_r)$ which is smooth, and such that any other smooth stack with the same moduli factors through $\boldsymbol{\mathcal{P}}$, cf. the proof of [V] 2.5. Furthermore on $\boldsymbol{\mathcal{P}}$ there is a smooth connected divisor $\boldsymbol{\mathcal{P}}_\infty$, the hyperplane at ∞, corresponding to the coordinate $X_0 = 0$, we extract an a^{th}-root of this divisor, where $a = gcd(a_1, \ldots, a_r)$, and the stack so obtained will be called a weighted projective stack, and denoted $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$. Explicitly in coordinates, we have for each $0 \leq j \leq r$, the j^{th}-coordinate patch isomorphic to $\mathbb{A}^r$ with standard coordinate functions, $\xi_{ij} = (X_i/X_j)^{a_i}$, $0 \leq i \leq r$, $i \neq j$, and $a_0 = 1$, together with a μ_{a_j}-action according to the rule,

$$\mu_{a_j} \times \mathbb{A}^r : \zeta_j \times (\xi_{ij}) \mapsto (\zeta_j^{-a_i} \xi_{ij})$$

which yield Zariski open substacks $[\mathbb{A}^r/\mu_{a_j}]$ of $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$. The patching maps are the obvious ones deduced from these on $\mathbb{P}^r$, or equivalently the normalised fibre product over $\mathbb{P}(1, a_1, \ldots, a_r)$.

Pictures of $\boldsymbol{\mathcal{P}}(a_1)$, and $\boldsymbol{\mathcal{P}}(a_1, a_2)$ should illustrate the structure pretty clearly,



Basically the weighted projective stacks $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$ enjoy just about every property of a standard projective space $\mathbb{P}^r$ with the obvious exception of scheme-ness. A relevant place to start, which in turn helps in other ways is,

I.9.2 Fact. *With coordinates as above, the foliation by curves on $\mathbb{P}^r$ corresponding to the pencil of lines through the origin $[1, 0, \ldots, 0]$ descends to a foliation denoted $\mathcal{R}$ (for radial in foliation parlance, or perhaps rational in an algebraic context) on $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$, and the resulting foliated stack, $(\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r), \mathcal{R})$, will be called a linearly foliated weighted projective stack. The foliation in question has a unique singular point at the necessarily scheme like origin, where in the above coordinate system on the $X_0 \neq 0$ patch it is generated by the vector field,*

$$\partial = \sum_{i=1}^{r} a_i \, \xi_{i0} \, \frac{\partial}{\partial \xi_{i0}} \, .$$

The tangent bundle of this foliation (strictly speaking the pre-foliation unsaturated at the origin if $r = 1$) will be denoted $\mathcal{O}_{\underline{a}}(1)$. Notice further that if we introduce the linear coordinate stacks $\mathcal{L}_j$ (i.e. $X_i = 0$, for $i \neq j$, $1 \leq j \leq r$) then by the adjunction formula,

$$\mathcal{O}_{\underline{a}}(1) . \, \mathcal{L}_j = \frac{1}{a_j} \, .$$

Unsurprisingly our previous considerations on intersection theory yield,

I.9.3 Fact.

$$\mathrm{Pic}(\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)) = \mathbb{Z} \cdot \mathcal{O}_{\underline{a}}(1)\,.$$

Proof. In the first place consider the moduli space $P = \mathbb{P}(1, a_1, \ldots, a_r)$. This has quotient singularities, and is in fact $\mathbb{Z}\left[\frac{1}{m}\right]$ factorial, where m is the lowest common multiple of the a_i/a. Better still the complement of the hyperplane at ∞, P_∞, say is an $\mathbb{A}^r$, so the usual short exact sequence of Chow groups yields,

$$\mathrm{CH}^0(P_\infty) \twoheadrightarrow \mathrm{CH}^1(P)$$

so that up to m torsion, $\mathrm{Pic}(P)$ is generated by P_∞, and so up to ma torsion $\mathrm{Pic}(\boldsymbol{\mathcal{P}}(a_1, \ldots a_r))$ is generated by $\mathcal{O}_{\underline{a}}(1)$.
To proceed to a conclusion, we avail ourselves of the natural maps (as we will do throughout this section, so we'll even fix notation),

$$\mathbb{P}^r \xrightarrow[\sigma]{} [\mathbb{P}^r/\mu_{\underline{a}}] \xrightarrow[\rho]{} \boldsymbol{\mathcal{P}}(\underline{a})\,.$$

In the first place we can relate $\mathrm{Pic}\,\boldsymbol{\mathcal{P}}(\underline{a})$ to $\mathrm{Pic}[\mathbb{P}^r/\mu_{\underline{a}}]$ by the Leray spectral sequence, and $\mathrm{Pic}[\mathbb{P}^r/\mu_{\underline{a}}]$ to $\mathrm{Pic}\,\mathbb{P}^r = \mathbb{Z}\,\mathcal{O}_{\mathbb{P}^r}(1)$ (which incidentally is easily seen to be $\sigma^*\rho^*\mathcal{O}_{\underline{a}}(1)$ by virtue of the intersection numbers with coordinate lines) by the Hochschild-Serre spectral sequence. Putting this together amounts to a commutative diagram with exact rows and columns of the form,

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & \mathrm{Hom}(\mu_{\underline{a}}, \mathbb{C}^\times) \arrow[d] & \\
0 \arrow[r] & \mathrm{Pic}\,\boldsymbol{\mathcal{P}}(\underline{a}) \arrow[r] & \mathrm{Pic}([\mathbb{P}^r/\mu_{\underline{a}}]) \arrow[r] \arrow[d] & H^0_{\text{ét}}(\boldsymbol{\mathcal{P}}(\underline{a}), R^1_{\rho_*}\mathbb{G}_m) \\
 & & \mathrm{Pic}(\mathbb{P}^r) \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

In particular $\mathrm{Pic}\,\boldsymbol{\mathcal{P}}(\underline{a})$ modulo its torsion is certainly generated by $\mathcal{O}_{\underline{a}}(1)$. Equally if L is a torsion bundle on $\boldsymbol{\mathcal{P}}(\underline{a})$, then ρ^*L comes from a character of $\mu_{\underline{a}}$. Now the bundles that come from characters, which we'll slightly abusively denote $\mathrm{Hom}(\mu_{\underline{a}}, \mathbb{C}^\times)$, restrict injectively to the bundles on the distinguished affine $\mathbb{A}^r$ given by $X_0 \neq 0$, indeed we even have,

$$\mathrm{Res}^* : \mathrm{Hom}(\mu_{\underline{a}}, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Pic}[\mathbb{A}^r/\mu_{\underline{a}}]\,.$$

On the other hand, $\mathrm{Res}^*\rho^* = \rho^*\,\mathrm{Res}^*$, and the corresponding $\mathbb{A}^r$ in $\boldsymbol{\mathcal{P}}(\underline{a})$ is wholly scheme like, so there are no torsion bundles on $\boldsymbol{\mathcal{P}}(\underline{a})$, from which we conclude. $\square$

In a similar vein, albeit rather more specific to characteristic zero, we have:

I.9.4 Fact. *$\mathcal{P}(\underline{a})$ is simply connected.*

Proof. For $\pi_1^{\text{ét}}$ this is clear, since $\mathbb{A}^r$ is a Zariski open, and $\mathcal{P}(\underline{a})$ is smooth, so any connected étale cover has the same generic point. More generally for a possibly analytic cover $h : \mathcal{X} \to \mathcal{P}(\underline{a})$ with covering group Γ, we have $\# \Gamma$ disjoint copies of our $\mathbb{A}^r$. A priori these might not admit a Zariski closure, but by the simple expedient of lifting to $\mathcal{X}$ the 1-dimensional and indeed simply connected, stacks invariant by $\mathcal{R}$, then taking the stack covered by those through a given lifting of the origin, we see that they do, so again $\mathcal{X}$ would have too many generic points if it were non-trivial. $\square$

Now let's proceed to some cohomological calculations. The best place of course to start is the H^0, where we'll state the result by way of,

I.9.5 Fact. *We have an isomorphism of graded rings,*

$$\bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{P}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) \xrightarrow{\sim} \mathbb{C}\,[Y_0, \ldots, Y_r]$$

where on the right Y_i corresponds to a section of $\mathcal{O}_{\underline{a}}(a_i)$, so in particular has degree a_i.

Proof. Arguing as in I.3.4, we see that σ_* is exact, supposing, as ever, I.2.6 so indeed for any i and n,

$$H^i(\mathcal{P}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) = H^i([\mathbb{P}^r/\mu_{\underline{a}}], \sigma^*\mathcal{O}_{\underline{a}}(n))$$

and we can calculate the latter using the Hochschild-Serre spectral sequence,

$$H^i(\mu_{\underline{a}}, H^j(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))) \Rightarrow H^{i+j}([\mathbb{P}^r/\mu_{\underline{a}}], \sigma^*\mathcal{O}_{\underline{a}}(n))$$

which also degenerates, so that unsurprisingly,

$$H^i(\mathcal{P}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) = H^i(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))^{\mu_{\underline{a}}}.$$

As such in terms of standard coordinates, we obtain our assertion by putting $Y_i = X_i^{a_i}$. $\square$

Manifestly we also have

I.9.6 Further Fact.

$$H^i(\mathcal{P}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) = 0\,, \quad 1 < i < r\,, \quad \forall\, n \in \mathbb{Z}\,.$$

As to the top dimensional cohomology we have, of course, an explicit description in terms of negative monomials, i.e.

I.9.7 Fact. *In terms of the monomials Y_i of I.9.5, the top cohomology is, $n \in \mathbb{Z}$,*

$$H^r(\mathcal{P}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) = \bigoplus_{\substack{\sum_{i=0}^{r} a_i s_i = n \\ s_j \in \mathbb{Z}_{<0}}} \mathbb{C} \cdot Y_0^{s_0} \ldots Y_r^{s_r}\,.$$

In particular we have the re-assuring fact that the largest n for which cohomology occurs is $-\sum_i a_i$, that the dimension of the said group is 1, and indeed $\omega_{\mathcal{P}(\underline{a})} = \mathcal{O}_{\underline{a}}\left(-\sum_i a_i\right)$. Consequently we have in the usual and wholly explicit fashion,

I.9.8 Fact. *The natural pairing of weighted monomials,*

$$H^0(\mathcal{P}(\underline{a}), \mathcal{O}_{\underline{a}}(n)) \times H^r(\mathcal{P}(\underline{a}), \omega_{\mathcal{P}(\underline{a})}(-n)) \to H^r(\mathcal{P}(\underline{a}), \omega_{\mathcal{P}(\underline{a})})$$

is a perfect pairing.

Probably the most important example in what follows is the so called bad orbifold $\mathcal{P}(a_1)$, so let's consider:

I.9.9 Example. $\mathcal{P}(a_1)$ in detail. In this case we'll write $\mathcal{O}\left(\frac{1}{a}\right)$ for $\mathcal{O}_{\underline{a}}(1)$, $a = a_1$, evidently. So the cohomology calculation yields,

$$h^0\left(\mathcal{P}(\underline{a}), \mathcal{O}\left(\frac{n}{a}\right)\right) = \left[\frac{n}{a}\right] + 1\,, n \geq 0$$
$$h^1\left(\mathcal{P}(\underline{a}), \mathcal{O}\left(\frac{n}{a}\right)\right) = -\left[\frac{n}{a}\right] - 1\,, n \leq -(a+1)\,.$$

It will be useful to be able to recognise global sections of $\mathcal{O}\left(\frac{n}{a}\right)$ in terms of the standard $\mathbb{A}^1$-coordinate patches U, scheme like neighbourhood of 0, V neighbourhood of the non-scheme like point at ∞. To this end let x be a standard coordinate on the former, and ξ on the latter, and identify $U \times_{\mathcal{P}(a)} V$ with $\mathbb{G}_m$ having coordinate t such that, $x = t^a$, and $\xi = 1/t$, then in standard notation, the transition function for $\mathcal{O}\left(\frac{n}{a}\right)$ is

$$g_{VU} = t^n\,.$$

Consequently a global section of $\mathcal{O}\left(\frac{n}{a}\right)$ amounts to two functions f_0, and f_∞ such that,

$$f_0(t^a) = t^n f_\infty\left(\frac{1}{t}\right)\,.$$

There is also a patching condition on $V \times_{\mathcal{P}(a)} V$, but this is implied by the above. In any case the important thing to recognise is that in terms of x, the transition function is $x^{1/a}$. The not unuseful exercise of unravelling definitions out of the way, let's consider a vector bundle E on $\mathcal{P}(\underline{a})$, and proceed by the usual procedure of induction on the rank to prove that it splits. By the exactness of $\pi : \mathcal{P}(\underline{a}) \to \mathbb{P}^1$, E has meromorphic sections so we can find,

$$s \in H^0\left(\mathcal{P}(\underline{a}), E \otimes \mathcal{O}\left(-\frac{n}{a}\right)\right)$$

with n maximal. Everything is smooth, so by induction we have an exact sequence,

$$0 \to \mathcal{O}\left(\frac{n}{a}\right) \to E \to \bigoplus_i \mathcal{O}\left(\frac{m_i}{a}\right) \to 0\,.$$

Now suppose $m_j \geq n+1$, for some j, then we would have an exact sequence,

$$H^0\left(\mathcal{P}(a), E\left(-\frac{(n+1)}{a}\right)\right) \to \bigoplus_i H^0\left(\mathcal{P}(a), \mathcal{O}\left(\frac{m_i - a - 1}{a}\right)\right)$$
$$\to H^1\left(\mathcal{P}(a), \mathcal{O}\left(-\frac{1}{a}\right)\right)$$

however the latter group is zero, so this would contradict the maximality of n, and we deduce that E splits by virtue of I.9.7. Of course just as in the $\mathbb{P}^1$-case, there's nothing unique about the splitting, but rather the unique thing is the $H-N$ filtration, or what amounts to the same thing in this particular case, the weights,

$$\frac{n}{a} = \frac{m_1}{a} \geq \frac{m_2}{a} \geq \cdots \geq \frac{m_k}{a}$$

or equivalent if d_i/a, $i \leq 1$ is a repetition free list of the same and ι_i the number of occurences, the filtration,

$$E = E^{\iota} \supset E^{\iota-1} = \bigoplus_{i \leq \iota - 1} \mathcal{O}\left(\frac{d_i}{a}\right)^{\oplus \iota_i} \cdots \supset E^1 = \mathcal{O}\left(\frac{d_1}{a}\right)^{\oplus \iota_1} \supset E^0 = 0$$

In any case the example over, this pretty much completes what we need to know about weighted projective stacks, beyond:

I.9.10 Fact. *All of this goes through verbatim (excepting* I.9.4*) in the relative situation of a bundle, $\nu : \mathcal{P} \to S$ of weighted projective stacks over a scheme, or algebraic space, or for that matter stack, provided S itself is over $\mathbb{C}$ (or a field of characteristic prime to the weights) i.e. $R^1\nu_*\mathbb{G}_m$ is generated by some $\mathcal{O}_{\underline{a}}(1)$, $\bigoplus_n R^0\nu_*\mathcal{O}_{\underline{a}}(n)$ is a coherent sheaf of weighted graded rings, etc.*

I.9.11 Proof/Remarks/Warning. Granted the calculation of Pic used CH at some point, so strictly speaking its only verbatim over a $\mathbb{Q}$-factorial base. Nevertheless, what's at stake is only the calculation mod torsion so [K1] II.4.1.9 could be used instead.

Note however,

There may fail to exist a global covering by a $\mathbb{P}^r$-bundle. Our assertion is, on the other hand, of a purely local character where such a covering will certainly exist, so this is irrelevant to the claim.

Now let's turn to the general case. We'll retake the notation of Definition I.9.1, up to some superficial changes to encourage consistency with further chapters and thus emphasise the different roles that the two definitions enjoy in the minimal model theory, namely:

I.9.12 Definition. Let $b_1, \ldots, b_t \in \mathbb{N}$ be given, put $\mu_{\underline{b}} = \mu_{b_1} \times \cdots \times \mu_{b_t}$, and consider the standard coordinate action on $\mathbb{P}^{t-1}$, i.e.

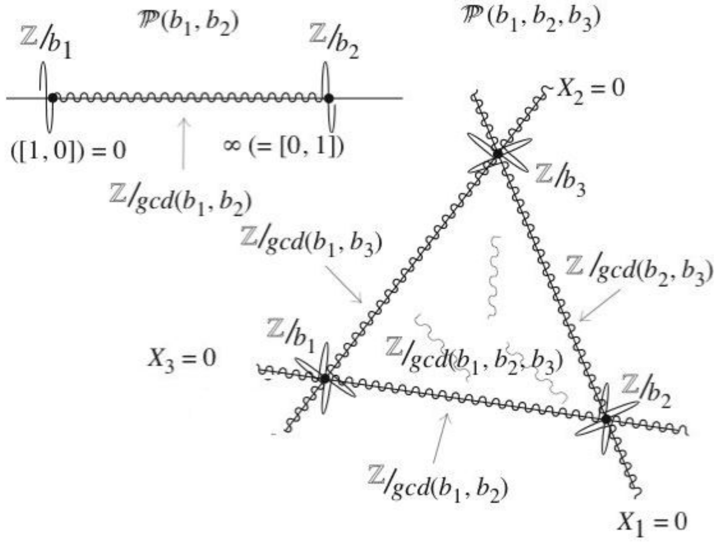
$$\mu_{\underline{b}} \times \mathbb{P}^{t-1} \to \mathbb{P}^{t-1} : (\zeta_1, \ldots, \zeta_t) \times [X_1, \ldots, X_t] \mapsto [\zeta_1 X_1, \ldots, \zeta_t X_t]$$

then the generalised weighted projective stack $\mathcal{P}^{t-1}(b_1, \ldots, b_t)$ has as moduli the weighted projective space $\mathbb{P}(b_1, \ldots, b_t)$, while the system of affine coordinate patches $\mathbb{A}^{t-1}$, $1 \leq j \leq t$, with standard coordinates $\xi_{ij} = (X_i/X_j)^{a_i}$, $i \neq j$, together with μ_{b_j}-action,

$$\mu_{b_j} \times \mathbb{A}^{t-1} : \zeta_j \times \xi_{ij} \mapsto \zeta_j^{-b_i} \xi_{ij}$$

yield Zariski open substacks $[\mathbb{A}^{t-1}/\mu_{b_j}]$, together with the not wholly obvious patching maps, i.e. strictly speaking to turn this data into a stack requires some choices (this was also true in op. cit., but the effect was negligible) amongst which is a splitting of, $\mu_b \to \mu_{\underline{b}}$, where $b = gcd(b_1, \ldots, b_t)$, and we'll denote the corresponding character of $\mu_{\underline{b}}$ by χ.

Again let's draw the pictures in dimensions 1 and 2, i.e.



So in particular a generalised weighted projective stack $\mathcal{P}^{t-1}(b_1, \ldots, b_t)$ is permitted to be everywhere non-scheme like with stabiliser group $\mathbb{Z}/b$, $b = gcd(b_1,..\, , b_t)$. We list the salient facts, whose proofs are just a minor variation of the previous ones, viz:

I.9.13 Facts. (Characteristic 0, or I.2.6) *Let $\mathcal{P}^{t-1}(b_1, \ldots b_t)$ be a generalised weighted projective stack, with χ the character implicit in the patching data of the definition. Then,*

(i) *There is a map, $\sigma : [\mathbb{P}^{t-1}/\mu_{\underline{b}}] \to \mathcal{P}^{t-1}(b_1, \ldots, b_t)$ and a bundle $\mathcal{O}_{\underline{b}}(1)$ on the latter, such that the free $\oplus$ torsion decomposition of the Picard group is,*

$$\mathrm{Pic}(\mathcal{P}^{t-1}(b_1, \ldots, b_t)) = \mathbb{Z}\mathcal{O}_{\underline{b}}(1) \oplus \mathbb{Z}\chi$$

and indeed $\sigma^*\mathcal{O}_{\underline{b}}(1) = \mathcal{O}(b)$, where $\mathcal{O}(1)$ is the standard tautological bundle on $\mathbb{P}^{t-1}$ descended to the classifying stack.

(ii) *The fundamental group of $\mathcal{P}^{t-1}(b_1, \ldots, b_t)$ is $\mathbb{Z}/b$.*

(iii) *If $Y_i = X_i^{b_i}$, and $S = \mathbb{C}[Y_1, \ldots, Y_t]$ the graded algebra assigning to Y_i weight b_i, then for $n \in \mathbb{Z}$,*

$$\bigoplus_{m\in\mathbb{Z}} H^0(\mathcal{P}^{t-1}(b_1, \ldots, b_t), \mathcal{O}_{\underline{b}}(m) \otimes \chi^n) = \begin{cases} S, & n \equiv 0(b) \\ 0, & \textit{otherwise} \end{cases}$$

(iv) *For any $0 < q < t-1$, and $m, n \in \mathbb{Z}$,*

$$H^q(\mathcal{P}^{t-1}(b_1, \ldots, b_t), \mathcal{O}_{\underline{b}}(m) \otimes \chi^n) = 0\,.$$

(v) *If ω is the canonical bundle, then $\omega = \mathcal{O}_{\underline{b}}(-b_1/b \cdots - b_t/b)$, so in particular, $\sigma^*\omega = \mathcal{O}(-b_1 \cdots - b_t)$.*

(vi) *The cohomology is given by the negative monomials, i.e. for* $m, n \in \mathbb{Z}$

$$H^{t-1}(\boldsymbol{\mathcal{P}}^{t-1}(\underline{b}), \mathcal{O}_{\underline{b}}(m) \otimes \chi^n) = \begin{cases} \bigoplus_{s_i < 0} \mathbb{C}\, Y_1^{s_1} - Y_m^{s_m}, & \sum b_i s_i = mb < 0 \\ & n \equiv 0(b) \\ 0, & \textit{otherwise} \end{cases}$$

(vii) *In particular, we have an explicit duality pairing,* $m, n \in \mathbb{Z}$*,*

$$H^0(\boldsymbol{\mathcal{P}}^{t-1}(\underline{b}), \mathcal{O}_{\underline{b}}(m) \otimes \chi^n) \times H^{t-1}(\boldsymbol{\mathcal{P}}^{t-1}(\underline{b}), \mathcal{O}_{\underline{b}}(-m) \otimes \omega \otimes \chi^{-n})$$

$$\to H^{t-1}(\boldsymbol{\mathcal{P}}^{t-1}(\underline{b}), \omega)\,.$$

II. $K_{\mathcal{F}}$ NEGATIVE CURVES

II.1. Foliations as birational groupoids

In reality the point of view of a foliation as an integrable quotient of the cotangent sheaf is completely wrong. Rather a foliation should be considered as an infinitesimal equivalence relation outside of its singularities, and the equivalence of this definition to that involving linear 1^{st} order data as a non-trivial theorem (not withstanding the triviality of the proof) specific to characteristic zero. In any case let us begin by reviewing the equivalence, whence let X be a normal affine variety over $\mathbb{C}$ and $\mathcal{F}$ a smooth foliation on X. Notice that X may be singular, so $\mathcal{F}$ smooth means that for some (and indeed any) embedding of X in a smooth variety M the composition,

$$T_{\mathcal{F}} \to \mathcal{J}_X \to T_M \otimes \mathcal{O}_X$$

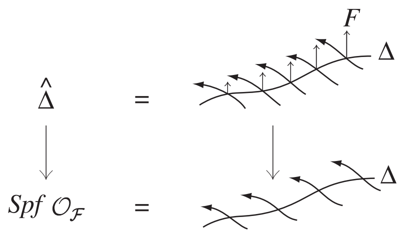
is an injection of bundles. Now consider the diagonal Δ in $X \times X$, with p_i the projections, and $p_2^* T_{\mathcal{F}}$ the foliation obtained by pull-back from the 2^{nd} direction. Dualising commutes with flat pull-back so this is notationally unambiguous, so that shrinking X as necessary we can find a local generator ∂ of $T_{\mathcal{F}}$ and $f \in I_\Delta$ such that $p_2^*\partial(f)$ is non-zero on X. We put $\delta = (p_2^*\partial(f))^{-1}\partial$, and for any function g on $X \times X$ define,

$$\tilde{g} := \sum_{n=0}^{\infty} (-1)^n \frac{f^n \delta^n(g)}{n!} \in \hat{\mathcal{O}}_\Delta := \varprojlim_n \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^n$$

then $\delta\tilde{g} = 0$, and better still if $\hat{\Delta}$ is the completion of $X \times X$ in Δ then the inclusion of rings,

$$\mathcal{O}_{\mathcal{F}} := \{h \in \hat{\mathcal{O}}_\Delta : \partial h = 0\} \subset \hat{\mathcal{O}}_\Delta$$

corresponds to a relatively smooth fibration of formal schemes,



such that the pull-back of the image of Δ in Spf $\mathcal{O}_{\mathcal{F}}$ is the corresponding infinitesimal equivalence relation, i.e. the formal sub-scheme of $\hat{\Delta}$ defined by the ideal generated by $\mathcal{O}_{\mathcal{F}} \cap \mathcal{I}_\Delta$ or equivalently the maximal sub-ideal of $\mathcal{I}_\Delta$ invariant by $\mathcal{F}$. Rather more picturesquelly what we have done is add a small germ in the $p_2^* T_{\mathcal{F}}$ direction for each point in the diagonal.

To extend this to stacks, even separated ones, is a little delicate since unless the stack is in fact an algebraic space the diagonal will fail to be an embedding. To remedy this it suffices to observe that we've actually been working in,

$$\mathfrak{P}_X := \operatorname{Spf} \mathcal{P}_X\,, \quad \mathcal{P}_X = \varprojlim_n \mathcal{P}_X^{(n)}$$

where $\mathcal{P}_X^{(n)}$ is Grothendieck's sheaf of n-jets consider as a nilpotent $\mathcal{O}_X$-algebra by way of the 1st-projection. Being by supposition of Deligne-Mumford type a stack $\mathcal{X}$ is defined by étale equivalence relations so there are well defined sheaves of nilpotent $\mathcal{O}_{\mathcal{X}}$-algebras, $\mathcal{P}_{\mathcal{X}}^{(n)}$ of n-jets, and of course idem, modulo replacing nilpotent by topologically so, for the inverse limit $\mathcal{P}_{\mathcal{X}}$. Equally the formation of the formal spectrum is a local construction, while both the projectors and the diagonal embedding patch, so we obtain an object which we summarise by way of,

II.1.1 Definition. The jet groupoid of a stack $\mathcal{X}$ is the formal stack,

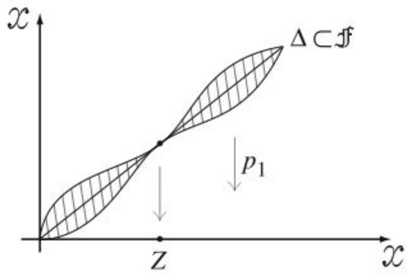
$$\mathfrak{P}_{\mathcal{X}} = \mathrm{Spf}\, \mathcal{P}_{\mathcal{X}} \rightrightarrows \mathcal{X}$$

with source map p_1, sink p_2, and identify the diagonal.

Notice in particular that the diagonal is actually embedded in the jet groupoid, so its worth emphasising what's happening. Specifically for a geometric diagonal point $x \times x$ in $\mathcal{X} \times \mathcal{X}$, its automorphism group is simply $\mathrm{Aut}(x) \times \mathrm{Aut}(x)$. Inside this group we have a copy of $\mathrm{Aut}(x)$ sitting diagonally. Now any attempt to define diagonal type subgroups of automorphisms for off diagonal points, and whence define an actual étale "neighbourhood" in which $\mathcal{X}$ embeds in some sort of diagonal way, is doomed to failure. At the infinitesimal level this can, however, be achieved.

Turning then to stacks foliated by curves, or indeed even foliated full stop, the corresponding foliations on étale neighbourhoods of the stack are again by supposition invariant by the corresponding étale groupoid so that we may once again apply the expedient of summary by way of definition, i.e.

II.1.2 Definition. Let $(\mathcal{X}, \mathcal{F})$ be a foliated stack, $\mathcal{Z}$ its singular locus, and $\mathcal{U} = \mathcal{X} \backslash \mathcal{Z}$ the smooth locus then the infinitesimal equivalence relation $\mathfrak{f} \rightrightarrows \mathcal{U}$ defined according to the correspondence which associates to $\mathcal{F}$ a formal subscheme of the jet groupoid will be denoted the smooth infinitesimal groupoid of $\mathcal{F}$. This construction may, however, fail catastrophically over $\mathcal{Z}$, i.e. consider:



then over $\mathcal{Z}$ we may have an essential singularity, so that the smallest closed formal sub-stack of $\mathfrak{P}_{\mathcal{X}}$ containing $\mathfrak{f}$ is $\mathfrak{P}_{\mathcal{X}}$ itself.

To remedy this latest difficulty we allow the possibility of birational groupoids, i.e. such that the identity map is simply birational. With this extra flexibility we can complete across the singularities. Specifically let $\tilde{\mathfrak{P}}_{\mathcal{X}}$ be the blow up of $\mathfrak{P}_{\mathcal{X}}$ in the diagonal embedding $\Delta(\mathcal{Z})$ of $\mathcal{Z}$ understood with any implied nilpotent structure on the singular locus. Now let $U \to \mathcal{X}$ be an étale neighbourhood of a geometric point $z \in \mathcal{Z}$ with $U \hookrightarrow M$ an embedding into a smooth. Consider coordinates $x_1, \ldots, x_n$ on M restricting to functions on U, then for $\mathcal{F}|_U$ Gorenstien, and shrinking U as necessary we may suppose that the foliation is defined by a vector field ∂ on U, which we write using the summation convention as,

$$\partial = a_i \frac{\partial}{\partial x_i}$$

so that $\mathcal{I}_{\mathcal{Z}}|_{\mathcal{U}} = (a_i)$. Now introduce x_i, y_i as coordinates on $U \times U$ obtained from our initial coordinates by way of 1^{st} and 2^{nd} pull-back respectively, and put $z_i = x_i - y_i$, then in z_i, x_i coordinates,

$$p_2^* \partial = p_2^* a_i \frac{\partial}{\partial y_i} = -p_2^* a_i \frac{\partial}{\partial z_i} .$$

Consequently if $\pi : \tilde{\mathfrak{P}}_{\mathcal{X}} \to \mathfrak{P}_{\mathcal{X}}$ is the projection, restricted to U on the $p_1^* a_i \neq 0$ patch,

$$\partial \left(\frac{z_i}{p_1^* a_i} \right) = \frac{-p_2^* a_i}{p_1^* a_i} = 1 + \frac{(p_1^* a_i - p_2^* a_i)}{p_1^* a_i} .$$

On the other hand the ideal of the diagonal embedding of $\mathcal{Z} \times_{\mathcal{X}} U$ has ideal $(p_1^* a_i, z_i)$ so on the proper transform $\tilde{\Delta}$ of Δ in $\tilde{\mathfrak{P}}_{\mathcal{X}}$ not only can we locate each point in some $p_1^* a_i \neq 0$ patch for an appropriate i, but indeed the function $z_i / p_1^* a_i$ in $I_{\tilde{\Delta}}$ enjoys a non-zero derivation with respect to $\pi^* \partial$. Better still we have blown up in a centre invariant by $p_2^* \mathcal{F}$ so the induced foliation $\widetilde{p_2^* \mathcal{F}}$ on $\tilde{\mathfrak{P}}_{\mathcal{X}}$ is both smooth in a neighbourhood of $\tilde{\Delta}$ and everywhere transverse to it. Whence we can just repeat our minor variant of the classical Frobenius theorem to obtain,

II.1.3 Fact/Definition/Summary. Let $(\mathcal{X}, \mathcal{F})$ be a foliated Gorenstien stack, then there is a formal substack $\tilde{\mathfrak{F}}$ of $\tilde{\mathfrak{P}}_{\mathcal{X}}$ together with projection maps, $p_i \circ \pi$, $i = 1$ or 2 defining a birational groupoid, i.e.

$$\tilde{\mathfrak{F}} \rightrightarrows \mathcal{X}$$

where the identity and composition are rational maps. In addition the projection $p_1 \circ \pi$ factors as,

$$\tilde{\mathfrak{F}} \to \tilde{\Delta} \to \mathcal{X}$$

with the former map relatively smooth of dimension 1. We call this structure the infinitesimal birational groupoid.

Notice in particular,

II.1.4 Fact. *There is an isomorphism,* $N_{\tilde{\Delta}/\tilde{\mathfrak{F}}} \xrightarrow{\sim} \mathcal{O}_{\tilde{\Delta}}(p_2^* T_{\mathcal{F}})$.

II.2. Chow's Lemma

We'll confine ourselves to that which is strictly necessary for applications. Our interest centres on smooth formal stacks $\mathfrak{F}$ whose underlying stack $\mathcal{C}$ is smooth of dimension 1. From our utilitarian point of view we'll confine ourselves to the case where $\dim \mathfrak{F} = 2$. Irrespectively there is a well defined normal bundle $N_{\mathcal{C}/\mathfrak{F}}$, and we make,

II.2.1 Definition. $\mathfrak{F}$ is a concave formal neighbourhood of $\mathcal{C}$ if $\deg(N_{\mathcal{C}/\mathfrak{F}}) > 0$.

Unsurprisingly the classical Chow lemma continues to hold, i.e.

II.2.2 Lemma. (Chow, Grauert et al.) *Let L be a line bundle on $\mathfrak{F}$ then there is a quadratic polynomial P_L, depending on L, such that for all $n \in \mathbb{N}$,*

$$h^0(\mathfrak{F}, L^{\otimes n}) \leq P_L(n)\,.$$

Proof. Let $\mathfrak{F}_m$ be the m^{th}-thickening of $\mathcal{C}$ then we have an exact sequence,

$$0 \to \mathrm{Sym}^m N^{\vee}_{\mathcal{C}/\mathfrak{F}} \to \mathcal{O}_{\mathfrak{F}_{m+1}} \to \mathcal{O}_{\mathfrak{F}_m} \to 0\,.$$

On the other hand if $h^0(\mathcal{C}, L^n \otimes \mathrm{Sym}^m N^{\vee}_{\mathcal{C}/\mathfrak{F}}) \neq 0$, then,

$$m \deg(N_{\mathcal{C}/\mathfrak{F}}) \leq n \deg_{\mathcal{C}}(L)\,.$$

Consequenlty for any $n \in \mathbb{N}$,

$$H^0(\mathcal{O}_{\mathfrak{F}_{m+1}} \otimes L^n) \hookrightarrow H^0(\mathcal{O}_{\mathfrak{F}_m} \otimes L^n)$$

is injective, provided $m > \dfrac{n \deg_{\mathcal{C}}(L)}{\deg(N_{\mathcal{C}/\mathfrak{F}})}$, so that if M is the aforesaid lower bound then,

$$h^0(\mathfrak{F}, L^{\otimes n}) = \varprojlim_m h^0(\mathfrak{F}_m, L^{\otimes n}) \leq \sum_{k=0}^{M} h^0(\mathcal{C}, L^n \otimes N^{-k}_{\mathcal{C}/\mathfrak{F}})\,.$$

Moreover we can find a map, $\rho : C \to \mathcal{C}$ from an honest curve, while for any bundle E, $h^0(\mathcal{C}, E) \leq h^0(C, \rho^* E)$, so we conclude by Riemann-Roch. □

II.3. Bend & Break

We are now in a position to extend the results of [I], so to this end let $(\mathcal{X}, \mathcal{F})$ be a foliated Gorenstien stack with projective moduli space $\pi : \mathcal{X} \to X$, and H an ample bundle on the latter. As ever the basic object of study is $K_{\mathcal{F}}$ negative curves on $\mathcal{X}$, i.e. maps $f : \mathcal{C} \to \mathcal{X}$ from smooth 1-dimensional stacks such that $K_{\mathcal{F}} \cdot_f \mathcal{C} < 0$. We impose further the condition that f does not factor through the singular locus $\mathcal{Z} = \mathrm{sing}(\mathcal{F})$. Consequently if we consider the infinitesimal birational groupoid as fibered over $\tilde{\mathcal{X}} = \mathrm{Bl}_{\mathcal{Z}}(\mathcal{X})$ by way of the 1st projection, then f admits a lifting $\tilde{f} : \mathcal{C} \to \tilde{\mathcal{X}}$ and we may form the fibre square,

$$\begin{tikzcd} \tilde{\mathfrak{F}} \arrow[d, "p"'] & \tilde{\mathfrak{F}}_{\mathcal{C}} \arrow[l] \arrow[d, "p"] \\ \tilde{\mathfrak{X}} & \tilde{\mathcal{C}} \arrow[l, "\tilde{f}"] \end{tikzcd}$$

In addition the identity map of the groupoid gives a section s of p of the left, so a fortiori of the right, vertical arrow, which is everywhere well defined since we're working with $\tilde{\mathcal{X}}$ rather than $\mathcal{X}$. Consequently $\tilde{\mathfrak{F}}_{\mathcal{C}}$ is a concave neighbourhood of $s(\mathcal{C})$. In addition if C is the modulus of $\mathcal{C}$ then we have maps,

$$\tilde{\mathfrak{F}}_{\mathcal{C}} \hookrightarrow \mathcal{C} \times \tilde{\mathfrak{P}}_{\mathcal{X}} \to \mathcal{C} \times \mathfrak{P}_{\mathcal{X}} \to C \times \mathfrak{P}_X \hookrightarrow C \times X \times X$$

where $\tilde{\mathfrak{P}}_{\mathcal{X}}$ is as the prequel to II.1.3, and we claim,

II.3.1 Fact. *The Zariski closure of the image of* $\tilde{\mathfrak{F}}_C$ *in* $C \times X \times X$ *is integral of dimension* 2.

Proof. Indeed let Y be the Zariski closure, which is integral since $\tilde{\mathfrak{F}}_C$ is integral. Moreover if L is an ample line bundle on $C \times X \times X$ then by definition,

$$H^0(Y, L) \to H^0(\tilde{\mathfrak{F}}_C, L)$$

is injective by the definition of Y, so we're done by the Chow lemma. □

Now certainly we have that the projection of Y to X by the second factor of $X \times X$ is a $\mathcal{F}$-invariant subvariety, but equally its projection to C is a 1st integral, so $(Y, \mathcal{F})$ is a pencil of curves, albeit perhaps a rather singular one.

Unfortunately since the diagonal of $\mathcal{X}$ will only be an embedding if $\mathcal{X}$ is a scheme to proceed from here to a well defined family of invariant stacks over $\mathcal{C}$ which is infinitesimally $\tilde{\mathfrak{F}}_C$, or something close to it, requires a little care. Now what we can certainly do is to consider the $\mathcal{F}$-invariant 2-dimensional integral substack $\mathcal{Y}_0$ of $\mathcal{X}$ obtained by projecting Y to X then pulling back to $\mathcal{X}$. The induced foliation $(\mathcal{Y}_0, \mathcal{F})$ with tangent bundle $\mathcal{O}_{\mathcal{Y}_0}(T_{\mathcal{F}})$ may only be a pre-foliation, but this is actually better rather than worse. In any case $\mathcal{F}$ is by hypothesis generically smooth around $f(\mathcal{C})$, so by algorithmic resolution (or more precisely the invariance of the resolving centres under vector fields) we know that the said image is not contained in the singular locus of $\mathcal{Y}_0$. Consequently if $\rho : \tilde{\mathcal{Y}} \to \mathcal{Y}_0$ is the algorithmic resolution and $\tilde{\mathcal{F}}$ the induced saturated foliation then $f : \mathcal{C} \to \mathcal{Y}_0$, lifts to $\tilde{f} : \mathcal{C} \to \tilde{\mathcal{Y}}$, say, and $K_{\tilde{\mathcal{F}}} \cdot_{\tilde{f}} \mathcal{C} \leq K_{\mathcal{F}} \cdot_f \mathcal{C}$. Additionally $(\tilde{\mathcal{Y}}, \tilde{\mathcal{F}})$ admits a meromorphic 1st integral $\varphi : \tilde{\mathcal{Y}} \dashrightarrow \mathbb{P}^1$, and we can blow up in the crossings of the image of $\tilde{f}$ with the singular locus of $\tilde{\mathcal{F}}$ to ensure that $\tilde{\mathcal{F}}$ is everywhere smooth in a neighbourhood of $\tilde{f}(\mathcal{C})$, and whence φ is defined on all of the same. Again such blow ups are in invariant centres, so the canonical degree is certainly still bounded above by $K_{\mathcal{F}} \cdot_f \mathcal{C}$, whence we may as well suppose that we have these additional hypothesis. However at this point the indeterminacy points of φ are all of dimension 0, from which some more blowing up all of which takes place outside a neighbourhood of $\tilde{f}(\mathcal{C})$ leads to φ being defined everywhere. Abusing notation slightly let's continue to denote this modification by $\tilde{\mathcal{Y}}$ and consider the fibre square,

$$\begin{tikzcd}
\tilde{\mathcal{Y}} \arrow[d, "\varphi"'] & \tilde{\mathcal{Y}}_{\mathcal{C}} \arrow[l] \arrow[d] & \mathcal{C} \arrow[l, "s=\tilde{f}\times\mathrm{id}"'] \\
\mathbb{P}^1 & \mathcal{C} \arrow[l, "\varphi\circ f"] &
\end{tikzcd}$$

It may, of course, happen that $\tilde{\mathcal{Y}}_{\mathcal{C}}$ is not irreducible, but it is generically smooth about the image of s, since φ itself is generically smooth. Consequently there is a unique irreducible component $\mathcal{Y}$ containing the image of s. Better still $\mathcal{Y}$ is actually smooth, albeit that it's evidently only ramified over $\tilde{\mathcal{Y}}$ in the fibre direction, and this would be wholly sufficient. Regardless to see that it's smooth we can work with small étale neighbourhoods U and V of $\tilde{\mathcal{Y}}$ and $\mathcal{C}$ in the analytic topology. Around $\tilde{f}(\mathcal{C})$ the foliation is smooth with transverse coordinate z, say, and parallel coordinate x. Without loss of generality we can write $\varphi = z^m$, so that for c a coordinate on $\mathcal{C}$, $\tilde{\mathcal{Y}}_{\mathcal{C}}$ is, around s, covered by étale neighbourhoods of the form,

$$\frac{\mathbb{C}\,[x, z, c]}{z^m - z(c)^m}$$

where of course, $f(c) = (x(c), z(c))$. Thus around the ramification of φ, an étale neighbourhood of $\tilde{\mathcal{Y}}_{\mathcal{C}}$ has the form,

$$\frac{\mathbb{C}\,[x, z, c]}{z - z(c)}$$

which is certainly smooth. Moreover the ramification of $\mathcal{Y} \to \tilde{\mathcal{Y}}$ is precisely along the fibres, so in a further abuse of notation if $(\mathcal{Y}, \mathcal{F})$ is the induced foliation then $K_{\mathcal{F}}$ is just the pull-back of $K_{\tilde{\mathcal{F}}}$. Whence to summarise,

II.3.2 Fact. *There is a proper family $p : \mathcal{Y} \to \mathcal{C}$ of stacks, together with a section s such that the induced foliation $\mathcal{F}$ on $\mathcal{Y}$ is the fibration p. Better still,*

(a) $K_{\mathcal{F} \cdot_s} \mathcal{C} \leq K_{\mathcal{F} \cdot_f} \mathcal{C}$.

(b) $(\mathcal{Y}, \mathcal{F}) \to (\mathcal{X}, \mathcal{F})$ *is everywhere defined, and respects the various foliations.*

Armed with our family $\mathcal{Y}$ we've almost constructed invariant rational curves through every point of $\mathcal{C}$. Indeed we may apply, provided modulo its generic stabiliser the generic fibre of $\mathcal{Y}$ is not a bad orbifold, our combination of semi-stable reduction and uniformisation, I.4.4, to find an honest curve B, a semi-stable family $\mathcal{S} \to B$ scheme like in codimension 2 along with a commutative diagram,

$$\begin{tikzcd} \mathcal{Y} \arrow[d] & \mathcal{S} \arrow[l, "\rho"'] \arrow[d] \\ \mathcal{C} & B \arrow[l] \end{tikzcd}$$

such that $\mathcal{S} \to \mathcal{Y}$ is ramified only on fibres, from which $\omega_{\mathcal{S}/B} = \rho^* K_{\mathcal{F}}$. In addition our section $s : \mathcal{C} \to \tilde{\mathfrak{F}}_{\mathcal{C}} \to \mathcal{Y}$, can be lifted, at least after a possibly larger base extension, to $\tilde{s} : B \to \mathcal{S}$, and $\omega_{\mathcal{S}/B} \cdot_{\tilde{s}} B = (B : \mathcal{C})\, K_{\mathcal{F}}.\mathcal{C} < 0$. Consequently to prove that the generic fibre of $\mathcal{S} \to B$ is a rational curve it suffices by [I] (or for that matter a classical theorem of Arakelov, [S1]) to show that $\mathcal{S}$ is scheme like around $\tilde{s}(B)$. Suppose, then, this fails at some point b, with x a coordinate coming from the base, and $y = 0$ defining the section on some étale neighbourhood V. Blowing up, if necessary, we can suppose x, y are a coordinate system. For the situation to be non-scheme like we would have to be able to find a finite group G acting on V leaving x invariant as well as the ideal (y). However in $\hat{\mathcal{O}}_{V,b}$ this can be linearised for a possibly different choice of y (essentially since $H^1(H, L) = 0$ for any cyclic group H, and coherent $\mathbb{Q}\,[H]$ module L) as,

$$(x, y) \mapsto (x, y)^g = (x, \chi(g)y)$$

for $\chi : G \to \mathbb{C}^\times$ some character. This would imply, however, non-scheme like behaviour around all of the section, and so we have almost arrived at,

II.3.3 Proposition. *Let $(\mathcal{X}, \mathcal{F})$ be a foliated Gorenstien stack with projective moduli X, and $f : \mathcal{C} \to \mathcal{X}$ a map from a 1-dimensional smooth stack not factoring through the singular locus of $\mathcal{F}$, with $K_{\mathcal{F}} \cdot_f \mathcal{C} < 0$ then for every geometric point $c \in \mathcal{C}$ there is a map $g_c : \mathcal{L}_c \to \mathcal{X}$ with $\mathcal{L}_c$ a stack of positive Euler characteristic such that for M any* nef. *$\mathbb{R}$-divisor on $\mathcal{X}$,*

$$M \cdot_{g_c} \mathcal{L}_c \leq 2 \, \frac{M \cdot_f \mathcal{C}}{-K_{\mathcal{F}} \cdot \mathcal{C}}$$

and $g_c : \mathcal{L}_c \to \mathcal{X}$ is invariant by $\mathcal{F}$.

Proof. According to our previous set up, if the generic fibre of $\mathcal{Y}$ modulo the generic stabiliser is not a bad orbifold, then for each $b \in \tilde{s}(B)$ we may find a rational curve $g_b : \mathbb{P}^1 \to \mathcal{S}$ factoring through the fibres of $\mathcal{S} \to B$ such that,

$$M \cdot_{g_b} \mathbb{P}^1 \leq 2 \, \frac{\rho^* M \cdot_{\tilde{s}} B}{-\omega_{\mathcal{S}/B} \cdot_{\tilde{s}} B} = 2 \, \frac{M \cdot_f \mathcal{C}}{-K_{\mathcal{F}} \cdot_f \mathcal{C}} \, .$$

Otherwise generically $\mathcal{Y}/G$, with G its generic stabiliser, is a bad orbifold so certainly of positive Euler characteristic. Consequently for each geometric point $c \in \mathcal{C}$, the component of the fibre of $\mathcal{Y}$ over $\mathcal{C}$ meeting s gives a map from a stack $g_c : \mathcal{L}_c \to \mathcal{X}$ which is invariant by the foliation. The degree bound on the other hand is, therefore, just that for the fibres of $\mathcal{Y} \to \mathcal{C}$, which is equally that for the fibres of the moduli $Y \to C$, say, which in turn is just the index theorem for a 2-dimensional normal algebraic space. ☐

II.3.4 Remark. The same proof works, in more complete generality, if $f(\mathcal{C})$ meets the pre-image of the open subset where the moduli are algebraic.

II.4. The Cone of Curves

We may now apply the basic estimate II.3.3 to the cone of curves of a $\mathbb{Q}$-foliated Gorenstien variety $(X, \mathcal{F})$ over $\mathbb{C}$. Indeed more precisely we have,

II.4.1 Fact. *Let $(X, \mathcal{F})$ be a $\mathbb{Q}$-foliated Gorenstien variety with log-canonical singularities then there are countably many rational curves L_i invariant by $\mathcal{F}$ with, $0 < -K_{\mathcal{F}} \cdot L_i \leq 2$ and,*

$$\overline{\mathrm{NE}}\,(X)_{\mathbb{R}} = \overline{\mathrm{NE}}\,(X)_{K_{\mathcal{F}} \geq 0} + \sum_i \mathbb{R}_+ \, L_i$$

where $\overline{\mathrm{NE}}\,(X)_{K_{\mathcal{F}} \geq 0}$ is the sub-cone of the closed cone of curves on which $K_{\mathcal{F}}$ is non-negative. Better still the $\mathbb{R}_+ \, L_i$ are locally discrete, and if $R \subset \overline{\mathrm{NE}}\,(X)_{\mathbb{R}}$ is an extremal ray in the half space $\mathrm{NE}_{K_{\mathcal{F}} < 0}$ then it is of the form $\mathbb{R}_+ \, L_i$.

Proof. This is a wholly formal consequence, i.e. modulo the existence of Hilbert schemes linear algebra and cf. [K1] III.1.2, of the following variant of II.3.3, viz:

II.4.2 Sub-Fact. *Let $(X, \mathcal{F})$ be as above, $f : C \to X$ a curve with $K_{\mathcal{F}} \cdot_f C < 0$ then for all $c \in C$ there is a rational curve $L_c \ni f(c)$ invariant by $\mathcal{F}$ such that for all* nef. *$\mathbb{R}$-divisors M on X,*

$$M \cdot L_c \leq 2 \, \frac{(M \cdot C)}{-K_{\mathcal{F}} \cdot C} \, .$$

Sub-Proof. Let $\pi : \mathcal{X} \to X$ be the corresponding Gorenstien stack, then provided $f(c)$ is not contained in the image of the singular locus of $(\mathcal{X}, \mathcal{F})$ then we simply apply II.3.3, and take L_c to be the moduli of the corresponding parabolic stack $\mathcal{L}_c$. Otherwise consider the linearisation map, i.e. the composition of,

$$D : \mathcal{I}_{\mathcal{Z}}/\mathcal{I}_{\mathcal{Z}}^2 \underset{d}{\longrightarrow} \Omega_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{Z}} \longrightarrow K_{\mathcal{F}} \otimes \mathcal{I}_{\mathcal{Z}}/\mathcal{I}_{\mathcal{Z}}^2$$

where $\mathcal{Z} \subset \mathcal{X}$ is the singular locus of $\mathcal{F}$. The linearisation is $\mathcal{O}_{\mathcal{Z}}$ linear, and since the singularities are canonical for any $z \in \mathcal{Z}$ some symmetric function of D defines a global section of some power $K_{\mathcal{F}}^{\otimes n}$ of $K_{\mathcal{F}}$ over $\mathcal{Z}$ which is non-vanishing at z, so for a curve contained in $\pi(\mathcal{Z})$, $K_{\mathcal{F}}. C \geq 0$, cf. [I]. $\square$

While in many ways highly satisfactory one should bear in mind the following facts about the cone theorem, viz:

II.4.3 Caveats. (i) It is a relatively simple consequence of the bend & break estimate II.3.3. Moreover the existence of a stack $\mathcal{L}_0$ with positive Euler characteristic mapping to the rational curve L_c is substantially stronger than the assertion of rationality, and in due course is what rather more refined estimates will be based on.

(ii) The need for canonical singularities is slightly artificial since they're only used to prove the non-negativity of $K_{\mathcal{F}}$ restricted to the singular sub-stack. As such the cone theorem could have been stated with this weaker hypothesis, which, incidentally, is implied by log-canonical in dimension 1.

Finally let us examine the possibilities for $K_{\mathcal{F}}$ negative invariant parabolic stacks under the additional hypothesis that the foliation is smooth and terminal at the non-scheme like points of $\mathcal{X}$. Whence let $f : \mathcal{L} \to \mathcal{X}$ be the normalisation of such a stack, then since it can't factor through the non-scheme like locus of $\mathcal{X}$ it's actually an orbifold. Better still adjunction, I.8.7, gives

$$K_{\mathcal{F}} \cdot_f \mathcal{L} = -\chi(\mathcal{L}) + s_{\mathcal{Z}}(\mathcal{L}) - R_f$$

and all points of $\mathcal{Z}$ are scheme like, so: $s_{\mathcal{Z}}(\mathcal{L}) - R_f \geq \# f^{-1}(\mathcal{Z})$. In addition if l_i are the non-scheme like geometric points and d_i the order of their stabilisers then,

$$-\chi = -2 + \sum_i \left(1 - \frac{1}{d_i}\right) .$$

So the only possibility is that there is at most one non-scheme like point $l \in \mathcal{L}$ with stabiliser $\mathbb{Z}/d$, and exactly one singular point $z \in f^{-1}(\mathcal{Z})$. Consequently, unless $f^{-1}(\mathcal{Z}) = \varphi$, and whence $\mathcal{F}$ is a pencil of rational curves, then we have,

II.4.4 Fact/Definition. Hypothesis as per the above discussion, then a $K_{\mathcal{F}}$-negative $\mathcal{F}$-invariant curve, $f : \mathcal{L} \to \mathcal{X}$ is $-1/d\,\mathbb{F}$, i.e. the automorphism group at the unique (if it exists) non-scheme like point l of $\mathcal{L}$, as well as the automorhism group of $f(l)$ is $\mathbb{Z}/d$. In particular $\mathcal{L}$ is the orbifold $\mathbb{P}^1$ with signature d at the said point, and f is an embedding everywhere except possibly at the scheme like point $f^{-1}(\mathcal{Z})$ where it may have a cusp. In any case,

$$K_{\mathcal{F}}. \mathcal{L} = -1/d\,,$$

and indeed $\mathbb{Z}/d$ is the holonomy, as per I.5.7, if $\mathcal{X}^f$ is the Gorenstien covering stack of its moduli.

II.5. Linear Holonomy of Smooth $-\frac{1}{d}\,\mathbb{F}$ Curves

To extract more subtle information on the structure of $-\frac{1}{d}\,\mathbb{F}$ curves, we consider the consequences for the singularities of $(\mathcal{X},\mathcal{F})$ under the hypothesis of smoothness and terminality of $\mathcal{F}$ at the non-scheme like points of $\mathcal{X}$. We begin with the linear holonomy for smooth $-\frac{1}{d}\,\mathbb{F}$ curves. Independently of this observe quite generally that if $\mathcal{C}$ is a smooth embedded substack of $\mathcal{X}$ then descent yields a well defined normal bundle $N_{\mathcal{C}/\mathcal{X}}$ to which we can specialise $\mathcal{F}$. Indeed if $V\to\mathcal{X}$ is étale, and U the pull-back of $\mathcal{C}$ then the pull-back of the associated cone is,

$$\operatorname{Spec} S := \bigoplus_{n=0}^{\infty} \frac{I_{U,V}^{n}}{I_{U,V}^{n+1}}\,.$$

On the other hand the foliation leaves $I_{U,V}$ invariant, so a local generator ∂ of $T_{\mathcal{F}}$ passes to a graded derivation of S by way of applying it to any lifting of an element in the n^{th}-graded piece, and then reducing modulo $I_{U,V}^{n+1}$. This process may not immediately lead to a foliation, but only a pre-foliation, i.e. the specialisation may not be saturated. Let us therefore distinguish the naively specialised foliation by the subscript p, i.e. $\mathcal{F}^p$, should it only be a pre-foliation, and reserve the letter $\mathcal{F}$ for saturated foliations. On the other hand the only way to occasion a pre-foliation would be at the singularities of Z, while quite generally if y is a coordinate along U, and x_i normal coordinates then the specialisation of ∂ takes the form,

$$\partial : y\mapsto b(y)=\partial y(\operatorname{mod} I_{U,V})\,,\qquad x_i\mapsto a_{ij}(y)\,x_j=\partial x_i(\operatorname{mod} I_{U,V})$$

where we employ the summation convention. Consequently $\mathcal{F}^p\neq\mathcal{F}$ at a singular point z iff the linearisation ∇_z of ∂ in $\operatorname{End}(N_{U,V}\otimes\mathbb{C}(z))$ is zero. Now consider the particular case of a smooth $-\frac{1}{d}\,\mathbb{F}$ curve $\mathcal{L}$ then by I.5.6 and I.6.11 the linear holonomy is $\mathbb{Z}/e$, for some $e\mid d$, while in an affine, i.e. $\xrightarrow{\sim}\mathbb{A}^1$, neighbourhood of the singular point considered as the origin we can normalise our specialised derivation so that $b(y)=y$. As a first step let us observe,

II.5.1 Fact. *The specialised foliation is in fact saturated.*

Proof. Indeed suppose otherwise, then on the foliated stack $(N_{\mathcal{L}/\mathcal{X}},\mathcal{F})$ the zero section has flat normal bundle. In particular $e=1$, so $\mathcal{L}\xrightarrow{\sim}\mathbb{P}(d)$, and there are analytic coordinate functions x_i normal to $\mathcal{L}$ in the neighbourhood of our given affine, V, such that $\partial x_i=0$, so in fact around the singularity the foliation we can take a scheme like open analytic neighbourhood Δ with coordinate functions x_i, y normal and parallel to $\mathcal{L}$ such that,

$$\partial x_i = 0(\operatorname{mod} I_{\mathcal{L}}^2)\,.$$

On the other hand the triviality of the holonomy equates to the existence of functions ξ_i on an analytic étale neighbourhood U of $\mathcal{L}\backslash\{0\}$ such that,

$$\partial\xi_i\equiv 0\,.$$

So without loss of generality we can say that $\xi_i|_V\equiv x_i|_U(\operatorname{mod} I_{\mathcal{L}}^2)$. Now proceed inductively supposing that we've found coordinates x_i such that this holds over U, $\operatorname{mod} I_{\mathcal{L}}^n$ for $n\geq 2$, and write,

$$\xi_i = x_i + b_{iJ}\,x^J$$

for b_{iJ} analytic functions on $U \times_{\mathcal{L}} V$, x^J monomials of degree n, and employ summation convention. On the other hand $\partial x_i|_U$ is zero $\mathrm{mod}\, I_{\mathcal{L}}^n$, so the same holds over Δ, whence,

$$\partial x_i = c_{iJ}\, x^J$$

where now the c_{iJ} are holomorphic on all of $\mathcal{L} \cap \Delta$. Combining we obtain, for $y = t^d$,

$$b'_{iJ}(t) = -d\, c_{iJ}(t^d)\, t^{-1}\,, \quad \forall\, i, J$$

thus the $c_{iJ}(0) = 0$ for all i, J, so the b_{iJ} are in fact holomorphic on $\mathcal{L} \cap \Delta$. As a result on the completion $\hat{\Delta}$ of Δ in $\mathcal{L}$, the foliation is given by a formal fibration $\xi_1 \times \cdots \times \xi_n = x_1 \times \cdots \times x_n$ over a formal affine space, whence is smooth, which is absurd. □

As such we can cease to worry about the difference between foliations and pre-foliations. Furthermore the divisor $g = 0$ in $N_{\mathcal{L}/\mathcal{X}}$ is invariant by $\mathcal{F}$, and $t \mapsto t^d = y$ defines a covering $\nu : \mathbb{P}^1 \to \mathcal{L}$, ramified only in $y = 0$, so we even have an induced foliation on $\nu^* N_{\mathcal{L}/\mathcal{X}}$ whose canonical bundle $K_{\mathcal{F}}$ is not only the pull-back of the same on $\mathcal{X}$ but, $\mathcal{O}_{\mathbb{P}^1}(-1)$. Additionally there is an action of $\mathbb{Z}/d$ on this foliation which we identify with an action of μ_d in the usual way by choosing a d-th root of unity ζ, acting on t by, $t \mapsto \zeta t$. Now over the coordinates $\mathbb{A}^1$ containing ∞, whose image we identify with the non-scheme like point if it exists, we have a basis ξ_i, in the analytic topology, of sections of $\nu^* N^{\vee}_{\mathcal{L}/\mathcal{X}}$ satisfying $\partial_\infty(\xi_i) = 0$, for ∂_∞ a generator of the specialised foliation over the said patch. Such sections are permuted by the Galois group, so we may as well say that they've been chosen so that the action is linear, i.e.

$$(\zeta, \xi_i) \mapsto \zeta^{-\frac{b_i d}{e}} \xi_i\,, \quad b_i \in \mathbb{Z}$$

where necessarily $\gcd(b_1, \ldots, b_n, e) = 1$, for n the codimension of $\mathcal{L}$. Over our original patch containing the origin, however, we can express the ξ_i as functions of the x_i's, i.e.

$$\xi_i = f_{ij}\, x_j$$

where the f_{ij} are analytic functions on $\mathbb{G}_m = \mathbb{P}^1 \backslash \{0, \infty\}$, necessarily meromorphic at ∞, and we employ the summation convention. The matrix $F = [f_{ij}]$ must, however, satisfy the differential equation,

$$F^{-1} \dot{F} = -\frac{d}{t}\, A$$

where $A = [a_{ij}(y)]$, while by [D], the limit of the holonomy about a circle of radius r around the origin as $r \to 0$ is $\exp(-A(0))$, so in the first place, without loss of generality, $A(0)$ is diagonal with eigenvalues α_i/e, $\alpha_i \in \mathbb{Z}$. Better still an explicit solution of this equation is furnished by,

$$\exp(-d \log t\, A(0) + B)$$

where tB is a primitive of $-\frac{d}{t}\,(A - A(0))$, so certainly holomorphic. Furthermore any other solution is a constant, i.e. $\mathrm{GL}(n, \mathbb{C})$, multiple of this one so if $-\lambda$ is the smallest eigenvalue then F satisfies an estimate of the form,

$$\|F\| \ll \frac{1}{|t|^{\lambda}}$$

in a neighbourhood of the origin, so the ξ_i are actually meromorphic sections of $\nu^* N^\vee_{\mathcal{L}/\mathcal{X}}$. Naturally we introduce their valuation, i.e.

$$v_i = v(\xi_i) := \min_j v_{ij} := v(f_{ij}) \in \mathbb{Z}$$

where v is the order of vanishing/polarity at the origin. Observe that $t^{-v_i}\xi_i$ is a regular section of $\nu^* N^\vee_{\mathcal{L}/\mathcal{X}}$ over our $\mathbb{A}^1 \ni 0$ which is non-zero in $N^\vee_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(z)$, and satisfies, $\partial(t^{-v_i}\xi_i) = -v_i/d\,(t^{-v_i}\xi_i)$. Indeed we can certainly order things to insist that,

$$v_1 \leq v_2 \leq \cdots \leq v_d$$

while we may also note that the Taylor expansion of each f_{ij} is of the form,

$$f_{ij}(t) = \sum_{n+\frac{b_i d}{e} \equiv 0(d)} c_{ijn}\, t^n$$

whence $v_i \equiv -b_i(d)$, and $t^{-v_i}\xi_i$ descends. With this in mind, we assert,

II.5.2 Claim. *For a possibly different μ_d linear basis ξ_i of solutions of $\partial_\infty = 0$, the sections $t^{-v_i}\xi_i$ of $\nu^* N_{\mathcal{L}/\mathcal{X}}$ form a basis of nowhere vanishing sections of $\nu^* N_{\mathcal{L}/\mathcal{X}}$ at the origin, and whence, indeed, everywhere over the $\mathbb{A}^1 \ni 0$.*

Proof. The indeed part is clear. We proceed inductively to the effect that we've found a G-linear basis ξ_i which respects the ordering of the valuation such that $t^{-v_1}\xi_1, \ldots, t^{-v_k}\xi_k$, $1 \leq k < n$ are linearly independent at the origin. The case $k = 1$ is clear. Furthermore having the inductive hypothesis in hand we can suppose without loss of generality that $x_i = t^{-v_i}\xi_i$ for $1 \leq i \leq k$. Now, $t^{-v_j}\xi_j(0)$ is an eigenfactor of the linearisation of $\partial \in \mathrm{End}\,(N_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(0))$ with eigenvalue v_j/d, so we're certainly done unless $v_k = v_{k+1}$. Writing things out in terms of our basis x_i, we have for any $i > k$,

$$\xi_i = \sum_{v_j < v_k} f_{ij}\, x_j + \sum_{\substack{v_j = v_k \\ j \leq k}} f_{ij}\, x_j + \sum_{j > k} f_{ij}\, x_j$$

we again see that we're done unless $v_{ij} > v_k$ for $j > k$, and every i such that $v_i = v_k$. So suppose this happens then for suitable constant functions c_{ij} we can replace ξ_i for $v_i = v_k$ by,

$$\tilde{\xi}_i := \xi_i + \sum_{\substack{v_j = v_k \\ j \leq k}} c_{ij}\, \xi_j$$

so that $v(\tilde{\xi}_i) > v_k$ if $v_i = v_k$. Indeed we need only check that $v_{ij} > v_k$ if $v_i = v_k$ and $v_j < v_k$, but this is clear since $t^{-v_i}\xi_i(0)$ is an eigenvector with eigenvalue v_k/d, while for $v_j < v_k$, $x_j(0)$ is an eigenvector of smaller eigenvalue. Better still this is a G-linear operation, so re-ordering the new basis, ξ_i, $i \leq k$, $\tilde{\xi}_i$, $v_i = v_k$, $i > k$, ξ_i, $v_i > v_k$ according to increasing valuation we're done. □

Now certainly $-v_i/d$ are the eigenvalues α_i/e of the linearisation of ∂ in $\mathrm{End}($ $\nu^* N_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(0))$, i.e. $A(0)$, but we've also constructed an isomorphism,

$$\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(v_i) \xrightarrow[\oplus_i t^{-v_i}\xi_i]{} \nu^* N^\vee_{\mathcal{L}/\mathcal{X}}$$

so in fact $N_{\mathcal{L}/\mathcal{X}} \xrightarrow{\sim} \bigoplus_i \mathcal{O}_{\mathcal{L}}(a_i/d)$, where $a_i = \alpha_i \frac{d}{e}$. One should however bear in mind that we have much more, i.e. a meromorphic 1^{st} integral $\xi_1 \times \cdots \times \xi_n$ of the foliation, where ξ_i is thought of as a meromorphic function on the cone, $\rho : \nu^* N_{\mathcal{L}/\mathcal{X}} \to \mathbb{P}^1$, or perhaps better, $t^{-v_i}\xi_i$ is thought of as a global section of $\rho^*\mathcal{O}_{\mathcal{L}}(a_i/d)$, defining an invariant divisor of $N_{\mathcal{L}/\mathcal{X}}$ isomorphic to,

$$\bigoplus_{j \neq i} \rho^* \mathcal{O}_{\mathcal{L}} \left(\frac{a_j}{d} \right) .$$

In particular the canonical or Harder-Narismhan filtration of $N_{\mathcal{L}/\mathcal{X}}$ corresponds to an increasing filtration of invariant sub-cones,

$$[0] = N_0 \subsetneqq N_1 \subsetneqq N_2 \subsetneqq \cdots \subsetneqq N_k = N_{\mathcal{L}/\mathcal{X}}$$

such that the normal bundle of N_i in N_{i+1} restricted to the zero section is a trivial bundle twisted by some $\mathcal{O}_{\mathcal{L}}(a_i/e)$ where a_i decreases as one proceeds up the chain.

II.6. Formal Holonomy

We wish to extend the previous discussion of linear holonomy of smooth $-\frac{1}{d}\mathbb{F}$ stacks to the rather more delicate case of formal holonomy. We retake verbatim the notation of the previous section with $(\mathcal{X}, \mathcal{F})$ a non-singular Gorenstien foliated stack such that $\mathcal{F}$ is terminal at the non-scheme like points, and $f : \mathcal{L} \to \mathcal{X}$ a smooth $-\frac{1}{d}\mathbb{F}$ curve. Consequently by hypothesis there is a formal analytic neighbourhood V of the $\mathbb{A}^1$ over ∞ together with a μ_d action on the same such that the formal classifying stack $\nu := [V/\mu_d]$ is a formal analytic neighbourhood of $[\mathbb{A}^1/\mu_d]$ over which the foliation trivialises, i.e. on V we have analytic coordinate functions ξ_i, s normal and parallel to our $\mathbb{A}^1$ respectively such that over V the foliation is just the formal fibration $\xi_1 \times \cdots \times \xi_n : V \to \hat{\Delta}^n$, where the latter space is a n-polydisc completed in the origin. Plainly we can take s to be meromorphic on a formal neighbourhood of $\mathcal{L}$, while the algebra $\mathbb{C}[[\xi_1, \ldots, \xi_n]]$ comes equipped with a μ_d action which, modulo the maximal ideal, is nothing other than that of the linear holonomy. In addition, the said algebra is an inverse limit of finite dimensional vector spaces so the action may be supposed linear and the ξ_i compatible with the basis of II.5.2.

Now suppose, to begin with, that we can choose ∂ on a formal neighbourhood U of the $\mathbb{A}^1$ patch around 0 together with a coordinate function y on U restricting to the same on $\mathbb{A}^1$ such that $\partial y \equiv y(\mathcal{I}_{\mathcal{L}}^m)$. Observe, additionally, that on any formal space an element congruent to 1 modulo an ideal of definition has a logarithm. Consequently if $\mathcal{X}_m$, U_m, V_m, etc. denote the reduction of our various spaces modulo $\mathcal{I}_{\mathcal{L}}^m$, then on extracting a d^{th}-root t of y, on which μ_d acts by multiplication by ζ, we obtain a covering $\mu : \tilde{U}_m \to U_m$ to which ∂ lifts and of which $U_m \times_{\mathcal{X}} V_m$ may be identified with an open formal analytic subspace. With this in mind we assert,

II.6.1 Claim. *For a possibly different basis ξ_i of the algebra $\mathbb{C}[[\xi_1, \ldots, \xi_n]]$, but which is nevertheless compatible with any previous choice of the same modulo $\mathcal{I}_{\mathcal{L}}^{m'}$ for $m' < m$, and which in any case is compatible with the above considerations,* *the functions $t^{a_i}\xi_i$ define global sections of the $\mathcal{O}_{\mathcal{X}_m}$ module $\mathcal{I}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}^{m+1}(+a_i/d)$, where $\mathcal{O}_{\mathcal{X}_m}\left(\frac{1}{d}\right)$ is the bundle with transition function t on $U_m \times_{\mathcal{X}} V_m$.*

Proof. We proceed inductively on m, the case $m=1$ being the already established linear result. Our first concern is what happens over $\tilde{U}_m$. By the induction hypothesis for $m-1$, $m \geq 2$, we can find coordinate functions x_i normal to $\mathcal{L}$ whose reduction modulo $\mathcal{I}_{\mathcal{L}}^2$ are a basis of the normal bundle over U and such that,

$$t^{a_i}\xi_i - x_i \mid_{U_m \times_{\mathcal{X}} V_m} \in \Gamma(U_m \times_{\mathcal{X}} V_m,\, \mathcal{I}_{\mathcal{L}}^m) = 0\,.$$

Consequently, $\partial x_i - a_i\, x_i$ restricted to $U_m \times_{\mathcal{X}} V_m$ is zero, so in fact it's zero on $\tilde{U}_m$. Whence if we consider the x_i as elements of the $\mathcal{O}_{U_m}$ module $\mathcal{I}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}^{m+1} \mid_{U_m}$, then,

$$\partial x_i = \frac{a_i}{d}\, x_i + a_{iJ}(t)\, x^J$$

where x^J is the monomial $x_1^{j_1}\ldots x_n^{j_n}$, $j_1+\cdots+j_n = m$, the summation convention is employed, and $a_{iJ}(t)$ is a holomorphic function of t. On the other hand,

$$t^{a_i}\xi_i = x_i + b_{iJ}(t)\, x^J$$

with the same conventions, but where, now, b_{iJ} is only holomorphic over our $\mathbb{A}^1$ minus the origin, which for convenience we'll denote $\mathbb{G}_m$. Combining we obtain,

$$t\,\dot{b}_{iJ} + b_{iJ}(a_J - a_i) = -a_{iJ}\, d \in \mathcal{O}_{\mathbb{G}_m}$$

where $a_J = \sum_i j_i\, a_i$, and no summation is implied. Again we can explicitly integrate this, by way of,

$$\frac{d}{dt}\,(t^{(a_J-a_i)}\, b_{iJ}) = -d\, a_{iJ}\, t^{a_J-a_i-1}\,.$$

As such the b_{iJ} are certainly meromorphic, and no $a_{iJ}\, t^{a_J-a_i-1}$ has a residue, from which we deduce,

$$b_{iJ} = h_{iJ}(t) + \frac{\lambda_{iJ}}{t^{a_i-a_J}}$$

where h_{iJ} is holomorphic, and λ_{iJ} is a constant. As a result if we replace ξ_i by,

$$\tilde{\xi}_i := \xi_i - \lambda_{iJ}\xi^J$$

where we sum over J with $a_i > a_J$, and observe that $a_J \equiv 0(d)$ if $\lambda_{iJ} \neq 0$, we see that the $\tilde{\xi}_i$ form a μ_d-linear basis of $\mathcal{I}_{\mathcal{L}}/\mathcal{I}_{\mathcal{L}}^{n+1} \mid_{V_m}$ compatible with our previous choices such that $t^{a_i}\tilde{\xi}_i$ are holomorphic on $\tilde{U}_m$. Better still on replacing the ξ_i by $\tilde{\xi}_i$, the $t^{a_i}\xi_i = x_i$ are invariant by μ_d, so if $\mathcal{O}_{\mathcal{X}_m}\left(\frac{1}{d}\right)$ is the bundle on the m^{th}-thickening of $\mathcal{L}$ with transition function t on the overlap $U_m \times_{\mathcal{X}} V_m$ then we have global sections as claimed. □

Before proceeding further let's clear up what's analytic, and what's algebraic. In the first place provided t is algebraic, the bundle $\mathcal{O}_{\mathcal{X}_m}\left(\frac{1}{d}\right)$ is algebraic, i.e. given by a transition function in $\Gamma(\mathcal{O}^*_{U_m \times_{\mathcal{X}} V_m})$, albeit that the said function may not lead in the limit to a meromorphic function on the completion of $\mathcal{X}$ in $\mathcal{L}$. Now given the algebraicity of t, and using GAGA for the case $m=1$, the proof shows that the $t^{a_i}\xi_i$ are algebraic on U_m while a similar, and rather easier, induction at ∞ shows that the ξ_i are algebraic on V_m. In any case let us proceed to find an appropriate t by re-considering the situation mod $\mathcal{I}_{\mathcal{L}}^{m+1}$, $m \geq 1$. Over U we have, in the notation/spirit of the proof,

$$\partial y = y + c_J\, x^J (\mathcal{I}_{\mathcal{L}}^{m+1})\,, \quad \partial x_i = \frac{a_i}{d}\, x_i + c_{iK}\, x^K (\mathcal{I}_{\mathcal{L}}^{m+2})$$

where the summation convention is back in force, with respect to multi-indices J and K of degrees m, $m+1$ respectively, and all the c_*'s are regular algebraic functions of y. Over $U_{m+2} \times_{\mathcal{X}} V_{m+2}$, y has a d^{th}-root, as ever denoted t, with standard μ_d action, so without loss of generality,

$$\xi_i\, |_{U_{m+2}\times_{\mathcal{X}} V_{m+2}} = t^{-a_i}\, x_i + b_{iK}\, x^K\, |_{U_{m+2}\times_{\mathcal{X}} V_{m+2}}\ (\mathrm{mod}\, \mathcal{I}_{\mathcal{L}}^{m+1})$$

where of course b_{iK} may only be analytic functions of t defined on our $\mathbb{G}_m$, and summation over the multi-index K of degree $m+1$ is implied. Combining these, yields for each multi-index J of weight m, and any i,

$$a_i\, c_J\, t^{a_J-(d+1)} = \sum_{x^K = x^J x_i} d\, c_{iK}\, t^{(a_J-1)} + \frac{d}{dt}\,(t^{a_K}\, b_{iK})\,.$$

Consequently, given we know the existence of a non-zero eigenvalue by II.5.1, if $a_J = d \geq 1$, then $c_J(0) = 0$. Consider on the other hand the obstruction to finding a coordinate $\tilde{y}$ over U restricting to the same on $\mathcal{L}$ such that,

$$\partial \tilde{y} = (1+\lambda)\, \tilde{y}\, (\mathcal{I}_{\mathcal{L}}^{m+1})\,, \quad \lambda = \lambda_J\, x^J \in \mathcal{I}_{\mathcal{L}}^m\,, \quad \lambda_J \in \mathcal{O}_{U\cap\mathcal{L}}\,.$$

If we look for such a $\tilde{y}$ in the form, $y + \Lambda_J\, x^J$, with Λ_J constants, then we require to solve,

$$\left(\frac{a_J}{d} - 1\right) \Lambda_J - \lambda_J\, y = -c_J$$

for all J. However if $a_J \neq d$, then $\Lambda_J = -c_J(0)\left(\frac{a_J}{d} - 1\right)^{-1}$, and λ_J whatever, will do, while if $a_J = d$, then we can freely take $\Lambda_J = 0$, and $\lambda_J = c_J\, y^{-1}$. Whence we inductively obtain a compatible system of coordinates y_m, and generators ∂_m of the foliation over U_m such that $\partial_m\, y_m = y_m$. In consequence passing to the limit we obtain that the canonical/Harder-Narismhan filtration of $N_{\mathcal{L}/\mathcal{X}}$ extends to the entire formal neighbourhood, i.e.

II.6.2 Proposition/Summary. *Let $\hat{\mathcal{X}}$ be the formal stack obtained by completing $\mathcal{X}$ in $\mathcal{L}$ then there is a bundle $\mathcal{O}_{\hat{\mathcal{X}}}\left(\frac{1}{d}\right)$ extending $\mathcal{O}_{\mathcal{L}}\left(\frac{1}{d}\right)$ and indeed $\mathcal{O}_{\hat{\mathcal{X}}}(D) = \mathcal{O}_{\hat{\mathcal{X}}}(1)$ for D a smooth formal invariant divisor transverse to $\mathcal{L}$ passing through the unique point z of $\mathcal{L} \cap \mathrm{sing}\,(\mathcal{F})$, together with a filtration of formal invariant sub-stacks,*

$$\mathcal{L} = \mathcal{X}_0 \subsetneqq \mathcal{X}_1 \subsetneqq \cdots \subsetneqq \mathcal{X}_k = \hat{\mathcal{X}}$$

such that if $\frac{\alpha_1}{e} > \cdots > \frac{\alpha_k}{e}$ are the distinct eigenvalues of ∂ considered linearised in $\mathrm{End}\,(N_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(z))$ with $n_1, \ldots, n_k$ the dimensions of the corresponding eigenspaces then $\mathcal{X}_i$ is defined by $\mathcal{F}$-invariant global sections γ_j of $\mathcal{O}_{\hat{\mathcal{X}}}(\frac{\alpha_j}{e})$, $j > i$, and n_j-sections for each j. In particular,

$$N_{\mathcal{L}/\mathcal{X}_i} \xrightarrow{\sim} \bigoplus_{j\leq i} \mathcal{O}_{\mathcal{L}}\left(\frac{\alpha_j}{e}\right)$$

and for $\mathcal{X}$ almost étale over its moduli, the automorphism group of the non-scheme like point of $\mathcal{L}$ is generated by $\prod_j \exp(2\pi\sqrt{-1}\,\alpha_j/d) \subset (\mathbb{C}^\times)^k$.

II.7. Jordan Decomposition

We briefly interupt our discussion of $K_{\mathcal{F}}$-negative invariant stacks to recall some saliant facts on Jordan decomposition which will be relevant both to our study of cusps, and the local uniqueness of the Harder-Narismhan filtration. The situation is entirely local and scheme-like, i.e. $\mathcal{O}$ is the ring of formal power series $\mathbb{C}[[x_1, \ldots, x_n]]$, $\mathfrak{m}$ its maximal ideal, and ∂ a $\mathbb{C}$-derivation of $\mathcal{O}$ with a singularity at the origin. Recall that since $\mathcal{O}$ is an inverse limit of finite dimensional vector spaces ∂ admits a Jordan decomposition, i.e. $\partial = \partial_S + \partial_N$, where the semi-simple part ∂_S acts as a semi-simple matrix on each $\mathcal{O}/\mathfrak{m}^n$, $n \in \mathbb{N}$, ∂_N is nilpotent, and of course $[\partial_S,\, \partial_N] = 0$. In particular if $\partial_S = \lambda_i\, x_i\, \frac{\partial}{\partial x_i}$, summation convention, then a conventional choice of basis for the nilpotent fields commuting with ∂_S is,

II.7.1 Revision (cf. [Ma]) Notations as above then $\partial_N = \sum_{i=1}^{n} \sum_{Q_i} a_{Q_i} x^{Q_i} x_i \frac{\partial}{\partial x_i}$, $a_{Q_i} \in \mathbb{C}$, where either,

(i) $Q_i = (q_1, \ldots, q_n)$, $q_j \in \mathbb{N} \cup \{0\}$, $x^{Q_i} = x_1^{q_1} \ldots x_n^{q_n}$, $\Lambda \cdot Q_i = 0$, or

(ii) $Q_i = (q_1, \ldots, q_n)$, $q_i = -1$, $q_j \in \mathbb{N} \cup \{0\}$, $j \neq i$, $x^{Q_i} = x_1^{q_1} \ldots x_n^{q_n}$, $\Lambda \cdot Q_i = 0$.

Now the Jordan decomposition of a vector field is certainly unique, and whence the property of semi-simplicity of a vector field is wholly unambiguous. For a foliation however the situation is rather more delicate since there is a question of rescaling by units. Whence suppose our field ∂ is semi-simple, and consider a field $\tilde{\partial} = u\partial$, where $u \equiv 1(\mathfrak{m})$ to avoid stupidity. Furthermore let's say, without loss of generality, that $\partial = \lambda_i\, x_i\, \frac{\partial}{\partial x_i}$ then we assert,

II.7.2 Claim. *There is a change of coordinates of the form, $\xi_i = u_i\, x_i$, $u_i \equiv 1(\mathfrak{m})$, and $\varepsilon \equiv 0(\mathfrak{m})$ with $\partial\varepsilon = 0$ such that the Jordan decomposition of $\tilde{\partial}$ is,*

$$\tilde{\partial} = \lambda_i\, \xi_i\, \frac{\partial}{\partial \xi_i} + \varepsilon\, \lambda_i\, \xi_i\, \frac{\partial}{\partial \xi_i}$$

i.e. $\tilde{\partial}$ may not be semi-simple, but the extent to which it is not is very particular.

Proof. Consider the following inductive proposition for $k \in \mathbb{N}$,
there are coordinates $x_{ik} = u_{ik}\, x_i$, $u_{ik} \equiv 1(\mathfrak{m})$, $\tilde{\partial} = u_k\, \partial_k$, $\partial_k = \lambda_i\, x_{ik}\, \frac{\partial}{\partial x_{ik}}$, $u_k \equiv 1(\mathfrak{m})$ such that $u_k^{-1} = 1 + \varepsilon_k + \delta_k$, where ε_k, δ_k are defined by way of the Jordan decomposition of $\mathfrak{m}$ as $\mathrm{Ker}\, \partial_k \oplus \mathrm{Im}\, \partial_k$, and $\delta_k \in \mathfrak{m}^k$.
The case $k = 1$ is simply our given data. Otherwise consider trying to improve the situation by putting, $x_{ik+1} = v_{ik}\, x_{ik}$, $v_{ik} \equiv 1(\mathfrak{m})$ to be chosen. If such a change were to actually render the situation semi-simple then we would have to solve,

$$\partial_k \log v_{ik} = \lambda_i \left(\frac{1}{u_k} - 1 \right) = \lambda_i(\varepsilon_k + \delta_k)$$

which plainly may not be possible if $\lambda_i \neq 0$, and $\varepsilon_k \neq 0$. However we can solve $\partial_k \log v_{ik} = \lambda_i\, \delta_k$, so that in particular, $v_{ik} \equiv 1(\mathfrak{m}^k)$, while in the new coordinates,

$$\tilde{\partial} = \frac{1 + \delta_k}{1 + \varepsilon_k + \delta_k}\, \lambda_i\, x_{ik+1}\, \frac{\partial}{\partial x_{ik+1}}$$

which is indeed what we're looking for, since putting $u_{k+1} = (1+\delta_k)\, u_k$ then,

$$u_{k+1}^{-1} = 1 + \varepsilon_k (1+\delta_k)^{-1} = 1 + \varepsilon_k + \sum_{n=1}^{\infty} (-1)^n \, \varepsilon_k \, \delta_k^n$$

so that $\delta_{k+1} \in \mathfrak{m}^{k+1}$.

Certainly therefore the $\delta_k \to 0$, but the proof also shows that for each i the infinite product, $\prod_k v_{ik}$ converges to some u_i, so putting $\xi_i = u_i\, x_i$ we're certainly done on observing that $\partial\varepsilon = 0$ obliges,

$$\left[\lambda_i \, \xi_i \, \frac{\partial}{\partial \xi_i} \, , \, \varepsilon \, \lambda_i \, \xi_i \, \frac{\partial}{\partial \xi_i}\right] = 0 \, .$$

□

The consequence of the fact that not only can Jordan decomposition of a rescaling of semi-simple only fail in a very controlled way, but also that Jordan decompositions of rescalings are related in a very simple way suggests that we introduce,

II.7.3 Definition. A germ of a foliation $(\hat{\mathbb{A}}^n, \mathcal{F})$ on a formal affine space with a not necessarily isolated singularity at the origin is said to be semi-simple, if $T_{\mathcal{F}} = \mathcal{O}_{\hat{\mathbb{A}}^n} \partial$ for some semi-simple vector field ∂.

As an important example/application consider the situation of blowing up in the origin, i.e. $\rho : (X, \tilde{\mathcal{F}}) \to (\hat{\mathbb{A}}^n, \mathcal{F})$ is the said modification with induced foliation and X is the completion in the exceptional divisor of the blow up of $\operatorname{Spec} \mathcal{O}$. Denoting by, $\partial = \partial_S + \partial_N$ a Jordan decomposition of any generator $T_{\mathcal{F}}$ we have,

II.7.4 Further Fact. *Suppose $\partial_S \neq 0$ and $(X, \tilde{\mathcal{F}})$ is not everywhere smooth (which in any case could only happen if in suitable coordinates $\partial = x_i \frac{\partial}{\partial x_i}$) then the following are equivalent,*

(1) $(\hat{\mathbb{A}}^n, \mathcal{F})$ *is semi-simple.*

(2) $(X, \tilde{\mathcal{F}})$ *is semi-simple at all of its singular points.*

(3) $(X, \tilde{\mathcal{F}})$ *is semi-simple at one of its singular points, and* $(\hat{\mathbb{A}}^n, \mathcal{F})$ *is semi-simple modulo* $\mathfrak{m}^2$.

(4) $(X, \tilde{\mathcal{F}})$ *is semi-simple at one of its singular points, which is itself a singular point of $\rho^* \partial_S$, and $\rho^* \partial_S$ is semi-simple at the said point.*

Proof. Since $(X, \tilde{\mathcal{F}})$ is not everywhere smooth the induced foliation is given everywhere by $\rho^* \partial$ (cf. I.6.2) so trivially (1) ⇒ (2) ⇒ (3) ⇒ (4). Consider therefore (4) ⇒ (1). Write $\partial_S = \lambda_i \, x_i \, \frac{\partial}{\partial x_i}$, and say, without loss of generality, the singularity finds itself on the $x_1 \neq 0$ patch, at a point p. At the completion of the local ring $\mathcal{O}_{X,p}$ in $\mathfrak{m}(p)$, by hypothesis $\rho^* \partial_S + \rho^* \partial_N$ is a Jordan decomposition of $\rho^* \partial$. However, there is some generator $\tilde{\partial}$ at p which is semi-simple, so an application of II.7.2 yields $\varepsilon \in \hat{\mathcal{O}}_{X,p}$ such that $\rho^* \partial(\varepsilon) = 0$, and,

$$\varepsilon \, \rho^* \, \partial_S = \rho^* \, \partial_N \, .$$

Consider, therefore the following cases,

(a) $\lambda_1 \neq 0$, with $\partial_N(x_1) = f$, then, $\lambda_1\, x_1\, \varepsilon = \rho^* f$, from which,

$$0 = \partial(f/x_1) = \frac{\partial f}{x_1} - f/x_1^2(\lambda_1\, x_1 + f)$$

so $x_1 \mid f$, and ε is actually a function on $\hat{\mathbb{A}}^n$, from which we conclude, or

(b) $\lambda_1 = 0$, and without loss of generality $\lambda_2 \neq 0$, with $\partial_N(x_2) = f$, from which $\varepsilon\, \lambda_2\, x_2 = f$, so as before $x_2 \mid f$, and ε comes from $\hat{\mathbb{A}}^n$. $\square$

A further question which we may reasonably address here is the uniqueness, or lack thereof, of the Jordan decomposition. Even without rescaling the particular choice of coordinates in which we may write a semi-simple field as $\lambda_i\, x_i\, \partial/\partial x_i$ may be catastrophically non-unique. Plainly the worst possible case is when all the λ_i are rational, or equivalently up to a harmless rescaling integers. Even this is of course not unique but it's not too bad since of course any rational point in some $\mathbb{P}^N(\mathbb{Q})$ is up to multiplication by ± 1 uniquely represented by a tuple of relatively prime integers, consequently let's establish some notation,

II.7.5 Notation. Let ∂ be a semi-simple derivation of $\mathcal{O}$ with integer eigenvalues $a_1, \ldots, a_r, -b_1, \ldots, -b_t$, $a_i, b_j \in \mathbb{N}$, s zeroes, $r \geq 1$, although possibly $t = 0$, i.e. no negatives, and $(a_1, \ldots, a_r, b_1, \ldots, b_t) = 1$, then we will suppose these ordered by decreasing size, i.e.

$$a_1 \geq a_2 \geq \cdots \geq a_r \geq 0 \geq -b_1 \geq \cdots \geq -b_t$$

and by $\alpha_1, \ldots, \alpha_k$, $k \leq r$, $\beta_1, \ldots, \beta_l$, $l \leq t$ a complete repetition free list of the same, so that,

$$a_1 = \alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$$

$$0 > -b_1 = -\beta_1 > -\beta_2 > \cdots > -\beta_l\,.$$

Now for a given choice of basis of a semi-simple derivation ∂ with the said eigenvalues i.e. a particular way of writing it as $a_i\, y_i\, \frac{\partial}{\partial y_i} - b_j\, x_j\, \frac{\partial}{\partial x_j}$, we can introduce,

II.7.6 Definition. The Harder-Narismhan pair of $(\hat{\mathbb{A}}^n, \mathcal{F})$ with respect to the data (∂, y_i, x_j) is the invariant formal sub-schemes, X_+, X_- whose ideals are generated by the non-positive, respectively non-negative, eigenvectors of ∂. If instead we take strictly negative, respectively strictly positive, eigenvectors then the resulting subschemes, denoted $X_+^{\geq 0}$, $X_-^{\leq 0}$, will be called the non-negative Harder-Narismhan pair.

Manifestly, apart from abreviating Harder-Narismhan to H-N, what's important is that the H-N pairs are well defined up to ± 1, i.e.

II.7.7 Fact. *Fix a choice of semi-simple ∂ with integer eigenvalues normalised as per II.7.5, then the following are equivalent,*

(1) $\{X_+, X_-\}$*, respectively* $\{X_+^{\geq 0}, X_-^{\leq 0}\}$*, is the H-N, resp. non-negative, pair with respect to ∂ in the basis* $\{x_i, y_j\}$.

(2) $\{X_+, X_-\}$*, resp.* $\{X_+^{\geq 0}, X_-^{\leq 0}\}$*, is the H-N, resp. non-negative, pair with respect to ∂ in any semi-simple basis.*

(3) $\{X_+,\, X_-\}$, *resp.* $\{X_+^{\geq 0},\, X_-^{\leq 0}\}$*, is the H-N, resp. non-negative, pair of any semi-simple* $\tilde{\partial} = u\partial$ *in any semi-simple basis for the same, where* $u \equiv 1(\mathfrak{m})$.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are all trivial, so consider (1) $\Rightarrow$ (3). By II.7.2, we know that we can find units $u_i,\, v_j \equiv 1(\mathfrak{m})$ such that if $\eta_i = u_i\, y_i,\, \eta_j = v_j\, x_j$ then $\tilde{\partial} = a_i\, \eta_i\, \frac{\partial}{\partial \eta_i} - b_j\, \xi_j\, \frac{\partial}{\partial \xi_j}$. As such $\{X_+,\, X_-\}$, resp. $\{X_+^{\geq 0},\, X_-^{\leq 0}\}$, is the H-N, resp. strict, pair of $\tilde{\partial}$ in the basis $\{\xi_i,\, \eta_j\}$. Now suppose $\tilde{\partial} = a_i\, f_i\, \frac{\partial}{\partial f_i} - b_j\, g_j\, \frac{\partial}{\partial g_j}$ in some other basis $f_i,\, g_j$. At the mod $\mathfrak{m}^2$ level this is just a question of the uniqueness of diagonalisation/the commutator of a diagonal matrix, so without loss of generality let's say $f_i \equiv \xi_i$, and $g_j \equiv \eta_j(\mathfrak{m}^2)$. For higher order terms, consider the Taylor expansion,

$$f_i = \xi_i + \sum_{\#J + \#K \geq 2} c_{iJKL}\, \xi^J\, \eta^K\, \zeta^L\,,$$

where, as ever, ξ^J etc. is the monomial $\xi_1^{j_1} \ldots \xi_r^{j_r}$, and $\zeta_1, \ldots, \zeta_s$ are the null vectors. Now $\tilde{\partial}\, f_i = a_i\, f_i$ so $\tilde{\partial}\, \xi^J = a_i$, and whence,

$$c_{iJKL} \neq 0 \Rightarrow \sum_\alpha a_\alpha\, j_\alpha - \sum_b b_\beta\, k_\beta = a_i\,.$$

Consequently if $f_i \notin (\xi_1, \ldots, \xi_r)$, then we have a manifest absurdity, and so conclude by symmetry. $\square$

The dependence on ± 1 is, however, unavoidable, except in the sense of,

II.7.8 Summary/Definition. Let $(\hat{\mathbb{A}}^n,\, \mathcal{F})$ be a germ of a singular semi-simple foliation such that the eigenvalues of a linearisation in $\mathfrak{m}/\mathfrak{m}^2$ are in $\mathbb{P}^{n-1}(\mathbb{Q})$ then there are two canonical pairs (as opposed to two pairs of canonical) of invariant subschemes in the form of the H-N pair, $\{X_+,\, X_-\}$, and the non-negative H-N pair $\{X_+^{\geq 0},\, X_-^{\leq 0}\}$. The former intersecting in the origin, the latter in the whole singular locus. If no-confusion is likely, the suffices may be dropped.

As to why this is in any way of interest, the point is,

II.7.9 Reason for the above discussion. Let $(\mathcal{X},\, \tilde{\mathcal{F}})$ be a smooth foliated stack with terminal singularities at the non-scheme like points, and $f : \mathcal{L} \to \mathcal{X}$ any smooth $-1/d\,\mathbb{F}$ curve, containing the wholly scheme like singular point z, then after completion at z, $\mathcal{L}$ lies in precisely one of $\mathcal{X}_+$ or $\mathcal{X}_-$, respectively, $\mathcal{X}_+^{\geq 0}$, $\mathcal{X}_-^{\leq 0}$. An important further task will be to extend this to cusps.

It's also the case, and for more or less the same reason, that X_+, say, can be canonically filtered as,

$$0 = X_+^0 \subsetneqq X_+^1 \subsetneqq \cdots \subsetneqq X_k^+ \subsetneqq X_{k+1}^+ = X^+$$

where X_+^i has for ideal eigenvectors of eigenvalue $> \alpha_i$, and similarly for X_-, $X_+^{\geq 0}$, $X_-^{\leq 0}$. This is, however, not so important, nor is it true that the filtration could be extended to all of $\hat{\mathbb{A}}^n$.

II.8. Cusps

We consider the consequences of the previous discussion for cuspidal $-\frac{1}{d}\,\mathbb{F}$ curves, so as ever $(\mathcal{X},\, \mathcal{F})$ is a non-singular foliated Gorenstien stack, with $\mathcal{F}$ terminal at the non-scheme like points. As such a cusp can only occur at a scheme like point. Consequently to begin with let us simply consider the situation of a map $f : \mathbb{A}^1 \to X$ to a smooth scheme having a cusp at the origin. The important thing is the embedding dimension, k, say, so actually we simply want to consider a formal $\hat{\mathbb{A}}^1$ mapping to a formal $\hat{\mathbb{A}}^k$, by way of a cusp of embedding dimension k. Denoting by v the order function at the origin of the $\hat{\mathbb{A}}^1$ we can associate to such a data a prefered system of coordinates $y_1, \dots, y_k$ with valuations $v_1, \dots, v_k$ such that,

$$v_1 < v_2 < \cdots < v_k\,, \quad \text{and } v_{i+1} \notin \mathbb{Z}_{\geq 0}\, v_1 + \cdots + \mathbb{Z}_{\geq 0}\, v_k\,.$$

Indeed the only non-trivial point is that the given topology on $\mathcal{O}_{\hat{\mathbb{A}}^k}$ is the same as the valuation topology, but this is a theorem of Chevalley. Now suppose, further, that the situation is invariant by a formal vector field δ then by algorithmic resolution $f^*\delta$ is again a vector field, so for η a coordinate function on the $\hat{\mathbb{A}}^1$ we can normalise δ, up to scalar multiplication, in such a way that,

$$v(f^*\delta\,\eta - \eta) \geq 2$$

provided that the Segre class of δ with respect to f at the origin is 1. A fact that we will, naturally enough, be able to suppose. As such the basis y_i enjoys the property that, $v(\delta y_i) = v_i$, so if $a_{ij} \in gl(k, \mathbb{C})$ is the matrix corresponding to the linearisation of δ at the origin in this basis then the property that, $v_{i+1} \notin \mathbb{Z}_{\geq 0}\, v_1 + \cdots + \mathbb{Z}_{\geq 0}\, v_k$ imply that a_{ij} is upper semi-triangular with diagonal $v_1, \dots, v_k$, so without loss of generality we may actually suppose that it's diagonal. Better still the said condition that v_{i+1} is not a non-negative sum of the v_i is exactly what's required to guarantee a linearisation of δ at the origin, so we may as well say that $\delta y_i = v_i\, y_i$ in $\mathcal{O}_{\hat{\mathbb{A}}^k}$. Consequently after a linear diagonal change of coordinates we can even assert that the cusp takes the form,

$$\eta \mapsto (\eta^{v_1}, \dots, \eta^{v_k})\,.$$

As such the Euclidean algorithm for v_i resolves the cusp by a sequence of blow ups such that the order of vanishing of the proper transform of f at the unique point where it crosses the exceptional divisor is the gcd of $v_1, \dots, v_k$, so in fact the $v_1, \dots, v_k$ are relatively prime.

Now let's return to the specifics of $-1/d\,\mathbb{F}$ curves which are cusps of embedding dimension k. Locally about the singular point we can find coordinate functions $x_1, \dots, x_\ell$ vanishing on the cusp, and embedding coordinates $y_1, \dots, y_k$. As such the $\mathcal{F}$-invariant ideal of the cusp is of the form $I = (x_i,\, g_i(y))$, while a local generator ∂ of the foliation takes the form,

$$\partial = a_i\,\frac{\partial}{\partial x_i} + (b_j + c_j)\,\frac{\partial}{\partial y_j}$$

where $a_i \in I$, $b_j \in (x_1, \dots, x_\ell)$, c_j is a pure function of the y's, and we employ the summation convention. Consequently the projection of the cusp onto the y-plane is invariant by the field $\delta = c_j\,\frac{\partial}{\partial y_j}$, which of course satisfies our Segre class 1-assumption. Consequently in the completion $\hat{\mathcal{O}}_{\mathcal{X},z}$ of $\mathcal{O}_{\mathcal{X},z}$ in $\mathfrak{m}_{\mathcal{X}}(z)$ we can suppose that the y_i and their valuations v_i have the form of the previous discussion, and of course we normalise our generator ∂ so that $v(\partial\eta - \eta) \geq 2$ for η a coordinate on the normalisation of the cusp.

Unsurprisingly our immediate concern is to prove that $\mathcal{F}$ is semi-simple at z. However if we are to do this by way of II.7.4, then we'll require a certain minimum amount of information about the comportment mod $\mathfrak{m}_{\mathcal{X}}(z)^2$. To this end consider the exact sequence,

$$0 \to N_f \to f^* \Omega_{\mathcal{X}} \to \omega_{\mathcal{L}}(-R_f) \to 0$$

where R_f is the ramification, and N_f by definition the kernel of the map on the right. If we denote by $\mathcal{L}^*$, $\mathcal{L} - 0$, where $f(0) = z$, then we know that specialising the foliation corresponds to a sort of connection,

$$\nabla : N_f \otimes \mathcal{L}^* \to N_f \otimes \mathcal{L}^*$$

we say sort of since usually a connection satisfies, $\nabla(gn) = dg \cdot n + g\nabla n$, for g a function and n a section, whereas locally ours satisfies $\nabla(gn) = f^*\partial(g) \cdot n + g\nabla n$, where $f^*\partial$ is the necessarily regular (by virtue of algorithmic resolution) derivation of $\mathcal{L}$ obtained by restricting a generator of $\mathcal{F}$. In any case what's required is that ∇ should extend over the puncture. Now by virtue of considering the situation on a resolution $\rho : \tilde{X} \to X$ obtained by blowing up in points according to the Euclidean algorithm it's certainly the case that the situation is at worst meromorphic, i.e. there is a map,

$$\bar{\nabla} : N_f \to N_f \otimes \mathcal{O}_{\mathcal{L}}(n0)$$

for $n \in \mathbb{N}$, satisfying, $\bar{\nabla}(gn) = f^*\partial(g)\, n + g\bar{\nabla} n$, with g, n, $f^*\partial$ etc. as before. Consequently to check that we can actually take $n = 0$ we can simply work in the complete local rings $\hat{\mathcal{O}}_{\mathcal{X},z}$ and $\hat{\mathcal{O}}_{\mathcal{L},0}$. We proceed by brute force, viz:

II.8.1 Step 1. Just as in the case where the embedding dimension is the dimension we can for possibly different y_i's but which nevertheless have the same valuations achieve a linearisation of ∂ in $\hat{\mathcal{O}}_{\mathcal{X},z}$ such that, $c_j = v_j\, y_j$.

II.8.2 Step 2. Again as per the case $l = 0$, we may as well therefore say that the cusp has the form,

$$\eta \mapsto (\eta^{v_1}, \dots, \eta^{v_k}, 0, \dots, 0)$$

so that generators of $N_f \otimes \hat{\mathcal{O}}_{\mathcal{L},0}$ take the form, $dx_1, \dots, dx_\ell$, $dy_i - \frac{v_i}{v_1}\, \eta^{v_i - v_1}\, dy_1$, $2 \leq i \leq k$, and $f^*\partial(\eta) = \eta$.

II.8.3 Step 3. $f^*\, I/I^2$ certainly maps to N_f, so just calculate. The case of the x_i's is automatic since, as we've noted, $da_i \in N_f$, whereas,

$$\begin{aligned}\nabla\left(dy_i - \frac{v_i}{v_1}\, \eta^{v_i - v_1} dy_1\right) &= f^*(db_i + v_i\, dy_i) - \frac{v_i}{v_1}\,(v_i - v_1)\, \eta^{v_i - v_1} dy_1 \\ &\quad - \frac{v_i}{v_1} \eta^{v_i - v_1} f^*(db_1 + v_1 dy_1) \\ &= v_i \left(dy_i - \frac{v_i}{v_1}\, \eta^{v_i - v_1} dy_1\right) \operatorname{mod} f^* I/I^2\end{aligned}$$

with this out of the way we can therefore use our knowledge of the linear holonomy to conclude that the induced linearisation $\bar{\nabla}_0 \in \operatorname{End}(N_f \otimes \mathbb{C}(0))$ is semi-simple, and whence for possibly different coordinates x_i, y_i but nevertheless satisfying the same hypothesis on embedding and valuation, the Jordan decomposition of ∂ in $\hat{\mathcal{O}}_{\mathcal{X},z}$ takes the form $\partial = \partial_S + \partial_N$, where

$$\partial_S = v_i\, y_i\, \frac{\partial}{\partial y_i} + a_j\, x_j\, \frac{\partial}{\partial x_j}\,, \quad \partial_N = \varepsilon\, x_1\, \frac{\partial}{\partial y_1} + \delta\,, \quad \delta \in \mathfrak{m}^2\, \mathrm{Der}(\hat{\mathcal{O}}_{\mathcal{X},z})$$

and $\varepsilon = 0$ or 1, so in the latter case $a_1 = v_1$. As such we're already pretty much done by II.7.4, unless indeed $\varepsilon = 1$. In any case consider a blow up $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$, formal or otherwise, in z then the proper transform $\tilde{f}$ finds itself at the origin $\tilde{z}$, say, in the standard coordinate system, of the $y_1 \neq 0$ patch. In particular, by direct calculation, $\rho^*\partial_S + \rho^*\partial_N$ is still the Jordan decomposition of $\rho^*\partial$, so by a descending induction on the number of blow ups required to resolve the cusp, we obtain,

II.8.4 Fact. *If $f : \mathcal{L} \to \mathcal{X}$ is a $-1/d\,\mathbb{F}$ cusp then there are formal coordinates x_i, y_j about the singular point z, with the latter embedding coordinates of valuation $v_j \notin \mathbb{Z}_{\geq 0}\, v_1 + \cdots + \mathbb{Z}_{\geq 0}\, v_{j-1}$, $1 \leq j \leq k$, $\gcd(v_1, \ldots, v_k) = 1$, along with integers $a_i\, e$ enjoying $\gcd(e,\, a_1, \ldots,\, a_\ell) = 1$, such that a suitable formal generator ∂ of $T_{\mathcal{F}}$ takes the form,*

$$\partial = \frac{a_i}{e}\, x_i\, \frac{\partial}{\partial x_i} + v_j\, y_i\, \frac{\partial}{\partial y_j}\,.$$

In particular, if in the notation of II.7.6, X_+, X_- are the H-N pair of formal subschemes of the completion of $\mathcal{X}$ in z, then the formal germ of the cusp at z is in either X_+ or X_-, while if $\mathcal{X}$ is the Gorenstien cover of its moduli then $e = d$.

This is already sufficient to avoid much further study of cusps, but for completeness let's do a little better. As such let $\hat{X}$ be a smooth formal affine neighbourhood of a $-1/d\,\mathbb{F}$-cusp minus the non-scheme like point. The Euclidean algorithm determines a sequence of formal blow ups,

$$\hat{X}_n \to \hat{X}_{n-1} \to \cdots \to \hat{X}_1 \to \hat{X}_0 = \hat{X}$$

where the final $\hat{X}_n$ is the completion, without loss of generality, of our previous $\tilde{X}$ in the proper transform of the cusp. Additionally on each $\hat{X}$ we have a distinguished singular point z_i be it of the corresponding foliation or the cusp itself provided in the latter case $i < n$. Bearing in mind our previous normalisation of a generator ∂ for the foliation at z_i, and noting that we've already proved that the eigenvalues, at the origin, not amongst the v_i are of the form a_j/e, $a_j \in \mathbb{Z}$, with $\gcd(a_1, \ldots,\, a_\ell,\, e) = 1$, we wish to prove,

II.8.5 Curiosity. The formal holonomy is of the form $\mathbb{Z}/e$ iff we can find coordinates $y_1, \ldots,\, y_k,\, x_1\, , \ldots,\, x_\ell$ with the y_i's embedding coordinates x_i vanishing on the cusp and a generator ∂ for the foliation on $\hat{X}$ such that,

$$\partial = v_i\, y_i\, \frac{\partial}{\partial y_i} + \frac{a_j}{e}\, x_j\, \frac{\partial}{\partial x_j}$$

with $\gcd(a_1, \ldots,\, a_\ell,\, e) = 1$, and as ever summation covention.

Proof. The if direction is trivial, the affine cusp has no Picard group, so neither does $\hat{X}$, so generators ∂_i over $\hat{X}_i$ exist for each i, and we proceed by descending induction to choose them in the required form.

To begin with simplify the notations to consider a formal blow up $\rho : \tilde{X} \to \hat{X}$ in $z \in \hat{X}$, with $\tilde{\partial}$ on $\tilde{X}$ of the required form with respect to coordinates $\tilde{y}_i$, $\tilde{x}_i$. We know that in $\hat{\mathcal{O}}_{\hat{X},z}$ i.e. $\mathcal{O}_{\hat{X},z}$ completed in $\mathfrak{m}(z)$, we can find an eigenfunction y_1 of a formal generator $\hat{\partial}$, with eigenfunction v_1, such $\rho^* y_1$ has the same property in $\hat{\mathcal{O}}_{\tilde{X},\tilde{z}}$, where, obviously, $\tilde{z}$ is the lifting of z. Equally, up to permutations of the y_i, and noting that the valuation of $\rho^* y_1$ is still v_1, we may as well say that $\tilde{y}_1$ has the same property, including that of valuation v_1.

However, by general non sense about eigenvalues we can write,

$$\rho^* y_1 = \tilde{y}_1 + \sum_{I,J} c_{IJ}\, \tilde{y}^I\, \tilde{x}^J$$

for $\tilde{y}^I = \tilde{y}_1^{i_1} \ldots \tilde{y}_k^{i_k}$, etc., c_{IJ} constants, $\# I \geq 2$, and $\tilde{y}^I\, \tilde{x}^J$ of eigenvalue v_1. However for $\# J$ fixed there are only finitely many possibilities for I, so the sum is convergent in all of $\tilde{X}$, i.e. $\rho^* y_1$ extends to a function on $\tilde{X}$ which we may as well say is $\tilde{y}_1$. Armed with this identification the same convergence argument shows that without loss of generality, $\tilde{y}_i = y_i/y_1$, $\tilde{x}_j = x_j/y_1$, $i \geq 2$, all j, for some formal coordinates in $\hat{\mathcal{O}}_{\hat{X},z}$. On the other hand $\mathcal{O}_{\hat{X},z} \to \mathcal{O}_{\tilde{X},z}$ is finite, and completion in $\mathfrak{m}(z)$ is faithfully flat, so in fact the said y_i, x_j are actually coordinate functions on $\hat{X}$. $\square$

Profiting from the Euclidean algorithm to solve $c_1\, v_1 + \cdots + c_k\, v_k = 1$, for some integers c_i, we could make a more strict analogue of II.6.2. Indeed in the above notation, $\phi = y_1^{c_1} \ldots y_k^{c_k}$ is a rational function on the affine cusp of degree 1, which we can use to form an explicit patch with an étale affine neighbourhood of the non-scheme like point at infinity according to the relation $\phi^d = s^{-1}$, for s a standard coordinate on the latter. Equally there is a bundle on the completion in $\mathcal{L}$, say, $\mathcal{O}(\frac{1}{d})$, with transition function $\phi^{1/d}$, which we can use to encode the poles and zeroes of the trivialised holonomy at infinity. Nevertheless a precise statement is irrelevant to what follows, so we'll omit it.

III. EXTREMAL SUBVARIETIES

III.1. Generalities

Unless specified otherwise, throughout this chapter $(\mathcal{X}, \mathcal{F})$ will be a foliated non-singular stack, with $\mathcal{F}$ smooth and terminal at the non-scheme like, together with $\log$-canonical foliation singularities elsewhere. We switch our attention from $K_{\mathcal{F}}$ negative curves, to $K_{\mathcal{F}}$ negative extremal rays R. The moduli X is of course supposed projective so if H_R is a nef. Cartier divisor supporting the ray, i.e. $H_R.\,\alpha = 0$, and α in the closed cone of curves iff $\alpha \in R$, then for sufficiently large $m \in \mathbb{N}$, $A_R := m\,H_R - K_{\mathcal{F}}$ is ample. In any case following Kollàr, Mori, et al., cf. [K1] III.1, we introduce our main object of study, by way of,

III.1.1 Definition. The locus of R, $\mathrm{Loc}\,(R)$ is the set of closed points $x \in X(\mathbb{C})$ such that there is a curve $x \in C \subset X$ with $[C] \in R \subset NS_1(X)$.

Observe that a priori $\mathrm{Loc}\,(R)$ is not a subvariety of X. Indeed for $m \in \mathbb{N}$, we can filter $\mathrm{Loc}\,(R)$ by sub-schemes $\mathrm{Loc}_m\,(R)$ on demanding that $x \in \mathrm{Loc}_m\,(R)$ if we can take the curve C of the definition to have $A_R.\,C \leq m$. That $\mathrm{Loc}_m\,(R)$ is a sub-scheme is immediate from the existence of the Hilbert scheme. To remedy this let us consider,

III.1.2 Further Definition. A R-pre-extremal subvariety is an irreducible subvariety $Y \subset \mathrm{Loc}\,(R)$ maximal amongst the set of irreducible varieties contained in the locus.

Trivially, the dimension in chains of proper inclusions of irreducible varieties must increase so R-pre-extremal subvarieties exist, any $x \in \mathrm{Loc}\,(R)$ is contained in one, and $\mathrm{Loc}\,(R)$ is the a priori countable union of all of them. Now if Y is R-pre-extremal, and $y \in Y$ then there is a C_y with $[C_y] \subset R$ containing y. However applying II.4.2, we know there is an invariant rational curve $L_y \ni y$ such that,

$$H_R.\,L_y \leq 2\,\frac{H_R.\,C_y}{-K_{\mathcal{F}}.\,C_y} = 0\,.$$

So in fact $L_y \in R$, and $A_R.\,L_y \leq 2$. Additionally L_y cannot be contained in $\mathrm{sing}\,(\mathcal{F})$ since it has $K_{\mathcal{F}}$-negative degree, so we can make a $\mathcal{F}$-invariant subvariety W by adding to generic points of Y an appropriate L_y. On the other hand Y is by hypothesis R-pre-extremal, so $W = Y$, i.e. Y is $\mathcal{F}$ invariant, with the induced foliated variety $(Y, \mathcal{F})$ being a pencil of rational curves of A_R degree at most 2. Hilbert schemes, however, exist, and being invariant is a closed condition so in fact there are at most finitely many R-pre-extremal subvarieties for a given R. Better still the Hilbert scheme yields for any R-pre-extremal subvariety Y a flat family, $L \to T$, for some irreducible sub-scheme T of the Hilbert scheme such that the projection of L to X factors as a generically finite map over Y. Surprisingly the slightly awkward case arises when X is itself a R-pre-extremal subvariety, i.e. $(X, \mathcal{F})$ is a pencil of rational curves. As a result we introduce,

III.1.3 Definition/More Terminology. A R-extremal subvariety Y is a subvariety of a R-pre-extremal subvariety Y' which is maximal amongst the subvarieties of Y' which are covered by invariant curves passing through at least one point of the image in X of the singular locus of $(\mathcal{X}, \mathcal{F})$.

So indeed unless $(X, \mathcal{F})$ is a pencil of rational curves then extremal and pre-extremal coincide, while in the awkward case an extremal variety will be specified by taking the invariant curves passing through an appropriate component of the singular locus. Now pulling everything back by the moduli map, $\pi : \mathcal{X} \to X$, define R-extremal stack in the obvious way, idem for the locus, denoted $\mathcal{L}\mathrm{oc}\,(R)$, and observe,

III.1.4 Fact. *The locus $\mathcal{L}\mathrm{oc}\,(R)$ of an extremal ray, is a finite union of R-pre-extremal stacks. Denote by $\mathcal{L}\mathrm{oc}'\,(R)$ the subvariety which is the union of R-extremal stacks, then any $\mathcal{Y} \subset \mathcal{L}\mathrm{oc}'\,(R)$ making up this union is covered by $-1/d\,\mathbb{F}$ curves, where d may vary from curve to curve. There is however a family $\mathcal{L} \to T$ of stacks, possibly non-flat at the non-scheme like points, such that, $(\mathcal{L}, \mathcal{L}/T) \to (\mathcal{Y}, \mathcal{F})$ is a generically finite map of foliated stacks, where of course $(\mathcal{L}, \mathcal{L}/T)$ is the foliation corresponding to the fibration $\mathcal{L} \to T$.*

III.2. Digression on Deformation

As remarked in the introduction the Hilbert scheme of a stack is not known to exist in general, and whence we're a priori missing some natural way to go from formal positivity to actualy being able to move in algebraic families. Fortunately however our stacks are all orbifolds over schemes so following Kollàr, [K2], we may easily remedy this. Specifically suppose $\pi : \mathcal{X} \to X$ is an integral stack with projective moduli, and generically finite stabiliser, the for $U = \coprod_\alpha U_\alpha \to X$ a sufficiently fine étale cover of X by integral affines then we know I.2.5 that $\mathcal{X} \times_X U_\alpha = [V_\alpha / G_\alpha]$, for G_α a finite group of automorphisms of V_α. Put $V = \coprod_\alpha V_\alpha$, and let $V^{\text{ét}}$ be the dense open subset where $V \to X$ is étale, then we may consider the étale equivalence relation,

$$R = V \cup (V^{\text{ét}} \times_X V^{\text{ét}} \backslash \Delta) \quad \underset{\mathrm{id} \cup p_2}{\overset{\mathrm{id} \cup p_1}{\rightrightarrows}} \quad V$$

where p_i are the projections. By [A2], the quotient exists as an algebraic space, and is what Kollàr calls a bug-eyed cover of X, since unless $V \to X$ were actually étale, equivalently $\mathcal{X} = X$, then it is non-separated by virtue of having too many tangent directions. In any case we denote the said quotient $\pi^b : X^b \to X$.

Now if Y is an irreducible proper, so in particular separated, algebraic space then $\mathrm{Hom}\,(Y,\, X^b)$ exists as an algebraic space, and satisfies the usual kind of deformation estimates. Furthermore inside X^b we have a rather strict analogue of the non-scheme like points of $\mathcal{X}$, namely the non-separated points, which we'll denote N^b and $\mathcal{N}$ respectively. Equally although we may know little about it we nevertheless have a functor $\mathrm{Hom}\,(Y,\, \mathcal{X})$ of algebraic spaces to sets given by,

$$\mathrm{Hom}\,(Y,\, \mathcal{X})(T) = \{T \text{ morphisms } Y \times T \to \mathcal{X} \times T\}\,.$$

Inside both these functors we can consider the sub-functors $\mathrm{Hom}'(Y,\, X^b)$, and $\mathrm{Hom}'(Y,\, \mathcal{X})$ of maps which don't factor through the non-separated, respectively non-scheme like points. Unsurprisingly we have,

III.2.1 Fact. *Notations as above then, $\mathrm{Hom}'(Y,\, X^b) = \mathrm{Hom}'(Y,\, \mathcal{X})$, not just as functors but as an actual natural identification between elements.*

Proof. To give an element f^b_T, or f_T is simply to give a T-morphism $f_T : Y \times T \to X \times T$ which can be lifted be it to the bug eyed cover or the stack, and which of course doesn't factor through the closed sub-scheme, $\pi(N^b) = \pi(\mathcal{N})$. Lifting is however a local discussion, so if $Y_\alpha = f_T^{-1}(U_\alpha \times T)$ then we require to lift, $f_\alpha : Y_\alpha \to U_\alpha \times T$ to $V^b_\alpha \times T$, respectively the classifying stack $[V_\alpha/G_\alpha] \times T$. On the other hand both of these admit $V_\alpha \times T$ as an étale cover; so in either case this amounts to the fibre product,

$$\begin{array}{ccc} V_\alpha \times T & \xleftarrow[\tilde{f}_\alpha]{} & \tilde{Y}_\alpha \\ \downarrow & & \downarrow \\ U_\alpha \times T & \xleftarrow[f_\alpha]{} & Y_\alpha \end{array}$$

having étale right hand arrow. Any lifting, if it exists, is under the given hypothesis unique, so we're done.

□

Of course, this is very much a case of better still, since the co-tangent bundles of X^b and $\mathcal{X}$, respectively, are defined by the identical subterfuge of the respective étale coverings by V, so that if f^b, f are liftings of the same map $f : Y \to X$, as ever not factoring through $\pi(N^b) = \pi(\mathcal{N})$, then $(f^b)^* \Omega^1_{X^b} = f^* \Omega^1_{\mathcal{X}}$. Putting these observations together with op. cit. 2.9, we therefore obtain,

III.2.2 Better Fact. *Suppose $\mathcal{X}$ is a smooth stack, and $f : Y \to \mathcal{X}$ a map not factoring through the non-scheme like points then there is an algebraic space T, possibly non-separated, of dimension at least,*

$$h^0(f^* T_{\mathcal{X}}) - h^1(f^* T_{\mathcal{X}})$$

together with a deformation $F : Y \times T \to \mathcal{X} \times T$ of f such that the corresponding map, $T \to \mathrm{Hom}\,(Y, X)$ is injective.

The better fact does, of course, suggest that T comes equipped with a natural separated stack structure. This could probably be proved by going through the details of [A4]. In cases where it may be of relevance, however, we'll easily be able to prove this, and more besides, wholly directly.

III.3. Finding Weighted Projective Spaces

As ever let $(\mathcal{X}, \mathcal{F})$ be a smooth foliated stack with log canonical foliation singularities, terminal at the non-scheme like points. Now let $f : \mathcal{L} \to \mathcal{X}$ be a smooth $-\frac{1}{d}\,\mathbb{F}$ curve with eigenvalues $\frac{a_1}{d} \geq \frac{a_2}{d} \geq \cdots \geq \frac{a_n}{d}$ at the unique point z where f meets the singular locus where of course eigenvalue refers to the eigenvalues of the habitual normalisation of a linearisation ∇_z of a local generator ∂ of $T_{\mathcal{F}}$ about z in $\mathrm{End}\,(N_{\mathcal{L}/\mathcal{X}} \otimes \mathbb{C}(z))$. If $a_1 \leq 0$, then we simply have nothing to say for the moment. Otherwise there is a formal invariant sub-stack $\hat{\mathcal{Y}}$ of the completion of $\mathcal{L}$ in $\mathcal{X}$ such that $N_{\mathcal{L}/\hat{\mathcal{Y}}}$ is ample. Consequently we may apply the Chow lemma to conclude that the minimal integral sub-stack $\mathcal{Y}$ of $\mathcal{X}$ containing $\hat{\mathcal{Y}}$ has the same dimension as $\hat{\mathcal{Y}}$, so in fact $\hat{\mathcal{Y}}$ is the completion of $\mathcal{Y}$ in $\mathcal{L}$. By construction $\mathcal{Y}$ is smooth in a neighbourhood of $\mathcal{L}$, so we can apply the deformation estimates to the cover $\mu : \mathbb{P}^1 \to \mathcal{L}$, $y \mapsto y^d$, ramified in the origin z with of course y^d a coordinate on the same. Now a priori the deformation guaranteed by III.2.2 is not complete, nor are deformations obtained by pulling back from the moduli deformations stricta dictum since we may lose flatness. However $\mathcal{Y}$ is invariant so we can apply a minor variant of the cone theorem to conclude,

III.3.1 Initial Fact. *If $\mathcal{L}$ corresponds to an extremal ray R in Néron-Severi, with supporting function H_R, and ample bundle $A_R = mH_R - K_{\mathcal{F}}$, then for all $y \in \mathcal{Y}$, there is a $-1/d(y)\,\mathbb{F}$ stack $\mathcal{L}_y \ni y$ in $\mathcal{Y}$ parallel to R in Néron-Severi.*

Proof. Indeed if $\mathcal{C} \subset \mathcal{X}$ is any integral 1-dimensional sub-stack lying over a curve C in X then $\pi_*[\mathcal{C}]$ is a multiple of C in Néron-Severi. On the other hand our initial deformation in $\mathcal{X}$ can be completed to a deformation of $\pi \circ f \circ \mu$ in X, and of course if $\sum_i a_i\, C_i$ is some effective 1-cycle numerically equivalent to $\pi_*[\mathcal{L}]$ then every C_i is so equivalent by virtue of the extremality. $\Box$

Now the singular point is scheme like, so if $L \hookrightarrow X \times \mathbb{P}^1 \times T$ is a flat family in $X \times \mathbb{P}^1$ completing our initial deformation, with p to X the projection then the completion in $p^{-1}(z)$ is a formal subscheme mapping properly by p to the completion of X (or equally $\mathcal{X}$) in z. Profiting from the closed nature of invariance we can insist that $p(L_t)$ is $\mathcal{F}$-invariant for every t, and so by a minor variant of our initial fact we may deduce for $\mathcal{L}$ parallel to some extremal R in Néron-Severi (which, incidentally, we'll suppose without further comment).

III.3.2 Further Fact. *For each eigendirection $\frac{\partial}{\partial x_i}$ of ∇_z in $\mathrm{End}\,(N_{\mathcal{L}/\mathcal{Y}} \otimes \mathbb{C}(z))$ there is a smooth $-1/d_i\,\mathbb{F}$ invariant stack $f_i : \mathcal{L} \to \mathcal{X}$ through z and parallel to R in Néron-Severi.*

Additionally points in $\mathbb{P}^t(\mathbb{Q})$, $t \in \mathbb{N}$, are, up to ± 1, uniquely represented by $t+1$ tuples of integers with $\gcd = 1$, so if we change to a more homogeneous notation, viz:

III.3.3 New Notation. Linearise a local generator ∂ of $T_{\mathcal{F}}$ in the completion of $\hat{\mathcal{O}}_{\mathcal{X},z}$ of $\mathcal{O}_{\mathcal{X},z}$ in $\mathfrak{m}_{\mathcal{X}}(z)$ by way of, $\partial = a_1\, y_1\, \frac{\partial}{\partial y_1} + \cdots + a_r\, \frac{\partial}{\partial y_r} - b_i\, x_i\, \frac{\partial}{\partial x_i}$, $a_i \in \mathbb{N}$, $b_i \in \mathbb{N} \cup \{0\}$, $(a_1, \ldots, a_r, b_1, \ldots, b_s) = 1$, with x_i local equations for $\mathcal{Y}$, and the summation convention in the obvious way, with s the codimension of $\mathcal{Y}$.

Then for each eigendirection $\frac{\partial}{\partial y_i}$ we have a smooth invariant $-1/d_i\,\mathbb{F}$ stack $f_i : \mathcal{L}_i \to \mathcal{X}$ parallel to R in Néron-Severi, with $a_i \mid d_i$. Observe further that if a is the gcd of $a_1, \ldots, a_r$, then the generic invariant stack in $\mathcal{Y}$ is a $-1/da\,\mathbb{F}$ curve, so in particular,

III.3.4 Intermediary Fact. *The fundamental group of a leaf of the induced smooth foliation $(\mathcal{Y}\backslash z, \mathcal{F})$ is generically $\mathbb{Z}/da$ for some $d \in \mathbb{N}$.*

Note, additionally, that $\mathcal{Y}$ is a smooth stack. Indeed if it were not then there would be a smooth sub-stack $\mathcal{B}$ in the singular locus invariant by the induced foliation. On the other hand the said foliation is smooth on $\mathcal{Y}\backslash z$, so $\mathcal{B}$ meets any punctured neighbourhood of z. However by construction $\mathcal{Y}$ is smooth at z, and smoothness is open, so this is nonsense. Continuing in this vein let Y be the moduli of $\mathcal{Y}$, and Y_0 the image of $\mathcal{Y}$ in $\mathcal{X}$ then, $Y \to Y_0$ is 1 to 1 on closed points, $\mathcal{Y}$ is a closed sub-stack of a separated stack, so Y is separated whence it's actually a scheme, and a projective one at that, while by [V] 2.8 the smoothness of $\mathcal{Y}$ implies that Y is geometrically uni-branch with at worst quotient singularities.

Now let's put this together to calculate the Picard group of $\mathcal{Y}$. Quite generally we have that $\mathrm{Pic}\,(\mathcal{Y})_{\mathbb{Q}} = \mathrm{Pic}\,(Y)_{\mathbb{Q}}$, so let's start with $\mathrm{Pic}\,(Y)$. Since Y has only quotient singularities they are in particular rational so if $\rho : \tilde{Y} \to Y$ is a resolution of singularities then $H^1(Y, \mathcal{O}_Y) = H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$. However $\tilde{Y}$ is covered by rational curves through a point, so $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$ and whence we have a commutative diagram,

$$\begin{array}{ccc} \mathrm{Pic}\,(Y)_{\mathbb{Q}} & \xrightarrow[\rho^*]{} & \mathrm{Pic}\,(\tilde{Y})_{\mathbb{Q}} \\ \Big\downarrow & & \Big\downarrow \\ NS^1(Y)_{\mathbb{Q}} & \xrightarrow[\rho^*]{} & NS^1(\tilde{Y})_{\mathbb{Q}} \end{array}$$

where the horizontal arrows are quite generally injective, and we've established the injectivity of the right vertical, whence (rather obviously, but a good reference wasn't clear), $\mathrm{Pic}\,(Y)_{\mathbb{Q}} \xrightarrow{\sim} NS^1(Y)_{\mathbb{Q}}$. However $NS_1(Y)_{\mathbb{Q}}$ is known, e.g. [K1] II.4.21, so: $\mathrm{Pic}\,(\mathcal{Y})_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}$, and the remaining issue is the possibility of torsion. So suppose there was torsion, then we would obtain an irreducible étale cover $\tau : \mathcal{Y}' \to \mathcal{Y}$, with fibre over $z, z_1, \ldots, z_d$, say. On the other hand $\mathcal{Y}$ is covered by simply connected stacks, so every invariant stack $\mathcal{M} \ni z$, is covered by d-copies of the same, with exactly one through each z_i, which is obvious nonsense since the family of invariant curves through any z_i would define an irreducible sub-stack of $\mathcal{Y}'$ not equal to it, i.e. $\mathrm{Pic}\,(\mathcal{Y}) \xrightarrow{\sim} \mathbb{Z}$. It remains to establish the exact relation of the divisors $\mathcal{D}_i$ in $\mathcal{Y}$ which are infinitesimally $y_i = 0$ in our coordinate system at z, where the existence of the $\mathcal{D}_i$ as well-defined divisorial sub-stacks of $\mathcal{Y}$ is just another application of the Chow lemma in the spirit of III.3.1. The said infinitesimal calculation already yields, however, for $\mathcal{L}_j$ the invariant stack tangent to $\frac{\partial}{\partial y_j}$, $\mathcal{D}_i \cdot \mathcal{L}_j = \frac{a_i}{a_j}$, and we know that $\mathcal{O}_{\mathcal{L}_j}(K_{\mathcal{F}})$ generates $\mathrm{Pic}\,(\mathcal{L}_j)$ for any j, by I.9.3. Consequently $K_{\mathcal{F}}$ generates $\mathrm{Pic}(\mathcal{Y})$, and $d_i = a_i d$, for some $d \in \mathbb{N}$ independent of i, with $\mathcal{D}_i = -d_i\, K_{\mathcal{F}}$.

With this behind us we can now proceed to extract roots of the sections, $\gamma_i \in \mathcal{O}_{\mathcal{Y}}(-d_i\, K_{\mathcal{F}})$ defining the $\mathcal{D}_i$ by the usual procedure. We do this systematically, observe that no $y_j|_{\mathcal{L}_j}$ has a root of any sort around z, so the d_1-th root cover of γ_1 in $\mathbb{V}(d_1\, K_{\mathcal{F}})$ is smooth and irreducible, and since $\mathcal{D}_1$ is invariant by $\mathcal{F}$, the induced foliation on the cover has canonical bundle isomorphic to the pull-back of $K_{\mathcal{F}}$. On the other hand the reduced pull-back of $\mathcal{D}_1$ is isomorphic to $\mathcal{D}_1$, so the $\mathcal{L}_j$, $j \geq 2$ are unchanged, while equally the γ_i are still sections of $K_{\mathcal{F}}^{-d_i}$ without d_i-th roots for $i \geq 2$, so continuing in this way we obtain a cover $\rho : \tilde{\mathcal{Y}} \to \mathcal{Y}$ ramified only in the $\mathcal{D}_i$ such that the induced foliation $(\tilde{\mathcal{Y}}, \tilde{\mathcal{F}})$ has canonical bundle $\rho^* K_{\mathcal{F}}$. Notice that $\tilde{\mathcal{Y}}$ is separated, whence so is its moduli space $\tilde{Y}$ which in turn is finite over Y, so $\tilde{Y}$ is projective. Now if $z \in \mathcal{M} \subset \mathcal{X}$ is a generic invariant stack through the origin, then for any j, $\mathcal{D}_j. \mathcal{M} = +\, d_j/d$, which is just the multiplicity of $\mathcal{Y}_j$ on the cusp, so apart from z, $\mathcal{M}$ doesn't meet the ramification locus, whence if $\tilde{\mathcal{M}} \to \mathcal{M}$ is a lifting then over $\mathcal{M} - z$, g is unramified, thus $(\tilde{\mathcal{M}} : \mathcal{M})$ divides da. Notice, however, if H_j is the reduced pull-back of $\mathcal{D}_j$ then the intersection of H_j with $\tilde{\mathcal{M}}$ is supported only in the unique point $\tilde{z}$ of z around which everything is scheme like, so:

$$\mathbb{N} \ni H_j.\, \tilde{\mathcal{M}} = \frac{1}{d_j}\, \rho^*\, \mathcal{D}_j.\, \tilde{\mathcal{M}} = \frac{(\tilde{\mathcal{M}} : \mathcal{M})}{da}$$

whence $H_j.\, \tilde{\mathcal{M}} = 1$, for all j, and $(\tilde{\mathcal{M}} : \mathcal{M}) = da$, so in fact $\tilde{\mathcal{M}} \xrightarrow{\sim} \mathbb{P}^1$. Repeating this calculation for the generic invariant stack in $\mathcal{D}_j$, then the $\mathcal{D}_j \cap \mathcal{D}_i$, right down to the $\mathcal{L}_k$, we conclude that $\tilde{\mathcal{Y}}$ is smoothly foliated by $\mathbb{P}^1$'s. In particular, away from $\tilde{z}$, $(\tilde{\mathcal{Y}}, \tilde{\mathcal{F}})$ is locally a product of the form $\Delta \times [\Delta^{r-1}/G]$ of a disc with a smooth classifying stack, with $\tilde{\mathcal{F}}$ the projection to the classifying stack, from which the non-scheme like locus of $\tilde{\mathcal{Y}}$ is invariant by $\tilde{\mathcal{F}}$, so the image under ρ in $\mathcal{Y}$ of the non-scheme like locus gives a non-scheme like sub-stack of $\mathcal{X}$ invariant by the foliation, contrary to the hypothesis of terminal singularities everywhere along the non-scheme like locus. Consequently $\tilde{\mathcal{Y}} = \tilde{Y}$ is projective, with leaf space isomorphic to $P(T_{\tilde{\mathcal{Y}}} \otimes \mathbb{C}(\tilde{z})) \xrightarrow{\sim} \mathbb{P}^{r-1}$. Better still if $y_i = \eta_i^{d_i}$, for $\eta_1, \ldots, \eta_r$ coordinates on $\tilde{Y}$ around $\tilde{z}$ then $d\rho^*\partial$ is precisely $\eta_1 \frac{\partial}{\partial \eta_1} + \cdots + \eta_r \frac{\partial}{\partial \eta_r}$ in the completion of $\mathcal{O}_{\tilde{Y},\tilde{z}}$ in $\mathfrak{m}(\tilde{z})$, so blowing up in $\tilde{z}$ resolves the singularity, and the holonomy of every leaf is trivial, so in fact the said blow up is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{r-1}$ with a section, the exceptional divisor, whence, cf. [K1] V.3, $(\tilde{\mathcal{Y}}, \tilde{\mathcal{F}})$ is actually $(\boldsymbol{\mathcal{P}}^r, \mathcal{R})$ i.e. $\mathbb{P}^r$ with the foliation the pencil of lines through some point. To conclude to the final result observe that the choice of linearising coordinates for a field with positive eigenvalues is unique up to the centraliser of the associated diagonal matrix, so $\mathrm{GL}(r, \mathbb{C})$ on $\mathbb{P}^r$ and the centraliser of $\mathrm{diag}\,\{d_1, \ldots, d_r\}$, on $\mathcal{Y}$. In either case $\eta_1, \ldots, \eta_r$ are not just local, but actual standard affine coordinates on some standard affine patch $\mathbb{A}^r$ on $\mathbb{P}^r$. Additionally our knowledge of the completion at $\tilde{z}$ even tells us that the relative Galois group $G = \mu_{a_1} \times \cdots \times \mu_{a_r}$ acts on this patch by, $(\eta_1, \ldots, \eta_r) \times (\zeta_1, \ldots, \zeta_r) \mapsto (\eta_1\, \zeta_1, \ldots, \eta_r\, \zeta_r)$. As such the moduli is certainly a weighted projective space with weights $\left(1, \frac{a_1}{a}, \ldots, \frac{a_r}{a}\right)$ in standard coordinates. Now let $(\boldsymbol{\mathcal{P}}\left(\frac{a_1}{a}, \ldots, \frac{a_r}{a}\right), \mathcal{R})$ be the Gorenstien covering stack of the induced foliation by lines then since $\mathcal{Y}$ is already Gorenstien the map $\mathcal{Y} \to Y$ must factor through the covering stack. Equally our standard coordinate patch $\mathrm{mod}\, G$ gives a smooth affine neighbourhood of Y, and by our habitual intersection theory considerations we know the pull-back to $\mathbb{P}^r$ from $\mathcal{Y}$ of the non-scheme like locus is a plane, which in these coordinates is necessarily the hyperplane at infinity. As such $\mathcal{Y} \to \boldsymbol{\mathcal{P}}\left(\frac{a_1}{a}, \ldots, \frac{a_r}{a}\right)$ is a map of smooth stacks one to one on closed points and with ramification only over the plane at infinity, necessarily corresponding to an automorphism group $\mathbb{Z}/da$ by [V] 2.8, generically over the same, whence, in fact,

III.3.5 Proposition/Summary. *Let $f : \mathcal{L} \to \mathcal{X}$ be a smooth $-?\,\mathbb{F}$ extremal ray, with $a_1, \ldots, a_r$ the positive eigenvalues of a linearisation in $\mathrm{End}\,(T_{\mathcal{X}} \otimes \mathbb{C}(z))$ of a generator ∂ of the foliation around the unique point where $\mathcal{L}$ meets the singular locus, normalised so that the eigenvalue tangent to f is positive, and the totality of positive and negative values is relatively prime, then there is in fact a smooth invariant $(\boldsymbol{\mathcal{P}}(a_1 d, \ldots, a_r d), \mathcal{R})$, some $d \in \mathbb{N}$, centered on z whose lines are parallel to $\mathcal{L}$ in Néron-Severi, and of course $N_{\mathcal{L}/\boldsymbol{\mathcal{P}}_l(da_1, \ldots, da_r)}$ is the ample part of $N_{\mathcal{L}/\mathcal{X}}$.*

III.4. Ignoring Cusps

So far we haven't discussed what may happen if our extremal ray R is represented by an invariant stack $f : \mathcal{L} \to \mathcal{X}$ which has a cusp at the unique singular point z where f meets $\operatorname{sing}(\mathcal{F})$. Thus, to that end, let f indeed be such, and indeed let's say that it's $-1/d\,\mathbb{F}$, so $\mathcal{L}$ is the bad orbifold with signature d at ∞. As ever a resolution of the cusp is dictated by the Euclidean algorithm for $v_1, \ldots, v_k$ where we normalise a local generator ∂ to $T_{\mathcal{F}}$ in the completion of $\hat{\mathcal{O}}_{\mathcal{X},z}$ of $\mathcal{O}_{\mathcal{X},z}$ in $\mathfrak{m}_{\mathcal{X}(z)}$ as per II.8, and indeed precisely retake the notations therein. In particular if $\tilde{f} : \mathcal{L} \to \tilde{\mathcal{X}}$ is the resolution obtained via blowing up in points according to the Euclidean algorithm, then there is a part of the Harder-Narismhan filtration (or perhaps more correctly a sub-quotient) of $N_{\mathcal{L}/\tilde{\mathcal{X}}}$ of the form $\bigoplus_{i=1}^{k} \mathcal{O}_{\mathcal{L}}\left(\frac{\tilde{v}_i}{e}\right)$ where $\tilde{v}_1 = 1$, and $\tilde{v}_2, \ldots, \tilde{v}_k > 1$ with $e \mid d$ the order of the holonomy at infinity. Now it may happen that $\tilde{f}$ is not extremal, nevertheless the ubiquitus Chow lemma combined with the full force of II.6.2, tell us that we can find a closed invariant sub-stack $\tilde{\mathcal{Y}}$ of $\tilde{\mathcal{X}}$ containing $\mathcal{L}$ with $N_{\mathcal{L}/\tilde{\mathcal{Y}}} \xrightarrow{\sim} \bigoplus_{i=1}^{k} \mathcal{O}_{\mathcal{L}}\left(\frac{\tilde{v}_i}{e}\right)$. Necessarily the standard covering map, $\nu^*\tilde{f} : \mathbb{P}^1 \to \tilde{\mathcal{Y}}$ moves as it should thanks to III.2.2, and by way of the moduli of $\tilde{\mathcal{X}}$ we can complete this in the weak sense of being able to cover $\tilde{\mathcal{Y}}$ by invariant stacks. If we project everything down to $\mathcal{X}$, with $\mathcal{Y}$ the image of $\tilde{\mathcal{Y}}$, the situation is much better. By construction the completion of $\mathcal{Y}$ at z contains a formal affine of the same dimension so $\mathcal{Y}$ is smooth everywhere around $\mathcal{L}$. Better still our deformation of $\nu^*\tilde{f} : \mathbb{P} \to \tilde{\mathcal{Y}} \to \mathcal{Y}$, completed by way of the moduli of $\mathcal{X}$, assures that, at not just every point of $\mathcal{Y}$, but at every formal invariant curve in $\hat{\mathcal{O}}_{\mathcal{Y},z}$ there is a $-1/d\,\mathbb{F}$ curve, evidently for varying d, parallel to R in Néron-Severi. Thus as per the smooth case, algorithmic resolution implies that $\mathcal{Y}$ is smooth, while for each eigen direction $\frac{\partial}{\partial y_i}$ in $T_y \otimes \mathbb{C}(z)$, there is even a smooth $-1/d_i\,\mathbb{F}$ curve, $f_i : \mathcal{L}_i \to \mathcal{X}$, parallel to R in Néron-Severi. Bearing in mind, however, that in the notation of II.8.4, $\gcd(ev_1, \ldots, ev_k, a_1, \ldots, a_l) = 1$, the completion $\hat{\mathcal{Y}}$ of $\mathcal{Y}$ at z must be contained in the unique formal invariant subvariety determined by the positive eigen-directions. On the other hand we already know that this is a weighted projective space by III.3.5, so in fact $(\mathcal{Y}, \mathcal{F})$ with the induced foliation is $(\boldsymbol{\mathcal{P}}(dv_1, \ldots, dv_k), \mathcal{R})$. So a d-th multiple of f, can certainly be moved into a dv_i-th multiple of f_i, or perhaps more correctly $\nu^* f : \mathbb{P}^1 \to \mathcal{X}$ to $\nu_i^* f_i : \mathbb{P}^1 \to \mathcal{X}$, for $\nu_* : \mathbb{P}^1 \to \mathcal{L}_*$ the standard covering appear in the same family. In any case, we can certainly move a cusp to a smooth, and so ignore the former.

III.5. Structure of Extremal Stacks

To begin with let $f : \mathcal{L} \to \mathcal{X}$ be a smooth $-\frac{1}{d_1}\,\mathbb{F}$ stack parallel to an extremal ray R in Néron-Severi. As ever we normalise a local generator ∂ of the foliation in the complete local ring $\hat{\mathcal{O}}_{\mathcal{X},z}$, for $z = f^{-1}(\operatorname{sing}\mathcal{F})$, according to II.3.3 with $d_1 = a_1 d$. Furthermore we may take the standard covering map, $\mu : \mathbb{P}^1 \to \mathcal{L} : y \mapsto y^{d_1}$, and apply a minor variant of the deformation estimates of III.2.2, to $\mu^* f : \mathbb{P}^1 \to \mathcal{Y}_{\geq 0}$, where $\mathcal{Y}_{\geq 0}$ is the formal sub-stack of the completion of $\mathcal{X}$ in $\mathcal{L}$ whose normal bundle is,

$$\bigoplus_{i=2}^{r} \mathcal{O}_{\mathcal{L}}\left(\frac{a_i}{a_1}\right) \oplus \mathcal{O}_{\mathcal{L}}^{\oplus s}$$

with s the dimension of the kernel of ∂ reduced mod $\mathfrak{m}_{\mathcal{X}}(z)^2$. Consequently deforming $\mu^* f$ in $\mathcal{Y}_{\geq 0}$ is unobstructed and gives a formal sub-algebraic space supported at $\mu^* f$ of $\mathrm{Hom}(\mathbb{P}^1, X^b)$, for X^b the associated bug eyed cover, with the same tangent space, so it is in fact all of the deformation space completed at $\mu^* f$. Profiting from the habitual subterfuge of completing the deformation in the moduli X, then passing to the associated closed integral sub-stack, we obtain an integral invariant sub-stack $\mathcal{Y}$ of $\mathcal{X}$ whose completion at $\mathcal{L}$ is $\mathcal{Y}_{\geq 0}$, and through every point of which there is a $-1/e\,\mathbb{F}$ stack, for varying e, parallel to R in Néron-Severi. Now let Z be the necessarily scheme like intersection of $\mathcal{Y}$ with the singular locus of $\mathcal{F}$, and suppose that Z isn't connected, say a disjoint union of components Z', Z'', then we may consider the sub-stacks $\mathcal{Y}'$, $\mathcal{Y}''$ covered by $K_{\mathcal{F}}$-negative extremal 1-dimensional stacks through Z' and Z'' respectively, consequently if $y \in \mathcal{Y}' \cap \mathcal{Y}''$ it is a singular point of some extremal 1-dimensional invariants stacks $\mathcal{L}'$, $\mathcal{L}''$, so in $Z' \cap Z''$, which is nonsense, so in fact Z is connected. Better still at z we know that Z is irreducible and smooth of $\dim = s$, so there is some irreducible component Z_0 of $\mathrm{sing}(\mathcal{F})$ of dimension s contained wholly in Z. However for any $\zeta \in Z$, there is a $-1/e(\zeta)\,\mathbb{F}$ stack $\mathcal{L}_\zeta \ni \zeta$, and indeed contained in $\mathcal{Y}$, so $\mathrm{sing}(\mathcal{F})$ is smooth at ζ by II.8.4, and whence $\zeta \mapsto \dim_\zeta \mathrm{sing}(\mathcal{F})$ is not just upper semi-continuous but continuous, i.e. constant $= s$, since Z is connected. Consequently $Z_0 = Z$, and is smooth irreducible of dimension s.

Now consider the ideal I_Z of Z in $\mathcal{X}$, along with the ideal $I_{Z,\mathcal{Y}}$ of Z in $\mathcal{Y}$ then our linearisation procedure for vector fields at singular points globalises to $\mathcal{O}_Z$-linear maps, fitting into a diagram,

$$\begin{array}{ccc} I_Z/I_Z^2 & \xrightarrow[D_Z]{} & I_Z/I_Z^2 \otimes K_{\mathcal{F}} \\ \big\downarrow & & \big\downarrow \\ I_{Z,\mathcal{Y}}/I_{Z,\mathcal{Y}}^2 & \xrightarrow[D_{Z,\mathcal{Y}}]{} & I_{Z,\mathcal{Y}}/I_{Z,\mathcal{Y}}^2 \otimes K_{\mathcal{F}} \end{array}$$

and defined initially, in a neighbourhood of Z, as the composition of the maps,

$$I_Z \xrightarrow[d]{} \Omega_X \longrightarrow K_{\mathcal{F}}.I_Z\,.$$

If we double dualise the lower map we get a map of reflexive sheaves,

$$D_{Z,\mathcal{Y}}^{\vee} : (I_{Z,\mathcal{Y}}/I_{Z,\mathcal{Y}}^2)^{\vee\vee} \to (I_{Z,\mathcal{Y}}/I_{Z,\mathcal{Y}}^2)^{\vee\vee} \otimes K_{\mathcal{F}}$$

so that the trace gives a section of $\mathcal{O}_Z(K_{\mathcal{F}})$ in codimension 2, and Z is smooth so in fact a section everywhere, which is certainly non-zero if the trace is non-zero at z. On the other hand at every point ζ of Z, D_Z is an isomorphism, since it's residue is nothing other than the identity on the non-zero eigenspaces up to a multiplicative constant. Consequently, $\mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_Z(\{\mathrm{codim}\, Z\}K_{\mathcal{F}})$, so $K_{\mathcal{F}}\mid_Z$ is in fact $\mathcal{O}_Z$ if the trace is non-zero somewhere. More generally for a matrix with eigenvalues a_i,

$-b_j$, a_i, $b_j > 0$, the second symmetric function is given by,

$$\frac{1}{2}\left(\sum_i a_i - \sum_j b_j\right)^2 - \frac{1}{2}\left(\sum_i a_i^2 - \sum_j b_j^2\right)$$

so at worst there is an étale double cover $Z^{+-} \to Z$ such that $K_{\mathcal{F}} \mid_{Z^{+-}}$ is trvial. As a result the eigenvalues of D_Z are well defined constant functions up to a choice of generator of $\mathcal{O}_Z(K_{\mathcal{F}})$ when this is possible, and otherwise they're well defined on Z^{+-}. This if necessary we choose a lifting z^+ of z to the double cover, and subsequently choose our local generator in such a way to have compatibility with our formal linearisation at z (identified locally with z_+ if necessary, i.e. the eigenvalues of D_Z are everywhere $a_1, \dots, a_r, -b_1, \dots, -b_t$, with $a_i, b_j \in \mathbb{N}$, and $\gcd(a_1, \dots, a_r, b_1, \dots, b_t) = 1$. In any case, for every $\zeta \in Z$, there is a well defined pair of eigenspaces, $\{T_{+(\zeta)}, T_{-(\zeta)}\}$ of $T_{\mathcal{X}} \otimes \mathbb{C}(\zeta)$, and every $K_{\mathcal{F}}$-negative 1-dimensional invariant stack has tangent space at ζ contained in precisely one of these. To fully profit from this we will have to extend from the normal bundle to a formal neighbourhood of Z, which probably shows that being lazy about convergence wasn't perhaps an optimal use of time. The discussion is local over affine neighbourhoods U of Z over which the normal bundle and $K_{\mathcal{F}}$ trivialise, and which we consider centered on a point ζ of Z. To momentarily simplify the notations let λ_i denote the necessarily non-zero eigenvalues of the normal bundle, and consider the following inductive proposition,

Let $\hat{\mathcal{O}}_U$ be the completion of $\mathcal{O}_U$ in $\mathfrak{m}_{\mathcal{X}}(\zeta)$, and for $k \in \mathbb{N}$, we have coordinates x_i normal to Z (evidently giving a basis for $N^{\vee}_{Z/x}$) and a generator ∂ of $\mathcal{F}$ over U such that,

(1) $\partial x_i = \lambda_i \, x_i \,(\mathrm{mod} I_Z^k)$

(2) There is a semi-simple generator $\hat{\partial}$ of $T_{\mathcal{F}} \otimes \hat{\mathcal{O}}_{U,\zeta}$ of the form $\lambda_i \, \xi_i \, \frac{\partial}{\partial \xi_i}$, for $\xi_i \in \hat{\mathcal{O}}_{U,\zeta}$ and $\xi_i = x_i \,(\mathrm{mod} I_Z^k)$.

The case $k = 1$ trivially follows from the previous discussion, so consider going from k to $k+1$, which evidently we wish to be compatible with restriction so that things converge. In any case, in terms of our usual notations about monomials and summation conventions we have, $\mathrm{mod}\, I_Z^{k+1}$,

$$\partial x_i = \lambda_i \, x_i + a_{iJ} \, x^J \, , \; a_{iJ} \in \mathcal{O}_U \, , \; \xi_i = x_i + b_{iJ} \, x^J \, , \; b_{iJ} \in \hat{\mathcal{O}}_{U,\zeta} \, .$$

Furthermore, $\hat{\partial} = u\partial$, $u \in \hat{\mathcal{O}}_{U,\zeta}$, and, $u = 1 + u_{iK} \, x^K$, $u_{iK} \in \hat{\mathcal{O}}_{U,\zeta}$, with $\# J = k$, $\# K = k - 1$. Now if we just put these equations together then we obtain,

$$a_{iJ} = (\lambda_i - \lambda_J) \, b_{iJ} - u_{iK} \, \lambda_i \, , \qquad \text{if } x^K \, x_i = x^J$$

$$a_{iJ} = (\lambda_i - \lambda_J) \, b_{iJ} \, , \qquad \text{otherwise}$$

without any summations. The second case is rather good since if $\lambda_i \neq \lambda_J := j_p \, \lambda_p$ we conclude that the b_{iJ} are algebraic, so if without loss of generality we replace

x_i, by,

$$x_i \mapsto x_i + \sum_{\substack{\lambda_i \neq \lambda_J \\ x_i \nmid x^J}} b_{iJ}\, x^J$$

then in fact we conclude that $a_{iJ} = 0$ if $x_i \nmid x^J$. As for the 1^{st}-case we do what we can. Specifically, again without loss of generality we can replace x_i by,

$$x_i \mapsto x_i + \sum_{\lambda_i \neq \lambda_J} \frac{a_{iJ}}{\lambda_i - \lambda_J}\, x^J$$

so that $a_{iJ} = 0$ if $\lambda_i \neq \lambda_J$, while if $\lambda_i = \lambda_J$ we conclude that u_{iK} is algebraic. Thus if we replace ∂ by,

$$\partial \mapsto \left(1 + \sum_{\lambda_K = 0} u_{iK}\, x^K\right) \partial$$

then $u_{iK} = 0$ if $\lambda_K = 0$, so in fact we can suppose $a_{iJ} = 0$ for all J. Consequently, $\hat{\partial}$ has the form,

$$\left(1 + \sum_{\lambda_K \neq 0} u_{iK}\, x^K\right) \partial\,.$$

However if we replace $\hat{\partial}$ by,

$$\tilde{\partial} = \left(1 + \sum_{\lambda_K \neq 0} \tilde{u}_{iK}\, \xi^K\right)^{-1} \hat{\partial}$$

for $\tilde{u}_{iK}$ appropriate functions of coordinates z in $\hat{\mathcal{O}}_{Z,\zeta}$ which restrict from coordinates in $\hat{\mathcal{O}}_{U,\zeta}$ annihilated by ∂, and of course $\tilde{u}_{iK} = u_{iK} \pmod{I_Z}$, then by II.7.2 $\tilde{\partial}$ is still semi-simple, with respect to a possibly different basis $\tilde{\xi}_i$ of the form $v_i\, \xi_i$, $v_i \equiv 1\,(I_Z^{k-1})$. To complete the induction, therefore, it suffices to observe, on supposing without loss of generality that $\xi_i = \tilde{\xi}_i$, that,

$$\xi_i \mapsto \xi_i - \sum_{\substack{\lambda_K = 0 \\ x^J = x_i\, x^K}} \tilde{b}_{iJ}(z)\, \xi^J$$

for $\tilde{b}_{iJ}$ satisfying much the same prescriptions as the $\tilde{u}_{iK}$ is still a trivialising basis for $\hat{\partial}$.

Consequently over an appropriately small affine U containing ζ, and bearing in mind that for any $\zeta' \in Z$ we know we can find appropriate coordinates in $\hat{\mathcal{O}}_{\mathcal{X},\zeta'}$ annihilated by ∂, we obtain formal subschemes U_+, U_- of the completion $\hat{U}$ of U in Z, whose subsequent completion at any $\zeta' \in Z \cap U$ is the non-negative Harder-Narismhan pair of II.7.8. The monodromy of the pair $\{U_+, U_-\}$ is precisely the monodromy of the pair $\{T_{X_+}, T_{X_-}\}$, so either these patch to formal subschemes, X_+, X_- of the completion of $\mathcal{X}$ in Z, which completed at any point is the non-negative H-N pair, and of course we normalize so that $\forall\, \zeta \in Z$, $T_+(\zeta) = T_{X_+} \otimes$

$\mathbb{C}(\zeta)$, $T_-(\zeta) = T_{X_-} \otimes \mathbb{C}(\zeta)$, or we get the same conclusion on a double covering of the completion.

With this out of the way we can quickly proceed to a conclusion. To begin with complete $\mathcal{Y}$ in Z, call it $\hat{\mathcal{Y}}$, say, then this is an integral formal scheme supported on Z whose completion at z is either X_+, or, $X_+ \cup X_-$, which is always well defined even if X_+, X_- are only well defined on a cover. Regardless by the usual propagation of singularities along leaves/algorithmic resolution argument $\mathcal{Y}$ is not smooth, except possibly around Z. Next consider projecting $\mathcal{Y}$ to Z, by sending an invariant 1-dimensional stack to its unique singular point, which we'll call p. Now if the mondoromy is non-trivial form the fibre product,

$$\mathcal{Y}^{+-} := \mathcal{Y}_p \times_Z Z^{+-}$$

then since the invariant rational curves covering it can only meet in Z^{+-} we see that the normalisation is disconnected, so there are irreducible stacks $\mathcal{Y}_+$, $\mathcal{Y}_-$ whose union is $\mathcal{Y}_{+-}$, and whose completion in Z^{+-} are X_+, X_- respectively. Equally, a minor variant of this reasoning shows that if the H-N pair has no monodromy, then the completion of $\mathcal{Y}$ at Z must be X_+, and so be it $\mathcal{Y}$ or either of the $\mathcal{Y}_+$, $\mathcal{Y}_-$ are everywhere smooth. Turning to the fibres $\mathcal{Y}_\zeta$ be it of $\mathcal{Y} \to Z$ in the monodromy free case, or more generally of $\mathcal{Y}_+ \to Z^{+-}$, so $\zeta \in Z$ or Z^{+-} as approriate, we see that it's completion $\hat{\mathcal{Y}}_\zeta$ is nothing other than, without loss of generality, the plus of the positive H-N pair. On the other hand $\mathcal{Y}_\zeta$ is covered by $-\mathbb{F}$ stacks, so $\hat{\mathcal{Y}}_\zeta$ is nothing other than the completion at ζ of a weighted projective stack $\mathcal{P}(da_1, \dots, da_r)$ obtained by first deforming from cuspidal $-\mathbb{F}$ stacks to smooth ones, then applying the recipe of III.3.5. Equally $\mathcal{Y}_\zeta$ is a closed stack, so it certainly contains the $\mathcal{P}(da_1, \dots, da_r)$, and since it cannot contain any other $-\mathbb{F}$-stacks except those in the weighted projective stack at worst its reduction is the said stack. $\mathcal{Y}_\zeta$ is, however, regularly embedded, since Z is smooth, and equal to the $\mathcal{P}(da_1, \dots, da_r)$ in a neighbourhood of ζ, so in fact they are equal everywhere. Consequently, if there is no monodromy, $p : \mathcal{Y} \to Z$ is a relatively smooth fibration, otherwise $p : \mathcal{Y}_{+\mathrm{or}-} \to Z^{+-}$ is, as is the non-scheme locus, whence d is independent of ζ, giving a fibration in weighted projective stacks, and as we've seen our particular weighted projective stacks are simply connected so the fibration is actually a bundle in the analytic topology, and in actual fact, as we'll establish later, in the Zariski. In any case to summarise we've established,

III.5.1 Large Fact. *Given a $-1/d_1\,\mathbb{F}$ curve $f : \mathcal{L} \to \mathcal{X}$ parallel to an extremal ray R in Néron-Severi, with z the unique singular point, then after multiplication by a suitable constant, a linearisation in $\mathrm{End}(\Omega_{\mathcal{X}} \otimes \mathbb{C}(z))$ of a generator ∂ of the foliation is a diagonal matrix $\mathrm{diag}\{a_1, \dots, a_r, 0, -b_1, \dots, -b_t\}$, $a_i, b_j \in \mathbb{N}$ without common divisor and s zeroes. Better still, normalising so that the tangent space to $f(\mathcal{L})$ lies in the positive eigenspace, there is an R-extremal stack $\mathcal{Y}$ containing f such that,*

(a) *$\mathcal{Y}$ contains a unique, smooth s-dimensional component Z of the singular locus of $\mathcal{F}$.*

(b) *$\mathcal{Y}$ retracts onto Z by p, and we have exactly one of,*

(i) $K_{\mathcal{F}} \mid_Z$ *is trivial, and* $p : \mathcal{Y} \to Z$ *is a bundle in the analytic topology of weighted projective stacks of the form* $\mathbf{\mathcal{P}}(da_1, \dots, da_r)$*, for some* $d \in \mathbb{N}$*, with* $d_1 = da_1$ *if* $\mathcal{L}$ *is tangent to the* a_1 *eigenvector.*

(ii) $K_{\mathcal{F}}^{\otimes 2} \mid_Z$ *is trivial, but* $K_{\mathcal{F}} \mid Z$ *is not, then there is an étale* $\mathbb{Z}/2$ *covering* $\mathcal{Y}_{+-} \to \mathcal{Y}$ *obtained by pull-back along* p *of the double covering* $Z^{+-} \to Z$ *defined by* $K_{\mathcal{F}} \mid_Z$ *such that* $\mathcal{Y}_{+-}$ *is a union of two stacks* $\mathcal{Y}_+$*,* $\mathcal{Y}_-$ *meeting in a simple node at* Z^{+-} *which considered as fibrings overs* Z^{+-} *are exactly as per* (i).

(c) *This holds good not just at the level of spaces, but for the induced foliated structure too, i.e.* $(\mathcal{Y}, \mathcal{F}|_y) \xrightarrow{\sim} (\mathbf{\mathcal{P}}(da_1, \dots, da_r), \mathcal{R})$.

(d) *Every extremal stack is of this form.*

It only remains to check the various assertions on extremality, and since $\mathcal{Y}$ is certainly contained in an extremal subvariety $\mathcal{Y}'$ we may as well just check (d) for arbitrary $\mathcal{Y}'$. Now the argument that the intersection of $\mathcal{Y}$ with $\mathrm{sing}(\mathcal{F})$ was a smooth component of the singular locus works for $\mathcal{Y}'$ verbatim, so call the said component Z' (necessarily Z if $\mathcal{Y}' \supset \mathcal{Y}$). Additionally for some (and indeed any) $z' \in Z'$, there is a $-\mathbb{F}$-stack $f' : \mathcal{L} \to \mathcal{Y}'$ containing z'. Whence by III.4 there is a smooth $-\mathbb{F}$-stack $g : \mathcal{L} \to \mathcal{X}$, perhaps not a priori in $\mathcal{Y}'$, containing Z', so the previous discussion about the H-N pair all goes through to establish that there are formal sub-schemes X'_+, X'_- of either the completion of $\mathcal{X}$ in Z', or a double cover as appropriate, which contain the germ of any $-\mathbb{F}$-stack (extremal or otherwise) passing through Z'. As such the completion of $\mathcal{Y}'$ in Z', say $\hat{\mathcal{Y}}'$, is contained in either X'_+ or X'_-, if this is well defined, and $X'_+ \cup X'_-$ otherwise. Whence we're already done if $\mathcal{Y} \subset \mathcal{Y}'$, since everything is irreducible. Otherwise, say without loss of generality $\hat{\mathcal{Y}}'$ is X'_+, so we can even take our smooth $-\mathbb{F}$-stack g to have germ at z' factoring through X'_+. At this point however, we know that X'_+ is the completion of an appropriate $\mathbf{\mathcal{P}}(a'_1, \dots, a'_r)$ bundle over Z' in a section, so indeed $\mathcal{Y}'$ is the said bundle. Note in particular, therefore, that the number of extremal subvarieties (independent of what rays are involved) is no more than twice the number of smooth components of the singular locus, and they're either all disjoint, or if they do intersect they do so at exactly everywhere in a mutual smooth component of $\mathrm{sing}(\mathcal{F})$, locally about which, of course, their completions would be a monodromy free H-N pair. As it happens we'll subsequently see that this latter possibility, and for that matter non-trivial $\mathbb{Z}/2$ monodromy of the pair, only occurs for foliations which are pencils of rational curves, although for the moment we'll content ourselves with observing,

III.5.2 Corollary to the above discussion. *The number of extremal rays in the half space,* $\mathrm{NE}_{K_{\mathcal{F}}<0}$ *is finite.*

Finally let us introduce,

III.5.3 More Notation. A smooth extremal substack which at one point z (and whence any) of its intersection with the singular locus $\mathcal{F}$ affords an appropriately normalised linearisation of a generator ∂ in $\mathrm{End}(\Omega_{\mathcal{X}} \otimes \mathbb{C}(z))$ as $\mathrm{diag}(a_1, \dots, a_r, 0, -b_1, \dots, -b_t)$ will be denoted $\mathcal{Y}(da_i, s, b_j)$ where s is the dimension of the kernel, and of course $a_i,\, b_j \in \mathbb{N}$ satisfy $\gcd(a_1, \dots, a_r,\, b_1, \dots,\, b_t) = 1$, with,

$$a_1 \geq a_2 \geq \cdots \geq a_r \geq 0 \geq -b_1 \geq \cdots \geq -b_t\,, \quad d \in \mathbb{N}\,.$$

III.6. More on the H-N filtration

Our first task will be to establish that a smooth extremal stack is actually a bundle of weighted projective stacks in the Zariski topology. To this end let $\mathcal{Y}(da_i,\, s,\, b_j)$ be the said stack, and Z its intersection with the singular locus $\mathcal{F}$, then wholly unsurprisingly we have,

III.6.1 Minor Fact. *Let* $\pi : \mathcal{Y}(da_i,\, s,\, b_j) \to Y(da_i,\, s,\, b_j)$ *be the moduli, then* $Y(da_i,\, s,\, b_j)$ *is projective.*

Proof. If Y' in X, the moduli of $\mathcal{X}$, is the image of $\mathcal{Y}(a_i,\, s,\, b_j)$ then,

$$Y(a_i,\, s,\, b_j) \to Y'$$

is quasi-finite. Moreover $\mathcal{Y}(a_i,\, s,\, b_j)$ is separated, whence so is $Y(a_i,\, s,\, b_j)$ and a fortiori $Y(a_i,\, s,\, b_j) \to Y'$, whence by [A1] 3.3, $Y(a_i,\, s,\, b_j)$ is a scheme. On the other hand $Y(a_i,\, s,\, b_j)$ is proper, so $Y(a_i,\, s,\, b_j) \to Y'$ is finite, and $Y(a_i,\, s,\, b_j)$ is projective.

Now let $\mathbb{P}\left(1, \frac{a_i}{a}\right)$, $a = \gcd(a_1, \dots,\, a_r)$ be the weighted projective space $\mathbb{P}(1, \frac{a_1}{a}, \dots, \frac{a_r}{a})$ as opposed to a weighted projective stack, and consider the functor on Z schemes,

$$\begin{aligned} T \longmapsto & \operatorname{Hom}\left(Y(da_i, s, b_j),\, \mathbb{P}\left(1, \frac{a_i}{a}\right)\right)(T) \\ = \ & \operatorname{Hom}\left(Y(da_i, s, b_j) \times_Z T, \mathbb{P}\left(1, \frac{a_i}{a}\right) \times T\right) \end{aligned}$$

then $\operatorname{Hom}\left(Y(da_i,\, s,\, b_j),\, \mathbb{P}\left(1, \frac{a_i}{a}\right)\right)$ is locally of finite type, and for $z \in Z$, with $\hat{Z}$ the completion at z, we have by virtue of $\mathcal{Y}(a_i,\, s,\, b_j)$ being an analytic bundle and/or our reflections on deformations, sections of the said Hom scheme over $\hat{Z}$. Consequently we may apply the implicit function theorem of [A1], to find an étale neighbourhood U of Z such that we have maps,

$$Y(a_i,\, s,\, b_j) \times_Z U \to \mathbb{P}(1,\, a_i/d) \times U$$

which are isomorphisms on an appropriately large thickening of Z at z, so by shrinking U as necessary, we conclude that $Y(da_i,\, s,\, b_j)$ is a bundle of weighted projective spaces in the étale topology.

The next thing to consider is another minor variant of the Harder-Narismhan filtration, this time of $\mathcal{Y}(da_i,\, s,\, b_j)$ completed in Z. Specifically we have a filtration of formal schemes supported on Z,

$$Z_1 = \hat{Y}_0 \subset \hat{Y}_1 \subset \cdots \subset \hat{Y}_k = \mathcal{Y}(\widehat{a_i,\, s,\, b}_j)$$

according to a complete repetition free list of eigenvalues $\alpha_1, \dots,\, \alpha_k$, which when completed at any $z \in Z$ is for some (and indeed any) semi-simple generator of the foliation is in the i^{th} place cut out by eigenfunctions of eigenvalue $> \alpha_i$. The existence of such of filtration at a point has already been noted in II.7.9, and the extension over Z is a small perturbation of the discussion in the previous section. In any case the germ of a $-\mathbb{F}$ stack at its singular point factoring through any given $\hat{Y}_i$ is a closed condition on the moduli of the moduli of such stacks, while we know that at any z the fibre of $\hat{Y}_i$ over z is covered by the germs of the same, so in fact the $\hat{Y}_i$ are actually completions of closed sub-stacks $\mathcal{Y}_i$ of $\mathcal{Y}$ in Z. Argueing precisely as before, i.e. $\mathcal{Y}_i$ is formally smooth around Z, so its smooth etc. we conclude that each $\mathcal{Y}_i$ is a bundle of weighted projective stacks of the form $\mathcal{P}(da_1, \ldots, da_{r_i})$ for $a_1, \ldots, a_{r_i}$ the eigenvalues at least α_i, whose moduli Y_i is a bundle of weighted projective spaces in the étale topology. At this point we can simply proceed by induction. Indeed let S_i be the sheaf of $\mathcal{O}_Z$-algebras,

$$\bigoplus_e (p_i)_* K_{\mathcal{F}}^{-e}$$

where $p_i : \mathcal{Y}_i \to Z$ is the projection. Now the moduli Y_i of each $\mathcal{Y}_i$ certainly maps to $\operatorname{Proj} S_i$, and the map is everywhere an étale isomorphism, so it's a Zariski isomorphism, and it remains to find appropriate weighted generators in the Zariski topology, which we do by induction. The case of $i = 1$ is essentially that of projective bundles, so there's nothing to do (except to note that the smallest $d_1 \in \mathbb{N}$, such that $p_* K_{\mathcal{F}}^{-d_1}$ has rank 2 gives us the normalising factor d, by way of, $d_1 = d\alpha_1$). Furthermore étale maps are faithfully flat, so we have an exact sequence,

$$0 \to J_i \to S_{i+1} \to S_i \to 0$$

where the last map is a map of graded algebras, all of whose graded pieces are, by a standard descent arguement, locally free $\mathcal{O}_Z$-modules. Consider therefore the $d\alpha_{i+1}$-th graded piece, then $J_i(d\alpha_{i+1})$ is locally free on generators $X_{i+1,p}$, $1 \leq p \leq k_{i+1}$, say, where k_{i+1} is the dimension of the eigenspace for α_{i+1}. Moreover the sequence of graded pieces can be locally split in the Zariski topology, so inductively we obtain a collection X_{qp}, $q \leq i+1$, $1 \leq p \leq k_q$ of local Zariski sections of degree $d\alpha_1$ which freely generate S_{i+1}. As such we're established that each Y_i, and in particular, $Y(da_i,\, s,\, b_j)$ is a bundle of weighted projective spaces in the Zariski topology. To conclude from this that $\mathcal{Y}(da_i,\, s,\, b_j)$ is a bundle, we can first put the minimal smooth stack structure $\mathcal{Y}'$, say, with moduli $Y(da_i, s, b_j)$ – which is in fact the Gorenstien cover of the induced radial foliation – and argue as above to identify the ramification locus of $\mathcal{Y}(da_i,\, s,\, b_j) \to \mathcal{Y}'$ with a Zariski bundle of weighted projective stacks, and so deduce,

III.6.2 Concluding Fact. *We can add to* III.5.1 *and* III.6.1 *the precision, the map $p : (\mathcal{Y}, \mathcal{F}) \to Z$, or for that matter $p : (\mathcal{Y}_{+\text{ or }-}, \mathcal{F}) \to Z^{+-}$ is a Zariski bundle of $(\mathcal{P}(da_1, \ldots, da_r), \mathcal{R})$'s.*

IV. FLIP, FLAP, FLOP

IV.1. Contractions

The basis of the flip theorem will actually be nothing other than a minor variant of Castelnuovo's contraction theorem. To motivate this consider,

IV.1.1 Data. *$\pi : \mathcal{X} \to X$, a smooth stack with projective moduli, $\mathcal{D}$ an integral Cartier divisor on $\mathcal{X}$ and D the corresponding $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$.*

Now it may not be true that $\mathcal{D}$ is the fibre product $\pi^{-1}(D)$, however if eD, $e \in \mathbb{N}$, is actually Cartier, then we have,

IV.1.2 Fact. *If $\mathcal{D}$ is not contained in the ramification locus $\pi^{-1}\,\mathcal{O}_X(-eD) = \mathcal{O}_{\mathcal{X}}(-e\mathcal{D})$, i.e. with natural scheme structures, $e\mathcal{D} = \pi^{-1}(eD)$.*

Proof. This is just the usual vanishing of local cohomology in the presence of the S_2-condition, so it would even work with $\mathcal{X}$ normal.

Observe moreover that since π_* is exact, we have an exact sequence,

$$0 \to \pi_*\mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \to \mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{D}} \to 0\,.$$

Thus $\mathcal{O}_D = \pi_*\mathcal{O}_{\mathcal{D}}$, and whence for any line bundle B on X,

IV.1.3 Further Small Fact. $H^0(\mathcal{D}, \pi^*B) = H^0(D,\, B)$.

As such following [K1] III.1.8.1, we can combine all of this to form a contraction criteria, viz:

IV.1.4 Criteria. Let A be a very ample bundle on X, and suppose for $k \in \mathbb{N}$, divisible by e,

(a) $H^1(X,\, A) = 0$

(b) $H^1(\mathcal{D},\, \mathcal{O}_{\mathcal{D}} \otimes \pi^*A(j\mathcal{D})) = 0$, $1 \le j \le k-1$

(c) $\mathcal{O}_{\mathcal{D}} \otimes \pi^*A(k\mathcal{D})$, or equivalently by, $\mathcal{O}_D \otimes A\left(\frac{k}{e}\,D\right)$, is generated by global sections, with $\mathrm{cont} : D \to D_0$ the corresponding map.

Then in fact $A(kD)$ is generated by global sections, so that if $\mathrm{Cont} : X \to X_0$ is the corresponding map, then,

(a)' $\mathrm{Cont} : X\backslash D \xrightarrow{\sim} X_0\backslash D_0$

(b)' $\mathrm{Cont}|_D = \mathrm{cont} : D \to D_0$

(c)' X_0 is projective.

Proof. Just use the exactness of π_*. □

To apply the criteria we'll need appropriate cohomological vanishing. The following minor variant of Serre vanishing, cf. [E], will suffice, viz:

IV.1.5 Vanishing Fact. *Let $\mathcal{F}$ be a coherent sheaf on a scheme W with, A an ample bundle, and $H_1, \ldots,\, H_n$ semi-ample bundles then for all but finitely many integers $p_1, \ldots,\, p_n \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$,*

$$H^i(W,\, \mathcal{F} \otimes A^p \otimes H_1^{p_1} \otimes \cdots \otimes H_n^{p_n}) = 0\,, \quad i \ge 1\,.$$

Proof. We proceed exactly as per Serre vanishing, i.e. on replacing A and the H_i by suitable tensor powers d, d_i, respectively, not to mention applying the desired outcome to $\mathcal{F} \otimes A^q \otimes H_1^{q_1} \otimes \cdots \otimes H_n^{q_n}$, $1 \le q < d$, $0 \le q_i < d_i$, we can assume without loss of generality that A is very ample, and the H_i are generated by global sections. As such we can reduce to supposing that W is a product of projective spaces provided we prove a slightly different fact, i.e.

IV.1.6 Similar Fact. *Let $P = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a product of projective spaces, L_i the pull-back of the tautological bundle on the i^{th}-factor, and $\mathcal{F}$ a coherent sheaf on P then for all but finitely many $p_1, \ldots, p_k \in \mathbb{N}$,*

$$H^i(P, \mathcal{F} \otimes L_1^{p_1} \otimes \cdots \otimes L_k^{p_k}) = 0\,, \qquad i \ge 1$$

which of course we just do by the usual decreasing induction on i, which itself is trivially true for $i \ge n_1 + \cdots + n_k$. Otherwise we have an exact sequence,

$$0 \to \mathcal{E} \to (L^{-m})^{\oplus a} \to \mathcal{F} \to 0$$

where $L = L_1 \otimes \cdots \otimes L_k$, $m \in \mathbb{N}$ is sufficiently large, and $a \in \mathbb{N}$. On the other hand, $H^i(P, L_1^{p_1} \otimes \cdots \otimes L_k^{p_k}) = 0$, for say, $i \ge 1$, and $p_j \ge 0$, $1 \le j \le k$, so we're done.

Observe in particular that we have,

IV.1.7 Corollary. *Let A, $L_1, \ldots, L_n$ be bundles on a stack $\mathcal{W}$ such that a suitable multiple of A is ample on the moduli $\pi : \mathcal{W} \to W$ and such that multiples of the L_i are generated by global sections, then for all but finitely many $p \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$,*

$$H^i(\mathcal{W}, A^p \otimes L_1^{p_1} \otimes \cdots \otimes L_n^{p_n}) = 0\,, \qquad i \ge 1\,.$$

Proof. Let d, $d_1, \ldots, d_n$ be the said multiples, then necessarily $A^{\otimes d}$, $L_i^{\otimes d_i}$, $1 \le i \le n$, are bundles on W. Consequently putting $p = nd + q$, $p_i = n_i d_i + q_i$, $0 \le q < d$, $0 \le q_i < d$, we can just use the exactness of π_* and apply IV.1.5, with $\mathcal{F} = \pi_*(A^q \otimes L_1^{q_1} \otimes \cdots \otimes L_n^{q_n})$, $H_i = L_i^{\otimes d_i}$, and of course $A = A^{\otimes d}$ for want of having been more notationally careful. □

Now let's apply all of this to an extremal smooth sub-stack $\mathcal{Y}(da_1, \ldots, da_r, s, b)$ which is a divisor, and which for convenience we'll denote $\mathcal{D}$. As ever $(\mathcal{X}, \mathcal{F})$ is our ambient smooth stack with terminal singularities at the non-scheme like points, and $\pi : \mathcal{X} \to X$ its moduli. To keep things clear D will be the image on $\mathcal{D}$ in X, which may be distinct from its moduli $|\mathcal{D}|$. In any case, associated to the implicit extremal ray R, there is a supporting Cartier divisor H_R, and without loss of generality we may suppose that $A = H_R - K_{\mathcal{F}}$ is an ample $\mathbb{Q}$-divisor on X. As such if $i : Z \hookrightarrow X$ is the embedding of the intersection of $\mathcal{D}$ with the singular locus of $\mathcal{F}$, then we already know that $i^* K_{\mathcal{F}}$ is trivial, so in fact $i^* H_R$ is ample. Moreover for some bundle B on Z we have by I.9.10, that $\mathcal{O}_{\mathcal{D}}(\mathcal{D}) = p^* B \otimes K_{\mathcal{F}}^b$, whence up to replacing H_R by a sufficiently large multiple, we can suppose, without loss of generality, that $i_Z^* H_R$ and $i_Z^* H_R + B$ are very ample. We wish to apply the criteria IV.1.4 to a sufficiently large and divisible multiple of $A + qH$, where $q \in \mathbb{N}$ is itself a sufficiently large integer to be chosen. Whence in reverse order we observe,

(c) There is a $m_0 \in \mathbb{N}$ such that for all q, $\mathcal{O}_{\mathcal{D}}\left(m\,A + mq\,H_R + \frac{m}{b}\mathcal{D}\right)$ is generated by global sections provided $m \ge m_0$ and sufficiently divisible, i.e. the bundle in question is a bundle not just on $\mathcal{D}$ but on $|\mathcal{D}|$.

Indeed if $m = bm'$, then the bundle is just, $m'(H + B + (q + b - 1)H)$, which is a sum of very ample bundles, so this is obvious, as is the fact that, $\mathrm{cont} : \mathcal{D} \to Z$ is nothing other than the projection p.

(b) There is a $q_0 \in \mathbb{N}$ such that for all m divisible by b,

$$H^1(\mathcal{D},\, \mathcal{O}_{\mathcal{D}}(m\,A + mq\,p^* H_R + j\mathcal{D})) = 0\,, \quad q \geq q_0\,, \quad 1 \leq j \leq \frac{m}{b} - 1\,.$$

Here we can simply rewrite the bundle in question as,

$$(m - jb)\,A + (mq + j(b-1))\,p^* H_R + j\,p^*(H_R + B)$$

so that IV.1.7, guarantees the existence of such a q_0.

(a) Take $q \geq q_0$ as in (b), and $m \geq m_0$ sufficiently large and divisible to guarantee that $m(A + q\,H_R)$ is very ample and without cohomology.

Consequently we've certainly constructed a contraction $\rho : X \to X_0$, where to be on the safe side we take the Stein factorisation of the Cont of in order to guarantee that X_0 is unique. The criteria also gives for free,

IV.1.8 Fact. *X_0 is $\mathbb{Q}$-factorial.*

Proof. Indeed a minor variant of the above argument works with an essentially arbitrary A, or to be more precise if F is any ample bundle on X then we can construct X_0 by way of the global sections of sufficiently large and divisible multiples of $F + \alpha D + qH$, where $\alpha \in \mathbb{Q}$ is determined by F, and any q bigger than some q_F depending on F will do. In particular, $\rho_* F + q\rho_* H$ is a $\mathbb{Q}$-divisor on X_0, for all $q \geq q_F$. However if E is any divisor on X_0, then its proper transform is the difference $F - F'$ of a pair of amples, so that taking an appropriately large q, e.g. $\max\{q_F,\, q_{F'}\}$, we see that E is $\mathbb{Q}$-Cartier as required. □

So while this is perfectly sufficient for applications, let's restore the stack structure. By I.7.4 we know that the stack structure of $\mathcal{X}$ is uniquely defined by the log-structure $(X,\, B,\, \mathcal{F})$ associated to the divisorial ramification of $\mathcal{X}$ over X. Certainly we have a contraction $(X_0,\, B_0,\, \mathcal{F}_0)$ of this data, and we require to prove,

IV.1.9 Claim. *$(X_0,\, B_0,\, \mathcal{F}_0)$ has log-canonical singularities, which are in fact terminal at $|B_0| \cup \mathrm{sing}\,(X_0)$.*

Proof. Putting, $K_{\tilde{\mathcal{F}}} = K_{\mathcal{F}} + \sum(1 - \frac{1}{e_i})B_i$, and $K_{\tilde{\mathcal{F}}_0} = K_{\mathcal{F}_0} + \sum(1 - \frac{1}{e_i})B_{0i}$, and denoting the exceptional divisor now by E, then by construction,

$$K_{\tilde{\mathcal{F}}} = \rho^* K_{\tilde{\mathcal{F}}_0} + \frac{1}{db}\,E\,.$$

Consequently if F is an exceptional rank 1 discrete valuation of the function field, distinct from E with things modeled on X_0, then,

$$a_{\mathcal{F}_0, D_0}(F) = a_{\mathcal{F}, D}(F) + \frac{\mu(F)}{eb}$$

where μ is the multiplicity of E along F. As such $(X_0,\, D_0,\, \mathcal{F}_0)$ certainly has log-canonical singularities. Better still if the centre of F in X_0 is contained in the locus Z_0 to which E is contracted, then its center on X is contained in E, and $\mu(F) > 0$. Whence, at every point of Z_0, $(X_0,\, D_0,\, \mathcal{F}_0)$ is log-terminal, and thus in fact terminal by I.6.11, while terminality along $|D_0|$ is obvious from the same at $|D|$. On the other hand $\mathrm{sing}(X_0) \subset Z_0 \cup \rho(\mathrm{sing}\, X)$, and terminality along $\mathrm{sing}(X)$ is a priori given by our hypothesis on $(\mathcal{X}, \mathcal{F})$ thanks to I.7.3. □

With this out of the way we can simply appeal to I.7.4, to restore the stack structure, i.e. we have a foliated smooth stack $(\mathcal{X}_0, \tilde{\mathcal{F}}_0)$ determined by the log-data (X_0, D_0, F_0) which fits into a diagram,

$$\begin{tikzcd} (\mathcal{X}, \tilde{\mathcal{F}}) \arrow[r, dashed, "\rho"'] \arrow[d, "\pi"'] & (\mathcal{X}_0, \tilde{\mathcal{F}}_0) \arrow[d, "\pi"] \\ (X, D, F) \arrow[r, "\rho"'] & (X_0, D_0, F_0) \end{tikzcd}$$

Note however, that the contraction map on stacks cannot in general be everywhere defined, whether at the stack level $\mathcal{Y}(da_1, \ldots, da_r, s, b)$ could be contracted to a singular stack is a mute point of little relevance which will be ignored. It is, therefore, perhaps useful to summarise by way of,

IV.1.10 Summary/Remarks/Terminology. Given a divisorial extremal smooth substack of a foliated smooth stack with terminal singularities at the non-scheme like points and log-canonical singularities elsewhere there is a flop, $\rho : (\mathcal{X}, \tilde{\mathcal{F}}) \dashrightarrow (\mathcal{X}_0, \tilde{\mathcal{F}}_0)$ to a foliated smooth stack enjoying the same prescriptions on the singularities. Here the definition of flop is not the usual one, but rather the data of the above diagram, where we think of the rightmost π as being almost log-étale, $\rho \circ \pi$ as a "bad" contraction, and note that ρ is well defined on objects but not on arrows.

IV.2. A Formal Approach

Although rich in information, and adequate for applications to foliations which are not fibrations in rational curves, the contraction theorem of IV.1 uses global information on extremal rays to prove something which is in fact of a purely local character. Not surprisingly this is not wholly desirable, since the requirement of verifying global hypothesis before concluding a local statement is just plain stupid. As such we wish to consider the following contraction problem,

IV.2.1 Problem. *Let $\mathfrak{X}$ be a smooth formal stack, with underlying stack $\mathcal{X}_0$, such that $\mathcal{X}_0 \hookrightarrow \mathfrak{X}$ is a divisor with $\mathcal{X}_0$ a bundle of weighted projective stacks $\mathbb{P}(\underline{a})$ over a smooth scheme (or indeed separated algebraic space) Z, then can we contract $\mathcal{X}_0$ if $\mathcal{O}_{\mathcal{X}_0}(-\mathcal{X}_0)$ is ample relative to Z.*

The precise meaning of contraction is as follows. The formal stack $\mathfrak{X}$ will have some moduli, $\pi : \mathfrak{X} \to \Xi$, where Ξ is a separated formal algebraic space (notice, as we've said stacks, formal or otherwise, are always separated so the existence of Ξ follows from a minor variant of [KM], or alternatively in applications everything is actually a completion) with underlying space Ξ_0, so that in the spirit of IV.1.10, what we'll actually do is construct a formal contraction of Ξ_0 (in the sense of [A3]) with a view to a leisurely restoration of the stack structure.

We begin by discussing the case where Z is affine, and denote by $\mathcal{O}(1)$ the tautological bundle on $\mathcal{X}_0$, so that by hypothesis $\mathcal{O}_{\mathcal{X}_0}(\mathcal{X}_0) = \mathcal{O}(-b)$ for some $b \in \mathbb{N}$. In so much as it exists the contraction is basically the Stein factorisation of

$\mathfrak{x}$ over its functions so we simply compute this. We could of course work at the Ξ level, however this involves working with symmetric powers of ideals, rather than powers, and would in any case just be a complicated way of obscuring the correct structure. In any case the appropriate diagram, with exact rows and columns, is as follows,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathcal{X}_0}(-n\,\mathcal{X}_0) \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathcal{X}_n}(-\mathcal{X}_0) \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{X}_{n-1}}(-\mathcal{X}_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathcal{X}_0}(-n\,\mathcal{X}_0) \arrow[r] & \mathcal{O}_{\mathcal{X}_{n+1}} \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{X}_n} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_{\mathcal{X}_0} \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathcal{X}_0} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Now we know by I.9.7, that $\mathcal{O}_{\mathcal{X}_0}(-n\,\mathcal{X}_0)$, $n \geq 0$, is acyclic, so the same is true of $\mathcal{O}_{\mathcal{X}_m}(-n\,\mathcal{X}_0)$, for any $m,\, n \geq 0$ by an obvious induction. Consequently on taking global sections everything is still exact, i.e. we have a diagram, with the notation established by the same,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_{bn} \arrow[r] \arrow[d, equal] & I_{n+1} \arrow[r] \arrow[d] & I_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_{bn} \arrow[r] & \Gamma_{n+1} \arrow[r] \arrow[d] & \Gamma_n \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_Z \arrow[r, equal] \arrow[d] & \mathcal{O}_Z \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where $S = \bigoplus_k S_k$ is the weighted graded algebra of I.9.5, and we confuse (in the tradition of the subject) functions on Z with the structure sheaf. The important thing however is that Mittag-Leffler is valid for the directed systems defined by I and Γ so in fact we have an exact sequence,

$$0 \longrightarrow I = \varprojlim_n I_n \longrightarrow \Gamma = \varprojlim_n \Gamma_n \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

with associated graded algebra, $\operatorname{gr}\Gamma = \bigoplus_{n=0}^{\infty} S_{bn}$, where of course we regard Γ as an admissible ring for the topology defined by,

$$0 \longrightarrow I^{(n)} \longrightarrow \Gamma \longrightarrow \Gamma_n \longrightarrow 0\,.$$

There is equally no difficulty in making similar diagrams with $\mathcal{O}_{\mathcal{X}_m}$ on the bottom row in place of $\mathcal{O}_{\mathcal{X}_0}$, and thus identifying $I^{(n)}$ with $\varprojlim_{p\geq n} H^0(\mathcal{O}_{\mathcal{X}_p}(-n\,\mathcal{X}_0))$. Better still $\operatorname{gr}\Gamma$ is a finitely generated $\mathcal{O}_Z$ algebra, so Γ is noetherian (indeed formally of essentially finite type) and we could of course identify the $I^{(n)}$ topology with the I^n topology. However formal schemes are really built from their nilpotent pieces, so it's unsurprisingly more appropriate to stick with the $I^{(n)}$, and identify the potential contraction with $\mathfrak{Y} := \operatorname{Spf}\Gamma$. In addition we have a free $\mathcal{O}_Z$-basis of $I^{(n)}/I$ which we may identify with monomials of the form $X_0^{t_0}\,X_1^{t_1}\dots X_r^{t_r}$ such that $t_0 + a_1t_1 + \cdots + a_rt_r = bn$. Indeed choosing a quasi-coefficient ring for $\mathcal{O}_Z$, or for that matter just shrinking Z since we're working in the étale topology, we can identify Γ with the subring of $\mathcal{O}_Z[[X_0,\dots,\,X_r]]$ generated by monomials of this form. Regardless, the important thing is that we define ideals $J^{(n)}$ of Γ as the ideals generated by the said free basis of $I^{(n)}/I$, and form the graded algebra $T := \bigoplus_n J^{(n)}$, with linear topology defined by the ideals $I^{(n)}T$. With this in mind, we assert,

IV.2.2 Claim. *Ξ is isomorphic to the weighted formal blow up* $\operatorname{Prof} T$.

Before proceeding to a proof, let's explain what Prof is since for some reason it's one of those notions that didn't make it into EGA.

IV.2.3 Intermission (on formal Proj). Quite generally let $X = \operatorname{Spf} A$ be an affine formal scheme, and T a graded A-module, with an admissible linear topology T_λ, $\lambda \in \Lambda$, of graded ideals such that $j : A \to T$ is continuous. We can subsequently form the schemes, $\operatorname{Proj}(T/T_\lambda) \to \operatorname{Spec}(A/j^{-1}T_\lambda)$, and take the direct limit in the category of formal schemes (or better formal formal schemes) to obtain a formal scheme $\operatorname{Prof} T$ over X. As such there are actually several different notions of formal/weighted formal blowing up according to the topology employed. The particular choice in IV.2.2 corresponds to completing a usual weighted blow up in the pre-image of the support of the irrelevant ideal. Consequently since we're local and everything is algebraisable (i.e. $\operatorname{Spf}\Gamma$ is the completion of $\operatorname{Spec}\Gamma$ in I) the claim automatically implies that $\Xi \to \operatorname{Spf}\Gamma$ is a formal modification in the sense of Artin.

To prove the claim observe that we have a map,

$$\pi : \mathfrak{X} \to \operatorname{Proj}\,\bigoplus_n H^0(\mathfrak{X},\,\mathcal{O}_{\mathfrak{X}}(-n\,\mathcal{X}_0))$$

which for the sake of precision could be identified with Prof in the discrete topology. Equally we have such a map for $\operatorname{Prof} T$, and even an embedding,

$$\operatorname{Prof} T \hookrightarrow \mathfrak{Y} \times \operatorname{Proj}\bigoplus_n H^0(\mathfrak{X},\,\mathcal{O}_{\mathfrak{X}}(-n\,\mathcal{X}_0))$$

and of course the projection $\rho : \mathfrak{X} \to \mathfrak{P}$. Manifestly $\rho \times \pi$ factors through $\mathrm{Prof}\, T$, and whence by the universal property of Ξ we obtain a map,

$$\rho \times \pi : \Xi \to \mathrm{Prof}\, T$$

and it remains to check that this is indeed an isomorphism. To this end an alternative description of $\mathrm{Prof}\, T$ will be useful. In the fist place consider the cone over $\mathbb{P}^r_Z$ embedded in $|\mathcal{O}_{\mathbb{P}^r_Z}(b)|$ by the full linear system, and let $\mathfrak{P}^\#$ be the completion of the said cone in the singular section, so in fact $\mathfrak{P}^\#$ is the formal quotient $\hat{\mathbb{A}}^{r+1}_Z/\mu_b$, where μ_b acts diagonally on a standard formal affine coordinate system. Better still we have a $\mu_{a_1} \times \cdots \times \mu_{a_r}$ action on the final r-coordinates in the standard way, so that if $\lambda : \tilde{\mathfrak{P}} \to \mathfrak{P}^\#$ is the formal blow up of $\mathfrak{P}^\#$ in the singular section then the $\mu_{a_1} \times \cdots \times \mu_{a_r}$ action-extends, Prof commutes with group action, and we conclude to a commutative diagram,

$$\begin{array}{ccc} \mathrm{Prof}\, T = \tilde{\mathfrak{P}}/\mu_{a_1} \times \cdots \times \mu_{a_r} & \longleftarrow & \tilde{\mathfrak{P}} \\ \downarrow \lambda & & \downarrow \lambda \\ \mathfrak{P} = \mathfrak{P}^\#/\mu_{a_1} \times \cdots \times \mu_{a_r} & \longleftarrow & \mathfrak{P}^\# \end{array}$$

We can also do a similar thing on the Ξ side. Indeed any non-negative multiple of $\mathcal{O}_{\mathcal{X}_0}(1)$ is acyclic by I.9.6/7, so there is no difficulty in lifting to $\mathfrak{X}$ the sections defining coordinate hyperplanes à la III.5. Having made such liftings as per op. cit. we may extract their roots to obtain a formal covering stack $\tilde{\mathfrak{X}} \to \mathfrak{X}$ whose underlying stack is now the classifying stack $[\mathbb{P}^r/\mu_{a_1} \times \cdots \times \mu_{a_r}]$, with exactly the same moduli Ξ. Equally the various $\mu_{a_1} \times \cdots \times \mu_{a_r}$ actions are compatible, and $\mathbb{P}^r \to [\mathbb{P}^r/\mu_{a_1} \times \cdots \times \mu_{a_r}]$ is étale, which in turn yields a formal covering stack with underlying stack $\mathbb{P}^r$, so that we're reduced to the case of $\mathcal{X}_0 \xrightarrow{\sim} \mathbb{P}^r$. However in this case $\mathrm{Prof}\, T = \tilde{\mathfrak{P}}$ is smooth as is Ξ, while if E is the exceptional divisor on $\tilde{\mathfrak{P}}$ then $(\rho \times \pi)^* E = \Xi_0$ by construction, so $\Xi \to \tilde{\mathfrak{P}}$ is étale, and Ξ is connected, so we're done.

Fortunately contraction is unique not just up to isomorphism, but up to the identity, so on covering our original formal stack $\mathfrak{X}$ of IV.2.1 by sufficiently small affines we conclude to a positive solution of our original problem. The problem solved we can appeal to Artin's convergence theorem to obtain,

IV.2.4 Fact/Summary (according to the generality in which the above works, and not just the special subcase that was discussed). Let X be an algebraic (or complex analytic) space with quotient singularities, and $D \subset X$ a divisor which is a bundle of weighted projective spaces over some smooth Z, which becomes a bundle of weighted projective stacks, or indeed generalised weighted projective stacks, on the natural smooth almost étale covering $\mathcal{X} \to X$ of [V] 2.5, then there is a weighted blow down $\rho : X \to X_0$ of D to a unique algebraic (or complex analytic) space.

Better still our discussion of foliation singularities I.7 in no way employs projectivity, and so we obtain,

IV.2.5 Corollary. *Let $(\mathcal{X}, \tilde{\mathcal{F}})$ be a foliated smooth stack with log-canonical singularities which are terminal at the non-scheme like points and $\pi : (\mathcal{X}, \tilde{\mathcal{F}}) \to (X, B, \mathcal{F})$ it's not necessarily projective logarithmic moduli then if $\mathcal{Y}(a_i d, s, b)$*

is an invariant divisor of the type encountered in III.3.3, *there is in the sense of* IV.1.10 *a flop of* $\mathcal{Y}(a_i d,\, s,\, b)$ *to a smooth stack* $(\mathcal{X}_0,\, \tilde{\mathcal{F}}_0)$.

A postiori it will emerge that such a $\mathcal{Y}(a_i d,\, s,\, b)$ is always extremal, provided that $(\mathcal{X},\, \mathcal{F})$ is not a pencil of rational curves, but we are now in the reasonably advantageous position that we can construct our minimal model as an algebraic space from purely local considerations, which themselves will subsequently be sufficient to imply that projectivity was never lost in the first place.

IV.3. The H-N Filtration again

We will require knowledge of the normal bundle to an extremal smooth sub-stack. Consequently, as ever, $(\mathcal{X},\, \mathcal{F})$ is a foliated smooth stack with log-canonical singularities, projective moduli $\pi : (\mathcal{X},\, \mathcal{F}) \to (X,\, \mathcal{F})$, and $\mathcal{Y}$ an extremal smooth sub-stack of the form $\mathcal{Y}(da_1, \ldots, da_r,\, s,\, b_1, \ldots,\, b_t)$. Given the said data we may specialise to the projective normal cone, $\mathcal{P} := \mathbb{P}(N^\vee_{\mathcal{Y}/\mathcal{X}}) = P(N_{\mathcal{Y}/\mathcal{X}})$, to obtain another foliated smooth stack $(\mathcal{P},\, \tilde{\mathcal{F}})$ again with log-canonical singularities.

Our primary interest is the local variation of $N_{\mathcal{Y}/\mathcal{X}}$ over the base/singular locus Z, so to begin with, and essentially without loss of generality, we'll restrict attention to the case $s = 0$. Naturally there are two tautological bundles of relevance, namely that on the weighted projective stack $\mathcal{Y}$, which is in fact $K^\vee_{\mathcal{F}}$, and the relative tautological bundle of $\rho : \mathcal{P} \to \mathcal{Y}$ which we'll denote H, while P, Y etc. will be the corresponding moduli. To begin with we seek to bound the effective cone of divisors on $\mathcal{P}$ so suppose that $H + x\rho^* K^\vee_{\mathcal{F}}$, $x \in \mathbb{R}$, is nef. Now suppose that $\mathcal{L} \to \mathcal{Y}$ is a smooth invariant substack, then of course,

$$\mathcal{P} \times_{\mathcal{Y}} \mathcal{L} \xrightarrow{\sim} P\left(\bigoplus_{j=1}^{t} \mathcal{O}_{\mathcal{L}}(b_j\, K_{\mathcal{F}})\right)$$

where as ever we order things according to $b_1 \leq b_2 \leq \ldots \leq b_t$. In particular if b is the smallest of the b_j, and is supposed to occur q times amongst the b_j, then we have an embedding,

$$\mathcal{P} \times_{\mathcal{Y}} \mathcal{L} \hookleftarrow \mathcal{L} \times \mathbb{P}^{q-1} : i\,.$$

Moreover this occurs in such a way, that if H_0 is the standard normalisation of the tautological bundle on $\mathbb{P}^{q-1}$ pulled back to the said product then, $i^* H = H_0 + b\,\rho^* K^\vee_{\mathcal{F}}$, so in fact $x \geq -b_1$. Now observe that either $K_{\tilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$ or $q = t$, $r = 1$ and all the b_j are equal. In the latter case we already know the answer, so ignoring it for simplicity, we can certainly appeal to the cone theorem to conclude the existence of a $K_{\tilde{\mathcal{F}}}$-negative extremal ray R, or better a smooth invariant parabolic stack $\mathcal{R}$ representing the same. Indeed taking $\mathcal{R}$ to be what might be termed a coordinate line in terms of the explicit coordinate system guaranteed by our holonomy considerations of we see that $\rho(\mathcal{R})$ is without loss of generality equal to $\mathcal{L}$, so, in fact, $\mathcal{R}$ is extremal in $\mathcal{P} \times_{\mathcal{Y}} \mathcal{L}$. Consequently for p an appropriate geometric point in our $\mathbb{P}^{q-1}$, $\mathcal{R} = \mathcal{L} \times p$, and $x = -b_1$ whenever $H + x\,\rho^* K^\vee_{\mathcal{F}}$ is both extremal and effective.

In addition if for our standard linearisation of a generator of the foliation about the singularity z, we denote by V the eigenspace of $b = b_1$ contained in $N_{\mathcal{Y}/\mathcal{X}} \otimes$

$\mathbb{C}(z)$ then canonically we may identify the aforesaid $\mathbb{P}^{q-1}$ with $P(V)$, which in turn is the singular locus of $\tilde{\mathcal{F}}$. As such our considerations on extremal subvarieties imply that the extremal rays sweep out a $\boldsymbol{\mathcal{P}}(da_1, \ldots, da_r)$ bundle Q parametrised by the said $\mathbb{P}^{q-1}$. Conversely we also have at our disposal the projection of Q to $\mathcal{Y}$, so in fact $Q \xrightarrow{\sim} \mathcal{Y} \times \mathbb{P}^{q-1}$, and embeds in $\mathcal{P}$ as a sub-projective bundle. Consequently applying ρ_* gives a surjection,

$$N^{\vee}_{\mathcal{Y}/\mathcal{X}} = \rho_* \mathcal{O}_{\mathcal{P}}(H) \twoheadrightarrow \rho_* \mathcal{O}_Q(H) = \mathcal{O}_{\mathcal{L}}(-bK_{\mathcal{F}})^{\oplus q} .$$

To deduce from here the structure of the H-N filtration of $N_{\mathcal{Y}/\mathcal{X}}$ it suffices to observe,

(a) Let N be the sub-bundle of $N^{\vee}_{\mathcal{Y}/\mathcal{X}}$ corresponding to the kernel of the above map, and p any geometric point of our $\mathbb{P}^{q-1}$, then we have a natural isomorphism,

$$N^{\vee}_{Q/\mathcal{P}} \otimes \mathcal{O}_{\mathcal{Y}\times p} \xrightarrow{\sim} N .$$

(b) The hypothesis $s = 0$ plays no essential role. Indeed for arbitrary s, Q is globally defined as an extremal subvariety of $\mathcal{P}$, and its local structure over the base of $\mathcal{Y} \to Z$ requires only that the base is a sufficiently small affine.

As such appealing inductively to the above considerations by way of (a) we conclude,

IV.3.1 Fact. *There is a canonical filtration of $N_{\mathcal{Y}/\mathcal{X}}$ by invariant sub-bundles for the induced foliation,*

$$\mathcal{O} = N_0 \subsetneqq N_1 \subsetneqq N_2 \subsetneqq \cdots \subsetneqq N_k = N_{\mathcal{Y}/\mathcal{X}}$$

such that if $\beta_1 < \cdots < \beta_k$ is a complete repetition free list of the $b_1, \ldots, b_t$, and q_j, $1 \le j \le k$ the corresponding multiplicities, then locally over Z,

$$N_j/N_{j-1} \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}}(+\beta_j K_{\mathcal{F}})^{\oplus q_j} .$$

Unsurprisingly we continue to refer to this as the H-N filtration, and observe that we even have a splitting (at least locally over Z), albeit non-canonically, of the form,

$$N_{\mathcal{Y}/\mathcal{X}} \xrightarrow{\sim} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathcal{Y}}(b_j K_{\mathcal{F}}) .$$

IV.4. Split Neighbourhoods & Weighted Blow Ups

Let us consider in more detail the rather special situation presented to us by the H-N filtration of the normal bundle of an extremal smooth subvariety $\mathcal{Y}$ of a foliated smooth stack $(\mathcal{X}, \mathcal{F})$ with of course log-canonical singularities, terminal at the non-scheme like points. In the particular case where $\mathcal{Y}$ has dimension 1, we already know that the corresponding formal neighbourhood $\hat{\mathcal{X}}$ of $\mathcal{Y}$ is actually *split*, i.e. there are invariant formal divisorial, substacks $\mathfrak{D}_1, \ldots, \mathfrak{D}_t$, such that $\mathcal{O}_{\mathcal{Y}}(\mathfrak{D}_j)$, or

$N_{\mathcal{Y}/\mathfrak{D}_j}$, define the various terms in the H-N filtration. Such a splitting appears to be specific to the additional structure provided by the foliation, but if $\mathcal{Y}$ is a $\mathcal{P}(a_1, \dots, a_r)$ for $r \geq 2$ with $N_{\mathcal{Y}/\mathcal{X}} \xrightarrow{\sim} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathcal{Y}}(-b_j)$, $b_1 \leq \cdots \leq b_t$ positive integers then such a splitting continues to hold without supposing the existence of any such foliation, albeit that the presence of a foliation is precisely what guarantees such simplicity of the H-N filtration. Regardless suppose indeed we have a H-N decomposition for the normal bundle of a bundle $\mathcal{X}_0$ of weighted projective stacks $\mathcal{P}(a_1, \dots, a_r)$ over a smooth scheme (or algebraic space) Z, or at least locally over Z, inside a formal neighbourhood $\mathfrak{X}$ of the same. Certainly, cf. IV.5, we have, on supposing Z a sufficiently small affine, that $\mathrm{Pic}(\mathfrak{X}) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{X}_0)$, and the liftability to $\mathfrak{X}$ of the divisors,

$$\mathcal{D}_j := \bigoplus_{\substack{k=1 \\ k \neq j}}^{t} \mathcal{O}_{\mathcal{Y}}(-b_k) \subset N_{\mathcal{X}_0/\mathfrak{X}}$$

for each $1 \leq j \leq t$, follows immediately from the acyclicity of $\mathcal{O}_{\mathcal{Y}}(n)$, for each $n \in \mathbb{Z}$. Let us therefore summarise this discussion by way of,

IV.4.1 Summary/Definition. A smooth formal neighbourhood $\mathfrak{X}$ of a bundle of weighted projective stacks $\mathcal{X}_0$ will be called split if there are smooth formal divisorial stacks $\mathfrak{D}_1, \dots, \mathfrak{D}_t$, $t = \mathrm{codim}\, \mathcal{X}_0$, such that,

$$\mathcal{X}_0 = \mathfrak{D}_1 \cap \cdots \cap \mathfrak{D}_t$$

where the implied stack structure is the stack/scheme theoretic one. Under the hypothesis discussed above (which will certainly be sufficient for applications) we are indeed in the situation of a split neighbourhood.

Observe that for a split neighbourhood, with $\mathcal{O}_{\mathcal{X}_0}(\mathfrak{D}_j) = \mathcal{O}_{\mathcal{X}_0}(-b_j)$, at least relative to Z, we can form a weighted blow up with weights $b = (b_1, \dots, b_t)$. Indeed for each $n \in \mathbb{N} \cup \{0\}$, and x_j a local equation for $\mathfrak{D}_j$ define sheaves of ideals by,

$$\mathcal{I}^{(n)} = (x_1^{s_1} \cdots x_t^{s_t} \mid s_1 b_1 + \cdots + s_t b_t \geq n)$$

and consider the graded algebra,

$$T = \bigoplus_n \mathcal{I}^{(n)}$$

with linear topology $\mathcal{I}^{(m)} T$, $m \in \mathbb{N}$, then we have the weighted formal blow up defined by,

$$\tilde{\mathfrak{X}} := \mathrm{Prof}\, T\,.$$

Now a priori the weighted blow up would seem to depend on two things,

(a) The particular splitting of $N_{\mathcal{X}_0/\mathfrak{X}}$, which unlike the H-N filtration is far from unique.

(b) The particular liftings of the divisors $\mathcal{D}_j$ from the normal bundle to the full neighbourhood.

In reality however it depends on neither. To begin with consider (a), and suppose $y_1, \ldots, y_t$ are the local equations of divisors on $\mathfrak{X}$ corresponding to a different splitting. Necessarily these are related to the equations $x_1, \ldots, x_t$ by a matrix of functions a_{ij} which is upper semi-triangular, i.e. $a_{ij} = 0$ if $j < i$, yet constant on the diagonal. In consequence not just $\tilde{\mathfrak{X}}$, but the actual graded algebra T is indepndent of (a). In a similar vein, if we fix the splitting and $y_1, \ldots, y_t$ correspond to a different lifting then for coefficient functions $c_l \in \mathcal{O}_Z$, $l = (l_1, \ldots, l_t)$ (i.e. shrink Z in the étale topology),

$$y_i = x_i + \sum_{l_1 b_1 + \cdots + l_t b_t \geq b_i} c_l\, x_1^{l_1} \ldots x_t^{l_t}\,.$$

So again even the algebra T is unchanged, and of course, a combination of this discussion removes any dependence should both (a) & (b) occur simultaneously. Consequently, in conclusion, we have:

IV.4.2 Fact/Summary. A split formal neighbourhood $\mathfrak{X}$ of a bundle of weighted projective stacks $\mathcal{X}_0$, admits a well defined weighted blow up $\sigma : \tilde{\mathfrak{X}} \to \mathfrak{X}$ with weights $b = (b_1, \ldots, b_t)$.

A more amenable description of the situation is, of course, provided in terms of an appropriate $\mu_{\underline{b}} := \mu_{b_1} \times \cdots \times \mu_{b_t}$ action. To begin with we work locally, so that $\mathfrak{U} \to \mathfrak{X}$ is a formal étale neighbourhood, and $x_1, \ldots, x_t$ local equations for the $\mathfrak{D}_j$. We can extract a b_j-th root, ξ_j, say, of each x_j, so as to realise $\mathfrak{U}$ as the coarse quotient $\mathfrak{V}/\mu_{\underline{b}}$ of some formal affine. The underlying formal scheme of $\mathfrak{V}$ is still the same as that of $\mathfrak{U}$, i.e. $U_0 = \mathfrak{U} \cap \mathcal{X}_0$, and indeed is the set theoretic pre-image of the same. Consequently if $\sigma : \tilde{\mathfrak{V}} \to \mathfrak{V}$ is the usual blow up of $\mathfrak{V}$ in U_0 then we have a not necessarily Cartesian diagram,

$$\begin{tikzcd}
\mathfrak{V} \arrow[d] & \tilde{\mathfrak{V}} \arrow[l, "\sigma"'] \arrow[d] \\
\mathfrak{U} = \mathfrak{V}/\mu_{\underline{b}} & \tilde{\mathfrak{X}} \times_{\mathfrak{X}} \mathfrak{U} = \tilde{\mathfrak{V}}/\mu_{\underline{b}} \arrow[l, "\sigma"']
\end{tikzcd}$$

This description admits various different globalisations, although the precise one in question is rather dependent on whether Z is affine or not. As such let us not singular out anyone in particular, and content ourselves with the conclusion,

IV.4.3 Fact. $\tilde{\mathfrak{X}}$ *has at worst quotient singularities; as such following* [V] 2.8, *we can, if we wish, associate to its moduli the minimal smooth stack structure associated to the same.*

Now let us turn to the issue of algebraicisability of the weighted blow up. Specifically suppose that in fact $\mathfrak{X}$ is the completion of some stack $\mathcal{X}$ in $\mathcal{X}_0 = \mathcal{Y}$. A priori as we've defined the algebra T affording $\mathfrak{X}$ it may not be an extension of an appropriate $\mathcal{O}_{\mathcal{X}_0}$-algebra. On the other hand the very argument employed to verify the dependence (or more precisely the lack thereof) of $\tilde{\mathfrak{X}}$ on the liftings of the $\mathcal{D}_j$ actually shows that if x_j is a local equation for $\mathfrak{D}_j$, and y_j any function such that,

$$y_j \equiv x_j (\mathcal{I}_{\mathcal{X}_0}^{b_j})$$

then in fact the weighted blow ups defined by way of the y_j and the x_j, or more precisely the ideals,

$$(y_1^{s_1} \ldots y_t^{s_t} \mid s_1 b_1 + \cdots + s_t b_t \geq n) \quad \text{and} \quad (x_1^{s_1} \ldots x_t^{s_t} \mid s_1 b_1 + \cdots + s_t b_t \geq n)$$

coincide. Better still if $\pi : \mathcal{X} \to X$ is the moduli of X, and $\pi : \tilde{\mathcal{X}} \to \tilde{X}$ that of $\tilde{\mathcal{X}}$, then by the invariance of the moduli under flat base change,

$$\tilde{X} = \mathrm{Proj}\,(\pi_* T)$$

while $\pi_* T$ is itself a graded $\mathcal{O}_X$-algebra generated by its pieces in degree at most n, for some sufficiently large n determined by the b_j's. As such we even have,

IV.4.4 Fact. *If $\mathfrak{X}$ is the completion of a stack $\mathcal{X}$ in some appropriate sub-stack $\mathcal{Y}$, then $\sigma : \tilde{\mathfrak{X}} \to \mathfrak{X}$ is the completion of a modification $\sigma : \tilde{\mathcal{X}} \to \mathcal{X}$ in the exceptional divisor. Moreover if $\mathcal{X}$ has projective moduli X, then $\tilde{\mathcal{X}}$ also enjoys a projective moduli space, say $\tilde{X}$.*

Finally let us note the ultimate motivation for this discussion. For simplicity let us suppose that $\mathcal{X}_0 = \mathbb{P}^r$, and write $\mathfrak{X} = \mathbf{X}'$ to distinguish the fact that our formal neighbourhood is now an honest formal scheme or algebraic space. Extracting the b_j-th root of the divisor $\mathfrak{D}_j$ yields a covering $\mathbf{X} \to \mathbf{X}'$, together with the accompanying μ_{b_j} action in such a way that $\mathbf{X}' = \mathbf{X}/\mu_{\underline{b}}$ and as we've said the weighted blow up $\tilde{\mathbf{X}}'$ is simply the quotient of the standard blow up of $\tilde{\mathbf{X}}$ in $\mathbb{P}^r$ modulo the induced $\mu_{\underline{b}}$ action. On the other hand the normal bundle of our $\mathbb{P}^r$, X_0, in $\mathbf{X}$ is $\mathcal{O}_{\mathbb{P}^r}(-1) \otimes \mathbb{C}^t$, so that the underlying sub-space of $\tilde{\mathbf{X}}$, say, is isomorphic to $\mathbb{P}^r \times \mathbb{P}^{t-1}$, and of course,

$$\mathcal{O}_{\tilde{\mathbf{X}}_0}(-\tilde{\mathbf{X}}_0) = \mathcal{O}_{\mathbb{P}^r}(1) \otimes \mathcal{O}_{\mathbb{P}^{t-1}}(1)\,.$$

As such, having formed the weighted blow up in the "b-direction" we're manifestly in the position where we'll be able to make a weighted blow down in the "a-direction".

IV.5. Formal Flipping

We could certainly deduce a flip theorem from the H-N filtration, the contraction theorem, the cone theorem, and the Euclidean algorithm. This would, however, as the previous mouthful suggests involve a certain, and as it happens unwaranted, degree of combinatorial complexity, not to mention an obfuscation of the correct structure. Consequently we'll proceed in the spirit of IV.2, beginning with,

IV.5.1 Problem. *Let $\mathfrak{X}$ be a smooth formal stack, with underlying stack $\mathcal{X}_0$ such that $\mathcal{X}_0 \hookrightarrow \mathfrak{X}$ is regularly embedded of codimension t. Suppose further,*

(a) *$\mathcal{X}_0$ is a bundle of weighted projective stacks $\boldsymbol{\mathcal{P}}(a_1, \ldots, a_r)$ over a smooth scheme (or separated algebraic/complex analytic space) Z.*

(b) *Locally over Z, $N_{\mathcal{X}_0/\mathfrak{X}} \xrightarrow{\sim} \bigoplus_{j=1}^{t} \mathcal{O}(-b_j)$, where $b_1 \leq \cdots \leq b_t$ are positive integers, and $\mathcal{O}(1)$ the tautological bundle on $\boldsymbol{\mathcal{P}}_l(a_1, \ldots, a_r)$.*

Then can we flip $\mathcal{X}_0$?

Again let us explain precisely what is intended. In the first place observe that by virtue of the exponential sequence,

$$0 \longrightarrow S^n I/I^2 \longrightarrow \mathcal{O}^*_{\mathcal{X}_{n+1}} \longrightarrow \mathcal{O}^*_{\mathcal{X}_n} \longrightarrow 0$$

and the relative acyclicity of symmetric powers of the co-normal bundle of $\mathcal{X}_0$ in $\mathfrak{X}$, that for Z affine,

$$\mathrm{Pic}(\mathfrak{X}) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{X}_0)\,.$$

As such we necessarily suppose Z a sufficiently small affine, and denote equally by $\mathcal{O}_{\mathfrak{X}}(1)$ the lifting of the tautological bundle of $\mathcal{O}_{\mathcal{X}_0}(1)$. Unsurprisingly flip is to be understood with respect to this bundle, so to this end let's actually write our initial data as $\mathfrak{X}^-$, $\mathcal{X}_0^-$, etc. with Ξ^-, Ξ_0^- etc., the formal moduli. For the sake of conformity with standard procedure we wish to first contract $\mathcal{X}_0$ to Z, i.e. find a formal modification,

$$\rho_- : \Xi^- \to \Xi$$

which restricted to $\mathcal{X}_0$ is nothing other than the projection map to Z.

The most convenient way to proceed is by changing the stack structure on $\mathfrak{X}^-$. Indeed as per the prequel to IV.2.4 we can extract roots of the coordinate hyperplanes not just on $\mathcal{X}_0$, but actually on $\mathfrak{X}^-$ to obtain a new formal stack, which rather abusively we will continue to denote by $\mathfrak{X}^-$ whose moduli continues to be Ξ^- but whose underlying stack $\mathcal{X}_0$ is actually the classifying stack $[\mathbb{P}^r_Z/\mu_{\underline{a}}]$, where of course $\mu_{\underline{a}} = \mu_{a_1} \times \cdots \times \mu_{a_r}$. Now in addition to IV.5.1, let us add to the discussion,

IV.5.2 Further Hypothesis. Suppose the neighbourhood $\mathfrak{X}$ of $\mathcal{X}_0$ of is actually split.

Consequently in the obvious extension of the definition of IV.4.1, the neighbourhood $\mathfrak{X}^-$ of the classifying stack $\mathcal{X}_0 = [\mathbb{P}^r_Z/\mu_a]$ is also split. Unsurprisingly we form a weighted blow up of $\mathfrak{X}^-$ by extracting b_j-th roots of the splitting divisors $\mathfrak{D}_j$ while additionally specifying a smooth stack structure $\tilde{\mathfrak{X}}$ in such a way that we have ramification of order b_j around the proper transforms of the $\mathfrak{D}_j$. The moduli of $\tilde{\mathfrak{X}}$ is itself, nothing other than the moduli of the weighted blow up of the original neighbourhood $\mathfrak{X}$, which in turn we denote $\tilde{\Xi}$. The advantage of this picture is of course that, $\mathbb{P}^r_Z \to \mathcal{X}_0$ is étale, so we may suppose that $\mathfrak{X}^-$ is the classifying stack, $[\mathbf{X}_1^-/\mu_a]$ for some honest formal algebraic space $\mathbf{X}_1^-$. Better still the weighted blow up $\tilde{\mathbf{X}}_1^-$ of $\mathbf{X}_1^-$ is also the moduli of a classifying stack, this time of the form $[\tilde{\mathbf{X}}/\mu_b]$ where $\tilde{\mathbf{X}}$ is the usual blow up of some formal neighbourhood $\mathbf{X}^-$ of $\mathbb{P}^r_Z$ with the property that $\mathbf{X}_1^- = \mathbf{X}^-/\mu_b$. All the stacks in question are of course smooth, and we arrive to the rather desirable position where the weighted blow up $\sigma : \tilde{\Xi} \to \Xi$ is the moduli of,

$$\begin{array}{ccc}
[\tilde{\mathbf{X}}/\mu_{\underline{a}} \times \mu_{\underline{b}}] & \xrightarrow[\pi]{} & \tilde{\Xi} \\
\big\downarrow{\scriptstyle \sigma_-} & & \big\downarrow{\scriptstyle \sigma_-} \\
[\mathbf{X}^-/\mu_{\underline{a}} \times \mu_{\underline{b}}] & \xrightarrow[\pi]{} & \Xi
\end{array}$$

for $\mathbf{X}^-$ a smooth split formal neighbourhood of $\mathbb{P}^r_Z$, with $\tilde{\mathbf{X}}$ the blowing up in the same, and the $\mu_{\underline{a}} \times \mu_{\underline{b}}$ acting in the natural way on the projective coordinates $X_0, \ldots, X_r$, and splitting divisors $\mathfrak{D}_1, \ldots, \mathfrak{D}_t$.

It almost goes without saying that a minor variant of the prequel to IV.2.4 as alluded to above establishes that the formal functions on $\tilde{\mathbf{X}}$ are simply those over the cone of the veronese embedding of $\mathbb{P}^r_Z \times_Z \mathbb{P}^{t-1}_Z$ considered as a formal sub-variety of the formal affine space $\hat{\mathbb{A}}^{(r+1)t}_Z$, and as a result in the natural topology determined by the underlying space,

$$\tilde{\mathbf{X}} \longrightarrow \operatorname{Spf}\Gamma(\mathcal{O}_{\tilde{\mathbf{X}}})$$

is not just a formal modification, but the blow down of the said underlying space to Z. Appealing, therefore, as ever to the exactness of π_* we conclude that,

$$\tilde{\Xi} \longrightarrow \operatorname{Spf}\Gamma(\mathcal{O}_{\tilde{\Xi}})(=\Gamma(\mathcal{O}_{\Xi})) := \Xi_0$$

is a formal modification, and in the obvious sense of the word, a bi-weighted blow down. Better still the argument works equally well for $\boldsymbol{\mathcal{P}}(a_1, \dots, a_r)$ as the classifying stack $[\mathbb{P}^r/\mu_{\underline{a}}]$, and is local for the étale topology with respect to which the underlying stack of $[\tilde{\mathbf{X}}/\mu_{\underline{a}}]$ is $[\mathbb{P}^r_Z/\mu_{\underline{a}}] \times_Z [\mathbb{P}^{t-1}_Z/\mu_{\underline{b}}]$, so that in fact we have a weighted blow down,

$$\tilde{\Xi} \longrightarrow \Xi^+$$

which at the stack level contracts $[\mathbb{P}^r_Z/\mu_{\underline{a}}] \times_Z [\mathbb{P}^{t-1}_Z/\mu_{\underline{b}}]$ to $[\mathbb{P}^{t-1}_Z/\mu_{\underline{b}}]$. We may summarise the situation by way of,

$$\begin{tikzcd}
 & \tilde{\Xi} \arrow[dl, "\text{weighted blow up} = \sigma_-"'] \arrow[dr, "\sigma_+ = \text{weighted blow down}"] & \\
\Xi^- \arrow[rr, dashed] \arrow[dr, "\rho_-"'] & & \Xi^+ \arrow[dl, "\rho_+"] \\
 & \Xi_0 &
\end{tikzcd}$$

and observe,

IV.5.3 Fact/Definition/Summary/Warning. The above diagram, or better still simply the upper triangle will be called a flap. Note that in terms of the $\mathbb{Q}$-Cartier divisor $\mathcal{O}(1)$ on Ξ we have not verified the irrelevant technical definition of formal flip, i.e.

$$\Xi_- = \operatorname{Prof}\bigoplus_{n=0}^{\infty} \mathcal{O}_{\Xi_0}(n(\rho_-)_*\,\mathcal{O}(-1))\,, \quad \Xi_+ = \operatorname{Prof}\bigoplus_{n=0}^{\infty} \mathcal{O}_{\Xi_0}(n(\rho_+)_*\,\mathcal{O}(1))$$

and that the given algebras are finitely generated, nor have we even verified that ρ_-, and ρ_+ are formal modifications in the technical sense of the word, albeit that we have done this for the composites $\rho_- \circ \sigma_- = \rho_+ \circ \sigma_+$. Nevertheless the divisors $\mathcal{O}(-1)$, and $\mathcal{O}(+1)$ are relatively ample at the level of the maps of the underlying spaces of ρ_-, and ρ_+, respectively, and whence we will not hesitate to equally employ the word flip to describe the lower triangle.

Turning to questions of convergence we may apply Artin's convergence theorem together with I.7 not to forget the uniqueness of contraction, to conclude,

IV.5.4 Corollary. *Let $\mathcal{X}^-$ be a smooth stack and $\mathcal{Y}$ a closed sub-stack which is a $\boldsymbol{\mathcal{P}}(a_1, \dots, a_r)$ bundle over a smooth scheme or algebraic space Z, such that locally over Z, the completion $\hat{\mathcal{X}}^-$ of $\mathcal{X}^-$ in $\mathcal{Y}$ is a split neighbourhood, then, employing as ever latin letters X^-, Y etc. for the moduli,*

(a) *The contraction, $\rho_- : X^- \to X_0$ of Y to Z exists as an algebraic space.*

(b) *There is an algebraic space, $\rho_+ : X^+ \to X_0$, which modulo the caveat of IV.5.3 is locally the formal flip of the unique generator of* $\mathrm{Pic}\,(\hat{X}^-)$.

(c) *All of this fits into a flip/flap diagram of algebraic spaces,*

$$\begin{array}{ccc} & \tilde{X} & \\ \text{weighted blow up} = \sigma_- \swarrow & & \searrow \sigma_+ = \text{weighted blow down} \\ X^- & - - - - \to & X^+ \\ \rho_- \searrow & & \swarrow \rho_+ \\ & X_0 & \end{array}$$

As already noted if X^- is projective then so is $\tilde{X}$, although we have no such a priori conlusions regarding X^+, and X_0. In addition due to our aforesaid lacadaisical study of $\hat{X}^-$, we even use the a priori existence of $\tilde{X}$ to conclude to the existence of X_0.

IV.6. Back at Foliations

We now wish to put our previous considerations into action in the case of foliated stacks. Consequently, as ever $(\mathcal{X}_-, \tilde{\mathcal{F}}_-)$ will be a smooth foliated stack with log-canonical singularities terminal at the non-scheme like points and projective moduli, $\pi : (\mathcal{X}_-, \tilde{\mathcal{F}}_-) \to (X_-, B_-, \mathcal{F}_-)$. Up to the laxism implicit in our definition of the same, IV.5.3, we wish to construct the flip of an extremal subvariety $\mathcal{Y}$, as a stack $(\mathcal{X}_+, \tilde{\mathcal{F}}_+)$ enjoying the same list of properties enunciated for $(\mathcal{X}_-, \tilde{\mathcal{F}}_-)$ with the exception of projectivity of the moduli. Unsurprisingly the Gorenstien covering stack, $\pi : (\mathcal{X}'_-, \mathcal{B}'_-, \mathcal{F}_-) \to (X_-, B_-, \mathcal{F}_-)$ will play an important intermediary role.

To begin with we consider the case $\mathcal{Y}$ is smooth, and we form the weighted blow up $\sigma_- : \tilde{X} \to X_-$, which we may certainly do since our split neighbourhood condition is guaranteed by II.6.2. In addition, as remarked in the prequel to IV.5.3, $\tilde{X}$ has at worst quotient singularities so there is a minimal smooth stack $\pi : \tilde{\mathcal{X}}' \to \tilde{X}$, étale in codimension 2. Note, however, that if $\mathcal{Y}$ is a $\mathcal{Y}(d\underline{a},\, s,\, \underline{b})$, for general $\underline{b}$, $\tilde{\mathcal{X}}'$ will be bigger than the Gorenstien covering stack of $\tilde{X}$ in the induced foliation $\mathcal{F}$. Nevertheless we have a map of stacks $\sigma_- : \tilde{\mathcal{X}}' \to \mathcal{X}'_-$, and by the various lack of dependences, cf. sequel to IV.4.1, of the weighted blowing on anything other than the H-N filtration, not to mention the identity of the same over a coordinate line in any $(\mathbb{P}(\underline{a}), \mathcal{R})$ with that on the $\mathbb{P}(\underline{a})$, and the fact that the former is defined by way of invariant divisors we conclude that,

$$K_{\mathcal{F}} = (\sigma_-)^* K_{\mathcal{F}_-} .$$

In addition if $x_1, \ldots, x_t$ are any local functions in the étale site of $\mathcal{X}'_-$ employed in the formulation of the weighted blow up, with y a local equation for $\mathcal{B}'_-$ then the $x_1, \ldots, x_t, y$ are simple normal crossing divisors. As such,

(a) The proper transform $\tilde{\mathcal{B}}'$ of $\mathcal{B}'_-$ is in fact its pull-back. Idem for any sufficiently large multiple of B_- which is Cartier, in terms of the proper transform $\tilde{B}$.

(b) There is no difficulty in forming a covering stack $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\tilde{\mathcal{X}}', \mathcal{F})$ ramified over the smooth divisor $\tilde{\mathcal{B}}'$ with appropriate weight implicit in the definition of B_-.

Necessarily we have a map, $\sigma_- : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}_-, \tilde{\mathcal{F}}_-)$, and, even $K_{\tilde{\mathcal{F}}} = (\sigma_-)^* K_{\tilde{\mathcal{F}}_-}$, from which we conclude that $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ also has log-canonical singularities. It may certainly happen, however, that $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ is not terminal at the non-scheme like points, so to go further requires a little care. To begin with observe that if E, $\mathcal{E}'$ are the exceptional divisors with respect to σ_- for $\tilde{X}$, and $\tilde{\mathcal{X}}'$ respectively then of course $\mathcal{E}' = \pi^* E$, while for $\mathcal{L} \subset \mathcal{E}'$ a $K_{\mathcal{F}}$-negative smooth stack,

$$\mathcal{O}_{\mathcal{L}}(b\, K_{\mathcal{F}}) = \mathcal{O}_{\mathcal{L}}(\mathcal{E}')$$

where b is the gcd of $b_1, \dots, b_t$. Even better since $\mathcal{B}'_-$ is transverse to $\mathcal{E}'$, then in fact for $\mathcal{E}$ the exceptional divisor on $\tilde{\mathcal{X}}$, $\mathcal{E} = \pi^* E$. As such if $K_{\tilde{\mathcal{F}}_+}$ is the canoncial class of the foliated logarithmic $\mathbb{Q}$-factorial algebraic space $(X_+, B_+, \mathcal{F}_+)$ whose existence is guaranteed by IV.5.4, then as per, IV.1.9,

$$K_{\tilde{\mathcal{F}}} = (\sigma_+)^* K_{\tilde{\mathcal{F}}_+} + \frac{1}{db} E\,.$$

Now we can argue exactly as per op. cit. The important being that any new singularities on X_+ are terminal, and whence by I.7.3,

IV.6.1 Fact/Summary. *There is a smooth foliated stack $(\mathcal{X}_+, \tilde{\mathcal{F}}_+)$ whose singularities are log-canonical, terminal at the non-scheme like points, albeit that its moduli $\pi : (\mathcal{X}_+, \tilde{\mathcal{F}}_+) \to (X_+, \mathcal{F}_+)$ may only be an algebraic space. Nevertheless it fits into a flip/flap diagram for the supposed extremal smooth $\mathcal{Y}$,*

$$\begin{array}{ccccc} & & \tilde{\mathcal{X}} & & \\ & \swarrow & & \searrow & \\ \mathcal{X}^- & & - - - \to & & \mathcal{X}^+ \\ & \searrow & & \swarrow & \\ & & \mathcal{X}_0 & & \end{array}$$

whose completion around $\mathcal{Y}$ is the formal flip/flap of IV.5.4. In particular, from the conformal surgery point of view, the bundle of weighted projective stacks $\mathbb{P}(da_1, .., da_r)$ over Z is flipped to a bundle of generalised projective stacks $\mathbb{P}_g(db_1, \dots, db_t)$ over the same.

Next let us consider the possibility that we have two extremal smooth sub-stacks, $\mathcal{Y}_+$, and $\mathcal{Y}_-$ meeting in Z, which, may or may not, be defined by the same extremal ray. The said stacks are necessarily of the form $\mathcal{Y}(d_+\underline{a}, s, \underline{b})$, respectively $\mathcal{Y}(\underline{a}, s, d_-\underline{b})$ for some $d_+, d_- \in \mathbb{N}$. Here we first profit from the split neighbourhood structure to make a weighted blow up, $\mathcal{X}_Z$ with support on Z, and minimal smooth structure around the exceptional divisor $\mathcal{E}_Z$. In our habitual local coordinates this will have weights a_i on the x_i, and b_j on the y_j. This gives rise to proper transforms $\tilde{\mathcal{Y}}_+$, $\tilde{\mathcal{Y}}_-$ which are now disjoint. In truth there is a moderate extra amount of monodromy around the induced singular locus, i.e. $\mathcal{E}_Z \cap \tilde{\mathcal{Y}}_+$ and $\mathcal{E}_Z \cap \tilde{\mathcal{Y}}_-$, but we can, nevertheless proceed mutatis-mutandis to flip/flap these as per VI.6.1. As such there is a weighted blow up $\tilde{\mathcal{X}} \to \mathcal{X}_Z$, with exceptional divisors $\mathcal{E}_+$, $\mathcal{E}_-$ which get blown down in the other direction by way of $\tilde{\mathcal{X}} \to \mathcal{X}^+$. The result is, however, rather different and up to at most a $\mathbb{Z}/2$ cover, the substack $\mathcal{Y}_+ \cap \mathcal{Y}_-$ is flapped to the proper transform $\mathcal{E}$ of $\mathcal{E}_Z$ which is itself a $\mathcal{P}_g(d_+, d_-)$ bundle over a $\mathcal{P}_g(\underline{a}) \times \mathcal{P}_g(\underline{b})$ bundle over Z. Consequently $\mathcal{X}^+$ may well not be terminal at the non-scheme like points, but the foliation is nevertheless still smooth around $\mathcal{E}$, on which the induced foliation is the said $\mathcal{P}_g(d_+, d_-)$ bundle. Consequently our foliation is, in fact, in conics and better still this construction is certainly local and invariant by any $\mathbb{Z}/2$-monodromy of the form encountered in III.5.1(b)(ii), so we summarise by way of,

IV.6.2 Fact *Suppose the situation presents itself where there are extremal substacks through both of the H-N directions at a component Z of the singular locus then there is a simultaenous flap of the extremal substack(s) through Z*

$$\begin{array}{ccc} & \tilde{\mathcal{X}} & \\ \swarrow & & \searrow \\ \mathcal{X}^- & -\,-\,-\to & \mathcal{X}^+ \end{array}$$

with the induced foliation on $\mathcal{X}_+$ a smooth foliation in rational stacks on and around the flapped locus.

It almost goes without saying, but let's say it anyway, that it is incorrect to describe IV.6.2 as a flip, since with these hypothesis the (total) extremal substack through Z cannot be contracted.

IV.7. Projectivity and Termination

Plainly the minimal model theorem for foliations is very close to being established. Indeed the only outstanding issues are the projectivity, not to mention termination, of flipping/flapping. Let us first concentrate on the non-conic case, which in any case is the sine qua non for a model with $K_{\mathcal{F}}$-nef. As such we only have to flip, and we re-take the notations prior to VI.6.1, while introducing the key invariant $Z = Z(\mathcal{Y})$, i.e. the unique smooth connected component of the singular locus of the foliation contained in the extremal smooth sub-stack $\mathcal{Y}$ which we have flipped. Observe that over the weighted blow up, $\sigma_- : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}_-, \tilde{\mathcal{F}}_-)$, the induced singular locus $\tilde{\mathcal{Z}}$, which may very well be non-scheme like, is the fibre of the exceptional divisor $\mathcal{E}$ over $\mathcal{Z}$, and so indeed forms a $\mathcal{P}_g(b_1, \ldots, b_t)$ bundle over Z transverse to the direction of the weighted blow down σ_+. As such, at least at the level of the moduli, $\mathcal{Y}$ is flipped to the image Z_+ of the moduli of $\tilde{\mathcal{Z}}$, and as we've said, $\sigma_+ : \tilde{\mathcal{Z}} \to Z_+$ is certainly one to one on closed points. In any case what's important is that the logarithmic moduli $(X_+, B_+, \mathcal{F}_+)$ is log-terminal at every scheme point of Z_+, so that for $\mathcal{Z}_+ \subset \mathcal{X}_+$, and $\mathcal{Z}'_+ \subset \mathcal{X}'_+$ the corresponding closed sub-stacks whether of the flip or the Gorenstien covering stack of $(X_+,\ \mathcal{F}_+)$ the induced foliations, are:

IV.7.1 Fact.

(a) *Smooth at every geometric point of $\mathcal{Z}_+$, and $\mathcal{Z}'_+$ respectively.*

(b) *The ramification, respectively branching, divisor $\mathcal{B}_+$, respectively $\mathcal{B}'_+$, are smooth and everywhere transverse to the induced foliation at every geometric point of $\mathcal{Z}_+$, respectively $\mathcal{Z}'_+$.*

Now let's consider the consequences of this observation for the projectivity of the moduli of $\mathcal{X}_+$. In the first place let us denote by R the extremal ray in Néron-Severi with respect to which $\mathcal{Y}$ is extremal. On forming the weighted blow up $\sigma_- : \tilde{\mathcal{X}} \to \mathcal{X}_-$, we have over each $K_{\tilde{\mathcal{F}}}$-negative curve in $\mathcal{Y}$, a $\mathbb{P}_g(b_1, \dots, b_t)$ of $K_{\tilde{\mathcal{F}}}$-negative curves in $\tilde{\mathcal{X}}$, all of which are parallel to a not necessarily extremal ray $\tilde{R}$ in the Néron-Severi group of $\tilde{\mathcal{X}}$. Indeed if L in Néron-Severi of $\tilde{\mathcal{X}}$ is the class of the image of an exceptional line, then,

$$\tilde{R} = \sigma_-^* \, R + L \, .$$

On the other hand $\tilde{\mathcal{X}}$ has projective moduli, so the cone theorem is valid over it, and whence by IV.6.2, there is only one possible obstructions to the extremality of $\tilde{R}$, namely :

IV.7.2 Obstruction. The ray $\tilde{\mathcal{R}}$ is not extremal in $\tilde{\mathcal{X}}$ iff there is some, necessarily finite, set of disjoint extremal subvarieties $\mathcal{Y}_1, \dots, \mathcal{Y}_k$, with say $\mathcal{Y} = \mathcal{Y}_1$, and $k \geq 2$, all of codimension at least 2, yet all swept out by $K_{\tilde{\mathcal{F}}}$-negative extremal stacks parallel to $\mathcal{R}$.

Now whether this could actually happen or not is open to question. Nevertheless, all the extremal stacks in question are disjoint, so it is easily dealt with. Specifically we form weighted blow up in each $\mathcal{Y}_p$, $1 \leq p \leq k$, in turn to get a smooth stack $\tilde{\mathcal{X}}(R)$ with projective moduli $\tilde{X}(R)$, naturally denoted σ_-, and exceptional divisors $\mathcal{E}_1, \dots, \mathcal{E}_k$. Furthermore in an obvious extension of our previous notations we have classes $\tilde{R}_1, \dots, \tilde{R}_k$, and $L_1, \dots, L_k$ in Néron-Severi of $\tilde{\mathcal{X}}(R)$ corresponding to $K_{\tilde{\mathcal{F}}(R)}$-negative stacks, and exceptional lines in the said divisors, with of course,

$$\tilde{R}_p = \sigma_-^* \, R + L_p \, , \qquad 1 \leq p \leq k \, .$$

As such each $\tilde{R}_p$ is extremal in $\tilde{\mathcal{X}}(R)$, and by IV.1.4 etc. we can make a weighted blow down of each in turn without loosing projectivity of the moduli. Unsurprisingly, therefore, this gives the flip $\mathcal{X}_+(R)$ of R as the graph of the various flips of the individual $\mathcal{Y}_p$'s. Better still if H_R is a supporting function for R, then the $\tilde{R}_p$'s and L_p's generate the extremal face defined by $\sigma_-^* \, H_R$ in Néron-Severi of $\tilde{\mathcal{X}}(R)$ so a minor variant of our considerations in IV.1, shows that the simultaneous blow down of the $\mathcal{E}_1, \dots, \mathcal{E}_k$ to the implied smooth components $Z_1, \dots, Z_k$ of the singular locus of $\mathcal{F}_-$ (which is of course the blow down of the $\mathcal{Y}_1, \dots, \mathcal{Y}_k$ to the same) yields a projective variety $X_0(R)$, so we even have;

IV.7.3 Fact. *The previous technical caveat about flips has been removed, i.e. in* *terms of the underlying moduli,*

$$\begin{tikzcd}
(X_-,\, B_-,\, \mathcal{F}_-) \arrow[rr, dashed] \arrow[dr, "\rho_-"'] & & (X_+(R),\, B_+(R),\, \mathcal{F}_+(R)) \arrow[dl, "\rho_+"] \\
& (X_0,\, B_0(R),\, F_0(R)) &
\end{tikzcd}$$

is not just a flap, but a flip of R in the proper technical sense of [K3], 2.26.

Better still, flipping manifestly terminates for the simple reason that the number of connected components of the singular locus decreases by at least 1 with the flip of any extremal ray, and so we conclude,

IV.7.4 Lemma. *Let $(\mathcal{X},\, \tilde{\mathcal{F}})$ be a foliated smooth stack with projective moduli $(X,\, \mathcal{F})$ and ramification divisor B. Suppose further that $(\mathcal{X},\, \tilde{\mathcal{F}})$ has log-canonical (respectively canonical) singularities which are terminal at the non-scheme like points, and that the said foliation is not a foliation in parabolic stacks then there is a sequence of flips and flops in the sense of IV.1.10 and IV.5/7.3 (or alternatively just flaps),*

$$(\mathcal{X},\, \tilde{\mathcal{F}}) = (\mathcal{X}_0,\, \tilde{\mathcal{F}}_0) \dashrightarrow (\mathcal{X}_1,\, \tilde{\mathcal{F}}_1) \dashrightarrow \cdots \dashrightarrow (\mathcal{X}_n,\, \tilde{\mathcal{F}}_n) = (\mathcal{X}_{\min},\, \tilde{\mathcal{F}}_{\min})$$

such that each $(\mathcal{X}_i,\, \tilde{\mathcal{F}}_i)$ enjoys all the properties enunciated above of $(\mathcal{X},\, \tilde{\mathcal{F}})$, and in addition $K_{\tilde{\mathcal{F}}_{\min}}$ is nef.

IV.8. Rational Foliations

We now wish to consider more precisely the case of smooth foliated stacks $(\mathcal{X},\, \tilde{\mathcal{F}})$ enjoying the property that through every geometric point there is a rational 1-dimensional stack, or alternatively a rational curve through every point of the moduli $\pi : (\mathcal{X},\, \tilde{\mathcal{F}}) \to (X,\, \mathcal{F})$. We'll begin with the case that π is actually the Gorenstien covering stack, so indeed $\tilde{\mathcal{F}} = \mathcal{F}$, and continue to suppose that the singularities are terminal at the non-scheme like points. We'll first consider the difference between canonical singularities and log-canonical, *i.e.*,

IV.8.1 Claim. *A foliation with canonical singularities $(X,\, \mathcal{F})$ on a normal variety X is a rational foliation iff through a generic closed point x of X there is a smooth invariant rational curve $L_x \ni x$ contained in the smooth locus of X and $\mathcal{F}$, so in particular with $K_{\mathcal{F}}.L_x = -2$.*

Proof. The if direction is clear by the existence of the Hilbert scheme, and the closedness of the invariance condition. Conversely let $(X,\, \mathcal{F})$ be a rational foliation, and denote by W and Z the singular loci of X and $\mathcal{F}$ respectively. Consider in the first instance the possibility that every invariant curve meets Z, so in fact the generic invariant curve meets some irreducible subvariety Z_0 in its generic point. Now call X, X_0, and blow up in Z_0, then either there is a subvariety Z_1 of the induced singular locus, where the generic rational invariant curve meets the generic point of the same, or there is not. In any case calling the blow up X_1, and proceeding inductively we eventually reach the situation that on some X_n, the exceptional divisor E over the generic point of Z_{n-1} meets the proper transform of a generic rational invariant curve generically, and whence cannot be invariant. However by the very definition, this is impossible if the singularities are canonical, so indeed the generic invariant curve is contained in the smooth locus of $\mathcal{F}$. Now suppose further that the generic invariant curve meets an irreducible subvariety W_0 of $\mathrm{sing}\,(X)$ in its generic point. By virtue of our previous discussion we can suppose $W_0 \not\subset \mathrm{sing}\,(\mathcal{F})$, so at the generic point of W_0, the foliation is a relatively smooth fibration. Consequently W_0 must be of codimension 1, which is nonsense given X normal. As such we obtain that the generic invariant rational curve is contained in $X\backslash\{\mathrm{sing}\, X \cup \mathrm{sing}\, \mathcal{F}\}$, from which we conclude. □

Notice, indeed, that the assumption of canonical is essentially optimal since,

IV.8.2 Clarification. There exist rational foliations with log-canonical singularities and $K_{\mathcal{F}}$ nef. For example on $\mathbb{P}^2$, the vector field given in standard affine coordinates by,

$$\partial = px\,\frac{\partial}{\partial x} + qy\,\frac{\partial}{\partial y}$$

$p, q \in \mathbb{N}$, relatively prime, affords such an example.

Now exactly as per IV.7.3: should we find a smooth extremal substack through a component of the singular locus then IV.6.1 is still an honest flip in the proper technical sense of [K3] 2.26 unless we come across,

IV.8.3 Obstruction. *For some smooth, and necessarily scheme like, component Z of $\mathrm{sing}\,(\mathcal{F})$ there are closed invariant weighted projective extremal sub-stacks $\mathcal{Y}_+ = \mathcal{Y}(d_+\underline{a},\, s,\, \underline{b})$, and $\mathcal{Y}_- = \mathcal{Y}(d_-\underline{b},\, s,\, \underline{a})$ corresponding to the H-N pair at Z, and swept out by $K_{\tilde{\mathcal{F}}}$-negative extremal stacks parallel to the same extremal ray R.*

As such, the first step towards a 'minimal model' for rational foliations is,

IV.8.4 Step 1 *Systematically flip all proper smooth extremal substacks à la IV.7.3/4 for which the obstruction IV.8.3 does not present itself.*

In carrying out step 1 we necessarily preserve our habitual hypothesis of projective moduli and terminal singularities at the non-scheme like points. Should, however, IV.8.3 occur we appeal to,

IV.8.5 Claim. *Suppose that we are in the situation where the obstruction of IV.8.3 to the projectivity of flipping presents itself with respect to an extremal ray R, or more generally that the union of proper extremal sub-stacks defined by R is non-smooth, i.e. throw in* III.5.1(b)(ii) *for good measure then* $\mathrm{Loc}\,(R) = X$.

Proof. In the notation of IV.6.2 with $\mathcal{X} = \mathcal{X}_-$, consider the possibly non-projective flap $\mathcal{X}_+$ be it of $\mathcal{Y}_+ \cup \mathcal{Y}_-$ or a singular extremal stack $\mathcal{Y}_{+-}$,

$$\begin{array}{ccc} & \tilde{\mathcal{X}} & \\ \sigma_- \swarrow & & \searrow \sigma_+ \\ \mathcal{X}_- & & \mathcal{X}_+ \end{array}$$

To begin with, let's suppose we're not in the $\mathbb{Z}/2$ monodromy case, and denote by C_+, C_- generic invariant curves in $\mathcal{Y}_+$ and $\mathcal{Y}_-$ respectively. Furthermore denoting by $\tilde{C}_+, \tilde{C}_-$ invariant curves in $\tilde{\mathcal{X}}$ lying over C_+ and C_- respectively, then for L the proper transform of an invariant curve in the flapped locus, we know that some positive rational combination of $\tilde{C}_+, \tilde{C}_-, L$ is equivalent to a smooth invariant rational curve inside the smooth locus of $\tilde{\mathcal{F}}$. By hypothesis, however, the push forwards under σ_- of $\tilde{C}_+, \tilde{C}_-$ are parallel to R, while that of L is zero. The non-trivial monodromy case is even easier up to a change of notation. □

Keeping the same notations, let us continue in this vein by way of,

IV.8.6 Further Claim. *Suppose that we're in the conditions of* IV.8.3 *then: every component Z of the singular locus $\mathcal{F}$ is smooth, and there are either proper extremal sub-stacks of codim ≥ 2, $\mathcal{Y}_+$, $\mathcal{Y}_- \supset Z$ or a singular extremal sub-stack $\mathcal{Y}_{+-} \supset Z$, which may actually have co-dimension 1, corresponding to the H-N pair swept out by invariant stacks parallel to R. In particular the singularities are canonical, not just log-canonical.*

Proof. Invariance by $\mathcal{F}$ is a closed condition on $\mathrm{Hilb}\,(X)$, so we can find a subvariety T of $\mathrm{Hilb}\,(X)$ containing a generic (and necessarily extremal) invariant curve, M, and smooth there of dimension: $\dim X - 1$. In particular if,

$$\begin{array}{ccc} C & \xrightarrow[q]{} & X \\ {\scriptstyle p}\big\downarrow & & \\ T & & \end{array}$$

is the universal family over T, then q is surjective. Now let $t \in T$, and $C_t = \Sigma\, a_i\, C_i$ the cycle theoretic fibre, then in $\mathrm{NS}_1(X)$,

$$R = M = q_*\, C_t = \Sigma\, a_i(q_*\, C_i)$$

so every component of C_t is a $K_{\tilde{\mathcal{F}}}$-negative extremal invariant curve parallel to R.

Now let Z be a component of $\mathrm{sing}\,(\tilde{\mathcal{F}})$, then for $z \in Z$, there is a $t \in T$ such that, $C_t \ni z$, so in fact $z \in \mathcal{L}$ for some smooth $K_{\tilde{\mathcal{F}}}$-negative extremal rational stack, which in turn defines an extremal subvariety $\mathcal{Y}_+(Z)$ or $\mathcal{Y}_{+-}(Z)$ à la III.5. Better still the existence of M forces $\mathcal{Y}_+(Z)$, respectively $\mathcal{Y}_{+-}(Z)$, to be proper of $\mathrm{codim} \geq 2$, and of course around Z we can find formal coordinates y_i, x_i defining Z, $1 \leq i \leq r$, $1 \leq j \leq t$, a_i, $b_j \in \mathbb{N}$ so that the foliation is of the form,

$$a_i\, y_i\, \frac{\partial}{\partial y_i} - b_j\, x_j\, \frac{\partial}{\partial x_j}$$

Whence, certainly Z is smooth, and the corresponding singularity of $\tilde{\mathcal{F}}$ canonical. As such if we have an irreducible $\mathcal{Y}_{+-}$ we're done, and otherwise we performing a naive flip of $\mathcal{Y}_+(Z)$ in the sense of IV.6.1. We thus obtain an algebraic space X_+ which still has a proper Hilbert-scheme so $\mathcal{Y}_-(Z)$ exists too, and we're done. $\square$

Consequently we deduce that a good move is,

IV.8.7 Step 2 *If the obstruction IV.8.3 is present, and/or there is a non-smooth extremal substack, then all invariant curves are parallel to a single extremal ray R, and we simultaenously flap all of these in the sense of IV.6.2.*

The operation of flapping manifestly maintains projectivity so we have established most of,

IV.8.8 Fact/Summary. *Let $\pi : (\mathcal{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ be a smooth foliated stack with projective moduli, and log-canonical singularities terminal at the non-scheme like points such that a sequence of flops and flips of proper extremal sub-stacks does not produce a foliated stack with nef. canonical class. In such circumstances let,*

$$(\mathcal{X}, \tilde{\mathcal{F}}) = (\mathcal{X}_0, \mathcal{F}_0) \dashrightarrow (\mathcal{X}_1, \mathcal{F}_1) \dashrightarrow \cdots \dashrightarrow (\mathcal{X}_n, \tilde{\mathcal{F}}_n) = (\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$$

be a maximal sequences of flops and flips of proper extremal sub-stacks, then, either,

(a) *The singularities of* $(\mathcal{X},\,\tilde{\mathcal{F}})$ *are not canonical, and* $(\tilde{\mathcal{X}},\,\tilde{\mathcal{F}})$ *is a bundle (in the Zariski topology) of linearly foliated projective stacks* $(\boldsymbol{\mathcal{P}}(da_1,\,\dots,\,da_r),\mathcal{R})$ *over a smooth projective variety* Z.

(b) *The singularities of* $(\mathcal{X},\,\tilde{\mathcal{F}})$ *are canonical, then either,*

(i)There is a separated stack $\tilde{\mathcal{X}}/\tilde{\mathcal{F}}$ *with projective moduli, such that* $\tilde{\mathcal{X}} \to \tilde{\mathcal{X}}/\tilde{\mathcal{F}}$ *is a* $\mathbb{P}^1(p,q)$ *bundle for some relatively prime* $p,\,q\in\mathbb{N}$ *in the étale topus of* $\tilde{\mathcal{X}}/\tilde{\mathcal{F}}$*, and* $\tilde{\mathcal{F}}$ *is the corresponding fibration.*

or

(ii) The foliation $\tilde{\mathcal{F}}$ *is still singular, and we perform the flap* $(\mathcal{X},\,\tilde{\mathcal{F}}) - \to (\mathcal{X}_+,\tilde{\mathcal{F}}_+)$ *of IV.8.7. We may thus obtain non-scheme like points which are non-terminal, but otherwise we have (i) albeit for* $\tilde{\mathcal{X}}_+/\tilde{\mathcal{F}}$

Proof. Pretty much everything has been done, and it's really just a matter of clearing up loose ends. We begin by observing that if $(\mathcal{X},\,\tilde{\mathcal{F}})$ has log-canonical singularities which are not canonical, then by I.6.12 there is an irreducible component Z of $\mathrm{sing}\,(\tilde{\mathcal{F}})$ around the generic point of which we can find formal coordinates $y_1,\dots,\,y_r$ defining Z, positive integers a_i, and a derivation ∂ defining the foliation, such that:

$$\partial = a_i\,y_i\,\frac{\partial}{\partial y_i} + f\cdot\delta$$

where f vanishes on Z, and δ is a derivation along Z. Manifstly such a singularity is not destroyed by flipping, and flopping proper sub-stacks, so in this situation a maximal (if not unique) chain of flips, and flops of proper sub-stacks is well defined, and terminates in some smooth foliated stack $(\tilde{\mathcal{X}},\,\tilde{\mathcal{F}})$ with projective moduli, and log-canonical singularities, terminal at the non-scheme like points. Nevertheless $K_{\tilde{\mathcal{F}}}$ is not nef., nor as the above discussion shows could an extremal ray be represented by an invariant stack in the smooth locus of $\tilde{\mathcal{F}}$. As such the only remaining possibility, by way of the obvious variant of III.5 is precisely (a), with a_i the above integers, provided $\gcd\,(a_1,\dots,\,a_r)=1$, and d the branching order of the hyperplane bundle at infinity.

This leaves us with the canonical case, where we've checked everything beyond the existence of the quotient and that $\mathcal{X}$ is a $\mathbb{P}^1(p,q)$ bundle over the same. To this end observe that the universal cover of the generic invariant curve is certainly a $\mathbb{P}^1(p,q)$, for some $p,\,q\in\mathbb{N}$. Now let $\mathcal{C}$ be any invariant curve in $\tilde{\mathcal{X}}$ or $\mathcal{X}_+$ according as whether we're in case (i) or (ii), then it's universal cover is again a $\mathbb{P}^1(p',q')$ for possibly different p', q'. On the other hand, the holonomy trivialises over the universal cover, so there is a polydisc Δ of dimension $\dim\mathcal{X}-1$, and an étale map, $\Delta\times\mathbb{P}^1(p',q')\to\mathcal{X}$, with the induced foliation the fibration over Δ. Consequently, we get a map from $\mathbb{P}^1(p',q')$ to $\mathbb{P}^1(p,q)$, so, without loss of generality, $p'=p$, and $q'=q$. Appealing to [A1] we therefore certainly get a bundle in the étale topus provided a fine quotient exists. We can, however, à la A.1.4 form a representable groupoid $\mathcal{F}_{\mathrm{hol}}\rightrightarrows\tilde{\mathcal{X}}$ or $\mathcal{X}_+$, as appropriate, whose stabaliser is proper, so the quotient is at worst a separated Deligne- Mumford stack. Nevertheless, be it $\tilde{\mathcal{X}}$ or $\mathcal{X}_+$ has projective moduli, and the said quotient must admit a finite map to some necessarily projective component of the chow scheme of the moduli, so we're done. □

IV.9. Logarithmic Remarks

If we translate back to varieties the minimalist type of stacks that we have discussed, viz: $(\mathcal{X}, \tilde{\mathcal{F}})$ smooth foliated with log-canonical singularities, terminal at the non-scheme like points, then in terms of the moduli $(X, B, \mathcal{F})$ and its branching divisor we know by I.7.4 that this translates as X normal, and log-canonical singularities terminal at $\mathrm{sing}\,(X) \cup |B|$. Translating once more, but this time to the Gorenstien covering stack, $(\mathcal{X}', \mathcal{B}, \mathcal{F})$ we have log-canonical singularities, terminal at the non-scheme like points, and $\mathcal{B}$ a smooth (albeit not necessarily connected) divisor everywhere transverse to $\mathcal{F}$ with finite weights. Now it's pretty clear that we can easily extend the programme to allowing infinite weights on whatever components of $\mathcal{B}$ we may choose by the simple expedient of introducing sufficiently large weights to be determined. There is, however, a rather easier strategy which quite generally produces better models, viz: introduce the 2-cover, $(\mathcal{X}_2, \mathcal{F}_2) \to (\mathcal{X}', \mathcal{F})$ by extracting a square root of every component of $\mathcal{B}$, and run the minimal model programme for $(\mathcal{X}_2, \mathcal{F}_2)$. This results in a smooth foliated stack $(\mathcal{X}_2^{\#}, \mathcal{F}_2^{\#})$ with all of the enunciated properties, and in particular a smooth divisor $\mathcal{B}^!$ of the Gorenstien covering stack of the moduli, $\mathcal{X}^!$, everywhere transverse to $\mathcal{F}^!$. In particular if we restore the full-stack structure $(\mathcal{X}^{\#}, \tilde{\mathcal{F}}^{\#})$, say, associated with an arbitrary system of finite weights b_i then in terms of $\mathbb{Q}$ divisors on $\mathcal{X}^!$,

$$K_{\tilde{\mathcal{F}}^{\#}} = K_{\mathcal{F}_2^{\#}} + \sum_i \left(\frac{1}{2} - \frac{1}{b_i}\right) \mathcal{B}_i^! \,.$$

If however $\mathcal{L}$ is an invariant sub-stack representing a $K_{\tilde{\mathcal{F}}^{\#}}$-negative extremal ray, then $\mathcal{B}_i^! \cdot \mathcal{L} \geq 0$ by transversality, so in fact $K_{\mathcal{F}_2^{\#}} \cdot \mathcal{L} < 0$, and indeed finiteness of the weights is irrrelevant, from which we deduce,

IV.9.1 One more Fact. Suppose more generally that we consider log-varieties $(X, B, \mathcal{F})$ with log-canonical singularities, terminal at $\mathrm{sing}\,(X)$, with B (on the Gorenstien cover) smooth and everywhere transverse to $\mathcal{F}$, not to mention possibly infinite weights. Then a maximal sequence of flips and flops for the 2-cover (in the above sense) yields a model $(X^{\#}, B^{\#}, \mathcal{F}^{\#})$ with either,

(a) $K_{\tilde{\mathcal{F}}^{\#}}$ nef., and all the said properties.

(b) A rational foliation, necessarily as per IV.8.8, except in the case of the existence of a genuinely infinite weight which considered as an appropriate affine stack is an $\mathbb{A}^1$-bundle, with, at least after a correction, projective moduli.

IV.10. Arbitrary Remarks

One might reasonably ask whether the previous body of theory is sufficient for the construction of minimal models of foliated varieties irrespective of the initial model. In this generality the question is slightly ambiguous, since "Mori theory" is a priori non-sensical without (log)-canonical singularities. We can, however, consider the question starting from a foliated stack $(\mathcal{X}, \mathcal{F})$ with projective moduli, $\pi : \mathcal{X} \to X$, and, of course log-canonical singularities. Manifestly, by a minor variant of I.5 we can, without loss of generality, suppose that $(\mathcal{X}, \mathcal{F})$ is foliated Gorenstien. Furthermore by algorithmic resolution we can blow up in $\mathcal{F}$-invariant centres to obtain a smooth modification $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ such that the canonical bundle of the induced foliation is still $\rho^* K_{\mathcal{F}}$. Indeed if the singularities are canonical this is clear, while if they're only log-canonical I.6.12 tells us that sing $(\mathcal{F})$ is regularly embedded where this holds, so this case is okay too. As such provided we're prepared to blow up, we may further suppose that $\mathcal{X}$ is smooth. Now associated to such a $\mathcal{X}$, we have not only the Gorenstien covering stack $(\mathcal{X}_0^*, \mathcal{F}_0)$, of the moduli, but the smallest stack $\mathcal{X}_1$ which is smooth over the moduli. In order to get from $\mathcal{X}$ to $\mathcal{X}_1$, we write the former locally as a classifying stack $[V/G]$, quotient out by the generic stabiliser, then systematically kill pseudo-reflections, cf. [V]. Necessarily a pseudo-reflection fixes a divisor, which itself is either invariant or not. Consequently we obtain a non-invariant divisor $\mathcal{B}$ on $\mathcal{X}_1$, with components $\mathcal{B}_i$ and weights b_i such that,

$$K_{\mathcal{F}} = K_{\mathcal{F}_0} + (1 - 1/b_i)\, \mathcal{B}_i$$

where, for simplicity of notation, we omit the implied pull-backs, and of course, employ the summation convention. Now we can appeal to the pre-quel to I.7.3 to conclude that all of the triples $(\mathcal{X}_1, \mathcal{B}, \mathcal{F}_0)$, $(\mathcal{X}_0, (\pi_{01})_* \mathcal{B}, \mathcal{F}_0)$, $(X, B, \mathcal{F}_0)$ have log-canonical singularities, where $\pi_{01} : \mathcal{X}_1 \to \mathcal{X}_0$, and latin letters denote moduli. Now the role of $\mathcal{B}$ here is pretty trivial except possibly if we're in I.6.14(2), but by hypothesis in this case we can blow up in the tangency locus between $\mathcal{F}_0$ and $\mathcal{B}$, followed by a blow up in the intersection of the proper transform of $\mathcal{B}$ with the exceptional divisor to obtain a model which is terminal at points of the proper transform of $\mathcal{B}$ and at worst canonical at any other points we may have introduced. This operation will raise the canonical class by $1/2$ of the proper transform of the 1^{st} exceptional divisor, but if we're really fussy about not changing $K_{\mathcal{F}}$ then we can blow this down, so, regardless, we may suppose that $\mathcal{F}_0$ is terminal at $\mathcal{B}$, with the latter everywhere transfer. As such what is critical is the difference between $\mathcal{X}_1$ and $\mathcal{X}_0$. To this end let us momentarily introduce a smooth resolution $\rho : \tilde{\mathcal{X}}_0 \to \mathcal{X}_0$ of the Gorenstien covering stack, then on replacing $\mathcal{X}_1$ by the smallest smooth stack covering the moduli of $\tilde{\mathcal{X}}_0$ we can suppose that every point of $\mathcal{X}_1$ admits a neighbourhood of the form $[V/G]$, with V smooth, and G-abelian without generic fixed point or pseudo-reflections. Better still if ζ is a non-scheme like geometric point of $\mathcal{X}_1$, then there is a character $\chi : G \to \mathbb{C}^\times$, without kernel, and a generator ∂ of the foliation, such that $\partial^\sigma = \chi(\sigma)\partial$, for all $\sigma \in G$. Consequently we can even suppose that $\mathcal{X}_0 = \mathcal{X}_1$, and we distinguish,

IV.10.1 Cases.

(a) $\mathcal{F}_0$ is terminal at ζ. Certainly this can happen, but obviously we're pretty happy if it does since this has been a habitual hypothesis throughout the preceeding ruminations. Notice additionally that since our temporarily introduced $\tilde{\mathcal{X}}_0$ was obtained by invariant modification and $\mathcal{F}_0$ was supposed terminal at $\mathcal{B}$, then à la I.7.3, $\tilde{\mathcal{X}}_0$ and $\mathcal{X}_0$ coincided around $\mathcal{B}$, and everything around $\mathcal{B}$ is still terminal.

(b) $\mathcal{F}_0$ is smooth at ζ, but ζ is invariant. Actually this can't happen, since χ is a faithful representation of the local stabiliser. In particular there are no invariant, $\mathcal{F}_0$ smooth, non-scheme like points.

(c) $\mathcal{F}_0$ is singular at ζ. This can happen, but it's difficult. For example, this is impossible not just in dimension 2, but codimension 2. As such the first examples are in codimension 3, and indeed since the local stabiliser preserves Jordan decomposition, these are easily seen to be,

(1) $G = \mathbb{Z}/3$, $\partial = x\frac{\partial}{\partial x} + \zeta y\frac{\partial}{\partial y} + \zeta^2 z\frac{\partial}{\partial z}$, $\zeta^3 = 1$, and ζ the origin, with the action being permutation of the eigenfuctions.

(2) $G = \mathbb{Z}/2$, and for some functions a, b, c of two variables, vanishing at the origin,

$$\partial = x\left\{1 + a(z, xy)\right\}\frac{\partial}{\partial x} - y\left\{1 + b(z, xy)\right\}\frac{\partial}{\partial y} + c(z^2, xy)\frac{\partial}{\partial z}$$

where as ever ζ is the origin, G acts by permuting x and y, but sending z to $-z$, and, of course, $a(z, xy) = b(-z, xy)$.

Now, the notable difference between the first and the second example is that the first on blowing up separates the fixed points of G from sing $(\mathcal{F}_0)$, whereas in the second case this isn't an option. Specifically, much as in [O], even though we have a perfectly smooth Gorenstien covering stack, no matter how much we blow up, we can never achieve that the induced Gorenstien covering stack is everywhere scheme like. In a different guise this example was brought to my attention by Felipe Cano, [C2]. Plainly it is necessarily a "final form" in any resolution game, but, rather more immediately such an example doesn't fit our standard hypothesis of "terminal at the non-scheme points", so let's look at what a $K_{\mathcal{F}}$-negative invariant stack must look like if indeed we have non-scheme like points at the singularities, so back at $(\mathcal{X}, \mathcal{F})$, albeit without loss of generality the Gorenstien covering stack is smooth, and $\mathcal{X} \to \mathcal{X}_0$ is unramified around sing $(\mathcal{F})$. As such let $f : \mathcal{L} \to \mathcal{X}$ be the normalisation of a $K_{\mathcal{F}}$-negative 1-dimensional stack, which by IV.10.1(b) is generically scheme like, so that the adjunction formula of I.8.7 takes the form,

$$\begin{aligned} K_{\mathcal{F}} \cdot_{\mathcal{F}} \mathcal{L} = & -2 + \sum_i \left(1 - \frac{1}{n_i}\right) \\ & + \sum_{\ell \in f^{-1}(\operatorname{sing} \mathcal{F})} \left\{ \frac{\operatorname{ord}_l(\operatorname{sing} \mathcal{F}) - \operatorname{ord}_l(\operatorname{Ram}_f)}{\#\operatorname{stab}_{\mathcal{L}}(\ell)} + \left(1 - \frac{1}{\#\operatorname{stab}_{\mathcal{L}}(\ell)}\right)\right\} \end{aligned}$$

where the n_i are the orders of the stabilisers of any non-scheme like points not in sing $(\mathcal{F})$, and we profit from the fact that the moduli of $\mathcal{L}$ must be rational. Consequently negativity obliges us to have at most one point in $f^{-1}(\operatorname{sing} \mathcal{F})$, counted without multiplicity, and unless there are actually no singular points (and whence we have a foliation in conics, which we've dealt with in IV.8) at most one non-scheme like points. Let us view, therefore, $\mathcal{L}$ over $\mathbb{P}^1$ as a stack with stabilisers $\mathbb{Z}/m$, and $\mathbb{Z}/n$ over 0 and ∞, with the former the point of $f^{-1}(\operatorname{sing}\mathcal{F})$. Manifestly, we may without loss of generality suppose $m > 1$, although we allow the possibility that $n = 1$ to avoid complicating the notation. Now the easy case, occurs when $\operatorname{ord}_l(\operatorname{sing}\mathcal{F}) - \operatorname{ord}_l(\operatorname{Ram}_f)$ is 1 (with the order function understood on a scheme like étale neighbourhood) so that by II.6.2 and II.8.4 or a very minor variant of the same, the eigenvalues (or better their ratios) of any generator ∂ of the foliation around $f(0)$ are rational. On the other hand $\mathcal{X}_0$ is equal to $\mathcal{X}$ around $f(0)$, and is the Gorenstien covering stack, so there is a faithful character $\chi : \mathbb{Z}/m \to \mathbb{C}^\times$, and an appropriate ∂, such that $\partial^\sigma = \chi(\sigma)\,\partial$, for $\sigma \in \mathbb{Z}/m$. Whence, working even just mod $\mathfrak{m}^2_{\mathcal{X},f(0)}$, we observe that $\mathbb{Z}/m$ permutes the eigenvalues, and the ratios within a permutation are the values of χ. On the other hand roots of unity are inevitably irrational unless $m = 2$, and there is at least one non-zero eigenvalue, so in fact one for every value of χ, i.e. at worst $m = 2$. However, even in this case the tangent space of $\mathcal{L}$ at $f(0)$ is both $\mathbb{Z}/2$ invariant since $\mathcal{L}$ is, yet not so since the sign of the associated eigenvector changes, which is complete nonsense, so under these hypothesis there are no $K_{\mathcal{F}}$-negative 1-dimensional stacks through sing $(\mathcal{F})$.

The remaining alternative is a bit more delicate, so we'll begin with the easy case that f is an isomorphism, and introduce a local coordinate t on an étale neighbourhood of 0 together with a generator ∂ of the foliation, such that, $\partial t = t^{p+1}\,u(t) \bmod I_{\mathcal{L}}$, where $p \in \mathbb{N}$, and u is a unit, such that, without loss of generality $u(0) = 1$. Now, certainly the holonomy around 0 is no worse than $\mathbb{Z}/n$, and it's possibly less if $\mathcal{X} \to \mathcal{X}_0$ is ramified at ∞. Furthermore, we may suppose that around 0, $\mathbb{Z}/m$ acts on t by a possibly different character ψ, i.e. $t^\sigma = \psi(\sigma)\,t$, $\psi : \mathbb{Z}/m \to \mathbb{C}^\times$, $\sigma \in \mathbb{Z}/m$, which is necessarily faithful, and whence: $\chi(\sigma)\,\psi(\sigma)^p = 1$, and $u = u(t^m)$. Equally there is at least one non-zero eigenvalue/vector in the normal direction, which is necessarily permuted by Im (χ), so independently of any Gorenstien covering hypothesis, the cardinality of the holonomy is at least m/p, and at most n, so $K_{\mathcal{F}}.{}_{\mathcal{F}}\,\mathcal{L} = p/m - 1/n \geq 0$. Thus although equality can occur, even in dimension 2, we cannot obtain a $K_{\mathcal{F}}$ negative curve in this way. Better still even if the image of $\mathcal{L}$ had a cusp around $f(0)$, then for our ubiquitous local generator ∂, $f^*\partial$ is a vector field on $\mathcal{L}$, such that for any uniformising parameter t at 0, $\operatorname{ord}_0(f^*\partial t) > 1$, and this condition persits under blowing up in the singularity of the cusp, so we reduce to the smooth case, and, whence, a contradiction. Consequently although IV.10.1(c)2 shows that we cannot guarantee our condition of "terminal at the non-scheme like points" everywhere, even after arbitrary modification, we can guarantee it around $K_{\mathcal{F}}$ negative invariant 1-dimensional stacks, so by II.4.1 etc. a fortiori around extremal rays, from which we arrive to,

IV.10.2 Corollary. *Let $(\mathcal{X}, \mathcal{F})$ be a $\mathbb{Q}$-Gorenstien foliated normal stack with projective moduli and (log)-canonical singularities (equivalently $(X, B, \mathcal{F})$ is a log-triple such that $K_{\mathcal{F}} + \left(1 - \frac{1}{b_i}\right) B_i$ is $\mathbb{Q}$-Cartier, X is normal projective, and the foliation singularities are log-canonical) then there is a modification $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ with $K_{\tilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$ together with a sequence of flips and flops in the sense* *of* IV.1.10 *and* IV.5/7.4,

$$(\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) = (\mathcal{X}_0, \mathcal{F}_0) \dashrightarrow (\mathcal{X}_1, \mathcal{F}_1) \dashrightarrow \ldots \dashrightarrow (\mathcal{X}_n, \mathcal{F}_n) = (\mathcal{X}_{\min}, \mathcal{F}_{\min})$$

such that $(\mathcal{X}_n, \mathcal{F}_n)$ *is either one of the rational foliations discussed in* IV.8.8, *or* $K_{\mathcal{F}_{\min}}$ *is nef.*

Indeed we've done everything, expect the rational case, and out with this case we could even offer the precision that the Gorenstien cover of the moduli of each $(\mathcal{X}_i, \mathcal{F}_i)$ is smooth. The rational case needs no real comment since the very difficult implicit in IV.10.1(c)2 cannot occur.

IV.11 Uniqueness

Historically, there is an alternative use of the word minimal in bi-rational geometry to the effect of something like, X is minimal provided every map $\tilde{X} \dashrightarrow X$ in its birational equivalence class is everywhere defined, *cf.* [W]. Evidently such a variety really would be minimal, and indeed the only candidate for such a model is, in current terminology, a canonical model. For foliations by curves, however, something quite close to the extension property continues to hold even if the model is no better than minimal. To begin with we'll need a couple of clarifying remarks, and even a little terminology so as to speed things up, *e.g.*

IV.11.1 Terminology. Call the singularities of a foliated log-variety $(X, B, \mathcal{F})$ with finite weights of minimalist (as opposed to minimal) Mori category if they are canonical $\mathbb{Q}$-foliated Gorenstien, no boundary component is invariant, terminal at every point of $\mathrm{sing}\,(X) \cup |B| \backslash \mathrm{sing}(\mathcal{F})$, with smooth index 1 cover at every point of $\mathrm{sing}(\mathcal{F})$, and of course X is normal.

By virtue of I.7.4, this just amounts to the smoothness of the Gorenstien covering stack $\pi : (\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$, as well as that of the branched covering $\beta : (\mathcal{X}, \tilde{\mathcal{F}}) \to (\mathcal{X}', \mathcal{F})$ defined by B, in conjunction with the habitual properties of terminal at the non-scheme like points and both $(\mathcal{X}, \tilde{\mathcal{F}})$, $(\mathcal{X}', \mathcal{F})$ enjoying canonical singularities. The reasons for introducing the new terminology are two fold. In the first place, we'd like to emphasise that we have an acceptable class of singularities in which we can run the minimal model programme, and whence, although we certainly could, there's no real practical benefit to be achieved from working in greater generality. More importantly, however, there are technical issues involved in defining rational maps between stacks that we'd rather avoid, as such we make,

IV.11.2 Definition. A rational map $(X, B, \mathcal{F}) \dashrightarrow (Y, D, \mathcal{G})$ between foliated log-varieties of minimalist Mori category is a rational map, $\psi : X \dashrightarrow Y$ such that,

(a) The image of the natural composition,

$$\psi^* N_{Y/\mathcal{G}} \to \psi^* \Omega_Y \dashrightarrow \Omega_X \to K_{\mathcal{F}}$$

is zero, where, of course, $N_{Y/\mathcal{G}} \hookrightarrow \Omega_Y$ is the co-normal sheaf of $\mathcal{G}$.

(b) For every prime Weil divisor F on X such that $\psi_* F \subset |D|$ either, F is invariant or its support is contained in that of B. In the latter case we further suppose that $\psi_* F$ is in fact a Weil divisor on Y, and the weight b of B on F

is related to the same, d of D on $\psi_* F$, by $d \mid b\nu$ where ν is the multiplicity of $\psi^* \psi_* F$ around F.

A modification $\rho : (X^{\#}, B^{\#}, \mathcal{F}^{\#}) \to (X, B, \mathcal{F})$ is of course an honest map with a rational inverse, which we a priori allow to have more singularities. Somewhat less trivially it is invariant if the log-canonical bundles coincide. With this out of the way we can now state,

IV.11.3 Lemma. *A projective log-canonical variety $(Y, D, \mathcal{G})$ of minimalist Mori category is a minimal model iff for every dominant rational map $\psi : (X, B, \mathcal{F})$- - $\to (Y, D, \mathcal{G})$ from a minimalist Mori category triple there is an invariant resolution,* i.e. *a diagram,*

$$\begin{tikzcd}
(X^{\#}, B^{\#}, \mathcal{F}^{\#}) \arrow[d, "\rho"] \arrow[dr, "\tilde{\psi}"] & \\
(X, B, \mathcal{F}) \arrow[r, dashed, "\psi"'] & (Y, D, \mathcal{G})
\end{tikzcd}$$

with ρ a proper invariant modification.

Notice that in the course of running the minimal model programme we've already proved that the condition is necessary. Specifically if $(Y, D, \mathcal{G})$ weren't minimal, then we could take ψ to be an anti-contraction or an anti-flip to obtain a contradiction. In the opposite direction we, somewhat inevitably, re-introduce our friends the Gorenstien covering stack $\pi : (\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$, and the branched covering $\beta : (\mathcal{X}, \tilde{\mathcal{F}}) \to (\mathcal{X}', \mathcal{F})$. Thanks to algorithmic resolution we can resolve the composite $\psi \circ \pi \circ \beta$ by a sequence, $\rho_{ij} : (\mathcal{X}_j, \tilde{\mathcal{F}}_j) \to (\mathcal{X}_i, \tilde{\mathcal{F}}_i)$ of blow ups in smooth centres $\mathcal{Z}_i$ say defining $\rho_{i+1,i}$ in such a way that the restriction of our rational map to $(\mathcal{X}_i, \tilde{\mathcal{F}}_i)$ is undefined at $\mathcal{Z}_i$. Now if every $\mathcal{Z}_i$ were invariant we'd be done, so let $\mathcal{Z}_n$ denote the last centre in our chain, if it exists, which is generically transverse to the induced foliation $\tilde{\mathcal{F}}_n$. By direct calculation we observe that the exceptional divisor $\mathcal{E}_{n+1}$ over $\mathcal{Z}_n$ is generically a bundle of weighted projective stacks. Better still $\mathcal{E}_{n+1}$ is invariant, and the induced foliation on the same is in fact a bundle of linear foliations on the said weighted projective stacks. Furthermore, by hypothesis, all subsequent blow ups are invariant so if $\mathcal{E}$ is the proper transform of $\mathcal{E}_{n+1}$ in our ultimate modification, $(\mathcal{X}_N, \tilde{\mathcal{F}}_N)$, which we'll denote $(\mathcal{X}^{\#}, \tilde{\mathcal{F}}^{\#})$, with E, $(X^{\#}, B^{\#}, \mathcal{F}^{\#})$ etc. the moduli, all of these salient features continue to be preserved, so, for example, E is covered by invariant rational curves in such a way that if C_t denotes a generic such, then $K_{\tilde{\mathcal{F}}^{\#}} \cdot C_t < 0$. Observe in addition that even though $(X^{\#}, B^{\#}, \mathcal{F}^{\#})$ may not be in our minimalist Mori category, every divisor contracted by ρ is invariant (by an obvious variation of the above discussion) so unambiguously $B^{\#}$ is the proper transform of B with the same weights b_i associated to the components $B_i^{\#}$ and, of course, $K_{\tilde{\mathcal{F}}^{\#}} = K_{\mathcal{F}^{\#}} + \sum_i (1 - 1/b_i)\, B_i$.

Manifestly, our next task is to relate $K_{\tilde{\mathcal{F}}^{\#}}$ to $K_{\tilde{\mathcal{G}}} := K_{\mathcal{G}} + \sum_i (1 - 1/d_i)\, D_i$, where d_i are the weights of our boundary on Y. To this end, we assert,

IV.11.4 Claim. *For some effective divisor F on $X^{\#}$, $K_{\tilde{\mathcal{F}}^{\#}} = \tilde{\psi}^* K_{\tilde{\mathcal{G}}} + F$.*

Proof. To begin with, note that IV.11.2(a) implies that we have a map,

$$\tilde{\psi}^*(K_{\mathcal{G}} \cdot \mathcal{I}_{\mathrm{sing}(\tilde{\mathcal{G}})}) \to K_{\mathcal{F}\#} \cdot \mathcal{I}_{\mathrm{sing}(\mathcal{F}\#)}$$

or more correctly the same but for some sheaves $\mathcal{K}_{\mathcal{G}}$, $\mathcal{K}_{\mathcal{F}\#}$ whose double duals are the respective canonical classes. Regardless, the map is certainly non-trivial since ψ is dominant, so at worst we find effective $\mathbb{Q}$-Cartier divisors F_+, F_- such that, $K_{\tilde{\mathcal{F}}\#} = \tilde{\psi}^* K_{\tilde{\mathcal{G}}} + F_+ - F_-$. Now quite generally, *cf.* [K1], every irreducible divisor F on $X^\#$ either dominates Y or defines a discrete prime rank 1-valuation v, say, which can be resolved in the standard way when Y is smooth, *i.e.* V has a centre W_0 on Y, blow up Y in W_0, call the new centre W_1 on $\mathrm{Bl}_{W_0}(Y) := Y_1$, and continue until we obtain a divisor, say F_0, all the while taking only an interest in the part of Y around the smooth locus of W_i, whence we obtain some normal modification $\tilde{Y} \to Y$ together with a map of germs,

$$\tilde{\psi}^* : \mathcal{O}_{\tilde{Y},F_0} \to \mathcal{O}_{X^\#,F} \, .$$

Evidently, in this situation, the canonical class around F is the canonical class around F_0 plus some effective divisor, and since the singularities of $(Y, D, \mathcal{F})$ are supposed canonical, we obtain,

IV.11.5 Sub-Claim. *Notations as above, if F_- exists and F is an irreduclible component of its support then F defines a rank 1 discrete valuation with centre contained in* $\mathrm{sing}\,(Y)$.

Indeed, outside of $\mathrm{sing}\,(\mathcal{G}) \cup \mathrm{sing}\,(Y)$ we have an honest map between canonical classes, so certainly $K_{\mathcal{F}\#} = \tilde{\psi}^* K_{\mathcal{G}} + F'_+ - F'_-$ where at worst F'_- defines a valuation with centre in $\mathrm{sing}\,(Y)$. If we subsequently throw in the boundary then the only further candidate for a divisor supported in F_- must be invariant on $X^\#$, and contract to a centre of codimension at least 2 on Y, since by hypothesis D is transverse. In such a scenario, however, the corresponding divisor F_0 would be invariant by I.6.6, so this cannot pose a problem either. The outstanding possibility that some F in the support of F_- could map to $\mathrm{sing}\,(Y)$ equally follows for the same reason on noting that in the resolution used to obtain F_0 smoothness of the ambient space around the centre is irrelevant provided that we're prepared to blow up in Weil divisors, and of course the valuation ring $\mathcal{O}_{\tilde{Y},F_0}$ must have invariant maximal ideal by I.6.7 and IV.11.1. □

Applying this to the particular situation where we further suppose $K_{\tilde{\mathcal{G}}}$ nef., we necessarily obtain for C_t a generic invariant curve in what we previously denoted E, $0 > \tilde{\psi}^* K_{\tilde{\mathcal{G}}} \cdot C_t + F \cdot C_t \geq F \cdot C_t$, so indeed E is in the support of F. Better still E cannot dominate Y, since otherwise the branched covering stack $(\mathcal{Y}, \tilde{\mathcal{G}})$ would be a foliation in rational stacks with nef. canonical bundle which is impossible by IV.8.1. As such E is certainly in some sense contracted by $\tilde{\psi}$, and what must be proved is that this doesn't happen in a direction opposite to that of ρ. Fortunately the direction in question is that of the foliation, so that unsurprisingly, we have,

IV.11.6 Further Claim. *The generic invariant rational curve, C_t, in E, maps to a point under $\tilde{\psi}$.*

Proof. Suppose the opposite, then $\tilde{\psi}(E)$ is invariant, so in particular it's generic point is not in $\mathrm{sing}\,(Y)$. Better still from the point of view of the resolution process of the corresponding discrete valuation as found in the proof of IV.11.4, this is in fact a sequence of blow ups in invariant centres. Consequently we obtain an invariant modification $\tilde{Y} \to Y$ with a map of germs, $\tilde{\psi}^* : \mathcal{O}_{\tilde{Y},E_0} \to \mathcal{O}_{X^\#,E}$ together with a map of canonical classes $\tilde{\psi}^* K_{\mathcal{G}} \to K_{\mathcal{F}^\#}$, which are, equally the log-canonical classes around the germ since everything is invariant. Furthermore, by hypothesis, this map vanishes on E, which leads to the absurdity that the C_t map to points. $\square$

Now at this point we're done, since the rigidity lemma, [K1], applies to show that when we were resolving ψ there was never any need to modify on account of what was actually defined in a neighbourhood of the last transverse centre in which we blew up, which is, of course, contrary to hypothesis. Consequently let's amuse ourselves by reviewing some special known cases of the lemma,

IV.11.7 Example 1. Let Y be a curve of positive genus, then every rational map $X - - \to Y$ from a smooth variety is, in fact, a map.

Proof. We consider Y as a foliation by curves whoses only leaf is itself, so the condition of a relative foliated mapping $(X, \mathcal{F}) - - \to Y$ is empty beyond the existence of the map itself. As such just choose an appropriately large number of foliations $\mathcal{F}_i$ on X, so that at least one of them is smooth and generically transverse to any points where $X - - \to Y$ is supposed not to exist. $\square$

IV.11.8 Example 2. The moduli stack $\mathcal{M}_g$ of curves of genus g is proper.

Proof. By definition this amounts to checking that if $X, Y \to \operatorname{Spec} R$ are families of stable curves of genus $g \geq 1$ over a DVR R with quotient field K which are generically isomorphic, then indeed they are isomorphic. Plainly we view the said families as foliations $\mathcal{F}$, $\mathcal{G}$ with relative canonical bundles $K_{\mathcal{F}}$, $K_{\mathcal{G}}$ respectively, and apply the lemma to obtain a diagram,

$$\begin{tikzcd} & (W, \mathcal{H}) \arrow[dl, "\theta"'] \arrow[dr, "\psi"] & \\ (X, \mathcal{F}) \arrow[rr, dashed, leftrightarrow] & & (Y, \mathcal{G}) \end{tikzcd}$$

with everything being Gorenstien, and $K_{\mathcal{H}} = \theta^* K_{\mathcal{F}} = \psi^* K_{\mathcal{G}}$. Now take a rational curve C in the special fibre blown down by θ, then $K_{\mathcal{G}} \cdot \psi_* C = 0$. On the other hand, the very definition of stable amounts to $K_{\mathcal{G}} \cdot E > 0$ for every rational curve in the singular fibre, so C must be blown down by ψ too. $\square$

IV.11.9 Remark. The "uniqueness lemma" IV.11.3 is rather close in spirit to the idea of the no bubbling lemma (1) of [Br4], or more correctly when the obstructions to the same in [CI] don't occur. Nevertheless they are very different. Specifically IV.11.3 is not valid in a non-algebraic setting, even for germs around DVR's of the function field which are non-prime. The no-bubbling lemma should, however, be local and analytic, with its validity arising from more restrictive hypothesis.

APPENDIX: FOLIATIONS AND GROUPOIDS

A.1 The Holonomy Groupoid

We wish to associate to a foliated stack $(\mathcal{X}, \mathcal{F})$ a representable groupoid in analytic stacks. As such the word analytic stack must be understood for the analytic étale topology, so that in particular étale maps are allowed to be of infinite degree. We'll restrict ourselves to the degree of generality impicit in IV.10.2, and in the traditional setting of working only over the smooth local $\mathcal{U}$ of the foliation. As such if we begin with the case where $\pi : (\mathcal{X}, \mathcal{F}) \to (X, \mathcal{F})$, is the Gorenstien covering stack with $\mathcal{X}$ smooth, then $\mathcal{F}$ is terminal at the non-scheme like points of $\mathcal{U}$ whose moduli we denote U.

To begin with we need to find a convergent version of the smooth infinitesimal groupoid $\mathfrak{F} \rightrightarrows \mathcal{U}$ of II.1.2. Technically it's more convenient to do this at the level of the moduli. Away from $\mathrm{sing}\,(U)$ there's no problem making a convergent neighbourhood, since the Frobenius theorem holds convergently. Equally, by hypothesis, around a complex point x of $\mathrm{sing}\,(U)$ we can find an étale polydisc Δ of x considered as a complex point of x, together with a smooth fibration $\Delta \to \Delta'$, compatible with the local monodromy, and whose fibres are the local leaves. Taking the moduli of the same we get a fibration, in possibly singular, analytic spaces $D \to D'$, and $D \times_{D'} D$ glues to our previous construction around the smooth points of both U and $\mathcal{F}$. Thus while not having done much, let us note this under,

A.1.1 Fact. *There is an analytic space F around the diagonal of U in $U \times U$, such that $F \rightrightarrows U$ extends convergently the moduli of the smooth infinitesimal groupoid $\mathfrak{F} \rightrightarrows \mathcal{U}$.*

We next wish to make F as big as possible, so we let $\mathbf{F}_{\mathrm{big}} \to U \times U$ be a domain of holomorphy for $F \to U \times U$. Such a space may in fact be rather too big, but it will usefully provide a vechile over which we'll be able to identify an appropriate open set, which in turn admits the right kind of stack structure. To this end we adopt the standard convention,

A.1.2 Definition. A leaf $\mathcal{L}$ of $(\mathcal{X}, \mathcal{F})$ is a maximal connected 1-dimensional invariant analytic substack of $\mathcal{U}$.

This definition certainly skates over several issues about singular points, but we'll leave remarking on these till later. Regardless, if we further suppose that $(X, \mathcal{F})$ is not a foliation in conics, which in any case we've dealt with in IV.8.8, then $\mathcal{L}$ is not a so called bad orbifold. Better still for $x \in \mathcal{L}$ non-scheme like we've already seen, I.5.7, that $\mathrm{Aut}(x)$ is naturally the local holonomy group. Consequently the holonomy covering $\tilde{L} \to \mathcal{L}$ is scheme like, and we denote by $\Gamma_{\mathcal{L}}$ the corresponding covering group. In particular, and essentially by definition, there is a representation $\rho_{\mathcal{L}} : \Gamma_{\mathcal{L}} \to \mathrm{Aut}(\Delta^n)$ of germs of convergent automorphisms fixing the origin of a complex polydisc of dimension $n := \dim \mathcal{X} - 1$, such that in the obvious notation, the foliation around $\mathcal{L}$ is the map of classifying stacks,

$$[\tilde{L} \times \Delta^n / \Gamma_{\mathcal{L}}] \to [\Delta^n / \Gamma_{\mathcal{L}}]$$

where the former action is the diagonal one. With this notation we assert,

A.1.3 Claim. *There is a natural open map,*

$$[\tilde{L} \times \Delta^n / \Gamma_{\mathcal{L}}] \times_{[\Delta^n / \Gamma_{\mathcal{L}}]} [\tilde{L} \times \Delta^n / \Gamma_{\mathcal{L}}] \to \mathbf{F}_{\mathrm{big}}\,.$$

Proof. Evidently, $[\Delta^n/\Gamma_{\mathcal{L}}]$ may have no moduli, although the classifying stack $[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}]$ does. Indeed it is a small open (even around the points of $\operatorname{sing}(U)$) neighbourhood Λ of the moduli L of $\mathcal{L}$ (which certainly does exist since the stabilisers are finite). Furthermore depending on whether we wish to emphasise the 1^{st} or 2^{nd} factors, say the first p_1 which is the usual for a leaf's source, we may identify the fibre product with more obvious things such as,

$$[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times \tilde{L}$$

with maps p_1, p_2 to U, where on the first factor the classifying stack goes to its moduli Λ, and p_2 is the inclusion of a leaf into $\mathcal{U}$ followed by projection to its moduli. If furthermore we choose a base complex geometric point $x \in \mathcal{L}$, and identify it with some $\tilde{x}$ on $\tilde{L}$, then locally around $\tilde{x} \times \tilde{x}$ the moduli of our fibre product is what we previously called $D \times_{D'} D$, with of course, compatible projections, which itself is an open subset of the moduli of $[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times \tilde{L}$. However, by definition, $\mathbf{F}_{\text{big}}$ is a domain of holomorphy, so the assertion follows. $\Box$

Unsurprisingly then, we replace $\mathbf{F}_{\text{big}}$ by the open set obtained by taking the union over all leaves of the images of the maps appearing in A.1.3, and call this $\mathbf{F}$. Better still the $[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}]$'s are open substacks of $\mathcal{X}$ and we have some patching data between them by way of the fibre product,

$$\begin{array}{ccc} [\tilde{M} \times \Delta^n/\Gamma_{\mathcal{M}}] \times_{\mathcal{X}} [\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] & \longrightarrow & [\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \\ \downarrow & & \downarrow \\ [\tilde{M} \times \Delta^n/\Gamma_{\mathcal{M}}] & \longrightarrow & \mathcal{X} \end{array}$$

where $\mathcal{M}$ is another leaf, the notation is the obvious one, and all the maps in question are both open and representable. Equally, and essentially by hypothesis, the said patching is compatible with the foliation, so we obtain an analytic stack, which rather abusively we'll denote $\mathcal{F}_{\text{hol}}$, with moduli $\mathbf{F}$ and containing as open substacks, the various fibre products,

$$[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times_{[\Delta^n/\Gamma_{\mathcal{L}}]} [\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \,.$$

In particular, not only does $\mathcal{F}_{\text{hol}}$ map to $U \times U$, but it maps to $\mathcal{U} \times \mathcal{U}$, and we obtain a groupoid $\mathcal{F}_{\text{hol}} \rightrightarrows \mathcal{U}$ in analytic stacks. Now the crucial technical point that avoids all of this degenerating into a discussion about how to define a 3-category is,

A.1.4 Fact. *The groupoid $\mathcal{F}_{\text{hol}} \rightrightarrows \mathcal{U}$ constructed above is representable.*

Proof. The question is certainly local, and to simplify matters, the various fibre products $[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times_{[\Delta^n/\Gamma_{\mathcal{L}}]} [\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}]$ are even open, so this just amounts to using various identities like,

$$[\tilde{L} \times \Delta^n/\Gamma_{\mathcal{L}}] \times_{[\Delta^n/\Gamma_{\mathcal{L}}]} \tilde{L} \times \Delta^n = [\tilde{L}_1 \times \tilde{L}_2 \times \Delta^n \times \Gamma_{\mathcal{L}}/\Gamma_{\mathcal{L}}]$$

where the ultimate action is $(x_1 \times x_2 \times z \times \gamma)^g = x_1 \times x_2^g \times z^g \times \gamma g^{-1}$, so the classifying stack in question is the wholly space like $\tilde{L}_1 \times \tilde{L}_2 \times \Delta^n$. $\Box$

The upshot of which is, of course,

A.1.5 Fact/Definition. *Suppose our foliation is not in rational curves, then there is a well defined classifying stack $[\mathcal{U}/\mathcal{F}_{\mathrm{hol}}]$ in the 2-category of smooth stacks in analytic spaces. Specifically it's the classifying stack deduced by étale "base change" $\coprod_{\alpha} U_\alpha \to \mathcal{U}$ for any scheme like covering U_α of the source of the smooth holonomy groupoid in analytic stacks $\mathcal{F}_{\mathrm{hol}} \rightrightarrows \mathcal{U}$. More precisely the inverted commas are to be understood in the sense of [KM] 2.6.*

Having completed the Gorenstien case, we may now, slightly more generally, consider a branched covering $\sigma : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ of the same, with say weighted branching divisor B in X, not containing any singularities of $\mathcal{F}$, or better the pull-back $\mathcal{B}$ of the said divisor to $\mathcal{X}$. Under these hypothesis the pull-backs $s^*\mathcal{B}$, $t^*\mathcal{B}$ by the source and sink of $\mathcal{F}_{\mathrm{hol}} \rightrightarrows \mathcal{U}$ fit together to a simple normal crossing divisor $s^*\mathcal{B} + t^*\mathcal{B}$, and indeed there is only ever at most 2-components through any geometric point. Consequently we can take an appropriate, *i.e.* dictated by the weights of $\mathcal{B}$, branched cover $\tilde{\mathcal{F}}_{\mathrm{hol}} \rightrightarrows \tilde{\mathcal{U}}$, where $\tilde{\mathcal{U}}$ is the smooth locus in $\tilde{\mathcal{X}}$, which we may equally note by way of,

A.1.6 Fact/Definition bis. *For $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ as above, we continue to have a representable groupoid $\tilde{\mathcal{F}}_{\mathrm{hol}} \rightrightarrows \tilde{\mathcal{U}}$ in analytic stacks, and as such a well defined classifying stack $[\tilde{\mathcal{U}}/\tilde{\mathcal{F}}_{\mathrm{hol}}]$ in the 2-category of smooth stacks in analytic spaces.*

This exercise in making sure that we haven't accidentally ran into an insurmountable problem in set theory out of the way, let's finish off in making,

A.1.7 Remarks/Caveats. 1) The entire content of the above simply amounts to verifying that the naive equivalence relation which identifies points belonging to the same leaf has sense. Manifestly the holonomy groupoid is the smallest object which captures this notion, *i.e.* any smooth stack in analytic spaces whose complex points are identified should they belong to the same leaf must factor through $[\mathcal{U}/\mathcal{F}_{\mathrm{hol}}]$, or more generally $[\tilde{\mathcal{U}}, \tilde{\mathcal{F}}_{\mathrm{hol}}]$.

2) In the smooth category, the holonomy groupoid may fail to be separated, *cf.* [Co]. Fortunately this is prevented by the unicity of analytic continuation.

3) There is something of an Ockham's razor in this definition, since if it were applied to the foliation associated to the fibration $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ of stable n-pointed curves of genus g over those with $(n-1)$ points, we won't get back $\mathcal{M}_{g,n-1}$ but rather some non-separated covering of it. The way to avoid this is, obviously, to profit from the whole of the domain of holonomy $\mathbf{F}_{\mathrm{big}}$.

The most pertinent of the above is (3). The reason why a slightly better groupoid profiting from $\mathbf{F}_{\mathrm{big}}$ hasn't been pursued is that outwith singularities which are in essence locally exactly as those of the relative singularities for $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ the situation doesn't improve that much. Unfortunately such singularities are highly particular. A more general completion across the singularities is provided by looking for an analytic extension of the infinitesimal birational groupoid $\tilde{\mathfrak{F}} \rightrightarrows \mathcal{X}$ of II.1.

To initially extend this to an analytic germ is, potentially, exactly the kind of area where stacks and spaces differ so we spell it out. To begin with, consider the situation at the level of the moduli, say, $\hat{F} \rightrightarrows X$. The convergent Frobenius theorem applied on étale neighbourhoods of $\tilde{\mathfrak{F}}$ followed by the taking of invariants shows that every point of $\hat{F}$ posseses an analytic extension, and these in turn, by the unicity of analytic continuation, glue to some analytic space $\tilde{F}$. Now go back up to a point $\xi \in \tilde{\mathfrak{F}}$ with $\hat{U}_\xi$ an etale neighbourhood of the same, such that $\tilde{\mathfrak{F}}$ is locally a formal classifying stack of the form $[\hat{U}_\xi/G_\xi]$ for some finite group G_ξ. Again the Frobenius theorem gives us an analytic extension U_ξ of $\hat{U}_\xi$, and we can consider the groupoid,

$$\coprod_{\xi,\eta} \widetilde{U_\xi \times_F U_\eta} \rightrightarrows \coprod_{\xi} U_\xi$$

where ˜ denotes normalisation. On completing in the proper transform $\tilde{\Delta}$ of the diagonal the classifying stack of this groupoid is just $\tilde{\mathfrak{F}}$, so indeed we have our desired analytic extension $\tilde{\mathfrak{F}}_{an}$ (or just $\tilde{\mathfrak{F}}$ if there is no danger of confusion), which a postiori can be locally expressed as $[U_\xi/G_\xi]$ for possibly smaller U_ξ.

Now plainly, this isn't going to lead to a groupoid with space of objects $\mathcal{X}$, since the identity is only bi-rational. On the other hand we have two honestly smooth foliations on $\tilde{\mathfrak{F}}_{an}$ corresponding to the source and sink directions. Projecting along these directions onto $\tilde{\Delta}$, therefore gives maps s, t from $\tilde{\mathfrak{F}}_{an}$ to the blow up, denoted slightly confusingly by $\tilde{\mathcal{X}}$, of $\mathcal{X}$ in the singularities. Taking domains of holomorphy, and throwing in ramification à la A.1.6 for good measure, we therefore obtain,
A.1.8 Fact *This time for* $(\tilde{\mathcal{X}}, \tilde{\mathcal{F}})$ *the induced (and actually posibly not saturated if the singularities aren't canonical) induced foliation on the blow up of a stack* $(\mathcal{X}, \mathcal{F})$ *foliated by curves, there is a smooth groupoid* $\tilde{\mathcal{F}}_{\mathrm{hol}} \rightrightarrows \tilde{\mathcal{X}}$ *completing either of the groupoids* A.1.5/6 *across the singularities.*

Needless to say this leads to,
A.1.9 Further Remarks and Caveats The groupoid so constructed is smooth, but it plainly, unless the foliation itself is smooth, cannot be, even infinitesimally, a smooth equivalence relation. Infact, even in the case of $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ it's still non-separated. Indeed quite generally it has the properties,

1) It is a smooth equivalence relation where the induced foliation on $\mathcal{X}$ is smooth.
2) Any point outwith this locus is equivalent to no other.
3) Consequently, such points have rather large stabalisers.

The stabalisers are nevertheless algebraic group schemes, and it's ultimately the smoothness of this groupoid that lies at the heart of the proof of the cone theorem, and the subsequent minimal model theory.

A.2 The Homotopy Groupoid

What we'd like to do now is to pass from the holonomy groupoid to a rather more simple object in which each leaf is replaced by its universal cover. The possibility that the universal cover is compact for even one leaf is completely dealt with by way of IV.8.8. Consequently we'll suppose that $(\mathcal{X}, \mathcal{F})$ is a minimal model of a foliated stack with log-canonical singularities terminal at the non-scheme like points outwith $\mathrm{sing}(\mathcal{F})$, with of course $\mathcal{X}$ smooth, $(X, \mathcal{F}_0)$ its moduli, and $\pi : (\mathcal{X}_0, \mathcal{F}_0) \to (X, \mathcal{F}_0)$ the Gorenstien cover of the same. To begin with let us consider the arrows of the holonomy groupoid of A.1.5 fibring over $\mathcal{X}$ by way of the source s, or more correctly its composite with the obvious inclusion. Quite generally when presented with such a smooth fibring $s : \mathcal{F}_{\mathrm{hol}} \to \mathcal{X}$, there is no difficulty in constructing an analytic stack whose fibres over $\mathcal{X}$ are the universal covers of the fibres of s, what, however, may be less than pleasant is that the stack so constructed is non-separated. As such if we replace the fibres of the sink by their universal covers then a priori we only get,

A.2.1 Fact/Definition. *Let $(\mathcal{X}, \mathcal{F})$ be as above then there is a representable (by a minor variant of A.1.5) smooth groupoid, the homotopy groupoid, $\mathcal{F}_{\mathrm{hom}} \rightrightarrows \mathcal{U} = \mathcal{X}\backslash \mathrm{sing}(\mathcal{F})$ whose fibres be it under the source or sink are isomorphic either to a disc or a complex line. In particular there is a well defined classifying stack, $[\mathcal{U}/\mathcal{F}_{\mathrm{hom}}]$ in the 2-category of smooth analytic stacks. Nevertheless it may occur that the space $\mathcal{F}_{\mathrm{hom}}$ is non-separated.*

If, however, we're on a minimal model, then morally the space $\mathcal{F}_{\mathrm{hom}}$ should be separated, cf. [Br4], Lemma 1, but also [CI]. This difficulty notwithstanding one expects the homotopy groupoid to be the better objecct. For example, unlike the holonomy groupoid it has good deformation invariance properties, as seen by,

A.2.2 Example. ("Non-Commutative Algebraic Tori") Consider the Kronecker type foliation on $\mathbb{G}_m^2$ with standard coordinates x, y given by $x\,\frac{\partial}{\partial x} + \lambda y\,\frac{\partial}{\partial y}$, $\lambda \in \mathbb{G}_m$. For $\lambda \notin \mathbb{Q}$ all the leaves are isomorphic to $\mathbb{C}$ so the homotopy and holonomy groupoids coincide, and have classifying stack isomorphic to $[\mathbb{G}_m/q^{\mathbb{Z}}]$ where $q = \exp(2\pi i\lambda)$, which is even an honest scheme, in fact an elliptic curve, for $\lambda \notin \mathbb{R}$. Alternatively for $\lambda \in \mathbb{Q}$, the leaves are $\mathbb{G}_m$'s and the classifying stack of the holonomy groupoid is $\mathbb{G}_m$ itself, which does not deform to an elliptic curve. In this respect the homotopy groupoid is better since its classifying stack is $[\mathbb{G}_m/\mathbb{Z}]$ with $\mathbb{Z}$ acting trivially, and this is indeed an honest deformation limit of $[\mathbb{G}_m/q^{\mathbb{Z}}]$ for $q \to 1$. The example becomes more amusing still if we compactify to $\mathbb{P}^1 \times \mathbb{P}^1$, where the induced foliation has canonical singularities for $\lambda \notin \mathbb{Q}$, and log-canonical singularities otherwise. In the former case the homotopy and holonomy groupoids continue to coincide with classifying stack $[\mathbb{P}^1/q^{\mathbb{Z}}]$, while in the latter case if $|\lambda| = p/q$ for (p, q) relatively prime, the classifying stack of the holonomy groupoid is the generalised weighted projective stack $\boldsymbol{\mathcal{P}}^1(p, q)$, which again fails to be a deformation limit of the irrational case, unlike its homotopy groupoid which is $[\mathbb{P}^1/\mathbb{Z}]$ for $\mathbb{Z}$ acting by way of,

$$\mathbb{Z} \to \mathrm{Aut}(\mathbb{P}^1) : [S, T]^n \mapsto [\zeta^{pn} S, \zeta^{qn} T]$$

in appropriate standard coordinates with ζ a $(pq)^{\mathrm{th}}$-root of unity. The classifying stacks of the homotopy groupoids in the compactified case are, therefore, essentially "non-commutative projective curves" in the sense of [AZ], while the $\mathbb{G}_m$-case is simply a variant on the "non-commutative real torus" of [Co]. As to why, however, one would wish to render a perfectly good étale classifying stack non-commutative is another matter. After all for H^* any reasonable cohomology theory, $H^*([\mathbb{P}^1/\mathbb{Z}])$ can be computed from the Hoschild-Serre spectral sequence, while it's equally geometrically false, albeit algebraically true, that the algebra $\mathbb{C}\,[x, y]/xy = qyx$ is a deformation of the homogeneous coordinate ring of $\mathbb{P}^1$. The correct geometric statement is that the trivial action deforms. Indeed since $\mathbb{P}^1$ doesn't deform, the deformation space of $[\mathbb{P}^1/\mathbb{Z}]$ under the trivial action, is $\mathrm{Aut}(\mathbb{P}^1)$.

A.3 Categorical Integration

We require to define a category of topological spaces in which integration has sense. As such we apologise for taking a rather minimalist approach to this whereby something sufficiently general rather than maximally general will be defined, a*i.e.*

A.3.1 Definition *Denote by $\underline{E}$QF(essentially quasi-finite) the category whose objects are separated topological spaces X whose topology satisfies,*

(a) There is a basis of opens, each one of which is contained in a compact set.

(b) The space itself is a countable union of compact sets.

The mappings between two objects are the continuous maps with discrete fibres.

In particular it follows that any map in $\underline{E}$*QF* has fibres which are at worst countable. Better still, there is a functor, $X \mapsto \Gamma_c(X)$ obtained by sending a space to its bounded Borel functions with compact support, and an arrow $f : X \to Y$ to,

$$f_* : \Gamma_c(X) \to \Gamma_c(Y) : F = F(x) \mapsto (f_*F)(y) := \sum_{x \in f^{-1}(y)} F(x)$$

As such $\Gamma_c(X)$ must be viewed as taking values in a sub-category $\underline{T}$ of possibly non-separated topological vector spaces which admits direct limits, so, definitely not Banach spaces. We give $\Gamma_c(X)$ the topology of the direct limit, so that f_* is continuous. In this respect one observes,

A.3.2 Remark/Warning If we denote by $\Gamma(X)$ the space of bounded Borel functions, then we can view this as either a Banach space or in the topology of compact convergence. The former is finer than the latter, and induces a topology on $\Gamma_c(X)$ by way of the natural inclusion. Nevertheless the $\Gamma_c(X)$ topology is finer still, unless X is compact. Specifically $f_n \to f$ in $\Gamma_c(X)$ iff there exists a compact set K such that f_n is identically f off K, while on K, $f_n|_K \to f|_K$ in the Banach space sense for $\Gamma(K)$.

Associated to $\underline{E}$*QF* there is, of course, its canonical topology which is probably finer than,

A.3.3 Definition *A quasi-canonical (qc) morphism in $\underline{E}$QFis an $\underline{E}$QFuniversally effective epimorphisms, which is universally open. The big site defined by the class of quasi-canonical maps will be written($\underline{E}$QF)$_{qc}$, and for X an object, $(X)_{qc}$ will be the small site arising from qc maps to open subsets of X.*

The relevance of the quasi-canonical topology to measure theory is provided by,

A.3.4 Fact *Let $f : X \to S$, and $g : Y \to S$ be open and surjective maps in $\underline{E}QF$ then the square,*

$$\begin{array}{ccc} \Gamma_c(X \times_S Y) & \longrightarrow & \Gamma_c(Y) \\ \Big\downarrow & & \Big\downarrow g_* \\ \Gamma_c(X) & \xrightarrow[f_*]{} & \Gamma_c(S) \end{array}$$

is Cartesian on the image of Γ_c. In particular in $\underline{T}$, there is an exact sequence,

$$\Gamma_c(X \times_S X) \rightrightarrows \Gamma_c(X) \to \Gamma_c(S) \to 0$$

proof Certainly the image of a compact set is compact, and since f is open, along with the precisions (a) & (b) of A.3.1, every compact set in S is the image of a compact set in X. Furthermore, directed limit is right exact, so not only does the second assertion follow from the first, but we we can suppose that everything, *i.e.* X, Y, and S, are compact. There is, however, a measurable bijection between, say, X/S and $\{(s, \#X_s)\} \subset S \times \mathbb{N}$, where $\#X_s$ is viewed as a subset of $\mathbb{N}$ consisting of the elements $\{1, .., n\}$, where n is the number of elements in X_s, so this just reduces to a calculation about matrices. □

We can, however identify Radon measures on X with the weak dual of $\Gamma_c(X)$, and whence,

A.3.5 Corollary/Definition *The functor, $X \mapsto M(X) := Hom(\Gamma_c(X), \mathbb{C})$, where the latter is understood in the weak topology defines the sheaf of (Radon) measures on $\underline{E}QF_{qc}$.*

In particular for $\mathcal{X} = [R \rightrightarrows X]$ the classifying stack of any qc groupoid in $\underline{EQF}$ the space of measures on $\mathcal{X}$ is given by the exact diagram,

$$0 \to M(\mathcal{X}) \to M(X) \rightrightarrows M(R)$$

To truly profit from this we need to be able to describe the 'functions with compact support on $\mathcal{X}$'. To this end consider the associated simplicial space $X_\bullet \in \mathrm{Hom}(\Delta^o, \underline{E}QF)$, where Δ is the simplicial category with initial object 1, *i.e.* $X = X_1$, not X_0, then the measure sequence is dual to the sequence given by,

A.3.6 Definition *Define the co-invariant bounded Borel functions on $\mathcal{X}$ by the exact diagram in the category of separated topological vector spaces,*

$$\Gamma_c(X_2) \rightrightarrows \Gamma_c(X_1) \to \Gamma_c(\mathcal{X}) \to 0$$

i.e. take the quotient of the middle term by the closure of the co-equaliser. Alternatively, the notation, $\Gamma_{c,i} := \Gamma_c(X_i)$ may be employed, with of course, $\Gamma_{c,0} := \Gamma_c(\mathcal{X})$, and indeed $\Gamma_i = \Gamma(X_i)$.

That this is independent of the presentation of $\mathcal{X}$ in its small qc site, reduces by A.3.4, and the usual diagram for base changing groupoids, to checking the case of an open embedding U in X, with $U \to \mathcal{X}$ an effective epimorphism. In this latter case, the square à la A.3.4 is still exact, so indeed things are well defined up to a possible lack of separatedness in the duality between Γ_c and measures. The ultimate resolution of this lacuna will take place in the course of our discussion, albeit that it is largely resolved in making the by no means trivial, but certainly simplifying assumption, that X_1 is compact. As such $\Gamma_c(X_1) = \Gamma(X_1)$ is actually a Banach space. Better still Banach space co-kernels, and co-kernels in the category of separated topological vector spaces coincide, so that,

A.3.7 Fact *If X_1 is compact then $\Gamma_c(\mathcal{X})$ is a Banach space, i.e. the co-kernel of the Banach space closure of the image of $\Gamma_{c,2}$, whence by Hahn-Banach/the axiom of choice is in separated duality with $M(\mathcal{X})$.*

All of which is quite promising, but what is really required is to put a ring structure on $\Gamma_c(\mathcal{X})$. Let's get the notation out of the way, *i.e.* let $p_{n,i}$, or just i if n is clear, be the projection from X_n to X_{n-1} which omits i, $1 \leq i \leq n$, so that we can concentrate on the key issue of finding amongst all invariant measures, those which are minimal. To this end we can certainly suppose that $M(\mathcal{X}) \neq 0$, while if $M(\mathcal{X}) = \mathbb{C}$ there is equally nothing to do. As such consider the situation where we have a positive invariant measure μ, where without loss of generality $\mu(1) = 1$.

In particular we get an ideal I_μ of Γ_1, or slightly more accurately of the continuous functions on X_1, defined by the condition that a function is zero μ-a.e. As such we may apply the axiom of choice to the set,

$$\{I_\mu \subset I \subset \Gamma_1 : 1 \notin I \quad \text{an invariant ideal}\}$$

Plainly the set in question is closed under unions of chains, so it has a maximal member J, say. Furthermore we can further apply the axiom of choice to find a maximal ideal $\mathfrak{m}$ of Γ_1 such that $J \subset \mathfrak{m}$. Necessarily $\mathfrak{m}$ is closed, so the closure of J doesn't contain 1, whence J is closed. We further assert,

A.3.8 Claim *There is an invariant measure ν such that J is the ideal of functions which are zero ν-a.e. Moreover, both the ν-a.e. co-invariants and invariants are $\mathbb{C}$.*

proof Denote by E the Banach space closure of $(1_* - 2_*)\Gamma_2$ in the space, slightly abusively written Γ_1, of continuous functions on X_1. By construction $1 \neq 0$ in $\Gamma_1/E + J$. Consequently there is a positive invariant measure ν vanishing on J. We normalise so that $\nu(1) = 1$, and denote by I_ν the functions which are zero ν-a.e., then by the maximality of J, $I_\nu = J$. Better still if λ is any other positive measure arising from a functional on $\Gamma_1/E + J$, similarly normalised, then $I_\lambda = J = I_\nu$. Consequently, λ, ν are absolutely continuous with respect to each other, *i.e.* $\lambda = f\nu$, $\nu = g\lambda$. Furthermore the measures are invariant, so f (respectively g) is an invariant function ν-a.e. (respectively λ-a.e.). The ring of ν-a.e. invariant functions is, however, a C*-algebra, and any maximal ideal in this ring leads to an invariant measure, so by the maximality of J, the ring of ν-a.e. invariant functions is reduced to $\mathbb{C}$. Consequently $\lambda = \nu$, and the co-invariant functions are also reduced to $\mathbb{C}$. $\Box$

As such let us make,

A.3.9 Definition *Call an invariant measure whose co-invariants are reduced to $\mathbb{C}$ a point.*

With this in mind it's rather helpful to introduce,

A.3.10 Notation *For an invariant measure μ, denote by $-\otimes\mathbb{C}(\mu)$ any of the previous constructions on replacing X_1 by the spectrum of functions modulo the relation of equivalence μ-a.e. Whence, most importantly, Γ_1, by it's completion in the semi-norm: essential supremum with respect to μ.*

Certainly, therefore, points ν satisfy $\Gamma_{c,0} \otimes \mathbb{C}(\nu) = \mathbb{C}$, and this ties up rather nicely with the cone structure of the space of invariant measures by way of another transfinite induction (or indeed the more precise versions of Hahn-Banach), *i.e.*

A.3.11 Fact *If for any invariant measure $\mu \in M(\mathcal{X})$, $dim_{\mathbb{C}}\Gamma_{c,0} \otimes \mathbb{C}(\mu) > 1$, then there is an invariant measure λ, linearly independent from μ, satisfying $\mu \geq \lambda$. In particular the points are precisely the extremal rays of the cone $M_+(\mathcal{X})$ of positive invariant measures.*

As such, points separate elements of $\Gamma_{c,0}$. Indeed for f represented by a notationally identical non-negative real valued function,

$$\|f\|_{\Gamma_{c,0}} = \sup_{\text{pts}} \nu(f)$$

Necessarily this leads to,

A.3.9 bis Definition *Write X_0 for the set of (invariant) points in the induced (i.e. as a subset of all measures) weak topology.*

Whence if ϵ_* is the natural map from Γ_1 to $\Gamma_{c,0}$, then each function $f \in \Gamma_1$ determines a function $\epsilon_*(f)$ on X_0, and these in turn generate a C*-sub-algebra A of $C^0(X_0)$ in which $\Gamma_{c,0}$ is isometrically embedded. Equally, the algebra of invariants Γ_0, considered as a sub-algebra of Γ_1, acts on co-invariants, and we assert,

A.3.12 Claim *For h invariant, and $g \in \Gamma_1$, the module action of Γ_0, and multiplication in A are compatible, i.e.*

$$\epsilon_*(h)\epsilon_*(g) = h\epsilon_*(g)$$

proof It suffices to check this at points $x \in X_0$. Whence it suffices to check that the ideal of functions I_x of functions in Γ_1 which are zero x-a.e. leads to a maximal ideal $I_x \cap \Gamma_0$ of Γ_0. Since, however, multiplication of an invariant measure by an invariant function leads to a new invariant measure this is indeed the case. □

Now for a given positive invariant measure $d\mu$, we can, by Hahn-Banach, extend it to a positive Radon measure, $d\tilde{\mu}(x)$, say, on A, and from there define for any $f \in \Gamma_1$ a new invariant measure by way of the formula,

$$\epsilon_*(f)d\tilde{\mu} : g \mapsto \int_{\mathrm{sp}\,A} \epsilon_*(f)\epsilon_*(g)(x)d\tilde{\mu}(x)$$

In particular, $|\epsilon_*(f)d\tilde{\mu}(g)| \leq \|\epsilon_* f\||d\mu(g)|$, so the new measure is absolutely continuous with respect to the old, and admits a derivative,

$$\epsilon^*_{\tilde{\mu}}\epsilon_* f := \frac{\epsilon_*(f)d\tilde{\mu}}{d\mu} \in \Gamma_1$$

Our measure is, however, invariant so the derivative is μ-a.e. invariant, *i.e.* takes values in $\Gamma_0 \otimes \mathbb{C}(\mu)$. As per A.3.12, this procedure is compatible with multiplication of $d\mu$ by invariant functions h, *i.e.*

A.3.13 Fact *For $f \in \Gamma_1$, and $h \in \Gamma_0 \otimes \mathbb{C}(\mu)$,*

$$\epsilon^*_{h\tilde{\mu}}\epsilon_* f = \epsilon^*_{\tilde{\mu}}(\epsilon_*(f))$$

Similarly we can define for any number of functions $f_1, \dots, f_n$ new invariant measures,

$$(\epsilon_* f_1 \dots \epsilon_* f_n)d\mu : g \mapsto \int_{\mathrm{sp}\,A} \epsilon_*(g)\epsilon_*(f_1)\dots\epsilon_*(f_n)d\tilde{\mu}(x)$$

and from there invariant derivatives,

$$\epsilon^*_{\tilde{\mu}}(\epsilon_* f_1 \dots \epsilon_* f_n) \in \Gamma_0 \otimes \mathbb{C}(\mu)$$

which are again compatible with $\tilde{\mu} \mapsto h\tilde{\mu}$ for h invariant. As such for any functions $f, g_1, \dots g_n$,

$$\begin{aligned}\int_{\mathrm{sp}\,A} \epsilon_*(f)(\epsilon_* g_1 \dots \epsilon_* g_n)d\tilde{\mu}(x) &= \int_{X_1} \epsilon^*_{\tilde{\mu}}(\epsilon_* g_1 \dots \epsilon_* g_n) f d\mu \\ &= \int_{X_1} \epsilon^*_{\tilde{\mu}}(\epsilon_* f)\epsilon^*_{\tilde{\mu}}(\epsilon_* g_1 \dots \epsilon_* g_n)d\mu\end{aligned}$$

Since $\epsilon_{\tilde\mu}^*\epsilon_* = 1$, and we have compatibilities A.3.12/13. Consequently, the injective representation arising by multiplication in A,

$$\Gamma_{c,0} \otimes \mathbb{C}(\mu) \longrightarrow \mathcal{L}^1_A(d\tilde\mu)$$

where the latter is the completion of A in the ℓ_1 norm, has the same range as the image in $\mathcal{L}^1_A(d\tilde\mu)$ of $\Gamma_0 \otimes \mathbb{C}(\mu)$, so the natural map arising from the restriction of ϵ_*,

$$\Gamma_0 \otimes \mathbb{C}(\mu) \longrightarrow \Gamma_{c,0} \otimes \mathbb{C}(\mu)$$

has dense range. Moreover, $\epsilon_{\tilde\mu}^*\epsilon_*$ is the identity, while $\epsilon_{\tilde\mu}^*$ has norm at most 1, so,

A.3.14 Fact *The above map is infact an isometry. In particular $\Gamma_{c,0} \otimes \mathbb{C}(\mu)$ is naturally a commutative* **C*** *algebra in the Banach space norm in the sense of* A.3.6 *on replacing X_1 by the spectrum $X_1 \otimes \mathbb{C}(\mu)$ of the ring of functions completed in the essential supremum norm of any invariant measure μ. Consequently every invariant measure λ on $X_1 \otimes \mathbb{C}(\mu)$, including μ itself, can be identified with a Radon measure $d\lambda(x)$ on $X_0 \otimes \mathbb{C}(\mu)$ satisfying,*

$$\lambda(f) = \int_{X_0 \otimes \mathbb{C}(\mu)} \epsilon_*(f) d\lambda(x), \quad f \in \Gamma_1$$

This, of course, leaves open;

A.3.15 Remark/Question In principle the full $\Gamma_{c,0}$ ought to have a commutative **C*** algebra structure, at least if one is prepared to consider 0 as such an algebra should there be no invariant measures. The basic, infact unique, difficulty in proving this amounts to trying to extend A.3.4 just a little to an appropriate diagram of Banach spaces rather than just topological vector spaces.

All of this relatives very well. Indeed,

A.3.16 Definition *For $\pi : Y \to \mathcal{X}$ a representable map from a space, not necessarily with discrete fibres, define a sheaf on the qc site of Y by,*

$$M_{Y/\mathcal{X}} : U \mapsto \mathrm{Hom}(\Gamma_c(U), \Gamma_c(\mathcal{X}))$$

Again by A.3.4 this is a sheaf as opposed to a pre-sheaf. Whence there is certainly a pairing,

$$\int_{Y/\mathcal{X}} : \Gamma(Y, M_{Y/\mathcal{X}}) \times \Gamma_c(Y) \longrightarrow \Gamma_c(\mathcal{X})$$

Unlike invariant measures, however, relative measures are rather easy to come by. Indeed for $Y_\bullet$ the simplicial space $Y \times_{\mathcal{X}} X_\bullet$ we have an exact diagram,

$$\Gamma_c(Y_2) \rightrightarrows \Gamma_c(Y_1) \to \Gamma_c(Y) \to 0$$

On the other hand, if to fix ideas we suppose that everything has a differentiable structure, and ω is a smooth differential form on Y of degree d =dim$Y$-dim$X$ standard relative integration yields maps,

$$\begin{array}{ccccccc}
\Gamma_c(Y_2) & \rightrightarrows & \Gamma_c(Y_1) & \longrightarrow & \Gamma_c(Y) & \longrightarrow & 0 \\
\downarrow & & \int_{Y_i/X_i} \omega \downarrow & & & & \\
\Gamma_c(X_2) & \rightrightarrows & \Gamma_c(X_1) & \longrightarrow & \Gamma_c(\mathcal{X}) & \longrightarrow & 0
\end{array}$$

Consequently,

A.3.17 Fact *if Y is compact, and things are differentiable of relative dimension d, then relative integration co-descends to a map, $\pi_* : A^d(Y) \to \Gamma_c(\mathcal{X})$. Better still if $d\mu \in M(\mathcal{X})$ is an invariant measure we can view $\pi^* d\mu$ as a distribution of dimension d, and Fubini's theorem holds, i.e.*

$$\int_Y \omega \pi^* d\mu = \int_{X_0 \otimes \mathbb{C}(\mu)} (\pi_* \omega) d\mu$$

proof The co-descent of π_* was the previous diagram, to which we just add another line to get Fubini, *i.e.*

$$\begin{array}{rccccccc}
 & \Gamma_c(Y_2) & \rightrightarrows & \Gamma_c(Y_1) & \longrightarrow & \Gamma_c(Y) & \longrightarrow & 0 \\
\displaystyle\int_{Y_i/X_i} \omega & \downarrow & & \downarrow & & \downarrow & & \\
 & \Gamma_c(X_2) & \rightrightarrows & \Gamma_c(X_1) & \longrightarrow & \Gamma_c(\mathcal{X}) & \longrightarrow & 0 \\
d\mu & \downarrow & & \downarrow & & \downarrow & & \\
 & \mathbb{C} & \underset{0}{\longrightarrow} & \mathbb{C} & \underset{\mathrm{id}}{\longrightarrow} & \mathbb{C} & \longrightarrow & 0
\end{array}$$

The composition of the middle row is just standard integration with respect to $\omega\pi^* d\mu$ by the usual Fubini theorem. The diagram commutes, so the co-descent of the composition on the right is the co-descent of standard integration from $Y_1 \to Y$, which itself is just standard integration. □

The simplicity of this shouldn't hide,

A.3.18 Remark The relative integral is wholly well defined even though it may very well be the case (consider foliations) that for every $X_\bullet$, the differential form $\omega|_{Y_\cdot}$ fails to be relatively integrable, as opposed to defining a relative measure.

The seeming generality of the discussion should also not obscure,

A.3.19 Warning The theory so developed while more than adequate to cover such dynamical systems as those defined by rational maps, or smooth differentiable foliations does not quite apply as stated to the immediate case of current interest, *i.e.* singular (holomorphic) foliations. The problem is that even after slicing a groupoid such as A.1.8 is not a groupoid in $\underline{E}QF$ spaces. Indeed the stabiliser over a singularity cannot, in general, be given a structure which is any better than that of a continuous group. As such no slice will ever be $\underline{E}QF$, and the key diagram A.3.4 is not a priori valid. It is, however, valid off the singularities. Better still, we could have just settled for making the definition A.3.6 after localising at a given transverse invariant measure $d\mu$, which plainly doesn't affect the validity of A.3.14. Consequently, provided that we proceed from a given transverse invariant measure on the smooth locus, which extends across the singular locus with finite mass (which will always be the case), and work throughout in the appropriate essential supremum norm, everything is applicable as stated.

Opus Tertium

Bloch Hyperbolicity

INTRODUCTION

There would appear to be two pre-dominant misconceptions in value distribution/ Nevanlinna theory,

• It solely pertains to entire mappings, or more generally holomorphic mappings from proper covers $p : Y \to \mathbb{C}$ of the line, to (quasi)-projective varieties.

• It is non-trivial.

The first of these should have been soundly laid to rest by Bloch's substantial generalisation, [B], of Montel's theorem on the normality of disc mappings to $\mathbb{P}^1\backslash\{0, 1, \infty\}$ to the case of $\mathbb{P}^2\backslash\{4\,\text{planes}\}$, and the subsequent extension to $\mathbb{P}^n\backslash\{n + 2\,\text{planes}\}$ of Cartan, [C]. Indeed lest there be any doubt, Bloch even spells it out with his dictum, *Nihil est in infinito quod non prius fuerit in finito*. Regrettably, however, the particular features of Bloch's problem, *e.g.* $\mathbb{P}^2\backslash\{4\,\text{planes}\}$ doesn't contain any $\mathbb{P}^1$'s or $\mathbb{A}^1$'s, served to create a sub-misconception,

• If disc mappings to a quasi-projective variety don't form a normal family, then there's nothing meaningful to be said.

Plainly, though, post Gromov's compactness theorem, [G2], that discs of bounded area converge to discs with bubbles, this is every bit as ridiculous as believing that value distribution theory is non-trivial. There is, however, a meaningful difference between area in the Nevanlinna theory sense, and it's classical manifestation, which allows us to fully profit from the complex structure, and even, in a quasi-projective situation, from a whole gambit of algebraic machinery. Specifically for ω an appropriately regular $(1, 1)$ form on the unit disc Δ one shouldn't simply consider,

$$\int_\Delta \omega$$

but rather,

$$\rlap{\kern0.2em/}\int_\Delta \omega := \int_0^1 \frac{dt}{t} \int_{\Delta(t)} \omega$$

where $\Delta(t)$ is the disc of radius t. It is, therefore, slightly unfair to say that Nevanlinna theory has no content, since despite having completely summarised it by way of the adjusted area function this subtle change in definition makes all the difference in the world, and allows us to turn a map f from a disc to a variety X into a more or less harmonic positive current, I.2.4, whose intersection with a given Cartier divisor is controlled by the proximity of its origin to the same, I.2.2. On the other hand, changing definitions usually comes at a price, so that while it's evident that a sequence of discs which became bigger and bigger in the sense of their area dominating the length of their boundary would actually define a closed positive current, it is not quite so evident that a sequence of discs whose Nevanlinna area went to infinity could be used to form a closed rather than just harmonic measure. Nevertheless, this is actually true I.2.5, and with a good bit of functoriality too, I.2.9. Indeed the harmony between the Nevanlinna area function and the complex structure is such that one can, in this special case, re-prove Gromov's compactness by a rather easy direct calculation, I.3.1-4. As such we're naturally led to enquire as to the exact relation between convergence of a sequence of discs to a disc with bubbles, and, so called normal convergence. To this end suppose we're looking at disc mappings to a projective variety X missing a boundary B whose components are B_i, then a lemma of Green, [G], shows that we have bubbling iff there are non-trivial (algebraic) mappings from $\mathbb{A}^1$ to either $X\backslash B$ or some $B_i\backslash \cup B_j$, $j \neq i$, I.3.5-7. Consequently if we augment the space Hom of disc mappings be it to X or $X\backslash B$ to $\overline{\mathrm{Hom}}$ by throwing in the discs with bubbles then the most generally valid extension of Bloch's theorem would be to ask whether there is a Zariski closed subset Z such that a sequence in $\overline{\mathrm{Hom}}(\Delta, X\backslash B)$ not enjoying a convergent subsequence in $\overline{\mathrm{Hom}}(\Delta, X)$ is arbitrarily close to $\bar{Z} \cup B$, *i.e.* is *hyperbolic modulo* Z, I.3.9(b). Nevertheless the Bloch/Cartan theorem is stronger still since in their case the $B_i\backslash \cup B_j$ contain no holomorphic lines, and in such a case we can even hope that such a divergent sequence would actually be arbitrarily close to $\overline{Z}$, *i.e.* is *complete hyperbolic modulo* Z, I.3.9(c). In arbitrary dimension, in fact 3 would do, it is unrealistic/false to assert that complete hyperbolic and hyperbolic coincide, but provided we work on the log-minimal model (X, B) perfectly within the bounds of credibility for surfaces.

More, or less, by definition complete hyperbolicity is quantative, and when Z is empty implies, [Kl], the isoperimetric inequality in Gromov's sense, and given effective quantification of the degeneration of the natural Kobayashi metric on $T_X(-\log B)$ around Z should, in principle, lead to a more general version of the same even should Z be non-empty. One could, however, ask for a priori a little more in the form of an isoperimetric inequality for the Nevanlinna area. For X of dimension 1, and genus ≥ 2 this is a triviality, and amounts to nothing other than integration by parts together with the concavity of the logarithm. Quite generally, the same integration by parts yields not an isoperimetric metric inequality stricta dicta, but a tautology, *i.e.* an explicit formula for the Nevanlinna degree of the tautological degree of the logarithmic derivative,

$$f' : \Delta \to P(T_X(-\log B))$$

with respect to the tautological line bundle L, I.4.1. It's strict analogue for a mapping f from a smooth compact curve C to X not factoring through B would be the formula,

$$L._{f'}C = 2g(C) - 2 - \mathrm{Ram}_f - \#f^{-1}(B)_{\mathrm{red}}$$

which, by any stretch of the imagination, is trivial. To be sure there is some minor subtlety occasioned by the fact that the natural metric on L arises from a complete metric on $X\backslash B$, which wouldn't be a smooth metric on the total space, but this is easily dealt with by the ubiquitous Jensen formula/integration by parts, I.4.5. As such we may roughly summarise the consequences of the change from area to Nevanlinna area, and the content of our preliminaries, by way of,

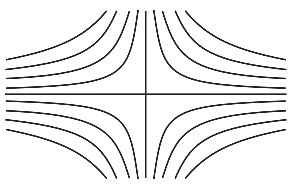
Summary of §I *Given a sequence of maps f_n from the disc to a quasi-projective variety $X\backslash B$ which don't admit a convergent (bubbling sense) subsequence, or, for that matter violate an appropriate version of the isoperimetric inequality, we can,*

- *Subsequence and associate to the limit a closed positive current T.*
- *Idem for the derivatives, so, say a current T' on $P = P(T_X(-\log B))$ pushing forward to T, and this satisfies, $L.T' \leq 0$.*
- *For effective Cartier divisors D at which the origins, or more generally compact sets, don't accumulate assert the non-negativity of the intersection numbers $D.T$, or $D.T'$.*
- *All of the above in a functorial way for more or less arbitrary birational modifications of the initial data.*

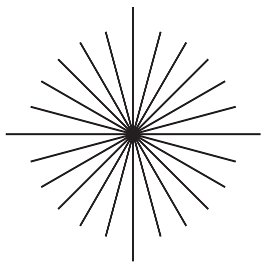
Beyond this, at least classically, value distribution theory is wholly sub-ordinate to algebraic geometry. For example Montel's theorem is immediate from I.4.2, and the fact that $\omega_{\mathbb{P}^1}(0+1+\infty)$ is ample, or resolving the singularities of the foliation on $P(T_{\mathbb{P}^2}(-\log 4 \text{ lines}))$ of which the generic leaf is the derivative of a line in Bloch's case, III.3, albeit that even in the latter case the closedness of the limiting measure is superfluous since the said foliation is algebraically integrable. The intervention, here, of foliations by curves should be unsurprising since the appropriate, and rather more ancient than Bloch's, maxim is *nemo det quo non habet.* Specifically it is absolutely clear that Nevanlinna theory is equidimensional in nature, and the most generally meaningful sense in which we could have a 1-dimensional problem regarding curves or discs on a variety X would be that they were related to a rank 1 foliation in some way. Indeed if they were actually invariant by the foliation $\mathcal{F}$ then our situation would be of honest relative dimension 1 for the 'fibring' $X \to [X/\mathcal{F}]$ of X over the classifying stack of the foliation. Supposing that X were smooth, we would then have a bundle of relative 1-forms $K_{\mathcal{F}}$ which would be very nearly equal to the restriction of the tautological bundle L to the quasi-section F of $P(T_X)$ which the foliation defines. Indeed the latter is the blow up in the foliation singularities considered with scheme structure, so, with exceptional divisor E, say, and for $\pi : F \to X$ the projection we have the relation,

$$L|_F = \pi^* K_{\mathcal{F}}(-E)$$

As such we almost have the situation where the current T is invariant by the foliation, and satisfies an inequality of the form $K_{\mathcal{F}}.T \leq 0$ into the bargain. This latter inequality is not, however, valid in the same way as its tautological counterpart. Indeed it holds if and only if the singularities are canonical. To see the principle, consider a typical picture of a canonical foliation singularity in dimension 2,



then it is basically true that the singularity cannot contribute to the intersection number unless the invariant curves/discs that we wish to study pass through the singularity. There are, however, only two such branches, and while they may be no better than formal, to find sections vanishing at the singularity in either of these directions is cheap, *i.e.* for sections of powers of an ample bundle $H^{\otimes n}$ to vanish to order p at a point on a surface costs around p^2, so there are no such unless p is around n, but vanishing to order p at a point and in one of finitely many directions costs only p. As such, provided that the curves/discs we're studying don't factor through the base of some linear systems implicit in the above counting we can indeed make the tautology of I.4, into a refined tautological inequality for the Nevanlinna degree of discs with respect to the canonical bundle of a foliation by curves with canonical singularities, and, indeed in all dimensions, II.2.10 & II.3.1. As the above indications suggest, what is required is an analysis of the curves through such singularities, II.1, and construction of appropriate linear systems II.2. Care though is required since already in dimension 2 it isn't difficult to construct counterexamples when the singularity is only log-canonical as is plain from the picture



i.e. given that the possible directions fill up the space there is no advantage to be had by the prescription, 'vanishing at the point in the foliation directions'. Worse still some invariant (formal) hyperplane through a singularity may only be log-canonical, so the argument uses the full force of I.4.2 together with a slightly arithmetic yet Noetherian induction, cf. II.1.2 (iii).

The machinery in place, we can make some low, *i.e.* 2-dimensional applications. For example the main theorem of [M1] asserts that a foliated surface $(X, \mathcal{F})$ with canonical (foliated) singularities and $K_X.K_{\mathcal{F}} > 0$ does not admit an entire map $f : \mathbb{C} \to X$ invariant by the foliation unless it factors through an algebraic curve. At the *in finito* level this manifests itself by way of the convergence of invariant discs modulo the algebraic invariant curves through the singularities, III.1.3, and, indeed, even the degeneration around the said curves may be quantified, III.1.4. Better still if we limit ourselves to discs which don't pass through the singularities, *i.e.* consider only the uniformising maps of leaves, then if we're on a minimal model of the foliation the convergence is modulo the necessarily finite number of leaves which are both algebraic and parabolic, III.1.6. A particular corollary of op. cit. was, however, that any map $f : \mathbb{C} \to X$ to a surface of general type and big co-tangent bundle factored through a rational or elliptic curve, and on searching for the version *in finito* we immediately come across the one substantive difference between the study of value distribution in the parabolic setting, *i.e.* holomorphic mapping from proper covers of the line, as opposed to the same for covers of discs.

Specifically, the only way in which the parabolic situation can lead to substantive negativity for the Nevanlinna intersection number with respect to an effective Cartier divisor is if the map in question factors through the divisor, while it's disc counterpart may be negative under the weaker property that the discs become arbitrarily close to the divisor. It follows that the basic reduction of the parabolic problem to a foliation problem is not quite valid, and we can only assume that if our discs were to diverge then on some ramified covering of our initial surface a sequence of mappings from a sequence of ramified coverings of the disc would, after derivation, be arbitrarily close to a divisor in the projective tangent space whose horizontal components were defined by foliations of the ramified cover. This leads naturally to the rather surprising fact, III.2.2, which, at least combinatorily speaking, asserts that a Kãhler-Einstein metric on a surface of general type has curvature ≤ -1 in the directions determined by an arbitrary 1st order ODE, i.e.

Main Lemma of $\S$**III** *Let $\mathcal{X}$ be a 2-dimensional Deligne-Mumford stack with projective moduli, and $\mathcal{B}$ a simple normal crossing divisor with, $f_k : \mathcal{C}_k/\Delta \to \mathcal{X}$ a sequence of maps, such that the derived current T' associated to the derivatives $f'_k : \mathcal{C}_k \to \mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{B}))$ is supported in a divisor all of whose horizontal components have degree* 1*. Then after a sufficiently large birational modification $\rho : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{B}}) \to (\mathcal{X}, \mathcal{B})$*[1]*, with $\widetilde{T}'$ the derived current associated to the proper transforms of the f_k and L the tautological bundle on $\mathbb{P}(\Omega_{\widetilde{\mathcal{X}}}(\log \widetilde{\mathcal{B}}))$ we have,*

$$L.\,\widetilde{T}' \geq (K_{\widetilde{\mathcal{X}}} + \widetilde{\mathcal{B}}).\,\widetilde{T}$$

provided $\widetilde{T}'$ is closed, and its push-forward $\widetilde{T}$ satisfies, $\mathcal{B}_1 \cdot \widetilde{T} \geq 0$ for all $\mathcal{B}_1$ in the support of $\widetilde{\mathcal{B}}$.

Just as the theorems of $\S$III.1 use the closedness of the current in an essential way to get some information on the intersection of the normal bundle to the foliation with the current by way of Baum-Bott's residue philosophy, [BB], so it is here, but in a strong quantative form due to M. Brunella, [Br1], albeit this is plainly un-necessary if there were only vertical divisors involved. Regardless we have,

Corollary *Let T' be a closed positive current associated to the derivatives of a holomorphic map $f : \mathbb{C} \to \mathcal{X}\backslash\mathcal{B}$ to a 2-dimensional log-stack of general type, then should there exist a proper sub-stack Z of $P(T_{\mathcal{X}}(-\log \mathcal{B}))$ such that $T' = \mathbb{1}_Z T'$ then f factors through a rational or elliptic curve, or, if one prefers: should f have Zariski dense image then so does T'.*

Indeed not only do we deduce from the above lemma , that a surface of general type and $c_1^2 > c_2$ is hyperbolic modulo some Zariski closed subset Z, but we can refine the technique to obtains that it is actually complete hyperbolic, III.5.3, with explicit, and essentially optimal, control on the degeneracy of Kobayashi metric. As such all that's really missing in this case is whether Z may be identified with the set of rational and elliptic curves, which the methodology only gives if Z has simple normal crossing.

Evidently I'm indebted to Brunella and Gromov, but even more so to Cécile.

[1]Sufficiently large means determined by $\mathcal{X}, \mathcal{B}$, the said divisor etc., but not the maps f_k, while $\widetilde{\mathcal{B}}$ is permitted to be very much bigger than $\rho^{-1}(\mathcal{B})$, although always with simple normal crossings.

I. VALUE DISTRIBUTION

I.1. Remarks on Intersection Theory

Our ultimate set up will be inverse to that of the uniformisation theorem, albeit that the key objects will remain $\mathbb{P}^1$, $\mathbb{C}$, and the unit disc Δ. Unsurprisingly, however, the theory over the unit disc encompasses its equivalents over $\mathbb{P}^1$ and $\mathbb{C}$, so we'll concentrate on the latter, while confining ourselves to parenthetical remarks on its parabolic companions.

In the first place, provided we fix a radius $r \in [0,1)$ we have a well defined notion of degree on Δ defined by way of Nevanlinna theory. Indeed if $\mathrm{Div}(\bar{\Delta})$ is the group of metricised Cartier divisors on Δ then we have,

I.1.1 Definition *Let $\bar{D}$ be a metricised Cartier divisor on Δ, with $\mathbb{1}_D \in \Gamma(\mathcal{O}_\Delta(D))$ the tautological section, then,*

$$\,\,⨍_{\Delta(r)} c_1(\bar{D}) := \sum_{0<|z|<r} \mathrm{ord}_z(D) \log \left|\frac{r}{z}\right| + \mathrm{ord}_0(D) \log r - \int_{\partial\Delta(r)} \log \|\mathbb{1}_D\| \cdot \frac{d\theta}{2\pi} + \lim_{|z|\to 0} \log \frac{\|\mathbb{1}_D\|}{|z|^{\mathrm{ord}_0(D)}}\,.$$

To those unfamiliar with Nevanlinna theory, this a priori may just look like sad rubbish. In reality, however, it is a delicate instrument which for effective Cartier divisors pulled back by maps to compact varieties will yield a positive intersection number under the condition that the image of the origin under the map is not "too close" to the divisor. The efficacity of the definition, however, results from an equivalent formulation via integration by parts/Jensen's formula, *i.e.*

I.1.2 Fact. *Identifying* $\mathrm{Div}(\bar{\Delta})$*, with the group of metricised line bundles* $\mathrm{Pic}(\bar{\Delta})$ *we have,*

$$⨍_{\Delta(r)} c_1(\bar{D}) = \int_0^r \frac{dt}{t} \int_{\Delta(t)} c_1(\bar{D})$$

where the integrals on the right are understood with respect to standard Lebesgue measure on $[0,r]$ and the disc $\Delta(t)$ of radius t respectively. In particular the Nevanlinna type integral $⨍$ extends to smooth $(1,1)$ forms by way of,

$$⨍_{\Delta(r)} : A^{1,1}(\Delta) \to \mathbb{C} : \tau \mapsto \int_0^r \frac{dt}{t} \int_{\Delta(t)} \tau$$

and more generally to $(1,1)$ forms of appropriate regularity, say $A^{1,1}_?(\Delta)$, to guarantee the finiteness of the right hand side.

The appropriate level of generality, however, involves finite coverings $p : \mathcal{C} \to \Delta$ of the unit disc by smooth analytic stacks. To this end observe,

I.1.3 Fact/Definition. *There is a well defined map,*

$$p_* : A^{1,1}(\mathcal{C}) \to A^{1,1}_?(\Delta)\,.$$

Proof/Justification. Indeed let, $q : \coprod_\alpha C_\alpha \to \mathcal{C}$ be an analytic atlas for $\mathcal{C}$, and τ a smooth $(1,1)$ form on $\mathcal{C}$. In addition as per [II] I.2.5, we can suppose that $\mathcal{C}$ has moduli C, with $[C_\alpha/G_\alpha] = \mathcal{C} \times_C U_\alpha$, for $\coprod_\alpha U_\alpha \to C$ a covering of C, and G_α finite groups acting on C_α. Denoting as ever by π the moduli map, we have in the first place,

$$\pi_* : A^{1,1}(C_\alpha) \to A^{1,1}_?(U_\alpha) : \sigma \mapsto \frac{1}{|G_\alpha|}(\pi\mid_{C_\alpha})_*(\sigma)$$

where now ? simply indicates some appropriate regularity class. Manifestly this extends to the whole of $\mathcal{C}$ according to,

$$\pi_* : A^{1,1}(\mathcal{C}) \to A^{1,1}_?(C) : \tau \to \pi_*(\tau\mid_{C_\alpha})$$

where the right hand side is understood to be the measurable function calculated at $c \in C$ by choosing a $U_\alpha \ni c$, and applying the previous local definition. As such, having got this far, we can just apply the standard push forward from $C \to \Delta$ in order to conclude, while, observing that this agrees with our previous degree conventions in [II] I.8.3.

Therefore, and essentially without need of comment, we make

I.1.4 Definition. *Let things be as above then we define,*

$$\oint_{\mathcal{C}(r)} : A^{1,1}(\mathcal{C}) \to \mathbb{C} : \tau \mapsto \oint_{\Delta(r)} p_*\,\tau\,.$$

What will ultimately be of paramount importance here is a Riemann-Hurwitz type formula. As such, to begin with we need to understand what is meant by "canonical degree of a disc", which in turn is nothing other than the Nevanlinna degree of the Poincaré metric, *i.e.*

I.1.5 Definition. *As ever for $r \in [0,1)$ put,*

$$\mathrm{discr}_\Delta(r) := \oint_{\Delta(r)} dd^c \log(1-r^2)^{-1} = \log(1-r^2)^{-1}\,.$$

Returning to finite covering stacks $p : \mathcal{C} \to \Delta$, with generic stabiliser G, observe that the Riemann-Hurwitz formula of [II] I.8.5 naturally suggests that we make,

I.1.6 Definition. *Let things be as above, with $C \to \Delta$ the moduli of p then,*

$$\mathrm{discr}_{\mathcal{C}}(r) := \frac{(C:\Delta)}{|G|}\log(1-r^2)^{-1} + \sum_{0<|p(c)|<r} \mathrm{ord}_c(\mathrm{Ram}_{\mathcal{C}/\Delta})\log\left|\frac{r}{p(c)}\right| + \sum_{c\in p^{-1}(0)} \mathrm{ord}_c(\mathrm{Ram}_{\mathcal{C}/\Delta})\log r$$

where the geometric closed points c of $\mathcal{C}$ are identified with those of C, but the order function at c is the order function on an étale neighbourhood divided by the order of the stabiliser of c.

While we will encounter in III.2 situations (perhaps surprisingly in light of the uniformisation theorem) that require this level of generality, most of the cases to be discussed will involve nothing more complicated than a stack whose moduli is the disc itself together with a generically trivial stabiliser. As such let us note,

I.1.7 Example. *Suppose $p : \mathcal{C} \to \Delta$ is in fact the moduli map, the stabiliser is generically trivially, and the non-scheme like points c_i of $\mathcal{C}$ have automorphism group $\mathbb{Z}/e_i\mathbb{Z}$ with no c_i lying over the origin, then,*

$$\mathrm{discr}_{\mathcal{C}}(r) = \log(1-r^2)^{-1} + \sum_{0<|p(c_i)|<r}\left(1-\frac{1}{e_i}\right)\log\left|\frac{r}{p(c_i)}\right|\,.$$

I.2. Positivity and Closedness

As already suggested post definition I.1.1, the crucial property of these Nevanlinna type definitions is that they lead to positivity of intersection between effective Cartier divisors and discs under relatively mild hypothesis. Indeed consider what will be the principle case of interest according to the following arrows, which in turn fixes our notation, *i.e.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"'] \arrow[d, "\pi"'] & \mathcal{X} \arrow[d, "\pi"] \\ C \arrow[r, "f"'] \arrow[d, "p"'] & X \\ \Delta & \end{tikzcd}$$

where as ever π is the moduli, and the stack $\mathcal{X}$ has a projective (or even just compact) moduli space X, and we equally denote by p what would be $p \circ \pi$ in the diagram. Note in addition that we will abusively employ,

I.2.1 Notation. *If α is a function on $\mathcal{C}$, then,*

$$\alpha(0) := (p_* \, \alpha)(0)$$

where p_* on functions is defined analogously to the same on forms as per I.1.3.

With this in mind, Jensen's formula yields,

I.2.2 Small (but critical) Fact. *Let $\bar{\mathcal{D}}$ be an effective metricised Cartier divisor on $\mathcal{X}$, with $\mathbb{1}_{\mathcal{D}} \in H^0(\mathcal{O}_{\mathcal{X}}(\mathcal{D}))$ the tautological section, then,*

$$\fint_{\mathcal{C}(r)} f^* \, c_1(\bar{\mathcal{D}}) \geq \log \|f^* \, \mathbb{1}_{\mathcal{D}}\| \, (0) \, .$$

One should, however, bear in mind,

I.2.3 Warning. *There is absolutely no a priori reason why* $\log \|f^* \, \mathbb{1}_{\mathcal{D}}\| \, (0)$ *need not be ridiculously small in comparison to the degree function. As such it will be a hypothesis appropriate to the situation which guarantees this.*

Warnings apart, the positivity implicit in I.2.2, is best combined with some further observations resulting from the equivalent formulation of the degree function given by I.1.2. Specifically,

I.2.4 Definition/Discussion. *Consider the situation of countably many maps $f_n : \mathcal{C}_n \to \mathcal{X}$, from countably many stacks $p_n : \mathcal{C}_n \to \Delta$ and suppose that for some $r_0 \in (0, 1)$,*

$$\frac{1}{(\mathcal{C}_n : \Delta)} \fint_{\mathcal{C}_n(r_0)} f_n^* \, c_1(\bar{H})$$

tends to infinity, where $\bar{H}$ is a metricised ample divisor on X. Independently of these hypothesis observe that for every n and r we can (suppose $\mathcal{X}$ smooth for simplicity) define a current,

$$T_n(r) : A^{1,1}(\mathcal{X}) \to \mathbb{C} : \tau \mapsto \left(\fint_{\mathcal{C}_n(r)} f_n^* \, c_1(\bar{H}) \right)^{-1} \fint_{\mathcal{C}_n(r)} f_n^* \, \tau \, .$$

Manifestly the said current is positive, and by Jensen's formula more or less harmonic. Under the initial hypothesis of our discussion, however, we have,

I.2.5 Fact. *Suppose infact the f_n are maps from discs to a stack $\mathcal{X}$ with projective moduli additionally that $f_n(0)$ (in the sense of* I.2.1*, so more precisely $f_n(p^{-1}(0))$) is bounded away from the branch locus of $\mathcal{X} \to X$ independently of n (or more generally an appropriate growth condition in n) then for $r \geq r_0$ outside a set of finite hyperbolic measure (*i.e. $(1-r^2)^{-1}\, dr$*) in $(0,1)$ any convergent weak limit in n, of the $T_n(r)$ defines a closed positive current. More generally for an appropriate condition on the distance of the p_n ramification from the origin in comparison with Nevanlinna degree the same holds true when the f_n are maps from covers $p_n : \mathcal{C}_n \to \Delta$. The said additional condition will be seen in the course of the proof, and will always be satisfied in applications.*

Before we can get underway with the proof we will need,

I.2.6 Lemma. *Let $f : \Delta \to \mathbb{P}^n$ be a holomorphic map to projective space, and fix $0 < r < 1$. Then for the Fubini-Study metric adapted to a particular choice of coordinates depending only on a fixed sufficiently small neighbourhood of $f(0)$,*

$$R|df(z)|_{\rm euc} \leq c_1 \fint\!\!\!\!\!\!\!\!\nabla_{\Delta(R)} f^*\, c_1(\mathcal{O}_{\mathbb{P}^n}(\bar{1})) + c_2$$

for $2|z| < R$, where $R = min\{\frac{r}{n}\exp(-\fint\!\!\!\!\nabla_{\Delta(r)} f^*\, c_1(\mathcal{O}_{\mathbb{P}^n}(\bar{1})), r\}$, and the implied constants depend only on the said neighbourhood of $f(0)$ with degrees being measured in the said metric.

proof. A disc is Stein, so fixing a trivialisation of $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ allows us to write $f = [f_0, \ldots, f_n]$ in projective coordinates. Everything is homogeneous under the unitary group so we may as well say $f(0) = [1, \ldots, 1]$. In particular if H_i is the $i^{\rm th}$-coordinate hyperplane then,

$$\fint\!\!\!\!\!\!\!\!\nabla_{\Delta(r_0)} f^* c_1(\mathcal{O}_{\mathbb{P}^n}(\bar{1})) = -\int_{|z|=r} \log \frac{\|f^*\mathbb{I}_{H_i}\|}{\|f^*\mathbb{I}_{H_i}\|(0)} + \sum_{0<|z|<r} {\rm ord}_z(f^*H_i) \log \frac{r}{|z|}$$

Now for convenience put, $T_f(r)$ to be the left hand side of the above, and observe that if $f(z) \in H_i$ with $|z| < r$, then:

$$|z| \geq \frac{r}{n} \exp(-T_f(r)) \,.$$

So put $R = \min\left\{r, \frac{r}{n}\exp(-T_f(r))\right\}$ then for $|z| < R$, $f_i(z) \neq 0$. In particular if we identify affine n-space $\mathbb{A}^n$ with $\mathbb{P}^n \setminus H_0$ then Δ_R maps to $\mathbb{A}^n$ under f with standard coordinates $g_i = f_i/f_0$ and each g_i is a unit on Δ_R.

Consequently Jensen's formula gives,

$$\frac{g_i'(z)}{g_i(z)} = \int_0^{2\pi} \log|g_i(Re^{i\theta})| \,\frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2} \frac{d\theta}{2\pi}$$

We require to control the modulus of the logarithm in the integrand, so to this end observe that inversion on $\mathbb{G}_m^n = \{x_1 \ldots x_n \neq 0\} \subset \mathbb{A}^n$ is well defined on a modification $\rho : W \to \mathbb{P}^n$ which is an isomorphism outside crossings of hyperplanes. In particular if $i : W \to \mathbb{P}^n$ is the extension of inversion then,

$$i^*\mathcal{O}_{\mathbb{P}^n}(1) = \rho^*\mathcal{O}_{\mathbb{P}^n}(n) - E$$

for E an effective divisor on W contracted by ρ. Over $f(\Delta_R)$ we know how to invert, so,

$$\frac{1}{2}\int_{|z|=R} \log\left(1+\sum_{i=1}^{n}\frac{1}{|g_i|^2}\right)\frac{d\theta}{2\pi} \leq n\,T_f(R)+C_n$$

where C_n is an absolute constant depending only on n. A similar bound for $\log\left(1+\sum |g_i|^2\right)$ is even easier, and so we obtain, for $|z| < R/2$,

$$\left|\frac{g_i'}{g_i}\right| \leq 8R^{-1}\left((n+1)\,T_f(R)+C_n\right)$$

with C_n as before up to its definition. Now consider the Fubini-Study metric for the particular choice of coordinates, with ω the corresponding $(1,1)$ form then,

$$\omega \leq 2\sum_i \frac{dd^c|x_i|^2}{(1+|x_i|^2)}$$

so that plugging in our estimates for g_i' gives the lemma. □

Notice, in addition, that if $\mathcal{X}$ is a stack with projective moduli, then we can profit from [II] I.3.2 to adapt the above discussion to the situation where $f : \Delta \to \mathcal{X}$. More precisely if by a metric $\omega_{\mathcal{X}}$ on $\mathcal{X}$ we mean a positive $(1,1)$ form induced by restriction of the same from a local embedding of an étale neighbourhood into a smooth variety, then for $U \to \mathcal{X}$ an étale scheme like neighbourhood of $f(0)$ we simply require a slightly smaller R in-order to guarantee that $f(z)$, $|z| < R$ does not approach the boundary of U, which, by op. cit., we may identify with the pull-back of the union of the hyperplanes H_i of the proof for an appropriate choice of projective embedding of the moduli. Consequently we have,

I.2.7 Corollary *Let $f : \Delta \to \mathcal{X}$ be a holomorphic map to a stack with projective moduli, then there is a metric $\omega_{\mathcal{X}}$ as above adapted to a fixed sufficiently small neighbourhood of $f(0)$ such that,*

$$R|df(z)|_{\text{euc}} \leq c_1 \fint_{\Delta(r_0)} f^*\omega_{\mathcal{X}} + c_2$$

for $|z| < c_3 R|\log R|^{-1}$, where $R = \min\{\frac{r}{n}\exp(-\fint_{\Delta(r)} f^*\omega_{\mathcal{X}}, r\}$, and, again, the, possibly different, implied constants depend only on the said neighbourhood of $f(0)$ with degrees being measured in the said metric.

Now certainly the lemma depends on the whereabouts of $f(0)$, but given a sequence of maps f_n we can always subsequence so that the $f_n(0)$ converge. At this point the choice of a metric adapted to $f(0)$ is irrelevant, given that the boundedness or otherwise of the degree, for H ample, or for that matter a Kãhler metric on a stack, cf. [IV] III.2, is wholly independent of the metric in the limit by the previous harmonicity remark. Let us proceed therefore to,

proof of I.2.5 for $\mathcal{C}_n$ a disc for each n.

Any sequence f_n has a subsequence such that

$$T_n(r) \underset{w}{\longrightarrow} T,$$

and, $T(c_1(\bar{H})) = 1$, so $T \neq 0$. The difficulty is to prove closure. Consequently for α a smooth $(1,0)$ form consider,

$$|\displaystyle\fint_{\Delta(r)} f_n^* \bar{\partial}\alpha| = \frac{1}{2}\left|\int_{\Delta_r} \bar{\partial}\log|z|^2\,\alpha\right| \le \|\alpha\|_\infty \int_{\Delta_R} |df_n|_{\mathrm{euc}} \frac{dt d\theta}{2\pi}$$

for a suitable determination of the sup-norm independent of f_n. Now observe,

$$\begin{aligned}\fint_{\Delta(r)} f_n^* c_1(\bar{H}) &= \int_0^r \frac{dt}{t}\int_0^t J_n(s) s ds \\ &= \int_0^r J_n(s) s ds \int_s^r \frac{dt}{t} = \int_0^r J_n(s) s \log\frac{r}{s} ds\end{aligned}$$

where $J_n(s) = \int_{|z|=s} |df_n|^2_{\mathrm{euc}} \frac{d\theta}{2\pi}$. Rearranging we have,

$$I_n(r) := \int_0^r J_n(t) t |\log t| dt = \fint_{\Delta(r)} f_n^*(c_1(\bar{H})) + |\log r| r \frac{d}{dr}\fint_{\Delta(r)} f_n^*(c_1(\bar{H}))$$

Consider for a suitable $\epsilon > 0$, depending on n, to be chosen,

$$\begin{aligned}\int_{r\ge|z|\ge\epsilon} |df_n|_{\mathrm{euc}} \frac{dt d\theta}{2\pi} &\le \int_\epsilon^r J_n(t)^{1/2}\, dt \\ &\le \left(\int_\epsilon^r t\,|\log t|\, J_n(t)\, dt\right)^{1/2}\left(\int_\epsilon^r \frac{dt}{t|\log t|}\right)^{1/2} \\ &\le I_n^{1/2}(r)\log^{1/2}\frac{|\log\epsilon|}{|\log r|}\end{aligned}$$

Now let's use the $\ll$ notation for inequality up to a constant that depends on what we can fix, i.e. r_0, X, H etc. but definitely not f_n. Then for any $\epsilon \ll \min\{r_0\exp(-T_f(r_0)\,(c_1(\bar{H})), r_0\}$ we may apply I.2.6/7 to conclude,

$$\begin{aligned}\int_{|z|\le r} |df_n|_{\mathrm{euc}} \frac{dt d\theta}{2\pi} &\ll I_n^{1/2}(r)\log^{1/2}\frac{|\log\epsilon|}{|\log r|} + \epsilon T_{f_n}(r_0)\exp(T_{f_n}(r_0)) \\ &\ll I_n^{1/2}(r)\log^{1/2}\left(\frac{T_f(r)}{|\log r|}\,(c_1(\bar{H}))\right)\end{aligned}$$

for a suitably optimal, depending on n, ϵ, supposing as we may that the Nevanlinna area $T_{f_n}(r_0)$ is sufficently large. It therefore remains to show that $I_n(r)$ is not that much bigger than $T_{f_n}(r)$, for an appropriate r independent of f_n. This is easily done by a standard piece of measure theory,

I.2.8 Claim *Let $\delta > 0$ and S_n a sequence of increasing differentiable functions on $[0,1)$ with $S_n(R_0) \ge n^{2/\delta}$ for some $R_0 \in (0,1)$ then the set,*

$$\{1 > R \ge R_0 : S_n'(R) \ge S_n^{1+\delta}(R)\,(1-R^2)^{-1},\ \text{for some } n\}$$

has finite measure in the induced hyperbolic measure on the unit interval.

proof. The set in question has measure bounded by,

$$\sum_n \int_{R_0}^1 \frac{S_n'(R)}{S_n^{1+\delta}(R)}\,dR \le \sum_n \int_{S_n(R_0)}^\infty \frac{dx}{x^{1+\delta}} = \frac{1}{\delta}\sum_n \frac{1}{S_n(R_0)}\,\delta = \frac{\pi^2}{6\delta} < \infty \;\square$$

Fixing $\delta > 0$ we can certainly subsequence to obtain, $T_{f_n}(r_0) \geq n^{2/\delta}$, consequently we can find $r \in (r_0, 1)$ with,

$$I_n(r) \leq \frac{|\log r|\, T_{f_n}^{1+\delta}(r)}{(1-r^2)}$$

for all n. Since $T_{f_n}(r) \geq T_{f_n}(r_0)$ we obtain a closed current as claimed after suitable sub-sequencing. □

Final comment on I.2.5 The precise generality in which we will need I.2.5 occurs in the situation when the maps f_n arise from lifting maps $g_n : \Delta \to W$, for W a projective variety, i.e. there is a commutative diagramme,

$$\begin{array}{ccc} \mathcal{C}_n & \xrightarrow{f_n} & \mathcal{X} \\ {\scriptstyle p_n}\downarrow & & {\scriptstyle p}\downarrow \\ \Delta & \xrightarrow{g_n} & W \end{array}$$

with $g_n(0)$ bounded away from the ramification, for n sufficiently large, of the supposed proper, and generically finite map p. Under these hypothesis we can avoid the slightly thorny question of trying to prove a version of I.2.6 when p is complicated around the origin, and argue as in I.2.7 to bound the derivatives of the f_n around $p_n^{-1}(0)$ by choosing an appropriate projective embedding such that the coordinate hyperlanes restricted to étale neighbourhoods of $\mathcal{X}$ contain the ramification divisor. Plainly, we could, more generally, give a precise relation about how far the p_n ramification must be from the origin to make the proof work, but since we'll have no need of this, we'll ignore it. □

Consequently we obtain as weak limits of the $T_n(r)$, closed positive currents $T(r)$ for r outside of some set of finite hyperbolic measure. In principle, of course, we're likely to have some dependence on some subset(s) of $\mathbb{N}$ over which we subsequence. In practice, however, this is never a problem (*cf.* [M3]), so we'll tend to use the notation $T(r)$, or even just T, as if it were wholly unambiguous. Indeed, such is the lack of ambiguity, that we even have a sort of functoriality, *i.e.*

I.2.9 Claim. *Let $\sigma_m : \mathcal{X}_m \to \mathcal{X}$ be a countable sequence of proper maps, f_{nm} liftings of f_n, then we can define closed positive currents T_m on $\mathcal{X}_m$, such that for some constants λ_m,*

$$(\sigma_m)_* T_m = \lambda_m T\,.$$

The proof amounts to a rather trivial variant of I.2.8, i.e.

Triviality I.2.10. *Let S_{nk} be a doubly indexed sequence of increasing differentiable functions then for $r \in (0,1)$ outside a set of finite hyperbolic measure,*

$$S'_{nk}(r) \leq \{S_{nk}(r) + 2e\} \log^{n^2+k^2+1}\{S_{nk}(r) + 2e\}\,(1-r^2)^{-1}\,.$$

Proof. Indeed writing $F_{nk}(r) = S_{nk}(r) + 2e$, we have $\log F_{nk}(0) > 1$, and the set in question has measure bounded by,

$$\sum_{n,k} \int_{F_{nk}(0)}^{\infty} \frac{dx}{x \log^{n^2+k^2+1} x} = \sum_{n,k} \frac{1}{(n^2+k^2)(\log F_{nk}(0))^{n^2+k^2}} < \infty \qquad \square$$

It may of course happen that the λ_m are zero, but this is usually what we want to know, so that to all intents and purposes, we may without loss of generality deploy the λ_m in the normalisation of the T_m in order to assert,

$$(\sigma_m)_* T_m = T .$$

Indeed, consider for example the situation where we lift maps from discs to a variety X to a stack $\mathcal{X}$ with the same moduli, so that things fit into a diagramme,

$$\begin{array}{ccc} \mathcal{C}_n & \xrightarrow{f_n} & \mathcal{X} \\ \big\downarrow & & \big\downarrow \\ \Delta & \xrightarrow{f_n} & X \end{array}$$

then, what we have to check is that if $U_\alpha \to \mathcal{X}$ is an étale neighbourhood, and ω_α a non-negative $(1,1)$ form on it with compact support and for a suitable constant (depending on ω_α),

$$\oint\!\!\!\!\!\!\!\nabla_{\Delta(r)} f_n^* \omega_\alpha \ll \oint\!\!\!\!\!\!\!\nabla_{\Delta(r)} f_n^* c_1(\bar{H}) .$$

To this end, consider a smooth modification $\rho : \tilde{X} \to X$, such that ρ^{-1} of the branch locus of π is a smooth normal crossing divisor, and denote by $\tilde{\mathcal{X}}$ the normalisation of $\tilde{X}$ in $\mathcal{X}$. Let us further introduce the divisors E_i contracted by ρ, and note that for an appropriately large constant α,

$$\omega := \alpha\, c_1(\tilde{H}) - \sum_i dd^c \log\log^2 \|1\!\!1_{E_i}\|$$

is a complete metric on $\tilde{X} \backslash \underset{i}{\cup} E_i$, where, say, $\tilde{H}$ is an ample divisor on $\tilde{X}$ in a Fubini-Study type metric and $\|1\!\!1_{E_i}\|$ is the norm of the tautological section of $\mathcal{O}_{\tilde{X}}(E_i)$ in some norm. As such if $\tilde{f}_n$ denotes a lifting of f_n to $\tilde{\mathcal{X}}$, then indeed, independently of n,

$$\oint\!\!\!\!\!\!\!\nabla_{\Delta(r)} f_n^* \omega_\alpha \ll \oint\!\!\!\!\!\!\!\nabla_{\Delta(r)} \tilde{f}_n^* \omega .$$

So that it only remains to verify,

$$\oint\!\!\!\!\!\!\!\nabla_{\Delta(r)} \tilde{f}_n^* \omega \ll \oint\!\!\!\!\!\!\!\nabla_{\Delta(r)} f_n^* c_1(\bar{H}) .$$

This, however, follows immediately from the fact that the class in Pic of $\tilde{H}$ is that of H minus some exceptional $\mathbb{Q}$-divisor together with our hypothesis on $f_n(0)$ and the ubiquitous Jensen formula.

We can also arrange all of the various subsequencing so that,

I.2.11 Fact. *Again let $\sigma_m : \mathcal{X}_m \to \mathcal{X}$ be a countable sequence of proper maps, and $\mathcal{Y} \subset \mathcal{X}$ a proper closed substack such that $f_n(0)$ is bounded away from $\mathcal{Y}$ independently of n. Suppose in addition that everything is as before (*i.e. I.2.5 *and* I.2.9 *hold) then for any effective Cartier divisor $\mathcal{D}$ on some $\mathcal{X}_m$ whose support is contained in $\sigma_m^{-1}(\mathcal{Y})$,*

$$\mathcal{D}.\, T_m \geq 0 .$$

Evidently the current T serves as a "homological limit" of the maps f_n, which we can think of as a homology class of the data $\{f_n\ |_{\Delta_r}\colon n \in \mathbb{N}\}$, with $r \geq r_0$ appropriately chosen. In order to get a little feeling for the definitions and their properties we may consider the analogues for $\mathbb{P}^1$, and $\mathbb{C}$, *i.e.*

I.2.12 Parenthetical Remark 1. Suppose our interest is some set of closed 1-dimensional irreducible substacks $\mathcal{C}_n$ of $\mathcal{X}$ or better maps, $f_n : \mathcal{C}_n \to \mathcal{X}$, with $\mathcal{C}_n$ smooth. Manifestly there is a priori no issue about boundaries, so that we already have closed positive currents,

$$T_n : A^{1,1}(\mathcal{X}) \to \mathbb{C} : \tau \mapsto (H._{f_n}\mathcal{C}_n)^{-1} \int_{\mathcal{C}_n} f_n^* \tau$$

or perhaps with a further normalising factor $(\mathcal{C}_n : \mathbb{P}^1)$ on choosing maps to $\mathbb{P}^1$, although this would only really be appropriate for rational points over function fields (and even in the general disc case $(\mathcal{C}_n : \Delta)$ is always bounded in applications). In any case there is certainly some weak limit T of some subsequence, enjoying the same sort of "functoriality" that we find in I.2.9. Better still the condition of I.2.11, that $f_n(0)$ be bounded away from $\mathcal{Y}$, may be replaced by $\mathrm{Im}(f_n) \not\subset \mathcal{Y}$, under which condition, and the hypothesis of I.2.5, we obtain once more,

$$\mathcal{D}.\, T_m \geq 0\,.$$

Furthermore, if in addition the curves $\mathcal{C}_n$ were Zariski dense, and the σ_m birational, then we would even have,

$$\mathcal{D}.\, T_m \geq 0$$

for every m, and every effective Cartier divisor $\mathcal{D}$ on $\mathcal{X}_m$. The main thing here, however, to reflect on is that in this case we have an object of study which is wholly well defined, even in positive characteristic, yet the nature of the limit is rather analytic, which in turn poses the question of whether is there a better way of taking a limit of algebraic cycles. Certainly we could work at the level of Néron-Severi, but unsurprisingly this has a tendancy to loose information in a way that is not entirely desirable.

I.2.13 Parenthetical Remark 2. Consider, this time, a proper covering stack $p : \mathcal{C} \to \mathbb{C}$, together with a map $f : \mathcal{C} \to \mathcal{X}$. Obviously, provided things are defined on the boundary all of the previous discussion continues to make sense, where appropriate, for $r = 1$, and in this way we may introduce the currents $T_f(r)$, or if one prefers $T_n := T_f(r_n)$, $r_n \to \infty$ defined as,

$$T_f(r) : A^{1,1}(\mathcal{X}) \to \mathbb{C} : \tau \mapsto \left(\int\!\!\!\!\!\!\nabla_{\mathcal{C}(r)} f^* \, c_1(\bar{H}) \right)^{-1} \int\!\!\!\!\!\!\nabla_{\mathcal{C}(r)} f^* \, \tau\,,$$

where for $r \in \mathbb{R}$ now, $\mathcal{C}(r)$ is regarded as a covering of the unit dist by way of,

$$\begin{array}{ccc} \mathcal{C}(r) & \longrightarrow & \mathcal{C} \\ {\scriptstyle p}\big\downarrow & \square & \big\downarrow{\scriptstyle p} \\ \Delta & \longrightarrow & \mathbb{C} \\ z & \mapsto & rz \end{array}$$

As per our earlier remark, the closure of an appropriate limit of the $T_f(r_n)$, $r_n \to \infty$ outside a set of finite Lebesgue measure is much easier (and indeed more intuitive) than its disc analogue, while again, as for complete curves, the hypothesis on $f_n(0)$ used to guarantee I.2.11, can be replaced by $f(\mathcal{C}) \not\subset \mathcal{Y}$, while again Zariski denseness of $\mathrm{Im}(f)$ guarantees the positivity of intersection with effective Cartier divisors on any birational modification.

A final remark is that although metrics are necessary in the definition of how we intersect divisors and discs, even without the various closure properties that we've discussed, the role of exactly which metric is employed is almost invariably without importance. As such, lest it is important, which in fact is only the case for rather delicate phenomenon related to quantifying the degeneracy of the Kobayashi metric, it will invariably be dropped from the notation.

I.3 Compactness

Our starting point is,

I.3.1 Lemma *Let $f_n \in \mathrm{H}om(\Delta, \mathbb{P}^N)$ be a sequence of maps with continuous extensions to the closed unit disc $\bar{\Delta}$ with,*

$$\fint_{\bar{\Delta}} f_n^* c_1(H)$$

bounded uniformly in n, then for any $r < 1$ we can subsequence in such a way that the f_n may be written in suitable homogeneous coordinates as $[f_{n0}, \ldots, f_{nN}]$ where the f_{ni} converge for all i.

proof. Without loss of generality we may suppose the $f_n(0)$ accumulate somewhere, so say $1 = [1, \ldots, 1]$ in an appropriate projective coordinate system with respect to which we may metricise H by way of the Fubini-Study metric. Consequently if $X_i \in \Gamma(\mathbb{P}^N, H)$ is the equation of a coordinate hyperplane then modulo further subsequencing and enlarging r a little as necessary we can suppose that for each i, the $f_n^{-1}(X_i) \cap \Delta_r$, counted with multiplicity are a convergent sequence of 0 cycles in $\bar{\Delta}_r$ which are actually contained in $\bar{\Delta}_s$ for some $s < r$, which we'll write as,

$$f_n^{-1}(X_i) = \sum_{z \in \bar{\Delta}_s} a_{ni}(z)[z] \to \sum_{z \in \bar{\Delta}_s} a_i(z)[z]$$

where as per the proof o I.2.6, $a_{ni}(z) = 0$ for all $|z| \leq t$, with t independent of i and n. The disc being Stein we can write $f_n \mid_{\Delta_r}$, for an appropriate $1 > R > r$ as,

$$\left[\prod_{w \in \bar{\Delta}_s} (z-w)^{a_{n0}(w)}, \prod_{w \in \bar{\Delta}_s} (z-w)^{a_{n1}(w)} u_1, \ldots, \prod_{w \in \bar{\Delta}_s} (z-w)^{a_{nN}(w)} u_N \right]$$

where the $u_1, \ldots, u_N$ are units. Now consider the meromorphic functions g_{ni} defined as $f_n^* X_i / f_n^* X_0$ and apply Jensen's formula, i.e. for $z \in \Delta_r$,

$$\log |g_{ni}(z)| \cdot |P_{ni}(z)| = \int_{|w|=R} \log |g_{ni}(w)| \, \Re \left\{ \frac{w+z}{w-z} \right\} d^c \log |w|^2$$

where $P_{ni}(z)$ is the Blaschke product,

$$P_{ni}(z) = \prod_{w \in \bar{\Delta}_s} \left\{ \frac{R^2 - \bar{w}\, z}{R(z-w)} \right\}^{a_{ni}(w) - a_{n0}(w)}$$

Consequently employing the technique of I.2.6, or more correctly its proof, to bound the integrals over $|w| = R$ of $\log(1 + |g_{ni}|)$ and $\log\left(1 + \frac{1}{|g_{ni}|}\right)$ independently of n we conclude that for some constant C independent of n, $z \in \Delta_r$,

$$|u_{ni}|(z) \leq C \cdot \frac{(R^2 - rs)^a}{R^b}$$

for non-negative integers a, b independent of n determined by the signs of $a_i(w)$ - $a_0(w)$, so that the u_{ni} converge, and we're done. □

Despite appearances, as Brunella pointed out to me, this doesn't guarantee the convergence of f_n in the compact open topology. The classic example is,

$$f_n : \Delta \to \mathbb{P}^1 : z \to \left[z - \zeta, z - \zeta + \frac{1}{n} \right]$$

for any ζ in the disc punctured at the origin. Evidently this problem presents itself quite generally at any point in the common zero locus of the limit $f_0, \ldots, f_N$. On the other hand the maps f_n manifestly "converge", and so the only real question is in what space. The answer is provided by the Bishop-Gromov compactness theorem which we summarise according to our needs. In the first place we have,

I.3.2 Definition *A disc with bubbles Δ^b is a connected 1-dimensional analytic stack with singularities at worst nodes exactly one of whose components is the unit disc Δ, and every connected component R_z, $z \in R(\Delta^b) := \Delta \cap \operatorname{sing}(\Delta^b)$, of the complement of Δ is a tree of smooth rational stacks, i.e. admitting ètale covering by a 1-dimensional orbifold of positive topological euler characteristic.*

Certainly for $f : \Delta^b \to \mathcal{X}$ without a bubble at the origin, and $\overline{L}$ a metricised line bundle we can easily extend the intersection product by way of,

$$\not\!\!\int_{\bar{\Delta}^b} f^* c_1(\bar{L}) := \not\!\!\int_{\bar{\Delta}} f^* c_1(\bar{L}) + \sum_{\substack{z \in R(\Delta^b) \\ |z| < r}} \log \frac{r}{|z|} \int_{R_z} f^* c_1(L)$$

so that necessarily we have similar positivity properties for intersections with effective divisors to the intersection product over discs. In addition we can define a graph Γ_f as,

$$\Gamma_f = (\mathrm{id} \times f)\,(\Delta) \bigcup_{z \in R(\Delta^b)} z \times f(R_z) \subset \Delta \times \mathcal{X}\,.$$

Moreover the compact open topology on $\operatorname{Hom}(\Delta, \mathcal{X})$ is equivalent to the Hausdorff topology on the graphs of maps, so that quite naturally we can consider $f : \Delta^b \to \mathcal{X}$ as a boundary point if its graph is a limit of graphs Γ_{f_n}, i.e.

I.3.3 Definition *Say that $f_n \in \operatorname{Hom}(\Delta, \mathcal{X})$ converges to $f : \Delta^b \to \mathcal{X}$ if for $r \in (0,1) \backslash \{|z| : z \in R(\Delta^b)\}$ the graphs $\Gamma_{f_n} \times_\Delta \overline{\Delta}_r$ converge to $\Gamma_f \times_\Delta \overline{\Delta}_r$ in the Hausdorff metric.*

We can certainly add such boundary points to our Hom sets, to get a new space $\overline{\mathrm{Hom}}(\Delta, \mathcal{X})$ topologised by embedding it in the space of substacks of $\Delta \times \mathcal{X}$ in the Hausdorff metric over compact subsets of Δ. With this in mind a convenient formulation of Gromov's compactness theorem on metricising our ample as $\bar{H}$ so that $c_1(\bar{H})$ is strictly positive on $\mathcal{X}$ -i.e. compatible above and below with the restriction of a strictly positive metric from a smooth into which a small étale neighbourhood of $\mathcal{X}$ may be embedded- becomes,

I.3.4 Fact ([G2]) *Let* $C : (0, 1) \to \mathbb{R}_+$ *be bounded,* $K \subset \mathrm{Aut}(\Delta)$ *compact then,*

$$\{f \in \overline{Hom}(\Delta, \mathcal{X}) : \fint_{\bar{\Delta}(r)} \alpha^* f^* c_1(\bar{H}) \leq C(r), \alpha \in K\}$$

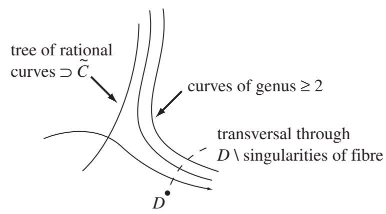
is compact. In particular $f_n \in \overline{Hom}(\Delta, \mathcal{X})$ *has a convergent subsequence iff for some subsequence* f_k*, radii* $r_m \to 1$ *and* $\alpha_k \in \mathrm{Aut}(\Delta)$ *convergent, the degrees of the* $\alpha_k^* f_k$ *at* r_m *are bounded independently of* k *for each* m*.*

Certainly we could appeal to the proof of [G2] to get this for Kähler stacks, but we can also get it by I.3.1 when $\mathcal{X}$ is a projective variety. More generally if we only have that the moduli X of $\mathcal{X}$ is projective then I.2.7 gives us honest, i.e. without bubbling, convergence in neighbourhoods of the origin, so that by I.3.4 on X together with [II] I.3.2, and the standard Gromov theorem of op. cit. for discs of bounded area we conclude to I.3.4.

As we've defined it as a subset of the set of substacks of $\Delta \times \mathcal{X}$ in the Hausdorff topology, $\overline{\mathrm{Hom}}(\Delta, \mathcal{X})$ is a minimal compactification of what may be termed bounded subsets of $\mathrm{Hom}(\Delta, \mathcal{X})$. Now certainly $\overline{\mathrm{Hom}}(\Delta, \mathcal{X}) = \mathrm{Hom}(\Delta, \mathcal{X})$ if $\mathcal{X}$ doesn't contain rational stacks but, conversely, we have,

I.3.5 Fact $\mathrm{Hom}(\Delta, \mathcal{X})$ *is a propersubset of* $\overline{\mathrm{Hom}}(\Delta, \mathcal{X})$ *if the smooth locus of* $\mathcal{X}$ *contains rational stacks.*

Proof. Taking hyperplane sections, it's clearly sufficient to treat the case of surfaces – actually even curves would do but we'd like to make examples of sequences of discs not wholly contained in the curve. Equally provided we make appropriately generic examples we can always blow them down so we might as well say C is a smooth rational stack in a smooth surface $\mathcal{X}$. Choose a point $\infty \in C$, then we can find an étale neighbourhood U of $C \backslash \infty$, such that $\mathcal{I}_C \otimes \mathcal{O}_U$ is generated by a single equation, x, say, which defines a pencil $x =$ constant. Passing to a modification $\rho : \widetilde{\mathcal{X}} \to \mathcal{X}$ we obtain a Lefschetz pencil. Denote by $\tilde{C}$ the proper transform of C, then for a sufficiently generic choice of pencil, the generic element is a hyperbolic curve, and some component D of the fibre containing $\tilde{C}$ minus the singularities is hyperbolic and does not blow down to a point under ρ. Around a chain of rational curves in the fibre containing $\tilde{C}$ as a component which without loss of generality we can suppose meets D our fibration looks like,



Consequently if we uniformise the hyperbolic stacks indicated in the figure by the disc with the origin centered on the indicated transversal then we obtain a sequence of discs that converges to a disc with bubbles on $\widetilde{\mathcal{X}}$, and by the particular choice of D this blows down to the same on $\mathcal{X}$. $\square$

Now suppose more generally that we're in a quasi-projective situation. The appropriate level of generality is to consider $\mathcal{X}\backslash\mathcal{B}$, for a pair $(\mathcal{X}, \mathcal{B})$ with log-canonical singularities (Mori sense) and the components of $\mathcal{B}$, $\mathbb{Q}$-Cartier divisors, although regardless of the singularities if we have a map from $\mathbb{P}^1$ or even just $\mathbb{A}^1$ to the smooth locus of $\mathcal{X}\backslash\mathcal{B}$ then I.3.5 goes through verbatim to show that $\mathrm{Hom}(\Delta, \mathcal{X})$ is a strict subset of $\overline{\mathrm{Hom}}(\Delta, \mathcal{X})$, while in the converse direction we may appeal to,

I.3.6 Lemma [G] *Let $f_n : Y \to \mathcal{X}\backslash\mathcal{B}$ be a sequence of maps converging uniformly on compact subsets of a Riemann surface to $f : Y \to \mathcal{X}$, then either, f maps to $\mathcal{X}\backslash\mathcal{B}$ or $\mathcal{B}$.*

Now necessarily if a sequence $f_n : \Delta \to \mathcal{X}$ with degrees uniformly bounded on a sequence of radii $r_k \to 1$ converges to a disc with bubbles, then for some $0 < |\zeta| < 1$, there are maps $\varphi_n : \Delta_{R_n} \to \{z \mid |z - \zeta| < \varepsilon_n\}$ with $R_n \to \infty$ and $\varepsilon_n \to 0$ such that $f_n \circ \varphi_n$ converges uniformly on compact subsets of a cover of $\mathbb{C}$ to a rational map $f : \mathcal{A} \to \mathcal{X}$ (cf. [Si]), where $\mathcal{A}$ is an affine stack, i.e. its moduli is $\mathbb{A}^1$, and its log-euler characteristic is positive. Whence if in fact the f_n are maps to $\mathcal{X}\backslash\mathcal{B}$ then either $\mathcal{X}\backslash\mathcal{B}$ or $\mathcal{B}$ must contain such affines. Better still if we write $\mathcal{B} = \Sigma\mathcal{B}_i$ in terms of its components then the line isn't just in $\mathcal{B}$ but $\mathcal{B}_i\backslash \bigcup_{j \neq i} \mathcal{B}_j$ for some i, and should this happen we have,

I.3.7 Fact *Notations as above, if $\mathcal{X}\backslash\mathcal{B}$ and $\mathcal{B}_i\backslash \bigcup_{j \neq i} \mathcal{B}_j$ do not contain any affine stacks then $\overline{\mathrm{Hom}}(\Delta, \mathcal{X}\backslash\mathcal{B}) = \mathrm{Hom}(\Delta, \mathcal{X})$.*

There are plenty of natural situations where the hypothesis that $\mathcal{B}_i\backslash \bigcup \mathcal{B}_j$, $j \neq i$ contains no lines is guaranteed, e.g. $\mathcal{X}$ a toric variety and $\mathcal{X}\backslash\mathcal{B}$ a torus, while for surfaces $\mathcal{S}$ we have the wholly general,

I.3.8 Fact/Definition (Log minimal model theorem) *Suppose $\mathcal{S}\backslash\mathcal{B}$ is not fibreed by lines then there is a contraction $\rho : (\mathcal{S}, \mathcal{B}) \to (\mathcal{S}', \mathcal{B}')$ within our given category of singularities with $\mathcal{S}\backslash\mathcal{B} = \mathcal{S}'\backslash\mathcal{B}'$ such that $\mathcal{B}'_i\backslash \cup \mathcal{B}'_j$, $j \neq i$, does not contain affine lines. Even better if $(\mathcal{S}, \mathcal{B})$ has general type then we may even insist that each $\mathcal{B}'_i\backslash \bigcup_{j \neq i} \mathcal{B}'_j$ is hyperbolic, in which case we say that the boundary $\mathcal{B}$ is* **hyperbolic**.

With these reflections in mind the proper reflection, "in finito", of some property such as the factorisation of entire maps to a projective variety through a Zariski closed set independent of the map is,

I.3.9 Definition (a) *For Z a proper substack of $\mathcal{X}$ we say $\mathcal{X}$ is hyperbolic modulo Z if a sequence of maps $f_n : \Delta \to \mathcal{X}$ of which a subsequence doesn't converge in $\overline{\mathrm{Hom}}(\Delta, \mathcal{X})$ is, in the compact open sense, contained in arbitrarily small neighbourhoods of Z.*

Observe that by I.3.5 the above definition is best possible. Indeed to fix ideas, say $\mathcal{X}$ is a surface then since everything which is Stein has a Stein neighbourhood at the very least Z must contain all the rational and elliptic curves, so in fact there must be a finite set of such. If there are only elliptic curves then we could replace $\overline{\mathrm{Hom}}$ by Hom, but this is probably a fairly special situation. In the quasi-projective situation the most generally valid definition is,

I.3.9 Definition (b) *Let $(\mathcal{X}, \mathcal{B})$ be a log-stack with projective moduli, and Z a proper substack of $\mathcal{X}\backslash\mathcal{B}$ then we say $(\mathcal{X}, \mathcal{B})$ is hyperbolic modulo Z if a sequence of maps $f_n : \Delta \to \mathcal{X}\backslash\mathcal{B}$ of which a subsequence doesn't converge in $\overline{\mathrm{Hom}}(\Delta, \mathcal{X})$ is, in the compact open sense, contained in arbitrarily small neighbourhoods of $Z \cup \mathcal{B}$.*

Again as we've seen this is best possible. On the other hand given that hyperbolicity necessitates general type at least for varieties, rather than foliations, we can propose a definition excluding the possibility of arbitrary proximity to $\mathcal{B}$, provided $\mathcal{B}$ is hyperbolic in the sense of I.3.8, i.e.

I.3.9 Definition (c) *Everything as per I.3.9(b), but suppose additionally that $\mathcal{B}$ is hyperbolic (I.3.8) then if f_n not enjoying a convergent subsequence obliges it to be arbitrarily close to Z, we say that $(\mathcal{X}, \mathcal{B})$ is complete hyperbolic modulo Z.*

One should bear in mind however that although such a compactification of $\mathcal{X}\backslash\mathcal{B}$ exists for surfaces, higher dimension seems anything but obvious. In any case the hyperbolicity of $\mathcal{X}$ – respectively $(\mathcal{X}, \mathcal{B})$-modulo Z leads to a natural metric, at least in the smooth case – respectively log-smooth – on $T_{\mathcal{X}}$, respectively $T_{\mathcal{X}}(-\log \mathcal{B})$, which we'll give in the more general log case, i.e.

I.3.10 Definition (Kobayashi-Royden) *Let $(\mathcal{X}, \mathcal{B})$ be a smooth projective variety, with simple normal crossing boundary, which is hyperbolic modulo Z then for $x \in \mathcal{X}\backslash\mathcal{B}\backslash Z$, and $t \in T_{\mathcal{X}}(-\log \mathcal{B}) \otimes \mathbb{C}(x)$,*

$$\|t\|_{\mathrm{Kob}} = \inf \left\{ \frac{1}{|R|} \mid f : (\Delta, 0) \to (\mathcal{X}\backslash\mathcal{B}, x) : f_* \left(\frac{\partial}{\partial z} \right) = Rt \right\} .$$

Now while the metric is certainly non-degenerate off Z under the hyperbolicity hypothesis, which motivates the same as Kobayashi's definition of hyperbolicity, hyperbolicity in the sense of I.3.9 implies more. Specifically,

I.3.11 Fact *Let $(\mathcal{X}, \mathcal{B})$ be hyperbolic modulo Z, then $\| \ \|_{\mathrm{Kob}}$ is continuous on the total space of $T_{\mathcal{X}}(-\log \mathcal{B})$ off $\mathcal{B} \cup Z$.*

proof. Upper semi-continuity is as ever just the property that everything Stein has a Stein neighbourhood. In the other direction, let $x \in \mathcal{X}\backslash\mathcal{B} \cup Z$, $x_n \to x$, and $(x_n, t_n) \to (y, t)$ in $T_{\mathcal{X}}(-\log \mathcal{B})$, with $(f_n)_* \left(\frac{\partial}{\partial z}\right) = R_n t_n$. Now f_n converges to a disc with bubbles, $f : \Delta^b \to \mathcal{X}$, which cannot bubble at the origin since $Z \cup \mathcal{B}$ necessarily contains the bubbling locus, and $x \notin Z \cup \mathcal{B}$, whence there is a fixed $\epsilon > 0$ such that the discs $f_n \mid_{\Delta_\epsilon}$ converge to $f \mid_{\Delta_\epsilon}$, thus if $f_* \left(\frac{\partial}{\partial z}\right) = R\partial$, then $\epsilon R_n \to \epsilon R$, and we're done. □

We will need one final preliminary, which for want of a better place we'll put here,

I.3.12 Lemma *Suppose a sequence $\{f_n\} \in \mathrm{Hom}(\Delta, \mathcal{X})$ is given which is not eventually arbitrarily close to a Zariski closed subset Z of $\mathcal{X}$, and that no subsequence of $\{f_n\}$ converges then after subsequencing there is a convergent sequence $\alpha_n \in \mathrm{Aut}(\Delta)$ such that $\mathrm{dist}(\alpha_n^* f_n(0), Z)$ and $\left\|(\alpha_n)_* (f_n)_* \left(\frac{\partial}{\partial z}\right)\right\| (0)$ are bounded away from zero independently of n.*

Proof. By hypothesis there is a disc of radius $r < 1$, with $f_n(\overline{\Delta}_r) \not\subset U$ for U

some open neighbourhood of Z, and $|df_n|_{\mathrm{euc}}$ large somewhere on $\overline{\Delta}_r$. Provided we simply take automorphisms that move the origin to somewhere in $\overline{\Delta}_r$ these will converge after subsequencing, so we might as well say that $|df_n|_{\mathrm{euc}}(0)$ is as large as we like. So consider a connected component U'_n of points in $\overline{\Delta}_r$ which are a suitable distance ϵ away from Z, which contains an open subset of points U''_n of points a distance 2ϵ away. Now take a small square in U''_n and extend it inwards as a rectangle until it touches U'_n (e.g. if the square is centered on the positive imaginary axis, the rectangle is given by taking the y coordinate as small as possible, remembering without loss of generality that $0 \notin U''_n$). This gives us a closed rectangle $\overline{R}_n$, say, in $\overline{\Delta}_r$ containing points u_n, v_n with $\mathrm{dist}(f_n(u_n), Z) \geq 2\epsilon$ and $\mathrm{dist}(f_n(v_n), Z) = \epsilon$. Equally we're done unless the derivative on $\overline{R}_n$ is very small, in particular bounded, so the same is true for the composite g_n of f_n with the obvious map $\overline{S} \to \overline{R}_n$ from the closed square of length 1. Denoting inverse image of this latter map by $\sim$ we can subsequence so that $g_n \to g$, $\tilde{u}_n \to u$, and $\tilde{v}_n \to v$, with of course $g_n\,(\tilde{u}_n) \to g(u)$ and $g\,(\tilde{v}_n) \to g(v)$, so g is non-constant. Take any point where dg is non-zero, and denote by z_n its image in R_n then $|d\,f_n(z_n)|_{\mathrm{euc}}$ is eventually bounded away from zero as required. □

I.4. Comparison of Conformal Structures

The minimal correct generality here is to consider smooth stacks $\mathcal{X}$ together with a simple normal crossing boundary $\mathcal{B}$. The natural conformal structure of the pair $(\mathcal{X}, \mathcal{B})$ amounts to a complete metric on $\mathcal{X}\backslash\mathcal{B}$, which in turn defines a metric on $T_{\mathcal{X}}(-\log \mathcal{B})$ with rather mild singularities, *i.e.* if on some étale neighbourhood $\mathcal{B}$ is defined by $x_1 \dots x_n = 0$ and $t = \lambda_i\, x_i\, \frac{\partial}{\partial x_i} + \mu_j\, \frac{\partial}{\partial y_j}$ with y_j the other coordinates, then,

$$\|t\|^2 \sim \sum_i |\lambda_i|^2 \log^{-2} |x_i|^2 + \sum_j |\mu_j|^2 \,.$$

In any case it's certainly sufficiently regular to permit us to metricise the tautological bundle L of $\mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{B}))$ in such a way that it makes sense to intersect the said metricised bundle, L^c, say, with maps from coverings of discs, provided that the latter don't factor through $\mathcal{B}$. Ironically, despite our previous protestations, this is a case where there is some importance in the metric (essentially because its singular rather than smooth) so we'll denote a smooth metric on the same by L without further need for notational precision. Supposing all of this, and introducing a sequence of maps $f_n : \mathcal{C}_n \to \mathcal{X}$ from covering stacks $p_n : \mathcal{C}_n \to \Delta$ of the disc satisfying the hypothesis that $f_n(0)$ is bounded away from the branching locus of $\pi : \mathcal{X} \to X$ over the moduli, we introduce an appropriate singular $(1,1)$ form ω on $\mathcal{X}$ which defines the metric on $T_{\mathcal{X}}(-\log \mathcal{B})$ and write,

$$f_n^*\,\omega = \frac{\|f'_n\|^2_{\log}}{|p'_n|^2}\, p_n^* dd^c |z|^2 = \frac{\|f'_n\|^2}{|p'_n|^2}\, p_n^*\, d\mu \,.$$

Now simply apply Jensen's formula to the logarithmic derivatives $f'_n : \mathcal{C}_n \to \mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{B}))$ to obtain,

I.4.1 Claim. *Suppose for simplicity f_n, p_n unramified over the origin, and $f_n(0) \not\subset \mathcal{B}$, then*

$$\fint_{\mathcal{C}_n(r)} f_n^* \, c_1(L^c) = -\log \frac{\|f_n'\|^2}{|p_n'|^2}(0) + \int_{\partial \mathcal{C}_n(r)} \log \frac{\|f_n'\|^2_{\log}}{|p_n'|^2} \, p_n^* \left(\frac{d\theta}{2\pi}\right)$$
$$+ \sum_{0<|p_n(c)|<r} \mathrm{ord}_c(\mathrm{Ram}_{p_n}) \log \left|\frac{r}{p_n(c)}\right|$$
$$+ \sum_{0<|p_n(c)|<r} \min\left\{\mathrm{ord}_c(f_n^* \, \mathcal{B}), \frac{1}{|\mathrm{Aut}(c)|}\right\} \log \left|\frac{r}{p_n(c)}\right|$$
$$- \sum_{\substack{0<|p_n(c)|<r \\ f_n(c)\notin\mathcal{B}}} \mathrm{ord}_c(\mathrm{Ram}_{f_n}) \log \left|\frac{r}{p_n(c)}\right|$$

where as ever ord_c *for* c *a closed geometric point is computed as the order function on an étale neighbourhood divided by* $|\mathrm{Aut}(c)|$.

To avoid long winded writing of complicated expressions let's give every term occuring in the sum a name. The term counting the p_n ramification is as per I.1.6, and equal to what we add to the Poincaré term to compute the discriminant so lets call it $\mathrm{discr}^{\#}_{\mathcal{C}_n}(r)$. The next term counts the intersection with $\mathcal{B}$ without multiplicity, so lets call it the radical, and write, $\mathrm{rad}_{f_n,\mathcal{B}}(r)$, while of course the last term counts what's left of the f_n ramification so we'll write $\mathrm{Ram}_{f_n}(r)$. To get a feeling for the term under the integral sign, we'll introduce the length in the complete metric, *i.e.*

$$\ell_{n,\log}(r) := \frac{1}{(\mathcal{C}_n : \Delta)} \int_{\partial\mathcal{C}_n(r)} \frac{\|f_n'\|^2}{|p_n'|^2} \, p_n^* \left(\frac{d\theta}{2\pi}\right)$$

where our conventions on relative degrees are as per [II] I.8. As such the Jensen type identity affords by way of the concavity of the logarithm,

I.4.2 Sub-Claim. *Everything as above,*

$$\log \frac{\|f_n'\|^2_{\log}}{|p_n'|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^* \, c_1(L^c) \leq (\mathcal{C}_n : \Delta) \log \ell_{n,\log}(r)$$
$$+ \mathrm{discr}^{\#}_{\mathcal{C}_n/\Delta}(r) + \mathrm{rad}_{f_n,\mathcal{B}}(r) - \mathrm{Ram}_{f_n}(r)$$

Notice, however, that notational improvements apart, I.4.2 is strictly less interesting than I.4.1, since we've now got a much less delicate measure of the "ramification of the boundary", whose exact measurement is critical to the difference between isoperimetric inequalities and curvature. Nevertheless what it is suggestive of is the basic observation of Nevanlinna theory, *i.e.* the lemma of the logarithmic derivative/tautological inequality,

I.4.3 Fact. *If we put* $S_n(r) = \fint_{\mathcal{C}_n(r)} f_n^* \, c_1(L^c)$ *then,*

$$\ell_{n,\log}(r) = \frac{1}{r(\mathcal{C}_n : \Delta)} \left(r \, S_n'(r)\right)'.$$

As such basic measure theory easily allows estimation, essentially independent of n, under hypothesis such as I.2.8 and I.2.11 with $\mathcal{B} \subset \mathcal{Y}$, and notations as per op. cit., of the form,

$$\ell_{n,\log}(r) \ll (1 - r^2)^{-1}$$

outside a set of finite hyperbolic measure. As such, in the spirit of I.2.8, albeit that one should bear in mind the obvious generalisations implicit in I.2.9/10 we have,

I.4.4 Example *Suppose that a sequence of maps $f_n : \Delta \to \mathcal{X}$ neither admits a convergent sequence nor do the origins $f_n(0)$ accumulate on $\mathcal{B}$ then after subsequencing,*

$$\log \frac{\|f_n'\|^2_{\log}}{|p_n'|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^* c_1(L^c) \leq_{\mathrm{exc}} \mathrm{discr}_{\mathcal{C}_n/\Delta}(r) + \mathrm{rad}_{f_n,\mathcal{B}}(r)$$
$$-\mathrm{Ram}_{f_n}(r) + o\left(\fint_{\mathcal{C}_n(r)} f_n^* c_1(H)\right)$$

where the subscript exc *denotes that r is excluded from a subset of the interval of finite hyperbolic measure.*

Unfortunately, there is a non-trivial difference between using the complete metric and the smooth metric. As it happens this is only of importance for computing rather delicate phenomenon about discs arbitrarily close to $\mathcal{B}$, but since we will investigate this let us observe,

I.4.5 Fact. *Denoting by $\| \ \|_{\log_c}$, $\| \ \|_{\log_{sm}}$ the complete and smooth metrics on $T_{\mathcal{X}}(-\log \mathcal{B})$ respectively, we have,*

$$\log \frac{\|f_n'\|^2_{\log_{sm}}}{|p_n'|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^* c_1(L) \leq \log \frac{\|f_n'\|^2_{\log_c}}{|p_n'|^2}(0) + \fint_{\mathcal{C}_n(r)} f_n^* c_1(L^c)$$
$$+ \log O\left(1, \fint_{\mathcal{C}_n(r)} f_n^* c_1(H), |\log f_n^* \| \mathrm{II}_{\mathcal{B}} \| (0)|\right)$$

So indeed, up to essentially, $\log\log^+ \|f_n^* \, \mathrm{II}_{\mathcal{B}}\| (0)$, there is no difference between working with L^c as opposed to L. Nevertheless examples do exist where this seemingly negligible term makes all the difference in the world, albeit that they require the f_n to be arbitrarily close to $\mathcal{B}$.

Anyway, be that as it may, the thing to retain from this discussion is,

I.4.6 Summary. *Given some bunch of maps $f_n : \mathcal{C}_n \to \mathcal{X}$ of unbounded degree we associate a limiting current T. Equally we may do the same for their logarithmic derivatives $f_n' : \mathcal{C}_n \to \mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{B}))$ and provided then $f_n(0)$ aren't too close to $\mathcal{B}$ we can do this so that the associated current T' satisfies a bound of the form,*

$$L.\, T' \leq \mathrm{d}iscr + \mathrm{r}ad - \mathrm{R}am$$

where disc, rad, ram are as per their counterparts in I.4.2 after dividing out by the degree with respect to an ample H. The right hand side may be thus be infinite, but that's inevitably even better than what we're trying to prove, so we may as well suppose that its finite and T' pushes forward to T.

II. REFINED TAUTOLOGY

II.1 Invariant curves and singularities

As a prelude to extending the previous, and essentially tautological, generalities on comparison of conformal structures to something rather more refined in the foliation context, we will require some knowledge of how invariant curves not wholly contained in the singular locus may meet the latter. To begin with our situation is completely local, *i.e.* if $\hat{\Delta}^n$ is formal affine space of dimension n, then we consider a derivation ∂ of $\mathcal{O}_{\hat{\Delta}^n}$ with a not necessarily isolated singularity at the origin, along with an invariant map, $f : \hat{\Delta}(= \hat{\Delta}^1) \to \hat{\Delta}^n$ not factoring through the singular locus. We suppose that the singularity of ∂ is not just log-canonical, but rather canonical, and introduce the Jordan decomposition $\partial_S + \partial_N$ of ∂ into its semi-simple and nilpotent part with respect to some appropriate coordinates $x_1, \dots, x_n$ which afford the standard representation of [II] I.7.1. Observe, in particular, that the naturality of the Jordan decomposition implies that f is not just invariant by ∂, but by both ∂_S and ∂_N. Invariance under a semi-simple field is, of course, rather easy to analyse, *i.e.*

II.1.1 Fact. *Let $\partial_S = \lambda_i \, x_i \, \frac{\partial}{\partial x_i}$ (summation convention, here, and throughout) be semi-simple with $f : \hat{\Delta} \to \hat{\Delta}^n$ ∂_S invariant, and $f^* x_1 \neq 0$, then after a re-ordering of coordinates, there is a coordinate z on $\hat{\Delta}$ such that,*

$$f(z) = (u_1 \, z^{e_1}, \dots, u_n \, z^{e_n}) \, , \; e_i \in \mathbb{N}$$

where $u_i \in \mathbb{C}$ is constant, and should it be non-zero then not only is $\lambda_i \neq 0$ but, $\lambda_i / \lambda_1 = e_i / e_1$.

Now let's keep the hypothesis and notation of the above fact, modulo re-ordering to obtain $\lambda_1 \leq \ldots \leq \lambda_m$, $f^* x_i \neq 0$, $1 \leq i \leq m$, $f^* x_i = 0$, $i > m$, for some $1 \leq m \leq n$, and proceed to an analysis which takes account of ∂_N. Necessarily ∂_N is supported on monomials of the form,

$$x_i \, x^Q \, , \; Q \in \mathbb{Z}^n_{\geq 0} \, , \text{ or } q_i = -1 \, , \; q_j \geq 0 \, , \; j \neq i \, , \text{ and in any case } \Lambda \cdot Q = 0 \, .$$

For such monomials, denote by $|Q|$ the integers k between 1 and n such that x_k occurs in $x_i \, x^Q$ to a positive power. Manifestly if $|Q| \cap \{m+1, \dots, n\} \neq \varphi$, then $f^*(x_i \, x^Q) = 0$. Furthermore for $1 \leq i \leq n$, and $Q \in \mathbb{Z}^n_{\geq 0}$, $\Lambda \cdot Q = 0$ is an impossibility without $|Q| \cap \{m+1, \dots, n\} \neq \varphi$, so indeed $f^*(x_i \, x^Q) = 0$ whenever $Q \in \mathbb{Z}^n_{\geq 0}$. As to the remaining cases, one notes that if $x_1, \dots, x_k$ are the non-zero under f^* functions with eigenvalue λ_1 then any $x_j \, x^Q$, $1 \leq j \leq k$, whose pull-back by f is non-zero, is of degree 1. Now consider a Jordan-block associated with λ_1, let's say,

$$\delta = (\lambda_1 \, y_1 + y_2) \, \frac{\partial}{\partial y_1} + \cdots + (\lambda_1 \, y_{r-1} + y_r) \, \frac{\partial}{\partial y_{r-1}} + \lambda_1 \, y_r \, \frac{\partial}{\partial y_r}$$

for some appropriate $\{y_1, \dots, y_r\} \subset \{x_1, \dots, x_n\}$, and suppse $1 \leq t \leq r$ is the largest integer such that $f^* y_t \neq 0$. Now for some meromorphic function $\varphi(z)$ we have, $y_t'(z) = \lambda_1 \, \varphi \, f^* \, y_t$, so in fact $\varphi(z) = \frac{e_1}{\lambda_1 z}$. As a result if $t > 1$, then, $\varphi \, f^* \, y_t = 0$, which is nonsense, and whence, $f^*(x_j \, x^Q) \equiv 0$ for any $1 \leq j \leq k$.

Moreover if we turn to the other eigenvalues, then we've established that $f_* \left(\frac{\partial}{\partial z}\right) = \frac{e_1}{\lambda_1 z} f^* \partial$, so that direct calculation yields $f^*\partial_N = 0$. Let's summarise by way of,

II.1.2 Fact. *Notations as per* [II] I.7.1, *but put* $\lambda_1 = 1$. *Furthermore let* $H(\Lambda)$ *be the big height in the standard* ℓ_1*-metric of the point* $\frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_M}{\lambda_1} \in \mathbb{A}^{M-1}(\mathbb{Q})$ *(including* $M = 1$*) where* $M \geq m$*, and* λ_i/λ_1 *are all the eigenvalues in* $\mathbb{Q}_{>0}$*, then,*

(i) *For any* $i > M$, $f^* x_i = 0$.

(ii) $f^*\partial_N \in f^* T_{\hat{\Delta}^n}$ *is identically zero.*

(iii) *The order of vanishing at the origin of* $f^*(x_1 \ldots x_M)$ *is at least* $H(\Lambda)$.

So to give an indication of where this is going, basically the point is that we're finding formal subvarieties which have a priori nothing to do with f, but in fact must vanish on it, or in a worst case scenario (viz: (iii)) vanish to high order unless Λ is some rather small rational point. We should, however, also consider the possibility that there is no such x_1, but this is precisely its own alternative, *i.e.*

II.1.3 Fact. *Things as before, but without the hypothesis on* $f^* x_1 \neq 0$*, nor the same after re-ordering, then in fact,*

(iv) $f^*\partial_S \in f^* T_{\hat{\Delta}^n}$ *is identically zero.*

Equally (iv) also holds if $f^* x_1 \neq 0$ but $\lambda_1 = 0$.

In order to subsequently profit from these observations we will require to understand how Jordan decomposition varies from point to point. Ultimately our discussion will be local for the étale topology in various formal neighbourhoods of the foliation singularities and so our set up is as follows: $\mathcal{O}$ will be a regular ring, complete in the adelic topology of an ideal I defined by coordinate functions $x_1, \ldots, x_m$ which in turn cut out a smooth connected subscheme Y of $\operatorname{Spec} \mathcal{O}$, and we'll even suppose that $\mathcal{O} = \mathcal{O}_Y[[x_1, \ldots, x_m]]$ with Y admitting an étale map to an affine space, so that there are functions $y_1, \ldots, y_n$ on Y such that dx_i, dy_j freely generate $\Omega_{\mathcal{O}}$. There will of course be a derivation ∂ of $\mathcal{O}$ with canonical singularities whose singular locus contains Y in a not necessarily isolated way. Furthermore we will suppose that the linearisation D of ∂ modulo I^2, which necessarily factors as,

$$D : \Omega_{\mathcal{O}} \otimes \mathcal{O}_Y \to I/I^2 \hookrightarrow \Omega_{\mathcal{O}} \otimes \mathcal{O}_Y$$

is already in Jordan normal form, so that indeed any nilpotent part is of the form, $x_i \frac{\partial}{\partial x_p}$ or $x_i \frac{\partial}{\partial y_q}$ for suitable i, p, q, and the eigenvalues $\lambda_1, \ldots, \lambda_m$ are well defined functions in $\mathcal{O}_Y$, having, without loss of generality, dx_i as the corresponding eigenfunctions under the semi-simple part. Unfortunately it will be necessary to distinguish several case, beginning with,

II.1.4 Case 1. *We single out a distinguished coordinate* x_1*, with everywhere non-zero eigenvalue* λ_1 *which is supposed an invariant hypersurface for* ∂*. Consequently we renormalise by a constant in order to suppose* $\lambda_1 = 1$*, and suppose that not all the remaining eigenvalues are in* $\mathbb{C}$*. As such let's say,* $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$*, and* $\lambda_{k+1}, \ldots, \lambda_m \in \mathcal{O}_Y$ *are non-constant functions for some* $1 \leq k < m$.

In this situation it will not be possible to produce a Jordan decomposition in $\mathcal{O}$ stricta dictum. However, what we will aim for is,

II.1.5 Definition/Warning. *In this context by an I-adic Jordan decomposition (or should it be necessary to be more precise, a Jordan decomposition secundum quid) of ∂ we will mean a decomposition of the form $\partial' + \partial''$ where for some functions ξ_i, η_j in an appropriate formal localisation $\mathcal{O}_{\{S\}}$, to be determined, in the directions normal and parallel to Y respectively,*

$$\partial' = \lambda_i \, \xi_i \, \frac{\partial}{\partial \xi_i} \, , \; \partial'' = a_{iQ} \, \xi_i \, \xi^Q \, \frac{\partial}{\partial \xi_i} + a_{jP} \, \xi^P \, \frac{\partial}{\partial \eta_j}$$

with $a_{iQ}, a_{jP} \in \mathcal{O}_{Y,S}$, $P \in \mathbb{Z}^m_{\geq 0}$, conventions on Q as before, and in any case $\Lambda \cdot Q = \Lambda \cdot P = 0$ in $\mathcal{O}_Y$. Although similar in appearance to Jordan decomposition ∂', ∂'' may not commute except in the directions normal to Y.

Bearing in mind that things are already Jordan modulo I^2, let's consider in more detail what's involved in this process of Jordanisation secundum quid. Manifestly we proceed inductively and suppose that we've found appropriate coordinates x_i, y_j modulo I^p, and look for a solution modulo I^{p+1} by way of a perturbation,

$$\tilde{x}_i = x_i + c_{iQ} \, x_i \, x^Q \, , \; \tilde{y}_j = y_j + c_{jP} \, x^P$$

with $c_{iQ}, c_{jP} \in \mathcal{O}_Z$, and the $x_i \, x^Q$ (possibly $q_i = -1$), x^P the obvious monomials in $x_1, \dots, x_m$ of degree p. Explicit computation shows that we require to solve linear equations of the form,

$$Lc = b$$

for $b \in I^p/I^{p+1} \otimes T_{\mathcal{O}}$, c_* as above, and $L \in \mathrm{End}(I^p/I^{p+1} \otimes T_{\mathcal{O}})$ some linear operator. Better still the mod I^2 decomposition shows that if $L_S + L_N$ is the Jordan decomposition of L into semi-simple and nilpotent parts, then L_S has eigenvectors, $x_i \, x^Q \otimes \frac{\partial}{\partial x_r}$, $x^P \otimes \frac{\partial}{\partial y_t}$ for suitable i, Q, r, P, t, with eigenvectors $\Lambda \cdot Q$ and $\Lambda \cdot P$ respectively. As such the said equations are solvable after localising by S_p, where S_p is the multiplicative set generated by the $\Lambda \cdot Q$ and $\Lambda \cdot P$ which are not identically zero in $\mathcal{O}_Y$, and which correspond to the appropriate monomials $x_i \, x^Q, x^P$.

Now the unfortunate thing is that a solution of such equations is not unique, albeit that at worst in the normal direction, say, any two solutions would differ by a power series in the $x_i \, x^Q$, $\Lambda \cdot Q = 0$. As such if we start with any two sets of normal coordinates, $\tilde{x}_i$, x_i, say, which agree modulo I^2, and proceed with the above Jordanisation algorithm to Jordan coordinates $\tilde{\xi}_i, \xi_i$ then,

$$\tilde{\xi}_i = \xi_i + \sum_Q a_{iQ} \, \xi_i \, \xi_Q \, , \; \Lambda \cdot Q = 0$$

with the $a_{iQ} \in \mathcal{O}_Y$.

We should, however, investigate the relation between topological and algebraic localisation, *i.e.* if S is the smallest (and, by the way, necessarily topologically non-nilpotent) multiplicative set containing all the S_p, then all of this is only defined in the formal localisation $\mathcal{O}_{\{S\}}$. As such it could perfectly well happen that the smallest ideal of $\mathcal{O}$ containing a Jordan coordinate $\xi_i \in \mathcal{O}_{\{S\}}$ is $\mathcal{O}$ itself. On the other hand for each such i, and $p \in \mathbb{N}$, there is a well defined Zariski closure, J_{ip}, say, in $\mathcal{O}$ of $\xi_j + I^p$ considered as an ideal of the usual localisation $\mathcal{O}_{S_p}$. In order to get to grips with what J_{ip} looks like around a closed point 0 of Y, let us begin by further supposing that $\mathcal{O}$ is $\mathfrak{m} = \mathfrak{m}(0)$-adically complete. As such we introduce Jordan coordinates stricta dictum x_i, y_j, and let ξ_i^p, η_j^p be their I-adic counterparts obtained by following the above procedure up to order p, then we further assert,

II.1.6 Claim. *Notations as above, then for any i, j, there are $s_i, s_j \in S_p$ and functions f_i, f_j of the form $\sum a_{iQ}\, x_i\, x^Q$, $\Lambda \cdot Q(0) = 0$, respectively $\sum a_{jP}\, x^P$, $\Lambda \cdot P(0) = 0$, $a_{iQ}\, a_{jQ} \in \mathcal{O}_Z$, such that,*

$$\xi_i^p = x_i + f_i/s_i \,,\ \eta_j^p = y_j + f_j/s_j \,.$$

Proof. We, of course, proceed inductively since everything is true for $p = 1$ by virtue of our set up. Now write, for example,

$$\partial\, \xi_i^p = \partial\, x_i + \frac{\partial f_i}{s_i} - f_i\, \frac{\partial s_i}{s_i^2}$$

and consider the I-adic expansion after localising by S_p. Certainly ∂x_i is $\lambda_i\, x_i$ plus a series of the form $\sum a_{iQ}\, x_i\, x^Q$, but the same is also true of ∂f_i, as one sees by explicit computation of expressions of the form,

$$x_j\, x^P\, \frac{\partial}{\partial x_j}\, (a_{iQ}\, x_i\, x^Q)\,,\ \text{and}\ x^R\, \frac{\partial}{\partial y}\, (a_{iQ}\, x_i\, x^Q)$$

where as ever $\Lambda \cdot P(0) = \Lambda \cdot R(0)$, and we permit the possibility of $p_j = -1$. As a result the element b of $I^p/I^{p+1} \otimes T_{\mathcal{O}}$ for which we require to solve $Lc = b$ is supported on elements of the form $x_i\, x^Q \otimes \frac{\partial}{\partial x_r}$, respectively $x^P \otimes \frac{\partial}{\partial y_t}$, from which we conclude. □

With this out of the way, we can quickly move to a conclusion by way of,

II.1.7 Fact. *Let things be as in Case 1, with no further hypothesis on $\mathcal{O}$ beyond I-adic completeness. Furthermore for $2 \le i \le m$, and $p \in \mathbb{N}$ let J_{ip} be the ideal of the Zariski closure of $\xi_i + I^p$ where $\xi_1, \ldots, \xi_m$ are I-adic Jordan coordinates secundum quid. Observe also that there is a perfectly well defined $J_{1p} = \xi_1 + I^p$ independently for any closure considerations by virtue of the aforesaid hypothesis. Finally, and bearing in mind that $\lambda_1 = 1$, for $y \in Y$ a closed point put $M_y \subset \{1, \ldots, m\}$ to be the set of indices such that $\lambda_i \in \mathbb{Q}_{>0}$, and denote by $H(\Lambda(y))$ the big height computed in the standard ℓ_∞-metric of $(\lambda_i)_{i \in M_y} \in \mathbb{A}^{M_y}(\mathbb{Q})$ for $M_y \ge 1$, and ∞ otherwise, then for $f : \hat{\Delta} \to \operatorname{Spf} \hat{\mathcal{O}}_{Y,y}$ (completion in $\mathfrak{m}(y)$) an invariant curve,*

$$\operatorname{ord}_0(f^{-1}\, J_{1p} \cap \ldots \cap J_{mp}) \ge \min\{p, H(\Lambda(y))\}\,.$$

Proof. Since the Zariski closure could only decrease under $\mathfrak{m}(y)$-adic completion, we may, more or less suppose that $\mathcal{O}$ is $\mathfrak{m}$-adically complete. We say more or less since we have to keep in mind the fact that the ξ_i are a priori given. However starting from an $\mathfrak{m}$-adic Jordan basis stricta dictum, x_i, y_j, we can Jordanise secundum quid to $\tilde{\xi}_i, \tilde{\eta}_j$, so that for S_p, S etc. as before we have the relations,

$$\tilde{\xi}_i = x_i + \sum a_{iQ}\, x_i\, x^Q,\ \Lambda \cdot Q(y) = 0\,,\ \xi_i = \tilde{\xi}_i + \sum b_{iP}\, \tilde{\xi}_i\, \tilde{\xi}^P,\ \Lambda \cdot P = 0 \in \mathcal{O}_Y$$

where $a_{iQ} \in \mathcal{O}_{Y,S}$, the $b_{iP} \in \mathcal{O}_Y$, and the monomials $x_i\, x^Q, \tilde{\xi}_i\, \tilde{\xi}^P$ which appear are, without loss of generality, supposed to have degree at least 2. Consequently for some $c_{iQ} \in \mathcal{O}_{Y,S}$ we obtain,

$$\xi_i = x_i + \sum c_{iQ}\, x_i\, x^Q\ ,\ \Lambda \cdot Q(y) = 0\,.$$

Consequently if $g \in J_{ip}$, $i \geq 2$, then modulo I_Y^p it is of the form, $d_i\, x_i + \sum d_{iQ}\, x_i\, x^Q$, $\Lambda \cdot Q(y) = 0$, where now $d_i, d_{iQ} \in \mathcal{O}_Y$. Consequently for e_i as per II.1.1, we obtain,

$$\mathrm{ord}_0(f^* g) \geq \min\{e_i, p\}$$

and the result follows. □

Fortunately the remaining cases are far more straightforward. We begin with,

II.1.8 Case 2. *Everything as per case 1, but with all the eigenvalues constant.*

From the arguments of the previous case, we see that there is no difficulty in finding I-adic Jordan coordinates $\xi_i, \eta_j \in \mathcal{O}$ stricta dictum, *i.e.* we have over the whole of $\mathcal{O}$ a decomposition $\partial = \partial_S + \partial_N$, where,

$$\partial_S = \lambda_i\, \xi_i\, \frac{\partial}{\partial \xi_i}\ ,\ \partial_N = a_{iQ}\, \xi_i\, \xi^Q\, \frac{\partial}{\partial \xi_i} + a_{jP}\, \xi^P\, \frac{\partial}{\partial \eta_j}$$

with as ever $P \in \mathbb{Z}^m_{\geq 0}$, much the same for Q excepting the possibility $q_i = -1$, and of course $\Lambda \cdot Q = \Lambda \cdot P = 0$, with $[\partial_S, \partial_N] = 0$. In particular for $y \in Y$ a closed point, on replacing the η_j by $\tilde{\eta}_j = \eta_j - \eta_j(y)$, we get an $\mathfrak{m}(y)$-adic Jordan decomposition of ∂. In addition as before we have ideals $J_{ip} = \xi_i + I^P$, although, fortunately these are defined over the whole of $\mathcal{O}$, and we distinguish sub-cases/sub-facts by way of,

II.1.9 Fact. *Everything as above, but suppose that not all the eigenvalues are in $\mathbb{Q}_{>0}$, then for $f : \hat{\Delta} \to \mathrm{Spf}\, \hat{\mathcal{O}}_{Y,y}$ a formal invariant curve through a closed point $y \in Y$,*

$$\mathrm{ord}_y(f^{-1}\, J_{1p} \cap \ldots \cap J_{mp}) \geq p\, \mathrm{ord}_y(f^{-1}\, I)\,.$$

There remains, of course, the sub-case that all of the eigenvalues are in $\mathbb{Q}_{>0}$. This could only happen, however, if Y were a generically isolated component of $\mathrm{sing}\,(\partial)$. Better still by [II] I.6.12, $\partial_N \neq 0$ (otherwise the singularity would not be canonical) consequently if J_p^{nil} is the ideal generated by the coefficients of $\partial_N(\xi_i)$, $1 \leq i \leq m$, and I_Z^p then J_p^{nil} is a proper ideal of $\mathcal{O}$ for $p \gg 0$, and by II.1.2, we have,

II.1.10 Fact bis. *Everything as per II.1.2(ii), but with all the eigenvalues in $\mathbb{Q}_{>0}$, then,*

$$\mathrm{ord}_y(f^{-1} J_p^{\mathrm{nil}}) \geq p\, \mathrm{ord}_y(f^{-1} I)\,.$$

The remaining case is, of course,

II.1.11 Case 3. *We continue to single out a distinguished coordinate x_1 corresponding to an invariant hypersurface for ∂, but suppose that it is nilpotent, i.e. $\frac{\partial x_1}{x_1} \in I$. The remaining non-zero eigenfunctions $\lambda_2, \ldots, \lambda_k$, say, may or may not be constant.*

As such we may need to face the problem of only having an I-adic Jordan decomposition secundum quid, not to mention the formal localisation issue. Nevertheless we're interested in a much simpler formal sub-scheme, W, say, cut out by $(\xi_2, \ldots, \xi_k)$ in $\mathcal{O}_{\{S\}}$ for ξ_i Jordan coordinates. As ever let J_W^p be the Zariski closure of $(\xi_2, \ldots, \xi_k) + I^P$, then it follows immediately by what we've seen in the discussion of case 1, that $J_W^p \subset J_{W(y)}^p$ where $y \in Y$ is a closed point and $W(y)$ is the sub-scheme of $\hat{\mathcal{O}}_{Y,y}$ ($\mathfrak{m}(y)$-adic completion) defined by the coefficients of the semi-simple part of the $\mathfrak{m}(y)$-adic jordan decomposition stricta dictum. Consequently on combining with II.1.2, we arrive to,

II.1.12 Fact. *Things as per case 3, with $f : \hat{\Delta} \to \operatorname{Spf} \hat{\mathcal{O}}_{Y,y}$ a formal invariant curve through a closed point $y \in Y$ such that $f^* x_1 \neq 0$, then,*

$$\operatorname{ord}_y(f^{-1} J_W^p) \geq p \operatorname{ord}_y(f^{-1} I) .$$

Before abandonning this topic, let us note,

II.1.13 Reason for the above discussion. Ultimately we'll wish to estimate the "size" of intersection of invariant curves with the singular locus, and what will lead to non-trivial estimates is the above series of facts which more or less say that such a curve must factor through a formal subscheme of dimension strictly smaller than the ambient dimension.

II.2 Counting

Our set up is as follows: $(\mathcal{X}, \mathcal{B}, \mathcal{F})$ is a foliated smooth stack with simple normal crossing non-invariant boundary $\mathcal{B}$, whose singularities are supposed canonical rather than just log-canonical. We will also suppose the existence of an invariant simple normal crossing divisor $\mathcal{E}$ which contains every component of $\operatorname{sing}(\mathcal{F})$. Notice that this latter hypothesis is very much without loss of generality, *i.e.* given $(\mathcal{X}, \mathcal{B}, \mathcal{F})$ satisfying the initial hypothesis then by [II] I.6, $\mathcal{B}$ is disjoint from the singular locus of $\mathcal{F}$. Furthermore by algorithmic resolution we can, by way of a sequence of blow ups in smooth invariant centres, find a modification $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ such that $\rho^{-1}(\operatorname{sing} \mathcal{F})$ is an invariant simple normal crossing divisor, with $K_{\tilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$, and of course $\rho^{-1}(\operatorname{sing} \mathcal{F}) \supset \operatorname{sing}(\tilde{\mathcal{F}})$. As a result $\mathcal{B} + \mathcal{E}$ is a simple normal crossing divisor, and we have a short exact sequence of the form,

$$0 \to \mathcal{N} \to \Omega_{\mathcal{X}}(\log \mathcal{B} + \mathcal{E}) \to K_{\mathcal{F}}(\mathcal{B}) \cdot I_{\mathcal{Z}} \to O$$

where $\mathcal{Z}$ is some closed invariant sub-stack contained in $\operatorname{sing}(\mathcal{F})$, and $\mathcal{N}$ a reflexive sheaf. Consequently if $f : \mathcal{C} \to \mathcal{X}$ is an invariant map from a proper covering stack $p : \mathcal{C} \to \Delta$, not factoring through $\mathcal{B}$ or $\mathcal{E}$ then we may apply I.4.2/5 to conclude,

II.2.1 Fact. *Suppose everything as above, then:*

$$\begin{aligned} \log \frac{\|f'\|_{\log}}{|p'|}(0) + \fint_{\mathcal{C}(r)} c_1(K_{\mathcal{F}} + \mathcal{B}) \leq{} & (\mathcal{C} : \Delta) \log \ell_{\log}(r) + \operatorname{discr}^{\#}_{\mathcal{C}/\Delta}(r) \\ & + \operatorname{rad}_{f, \mathcal{B}+\mathcal{E}}(r) + s_{f,\mathcal{Z}}(r) + 0(1) \\ & + 0 \left(\fint_{\mathcal{C}(r)} c_1(H), \log \log^+ f^* \| \mathrm{I\!I}_{\mathcal{B}+\mathcal{E}} \|(0) \right) \end{aligned}$$

where we compute everything with respect to smooth metrics, and the new term, i.e. *the Segre class of f around $\mathcal{Z}$ is the Nevanlinna counterpart of [II]* I.8.6*, or more precisely let $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow up in $\mathcal{Z}$, $\mathcal{E}_{\mathcal{Z}}$ the exceptional divisor, and $\tilde{f}$ the lifting of f, then $s_{f,\mathcal{Z}}(r) = \fint \tilde{f}^* c_1(\mathcal{E}_{\mathcal{Z}})$, so multiplicty in the sense of [IV] V.1.3.*

Now the things that we want to try and get rid of in the right hand side are $\mathrm{rad}_{f,\mathcal{E}}(r)$, and $s_{f,\mathcal{Z}}(r)$. There is of course an immediate improvement we can make, viz:

II.2.2 Further Fact. *Hypothesis as before then,*

$$\mathrm{rad}_{f,\mathcal{E}}(r) = \mathrm{rad}_{f,\mathrm{sing}(\mathcal{F})}(r)$$

where the radical of f around $\mathrm{sing}(\mathcal{F})$ *is computed in the obvious way by way of blowing up* $\mathrm{sing}(\mathcal{F})$.

Proof. $\mathcal{E}$ is invariant by $\mathcal{F}$.

Naturally we're going to try and make use of the previous sections, which leads us to pose:

II.2.3 Sub-Problem (of a purely scheme like nature). *Let X be a projective variety, H ample on X, Y a subvariety of X and $\mathfrak{V}$ a formal subscheme of the formally formal (*cf. [M4]*) generic point of the completion of X in Y. As such for each $p \in \mathbb{N}$, there is a Zariski neighbourhood U_p of Y, and subscheme V_p of $\mathcal{O}_{U_p/I_Y^p}$, or equivalently an ideal I_V of $\varprojlim_p \frac{\mathcal{O}_{X,Y}}{I_Y^p}$. In any case for $s, t \in \mathbb{N}$, $t \geq s$, we have ideals $I_V^s + I_Y^t$ with well defined Zariski closure, say, $I_{V,Y}^{(s,t)}$ in $\mathcal{O}_X$, and we wish to estimate, for $r \in \mathbb{N}$,*

$$h^0(X, H^r \cdot I_{V,Y}^{(s,t)}) .$$

For the purposes of this calculation, and indeed elsewhere, we'll use the symbol $\ll$ to denote less than or equal up to a linear sum in irrelevant constants. It will also be convenient, in the beginning, to fix r, s and let t be $s + n$, $n \in \mathbb{N}$. In any case for $k \in \mathbb{N} \cup \{0\}$, we certainly have a short exact sequence,

$$0 \to I_{V,Y}^{(s,s+k+1)} \to I_{V,Y}^{(s,s+k)} \to Q_{V,Y}^{(s,k)} \to 0$$

where by definition $Q_{V,Y}^{(s,k)}$ is nothing other than the quotient, with a more precise investigation of the same being only momentarily postponed. This apart, the standard exact sequence for H^0 certainly gives,

$$h^0(X, H^r \cdot I_{V,Y}^{(s,s+n)}) \geq h^0(X, H^r \cdot I_Y^s) - \sum_{k=0}^{n-1} h^0(X, H^r \otimes Q_{V,Y}^{(s,k)})$$

while a similar argument profiting from the short exact sequence,

$$0 \to I_Y^{k+1} \to I_Y^k \to I_Y^t / I_Y^{k+1} \to 0$$

eventually yields,

$$\begin{aligned} h^0(X, H^r I_{V,Y}^{(s,s+n)}) &\geq h^0(X, H^r) \\ &\quad - \sum_{k=0}^{s-1} h^0\left(X, H^r \cdot \frac{I_Y^{k-1}}{I_Y^k}\right) - \sum_{k=0}^{n-1} h^0(X, H^r \otimes Q_{V,Y}^{(s,k)}) . \end{aligned}$$

In order to proceed to a conclusion, we cut Y by $\dim Y$ very general hyperplanes under the further hypothesis that H is very ample. This gives exact sequences of the form,

$$0 \to H^i \otimes Q_V^{(s,k)} \to H^{i+1} \otimes Q_{V,Y}^{(s,k)} \to H^{i+1} \otimes Q_{V\cap H, Y\cap H}^{(s,k)} \to 0$$

$$0 \to H^i \otimes I_Y^k / I_Y^{k+1} \to H^{i+1} \otimes I_Y^k / I_Y^{k+1} \to H^{i+1} \otimes I_{Y\cap H}^k / I_{Y\cap H}^{k+1} \to 0$$

etc., so that ultimately it suffices to understand the case where Y is zero dimensional, or more precisely a finite number of reduced points. Certainly, however, we have estimates of the form,

$$h^0(X, H^i \otimes Q^{(s,k)}_{V\cap H_1 \cap \ldots \cap H_{\dim Y}, Y\cap H_1\cap\ldots\cap H_{\dim Y}}) =$$
$$h^0(X, Q^{(s,k)}_{V\cap H_1 \cap \ldots \cap H_{\dim Y}, Y\cap H_1\cap\ldots\cap H_{\dim Y}}) \ll s^{\dim X - \dim V}\,(s+k)^{\dim V - \dim Y - 1}$$

and

$$h^0\left(X, H^i \otimes \frac{I^k_{Y\cap H_1\cap\ldots\cap H_{\dim Y}, H_1\cap\ldots\cap H_{\dim Y}}}{I^{k+1}_{Y\cap H_1\cap\ldots\cap H_{\dim Y}, H_1\cap\ldots\cap H_{\dim Y}}}\right)$$
$$= h^0\left(X, \frac{I^k_{Y\cap H_1\cap\ldots\cap H_{\dim Y}, H_1\cap\ldots\cap H_{\dim Y}}}{I^{k+1}_{Y\cap H_1\cap\ldots\cap H_{\dim Y}, H_1\cap\ldots\cap H_{\dim Y}}}\right) \ll k^{\dim X - \dim Y - 1}$$

Notice in addition that we've been rather course with much of our estimation. In particular we don't actually need to bound above, $h^0(X, H^i \otimes Q_{V,Y}^{(s,k)})$, or for that matter the equivalent object after cutting by a hyperplane, but only things such as,

$$\dim\left\{\mathrm{Im}(H^0(X, H^i \otimes I_{V,Y}^{(s,k)}) \to H^0(X, H^i \otimes Q_{V,Y}^{(s,k)}))\right\}$$

or more generally,

$$\dim\left\{\mathrm{Im}(H^0(X, H^i \otimes I_{V\cap\Lambda,Y\cap\Lambda}^{(s,k)}) \to H^0(X, H^i \otimes Q_{V\cap\Lambda,Y\cap\Lambda}^{(s,k)}))\right\}$$

for Λ an intersection of very general hyperplanes. As such when applying the standard exact sequence and truncating, we can always pretend that if $\dim Y \cap \Lambda > 0$, $h^0(X, Q^{(s,k)}_{V\cap\Lambda, Y\cap\Lambda})$" $= 0$", where inverted comma means: in so far as it's important to our calculation. Consequently, we eventually obtain,

$$h^0(X, H^r \cdot I_{V,Y}^{(s,t)}) - h^0(X, H^r)$$
$$\gg -r^{\dim Y}(s^{\dim X - \dim Y} + s^{\dim X - \dim V} t^{\dim V - \dim Y})$$

Now let us apply these observations to the computation of the radical. Specifically,
II.2.4 Fact. *Let $\varepsilon > 0$, then there is a proper closed substack $\mathcal{Z}_\varepsilon$ of $\mathcal{X}$ such that for $f : \mathcal{C} \to \mathcal{X}$ an invariant map from a covering $p : \mathcal{C} \to \Delta$ of the disc which doesn't factor through $\mathcal{E}$,*

$$\mathrm{rad}_{f,\mathcal{E}}(r) \leq \varepsilon \fint_{\mathcal{C}(r)} f^* c_1(H) + O_\varepsilon(\log \mathrm{dist}\,(f(0), \mathcal{Z}_\varepsilon))$$

where as ever H is an ample bundle, on the, supposed projective, moduli X.

Proof. Fix some rational $\delta > 0$ to be chosen a posteriori. In addition stratify $\mathrm{sing}\,(\mathcal{F})$ by finitely many substacks $\mathcal{Y}_i$ and projections $\psi_i : \mathcal{Y}_i \to \mathbb{P}^{n_i}$ in such a way that every closed geometric point $y \in \mathrm{sing}\,(\mathcal{F})$ is contained in the open subset of some $\mathcal{Y}_i$ where,

(a) $\mathcal{Y}_i$ is smooth, and indeed ψ_i is étale.

(b) The roots of the characteristic polynomial of the linearisation of $\mathcal{F}$ around $\mathcal{Y}_i$ (*i.e.* the map $\Omega_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{Y}_i} \to \Omega_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{Y}_i}(K_{\mathcal{F}})$, defined pre II.1.4) define an étale covering.

(c) The Jordan decomposition of the map of (b) on the said étale neighbourhood is constant, or more correctly the dimension of the eigenspaces of the semi-simple part and the nilpotent part are constant.

Furthermore introduce a subindex set, I', say, of the set of such $\mathcal{Y}_i$'s, according to the rule, $i \in I'$ iff the eigenvalues (equivalently the characteristic polynomial) are non-constant, considered as elements of the appropriate projective space. We also let N denote a large number to be chosen, and consider the substacks of some appropriate étale neighbourhood of $\mathcal{Y}_i$ defined by,

$$\lambda_2 = e_2, \ldots, \lambda_M = e_m$$

provided there is some component $\mathcal{E}_1$ of $\mathcal{E}$ containing $\mathcal{Y}_i$, such that for x_1 a local equation of the same, and ∂ a local generator of $\mathcal{F}$, $\frac{\partial x_1}{x_1}$ is non-zero, so that $\lambda_2, \ldots, \lambda_M$ are the non-constant eigenvalues normalised in such a way that $\partial x_1 = x_1$ and $e_2, \ldots, e_M \in \mathbb{A}^M(\mathbb{Q}_{>0})$ is a positive rational point of big height at most N/δ. This introduces around $(N/\delta)^{M+1}$ additional stacks $\mathcal{Y}_{ij}$ which we further stratify to ensure (a), (b), (c) on each of them. We may also assume, and for convenience we shall, that we have such a stratification $\mathcal{Y}_i$, $i \in I'$, of $\mathrm{sing}\,(\mathcal{F}) \backslash \mathcal{Z}$, $I' \subset I$ as above, and $\mathcal{Y}_k$, $k \in K$, of $\mathcal{Z}$ in turn, with $\mathcal{Z}$ as defined prior to II.2.1.

We now introduce appropriate formal subschemes around the generic points of the various $\mathcal{Y}_i$'s, $\mathcal{Y}_{ij}$'s, $\mathcal{Y}_k$'s etc. as dictated by the analysis of II.1. In particular,

II.2.5 Case 1. *For $i \in I'$, take $\mathcal{V}_{i\alpha}$ to be the formal substack of the completion in $\mathcal{Y}_i$ at its generic point defined by a Jordan coordinate, in the sense of II.1.5, ξ_α in the normal direction to $\mathcal{Y}_i$.*

II.2.6 Case 2. *For $i \in I \backslash I'$, or $i \in I'$ and $\mathcal{Y}_{ij}$, $j \in J_i$, say, arising from the further stratification, again take $\mathcal{V}_{i\alpha}$ respectively $U_{ij\alpha}$ to be the formal substack defined by a Jordan coordinate unless all the eigenvalues are generically positive rational, where we take an appropriate component of $\mathcal{E}$ as the means to obtain a distinguished eigenfunction x_1.*

II.2.7 Case 2 bis. *If in fact all the eigenvalues are rational, with things as per Case 2, take $\mathcal{V}_i$, respectively $\mathcal{V}_{ij}$, to be cut out by the coefficients of the nilpotent part of the Jordanisation.*

II.2.8 Case 3. *$k \in K$, so in fact $\mathcal{Y}_k \subset \mathcal{Z}$, then take $\mathcal{W}_k$ to be the formal substack of the completion in $\mathcal{Y}_k$ defined generically by the vanishing of the coefficients of the semi-simple part of the Jordanisation.*

With these various choices, and extending our notations from the scheme like discussion of II.1, in the obvious way, we consider sections of,

$$H^0\left(\mathcal{X}, \pi^* H^p \otimes \bigcap_{\substack{i\alpha \\ i \in I'}} I^{(q,r)}_{\mathcal{U}_{i\alpha}, \mathcal{Y}_i} \bigcap_{\substack{i\alpha \\ i \notin I'}} I^{(1,s)}_{\mathcal{V}_{i\alpha}, \mathcal{Y}_i} \bigcap_{\substack{ij\alpha \\ i \in I'}} I^{(1,t)}_{\mathcal{V}_{ij\alpha}, \mathcal{Y}_{ij}} \bigcap_{\substack{k \\ k \in K}} I^{(1,u)}_{\mathcal{W}_k, \mathcal{Y}_k}\right)$$

whose dimension we'll denote $h(p, r, s, t, u)$. As such our previous dimension count, or more precisely codimension count, yields,

$$
\begin{aligned}
&h(p,r,s,t,u) - h^0(X, H^p) >> \\
&-\sum_{i\in I'} p^{\dim \mathcal{Y}_i}(q^{\dim\mathcal{X}-\dim\mathcal{Y}_i} + r^{\dim\mathcal{X}-\dim\mathcal{Y}_i-1}) - \sum_{i\in I\setminus I'} p^{\dim\mathcal{Y}_i}\, s^{\dim\mathcal{X}-\dim\mathcal{Y}_i-1} \\
&-\sum_{k\in K} p^{\dim\mathcal{Y}_k}\, u^{\dim\mathcal{W}_k-\dim\mathcal{Y}_k} - \sum_{i\in I'}\sum_{j\in J_i} p^{\dim\mathcal{Y}_{ij}}\, t^{\dim\mathcal{X}-\dim\mathcal{Y}_{ij}-1}
\end{aligned}
$$

We take, in consequence, $q = \delta p$, for p sufficiently divisible, so that necessarily we take $r = M_p$, for M about, $\delta^{-\frac{1}{\dim\mathcal{X}-\dim\mathcal{Y}_i-1}}$. Equally we can safely take s, u to be of the order $p^{1+\eta}$, for $\eta > 0$, depending only on $\dim\mathcal{X}$, and we can even do the same for t provided N, δ are fixed, say of orders about $1/\varepsilon$ and $\varepsilon^{\dim\mathcal{X}}$ respectively.

In particular with these kind of choices, $h(p, r, s, t, u) > 0$ (in fact even of order $p^{\dim\mathcal{X}}$) while for γ a global section of the corresponding H^0, and $c \in \mathcal{C}$ a closed point,

$$\mathrm{ord}_c(f^*\gamma) \geq p/\varepsilon \min\{\mathrm{ord}_c(f^{-1}\mathrm{sing}\,\mathcal{F}), 1\}$$

and whence the assertion by way of the ubiquitous Jensen formula. □

Notice also that in the particular example of case 3, the above discussion yields much more. Indeed for $(\mathcal{X}_i, \mathcal{F}_i)$ a foliated stack, with $\mathcal{E}_i, \mathcal{B}_i$ as before, let us call $\mathcal{Z}_i$ of an exact sequence such as that prior to II.2.1, the locus where things are not log-flat. Now blow up in $\mathcal{Z}_i$, and call the resulting stack $\tilde{\mathcal{X}}_i$ with $\mathcal{X}_{i+1}$ a resolution of the ideal $\mathcal{I}_{Z_i}$ by a sequence of blow ups in smooth invariant centres, which necessarily exists by algorithmic resolution, while $\mathcal{E}_{i+1}$ will be the reduced pull-back of $\mathcal{E}_i$, and $\mathcal{B}_{i+1}$ the proper transform of $\mathcal{B}_i$. Necessarily if we take $\mathcal{X} = \mathcal{X}_0$, then we obtain a tower of stacks of the form

$$
\begin{tikzcd}
 & & & {} \arrow[dl, "\rho_{23}"'] & \\
 & & (\mathcal{X}_2, \mathcal{F}_2) \arrow[dl, "\rho_{12}"'] \arrow[d, "\tau_{21}"] & {} \arrow[l, "\sigma_2"] \\
 & (\mathcal{X}_1, \mathcal{F}_1) \arrow[dl, "\rho_{01}"'] \arrow[d, "\tau_{10}"] & (\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{F}}_1) \arrow[l, "\sigma_1"] & \\
(\mathcal{X}, \mathcal{F}) = (\mathcal{X}_0, \mathcal{F}_0) & (\widetilde{\mathcal{X}}_0, \widetilde{\mathcal{F}}_0) \arrow[l, "\sigma_0"] & &
\end{tikzcd}
$$

Now the remaining term in that we have not so far got into the shape that we desire (*i.e.* small and/or intrinsic to $\mathcal{C}$) is the $s_{f,\mathcal{Z}}(r)$ term, and even here the nature of case 3 is such that we already know,

II.2.9 Intermediary Fact. *Let $\varepsilon > 0$ be given, then there is a proper closed sub-stack $\mathcal{W}_\varepsilon \supset \mathcal{Z}$ such that $s_{f,\mathcal{Z}}(r)$ is bounded by,*

$$\int_{\partial\mathcal{C}(r)} \log\frac{\|f^*\mathbb{I}_\mathcal{Z}\|}{\|f^*\mathbb{I}_\mathcal{Z}\|(0)}\, p^*\left(\frac{d\theta}{2\pi}\right) + \varepsilon \fint_{\mathcal{C}(r)} f^*c_1(H) + O_\varepsilon(\log\|f^*\mathbb{I}_{\mathcal{W}_\varepsilon}\|(0))$$

*where for $\mathcal{Y} \subset \mathcal{X}$ a closed substack, $1\!\!1_{\mathcal{Y}}$ is the tautological section of the exceptional divisor on the blow up in $\mathcal{Y}$, and we'll use the standard notation $m_{f,\mathcal{Z}}(r)$ for the 1*st *integral on the right,* i.e. *the so called proximity function.*

Proof. Indeed this is an immediate and minor variation of II.2.4, where, evidently in the notations of the proof of the same, we look at sections of,

$$H^0\left(\mathcal{X}, \pi^* H^p \bigcap_{k \in K} \mathcal{I}^{(1,t)}_{\mathcal{W}_k, \mathcal{Y}_k}\right)$$

which is certainly non-zero for t of order $p^{1+\eta}$, $\eta > 0$ depending only on dim $\mathcal{X}$.

In order to remove the last offending term, *i.e.* the proximity around $\mathcal{Z}$, we could, of course, repeat the arguments of II.1 at "infinity". On the other hand this involves just a little bit more analysis than we're at home to, so instead let us observe that for $\mathcal{X}' \to \mathcal{X}$ a blow up in a smooth centre contained in $\mathcal{Z}$, and $\mathcal{Z}'$ the new locus where things are not log-flat the proper transforms $\mathcal{W}'_k$, $k \in K$, or more correctly their union, contain at every closed point $z' \in \mathcal{Z}'$ the formal subscheme of $\mathcal{X}'$ completed in z' cut out by the Jordan coordinates corresponding to non-zero eigenfunctions of the semi-simple part of a local generator of the foliation. As such given a section of,

$$s \in H^0\left(\mathcal{X}, \pi^* H^p \bigcap_{k \in K} \mathcal{I}^{(1,t)}_{\mathcal{W}_k, \mathcal{Y}_k}\right)$$

with as ever t around $p^{1+\eta}$, and $p \gg 0$ to be chosen, there is some modification $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$, with $\tilde{\mathcal{X}}$ one of the $\mathcal{X}_N$'s prior to II.2.9 (with N probably about t, but this is unimportant) such that,

$$\mathrm{ord}_{\tilde{\mathcal{Z}}}(\rho^* s) \gg t$$

with as ever $\tilde{\mathcal{Z}}$ the new non-log flat locus. Consequently if we let $\tilde{\ell}_{\log}(r)$ be the length of some lifting $\tilde{f}$ of r, then finally we arrive to our goal,

II.2.10 Main Fact. *Let everything be as above (in particular, and critically, the singularities of $(\mathcal{X}, \mathcal{B}, \mathcal{F})$ are not just log-canonical, but canonical) then for $\varepsilon > 0$ there is a closed proper substack $\mathcal{Z}_\varepsilon$ of $\mathcal{X}$ and an invariant modification $\rho : \tilde{\mathcal{X}}_\varepsilon \to \mathcal{X}$ (with in fact $\tilde{\mathcal{X}}_\varepsilon$ some $\mathcal{X}_N$ as above) such that for $f : \mathcal{C} \to \mathcal{X}$ any invariant map from a covering $p : \mathcal{C} \to \Delta$ of the disc, the following isoperimetric type inequality holds,*

$$\begin{aligned}\log\left\|\frac{\tilde{f}'}{p'_{\log}}\right\|(0) + \fint_{\mathcal{C}(r)} f^* c_1(K_{\mathcal{F}} + \mathcal{B}) \leq (\mathcal{C} : \Delta)\log \tilde{\ell}_{\log}(r) + \mathrm{discr}^{\#}_{\mathcal{C}/\Delta}(r) \\ + \mathrm{rad}_{f,\mathcal{B}}(r) + \varepsilon \fint_{\mathcal{C}(r)} f^* c_1(H) \\ + O_\varepsilon(1, \log \mathrm{dist}(f(0), \mathcal{Z}_\varepsilon))\end{aligned}$$

II.3 Schematic Interpretation

Let us close this section by translating the main fact II.2.10 into schematic language. It will be convienient to suppose that $(\mathcal{X}, \tilde{\mathcal{B}}, \mathcal{F})$ is generically scheme like.

Naturally we note the moduli as $(X, \mathcal{B}, \mathcal{F})$, and, of course, we have our old friend the Gorenstien cover $\pi : (\mathcal{X}', \mathcal{B}', \mathcal{F}') \to (X, \mathcal{B}, \mathcal{F})$. We are now a priori interested in invariant maps $f : C \to X$, where $p : C \to \Delta$ is an everywhere space like proper covering of the disc. We have of course,

$$K_{\tilde{\mathcal{F}}} + \mathcal{B} = \pi^*(K_{\mathcal{F}} + \mathcal{B}') = \pi^*(K_{\mathcal{F}} + B)$$

where the latter is interpreted as the pull-back of a $\mathbb{Q}$-Cartier divisor in the usual way. Furthermore, we can/will suppose that f doesn't factor through the non-scheme like locus, so that the normalisation of an appropriate irrecubile component be it of $C \times_X \mathcal{X}'$, or $C \times_X \mathcal{X}$ affords a lifting of f from a smooth satck $\mathcal{C}'$ or $\mathcal{C}$ with moduli C. As such identifying a geometric point c of $\mathcal{C}'$ with the same on C we have identities such as,

$$\mathrm{ord}_{\mathcal{C}',c}(f^*\mathcal{B}') = \mathrm{ord}_{C,c}(f^*B)$$

where the former is understood as the order function calculated in $\mathcal{C}'$, *i.e.* divide by the stabiliser, and the latter is the standard order function extended to $\mathbb{Q}$-Cartier divisors. Consequently we define,

$$\mathrm{rad}_{f,B}(r) := \sum_{0<|p(r)|<r} \min\left\{\frac{1}{|\mathrm{A}ut_{\mathcal{X}'}(f(c))|}, \mathrm{ord}_c(f^*\, B)\right\} \log\left|\frac{r}{z}\right|$$

so that in fact, $\mathrm{rad}_{f,B}(r) = \mathrm{rad}_{f,\mathcal{B}'}(r)$. It may happen, however, that $\mathrm{discr}^{\#}_{\mathcal{C}'/\Delta}(r)$ is bigger than $\mathrm{discr}^{\#}_{C/\Delta}(r)$. In principle there could also be a similar difficulty at infinity, but this is already wholly dealt with by I.4.5, so it remains to control the difference between the said finite parts of the appropriate discriminants. This is, however, no worse than a radical type term around the non-scheme like points, and should these also be singularities of $\mathcal{F}$ we've completely dealt with this. Unsurprisingly what remains is rather easy, *i.e.* if $\mathcal{Y}$ in $\mathcal{X}'$ is irreducible, and generically non-scheme like yet meeting the smooth locus of $\mathcal{F}$, we can define, cf. [II] II.1, a formal substack $\mathcal{W}$ of the completion of $\mathcal{X}'$ in $\mathcal{Y}$ which is generically obtained by adding in the formal curves in the foliation direction. Subsequently we may easily estimate the difference in discriminats by way of sections of

$$H^0(\mathcal{X}', \pi^* H^p \cdot (\mathcal{I}_{\mathcal{W}} + I^t_{\mathcal{Y}}))$$

with as ever t around $p^{1+\eta}$, $\eta > 0$, in order to supplement what we would missing for a Noetherian induction in the spirit of II.1-2. Consequently, we arrive to,

II.3.1 Variant. *Things as above, and let $\varepsilon > 0$, then there is a proper closed subvariety Z_ε of X, and a modification $\rho : \tilde{X}_\varepsilon \to X$ (manifestly the moduli of that in* II.2.10*) such that,*

$$\begin{aligned} \log\left\|\frac{f'}{p'}\right\|_{\log}(0) + \fint_{C(r)} f^*c_1(K_{\mathcal{F}} + B) &\leq \log\tilde{\ell}_{\log}(r) + \mathrm{discr}^{\#}_{C/\Delta}(r) \\ &\quad + \mathrm{rad}_{f,B}(r) + \varepsilon \fint_{C(r)} f^*c_1(H) \\ &\quad + O_\varepsilon(1, \log \mathrm{dist}(f(0)\, Z_\varepsilon)) \end{aligned}$$

We can tie our considerations of the unicity of the minimal model in [II] IV.11 together with the above value distribution theory rather nicely in the case that our foliated variety has general type. Indeed if $(\mathcal{X}, \mathcal{F})$ is of minimalist Mori category, then the uniqueness lemma IV.11.3 guarantees that its numerical Kodaira dimension is well defined, i.e.

II.3.2 Definition. *If $(\mathcal{X}, \mathcal{F})$ is a foliated stack whose branched moduli $(X, B, \mathcal{F})$ is minimalist Mori category and is not a foliation in conics, then its numerical Kodaira dimension $\nu(\mathcal{F})$ is given by,*

$$\max, 0 \le i \le \dim \mathcal{X}, \text{ such that } K_{\mathcal{F}}^i \neq 0$$

where, here, $K_{\mathcal{F}}^i$ is viewed as a numerical equivalence class.

and, if this is maximal, i.e. $(\mathcal{X}, \mathcal{F})$ is of *general type*, equal to the actual Kodaira dimension. As such II.2.10, or better II.3.1, guarantees

II.3.3 Example *Let $f_n : \Delta \to \mathcal{X}$, or, $f_n : \Delta \to X$ a sequence of invariant discs with origins, i.e. the $f_n(0)$'s , bounded away from some algebraic set, determined by the base locus of $|K_{\mathcal{F}}|$ and its tensor powers, then some subsequenceconverges to a disc with bubbles.*

III. SURFACES

III.1 Foliated Surfaces

By a foliated surface we will mean a 2-dimensional algebraic stack $\mathcal{X}$ with projective moduli X, and needless to say a foliation by curves $\mathcal{F}$. It follows from [M2] that as soon as the latter has canonical $\mathbb{Q}$-gorenstein foliated singularities, then we may suppose that the former is smooth. Unsuprisingly the small dimension of this example means that we can give a rather better description of the linear systems occuring in II.2.3. This is most conveniently done in terms of blowing up, $(\mathcal{X}, \mathcal{D}, \mathcal{F})$, where $\mathcal{D}$ is the set of algebraic curves through the singularities of $\mathcal{F}$. To this end we digress on saddle nodes, i.e. singularities where the implied scheme structure is non-reduced. In a formal étale neighbourhood with coordinates x, y, say, the foliation is given by a field $\partial = x(1+\nu y^{e-1}) \frac{\partial}{\partial x} + y^e \frac{\partial}{\partial y}$ for some $e \geq 2$ and $\nu \in \mathbb{C}$. The so called weak branch given by $x = 0$ determines a formal germ at the origin, and after blowing up once one should blow up again in the intersection of the exceptional divisor with the proper transform of the weak branch, repeating this process e times to obtain the minimal resolution. Now given $(\mathcal{X}, \mathcal{D}, \mathcal{F}) = (\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$ we blow up in the singularity if $\mathcal{D}$ does not contain all the branches. This gives us a modification $\mathcal{X}_1$ of $\mathcal{X}$ with induced foliation $\mathcal{F}_1$ and a new boundary $\mathcal{D}_1$, which is the proper transform of $\mathcal{D}$ together with the exceptional curves. Now just iterate the whole procedure to get a sequence of modifications $(\mathcal{X}_k, \mathcal{D}_k, \mathcal{F}_k)$, denote by π_{mk} the projection from the m^{th}-modification to the k^{th}, for $m > k$. Over a singularity z of $\mathcal{X}_{k-1}$, let $\mathcal{E}_{kzj}$ be the basis of exceptional curves between $\mathcal{X}_k$ and $\mathcal{X}_{k-1}$ contracted to z such that $\mathcal{E}_{kzi} \cdot \mathcal{E}_{kzj} = -\delta_{ij}$. Evidently if z has no nilpotent structure there is just one such curve, otherwise there are $e(z)$ where e is the degree of the node, and we order these according to the natural way that they appear in the resolution process described above. It is convenient to put $\mathcal{E}_{kz} = \sum_{j=1}^{e(z)} \mathcal{E}_{kzj}$. Observe, in the spirit of II.1 that the lifting of f to $\mathcal{X}_k$ can only intersect the boundary at a singularity where there are not two algebraic branches and $k \geq 2$ call thesesingularities bad, then we can be rather more precise about the ideals intervening in II.1-2, which are none other than,

$$(\pi_k)_* \prod_{z \in sing(\mathcal{F}_{k-1})_{\mathrm{bad}}} \mathcal{O}_{\mathcal{X}}(-\mathcal{E}_{kz})$$

Consequently, we can considerably refine II.2.3 by way of,

III.1.1 Lemma *Let $(\mathcal{Y}, \mathcal{F})$ be a foliated non-singular surface with canonical singularities and b a formal branch of $\mathcal{F}$ through a canonical singularity z through which there is not an algebraic invariant curve. Let $\mathcal{Y}_1$ be obtained by blowing up $\mathcal{Y}_0 = \mathcal{Y}$ in z, with b_1 the proper transform of the branch $b_0 = b$, $\mathcal{Y}_2$ from $\mathcal{Y}_1$ by blowing up the crossing point of b_1 with the exceptional divisor etc., and $\mathcal{E}_k$ the exceptional curve on $\mathcal{Y}_k$ blown down by $\rho_{k,k-1}$ (ρ_{km} from $\mathcal{Y}_k$ to $\mathcal{Y}_m$, $m < k$ being the projection) then for H ample there is a positive rational constant α such that for all k, every sufficiently large and divisible multiple of $\alpha\,[\sqrt{k}]^{-1} \rho_{k0}^* H - \mathcal{E}_k$ is generated by its global sections outside the exceptional curves other than $\mathcal{E}_k$.*

proof. For any integer $m > 0$, let us consider the bundle $F_{k,m}$ on $\mathcal{Y}_k$ defined by,

$$F_{k,m} = m\,\rho_{k0}^*\,H - \sum_{i=1}^{k} \rho_{ki}^*\,\mathcal{E}_i$$

then $F_{k,m}^2 = m^2\,H^2 - ck > 0$ for $m \geq \sqrt{k}$, and c a constant depending on the monodromy of $\mathcal{Y}$ over its moduli. Further $\rho_{k0}^*\,H \cdot F_{k,m} = m\,H^2 > 0$, so $F_{k,m}$ is certainly big. To get appropriate global generation requires an understanding of the intersection of $F_{k,m}$ with various curves. We firstly consider the exceptional ones, so let $\tilde{\mathcal{E}}_{ki}$ be the proper transform of $\mathcal{E}_i$ on $\mathcal{Y}_k$, $i \leq k$, and note:

$$-\mathcal{E}_i \cdot (\rho_{ki})_*\,(\tilde{\mathcal{E}}_{kj}) = \begin{cases} > 0\,, & j = i \\ < 0\,, & j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

where the implied positive, respectively negative, numbers of the first two possibilities, have modulus bounded independently of k. Consequently, $F_{k,m} \cdot \tilde{\mathcal{E}}_{kj} = 0$ for $j < k$, and $F_{k,m} \cdot \mathcal{E}_k = \text{const} > 0$, it i.e. depending only on $\mathcal{Y}$. Next let $\tilde{C}$ be the proper transform of a curve C on $\mathcal{Y}$ then,

$$\tilde{C} = \rho_{k0}^*\,C - \sum_i a_i\,\rho_{ki}^*\,\mathcal{E}_i$$

where, $a_i = \tilde{C} \cdot \rho_{ki}^*\,\mathcal{E}_i = \mathcal{E}_i \cdot (\rho_{ki})_*\,\tilde{C} \in \mathbb{Q} \cup \{0\}$. Now suppose C is not invariant by the foliation. The canonical bundle of $\mathcal{F}$ on $\mathcal{Y}_k$ is precisely that of $\mathcal{F}$ on $\mathcal{Y}_0$, as ever say $K_{\mathcal{F}}$, so:

$$(\rho_{k0}^*\,K_{\mathcal{F}} + \tilde{C}) \cdot \tilde{C} = K_{\mathcal{F}} \cdot C + C^2 - \sum_i a_i^2 \geq 0\,.$$

While we also have,

$$F_{k,m} \cdot \tilde{C} = m\,H \cdot C - \sum_i a_i\,.$$

Not surprisingly we control $\sum_i a_i$ by way of the usual ℓ_1 versus ℓ_2 inequality, and the index theorem. Specifically:

$$\sum a_i \leq \sqrt{k}\left(\sum a_i^2\right)^{1/2} \leq \sqrt{k}\left(K_{\mathcal{F}} \cdot C + \frac{(H \cdot C)^2}{H^2}\right)^{1/2} \leq \alpha' \sqrt{k}\,H \cdot C$$

for a suitable constant α'. So that, $F_{k,m} \cdot \tilde{C} \geq (m - \alpha' \sqrt{k})\,H \cdot C$. Finally consider the possibility that $\tilde{C}$ is invariant. By our hypothesis there is at most one such and it is transverse to the given branch so for some $d \in \mathbb{N}$, idependent of m and , k

$$F_{k,m} \cdot \tilde{C} = m\,H \cdot C - \frac{1}{d} \geq (m-1)\,H \cdot C\,.$$

Consequently for an appropriate $\alpha > \alpha'$ and $\tilde{C}$ a non-exceptional curve, we have for $m = \alpha\,[\sqrt{k}]$, $F_{k,m} \cdot \tilde{C} > 0$, with $F_{k,m} \cdot \mathcal{E} = 0$ if and only if $\mathcal{E}$ is one of the exceptional curves $\tilde{\mathcal{E}}_{ki}$ for $i \neq k$. Evidently the curves $\underset{i<k}{\cup} \tilde{\mathcal{E}}_{ki}$ may be contracted to a $\mathbb{Q}$-factorial singularity so sufficiently large divisible multiples of $F_{k,m}$ are generated by global sections, while for a possibly slightly larger α,

$$\alpha\,[\sqrt{k}]^{-1}\,H - \mathcal{E}_k = \frac{1}{k}\,F_{k,m} + \sum_{i=1}^{k-1} \frac{i}{k}\,\tilde{\mathcal{E}}_{ki}\,. \qquad \square$$

Putting everything together, while bearing in mind [II] I.1.11-14, we obtain,

III.1.2 Example *On a foliated log-surface $(\mathcal{X}, \mathcal{B}, \mathcal{F})$ with canonical singularities, II.2.10 holds in the following strengthened, and explicit form,*

- *The modifications $\rho : \mathcal{X}_\epsilon \to \mathcal{X}$, are infact the above $\mathcal{X}_k$ with k of order ϵ^{-2}.*
- *The exceptional substack $\mathcal{Z}_\epsilon$ of op. cit., is effectively independent of ϵ, and indeed equal to the total transform of the invariant algebraic curves in $\mathcal{X}$ through the singularities.*

Idem for the schematic variant II.3.1.

As an application, we may consider the hyperbolicity of $(\mathcal{X}, \mathcal{F})$ in the obvious variation of I.3.9 with empty boundary divisor modulo the divisor $\mathcal{D}$ of invariant curves through the singularities, i.e. we consider only maps from the disc invariant by the foliation, and we have,

III.1.3 Claim *Let $(\mathcal{X}, \mathcal{F})$ be a foliated surafce with canonical singularities, $K_{\mathcal{F}}$ nef., with $\mathcal{D}$ the invraiant curves through the singularities, then if $K_{\mathcal{F}}.K_{\mathcal{X}} > 0$, $(\mathcal{X}, \mathcal{F})$ is hyperbolic modulo $\mathcal{D}$.*

proof Combining the example with I.2.5-11, we find on supposing that the assertion is false that there is a closed positive current, actually a transverse invariant measure $d\mu$ arising as the limit of bigger and bigger discs such that,

$$K_{\mathcal{F}}.d\mu \leq 0$$

whence indeed equal to zero since $K_{\mathcal{F}}$ is nef. Now we can appeal to Siu's decomposition theorem, [S], or the more generally valid [IV] VI.3, to write $d\mu$ as,

$$d\nu + \sum_i a_i C_i$$

where $d\nu$ is diffuse, and we confuse the invariant curves C_i, a priori countable in number, with their currents of integration, and, of course, $a_i \in \mathbb{R}_+$. In particular if $d\nu \neq 0$ then it defines the same line as $K_{\mathcal{F}}$ in co-homology by the index theorem, so $K_{\mathcal{F}}^2 = 0$ and $K_{\mathcal{X}}.d\nu \neq 0$. Appealing to [IV] IV.3.1 we see that for some singularity $z \in \mathrm{sing}(\mathcal{F})$, the Lelong number $\lambda_z(d\nu) \neq 0$, which, by a theorem of Demailly, [D1], gives $d\nu^2 \neq 0$, which is absurd. Consequently, $d\nu = 0$, while a minor variation of this reasoning shows that in fact the C_i must lie in $\mathcal{D}$.

To treat the remaining case we observe that our hypothesis imply that $C.d\mu \geq 0$ for every curve in $\mathcal{D}$ by I.2.11, whence the index theorem continues to give $K_{\mathcal{F}}^2 = 0$, and, of course $K_{\mathcal{F}}$ parallel to $d\mu$ in co-homology. As such we can appeal to the 'elliptic fibring' lemma of, for example, [M2] VI.3 or, in its original form [M1], to deduce,

$$0 < K_{\mathcal{X}}.d\mu \leq K_{\mathcal{F}}.d\mu \leq 0$$

which is nonsense. $\square$

Better still, we can actually quantify to what extent discs fail to converge around $\mathcal{D}$. The most convenient way to do this is to work on the canonical model, or even just it's moduli $(X_0, \mathcal{F}_0)$ of [M2], and to introduce an invariant Kobayashi type metric, *i.e.* for $x \notin \mathrm{sing}(X_0) \cup \mathrm{sing}(\mathcal{F}_0)$ and ∂ a local generator for $T_{\mathcal{F}_0}$,

$$\|\partial\|_{\mathrm{Kob}} = \inf\{\frac{1}{R} : f : (\Delta, 0) \to (X_0, x), f_* \left(\frac{\partial}{\partial z}\right)(0) = R\partial, f \text{ invariant}\}$$

As it happens all foliated surfaces not of general type are rather easily classifed once one has III.1.3 so the degeneration question is only of interest when $(X_0, \mathcal{F}_0)$ has general type, where we have,

III.1.4 Further Claim *Let* $(X_0, \mathcal{F}_0)$ *be as above, and of general type, with* Z *the union of the singularities of* X_0 *and* $\mathcal{F}_0$*, and* D_0 *the invariant algebraic curves through the singularities then there is constant* $\alpha > 0$ *such for all* $\varepsilon > 0$ *there exists* $\beta(\varepsilon) > 0$ *for which,*

$$\|\partial\|_{\mathrm{Kob}}(x) \geq \frac{\beta(\varepsilon)\mathrm{dist}(x, Z)^{\alpha}}{|\log \| 1\!\!1_{D_0} \| (x)|^{1+\varepsilon}} .$$

proof If the theorem is false then by I.2.6 there is a sequence of invariant maps f_n from the disc with derivatives at the origin going to infinity which must map any compact set to arbitrarily small neighbourhoods of D_0. Furthermore,

$$\fint_{\Delta(r)} f_n^* c_1(\bar{H}) \longrightarrow \infty, \text{ for any } r \in (0, 1)$$

where $\bar{H}$ is a positively metricised ample divisor. Thus we can combine, I.2.8/10; I.4.5 and III.1.1 together with the general type hypothesis to deduce that there are constants α and δ such that for all $\epsilon > 0$ we have constants $\beta(\epsilon)$, $\gamma(\epsilon) > 0$ for which,

$$\delta \fint_{\Delta(r)} f_n^* c_1(\bar{H}) + \log \|f_* \left(\frac{\partial}{\partial z}\right)\| \leq -\alpha \log \mathrm{dist}(\rho\, f(0), Z)$$

$$+ \log^+ |\log f^* \| 1\!\!1_{D_0} \||(0) + \gamma(\epsilon) \log \fint_{\Delta(r)} f_n^* c_1(\bar{H}) + \beta(\epsilon)$$

From which we conclude by letting n go to ∞. $\square$

The Kobayashi metric so defined is thus lower semi-continuous. It may, however, not be upper semi-continous since the invariant discs of it's definition are allowed to pass through singularities. Consequently there is a somewhat different result to be had by an appropriate, cf. [Br4], adjustment for this. Specifically any foliated surface admits a canonical model $(\mathcal{X}_0, \mathcal{F}_0)$, cf. [M2] without forgetting the moral of [II] IV.1.10, which is infact a smooth stack outwith a finite number of elliptic $(K_{\mathcal{X}_0})$-Gorenstein singularities, where the foliation may be modelled as per a cusp on a bi-disc quotient. In this situation the foliation will fail to be $\mathbb{Q}$-gorenstien, and one should understand by sing$\mathcal{F}_0$ not only the singularities of $\mathcal{F}_0$ over the gorenstien locus, but the union of the same with any elliptic $(K_{\mathcal{X}_0})$-Gorenstien singularities that may be present. Consequently, we may introduce,

III.1.5 Definition *Let $(\mathcal{X}_0, \mathcal{F}_0)$ be the canonical model of a foliated surface, then for $x \notin Z := \mathrm{sing}(\mathcal{F}_0)$ and ∂ a local generator for $T_{\mathcal{F}_0}$,*

$$\|\partial\|_{\mathrm{Pcr}} = \inf\{\frac{1}{R} : f : (\Delta, 0) \to (\mathcal{X}_0 \setminus Z, x), f_* \left(\frac{\partial}{\partial z}\right)(0) = R\partial, f \textit{ invariant}\}$$

is the invariant Poincaré metric, and the image of any maximal such f will be termed a leaf.

Unlike the invariant Kobayashi metric this certainly is upper semi-continuous, while by III.1.4 and I.3.11 it is also lower semi-continuous outwith the invariant curves through sing$\mathcal{F}_0$. Better still,

III.1.6 Final Claim *If all the leaves are not parabolic, then the Zariski closure of the set of such is a simple normal crossing divisor $\mathcal{P}_0$, and outwith $\mathcal{P}_0$ the Poincaré matric is continuous.*

proof Everything in the assertion either follows from some combination of I.3.11, III.1.2, or the classification theorem, [M2], with the exception of the reduction from the set $\mathcal{D}_0$ of algebraic curves through the singularities to the possibly proper, if all the leaves are not parabolic, subset $\mathcal{P}_0$. To this end, notice that any elliptic singularities are disjoint from $\mathcal{D}_0$, so introducing a minimal smooth resolution $\rho : \mathcal{X} \to \mathcal{X}_0$ resolving the cusps changes very little since we may assume that the f_n are arbitrarily close to $\mathcal{D}_0$.

Now suppose that a divergent sequence f_n isn't arbitrarily close to $\mathcal{P}_0$, then we can subsequence and move by bounded automorphisms of the disc à la I.3.12 so that the origins accumulate at a point of $\mathcal{C} \setminus Z$ where $\mathcal{C}$ is in $\mathcal{D}_0$ but $\mathcal{C} \setminus Z$ has negative Euler characteristic $\chi_{\mathcal{C}\setminus Z}$. Consequently we proceed much as before, but with a different strategy for singularities $z \in \mathrm{sing}(\mathcal{F}) \cap C$. Specifically if there is an algebraic curve through z other than $\mathcal{C}$ we proceed as before, i.e. do nothing, while if there is not we blow up once. Observe that in the first case f_n may be supposed not to pass through z, since otherwise it would factor through $\mathcal{C}$, nor does it in the second case by virtue of the definition of leaf. As a result we get an improvement on III.1.2, so that arguing as in III.1.3 we obtain in the limit a transverse invariant measure $d\mu$ with the following properties,

- For every D with generic point in $\mathcal{D} \setminus \mathcal{C}$, $D.d\mu \geq 0$, cf. I.2.5-11.
- If we decompose $d\mu$ as $d\nu + \lambda\mathcal{C}$ then, $K_{\mathcal{F}_0}.d\nu + \lambda\chi_{\mathcal{C}\setminus Z} \leq 0$.

As such, the nefness of $K_{\mathcal{F}_0}$ forces $\lambda = 0$ by the latter property, while the former property allows us to identify $d\mu$ with a nef. divisor (Mumford sense) on $\mathcal{X}_0$ consisting of invariant curves each of which meets $K_{\mathcal{F}_0}$ in 0. As a result the divisor can only be a sum of disjoint rational or elliptic stacks, in which case the foliation must be Ricatti or turbulent, cf [M2], i.e. the suspension of a representation of the fundamental group of some 1-dimensional orbifold in either $\mathrm{PSL}_2(\mathbb{Z})$ or $\mathrm{Aut}(E)$ for E an ellitic curve. In these latter cases, we conclude by inspection. □

III.2 Symmetric Tensors

Similar methodology can be applied to the study of the hyperbolicity of surfaces (*i.e.* a 2-dimensional algebraic stack with projective moduli) themselves, thanks to the following observation of Bogomolov,

III.2.1 Lemma ([B2]) *Let* $(\mathcal{X}, \mathcal{D})$ *be a smooth surface with simple normal crossing boundary of logarithmic general type and* $s_2(\mathcal{X}, \mathcal{D})$ $(= c_1^2(\mathcal{X}, \mathcal{D}) - c_2(\mathcal{X}, \mathcal{D})) > 0$ *then the tautological bundle* L *on* $\pi : \mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{D})) \to \mathcal{X}$ *is big.*

Now let's try and apply this in combination with I.4.2 and I.2.8 to the study of $\mathrm{Hom}(\Delta, \mathcal{X})$. Taking $\overline{H}$ to be a positively metricised ample divisor then the lemma implies (indeed it is equivalent) to the existence of some $b \in \mathbb{N}$ and a naturally metricised effective divisor $\overline{F}$ such that $\overline{L}$, smoothly metricised as in I.4.5, is equal to $\pi^* \overline{H}(\overline{F})$. So that for $f \in \mathrm{Hom}(\Delta, \mathcal{X})$ with $f(0) \notin F$,

$$b \fint_{\Delta(r)} (f')^* c_1(\bar{L}) \geq \fint_{\Delta(r)} f^* c_1(\bar{H}) + \log \|f'^* \, 1\!\!1_F\| (0) + \mathrm{const}$$

where the constant is just a lower bound for $-\log \|1\!\!1_F\|$. Now suppose further that $f_n \in \mathrm{Hom}(\Delta, \mathcal{X})$ don't converge in $\overline{\mathrm{Hom}}(\Delta, \mathcal{X})$ then we can subsequence to guarantee that the Nevanlinna degrees of the f_n with respect to $\overline{H}$ are for some fixed r_0, as large a function of n as we like, so that applying I.4.2 and I.2.8 we obtain for $r \in (0, 1)$ outside a set of finite hyperbolic measure,

$$\begin{aligned} \log \|df_n\| (0) + \fint_{\Delta(r)} (f_n')^* c_1(\bar{L}) \leq (1 + o(\fint_{\Delta(r)} f_n^* c_1(\bar{H})) \log(1 - r^2)^{-1} \\ + \mathrm{rad}_{f_n, \mathcal{D}}(r) + \log |\log f_n^* \, 1\!\!1_{\mathcal{D}}| (0) \end{aligned}$$

Certainly we can restrict to the case $\mathrm{rad}_{f_n, \mathcal{D}} = 0$, e.g. $f_n : \Delta \to \mathcal{X} \backslash \mathcal{D}$ or $\mathcal{D}$ empty, but the real problem concerns with what happens at zero. Equally it makes little difference whether we prove that the f_n converge in $\overline{\mathrm{Hom}}$ or that the $\alpha_n^* f_n$ converge, where α_n are a convergent sequence of automorphisms, so provided that the f_n' are not contained in arbitrarily small neighbourhoods of $F \cup \pi^{-1}(\mathcal{D})$, composing with a convergent sequence of automorphisms implies by I.3.12 that,

$$\fint_{\Delta(r)} f_n^* c_1(\bar{H}) \leq C(r)$$

and whence a contradiction. The real difficulty arises, therefore, when the sequence f_n becomes arbitrarily close to some divisor, which by the way contains as a strict, and relatively straightforward subcase, the possibility that the f_n are actually in the divisor, which modulo a simple reduction is wholly covered by the results of III.1.

The immediate difficulty is that the divisor F is potentially a complete mess. Concentrating on its horizontal components, these correspond to saturated maps,

$$0 \to M \to \mathrm{Sym}^d \, \Omega_{\mathcal{X}}(\log \mathcal{D})$$

for $d \in \mathbb{N}$ and M a line bundle, which don't factor through a lower symmetric power. Abusing notation slightly, and denoting the component equally by F, with U the universal sub-bundle over the same we have a factorisation,

$$\pi^* M \otimes \mathcal{O}_F \to U \otimes \pi^* \mathrm{Sym}^{d-1}\Omega_{\mathcal{X}}(\log \mathcal{D}) \otimes \mathcal{O}_F \to \pi^* \mathrm{Sym}^d \Omega_{\mathcal{X}}(\log \mathcal{D}) \otimes \mathcal{O}_F$$

So passing to a ramified cover $\rho : \mathcal{S} \to \mathcal{X}$ we have a fibre square,

$$\begin{array}{ccc} \mathbb{P}(\rho^*\Omega_{\mathcal{X}}(\log \mathcal{D})) & \xrightarrow{\rho} & \mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{D})) \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & \mathcal{X} \end{array}$$

with all the horizontal components of $\rho^* F$ enjoying degree 1. Further passing to a resolution of singularities not just of $\mathcal{S}$ but of $\rho^{-1}(\mathcal{D})$, which we'll denote $\rho : (\mathcal{S}, \mathcal{B}) \to (\mathcal{X}, \mathcal{D})$, we have the natural map,

$$\rho^*\Omega_{\mathcal{X}}(\log \mathcal{D}) \to \Omega_{\mathcal{S}}(\log \mathcal{B})$$

and a birational map, $\mathbb{P}(\Omega_{\mathcal{S}}(\log \mathcal{B}) \dashrightarrow \mathbb{P}(\rho^*\Omega_{\mathcal{X}}(\log \mathcal{D}))$. Resolve this latter map by, $\psi : P \to \mathbb{P}(\Omega_{\mathcal{S}}(\log \mathcal{B}))$, and consider what's become of the maps f_n. In the first place these do not necessarily lift to $\mathcal{S}$, but we do have a sequence of liftings to ramified coverings,

$$\begin{array}{ccc} \mathcal{C}_n & \xrightarrow{\tilde{f}_n} & \mathcal{S} \\ {\scriptstyle p_n}\downarrow & & \downarrow \\ \Delta & \xrightarrow{f_n} & \mathcal{X} \end{array}$$

with the derivatives $\widetilde{f_n'} : \mathcal{C}_n \to \mathbb{P}(\Omega_{\mathcal{S}}(\log \mathcal{B}))$ of the proper transforms $\widetilde{f_n}$ lifting $f_n' : \Delta \to \mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{D}))$ to P. Whence if the f_n are eventually arbitrarily close to some divisor on $\mathbb{P}(\Omega_{\mathcal{X}}(\log \mathcal{D}))$ then the $\widetilde{f_n'}$ are the same for its pull-back to P. Consequently since P is isomorphic to $\mathbb{P}(\Omega_{\mathcal{S}}(\log \mathcal{B}))$ over the generic point the $\widetilde{f_n'}$, defined on $\mathcal{C}_n$, are arbitrarily close to a divisor each horizontal component of which has degree 1.

This basically completes the reduction of the hyperbolicity of $\mathcal{X} \backslash \mathcal{D}$ to a foliation problem. Specifically we are led to consider sequence of triples,

$$\begin{array}{ccc} \mathcal{C}_n & \xrightarrow{\tilde{f}_n} & \mathcal{S} \\ {\scriptstyle p_n}\downarrow & & \\ \Delta & & \end{array}$$

of maps of finite covers of the disc whose ramification is not worse than the degree such that for some sufficiently large simple normal crossing divisor $\mathcal{B}$ on $\mathcal{S}$ and foliations $\mathcal{F}^i$, $1 \leq i \leq r$, the derivatives f_n' are arbitrarily close, in the compact open sense to the divisor in $\mathbb{P}(\Omega_{\mathcal{S}})$ cut out by $\mathcal{B}$ and the foliations. Naturally for some ample divisor H on $\mathcal{S}$ we suppose the degrees,

$$\fint_{\mathcal{C}_n(s)} f_n^* c_1(\bar{H})$$

unbounded for some $0 < s < 1$, and that the f_n – or more precisely, simply the $f_n(0)$ – are not arbitrarily close to some Zariski closed subset V to be specified. With this in mind the following lemma will do nicely

III.2.2 Lemma (Notations as above modulo replacing $(\mathcal{S},\mathcal{B})$ by $(\mathcal{X},\mathcal{D})$) *For any* $\varepsilon > 0$ *there is a proper birational modification* $\rho : (\widetilde{\mathcal{X}},\widetilde{\mathcal{D}}) \to (\mathcal{X},\mathcal{D})$ *such that if* $f'_n : \mathcal{C}_n \to \mathbb{P}(\Omega_{\widetilde{\mathcal{X}}}(\log \widetilde{\mathcal{D}}))$ *is the logarithmic derivative then for* $r \in (0,1)$ *outside of a set of finite hyperbolic measure,*

$$\fint_{\mathcal{C}_n(r)} f_n^* c_1(K_{\widetilde{\mathcal{X}}} + \widetilde{\mathcal{D}}) \leq \fint_{\mathcal{C}_n(r)} f_n^* c_1(L) + \epsilon \fint_{\mathcal{C}_n(r)} f_n^* c_1(H)$$

where L *is the tautological bundle.*

In order to conclude to a hyperbolicity statement from the lemma, one first notes the minnor fact that if the f_n are not arbitrarily close to some divisor ∂ in $\mathcal{X}$ then the same is true for the $\widetilde{f_n}$ at the $\mathcal{S}$ level with respect to $\rho^{-1}(\partial)$. Thus by a priori composing with convergent sequences of automorphisms of the disc we can guarantee,

$$\fint_{\mathcal{C}_n(r)} \tilde{f}_n^* c_1(E) \geq \mathrm{const} + \log \|\widetilde{f}_n^*\, \mathbb{1}_E\|\,(0) \geq \mathrm{const}$$

for any E in the support of $\rho^{-1}(\partial)$, and so guarantee the conditions of the lemma. In any case, given the lemma, we have for $\epsilon > 0$,

$$\begin{aligned}\fint_{\mathcal{C}_n(r)} f_n^* c_1(K_{\mathcal{S}} + \mathcal{B}) + \log \|df_n\|(0) &\leq \mathrm{rad}_{f_n,\mathcal{D}}(r) + \mathrm{discr}^{\#}_{\mathcal{C}_n/\Delta}(r) \\ &\quad + C(r) + \epsilon \fint_{\Delta(r)} f_n^* c_1(H)\end{aligned}$$

and by including the ramification in our excluded set, we can estimate $\mathrm{discr}^{\#}$ in the obvious way by the ramification divisor, so that restricting to one of the natural cases where the radical is zero, we'll obtain,

$$\fint_{\Delta(r)} f_n^* c_1(K_{\mathcal{X}} + \mathcal{D}) \leq C(r) + \epsilon \fint_{\Delta(r)} f_n^* c_1(H)$$

from which the hyperbolicity of $\mathcal{X}\backslash\mathcal{D}$ modulo some effectively computable, if rather unclear, proper Zariski closed subset.

The important things to draw from this discussion are in the first place that we need the full force of the main lemma, i.e. it would be wholly insufficient to simply bound the degree of the maps $f_n : \mathcal{C}_n \to \mathcal{S}$ in terms of the ramification, since this would not a priori imply a similar bound for the maps $f_n : \Delta \to \mathcal{X}$. Secondly, although in the case of $c_1^2 > c_2$ where the specific problem is that our maps may intersect some horizontal divisor negatively, we might hope to play some numerical game between $\mathcal{X}$ and S (all be it, that it's not clear what) to reduce to some easier problem involving less components, this isn't the point. More precisely if we hope to conclude to results on the hyperbolicity of algebraic surfaces, by use of analogues of III.2.1 for higher jet bundles, then we must be able to exclude systematically, by way of appropriate analogues of III.2.2, the degeneration at the level of lower order jets as next example illustrates.

III.3 The Bloch Theorem

The Bloch Theorem concerns the hyperbolicity of $(\mathbb{P}^2, D)$, where D is a divisor consisting of 4 lines in general position. The arguements of this section work quite generally for any number of lines, and leads to specific quantification of the value distribution of discs about the same. However the case of at least 5 lines (all be it with worse quantification) follows from the above combination of III.2.1 with III.2.2,, and things are combinatorially a bit involved so we'll stick to the case of 4.

To begin with denote by P, respectively $P_{\log}$, the bundles $\mu : \mathbb{P}(\Omega_{\mathbb{P}^2}) \to \mathbb{P}^2$, and $\nu : \mathbb{P}(\Omega_{\mathbb{P}^2}(\log D)) \to \mathbb{P}^2$. The bundle P is equally smoothly fibred by λ say, over $(\mathbb{P}^2)^\vee$, which we think of as a smooth connection on $\mathbb{P}^2$, or equivalently a smooth foliation $\mathcal{F}$ on P arising from a splitting of the bundle of 2-jets, J, i.e. the quotient of Ω_P by the universal sub-bundle. Equally if $J_{\log}$ is the quotient of $\Omega_{P_{\log}}(\log \nu^* D)$ by the corresponding universal sub-bundle we have an induced connection corresponding to a singular foliation $\mathcal{F}_{\log}$, with a splitting of the natural exact sequence for $J_{\log}$, i.e.

$$0 \to L_{\log} \to J_{\log} \underset{\overset{\text{split}}{\longleftarrow}}{\longrightarrow} \nu^* K_{\mathbb{P}^2}(D) \otimes L_{\log}^{-2} \to 0$$

where L, and $L_{\log}$ are the respective tautological bundles on P and $P_{\log}$. Now argueing as above this time with respect to the second derivatives $f_n'' : \Delta \to \mathbb{P}(J_{\log})$, for $f_n : \Delta \to \mathbb{P}^2 \backslash D$ a sequence of divergent maps which are not eventually arbitrarily proximate to D, it's immediate that we're done unless the f_n'' are eventually arbitrarily close to the divisor in $\mathbb{P}(J_{\log})$ defined by the connection. In order to reduce to III.2.2 we have to relate $P_{\log}$ to P. Observe that over a line C in D, both these bundles posses canonical sections, i.e.

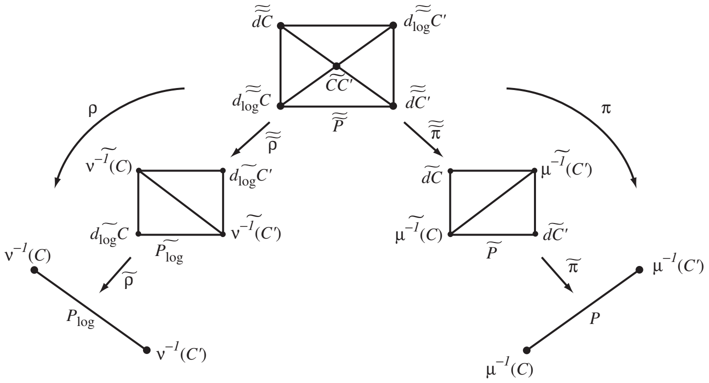
(i) The derivative, dC,

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-C) \to \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_C \to \omega_C \to 0$$

(ii) The residue, $d \log C$,

$$0 \to \mathcal{O}_C(K_X + D) \to \Omega_{\mathbb{P}^2}(\log D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to 0\,.$$

The possibilities for resolving the birational map $P_{\log} - - \to P$ are not unique. The correct one is to first blow up in all the $d \log C$, which by the way are disjoint, then blow in the crossings of the proper transforms of the $\nu^{-1}(C)$, $C \subset |D|$. The final exceptional divisors so obtained are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$'s, so blow down in the other direction, or alternatively they're the blow up in the crossings of the proper transforms of the $\mu^{-1}(C)$ after blowing up in the dC. Note also that the proper transforms of the $\nu^{-1}(C)$, respectively $\mu^{-1}(C)$ blow down in the other direction. Let's summarise all this in terms of some maps, and simplicial complex defined by the intersection of various divisors, which in turn fixes our notation, viz,



The canonical bundles of the various foliations are related by way of the formula,

$$\rho^* K_{\mathcal{F}_{\log}} - \sum_C \widetilde{\widetilde{d\log C}} = K_{\tilde{\tilde{\mathcal{F}}}} = \pi^* K_{\mathcal{F}} + \sum_{CC} \widetilde{CC'}$$

where the negative discrepancy on the left reflects the fact that the $\widetilde{d\log C}$ are not invariant, and of course the positive discrepancy on the right that the $\widetilde{CC'}$ are so. Consequently the natural divisor to work with, at least initially, is,

$$\tilde{\tilde{D}} := \sum_C \widetilde{\widetilde{dC}} + \sum_{CC} \widetilde{CC'}\,.$$

However we also have other interesting divisors arising from the pencils (foliations) defined by any pair of lines. In $(\mathbb{P}^2)^\vee$ these correspond to six lines, which we'll denote by F's, meeting in the 4 points, $[C]$, corresponding to our given curves. This information incorporates itself naturally into our data since $\widetilde{P}$ arises as the fibre product,

$$\begin{array}{ccccc}
\widetilde{P} & \xrightarrow{\tilde{\lambda}} & (\widetilde{\mathbb{P}^2})^\vee & \longleftarrow & \widetilde{\lambda}(\widetilde{dC}) \\
{\scriptstyle\tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\tilde{\pi}} & & \big\downarrow \\
P & \xrightarrow{\lambda} & (\mathbb{P}^2)^\vee & \longleftarrow & [C]
\end{array}$$

where the middle vertical arrow is the blow up in the 4, aforesaid points, and we denote by $\widetilde{F}$ the pull-back to $\widetilde{P}$ of the proper transform of the various F's, with $\tilde{\tilde{F}}$ its proper transform on $\tilde{\tilde{P}}$. At the $\tilde{\tilde{P}}$ level all divisors appearing so far are simple normal crossing, with the $\tilde{\tilde{\mathcal{F}}}$ invariant ones being $\widetilde{D}$ together with the $\tilde{\tilde{F}}$'s. Moreover the pull-back to $\tilde{\tilde{P}}$ of the universal subbundle $U_{\log}$ on $P_{\log}$ is still a subbundle of $\Omega_{\tilde{\tilde{P}}}(\log \widetilde{D} + \tilde{\tilde{F}})$ (although having a bundle is notationally more important than anything else) so we get an induced jet bundle $\tilde{\tilde{J}}$ with dévissage induced by $\tilde{\tilde{\mathcal{F}}}$,

$$0 \to \rho^*\nu^* K_{\mathbb{P}^2}(D) \otimes \rho^* L_{\log}^{-2}\left(\sum_C \widetilde{\widetilde{d\log C}} + \sum_F \tilde{\tilde{F}}\right) \to \tilde{\tilde{J}} \to K_{\tilde{\tilde{\mathcal{F}}}} \to 0\,.$$

Now suppose that the derivatives of our maps $f'_n : \Delta \to P_{\log}$ are not eventually arbitrary close to $\bigcup_F \rho(\widetilde{\widetilde{F}})$ which by the way contain the $d\log C$, then after composition with convergent sequences of automorphisms of the disc the radical of $\widetilde{\widetilde{F}}$ for the induced maps $\widetilde{\widetilde{f_n}} : \Delta \to \widetilde{\widetilde{P}}$ may comfortably be estimated by,

$$\mathrm{rad}_{\tilde{\tilde{f}}_n, \tilde{\tilde{F}}}(r) \leq \rlap{\,/}\int_{\Delta(r)} f_n^* c_1(\widetilde{\widetilde{F}}) + \mathrm{const}$$

where the implied constant is independent of n. Consequently applying the tautological inequality for $(\widetilde{\widetilde{P}}, \widetilde{\widetilde{D}} + \widetilde{\widetilde{F}})$ with respect to the induced maps $\widetilde{\widetilde{f_n}}$ and their derivatives, and argueing as in the applicability of III.2.1 we're certainly going to bound the degree of the $\widetilde{f_n}$, unless the derivatives are eventually arbitrarily proximate to the divisor in $\mathbb{P}(\widetilde{\widetilde{J}})$ defined by the above dévissage.

At this point let's work in terms of the induced currents in the sense of I.2.4. Call an appropriate such on $\mathbb{P}(\widetilde{\widetilde{J}}), T$. We may devide T into two parts, according to the singularities of $\widetilde{\widetilde{\mathcal{F}}}$ which occur at the intersections $\widetilde{\widetilde{F}} \cap E_{CC'}$. Denote the said singular curves S, and use the Skoda-El Mir theorem to write,

$$T = 1\!\!\mathrm{I}_S\, T + T_{\mathcal{F}}$$

where the closed current $T_{\mathcal{F}}$ is invariant by $\widetilde{\widetilde{\mathcal{F}}}$ in the strongest possible sense, i.e. $\widetilde{\lambda}_* \,\widetilde{\widetilde{\pi}}_* \,T_{\mathcal{F}} = 0$. Furthermore the full foliation on $\widetilde{\widetilde{P}}$ is given by,

$$0 \to \widetilde{\widetilde{\pi}}^* \,\widetilde{\lambda}^* \,\Omega_{(\widetilde{\mathbb{P}}^2)^\vee}(\log \widetilde{F} + \widetilde{dC}) \to \Omega_{\widetilde{\widetilde{P}}}(\log \widetilde{\widetilde{D}} + \widetilde{\widetilde{F}}) \to K_{\widetilde{\widetilde{\mathcal{F}}}} \to 0$$

and so,

$$K_{\widetilde{\widetilde{\mathcal{F}}}} \cdot T_{\mathcal{F}} = (K_{\widetilde{\widetilde{P}}} + \widetilde{\widetilde{D}} + \widetilde{\widetilde{F}})\, T_{\mathcal{F}}$$

while by accident more than design,

$$K_{\widetilde{\widetilde{\mathcal{F}}}} \cdot 1\!\!\mathrm{I}_S\, T = \sum_{CC'} (\lambda(CC') + E_{CC'} \cdot 1\!\!\mathrm{I}_S\, T) = (K_{\widetilde{\widetilde{P}}} + \widetilde{\widetilde{D}} + \widetilde{\widetilde{F}}).1\!\!\mathrm{I}_S\, T$$

where $\lambda(CC')$ is the Lelong number of T along the component of S determined by C and C'. Consequently on combining with the original tautological inequalities for $\mathbb{P}(J_{\log})$ and $\mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))$ we obtain,

$$0 \geq (K_{P_{\log}} + D).\rho_* T + \left(\sum_C d\log C + \widetilde{\widetilde{F}} \right) T \geq (K_{P_{\log}} + D).\rho_* T\,.$$

From which we deduce that the Bloch-Theorem reduces to III.2.2 in the form,

III.3.3 Fact *Bearing in mind I.3.7, if a sequence $f_n \in \mathrm{Hom}(\Delta, \mathbb{P}^2 \backslash D)$ which is not eventually arbitrarily close to D doesn't have a subsequence convergent in $\mathrm{Hom}(\Delta, \mathbb{P}^2)$, the derivatives $f'_n : \Delta \to \mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))$ are eventually arbitrarily close to the divisors defined by the pencils of lines defined by any two pairs of lines in D, together with the divisor D itself.*

III.4 The general 1st order ODE

Having hopefully illustrated its importance let us devote this section to the proof of III.2.2.

proof. Obviously we blow up until we reach the point where all of our foliations have canonical singularities. Thus we implicitly use the fact, specific to surfaces, that modifications of foliations with canonical singularities still enjoy the same. The same principle even allows us to assume the marginally stronger class of what are classically termed reduced singularities, and whence the technical convenience that any of our foliations has exactly two branches at a given singularity. Regardless we are now in a situation where any exceptional curve on a further blow up must be invariant by all the foliations, so that after further modifications we might as well suppose that we begin from a point where a curve $\mathcal{B}$ in the support of $\mathcal{D}$ with respect to a given foliation $\mathcal{F}^i$ is either invariant by it or nowhere tangent to it, and in addition that at any crossing of $\mathcal{D}$ there passes at least one curve invariant by all the foliations, while at any singularity of any foliation there is at least one curve in the support of $\mathcal{D}$ invariant by it.

Evidently the meaning of sufficiently large is in terms of modifications as employed to conclude III.1.2, which we flesh out in this multi-foliated and quasi-projective situation. We also insist that our divisor is sufficiently large, i.e. that $\mathcal{D}$ contains any curve which is invariant and passes through at least one singularity of some $\mathcal{F}^i$, and call this initial modification $(\mathcal{X}_0, \mathcal{D}_0)$. Now suppose our foliation $\mathcal{F}^i$ is given by a dévissage,

$$0 \to N_0^i \to \Omega_{\mathcal{X}_0} \to K_{\mathcal{F}_0^i} \cdot I_{Z_0^i} \to 0$$

We can divide our divisor $\mathcal{D}_0$ as $\Delta_0^i + \partial_0^i$ where Δ_0^i is the part invariant by $\mathcal{F}^i$, and consider the corresponding log dévissage,

$$0 \to N_0^i(\Delta_0^i) \to \Omega_{\mathcal{X}_0}(\log \mathcal{D}_0) \to K_{\mathcal{F}_0^i}(\partial_0^i) \cdot I_{W_0^i} \to 0\,.$$

Given that there is at least one $\mathcal{F}^i$ invariant curve through any crossing of $\mathcal{D}_0$, and $\mathcal{D}_0$ is everywhere transverse to $\mathcal{F}^i$ where it isn't invariant, $W_0^i \subset Z_0^i$. Better still since at every singular point there is at least one $\mathcal{F}^i$ invariant curve $\mathcal{B}$ in $\mathcal{D}_0$, if Z_0^i has degree 1 at this point, W_0^i is empty. The structure of W_0^i therefore comes from the saddle nodes, and W_0^i is empty or not according to whether or not, the weak branch is in $\mathcal{D}_0$.

Now certainly amongst the foliations singularities may coincide, but we might as well insist that we've blown up sufficiently that this will be the case iff the branches coincide. In particular if a point is a singularity of two foliations $\mathcal{F}^i$ and $\mathcal{F}^j$ then a given branch is algebraic (whence in the support of $\mathcal{D}_0$) for one iff it's algebraic for the other. Whence define Z_0 to be the set of points in $\mathcal{X}_0$ which are saddle nodes for some foliation $\mathcal{F}^i$ whose weak branch at the point is not algebraic. Define $\mathcal{X}_1$ to be the blow up of $\mathcal{X}_0$ in Z_0, $\mathcal{D}_1$ to be the pull-back of $\mathcal{D}_0$ (by construction this is automatically reduced and contains the exceptional divisor), Z_1 to be the induced saddle nodes with non-algebraic branches, and continue inductively to get a sequence of modifications,

$$(\mathcal{X}_n, \mathcal{D}_n) \to (\mathcal{X}_{n-1}, \mathcal{D}_{n-1}) \to \cdots \to (\mathcal{X}_1, \mathcal{D}_1) \to (\mathcal{X}_0, \mathcal{D}_0)$$

with $\rho_{pq} : (\mathcal{X}_q, \mathcal{D}_q) \to (\mathcal{X}_p, \mathcal{D}_p)$ the projection for $q > p$, and for $z \in Z_0$, E_{zq} the exceptional curve on $(\mathcal{X}_q, \mathcal{D}_q)$ over z blown down by $\rho_{q,q-1}$. Observe that our foliations $\mathcal{F}^i$ transform according to one of the following possibilities in a neighbourhood of $z \in Z_0$,

(i) $\mathcal{F}^i$ doesn't have a saddle node at z, then all induced singularities on $\mathcal{X}_q$ have degree 1. Whence if $\mathcal{F}^i_q$ is the induced foliation and $\mathcal{D}_q = \Delta^i_q + \partial^i_q$ the decomposition into invariant and non-invariant parts, $K_{\mathcal{F}^i_q}(\partial^i_q)$ is a quotient of $\Omega_{\mathcal{X}_q}(\log \mathcal{D}_q)$.

(ii) $\mathcal{F}^i$ has a saddle node at z, with non-algebraic weak branch b, then for a subcheme W^i_q supported at the crossing of $\widetilde{b}$ with the exceptional divisor, say z_q, $K_{\mathcal{F}^i_q}(\partial^i_q)\, I_{W^i_q}$ is a quotient of $\Omega_{\mathcal{X}_q}(\log \mathcal{D}_q)$.

Now let us turn to our maps $f_n : \mathcal{C}_n \to \mathcal{X}$, and their various proper transforms, also denoted f_n, to any $\mathcal{X}_q$. Certainly we have various logarithmic derivatives, $f'_n : \mathcal{C}_n \to \mathbb{P}(\Omega_{\mathcal{X}_q}(\log \mathcal{D}_q))$ which thanks to the log-tautological inequality, and the general considerations of I.2 give for all q and $r \in (0,1)$ outside a set of finite hyperbolic measure positive currents,

$$T'_{nq}(r) : A^{1,1}(\mathbb{P}(\Omega_{\mathcal{X}_q}(\log \mathcal{D}_q)) \to \mathbb{C} : \eta \mapsto \left(\fint_{\mathcal{C}_n(r)} f_n^* c_1(H) \right)^{-1} \fint_{\mathcal{C}_n(r)} f_n^* \eta$$

converging weakly, after suitable subsequencing, to closed positive currents T'_q, provided of course that we supposed lack of proximity to a sufficiently large Zariski subset in order to guarantee the degree on $\mathcal{X}_0$ measured with respect to H, dominates the degree with respect to an ample divisor on $\mathcal{X}_q$. Within $\mathbb{P}(\Omega_{\mathcal{X}_q}(\log \mathcal{D}_q))$, which we may as well denote P_q, with π to $\mathcal{X}_q$ the projection, we have two kinds of divisors that interest us, one being $\pi^*\mathcal{B}$ for $\mathcal{B}$ a component of $\mathcal{D}$, the other the closure of the section in codimension 2 defined by the foliation $\mathcal{F}^i$, which we'll denote $\mathcal{X}^i_q$. Necessarily $\mathcal{X}^i_q$ is isomorphic to the blow up of $\mathcal{X}_q$ in W^i_q, so it's a section away from saddle nodes z, but in general for a node of degree $d_i(z)$ at least 3, it will be a blow up in a non-reduced scheme, so it's singular, and its minimal smooth resolution $\widetilde{\mathcal{X}}^i_q$ is around the node isomorphic to $\mathcal{X}_{q+d_i(z)}$. By hypothesis T'_q is supported only on these divisors. Before breaking T'_q up according to its support, observe that for any $\mathcal{B}$ in $\mathcal{D}$, $(K_{\mathcal{X}} + \mathcal{D}).\, \mathcal{B} \le 0$ iff $\mathcal{D}\backslash\mathcal{B}$ has non-negative topological euler characteristic. Call such curves parabolic, and observe further that it is a harmless hypothesis to suppose that if $\mathcal{B}$ in $\mathcal{D}$ is not parabolic and $\mathcal{B}' \neq \mathcal{B}$ in $\mathcal{D}$ meets $\mathcal{B}$ then $\mathcal{B}'$ is parabolic, i.e. just blow up in the crossings of $\mathcal{D}$, which we might as well have done a priori so that this holds for all $\mathcal{D}_q$, $q \ge 0$. Now we can divide T'_q up into a sum of closed positive currents according to the rule,

$$T'_q = \sum_{\mathcal{B} \subset \mathcal{D}_q^{n\mathrm{par}}} \pi^* \, 1\!\!1_{\mathcal{B}}\, T'_q + \sum_{\mathcal{B} \subset \mathcal{D}_q^{\mathrm{par}}} \pi^* \, 1\!\!1_{\mathcal{B}\backslash \mathcal{D}_{np}}\, T'_q + \sum_i 1\!\!1_{\mathcal{X}^i_q \backslash \mathcal{D}_q}\, T'_q$$

where $\mathcal{D}_q = \mathcal{D}_q^{\mathrm{par}} + \mathcal{D}_q^{n\mathrm{par}}$ is the decomposition of $\mathcal{D}_q$ into its parabolic and non-parabolic components.

Given the nature of our preliminaries we take care of $1\!\!1_{\mathcal{X}^i_q \backslash \mathcal{D}_q}\, T'_q$ first. For simplicity let's write this as $1\!\!1_{\mathcal{F}^i_q}\, T_q$, by construction $\pi_*\, 1\!\!1_{\mathcal{F}^i_q}\, T_q$ is a transverse invariant measure for $\mathcal{F}^i_q$, which can have atomic components on at most invariant curves wholly contained in the smooth locus. Locally over the points z, if any, where $\mathcal{X}^i_q$

isn't a section we have an exceptional Cartier divisor $E_q^i(z)$ whose pull-back to the minimal smooth resolution $\mathcal{X}_{q+d(z)}$ is $\sum_{k=1}^{d_i(z)} \rho^*_{q+d_i,k} E_k(z)$. In any case if L_q is the tautological bundle on P_1, then,

$$L_q \otimes \mathcal{O}_{\mathcal{X}_q^i} = \mathcal{O}_{\mathcal{X}_q^i}\left(K_{\mathcal{F}_q^i} + \partial_q^i - \sum_z E_q^i(z)\right).$$

Furthermore by Brunella's extension of the elliptic fibring lemma to transverse invariant measures, [Br1], for T on $\mathcal{X}_q$ a transverse $\mathcal{F}_q^i$ invariant measure, without atomic support $(N_q^i + \Delta_q^i).\, T \leq 0$. However this inequality equally holds trivially for any invariant curve wholly contained in the smooth locus of $\mathcal{F}_q^i$, so it holds for $\pi_* \, \mathbb{1}_{\mathcal{F}_q^i} T_q$, which implies,

$$L_q \cdot \mathbb{1}_{\mathcal{F}_q^i} T_q \geq \pi^*(K_{\mathcal{X}_q} + \mathcal{D}_q)\, \mathbb{1}_{\mathcal{F}_q^i} T_q - \sum_z E_q^i(z) \cdot \mathbb{1}_{\mathcal{F}_q^i} T_q\,.$$

On bearing in mind that $\pi_* \, \mathbb{1}_{\mathcal{F}_q^i} T_q$ is nef. for any q, we can therefore apply III.1.1 to obtain,

$$\sum_z E_q^i(z) \cdot \mathbb{1}_{\mathcal{F}_q^i} T_q \leq (\mathrm{const})\, [\sqrt{q}]^{-1}$$

for some absolute constant depending only on $\mathcal{X}$, so that for some q sufficiently large, $L_q.\mathbb{1}_{\mathcal{F}_q^i} T_q \geq \pi^*(K_{\mathcal{X}_q} + \mathcal{D}_q).\mathbb{1}_{\mathcal{F}_q^i} T_q - \epsilon$, and we're reduced to obtain a similar inequality for the part of T_q' supported over $\mathcal{D}$.

To this end consider for $\mathcal{B}$ in $\mathcal{D}$ the canonical section, defined by the short exact sequence,

$$0 \to \mathcal{O}_{\mathcal{B}}(K_{\mathcal{X}} + \mathcal{D}) \to \Omega_{\mathcal{X}}(\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{B}} \to \mathcal{O}_{\mathcal{B}} \to 0$$

where the last map is the residue map. For simplicity of notation consider q as fixed, and let $P_{\mathcal{B}}$ be the fibre of P over $\mathcal{B}$ with S the said section. Certainly we have,

$$\mathcal{O}_{P_{\mathcal{B}}}(S) = \mathcal{O}_{P_{\mathcal{B}}}(L_q - \pi^*(K_{\mathcal{X}} + \mathcal{D}))$$

so an estimate of the form $S \cdot \mathbb{1}_{P_{\mathcal{B}}} T' \geq 0$ will serve us nicely. Better still for $\mathcal{B}$ parabolic, $S^2 \geq 0$, so such an estimate trivially holds for any current supported in $P_{\mathcal{B}}$, in particular for $\mathbb{1}_{P_{\mathcal{B}} \setminus \pi^* \mathcal{D}^{n\mathrm{par}}} T$, and we're reduced to study $\mathcal{B}$ non-parabolic.

At this point it becomes clear why we wished to simplify the notations. Specifically we consider a sequence of modifications,

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 = P$$

with P_1 the blow up of P_0 in S, so that the proper transform $P_{\mathcal{B}}^1$ of $P_{\mathcal{B}}$ is isomorphic to $P_{\mathcal{B}}$ whence in this sense we blow up P_1 in S and continue ad infinitum. Here we'll use π_{mn} for the projection between P_n and P_m, and F_n the exceptional divisor blown down by $\pi_{n-1,n}$.

To set the scene on P_1, $P_{\mathcal{B}}^1$ can be blown down, and the resulting variety is $\mathbb{P}(\Omega_{\mathcal{X}_q}(\log \mathcal{D}_q'))$ where $\mathcal{D}_q = \mathcal{D} \backslash \mathcal{B}$, with $P_{\mathcal{B}}^1$ blowing down to the derivative of $\mathcal{B}$. Now since a derived type current with non-trivial push-forward in $\mathbb{P}(\Omega_{\mathcal{X}_q}(\log \mathcal{D}_q')$

$\otimes\mathcal{O}_{\mathcal{B}}$) is supported on the derivative this shows that there are no non exceptional components of the induced current supported on F_q. However this holds much more generally, but let's first set up the notation and clarify the above remarks.

Our sequence $f'_k : \mathcal{C}_k \to P$, has a lifting to any P_n which leads to an induced current T_n (strictly speaking T_{nq}, but q will play no further role). We're interested in the decomposition,

$$T_n = \mathbb{I}_{P^n_{\mathcal{B}}} T_n + \mathbb{I}_{\pi^*_n\mathcal{B}\backslash P^n_{\mathcal{B}}} T_n + \mathbb{I}_{P_n\backslash\pi^*_n\mathcal{B}} T_n$$

and we've already remarked that $(\pi_{10})_*(\mathbb{I}_{\pi^*_{01}\mathcal{B}\backslash P^1_{\mathcal{B}}} T_1) = 0$, however suppose this held more generally, i.e., we have,

III.4.1 Claim. $(\pi_{n0})_* (\mathbb{I}_{\pi^*_n\mathcal{B}\backslash P^n_{\mathcal{B}}} T_n) = 0$.

Then we may conclude as follows. In the first place this forces $\mathbb{I}_{P^n_{\mathcal{B}}} T_n = \mathbb{I}_{P_{\mathcal{B}}} T_0$, so that $(F_1 + \cdots + F_n) \cdot \mathbb{I}_{P^n_{\mathcal{B}}} T_n = n\, S \cdot \mathbb{I}_{P_{\mathcal{B}}} T_0$. Furthermore $\mathbb{I}_{\pi^*_{0n}\mathcal{B}\backslash P^n_{\mathcal{B}}}$ T_n is homologous to a positive sum of curves contracted by π_{n0}, and $F_1 + \cdots + F_n$ is non-positive on such so,

$$0 \le (F_1 + \cdots + F_n) \cdot T_n \le n\, S \cdot \mathbb{I}_{P_{\mathcal{B}}} T_0 + \sum_{i=1}^{n} F_i \cdot \mathbb{I}_{P_n\backslash\pi^*_{0n}\mathcal{B}} T_n$$

where the first inequality holds since $f_n(0)$ is supposed bounded away uniformly from $\mathcal{B}$. Finally note that $P^n_{\mathcal{B}}$ intersects positively any current supported off it, so:

$$\sum_{i=1}^{n} F_i \cdot \mathbb{I}_{P_n\backslash\pi^*_{0n}\mathcal{B}} T_n \le P_{\mathcal{B}} \cdot \mathbb{I}_{P_0\backslash\pi^*_{01}\mathcal{B}} T_0$$

since the latter is fixed independently of n, our claim indeed implies that $S \cdot \mathbb{I}_{P_{\mathcal{B}}} T_0 \ge 0$.

It remains therefore to justify the claim. To this end suppose locally $\mathcal{B}$ is given by an equation $x = 0$, and y is the other coordinate. In a neighbourhood of S we have a coordinate $t = \frac{dy}{dx/x}$ (or $\frac{dy/y}{dx/x}$ if we're close to a crossing of $\mathcal{D}$) with respect to which $\pi_{0n} : P_n \to P$ is the minimal smooth resolution of the ideal $I = (t^n, x)$, i.e. $\bigotimes_{i=1}^{n} \pi^*_{in}\, \mathcal{O}_{P_i}(-F_i) = \pi^{-1}_{0n} I$, so to be an open neighbourhood a distance ϵ off $P^n_{\mathcal{B}}$ amounts to the condition,

$$\frac{|x|}{|x| + |t|^n} \ge \epsilon \Leftrightarrow |dy| \text{ (resp. } \left|\frac{dy}{y}\right| \text{ near a crossing)} \le \left(\frac{1-\epsilon}{\epsilon}\right)^{\frac{1}{n}} \frac{|dx|}{|x|^{1-1/n}}\,.$$

Consequently the claim will follow from an appropriate estimate for the measure, $T\left(\mathbb{I}_{\delta,\epsilon}\, \frac{dd^c|x|^2}{|x|^{2-2/n}}\right)$ for $\mathbb{I}_{\delta,\epsilon}$ a small neighbourhood a distance δ around $\mathcal{B}$, but ϵ off $P^n_{\mathcal{B}}$. More canonically in terms of the norm $\mathbb{I}_{\mathcal{B}}$, we may observe for δ sufficiently small, determined by n,

$$\frac{1}{n^2}\, \frac{dx\, d\overline{x}}{|x|^{2-2/n}} \gg\ll 4d\, \mathbb{I}_{\mathcal{B}}^{\frac{1}{n}}\, d^c \mathbb{I}_{\mathcal{B}}^{1/n} + |x|^{\frac{1}{n}}\, \omega_n$$

for ω_n a smooth form depending on n. Equally $4d\, \mathbb{I}_{\mathcal{B}}^{1/n}\, d^c \mathbb{I}_{\mathcal{B}}^{1/n}$ is just $dd^c \mathbb{I}_{\mathcal{B}}^{2/n}$ this time up to a factor of $\mathbb{I}_{\mathcal{B}}^{2/n} \times$ smooth, so the following Jensen formula, with $\mathbb{I}_\delta$ denoting points with $\mathbb{I}_{\mathcal{B}} \le \delta$, will do nicely,

$$
\begin{aligned}
\fint_{\mathcal{C}_k(r)} f_k^* \left(1\!\!\mathrm{I}_\delta \, \frac{dd^c \| 1\!\!\mathrm{I}_{\mathcal{B}} \|^{2/n}}{\delta^{2/n}} \right) = & \int_{\partial \mathcal{C}_k(r)} 1\!\!\mathrm{I}_\delta \left(\frac{\| 1\!\!\mathrm{I}_{\mathcal{B}} \|^{2/n}}{\delta^{2/n}} - 1 \right) p_k^* \left(\frac{d\theta}{2\pi} \right) \\
& - \frac{1}{n} \int_{\partial \mathcal{C}_k(r)} 1\!\!\mathrm{I}_\delta \log \frac{\| 1\!\!\mathrm{I}_{\mathcal{B}} \|}{\delta} \, p_k^* \left(\frac{d\theta}{2\pi} \right) \\
& - 1\!\!\mathrm{I}_\delta \left(\frac{\| 1\!\!\mathrm{I}_{\mathcal{B}} \|^{2/n}}{\delta^{2/n}} - 1 \right) (0) \\
& + \frac{1}{n} \, 1\!\!\mathrm{I}_\delta \log \frac{\| 1\!\!\mathrm{I}_{\mathcal{B}} \|}{\delta} \, (0) \\
& + \frac{1}{n} \sum_{0 < |p_k(y)| < r} \mathrm{ord}_y (f_k^* \mathcal{D}) \log \left| \frac{r}{p_k(y)} \right| \\
& - T_{f_k, \mathcal{C}_k}(r) \left(1\!\!\mathrm{I}_\delta \, c_1(\overline{\mathcal{B}}) \right)
\end{aligned}
$$

From which we see that the measure $T\left(1\!\!\mathrm{I}_\delta \, \frac{dd^c |x|^2}{|x|^{2-2/n}}\right)$ goes to zero like $\delta^{2/n}$. Whence if we take $\delta = \epsilon^2$, we've comfortably established our claim. $\square$

III.5 DEGENERATIONS

Manifestly therefore whenever the condition that a sequence of maps $f_n : \Delta \to \mathcal{X}$ or more generally $\mathcal{X} \backslash \mathcal{D}$ enjoys the property that their derivatives are eventually arbitrarily close to some divisor on $P(T_{\mathcal{X}})$ but not eventually arbitrarily close to a divisor in $\mathcal{X}$, III.2.2 and I.4.4/5 apply to bound the $K_{\mathcal{X}}$, respectively $K_{\mathcal{X}} + \mathcal{D}$, degree. Consequently provided $\mathcal{X}$ has general type and the divisor is taken to include the curves contracted to the canonical (not just minimal) model we obtain convergence of a subsequence in $\overline{\mathrm{Hom}}(\Delta, \mathcal{X})$, while evidently in the quasi-projective case we work with the log-canonical model.

The question arises therefore of what becomes of a sequence of maps $f_k : \Delta \to \mathcal{X}$ which are arbitrarily close to some divisor ∂ in $\mathcal{X}$. Apart from the need to assume simple normal crossings, the proof of III.2.2 points us to a wholly general estimate.

Under these more restrictive hypothesis, and in the notations of the proof it will be more than sufficient to prove $S.\ 1\!\!\mathrm{I}_{P_{\mathcal{B}}}\, T_0 \geq 0$ for S the residual section associated to a non-parabolic curve $\mathcal{B} \subset \partial$. The key to doing this is to work with the tautological bundle L on $\mathbb{P}(\Omega_{\mathcal{X}}(\log \partial))$ in a metric which is complete for $\mathcal{X} \backslash \partial$, whence to make this difference clear we'll use, as per I.4, L^c for L metricised in such a way, and L^{sm} for a smooth metric. Provided we work with L^c and evaluate derivatives of maps at the origin in the complete metric we have an arguably more satisfactory inequality with much the same sort of error but with the $\log\log^2 \| 1\!\!\mathrm{I}_\partial \|$ at the origin term implicit in the combination of I.4.4 and I.4.5 replaced by log of the degree of $-dd^c \log\log^2 \| 1\!\!\mathrm{I}_\partial \|$. Consequently if $T_k \to T$ is a sequence of normalised currents associated to the degree function for maps from various discs it is not, in general, true that $L^c \cdot T_k \to L^c \cdot T$ when the latter is calculated in the sense of currents, since it would necessarily be $L^{\mathrm{sm}} \cdot T$. The fact that this is so is why the L^c formulation has not been employed as it would be unsuitable for foliations where L must be compared with $K_{\mathcal{F}}$ by way of smooth metrics. For maps in a tubular neighbourhood of curves without other restrictions, however, it is ideal provided we bear in mind that we're working with $\lim_k L^c \cdot T_k$ everything goes through, and the key points are:

III.5.1 Facts (a) (Notations as above) *Let $f_k : \Delta \to \mathcal{X}$ be a sequence of maps of unbounded degree at some radius $r_0 \in (0,1)$ with $\mathcal{D}$ a divisor on $\mathcal{X}$ then, after subsequencing, for $r \in (0,1)$ outside a set of finite hyperbolic measure we have,*

$$\fint_{\Delta(r)} f_k^*(-dd^c \log\log^2 \|1\!\!\mathrm{I}_{\mathcal{D}}\|) \leq \log(1-r^2)^{-1} + 0(\log \fint_{\Delta(r)} f_k^* c_1(\overline{H}) + \log N_{f_k,\mathcal{D}}(r))$$

where, as ever, $\overline{H}$ is positive ample, and $N_{f_k,\mathcal{D}}$ is the counting function, i.e. as per the definition of the radical around $\mathcal{D}$ but with multiplicity

(b) *In particular if $f_k' : \Delta \to \mathbb{P}(\Omega_{\mathcal{X}}(\log \partial))$ are the logarithmic derivatives and S the residual section associated to a divisor omitted by the f_k with $\|1\!\!\mathrm{I}_S\|$ calculated in the complete metric then after subsequencing we can find a convergent sequence of automorphisms $\alpha_k \in \mathrm{Aut}(\Delta)$ such that $\alpha_k^*(f_k')^* \|1\!\!\mathrm{I}_S\|(0)$ is uniformly bounded away from 0 in k.*

Proof. (a) This is just an application of Jensen's formula used to yield integral representations of $\log f_k^* \|1\!\!\mathrm{I}_{\mathcal{D}}\|$. The model is the proof of the logarithmic derivative lemma in [L].

(b) In the coordinates x, y, t of the proof of III.2.2, and away from a crossing of ∂ (otherwise even easier) the condition $\|1\!\!\mathrm{I}_S\| \leq \epsilon$ amounts to,

$$\frac{|t|}{|t| + \frac{1}{|\log x|}} \leq \epsilon \Leftrightarrow |t| \leq \frac{\epsilon}{(1-\epsilon)} |\log x|^{-1}$$

and the statement follows from the Jensen type formula for $dd^c \log\log^2 \|1\!\!\mathrm{I}_{\mathcal{B}}\|$. $\square$

Applying this to our situation, or more correctly a minor variant to cope with several residual sections at the same time, and denoting by F_1^c the metricisation of the exceptional divisor over the blow up of S in the complete metric with $T_k \to T$ the limiting sequence of degree functions we obtain $\lim_k F_1^c \cdot T_k \geq 0$ regardless of the proximity of the maps f_k to $\mathcal{B}$ at the origin. Better still letting ρ_ϵ be a sequence of bump functions in a neighbourhood of $\mathcal{B}$ and unravelling the Skoda-El Mir theorem we have,

$$\lim_{\epsilon \to 0} \lim_{k \to \infty} T_k\{(1-\rho_\epsilon)\, c_1(F_1^c)\} = 0$$

since $\|1\!\!\mathrm{I}_S\|$ blows no worse than $|\log \|1\!\!\mathrm{I}_{\mathcal{B}}\||$ around $\mathcal{B}$, and whence,

III.5.2 Fact (Notations as above) *Suppose for $r_0 \in (0,1)$, the degrees of the f_k at radius r with respect to a positive metricised ample are unbounded in k and the f_k are arbitrarily close to ∂, then there is a constant $a > 0$ such that after subsequencing, and composing with convergent automorphisms, we have for $n \in \mathbb{N}$*

and $r \in (0,1)$ outside a set of finite hyperbolic measure,

$$\fint_{\Delta(r)} f_k^* c_1(L^c) \geq \fint_{\Delta(r)} f_k^* c_1((K_{\mathcal{X}} + \partial) - \frac{a}{n}\fint_{\Delta(r)} f_k^* c_1(H) + \frac{1}{n}\log \| f_k^*\, \mathbb{1}_Z \| (0)\,,$$

provided $k = k(0) \gg 0$, where Z is the part of the boundary not ommitted by the maps.

Proof. The fact that the error term at the origin only involves Z is the above discussion. Everything else is immediate from III.4 up to being careful about how the current is normalised. On observing however, and quite generally, that on any modification $\psi : \widetilde{V} \to V$ with exceptional divisors E_i, and H ample on the base, some $\psi^* H\left(-\sum_i \delta_i\, E_i\right)$ for δ_i as small as we like is still ample we have our required normalisation with any extra error absorbed into the inequality itself. □

Now let's turn this into an estimate for the degeneration of the Kobayashi metric on a quasi-projective surface $(\mathcal{X}, \mathcal{D})$ with big log co-tangent bundle, and indeed its complete hyperbolicity modulo Z. We necessarily suppose not only that $(\mathcal{X}, \mathcal{D})$ has log general type, but that $(\mathcal{X}, \mathcal{D})$ is the log-canonical model, or equivalently $K_{\mathcal{X}} + \mathcal{D}$ is ample. The unfortunate thing is that the set Z on $\mathcal{X}$ could be highly singular and for $\mathcal{D} \neq \varphi$, meet the boundary in an extremely messy way. Consequently to apply III.5.2 we must pass to a resolution $\rho : \widetilde{\mathcal{X}} \to \mathcal{X}$ such that $\widetilde{\mathcal{X}}$ is smooth, and the total pull-backs of $\mathcal{D}$ and Z have simple normal crossings. Our boundary ∂ then will be the union of $\widetilde{\mathcal{D}}$, the reduced total pull-back of $\mathcal{D}$, and $\widetilde{Z}$, any component of the reduced pull-back that isn't in $\widetilde{\mathcal{D}}$. An appropriate measure of the singularity of our initial situation is provided by the discrepancy, i.e.

$$K_{\widetilde{\mathcal{X}}} + \widetilde{\mathcal{D}} = \rho^*(K_{\mathcal{X}} + \mathcal{D}) + \sum_i a_i\, E_i$$

where the E_i are the curves contracted by ρ, and $a_i \in \mathbb{Q}_{\geq 0}$. Obviously to keep the situation as natural as possible ρ should be minimal, i.e. first pass to the unique minimal smooth resolution $(\mathcal{X}', \mathcal{D}')$ of $(\mathcal{X}, \mathcal{D})$, then take a minimal resolution for the divisor $\mathcal{D}' + Z'$, where the latter is the proper transform of Z. In any case denote by $\| \ \|^c_{\widetilde{\log}}$ a complete norm on $T_{\widetilde{\mathcal{X}}}(-\log \partial)$ then we have,

III.5.3 Corollary *Notations as above $(\mathcal{X}, \mathcal{D})$ is complete hyperbolic modulo Z, and for all $\epsilon > 0$ there is a constant $C(\epsilon)$ such that for any $f \in \mathrm{Hom}(\Delta, \mathcal{X}\backslash\mathcal{D})$,*

$$\prod_i \| \mathbb{1}_{E_i} \|^{a_i+\epsilon}(0)\, \| d\widetilde{f} \|^c_{\widetilde{\log}}(0) \leq \frac{C(\epsilon)}{\| \mathbb{1}_{\widetilde{Z}} \|^{1+\epsilon}}$$

where $\widetilde{f} : \Delta \to \widetilde{\mathcal{X}}$ is the proper transform.

Apart from bearing in mind that this and III.5.2 are the unique situations where the degree on $\mathcal{X}$ may not dominate that on $\widetilde{\mathcal{X}}$, and so far an ample bundle we must use $\rho^*(K_{\mathcal{X}} + \mathcal{D})\left(-\sum_i \delta_i\, E_i\right)$ for appropriately small δ_i, and whence the ε of the announcement, this is just an application of III.5.2 as per the methodology of III.1.4, with the various inequalities for the $K_{\mathcal{F}}$ degree replaced by the said bound for the $K_{\widetilde{\mathcal{X}}} + \widetilde{\mathcal{D}} + \widetilde{\partial}_0$ degree.

Opus Quartum

Uniform Uniformisation

INTRODUCTION

Most misunderstandings in mathematics start in dimension 1. Probably the best example is class field theory, but another good one arose with R. Nevanlinna's use of the uniformisation theorem to prove a trivial yet hard isoperemtric inequality for the punctured plane. Unlike a classical inequality between area and length, Nevanlinna averaged the former, and so was able to replace the latter by its logarithm. In so doing, though, he used the full power of the complex structure, and rendered his study ill suited to soft generalisation. Nevertheless he did create value distribution theory, or better the study of the isoperimetric inequality in his sense, VI.2.2, on quasi-projective varieties. Depending on whether one studies discs or parabolic (in the sense of Ahlfor's) Riemann surfaces on one's variety this is more, respectively less, general than the isoperimetric inequality in Gromov's sense, albeit that under reasonable hypothesis, [Kl], even the less general version will imply the latter, and of course, all of these inequalities should be viewed as an integrated or large scale version of negative curvature.

At first glance this invocation of hard methods appears pointless. A brief reflection shows that holomorphic sectional curvature, even stricta dicta, does not admit an a priori interpretation in terms of algebraic geometry, except, of course, for curves where it coincides with Ricci curvature, whence the equidimensional Nevanlinna theory of Griffiths, Stoll et al. The said equidimensional study, however, confined itself, for largely historical reasons, to smooth varieties as opposed to what is now known to be the natural condition for negative Ricci curvature, i.e. Mori/Reid's canonical singularities. Indeed these can even be defined as precisely the singularities for which local versions of any of the above isoperimetric inequalities hold, and whence a conjectural equivalence between the isoperemtric inequality for all balls and canonical models of algebraic varieties, with the conjecture being the reverse implication by way of the minimal model programme.

Combined with examples this does seem to suggest that any attempt to pursue algebraic criteria for the isoperemtric inequality in a non-equidimensional setting is by definition hopeless. Indeed unlike the 'logarithmic surface area' of a ball which is naturally and uniquely computed by Nevanlinna's height applied to the canonical class there are countably many bundles that one might look at for discs, i.e. the tautological bundles on spaces of jets. At this point, however, Green & Griffiths, [GG], made the key observation that the law of large numbers may come to the rescue, implying, as it does, at least for algebraic surfaces when combined with the more subtle properties of the canonical class -essentially the existence of a Kähler-Einstein metric-, that for high enough jets the bundles in question are big when the

Ricci curvature is negative. Developments in jet technology post op. cit. occasions a review of this in VI.1, but unlike the actual canonical class, it may not be possible to carry out surgery on either the surface or its jets to guarantee that these bundles are ample. Even worse there may be a base locus on which the restrictions of any of these bundles fails to be pseudo effective. On the positive side, though, the problem becomes equidimensional, albeit in the softest sense that current understanding of category theory permits, i.e. the linearisation of an nth order ODE corresponding to a component of the base is a foliation by curves, $\mathcal{F}$, on a $n + 1$ dimensional variety X. Whence, up to some technical issues of definition occasioned by it's singularities, an identification of negative curvature with the relative positivity of the canonical class $K_{\mathcal{F}}$ of X over its classifying stack $[X/\mathcal{F}]$.

The issues posed by the foliation singularities are two fold. The one of no immediate relevance is for the definition of the classifying stack itself. This is, however, no worse than the problem of trying to view a punctured disc as a stack in the analytic topology with an infinite punctual stabiliser, and so is easily swept under the carpet. More critically, just as for Ricci curvature on varieties, one needs canonical foliation singularities before negative curvature, even locally, can be identified with a positivity property of $K_{\mathcal{F}}$. Indeed without this, one has, even for X smooth, a bundle which is only really a relative dualising sheaf rather than a genuine canonical class. Again, this can be taken as a definition of canonical singularities, which in turn is equivalent to the functorial extension of the Mori/Reid definition to this context, [II]. Consequently the universality of our discussion depends on progress to a hypothesised resolution LCR, I.6.1, but this is rather promising, and while postponing our principle commentary on it till later, it is of immediate relevance to note that this cannot be achieved in the category of smooth spaces foliated by curves, but only in the 2-category of smooth algebraic (which will always mean Deligne-Mumford) stacks. Fortunately on replacing X by an algebraic stack $\mathcal{X}$ there are no more problems associated with the formation of $[\mathcal{X}/\mathcal{F}]$ than those we've already described, whence we will work with, and profit from, the soft flexibility that algebraic stacks afford, albeit that the presence of a taming form, i.e. the projectivity of the moduli, is essential.

Not all our initial preparation, however, is confined to blowing up, for we must also blow down, or more generally flip, the more obvious obstructions to negative curvature that we come across. In so doing we will be able to localise the refined tautological inequality of [III] II.2.10, with the resulting model $(\mathcal{X}, \mathcal{F})$ enjoying $K_{\mathcal{F}}$ nef. Such a, so called minimal, model, which by op. cit. exists whenever the foliation isn't a pencil of conics, only eliminates any positively curved invariant subvarieties, and what is really desired is to carry out algebraic surgery until such times that we arrive to, a *canonical model*, i.e. not only $K_{\mathcal{F}}$ nef. but any solution curve $\mathcal{C}$ with $K_{\mathcal{F}}.\mathcal{C} = 0$ meets every effective Cartier divisor non-negatively, or equivalently: if in an act of wild optimism we suppose that the obstructions to negative curvature along our foliation will concentrate on something algebraic, then the algebraic object should move, giving our foliation a special form. Unlike minimality this is a substantially more difficult condition to guarantee. Indeed, every type of singularity that one can conceive of in actual Mori theory has a variant, in fact several, in the theory of canonical models of foliations, with plenty more beside, e.g. of arbitrary embedding dimension on 3 folds. The lack then of such a model, and with it the implied possibility of large algebraic sets of nil curvature will be a rather major difficulty in addressing our curvature problem to which we finally give a concrete form,

Question *Presented with a minimal algebraic stack $(\mathcal{X}, \mathcal{F})$ foliated by curves, can we find algebraic criteria to guarantee that $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} > 0$ for all transverse invariant measures $d\mu_{\mathcal{X}/\mathcal{F}}$?*

Of course the measures associated to bigger and bigger discs are not arbitrary, indeed they will, without loss of generality, also be nef., i.e. intersect every Cartier divisor non-negatively, so $K_{\mathcal{F}}$ big (general type) is certainly sufficient. Unlike the Ricci case, however, bigness is not a sine qua non for negative curvature, but rather for 'full' modular variation of hyperbolic leaves. A fact that is already evident in the theory of algebraic curves, but whose manifestations are not confined to such integrable examples. If, however, we could quantify the lack of bigness by appropriate smooth metricisation of $K_{\mathcal{F}}$ then the Bochner technique, [M2], would allow us to assert that a lack of modular variation causes the Kähler structure to split, and whence resolve our problem. Unfortunately this is very much a catch 22 scenario since the key to realising such a metricisation is the question itself. What we do learn, though, is that in each dimension n we should expect exactly one new case, i.e. not pulled back or fibred over lower dimension, which is hyperbolic yet not of general type, and it should satisfy $K_{\mathcal{X}}.K_{\mathcal{F}}^{n-1} > 0$. We can thus refine our question by way of,

Antithesis *Can there exist a parabolic invariant measure, i.e. $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$, together with some naturality conditions, V.I.2, on a minimal foliated stack with $K_{\mathcal{F}}^n = 0$, but $K_{\mathcal{X}}.K_{\mathcal{F}}^{n-1} > 0$.*

To put the antithesis in perspective, one observes, quite generally, II.2.2, that a nef. bundle K on a n dimensional projective variety with $K^{n-1} \neq 0$ admits at most one class intersecting every effective Cartier divisor non-negatively, yet 0 with K, i.e. K^{n-1}. Needless to say such nefness is one of the naturality conditions on our measure $d\mu_{\mathcal{X}/\mathcal{F}}$, and so what our antithesis leads to is a special representative of $K_{\mathcal{F}}^{n-1}$. Plainly the particular thing about this representative is its invariance. It is important, however, to have some feeling for the nature of measures on proper stacks, which are a priori much better behaved than the simple patching of local data would seem to suggest. Indeed the algebraist who would be inclined to think of such as some sort of analytic 1-cycle will be much closer to the truth, than the analyst who carries a bag of local counterexamples to any reasonable statement that one might wish to make. In particular for every closed substack $\mathcal{Z}$ of $\mathcal{X}$ there are well defined segre classes $s_{\mathcal{Z},d\mu}$, or residual measures, on the projective normal cone, III.3, specialisation to the cone itself, III.5, and a well defined notion of *diffuseness*, i.e. lack of expressablity as a countable sum of direct images from lower dimensional stacks, V.3 & VI.3. The invariance, however, gives rise to a very different, a priori highly non-logarithmic, residue symbol, $Res(d\mu_{\mathcal{X}/\mathcal{F}})$, IV.1, expressing the lack of flatness of the bi-rational groupoid, IV.4, $\tilde{\mathfrak{F}} \rightrightarrows \mathcal{X}$ seen from $d\mu_{\mathcal{X}/\mathcal{F}}$.

In the presence of LCR, however, and essentially by definition, foliation singularities are logarithmic. The difficulty though, as is well known for saddles on surfaces, is that the logarithmic coordinate defines a sub-variety, the weak branch, which may be purely formal. In general, LCR implies, I.6.7, with the aid of what one might venture is the best theorem in [EGA], coherence of proper push forward for formal schemes, a higher dimensional theory of weak branches, defined as proper formal sub-stacks. There is, however, an important caveat that these branches may only be definable at completions in irreducible rather than connected components of the singular locus, and as the component changes, the nature of the weak branch changes radically. The first example of this, the 3-D beast, I.5.1, is presented in detail. If, then, we were to sweep the difference between convergence on formal, and analytic neighbourhoods under the carpet, the symbol Res is given by a logaithm: the residual measure around the weak bracnhes, IV.4.4. This leaves a problem of computing meromorphic residues within the weak bracnhes, IV.2.4, but the real difficulty is that the residual measure at weak branches enjoys no obvious relation with the residual measure around any algebraic sub-stack, and in particular the residual measure around $\mathrm{sing}(\mathcal{F})$.

For isolated singularities, IV.4.3, the trick of [M1] for computing residues at surface saddles continues to work, and provides a direct relation between the said measures. In arbitary dimension, however, any attempt to compare the residual measure around the weak branches with that around $\mathrm{sing}(\mathcal{F})$ is hampered by the curvature of the singular components. In the presence of many meromorphic functions, however, one can arrange for these curvature contributions to arise with a definite, and favourable, sign provided that the logarithms of meromorphic functions are ℓ_1 against a Kấhler form wedged with our invariant measure, IV.6.6. This may be the ravings of someone obsessed with integrating $\partial\overline{\partial}$, but it may also be an important reflection of the fact that nothing here is well defined unless $\mathrm{sing}(\mathcal{F})$ is proper, and, of course, meromorphicity is a substantial aid to the study of proper spaces. More, importantly though, the absolute integrability of logs of meromorphic functions against an invariant measure is a wholly non-trivial condition, which, in general, is quite false, even on supposing that the residual measure around the zeroes and poles of these functions is nil. The integrability of logarithms on Riemann-surfaces not withstanding, one can make all the counter-examples that one wants by just looking at algebraic fibrations. Amongst this myriad of counter-examples, one can identify the classical integrability of the logarithm as taking place when the invariant measure is a δ-function on the base of the fibration. This suggests that what we require to do is to identify amongst all transverse invariant measures those which are extremal, put a space structure on these, and realise our given invariant measure as a Radon measure on this space. This has been done, [II], A.3, and allows us to reduce the study of the residue symbol to that for extremal measures. Although functorially identical with δ-functions on the base of a fibration, such measures still do not lead to a verbatim generalisation of elliptic theory in the relative setting of $\mathcal{X} \to [\mathcal{X}/\mathcal{F}]$, since, at the end of the day, classical elliptic theory needs the product of compacts to be compact, which categorically would amount to universal closedness, which we plainly cannot suppose. The theory of extremal measures, however, does retain a tiny vestige of elliptic regularity: the ℓ_1 norm of a function is bounded by its ℓ_1 derivative. In principle, this appears to be a more or less universally true categorical statement valid in the generality of op. cit., although our applications of it are ad hoc, IV.6, and one should be careful to note that this principle is optimal, *e.g.* there are extremal measures, and functions whose Laplacian in the foliation direction is ℓ_∞, but which aren't much better than ℓ_p for all p. Nevertheless, combined with LCR it's enough to allow us to integrate log's of meromorphic functions against extremal measures, and deduce,

Lemma (IV.7.3) *Suppose* LCR^+ *and that the segre class of a transverse invariant measure around* $\mathrm{sing}(\mathcal{F})$ *is zero, then its residue class, Res, is zero.*

Here LCR^+ is a serious of plausible, *i.e.* compatible with actual theorems in dimensions 2 and 3, convergence assumptions over and above LCR, IV.4.3, which lead to a more refined application of the basic stratagem for computing Res in order to accomodate the failure, in general, of the formal description of weak branches to be excatly reflected at the analytic level. This though is what makes a class in Néron-Severi represented by an invariant measure numerically special, so that if in the antithesis we further suppose the measure is diffuse, then a simple minded counting argument with global sections of bundles, V.1.4, V.2.2, implies,

Corollary (VI.3.4/5) *For* $d\mu_{\mathcal{X}/\mathcal{F}}$ *diffuse the antithesis is rubbish. Indeed even* $K_{\mathcal{X}}.K_{\mathcal{F}}^{n-1} \neq 0$ *will suffice.*

One's immediate reaction, therefore, is that our question has been answered modulo an induction the size of an olympiad problem book (compendium edition). Unfortunately, though, a new higher dimensional difficulty, V.4 & VI.4, emerges: obstructions to negative curvature can hide in $\mathrm{sing}(\mathcal{F})$. This is not, for example, visible on the universal curve over the moduli stack of algebraic curves since it initially occurs when invariant discs on the smooth locus limit on $\mathrm{sing}(\mathcal{F})$, giving rise to singular solutions, I.4.10. As such it is a phenomenon that takes place only in components of $\mathrm{sing}(\mathcal{F})$, which are invariant for the induced foliation of some formal weak branching stack.

This ultimate difficulty, which is precisely the obstruction to inducting the parab olicity of a measure through smaller substacks merits some comment. Having got this far, there is a precise technical sense, numerical relevance VI.5.1/2, in which we can meaningfully discuss whether a measure is 'supported' (infinitely tangent is more accurate) in a weak branch or not, irrespective of any questions of analytic convergence of such branches, and a very general counting argument shows that any part of $d\mu_{\mathcal{X}/\mathcal{F}}$ which is relevant to a weak branch, VI.5.1/2, is in fact a diffuse measure on some honest substack in it's pseudo trace, i.e. the completion of the latter in appropriate components of $\mathrm{sing}(\mathcal{F})$ is non-empty and factors through the formal stack in question. The critical observation, V.4.1 & VI.5.3, is that the dynamics around the weak branch are so isolated from everything else that the totality of components held together by this infinitesimal glue inherits all the naturality properties, including nefness to the extent that it has sense, of a parabolic measure with the exception of the parabolicity condition itself. This latter requires further surgery, weak flops, specified by a final critical refinement, VI.6.5 & VI.7.3, of the cone theorem of [I]. Such weak flops are rather natural since they terminate in a model on which every sequence of 'solutions' to the foliation, i.e. the co-normal bundle vanishes, converging to a disc with bubbles, converges to an honest disc.

Flops, though, are to be understood as any non-contractile surgery from a minimal to a canonical model, as such, even weak ones are strictly more difficult than flips-klt foliated triples $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ are very rare. Thus to complete our machinery we could either make weak flops or extend the flip theorem to 'pseudo irreducible formal substacks'. The former is more conceptual, but leads to slightly worse ambient singularities, so, for expediency, we choose the latter, VI.6.10-12, to kill the last set of a priori obstructions to localising parabolicity.

Rather than undertaking a humongous induction to completely resolve our curvature query, let us take a couple of illustrative examples beginning with 3-folds, where the previous step in dimension 2 has been done, [M2], and we require to understand the hypothesised, necessarily non-diffuse, parabolic measure of the antithesis. The surface theory, combined with the above machinery, quickly allows us to deduce, V.2 & 4, that it must be a countable sum of measures of the following form: invariant measures on surfaces where the induced foliation is a conic pencil, rational curves which don't move, and divisors of elliptic fibre type inside a formal weak branching surface. The first and third possibilities are effectively finite, at least as far as the conic pencil itself is concerned. Plainly, we should really try and move the elliptic curve to show that it doesn't exist, but this is an aesthetic point, and what's essential is to exclude the possibility of infinitely many rigid rational curves invariant by the foliation and intersecting $K_{\mathcal{F}}$ in zero. A detailed local analysis, II.3-8, shows the curves in question can be flopped. If then it could be guaranteed that flopping preserved projectivity it would be easy to show that there were at most finitely many. The problem though is that it is rather difficult to relate this soft sense of flopping to the more generally accepted one involving a specific divisor. Indeed one could imagine being on an actual canonical model, but still finding infinitely many such contractible curves because the singularities became so bad that one simply ran out of Cartier divisors. The flopping operation, though, can only fail to be projective because there is no sense of positivity in Néron-Severi which if raised to the level of measures would be a contradiction since these must always have a sign irrespective of the existence of a taming form, and this is exactly what we do by way of some a priori soft, i.e. étale surgery, II.9, to ensure that the rigid curves are wholly isolated, and whence,

Finiteness in 3-D (V.2-4) *Let $d\mu_{\mathcal{X}/\mathcal{F}}$ be a parabolic invariant measure on a three dimensional minimal foliated stack $(\mathcal{X}, \mathcal{F})$ not of general type with $K_{\mathcal{X}}.K_{\mathcal{F}}^2 \neq 0$ then there is a decomposition,*

$$d\mu_{\mathcal{X}/\mathcal{F}} = \sum_i d\mu_i + \sum_j L_j + \sum_k E_k$$

where, $d\mu_i$ are invariant measures on the finitely many invariant surfaces where the foliation is in conics, and up to positive multiples, the L_j are a finite sum of invariant rational curves, and the E_k are the above, likely inexistent, elliptic curves.

In the case that $K_{\mathcal{X}}.K_{\mathcal{F}}^2 < 0$, $(\mathcal{X}, \mathcal{F})$ is actually fibred in rational curves over a foliated surface, so the elliptic bit doesn't exist, and the whole thing is an easily analysed Ricatti type object. In the antithesis case, the proof pretty much establishes that all the rational part can be killed by flopping, and contraction, so up to moving the elliptic object, and a slightly better understanding of the role $c_1^2 + c_2$ plays in Riemann-Roch, our curvature question is completely answered on 3-folds. To apply this to our initial problem on surfaces we need canonical resolution, albeit in local form, ILUT, I.1.3, together with [III] III.2.2, which instantly gives,

Corollary *Suppose ILUT, and let $(\mathcal{S}, \mathcal{D})$ be a 2-dimensional log-stack of general type with $c_1^2 > \frac{2}{3}c_2$ then the set of rational and elliptic curves (i.e. log-substacks étale covered by orbifolds of positive, respectively nil, topological Euler characteristic) form a proper substack $\mathcal{Z}$ outside of which the isoperimetric inequality holds in Nevanlinna's sense for arbitrary ramified covers of the line. In particular curves of genus g are bounded in moduli, and no holomorphic map from an affine algebraic curve can have Zariski dense image.*

More generally, our machinery is well adapted to investigating the isopermetric inequality on any surface. Indeed the semi-stability of the co-tangent bundle on a minimal surface with respect to it's canonical means that any ODE of order at least 1 once linearised and viewed as a foliation is either of general type, or satisfies the conditions of the antithesis. The extension to minimal models, [II], of the fact that any rational map to a curve of positive genus is a morphism allows us to move easily between ODEs, and sub-ODEs, even if the latter isn't finite over it's natural image in the jet space, and to induct all the way down to order 0, i.e. curves on surfaces, VI.3. Arriving to hyperbolic curves is impossible, but for rational or elliptic we need [III] III.2.2 (strictly speaking III.4.1), which in turn needs the number of these to be finite. Whence, our second application takes the form,

Theorem *Suppose* LCR^+, *then for $(\mathcal{S}, \mathcal{D})$ a 2-dimensional log-stack with canonical (Mumford $K_\mathcal{S} + \mathcal{D}$ sense) singularities and stable curve without elliptic connected components as boundary the following are equivalent,*

(a) *$(\mathcal{S}, \mathcal{D})$ is of general type without rational or elliptic curves.*

(b) *$(\mathcal{S}, \mathcal{D})$ satisfies the isoperimetric inequality in Nevanlinna's sense for ramified covers of the line.*

(c) *$(\mathcal{S}, \mathcal{D})$ satisfies the isoperimetric inequality in Gromov's sense for discs.*

More, or less, depending on your point of view, generally, the following are also equivalent under LCR,

(a) *$(\mathcal{S}, \mathcal{D})$ has general type, and the set of rational and elliptic curves form a proper closed substack $\mathcal{Z}$.*

(b) *The isoperimetric inequality for ramified covers of the line holds in Nevanlinna's sense on $(\mathcal{S}, \mathcal{D})$ outwith finitely many rational and elliptic curves*

(c) *The only ODEs on $(\mathcal{S}, \mathcal{D})$ admitting parabolic measures are rational and elliptic curves.*

(d) *The solution discs of any ODE on $(\mathcal{S}, \mathcal{D})$, other than a rational or elliptic curve, converge, in the compact open sense, modulo a proper algebraic set.*

To establish the a priori finiteness of elliptic curves on surfaces of general type, would appear to be a perfectly feasible refinement of the methodology. Despite the implied additional elegance, this isn't done because it won't change the basic fact that the methodology has a problem with the obstruction posed by parabolic algebraic curves. Indeed from the foliated point of view, a repeat of the 3-D finiteness theorem for rational curves looks quite hard, and plainly requires a fairly systematic attack in the direction of the construction of a canonical model, but with the same sort of caveats as for 3-folds that this isn't quite the same thing. In particular, while it's likely, it's not a given that the finiteness theorem holds for arbitrary foliated 4-folds, but at the same time it's unclear how to tailor things to the original surface.

Lack of LCR (log-canonical resolution) of foliation singularities should also be viewed as an obstruction rather than LCR as a hypothesis. Indeed if we have the convergence of invariant discs then a valuation for which LCR fails, which is necessarily approximable by discs outwith very special circumstances, will infact be a disc (strictly speaking a family of discs) and this gives the sort of contact structure which implies LCR. Plainly, making this rigorous is worth while, since we'd then arrive to a completely algebraic description of the validity or otherwise of the isoperimetric inequality on algebraic surfaces. One reason for not pursuing this is the untidy intervention of convergence issues in LCR^+, another is the on going work of F. Cano, of which I'll attempt a quick summary. In [C1], F. Cano proved the local uniformisation theorem in dimension 3, and whence one would have imagined LCR in dimension 3. Unfortunately the corollary didn't follow since although the local global argument here goes back to Zariski, the local version in question was not sufficiently functorial with respect to the ideas, i.e. not all the centres respected the foliation, and what was worse there was even a case where the last centre was defined by a differential rather than an algebraic condition, I.1.2. This ultimate problem is intrinsic, and whence Cano proposed that the correct strategy was to take a root. Consequently the globalisation procedure is necessarily in the 2-category of algebraic stacks and/or $\mathbb{Q}$ gorenstien foliations if one wants to stick to spaces. The content of I.1 is the verification that Zariski's local global argument still works in these circumstances, which indeed it does, albeit in a slightly more delicate way, i.e. first use it to 'prepare' the divisor where one needs to take a root, then pass to stacks, then run it again. This discussion is not particularly specific to dimension 3, since combined with [BM] it will globalise any reasonable local theorem. The critical step then of local uniformisation looks likely to be accomplished by Cano and his collaborators by explicitly exploiting the one-dimensional features of the problem. Indeed just as the minimal model theorem can be viewed/is a large scale generalisation of semi-stable reduction of curves, they intend to view the problem as a large scale generalisation of plane curve desingularisation. The key point is a theorem of J. Cano, [Cj], see also [GS], that the Newton-Puiseux description around a valuation retains its validity for an arbitrary plane ODE. Thus although of almost immediate relevance to our situation on algebraic surfaces, it quite generally implies by a generic projection argument the existence of a contact structure, and the lack of any unpleasant surprises such as some Diophantine condition between coefficients of power series. Given that he's worked hard for it, and the solution is in sight, I would anticipate that F. Cano's immediate goal is to complete the proof on 3-folds, so that my only fear for the validity of LCR is that he stops short of arbitrary dimension, which in no essential way differs from 4.

I am, therefore, particularly indebted to Cano for explaining this to me, along with Bogomlov, Bonk, Gromov, Kontsevich & Paun for several other key contributions.

I. SINGULARITIES

I.1. Invariant Local Uniformisation

An example of F. Cano, c.f. [C1], shows that canonical resolution of singularities cannot be achieved in dimension at least 3 by a sequence of blow ups in smooth centres. The difficulty, however, is resolved by working in the 2-category of stacks, and for convenience, we'll take this to mean with projective moduli. As such consider the following types of 'simple' morphisms between foliated logarithmic (i.e. with simple normal crossing boundary) stacks $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{D}, \mathcal{F})$, with, of course, $\tilde{\mathcal{D}} = \rho^{-1}\mathcal{D}_{red}$,

(P) *Blow up in a $\mathcal{F}_{\log}$ super-singular point, i.e. a negative discrepancy singular point 0 of $\mathcal{F}_{\log}$ such that the order of vanishing at 0 of a local generator is greater than that of any smooth invariant curve through it.*

(C) *Blow up in a smooth $\mathcal{F}_{\log}$-invariant curve, 'transverse' to $\mathcal{D}$, where, for want of a better word, 'transverse' means chosen so as so to preserve the simple normal crossing hypothesis, i.e. if the curve meets sing$(\mathcal{D})$ then it's actually contained therein.*

where $\mathcal{F}_{\log}$ refers to the logarithmic vector field in $T_{\mathcal{X}}(-\log \mathcal{D})$ generating the foliation. Manifestly there is a smallest 2-category $\mathcal{C}$ generated by 'simple' morphisms given an initial object $(\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$. The key to deducing a global resolution statement from a local one involving sequences of 'simple' morphisms is well known to experts, i.e.

I.1.1 Claim *The above 2-category $\mathcal{C}$ has fibre products or, easier, the stack fibre product can be dominated by an element of $\mathcal{C}$.*

proof Plainly this reduces to looking at diagrams of the form,

$$\begin{array}{ccc} (\mathcal{X}_1, \mathcal{D}_1, \mathcal{F}_1) & \xleftarrow[?]{} & ? \\ \Big\downarrow & & ?\Big\downarrow \\ (\mathcal{X}, \mathcal{D}, \mathcal{F}) & \longleftarrow & (\mathcal{X}_2, \mathcal{D}_2, \mathcal{F}_2) \end{array}$$

with 'simple' unquestioned arrows, and to show that we can take the questioned arrows in $\mathcal{C}$. Independently of the foliation this is absolutely trivial for unquestioned arrows of type (P) or (C). Indeed, everything reduces to the combination (P) & (C), so the questioned arrows are either blow up in the proper transform of the curve, or the fibre over the point. What is less trivial is to guarantee the invariance of the centres under $\mathcal{F}_{\log}$, or more correctly in the case of the fibres over the point. Nevertheless the definition of super-singular has been precisely constructed to guarantee this. $\square$

As we've already remarked the operations (P) & (C) are insufficient to achieve log-canonical resolution in dimension 3, as the following example of F. Cano illustrates, viz:

I.1.2 Example

$$\partial = y\frac{\partial}{\partial z} + xz\frac{\partial}{\partial y} + x^{p+1}\frac{\partial}{\partial x},\ p \in \mathbb{N},\ +\ h.o.t.$$

with precision on the higher order terms to be found in [FF]. As such consider the following additional operation,

(R) *Extract a root of a smooth component of $\mathcal{D}$ if around the said component no 'improvement' is to be had by operations of type* (P) *or* (C).

Manifestly the word 'improvement' merits amplification. Plainly if a generator in $T_{\mathcal{X}}(-\log \mathcal{D})$ vanishes to order at least 2 at some point, then $\mathcal{F}_{\log}$ has a centre of positive discrepancy around it, and so, this can be improved. Consequently the only candidate for being non-improvable is something nilpotent in its linear part, and, more precisely, Cano's example. The said example can, however, be taken without loss of generality, to have $x = 0$ a component of $\mathcal{D}$, and is subsequently resolvable on extracting a root. As such consider,

I.1.3 ILUT (Invariant local uniformisation theorem) *Let v be a valuation of the function field of $\mathcal{X}_0$, then there is a sequence,*

$$(\mathcal{X}_n, \mathcal{D}_n, \mathcal{F}_n) \underset{\rho_n}{\longrightarrow} (\mathcal{X}_{n-1}, \mathcal{D}_{n-1}, \mathcal{F}_{n-1}) \underset{\rho_{n-1}}{\longrightarrow} \ldots\ldots \underset{\rho_1}{\longrightarrow} (\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$$

with ρ_i of type (P), (C), *or* (R) *around the centre of v on $\mathcal{X}_{i-1}$ such that the centre of v on $\mathcal{X}_n$ has log-canonical foliation singularities.*

The consequence of ILUT as opposed to LUT is, or course,

I.1.4 Fact *Suppose* ILUT, *then for any $(\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$ there is a sequence,*

$$(\mathcal{X}_n, \mathcal{D}_n, \mathcal{F}_n) \underset{\rho_n}{\longrightarrow} (\mathcal{X}_{n-1}, \mathcal{D}_{n-1}, \mathcal{F}_{n-1}) \underset{\rho_{n-1}}{\longrightarrow} \ldots\ldots \underset{\rho_1}{\longrightarrow} (\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$$

with ρ_i globally of type (P), (C), *or* (R) *such that $(\mathcal{X}_n, \mathcal{D}_n, \mathcal{F}_n)$ has log-canonical foliation singularities.*

proof Suppose otherwise and augment $\mathcal{C}$ to the smallest 2-category, $\tilde{\mathcal{C}}$ generated by $\mathcal{C}$ and global (R). Now form,

$$(\hat{\mathcal{X}}, \hat{\mathcal{D}}, \hat{\mathcal{F}}) := \varprojlim_{\lambda \in \tilde{\mathcal{C}}} (\mathcal{X}_\lambda, \mathcal{D}_\lambda, \mathcal{F}_\lambda)$$

Of itself $(\hat{\mathcal{X}}, \hat{\mathcal{D}}, \hat{\mathcal{F}})$ is not dominated by the Zariski-Riemann surface of the function field of $\mathcal{X}_0$, but, its points are. Better still under morphisms in $\tilde{\mathcal{C}}$, non log-canonical points map to non-log-canonical points, so there must be a valuation v of $\mathbb{C}(\mathcal{X}_0)$ whose centre is never log-canonical on any $(\mathcal{X}_\lambda, \mathcal{D}_\lambda, \mathcal{F}_\lambda)$. This doesn't quite contradict ILUT since its statement permits (R) locally, and, we need global. ILUT does, however, tell us that there is a $(\mathcal{X}_{\lambda_1}, \mathcal{D}_{\lambda_1}, \mathcal{F}_{\lambda_1})$ such that the centre of $v_1(= v)$ on $\mathcal{X}_{\lambda_1}$ is the origin in a Cano example, and, without loss of generality, the $x = 0$ divisor, defined by I.1.2, is a component $\mathcal{B}$ of $\mathcal{D}_{\lambda_1}$. As such, we have to get ourselves into a position where we can take global (R). To this end consider,

I.1.5 Sub-claim *Suppose in the notation of* I.1.4 *we replace* $(\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0)$ *by its germ around* $\mathcal{B}$*, and each* $(\mathcal{X}_i, \mathcal{D}_i, \mathcal{F}_i)$ *with its germ around the proper transform* $\mathcal{B}_i$ *of* $\mathcal{B}$ *then we may suppose that* $(\mathcal{X}_n, \mathcal{D}_n, \mathcal{F}_n)$ *is log-canonical.*

Sub-proof The basic problem that we face is the strong restriction imposed by global (R), whereas, we, manifestly, want to augment $\mathcal{C}$ by,

Arbitrary (R)*: Extract a square root of a component of* $\mathcal{D}$

In a global situation this operation is inadmissible, since, if we augment $\mathcal{C}$ to $\mathcal{C}^*$ by way of adding in Arbitrary(R), we may no longer be able to form fibre products à la I.1.1. The problem, in the notation of op. cit., is that if $\rho_1 : \mathcal{X}_1 \to \mathcal{X}$ is the extraction of a root and, $\rho_2 : \mathcal{X}_2 \to \mathcal{X}$ a blow up in Z then after blowing up in $\rho_1^{-1}(Z)_{red}$ we may obtain additional non-scheme like structure not supported on the root, and, this locus, in which we must blow up to dominate $\mathcal{X}_2$, may not be invariant. Locally around $\mathcal{B}$, however, the only global root that we have to worry about is $\mathcal{B}$ itself, and, as we've said, there is no problem around the proper transform of $\mathcal{B}$. By way of detail let $\tilde{\mathcal{X}}_1$ be the blow up of $\mathcal{X}_1 = \mathcal{X}(\sqrt{\mathcal{B}})$ in $\rho_1^{-1}(Z)_{red}$, with $\mathcal{E}_1$ the exceptional divisor, then around the proper transform $\tilde{\mathcal{B}}$ of $\mathcal{B}$, $\tilde{\mathcal{X}}_1$ already maps to $\mathcal{X}_2$. To see this map is in $\mathcal{C}^*$, one notes that it's the same as blowing up in the intersection of $\mathcal{B}$ with the exceptional divisor on $\mathcal{X}_2$, then extracting a root of the proper transform of $\mathcal{B}$. Consequently, in this semi-local setting, $\mathcal{C}^*$ has fibre products, and, whence, the sub-claim by LUT, and the compactness of the Zariski-Riemann surface. $\square$

Now let's use the sub-claim to clean up $\mathcal{B}$, by supposing that for all $(\mathcal{X}_\nu, \mathcal{D}_\nu, \mathcal{F}_\nu)$ in $\tilde{\mathcal{C}}$ over $(\mathcal{X}_{\lambda_1}, \mathcal{D}_{\lambda_1}, \mathcal{F}_{\lambda_1})$ there are non-canonical points other than in the component of the singular locus through v_1. Again appealing to ILUT, we find, $(\mathcal{X}_{\lambda_2}, \mathcal{D}_{\lambda_2}, \mathcal{F}_{\lambda_2})$, and a valuation v_2 with centre the origin of another Cano example on the proper transform $\mathcal{B}_2$ of $\mathcal{B}$. This cannot, however, by the sub-claim, continue ad infinitum, so we may eventually suppose that everything on $\mathcal{B}$ is log-canonical apart from finitely many Cano examples. At which point we have many choices, but a convenient one is to separate the v_i for $i > 1$ from $\mathcal{B}$ by blowing up in points, so that everything is canonical apart from the component of the singular locus through v_1, which, itself, is necessarily isolated since the log-canonical points are open in sing$(\mathcal{F})$, and indeed, without loss of generality, both smooth with the field enjoying everywhere non-zero linear part by a minor variant of the above. Whence, we've eventually arrived to the situation where we may apply global (R), and so contradict ILUT for v_1. $\square$.

Having profited from the greater flexibility afforded by the 2-category of algebraic stacks to obtain a resolution even of an initially wholly scheme like object $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ there are many issues remaining, viz: is this really the kind of resolution we want ? how far away is it from being projective/a scheme etc. Fortunately, these have all been addressed in [II] I.3, I.6 & I.7, with a convenient conclusion being,

I.1.6 Corollary *Let* (X, D, F) *be a foliated 3-dimensional log-triple, then there is a projective bi-rational modification* $\rho : (\tilde{X}, \tilde{D}, \tilde{F}) \to (X, D, F)$ *such that the latter has log-canonical foliation singularities, and* $\tilde{X}$ *has at worst* $\mathbb{Z}/2$ *quotient singularities. In particular if* X *is projective (respectively a scheme),* $\tilde{X}$ *is projective (resp. a scheme.)*

proof The various projective/scheme conclusions arise from the representability of a stack with projective moduli as a groupoid with finite étale source and sink, [II] I.3.2, a condition which is plainly not altered by any number of applications of global (R). Everything else is in op. cit. I.6 & I.7 albeit that its worth re-visiting I.1.2 by way of another manifestation of the Cano example whereby we see that $\mathbb{Z}/2$ quotient singularities is best possible, i.e.

$$\partial = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z^{2p}\frac{\partial}{\partial z},\ x \leftrightarrow y,\ z \mapsto -z,\ p \in \mathbb{N} \qquad \square$$

I.2. Weak 3-D Branching

Even in the presence of canonical/log-canonical singularities sing($\mathcal{F}$) can still be complicated. The basic reason for this is that log-canonical singularities only assure the non-nilpotence of the 1st order linearisation of a local generator around a point, so for example, there may well be less than the co-dimension of sing($\mathcal{F}$) eigenvalues. In dimenson 3 we have two possibilities, i.e. 1 or 2 -always to be understood with multiplicity- eigenvalues around an isolated singularity, or 1 eigenvalue in the non-isolated case. To understand the former requires only Jordan decomposition, while in the latter we require to perform Jordan decomposition uniformly around sing($\mathcal{F}$). Plainly this is a local and purely scheme like question, so let Z be an irreducible component of sing($\mathcal{F}$), and $\hat{X}$ =Spf$\mathcal{O}$ a formal scheme complete in the I_Z adic topology, with ∂ a generator of the foliation on $\hat{X}$. Our objective is,

I.2.1 Claim *Suppose the formal germ $(\mathcal{O}, \partial)$ is log-canonical with 1 eigenvalue at the generic point of Z, then there is an invariant formal subscheme W of $\hat{X}$ which at every point $z \in Z(\mathbb{C})$ has for completion around z the formal subscheme defined by the equation $x = 0$, where, at z the semi-simple part of ∂ is $x\frac{\partial}{\partial x}$.*

Notice that at z, although the function x is non-unique, the sub-scheme $x = 0$ is unique, and dependent only on $\mathcal{F}$. To prove the claim observe,

I.2.2 Reduction *We may, without loss of generality, suppose that Z is smooth.*

proof Certainly we can find a resolution of singularities $\rho : \tilde{X} \to \hat{X}$, or better a formal blow up, c.f. [II] IV.2, such that ∂ lifts to $\tilde{X}$, and $\tilde{X}$ is complete around the proper transform $\tilde{Z}$ of Z. To deduce the claim from here one appeals to the co-herence of push-forward of co-herent sheaves under proper maps of formal schemes, or, slightly more correctly, the proof of the same, cf. [EGA],III, III.3.4.2. $\square$

Of course we've profited here, and previously, from the openness of the 2-eigenvalue condition, and all that remains is to make a few observations beginning with,

I.2.3 Fact *Let D be an endomorphism of a rank 2 vector bundle E over a curve Z which is rank 1 at every point $z \in Z$, then every point of Z has an open neighbourhood U over which we may write D as,*

$$\begin{bmatrix} \lambda(z) & 0 \\ 0 & 0 \end{bmatrix}$$

where $\lambda(z)$ is the eigenvector at z.

proof Both Ker$D$ and Ker$D - \lambda(z)$ are rank 1 sub-bundles of E which are everywhere disjoint. $\square$

Everything being local, we may therefore shrink Z so as to suppose mod I_Z^2, ∂ has the form,

$$\partial = x\frac{\partial}{\partial x} + f(z)y\frac{\partial}{\partial z}$$

with $x = y = 0$ the defining equations of Z. A simple induction mod I_Z^n therefore produces an element ξ of $\mathcal{O}$ vanishing on Z such that $\partial\xi = \xi(1+u)$, for $u \in I_Z$, and since the local weak branches of completions at points depend only on the foliation, we may, without loss of generality suppose that $\partial x = x$ in $\mathcal{O}$. This already proves the claim, but we may also note in passing,

I.2.4 Fact *If Z is smooth then every point $z \in Z(\mathbb{C})$ has a I_Z complete neighbourhood U over which we can find coordinates $x, y, z, \in \mathcal{O}$ such that the foliation is given by a field of the form,*

$$\partial = x\frac{\partial}{\partial x} + a(y,z)\frac{\partial}{\partial y} + b(y,z)\frac{\partial}{\partial z}$$

proof Proceed by induction modulo I_Z^n. $\square$

Notice, however,

I.2.5 Z Warning *Both I.2.1 & 1.2.4 are false for completions in* $\mathrm{sing}(\mathcal{F})$, *i.e.* if $\mathrm{sing}(\mathcal{F})$ *is not wholly contained in the weak scheme $x = 0$ of the Jordan decomposition of a point $z \in$* $\mathrm{sing}(\mathcal{F})$, *then we can neither find a uniform Jordan decomposition in the completion, nor even, the weak sub-scheme $x = 0$. The basic example of this is when the local Jordan decomposition at a point has the form $x\frac{\partial}{\partial x} + zy\frac{\partial}{\partial y}$.*

There are, of course, no such problems at isolated points. Furthermore, an isolated point with weak branching becomes a singular curve with weak branching after blowing up except,

I.2.6 Fact *Let 0 be an isolated singularity with weak branching, then, if after arbitrary blowing up in* $\mathrm{sing}(\mathcal{F})$ *and the proper transform thereof this phenomenon persists, we may, after blowing up, suppose for $\lambda \in \mathbb{C}^\times$ that the semi-simple part ∂_S of the Jordan decomposition of a local generator has the form,*

$$\partial_S = x\frac{\partial}{\partial x} + \lambda y\frac{\partial}{\partial y}$$

I.3 Log-Flatness

Let $(\mathcal{X}, \mathcal{F})$ be a foliated smooth stack with canonical singularities, then we may apply the algorithmic resolution procedure of [BM] to obtain a modification $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ by a sequence of blow ups in invariant centres to obtain an invariant simple normal crossing divisor $\mathcal{E}$ which contains every point of the induced singular locus. As such we have an ideal $\mathcal{I}_{nlf}$ of non-log flat points according to the surjection,

$$\Omega_{\mathcal{X}}(\log \mathcal{E}) \longrightarrow \rho^* K_{\mathcal{F}} \mathcal{I}_{nlf} \longrightarrow 0$$

For many problems, such further modification is neither here nor there, so we will adopt,

I.3.1 Convention *We will often assume with little or no warning, that such a divisor $\mathcal{E}$ is present*

In particular, and in so much as as we do permit such modification, we can bring the singularities of $\mathcal{F}$, especially in dimension 3, into better shape. The discussion is local, so without loss of generality scheme like, and we begin with,

I.3.2 Fact *If at $z \in \mathrm{sing}(\mathcal{F})$ the semi-simple part of the Jordan decomposition has 2 non-zero eigenvalues and $\mathrm{sing}(\mathcal{F})$ has dimension 1 then $\mathcal{F}$ is log-flat at Z and, indeed, $\mathrm{sing}(\mathcal{F})$ is smooth at z. Better still there is at most one other formal invariant hypersurface through z containing the generic point of $\mathrm{sing}(\mathcal{F})$.*

proof Notice that any generic point of $\mathrm{sing}(\mathcal{F})$ is contained in a component of $\mathcal{E}$, so there is at least 1 smooth invariant invariant hypersurface with local equation y, say, at z containing the said generic point.

Now let ξ, η, ζ be Jordan coordinates at z, with ∂ a local generator affording semi-simple part $\partial_S = \xi\frac{\partial}{\partial\xi} + \lambda\eta\frac{\partial}{\partial\eta}$, $\lambda \neq 0$. Plainly $\partial\xi$, $\partial\eta$ are in $I_{\mathrm{sing}(\mathcal{F})}$, while the former generate the ideal (ξ, η), so by our hypothesis on the dimension $\partial\zeta \in (\xi, \eta)$, and $\mathrm{sing}(\mathcal{F})$ is smooth at z.

Suppose, therefore, that $f : V \to \hat{X}$ is the normalisation of an invariant formal hypersurface containing $\mathrm{sing}(\mathcal{F})$, with $\hat{X}$ the completion at z, so that in particular $f^{-1}\mathrm{sing}(\mathcal{F})$ is generically a bunch of Cartier divisors around which V is smooth, and we pick one of these, W, say. A priori neither V nor $\hat{X}$ are schemes, but everything is local around z, and,

$$\mathrm{Hom}(\mathcal{O}_{\hat{X}}, \mathcal{O}_V) \supset \mathrm{Hom}(V, \hat{X})$$

so we still have a map on replacing $\mathrm{Spf}\mathcal{O}_{\hat{X}}$ by $\mathrm{Spec}\mathcal{O}_{\hat{X}}$ etc., and, better still, $\xi = 0$ or $\eta = 0$ are invariant equations of schemes if they are so in $\hat{X}$. Profiting from this let $w \in \mathrm{sing}(\mathcal{F})$ be a nearby point where V, $f^{-1}\mathrm{sing}(\mathcal{F})$ and the induced foliation are smooth around the pre-image of w, and let ${\partial_S}^w + {\partial_N}^w$ be a Jordan decomposition of ∂ in the completion at w. In particular if ${\partial_S}^w = \xi^w\frac{\partial}{\partial\xi^w} + \lambda(w)\eta^w\frac{\partial}{\partial\eta^w}$ then we observe,

I.3.3 Possibilities *In the completion at w either,*

(a) *$\lambda(w) \notin \mathbb{N}\cup\mathbb{N}^{-1}$ and V completed at w is one of the two necessarily invariant formal schemes $\xi^w = 0$ or $\eta^w = 0$*

(b) *$\lambda(w) \in \mathbb{N}\cup\mathbb{N}^{-1}$, so, say $\lambda(w) = n$, and $\xi^w = 0$ invariant with either*

(i) *There is no nilpotent part, and $\eta^w = 0$ is invariant.*

(ii) *A component of $\partial_N\eta^w = 0$ defines the only other invariant hypersurface through w containing $\mathrm{sing}(\mathcal{F})$, so something of the form, $t^p(\xi^w)^n + \eta^w$, or $(\xi^w)^n + t^p\eta^w$, for an appropriate local function $t = 0$.*

(iii) *There are no other invariant hypersurfaces, i.e. $\partial_N = \xi^w\frac{\partial}{\partial\eta^w}$.*

Similarly one has either,

(a)' $\lambda \notin \mathbb{N} \cup \mathbb{N}^{-1}$ *and both* $\xi = 0$, $\eta = 0$ *are invariant.*

(b)' $\lambda \in \mathbb{N} \cup \mathbb{N}^{-1}$ *and we have (i), (ii), or (iii) as above.*

As such if we have (a)' we cannot have (b)(iii), and, indeed, for very general w, without loss of generality $\xi = 0$, $\xi^w = 0$ (respectively $\eta = 0$, $\eta^w = 0$) define the same hypersurface, so V is either $\xi =)$ or $\eta = 0$. The same is true if we have (b)'(i), while if we have (b)'(ii) we know the form of an invariant hypersurface through z, and, likewise for (b)'(iii).

Consequently we're pretty much done, beyond noting that our initial equation $y = 0$ is smooth, so that it cannot be the singular hypersurface that may occur in (b)'(ii), as such $\frac{\partial y}{y}(z) \neq 0$, and even here things are log-flat. It's also worth noting down,

I.3.4 Sub-fact *In the notations of I.3.2 and its proof, any invariant hypersurface in the completion of* X *in* $\mathrm{sing}(\mathcal{F})$ *through* z *containing* $\mathrm{sing}(\mathcal{F})$*, is, on completion at a very general point* w *of the unique component of* $\mathrm{sing}(\mathcal{F})$ *through* z*, of one of the hypersurfaces* $\xi^w = 0$*, or* $\eta^w = 0$ *defined by* ∂_S^w.

Manifestly the situation presented by I.3.2 is not only reasonable, but, best possible. As such before proceeding let us consider to what extent we can improve the other singularities. To this end, we have,

I.3.5 Lemma *Let* $x = 0$ *be the weak branch* W *through a not necessarily isolated singularity* 0 *with* $\partial = \partial_S + \partial_N$ *the Jordan decomposition of a local generator, then necessarily* $\partial_S = x\frac{\partial}{\partial x}$ *and* ∂_N *is a non-log-canonical field in* y *and* z*. Suppose further that* ∂_N *is at least quadratic in* y *and* z *then for* $\rho : \tilde{X} \to X$ *the blow up in* 0*, a singularity* p *on the induced foliation satisfies either,*

(a) $\rho^*\partial$ *has an isolated singularity with semi-simple part of full rank, and* p *is not in the proper transform of* W.

(b) p *is non-isolated and the induced weak branch is the proper transform* $\tilde{W}$ *of* W *through* p.

proof Outside of the exceptional divisor the lemma is clear, and one just calculates the exceptional divisor by hand. □

The usefulness of the lemma derives from,

I.3.6 Further Lemma *Again let* $x = 0$ *be the weak branch* W*, but through a non-isolated singularity* 0 *such that the induced foliation together with* $\mathrm{sing}(\mathcal{F}) \cap \mathrm{W}$ *does not have log-canonical singularities then the nilpotent part of a local generator is indeed at least quadratic.*

proof We can write a generator as $\partial = x\frac{\partial}{\partial x} + f(y,z)\delta$ for δ a saturated plane field in y and z with $f(0) = 0$. As such either,

(i) δ is singular, and we're done.

(ii) δ is smooth, but f is not even simple normal crossing, whence at least quadratic. □

The only possibility not covered by this discussion is therefore,

I.3.7 Exceptional Lemma *As ever* $x = 0$ *the weak branch* W*, but here through an isolated singularity with* ∂_N *a saturated nilpotent plane field in* y *and* z*, then for* $\rho : \tilde{X} \to X$ *the blow up in the origin of the germ around* 0*, a singularity of the induced foliation is either,* *(a) Isolated with semi-simple part of full rank, and* p *not in the proper transform* $\tilde{W}$ *of* W.

(b) p *is in* $\tilde{W}$ *which is still the weak branch.*

proof Again proceed by direct calculation on the exceptional divisor. □

To describe the results of these lemmas, let us introduce,

I.3.8 Definition *The weak branching locus of* $(\mathcal{X}, \mathcal{F})$ *is the set of points in* $\mathrm{sing}(\mathcal{F})$ *where the rank of the semi-simple part of a local generator is less than the co-dimension of* $\mathrm{sing}(\mathcal{F})$*. We take only reduced scheme structure over the said locus and denote it* $WB(\mathcal{F})$.

Applying the lemmas we obtain,

I.3.9 Fact *Let* $(\mathcal{X}, \mathcal{F})$ *have canonical singularities and dimension 3, then there is a sequence of modifications in singular points of the foliation* $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ *such that,*

(a) Any isolated points in $WB(\tilde{\mathcal{F}})$ *have semi-simple part of rank 2.*

(b) Not just $WB(\tilde{\mathcal{F}})$ *but the non-isolated points of sing*$\tilde{\mathcal{F}}$ *form in their reduced structure a curve with singularities at worst nodes. Better still the induced foliation in the weak branch of* $\tilde{\mathcal{F}}$ *around the non-isolated points, together with the divisor sing*$\tilde{\mathcal{F}} \cap W$ *has log-canonical singularities.*

proof At an isolated singularity of rank 2 or 3 we do nothing. Likewise at a non-isolated singularity having rank 2. If, however, the singularity is isolated of rank 1, we blow up in points of W and its proper transform until we have a log-canonical situation on the same. By I.3.6 & I.3.7 we never introduce any other singularities, except isolated ones of full rank. Similarly if we begin at a non-isolated singularity by I.3.5 & I.3.7 the weak branch on a punctual blow up in the same is its own proper transform, so we proceed to a log-canonical situation, with any other singularity being isolated of full rank. This proves everything when combined with I.3.5-7 except possibly at points where the singular locus goes out of W, but there, W is smooth and the Jordan form looks like, $\partial = x\frac{\partial}{\partial x} + y^p z^q \frac{\partial}{\partial z}$, $p, q \in \mathbb{N}$, so $\mathrm{sing}(\mathcal{F})$ is at worst nodal. □

To describe the isolated points we have,

I.3.10 Further Fact *On blowing up a foliation* $(\mathcal{X}, \mathcal{F})$ *with canonical singularities,* $\mathcal{X}$ *smooth of dimension 3, in points we may suppose any isolated singularity has either,*

(a) Semi-simple part of full rank, and is log-flat.

(b) Semi-simple part $\partial = x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$, $\lambda \in \mathbb{C}^{\times}$*, with* $y = 0$ *in* $\mathcal{E}$ *if it is log-flat, or* $z = 0$ *the local equation for* $\mathcal{E}$ *otherwise.*

proof The co-tangent bundle of any component of $\mathcal{E}$ is an invariant subspace of the 1st order linearisation, so (a) is automatic, as is (b). □

The non-isolated points of full rank have already been described in 1.3.2-4, and so we have,

I.3.11 Final Fact *Let* $(\mathcal{X}, \mathcal{F})$ *be a smooth foliated stack with canonical singularities of dimension 3, then after a sequence of blow ups in points we obtain a modification* $\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{F})$ *such that any singularity not described by I.3.9 or I.3.10 is a point of a non-isolated component of* $WB(\tilde{\mathcal{F}})$*. Furthermore any connected component of* $WB(\tilde{\mathcal{F}})$ *is either everywhere log-flat or not at all. In either case if* $\mathcal{E}'$ *is the part of* $\mathcal{E}$ *other than components which are weak branches then as* *reduced stacks,* $\mathcal{E}' \cap W(\tilde{\mathcal{F}}) = WB(\tilde{\mathcal{F}}) = sing\tilde{\mathcal{F}} \cap W(\tilde{\mathcal{F}})$ *and is a simple normal crossing divisor in* $\tilde{W}$*. Finally for* $p \in \mathbb{N}$, $q \in \mathbb{N} \cup \{0\}$ *the Jordan form at such points is one of,*

(a) $\partial = x\frac{\partial}{\partial x} + \nu(y,z)y^p z^q \frac{\partial}{\partial y}$, $\nu(0) \neq 0$.

(b) $\partial = x\frac{\partial}{\partial x} + \nu(y,z)y^p z^q \frac{\partial}{\partial z}$, $\nu(0) \neq 0$.

(c) y, z *are Jordan coordinates for a canonical plane field* δ *with*

$$\partial = x\frac{\partial}{\partial x} + \nu(y,z)y^p z^q \delta\, ,\ \nu(0) \neq 0.$$

In all cases $y = 0$ *is a component of* $\mathcal{E}'$*, and* $z = 0$ *too when* $q > 0$*, except for (a) when* $p = 1$ *and the singularity is log-flat.*

To finish this section let us make,

I.3.12 Remark/Definition *Notice that in respect of the various descriptions 1.3.9-11 we do not claim, nor care, whether the said forms enjoy any sort of stability under further blowing up. The descriptions are, however, convenient, and so we introduce: Any of the singularities described in I.3.9-11 will be labeled convenient. Furthermore we will extend the definition to further suppose that a log-flat singularity of type I.3.10(b) also has the plane* $z = 0$ *in* $\mathcal{E}$*. This, as ever, can be achieved by blowing up in points, albeit that this may create singularities of type I.3.2 if* $\lambda \in \mathbb{N}$.

I.4. Invariant Curves in 3-D

We wish to tabulate the possibilities for invariant curves through the singularities of a foliated 3-fold $(\mathcal{X}, \mathcal{F})$ with convenient singularities. The discussion is local and even formal in the completion of points of $\mathrm{sing}(\mathcal{F})(\mathbb{C})$, so throughout let X be a complete scheme like étale neighbourhood of 0, with ∂ a local generator of the foliation affording a Jordan decomposition $\partial_S + \partial_N$. A more general discussion of the possibilities for a germ C of an invariant curve is already in [III] II.1, so we content ourselves to a quick resumé with a little extra detail, taking only an interest in invariant curves not factoring through $\mathcal{E}$. To begin with,

I.4.1 Case 1 0 *is isolated of full rank.*

Here things are log-flat, so take $y = 0$ in $\mathcal{E}$, and normalise by way of $\partial_S(y) = y = \partial(y) = y$ with λ, μ the eigenvalues of Jordan coordinates $x,\ z$. Notice,

I.4.2 Fact *If* λ*, respectively* μ*, is not in* $\mathbb{Q}_+$ *then* x*, respectively* z*, vanishes on* C*. In particular if* λ, $\mu \notin \mathbb{Q}$ *or* λ, $\mu \in \mathbb{Q}_{<0}$ *then* C *is the necessarily smooth curve* $x = z = 0$.

Plainly this leads to,

I.4.3 Sub-case 1(a) $\lambda \in \mathbb{Q}_+,\ \mu \in \mathbb{Q}_-$. *Renormalise by way of* $\partial_S(y) = \partial(y) = py$, $\partial_S(x) = qx$, $\partial_S(z) = -rz$; $(p,q,r) = 1$, $p, q, r \in \mathbb{N}$*. As such the plane* z *is invariant, and the induced foliation is either semi-simple or it is not. If not, whence* $p = 1$ *there are no curves, otherwise they're of the form,*

$$z = 0,\ y^{\tilde{q}} = cx^{\tilde{p}},\ c \in \mathbb{P}^1(\mathbb{C})\backslash\{0\},\ \tilde{p} = \frac{p}{(p,q)},\ \tilde{q} = \frac{q}{(p.q)}$$

Similarly we have,
I.4.4 Sub-case 1(b) $\lambda,\ \mu \in \mathbb{Q}_+$, *so re-normalise as above, but with* $\partial_S(z) = rz$. *The only canonical singularities admitting a curve not in* $\mathcal{E}$ *are,*

(I) $q = r$, *and* $\partial_N = x\frac{\partial}{\partial z}$.

(II) $r = ip + jq$, $i, j \in \mathbb{N}$, $(p, q) = 1$, *i minimal amongst such representations of r, and,* $\partial_N = f(x, y)\frac{\partial}{\partial z}$, *where,*

$$f(x, y) = \sum_{0 \le d \le [j/p]} a_d y^{i+dq} x^{j-dp} \quad \text{not a power of } y$$

In both cases C is contained in $\partial_N(z) = 0$, so 1(b)(I) is like 1(a) modulo interchanging x & z. On the other hand 1(b)(II) is a bit fastidious, so let's punctually blow up to improve it. Under such a modification, for $p \neq q$, there are 3 isolated singularities, two of type 1(a) and one of type 1(b). When eventually $p = q = 1$, we get one isolated singularity, and a non-isolated one without weak branching, so we can subordinate this to other cases to be discussed presently. Before this observe,
I.4.5 Case 2 ∂ *is isolated with* ∂_S *of rank 2. If it is log-flat, there are no curves outwith* $\mathcal{E}$*, otherwise the only such curve is the weak branch itself.*

As to the next case, which contains 1(b)(II) after blowing up,
I.4.6 Case 3 ∂ *is non-isolated of rank 2.*

Again we normalise in this necessarily log-flat situation by $\partial_S(y) = \partial(y) = y$, $y = 0$ in $\mathcal{E}$, take x to be the non-zero eigenfunction of ∂_S with eigenvalue λ, and z the nil-function. Consequently z is zero on every curve, and,
I.4.7 Fact *If* $\lambda \notin \mathbb{Q}_+$*, then C is the necessarily smooth and invariant curve* $x = z = 0$*. For* $\lambda \in \mathbb{Q}_+$*, the plane* $z = 0$ *is invariant, and for the normalisation* $\partial(y) = \partial_S(y) = py$, $\partial_S(x) = qx$; $p, q \in \mathbb{N}$, $(p, q,) = 1$*. The description is as per I.4.3.*

This leaves us with weak branching to consider around curves, i.e.
I.4.8 Case 4 ∂ *has a non-isolated singularity in the non-log-flat weak branching locus.*

Consequently the curves lie in the weak branch W, and are described by,
I.4.9 Fact *Things as above, with* C *not contained in* $\mathrm{sing}(\mathcal{F})$*, then* $\mathrm{sing}(\mathcal{F})$ *is smooth and everywhere tangent to the induced foliation in* W*, i.e. the Jordan form is* $x\frac{\partial}{\partial x} + \nu(y, z)y^{p+1}\frac{\partial}{\partial y}$, $p \in \mathbb{N}$, $\nu(0) \neq 0$*, and* C *is the curve* $x = z = 0$*. Plainly there is a unique such curve for every such point of* $\mathrm{sing}(\mathcal{F})$.

One should not, however, be fooled by this/develop a false sense of security, i.e.
I.4.10 Remark/Definition/Warning *The curves in the weak branching locus which are invariant by the foliation induced on* W *behave in many respects like the closure of invariant curves in the smooth part. Unlike other curves in* $\mathrm{sing}(\mathcal{F})$*, the map from the co-normal bundle of* $\mathcal{F}$ *to the curve vanishes, and they merit a name such*

as **singular solutions** *or* **singular leaves**. *At a generic point of such the Jordan decomposition is* $x\frac{\partial}{\partial x} + \nu(y,z)y^p\frac{\partial}{\partial z}$, $\nu(0) \neq 0$.

I.4.11 Case 5 *∂ has a non-isolated singularity at the weak branching locus, which is supposed log-flat.*

As ever we take $y = 0$ a component of $\mathcal{E}$, and normalise by $\partial(y) = \partial_S(y) = y$, with y, x, z Jordan coordinates. Consequently the curve is given by $x = z = 0$. A useful description of this curve may be given in terms of invariant hypersurfaces. Indeed suppose $f_n = 0$ is an irreducible local equation of such with the property that,

$$f_n = g_n(x,z) + y^n h_n,\ n \in \mathbb{N}$$

then by virtue of the invariance under ∂_S, we obtain,

$$y^n\{nh_n + y\frac{\partial h_n}{\partial y}\} \in (f_n)$$

Things are factorial, so $h_n = c_n f_n - y\frac{\partial h_n}{\partial y}$ for some function c_n, and whence,

$$f_{n+1} := f_n(1 - c_n y^n) = g_n + y^{n+1}h_{n+1}(= -\frac{\partial h_n}{\partial y})$$

the infinite product $\prod(1 - c_n y^n)$ comfortably converges at the formal level, so if our hypersurface isn't $y = 0$ it can be described by an equation $g(x,z) = 0$. This must be invariant under ∂_N, so we conclude,

I.4.12 Fact *Let the singularity be as per I.4.11 (or even I.3.11) then there is at least 1, and at most 2, invariant hypersurfaces through the singularity other than the weak branch itself. Together these hypersurfaces form a simple normal crossing divisor, so that, in particular, an invariant hypersurface is determined uniquely by the image of its co-normal sheaf in the residual co-tangent bundle. When there are two such hypersurfaces, their intersection is the curve not in $\mathcal{E}$.*

I.5. Convergence & the 3-D Beast

By definition a beast occurs when the foliation passes from weak branching to a semi-simple part of full rank, and in doing so fails to be log-flat. Consequently,

I.5.1 Definition *In the presence of convenient singularities the Jordan form of a 3-D beast in appropriate coordinates is $\partial = x\frac{\partial}{\partial x} + z^p y\frac{\partial}{\partial y}$, where $y = 0$ is a local equation for $\mathcal{E}$.*

At first glance this looks pretty trivial, but in reality it's anything but, since the Jordan form may fail to exist in anything other than the completion of a point. To see this, let us discuss in somewhat greater generality, the behaviour of the Jordan form around non-isolated, non-weak branching singularities. As ever the discussion is local, whence scheme like, and log-flat, so there is a coordinate y with $y = 0$ describing a component of $\mathcal{E}$, and x another coordinate such that $x = y = 0$ defines $\mathrm{sing}(\mathcal{F})$, or more accurately the non-weak branching component Z, with ∂ a local generator of $\mathcal{F}$. The preferable way to normalise is to take ∂ to be the identity on the co-normal bundle of Z in $y = 0$, so $\partial x = x + a(z)y$, $\partial y = \lambda(z)y$, supposing of course, $\lambda \neq \infty$. Consequently we can diagonalise the linearisation of ∂ in End$(N_{Z/X})$ iff $1 - \lambda | a$. In neighbourhoods of the beast this is not a problem since λ is close to zero, otherwise, and quite generally if v_p is the valuation at a point $p \in Z$, we can take a such that $v_p(a) < v_p(1 - \lambda)$. Similarly we want a good coordinate z on Z, so consider ∂ as an element of End$(\Omega_X \otimes \mathcal{O}_Z)$, then a priori, $\partial z = b(z)x + c(z)y$, mod$I_Z^2$. To find z with $\partial z = 0$ mod I_Z^2 is only obstructed if $\lambda = 0$, but since we're supposing that the weak branch is in convenient form this obstruction doesn't present itself, so that apart from around $\lambda = 1$, all points admit an étale neighbourhood such that ∂ is semi-simple in End$(\Omega_X \otimes \mathcal{O}_Z)$, while at points p with $\lambda(p) = 1$ we have the situation,

$$\begin{bmatrix} 1 & 0 & 0 \\ a & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with a locally polynomial of degree $< v_p(1 - \lambda)$.

Now let's consider the situation under completion in Z, with $\hat{X}$ the corresponding étale formal neighbourhood, and $\hat{X}_{an}$ the same in the analytic topology. On the latter we can consider a neighbourhood defined by $\lambda \notin \mathbb{Q}$, with for the moment λ non-constant, and we have,

I.5.2 Fact *On $\hat{X}_{an} \backslash \lambda^{-1}(\mathbb{Q})$ we may find coordinates x, y, z such that,*

$$\partial x = x,\ \partial z = 0,\ \textit{and}\ \partial y = \lambda(z)y$$

proof We've already done this mod I_Z^2, and one proceeds by induction mod I_Z^n. $\Box$

Apart from this, the situation is rather more complicated. There are of course, subcases determined by $\mathbb{Q}_-$ and $\mathbb{Q}_+$, so let V_- and V_+ be open sectors around either of these, then as per I.5.2 we obtain,

I.5.3 Fact *Over V_- there is a coordinate X defining a hypersurface containing* $\mathrm{sing}(\mathcal{F})$*, everywhere transverse to $y = 0$ such that $\frac{\partial x}{x} = 1$ (mod I_Z).*

Any improvement beyond this is highly local, i.e.

I.5.4 Fact *Let $p \in \hat{X}_{an}$ with $\lambda(p) = -m/n$, $m, n \in \mathbb{N}$, $(m.n) = 1$ then for $v = v_p(\lambda + m/n)$ we have Jordan coordinates x, y, z such that,*

$$\partial x = nx\{1 + N(z, x^m y^n)\},\ \partial y = -my\{1 + M(z, x^m y^n)\},\ \partial z = L(z, x^m y^n)$$

where L, M, N have degree $< v$ in z, and $x^m y^n$ divides L.

The situation around V_+ is both better and worse, i.e.

I.5.5 Fact *Around V_+ we can find coordinates x, y, z such that $\partial z = 0$, $\partial y = \lambda(z)y$, and,*

$$\partial x = x + \sum_{n=1}^{\infty} a_n(z) y^n$$

Furthermore if $p \in \mathrm{sing}(\mathcal{F})$*, then we can take* $\partial x = x$ *in a neighbourhood of* p *unless* $\lambda \in \mathbb{N}^{-1}$*. In the latter case with* $\lambda(p) = 1/n$*, this further simplification is possible iff* $v_p(a_n) \geq v_p(\lambda - 1/n)$*.*

Before interpreting these results let's observe that λ is defined globally on the component Z. Indeed $\frac{\partial y}{y}$ defines a global section of $H^0(Z, K_{\mathcal{F}})$ as does the trace, which gives a map $\lambda : Z \to \mathbb{P}^1$, and in neighbourhods of infinity we have,

I.5.6 Fact *Renormalise at infinity, so* $\delta(y) = y$ *for a possibly different field* δ*, then there is a smooth invariant hypersurface in the completion of* $\mathcal{X}$ *in* $\mathcal{Z}$ *given locally by* $x = 0$*, with* $x = y = 0$ *defining* $\mathcal{Z}$*.*

As such the existence of $x = 0$ in the completion $\hat{\mathcal{X}}$ of $\mathcal{X}$ around $\mathcal{Z}$ is obstructed only at 0 and $\mathbb{N}^{-1}$. Returning therefore to our local set up we observe that one of two things happens, i.e.

I.5.7 Fact *Suppose indeed* $\lambda(p) = 1/n$*,* $n \in \mathbb{N}$*, and in the notation of I.5.5,* $v_p(a_n) < v_p(\lambda - 1/n)$ *with* m *the difference, then on a neighbourhood* $\Delta \ni p$ *of* $\hat{X}$ *there is an invariant hypersurface through* p *of the form* $y^n + t^m x$ *for a suitable parameter* t *at* p*.*

proof In the situation as described we can certainly find Δ so that $\partial x = x + a_n y^n$ for a possibly different x to that in I.5.5, with $v_p(a_n) < v_p(\lambda - 1/n)$. The desired surface is then the component of,

$$\frac{x + a_n y^n}{\lambda(z)} - nx$$

which contains $\mathrm{sing}(\mathcal{F})$.

Consequently we arrive to,

I.5.8 Fact/Definition *Let* $\hat{\mathcal{X}}^*_{an}$ *be an appropriate analytic neighbourhood of* $\hat{\mathcal{X}}$ *minus its beasts, then there is a formal substack* $\mathcal{W}$ *of* $\hat{\mathcal{X}}^*_{an}$ *defined uniquely by the condition that its completion at some point of a real blow up in some beast is the completion of the weak branch of the beast in the same. Furthermore at a given beast, taken in the notations of I.5.5, to be* $x = y = z = 0$*, the following are equivalent,*

(a) $\mathcal{W}$ *extends to a neighbourhood of* 0 *in* $\hat{\mathcal{X}}_{an}$*, and we say that the beast is* **tame**.

(b) $a_n(p) = 0$*, for almost all* p*, with* $\lambda(p) = 1/n$*.*

(c) *For almost all* p *with* $\lambda(p) = 1/n$*, there are two invariant smooth hypersurfaces through* p *containing* $\mathrm{sing}(\mathcal{F})$*.*

(d) *For almost all* p *with* $\lambda(p) = 1/n$*, there is an invariant curve through* p *which does not lie in* $\mathcal{E}$*.*

proof That $\mathcal{W}$ exists in $\hat{\mathcal{X}}^*_{an}$ is a consequence of the uniqueness discussion in I.3.3. Condition (b) implies (a) by direct calculation. Conversely if we have (a) then I.3.2 gives (b), which is certainly equivalent to (c) by the same. As for (d), for almost all p with $\lambda(p) = 1/m$, $v_p(\lambda - 1/n) = 1$ so we have (c) iff (d) by I.3.3. □

Regrettably, therefore, most beasts will not be tame and we deduce,

I.5.9 Final Fact *If the beast is not tame, it is not possible to express it's Jordan form convergently. Not only does this apply (unsurprisingly) to convergence in the strict sense, it also applies after real blowing up. Indeed there are not even finitely many sectors determined by arg*(y)*, arg*(z) *(which modulo further blowing up, can always be supposed to define any asymptotic expansion about the origin) on which the weak branch* $x = 0$ *can be supposed convergent.*

I.6. Higher Convenience

We will proceed as per the convention of I.3.1, so that $(\mathcal{X}, \mathcal{F})$ is a foliated stack with log-canonical singularities, and $\mathcal{E} \supset \mathrm{sing}(\mathcal{F})$ an invariant divisor that we will augment at will by appropriate modification. The dimension is now arbitrary, and to understand this situation we will need a higher dimensional resolution theorem/hypothesis, i.e.

I.6.1 Hypothesis *In the 2-category of formal stacks over* $\mathbb{C}$ *consider the following simple modifications* $\rho : (\mathcal{X}, \mathcal{E}, \mathcal{F}) \to (\mathcal{Y}, \mathcal{D}, \mathcal{G})$ *between foliated log-stacks,*

(a) ρ *is a blow up in an invariant centre.*

(b) ρ *is the extraction of a root of a component of* $\mathcal{D}$*.*

Then for any formal stack $(\mathcal{Y}, \mathcal{D}, \mathcal{G})$ *there is a sequence of simple modifications,*

$$(\mathcal{X}, \mathcal{E}, \mathcal{F}) = (\mathcal{Y}_n, \mathcal{D}_n, \mathcal{G}_n) \underset{\rho_n}{\longrightarrow} \cdots\cdots \underset{\rho_1}{\longrightarrow} (\mathcal{Y}_0, \mathcal{D}_0, \mathcal{G}_0) = (\mathcal{Y}, \mathcal{D}, \mathcal{G})$$

such that $(\mathcal{X}, \mathcal{E}, \mathcal{F})$ *has log-canonical singularities with* $\mathcal{X}$ *smooth and* $\mathcal{E}$ *simple normal crossing.*

Notice that pseudo-reflecting monodromy around invariant divisors is irrelevant to any of the enunciated properties of $(\mathcal{X}, \mathcal{E}, \mathcal{F})$ so infact,

I.6.2 Fact *Given* $(\mathcal{X}, \mathcal{E}, \mathcal{F})$ *as above we may kill any extra pseudo-reflecting monodromy that we may have introduced to get a map* $\pi : (\mathcal{X}, \mathcal{E}, \mathcal{F}) \to (\mathcal{X}_0, \mathcal{E}_0, \mathcal{F}_0)$ *through which the composition of the* ρ*'s factors,* $(\mathcal{X}_0, \mathcal{E}_0, \mathcal{F}_0)$ *has all the desired properties, and in addition is an honest birational modification of* $(\mathcal{Y}, \mathcal{D}, \mathcal{G})$*.*

Just as per dimension 3, the critical object is the weak branching locus. From the higher dimensional standpoint, we only really care about points which are non-log flat so introduce,

I.6.3 Definition *A point* ξ *of* $\mathcal{X}$ *is said to be in the (very) weak branching locus,* $(V)WB(\mathcal{F})$ *if it is not log-flat, and the linearisation of a generator* ∂ *of the foliation at* ξ *has semi-simple part of rank strictly less than the co-dimension of* $\mathrm{sing}(\mathcal{F})$*.*

Notice that the use of the word very is simply intended to avoid confusion with the possibility that log-flat points may admit weak branching. In practice we're unconcerned about this so, as the parenthesis suggests, the word very may well be omitted. In any case, observe,

I.6.4 Fact *After further modification by blowing up in invariant centres the non-log flat locus and very weak branching locus are synonymous.*

proof Indeed suppose otherwise, then at a point ξ we have as many eigenvalues as the co-dimension, so the eigenfunctions actually define the singular locus, which must, therefore, be smooth and after blowing up in this centre every point over ξ

becomes log-flat. The global proposition reduces to the above local discussion by the compactness of the Zariski-Riemann surface and the existence of algorithmic resolution. □

The redeeming feature of the weak branching locus is,

I.6.5 Fact (cf. [III] II.1) *Let $\mathfrak{X}$ be an affine formal scheme complete in the I_Z-adic topology of the singular locus Z of a foliation by curves such that,*

(a) $\mathfrak{X}$ and Z are smooth.

(b) there is a local generator of the foliation whose linearisation mod I_Z^2 admits a non-nilpotent Jordan decomposition.

then there is a unique formal invariant subscheme $\mathfrak{W}$, the weak branch, whose co-normal bundle is given by the vanishing of the non-zero eigenvectors mod I_Z^2. In particular for $\hat{\mathcal{X}}$ the completion of $\mathcal{X}$ in a component Z of, say, $VWB(\mathcal{F})$ there is an open formal algebraic stack $\mathfrak{U}$ (containing at least the points of Z where the number of eigenvalues is that of the generic point) and a closed irreducible formal substack $\mathfrak{W}_{\mathbb{C}(\mathcal{Z})}$ of $\mathfrak{U}$, the weak branching stack, which has exactly this property at the generic point.

Observe, in particular, by the definition of formal localisation that $\mathfrak{W}_{\mathbb{C}(\mathcal{Z})}$ may very well be defined apriori by functions with essential singularities, and its Zariski closure could easily be not just all of $\mathcal{X}$ but even $\hat{\mathcal{X}}$ when closure is understood formally. Given LCR, however, this doesn't happen and we'll establish,

I.6.6 Claim *Suppose* LCR *then every weak branching stack has a formal Zariski closure of the same dimension, and by a sequence of simple maps we can achieve.*

I.6.7 Emb(beded)LCR *Not only may we suppose after a sequence of simple modifications that $(\mathcal{X}, \mathcal{E}, \mathcal{F})$ is smooth, $\mathcal{E}$ has simple normal crossings and $\mathcal{F}$ has log-canonical singularities, but we may suppose that every (very) weak branching stack considered as a formal substack of $\mathcal{X}$ completed in the appropriate component of $\mathrm{sing}(\mathcal{F})$ has the same properties.*

proof We proceed by a double induction in the 2-category of formal stacks. Firstly we induct on the dimension n say, of $\mathcal{X}$, so suppose the whole statement for anything of dimension at most n, and then induct by the co-dimension of the weak branching stack. In particular if $\mathcal{Z}$ is the substack of $WB(\mathcal{F})$ where there is exactly one eigenvalue then as per I.2.1, it's immediate that we can find a smooth invariant formal sub-stack $\mathcal{W}_1$ of $\mathcal{X}$ completed in $\mathcal{Z}$. Now apply EmbLCR to $\mathcal{W}_1$ and we're off and running, i.e. suppose we have not only proved that every weak branch of co-dimension p has a well defined equidimensional formal Zariski closure, but that we've also got the embedded resolution property for each of these.

Whence let $\mathcal{Z}$ be a component of $WB(\mathcal{F})$ where the weak branch has co-dimension $p+1$, and as per I.2.2, we can suppose that $\mathcal{Z}$ is smooth if our initial interest is to prove that it admits an equidimensional Zariski closure in the completion of $\mathcal{X}$ in $\mathcal{Z}$, which we may aswell suppose is synonymous with $\mathcal{X}$. The existence of the Zariski closure, however, is local at a point $\zeta \in \mathcal{Z}$. Furthermore, where the point is generic, in fact when the number of eigenvalues is exactly $p+1$, there's nothing to prove, so we may in fact suppose that the number of eigenvalues is $q \leq p$, so after, possibly additional, completion in $\mathcal{X}$ to say $\hat{\mathcal{X}}$ we have a well defined $\mathcal{W}_q \ni \zeta$. On the other hand $\mathcal{Z}$ is smooth, and the weak branch is already generically defined, so the existence of an equidimensional Zariski closure at ζ is really a question of the surjectivity of some maps between reflexive $\mathcal{O}_{\mathcal{Z}}$-modules, and since completion is faithfully flat we may suppose that $\mathcal{X}$ not only is $\hat{\mathcal{X}}$ but further complete as we please. In particular by way of EmbLCR for $\mathcal{W}_q$ we find a smooth formal substack of co-dimension $p+1-q$ which is precisely the Zariski closure of $\mathfrak{W}_{\mathbb{C}(\mathcal{Z})}$ at ζ. Plainly $\mathfrak{W}_{\mathbb{C}(\mathcal{Z})}$ has dimension smaller than $\mathcal{X}$, and we may apply EmbLCR to it to conclude. $\square$

II. RATIONAL CURVES IN 3-D

II.1. Cone Theorems

We proceed from the previous local discussion I.4 to the global study of curves in dimension 3, or more precisely the extremal sub-cone $\mathrm{NE}_{K_{\mathcal{F}}=0} := \{\alpha \mid K_{\mathcal{F}}.\alpha = 0\}$. Necessarily for $(\mathcal{X}, \mathcal{F})$ a gorenstein foliated 3-fold with projective moduli and $K_{\mathcal{F}}$ nef, any extremal ray in $\mathrm{NE}_{K_{\mathcal{F}}=0}$ is an extremal ray in $\mathrm{NE}_1(\mathcal{X})_{\mathbb{R}}$, e.g. [K1] II.4.10.3. We begin by filtering $\mathrm{NE}_{K_{\mathcal{F}}=0}$ by co-dimension, i.e. define cones by,
II.1.1 Definition *Let things be as above, and put,*

(1) $P_+ = \{\alpha \in \mathrm{NE}_{K_{\mathcal{F}}=0} \mid D.\alpha \geq 0, \forall D \in \mathrm{NE}^1(\mathcal{X})\}$

(2) $P_1 =$ linear span of $\{\alpha \in \mathrm{NE}_{K_{\mathcal{F}}=0} \mid \exists$ an invariant divisor D with $D.\alpha < 0\}$

(3) $P_1 =$ linear span of $\{\alpha \in \mathrm{NE}_{K_{\mathcal{F}}=0} \mid \exists$ a non-invariant divisor D with $D.\alpha < 0\}$

Plainly $P_+ + P_1 + P_2$ is $\mathrm{NE}_{K_{\mathcal{F}}=0}$. Furthermore a class in P_2, or more correctly an extremal ray R not in $P_+ + P_1$ is necessarily represented by the class of a map $f : \mathcal{L} \to \mathcal{X}$ with invariant image, albeit that we should apriori include the possibility that the image is completely singular. Indeed since the D of P_2 is non-invariant, the tangency locus $\mathcal{O}_D(K_{\mathcal{F}} + D)$ is a Cartier divisor on D which must contain the ray, then $K_{\mathcal{F}} + 2D$ is the tangency with the tangency, etc., so R must be an invariant curve with $K_{\mathcal{F}}.R = 0$. Plainly the number of such classes is countable, and we have to do similarly for P_1, i.e.
II.1.2 Claim *There are countably many invariant divisors D_i and invariant curves R_j such that,*

$$\mathrm{NE}_{K_{\mathcal{F}}=0} = P_+ + \sum_i (i_{D_i})_* \{\mathrm{NE}(D_i)_{K_{\mathcal{F}}=0}\} + \sum_j \mathbb{R}_+ R_j$$

proof Since the chow scheme of the projective moduli X of $\mathcal{X}$ exists, and has countably many components, with invariance being a closed condition, the only non-trivial case occurs when $(\mathcal{X}, \mathcal{F})$ has a meromorphic first integral. After a sequence of blow ups $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ we can assume that the integral is resolved by a map, so a generic member of the family of divisors in question has trivial normal bundle, whence on $\tilde{\mathcal{X}}$ we only need countable. However, $P_+(\tilde{\mathcal{X}})$ pushes forward to $P_+(\mathcal{X})$, so on $\mathcal{X}$ we only need countable too. □

Notice that we can further refine the situation within a given invariant divisor D_k or better invariant substack $\mathcal{D}$. To begin with, notice that,

$$\mathrm{NE}_{K_{\mathcal{F}}=0} = P_+ + \sum_i (i_{D_i})_* \{\mathrm{NE}(D_i)_{K_{\mathcal{F}}=0}\} + \sum_j \mathbb{R}_+ R_j$$

is again a closed cone, essentially because the intersection of divisors is a curve and the moduli has an ample divisor on it. Better still the cone $\mathrm{NE}(\mathcal{D})_{kf=0})$ is itself polyhedral, except in some degenerate cases that we proceed to investigate. A priori

$\mathcal{D}$ isn't even normal, so let $\tilde{\mathcal{D}}$ be it's normalisation with $\mathcal{G}$, $\tilde{\mathcal{G}}$ the induced foliations. For $\tilde{\mathcal{D}}$ it makes sense to talk about a relatively (foliated) minimal resolution $\rho : \mathcal{Y} \to \tilde{\mathcal{D}}$, so for $\mathcal{H}$ the foliation on $\mathcal{Y}$,

$$K_{\mathcal{H}} = \rho^* K_{\tilde{\mathcal{G}}} - \sum_i a_i E_i$$

with $\bigcup E_i$ are the curves contracted by ρ_i and $a_i \in \mathbb{Q}_+$. In particular we may thus go from $(\mathcal{Y}, \mathcal{H})$ to a canonical model $(\mathcal{Y}_0, \mathcal{H}_0)$, cf. [M2], by ρ_0 with,

$$K_{\mathcal{H}} = \rho_0^* K_{\mathcal{H}_0} + \sum_j b_j C_j$$

where $\bigcup C_j$ is contracted by ρ_0, and $b_j \in \mathbb{Q}_{\geq 0}$. Finally $K_{\tilde{\mathcal{G}}}$ is related to $K_{\mathcal{F}}$ by,

$$K_{\tilde{\mathcal{G}}} = K_{\mathcal{F}} - Z$$

for a possibly empty curve supported on $\mathrm{sing}(\mathcal{F})$, where here, and throughout, we'll be notationally a bit loose about the difference between a divisor on $\mathcal{D}$ and it's pull-back to $\tilde{\mathcal{D}}$.

Plainly we can immediately exclude the possibility that $\mathcal{H}$ is of general type, since although $\mathrm{NE}_{K_{\mathcal{H}}=0}$ may be non-trivial, it's generated by finitely many invariant curves. The other cases are more fastidious, so let's suppose as much as we can, namely: $\mathcal{D}$ is non-redundant, so that there is an extremal ray R supported in $\mathcal{D}$, and a supporting function H_R vanishing on it, but strictly positive on the closed sub-cone obtained by summing over the invariant divisors other than $\mathcal{D}$.

Now, since everything is gorenstein, any curves around which $\mathcal{D}$ isn't generically smooth are invariant, and since we can plainly suppose that R has no support on such curves, there is an unambiguous sense to R as a class in $\mathrm{NS}_1(\tilde{\mathcal{D}})$ understood in Mumford's sense, and we divide by cases, i.e.

II.1.3 case (I) $R^2 < 0$.

This is rather easy. Indeed as per [K1] 4.12.3, we may suppose that R is an honest curve C with non-invariant proper transform $\tilde{C}$. Consequently the adjunction formula of [M2] gives $(K_{\mathcal{H}} + \tilde{C}).\tilde{C} \geq 0$, so that for a possibly empty divisor E contracted by ρ,

$$(K_{\tilde{\mathcal{G}}} + C).C = K_{\tilde{\mathcal{G}}}.(\tilde{C} + E) + C^2 \geq K_{\tilde{\mathcal{G}}}.E + (C^2 - \tilde{C}^2) \geq 0$$

and we conclude to the absurdity, $K_{\mathcal{F}}.R \geq -R^2 > 0$.

II.1.4 Case (II) $R^2 > 0$

This can only happen if every curve in $\mathcal{D}$ becomes equivalent when pushed forward to $\mathcal{X}$, and of course $\mathcal{H}$ must have Kodaira dimension 0. Better still there can be no invariant curves on $(\mathcal{Y}_0, \mathcal{H}_0)$, $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D}$ has dimension at most 0, and $(\mathcal{Y}, \mathcal{H})$ coincides with $(\tilde{\mathcal{D}}, \tilde{\mathcal{G}})$. In particular $\mathbb{G}_a \times \mathbb{G}_m$ or $\mathbb{G}_m \times \mathbb{G}_m$ actions are excluded, or even extensions of elliptic curves by $\mathbb{G}_a$ or $\mathbb{G}_m$. Notice in addition that if $(\mathcal{Y}, \mathcal{H}) = (\mathcal{Y}_0, \mathcal{H}_0)$ is a smooth foliation (e.g. the remaining case of an abelian surface) then $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D} = \emptyset$. Indeed otherwise by blowing up in the singular point, and normalising around the exceptional divisor, we would construct a prime divisor of $\mathbb{C}(Y)$ of zero discrepancy contrary to the hypothesised terminality of $(\tilde{\mathcal{D}}, \tilde{\mathcal{G}})$. Furthermore singularities of $\mathcal{X}$ or $\mathcal{Y}$ (actually in this situation they coincide) where $\mathcal{F}$ is smooth must give rise to invariant curves, so everything may be supposed smooth. Observe, however, that either $\mathcal{O}_{\mathcal{D}}(\mathcal{D})$ or $\mathcal{O}_{\mathcal{D}}(-\mathcal{D})$ is effective, or both (i.e. $\mathcal{O}_{\mathcal{D}}(\mathcal{D})$ is numerically trivial, which gives us what we want anyway). To elaborate, there is at most one real class in co-homology tangent to the foliation, and since there is an effective such coming from invariant measures, we find that either $\mathcal{D}^2$ or $-\mathcal{D}^2$ is in P_+, which is nonsense.

II.1.5 Case (III) $R^2 = 0$.

This requires sub-division to discuss the behaviour,

sub-case (a) $(\mathcal{Y}_0, \mathcal{H}_0)$ *has numerical Kodaira dimension 1.*

The index theorem gives $K_{\mathcal{H}_0}, K_{\mathcal{F}}$ & R all parallel. Consequently if the foliation is parallel to the Kodaira fibration, R is elliptic and we're done. Furthermore R is extremal, so we conclude that $\mathcal{Y} = \tilde{\mathcal{D}}$ with $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D}$ of dimension at most 0. In addition if $\mathcal{Y} \neq \mathcal{Y}_0$, and the Kodaira fibration exists, then the fibration considered on $\mathcal{Y}$ has an invariant fibre, which is nonsense, so we reduce to the possibility that $(\mathcal{Y}, \mathcal{H}) = (\mathcal{Y}_0, \mathcal{H}_0)$ is perfectly smooth, and as above so are $\mathcal{D}$ and $\mathcal{X}$, since otherwise we would get invariant curves parallel to R. More than this we cannot really say except to note that $R = K_{\mathcal{F}}.\mathcal{D}$, and that H_R is a supporting function.

The sub-sub-case of Hilbert-Modular surfaces is less satisfactory, since $(\mathcal{Y}, \mathcal{H})$ may not coincide with $(\mathcal{Y}_0, \mathcal{H}_0)$, and $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D}$ or $\mathrm{sing}(\mathcal{Y})$ or $\mathrm{sing}(\mathcal{X})$ could have support on the curves contracted by ρ_0. Nevertheless R is still $K_{\mathcal{F}}.\mathcal{D}$, and again H_R is a supporting function.

sub-case (b) $(\mathcal{Y}_0, \mathcal{H}_0)$ *has Kodaira dimension 0.*

Arguing as above, and noting that the cones for $\mathbb{G}_a \times \mathbb{G}_m$ or $\mathbb{G}_m \times \mathbb{G}_m$ actions are generated by invariant curves, we reduce to $(\mathcal{Y}_0, \mathcal{H}_0) = (\mathcal{Y}, \mathcal{H}) = (\tilde{\mathcal{D}}, \tilde{\mathcal{G}})$, with $\mathcal{F}$ smooth around $\mathcal{D}$, and, if they exist, $\mathrm{sing}(\mathcal{X})$ and $\mathrm{sing}(\mathcal{Y})$ coinciding in an invariant curve. As such there are the following sub-sub-cases up to étale covering,

(i) *Things arise from the extension of an elliptic curve by $\mathbb{G}_a$ or $\mathbb{G}_m$, so R is a non-invariant rational curve.*

(ii) *The foliation is a product $E \times B$, with E elliptic, and R is a non-invariant section $e \times B$.*

(iii) *The foliation is of Kronecker type on an abelian surface. As such there is a risk that the cone may cease to be polyhedral if the surface isn't irreducible, or worse the product of a complex multiplication curve with itself.*

Consequently H_R is a supporting function except possibly in (iii) if the cone is round. In all these cases, however, we can play the $\mathcal{D}^2$ or $-\mathcal{D}^2$ is effective trick to find a special class in P_+, and this is what forces H_R to be a supporting function when the abelian surface of (iii) has a 2 dimensional cone.

sub-case (c) $(\mathcal{Y}_0, \mathcal{H}_0)$ *is a conic bundle.*

The statement itself is something of an over simplification, since what one should really do is divide Z_{red} into its parts $Z' + Z''$ which are invariant and non-invariant for $\tilde{\mathcal{G}}$, then work at the level of the log-canonical bundle $K_{\tilde{\mathcal{G}}} + Z''$. When the log-Kodaira dimension is non-negative, the essential deduction,

$$(\tilde{\mathcal{D}}, \tilde{\mathcal{G}}, Z'') = (\mathcal{Y}, \mathcal{H}, Z'') = (\mathcal{Y}_0, \mathcal{H}_0, Z'')$$

continues to work, and we deduce that we have an isotrivial family of quasi projective curves after an étale covering. Consequently if R isn't invariant its a section, and coincides with Z'' which is invariant in $\mathcal{X}$. In the remaining case, we observe that Z'' cannot be empty since $K_{\mathcal{F}}$ is nef, and by I.4.11 necessarily has just 1 component with invariant curves factoring through the weak branching scheme. As such, by op. cit., the degenerations of an irreducible fibre (which itself can only intersect $\mathrm{sing}(\mathcal{F})$ in Z'') only intersect $\mathrm{sing}(\mathcal{F})$ in Z''. In particular the quadratic form of the degeneration of the generic rational curve has no non-zero entries, so a minor variant of the standard lemma shows that a positive sum of Z'' and the components of degenerations is equal to R, since, incidentally, the extremality of R implies $K_{\mathcal{H}}.R \geq 0$.

To summarise, therefore, we have,

II.1.6 Fact *Let $\tilde{P}_+$ be the necessarily closed cone defined by,*

$$P_+ + \sum_k NE_1(\mathcal{D}_k)$$

where $\mathcal{D}_k$ is an invariant Weil divisor whose normalisation (equal to itself if $\mathcal{X}$ is smooth around $\mathcal{D}_k$)is an étale quotient of an abelian surface, with $\mathcal{D}_k \cap \mathrm{sing}(\mathcal{F}) = \emptyset$, and the induced foliation of Kronecker type, then,

$$NE_{K_{\mathcal{F}}=0} = \tilde{\mathcal{P}}_+ + \sum_i \mathbb{R}_+ R_i$$

where the R_i are extremal rays having one of the following forms,

(a) *R_i is a $\mathcal{F}$ invariant, including possibly completely singular, curve.*

(b) *$R_i = K_{\mathcal{F}}.\mathcal{D}_i$ for $\mathcal{D}_i$ a Weil divisor whose normalisation $\tilde{\mathcal{D}}_i$ is the minimal model of a foliated surface of numerical Kodaira dimension 1, and which intersects $\mathrm{sing}(\mathcal{F})$ in at most points. Indeed if $\tilde{\mathcal{D}}_i$ has actual Kodaira dimension then all of $\mathcal{X}$, $\mathcal{F}$, $(\mathcal{D}_i, \mathcal{F})$ are completely smooth.*

(c) *R_i is either a rational curve not invariant by $\mathcal{F}$, or transverse to the foliation on a divisor whose normalisation is a quotient of a trivial elliptic fibration co-inciding with the induced foliation.*

In addition the rays $R_i \notin \tilde{P}_+$ are locally discrete, i.e. the cone is locally polyhedral.

proof/elaboration Notice that all the closure/discreteness statements are based on the same tick that if $R \notin P_+$ then there is a divisor with $D.R < 0$. □

II.2. An Index Theorem

A good clarification of the nature of the cone P_+ arises from wholly general considerations about numerical Kodaira dimension. Indeed let X be a normal projective variety of dimension n, say over a field, but of arbitrary characteristic, H an ample bundle, and K a nef. divisor with $K^n = 0$, but $K^{n-1} \neq 0$. Necessarily K^{n-1} is a class in the cone $A_1(X)$ of nef. curves, and indeed in the extremal face $A_1(X)_{K=0}$. Now suppose that it's not everything in this face, then there is a not necessarily effective divisor L positive on K^{n-1}, vanishing on some ray R of $A_1(X)_{K=0}$. We'd like to believe that a small perturbation of K by L has lots of sections, and rather surprisingly this turns out to be easy, i.e.

II.2.1 Fact (cf. [B&] 4.4) *Suppose $\epsilon > 0$ is sufficiently small and rational, then for $d \in \mathbb{N}$ sufficiently large and divisible,*

$$h^0(X,\, K^d L^{\epsilon d}) \gtrsim \frac{d^n \epsilon}{(n-1)!} K^{n-1}.L$$

proof Choose H a priori so that $L_+ := H + L$ is ample, then iterated application of exact sequences, $p < \epsilon d$,

$$\begin{aligned} 0 \to H^0(X, \mathcal{O}_X(dK + \epsilon d L_+ - (p+1)H)) \to H^0(X, \mathcal{O}_X(dK + \epsilon d L_+ - pH)) \\ \to H^0(\mathcal{O}_H(dK + \epsilon d L_+ - pH)) \end{aligned}$$

gives the estimate,

$$\begin{aligned} h^0(X,\, K^d L^{\epsilon d}) &\geq h^0(X,\, K^d L_+^{\epsilon d}) - \sum_{p=0}^{\epsilon d - 1} h^0(H,\, K^d L_+^{\epsilon d}) \\ &\gtrsim \frac{d^n}{n!}(K + \epsilon L_+)^n - \frac{\epsilon d^n}{(n-1)!}(K + \epsilon L_+)^{n-1}.H \\ &= \frac{d^n \epsilon}{(n-1)!}(K^{n-1}.L + O(\epsilon)) \end{aligned}$$ □

Certainly therefore there is a $\delta > 0$ such that $dK + \epsilon L \geq \delta H$ as effective $\mathbb{Q}$ divisors whence the absurdity that $R = 0$, which we'll note explicitly by,

II.2.2 Fact If $\alpha \in A_1(X)$, and $K.\alpha = 0$, then $\alpha \in \mathbb{R}_{\geq 0} K^{n-1}$.

Notice also that an even more trivial variation of the same argument shows,

II.2.3 Claim *If infact K is nef, and $K^n > 0$, then $A_1(X) \cap (K = 0) = 0$.*

Presumably the general situation of numerical Kodaira dimension $\nu \in \{0, ..., n\}$ is described by $\alpha \in A_1(X) \cap (K = 0)$ iff $\alpha = K^\nu.\beta$, where β is a class of 'Demailly-Peternell' type, i.e. the intersection of $n - 1 - \nu$ amples on a modification $\tilde{X} \to X$. The other extremal case of $\nu = 0$ is of course [B&]. Regardless the discussion shows that in our main case of interest for foliations the cone is polyhedral, i.e.

II.2.4 Claim *Suppose that $(\mathcal{X}, \mathcal{F})$ is a foliated gorenstein 3-fold with $K_\mathcal{F}$ nef, and $\mathbb{Q}$-gorenstein in the usual sense, i.e. $K_\mathcal{X}$ is a $\mathbb{Q}$-bundle, and that the cone generated by $K_\mathcal{F}$ and $K_\mathcal{X}$ contains a big, then,*

$$\mathrm{NE}_{K_\mathcal{F}=0} = P_+ + \sum_i \mathbb{R}_+ K_\mathcal{F}.\mathcal{D}_i + \sum_j \mathbb{R}_+ R_j$$

where the R_j are invariant curves with $K_{\mathcal{F}}.R_j = 0$; the invariant divisors $\mathcal{D}_i$ are as described in II.1.6(b), and for $K_{\mathcal{F}}^2 \neq 0$, P_+ is nil if $K_{\mathcal{F}}^3 \neq 0$, and $\mathbb{R}_+ K_{\mathcal{F}}^2$ otherwise.

proof Notice that the condition on $K_{\mathcal{F}}$ & $K_{\mathcal{X}}$ implies that things like II.1.6(c) don't happen, i.e. in the notation of op. cit. one has $\mathcal{D}^2$ or $-\mathcal{D}^2 \in P_+$, but these classes intersect $K_{\mathcal{X}}$ in zero, which is nonsense. □

It's probably also worth observing that there's a nicer looking cone defined, for example, inside positive measures or co-homology, or even projected into Néron-Severi, but generated by transverse invariant measures. Let's give some notation to this such as $\mathrm{NE}(X/\mathcal{F})_{K_{\mathcal{F}}=0}$, then inside this cone the only terms that one finds in general are of the form P_+, $\mathbb{R}_+ R_j$ for R_j an invariant curve, V.2, proving this relies on some even more delicate results in dimension 2, but unfortunately it's a priori difficult to relate extremality here to extremality in $\mathrm{NE}_{K_{\mathcal{F}}=0}$, and the latter is what underpins flopping.

II.3. Tabulation

Plainly as one can see from II.1.6 or II.2.4 we aren't actually interested in all the invariant rational curves, but only those which intersect $K_{\mathcal{F}}$ in zero, and are not contained in the singular locus. Consequently let $(\mathcal{X}, \mathcal{F})$ be a foliated gorenstein stack with $K_{\mathcal{F}}$ nef, and $f : \mathcal{L} \to \mathcal{X}$ the normalisation of an invariant 1-dimensional substack intersecting $K_{\mathcal{F}}$ in zero. Furthermore let G be the generic stabiliser of $\mathcal{L}$, then there is a 1-dimensional orbifold $\tilde{\mathcal{L}}$ and a group stack $\mathbb{G}$ over $\mathcal{L}$ generically equal to G such that $\mathcal{L} = [\tilde{\mathcal{L}}/\mathbb{G}]$. On $\tilde{\mathcal{L}}$ we have two types of distinguished points, viz:

(a) $l_i \in \tilde{\mathcal{L}}(\mathbb{C})$ maps to $\mathrm{sing}(\mathcal{F})$, with m_i the order of it's stabiliser.

(b) $l_j \in \tilde{\mathcal{L}}(\mathbb{C})$ does not map to $\mathrm{sing}(\mathcal{F})$, but has non-trivial stabiliser of order n_j.

Consequently if we let g denote the genus of the moduli of $\mathcal{L}$, with I, J index sets for points of type (a) or (b), then the adjunction formula of [II] I.8 reads,

$$0 = \#G\, K_{\mathcal{F}}.{}_f\mathcal{L} = (2g-2) + \#I + \sum_{i \in I} \frac{s_i - R_i - 1}{m_i} + \sum_{j \in J} 1 - \frac{1}{n_j}$$

where s_i, R_i are the segre class of $\mathrm{sing}(\mathcal{F})$, and the ramification calculated on an étale scheme like neighbourhood of l_i, cf. op. cit. I.8.6. Necessarily, $s_i - R_i - 1 \geq 0$, so we deduce,

II.3.1 Fact *If the moduli isn't rational, $\tilde{\mathcal{L}}$ is an elliptic curve, with $\mathcal{L}$ wholly contained in the smooth locus.*

Thus, supposing rational moduli, we profit from the inequality $\#I + \#J \leq 2$. In particular,

II.3.2 Similar Fact *If $\#I = 0$, then $\mathcal{L}$ is wholly contained in the smooth locus, and admits an étale cover by an elliptic curve.*

The most important case though is,

II.3.3 Fact/Definition *If $\#I = 2$ (so $\#J = 0$), then $s_i = R_i + 1$, for both points l_i, $i \in I$, which will be refered to as the ends.*

As a quasi sub-case this contains,

II.3.4 Fact *If $\#J = 2$, and $\#I \neq 0$, whence $= 1$, $s_i = R_i + 1$ at the unique point l_i mapping to $\mathrm{sing}(\mathcal{F})$. Consequently the universal cover of $\mathcal{L}$ is $\mathbb{P}^1$, and we get a map $f : \mathbb{P}^1 \to V$, for $V \to \mathcal{X}$ étale such that f is as per II.3.3*

To take care of the remaining cases with $\#I = 1$ observe that a local generator ∂ of $\mathcal{F}$ around the singular point $f(l)$ lifts to a vector field on scheme like étale neighbourhoods of l on $\mathcal{L}$ and the order of vanishing v of ∂ applied to a parameter at l on the same is well defined and satisfies,

$$s_i - R_i - 1 \geq v - 1 \geq 0$$

In particular we cannot have $v = 1$, and [II] IV.10 gives further restrictions, i.e.

II.3.5 Fact *If $\#I = \#J = 1$, then $n_j | m_i$, and $v = m_i/n_j + 1$. As such, if the singular point is scheme like in $\mathcal{X}$, $\mathcal{L}$ is everywhere scheme like and $v = 2$.*

In dimension 3 we can of course make,

II.3.6 Assumption *$(\mathcal{X}, \mathcal{F})$ is a smooth foliated stack with convenient singularities with the only non-scheme like ones being of Cano type, i.e. with Jordan form*

$$x\{1 + a(xy, z)\}\frac{\partial}{\partial x} - y\{1 + b(xy, z)\}\frac{\partial}{\partial y} + z^{2p}\{1 + c(xy, z)\}\frac{\partial}{\partial z}$$

and $\mathbb{Z}/2$ acting via $x \leftrightarrow z$; $z \mapsto -z$, $p \in \mathbb{N}$. We will further assume that convention I.3.1 is in force.

Consequently we deduce,

II.3.7 Fact *Things as per II.3.6, then at least after further blowing up in points, and up to finitely many exceptions we may assume that any invariant curve, rational or otherwise, not factoring through $\mathcal{E}$ meets $\mathrm{sing}(\mathcal{F})$ in a non-isolated convenient singularity which is scheme like.*

proof The only isolated possibilities which may not be finite in number are I.4.3 & 4, and these occur in a unique formal hypersurface through the singularity in which the induced foliation is log-canonical rather than canonical. We can resolve this singularity to canonical, at which point the proper transform of the surface contains a non-isolated component of $\mathrm{sing}(\mathcal{F})$ through which all but finitely many of the original curves must pass. With a little more blowing up, we may suppose that this new non-isolated singularity is everywhere convenient. This reduces us to non-isolated singularities, where it is possible for non-scheme like singularities to exist, but these are easily removed. Specifically,

(a) The gorenstein cover of the moduli may have the property that non-isolated points of $\mathrm{sing}(\mathcal{F})$ meet the non-scheme like points in a point. This doesn't happen after blowing up in the appropriate component of $\mathrm{sing}(\mathcal{F})$, which is necessarily of the form II.3.6.

(b) $\mathcal{X}$ is ramified in invariant divisors over its gorenstein cover, in which case there is a $\mathcal{X}_0$ between $\mathcal{X}$ and the cover, cf. [II] I.7.4, with the same $K_{\mathcal{F}}$, so we'll work on $\mathcal{X}_0$ instead. $\square$

Putting this together with II.3.3-5 we obtain,

II.3.8 Fact *With finitely exceptions and* $(\mathcal{X}, \mathcal{F})$ *of dimension 3, any curve* $f : \mathcal{L} \to \mathcal{X}$ *invariant by* $\mathcal{F}$ *not factoring through* $\mathcal{E}$ *meeting* $\mathrm{sing}(\mathcal{F})$ *with* $K_{\mathcal{F}} \cdot_f \mathcal{L} = 0$ *has a scheme like analytic neighbourhood (formal would be wholly sufficient)* $X \to \mathcal{X}$ *such that* $\mathbb{P}^1$ *is the normalisation of the pre-image of* $\mathcal{L}$ *and indeed we have a fibre square,*

$$\begin{array}{ccc} (X, \mathcal{F}) & \xleftarrow[f]{} & \mathbb{P}^1 \\ \big\downarrow & & \big\downarrow \\ (\mathcal{X}, \mathcal{F}) & \xleftarrow[f]{} & \mathcal{L} \end{array}$$

Furthermore $f(\mathbb{P}^1)$ *meets* $\mathrm{sing}(\mathcal{F})$ *in either one end* 0*, or two ends* 0 *and* ∞ *both of which lie in non-isolated convenient components, and the the possibilities are described as follows,*

(I) *There is only one end* $f(0)$*, and* f *factors at* $f(0)$ *through a non-log flat weak branch* W*. As such* f *has smooth image and the Jordan form at* $f(0)$ *looks like,* $x\frac{\partial}{\partial x} + \nu(y,z)y^2\frac{\partial}{\partial y}$, $\nu(0) \neq 0$.

(II) *There are two ends albeit that* $f(0)$ *may equal* $f(\infty)$*. In the latter case the branches over* $f(0)$ *&* $f(\infty)$ *are no worse than plane cusps, and both these branches lie in the same plane. Equally both ends may be disjoint cusps, or disjoint smooth, or a combination. Regardless the singularities at* $f(0)$ *&* $f(\infty)$ *are log-flat, and the parameter* v *prior to* II.3.5 *is always 1.*

proof Applying II.3.4 we may assume that the only non-scheme like points on $f(\mathcal{L})$ have stabiliser $\mathbb{Z}/2$, and there are 2 of these at points other than the ends. Consequently the fundamental group of $f(\mathcal{L})$ is either $\mathbb{Z}/2$ or $< a, b : b^2 = 1 >$, and in either case the étale covering of $\mathcal{L}$ by $\mathbb{P}^1$ descends to $f(\mathcal{L})$. Furthermore (II) is valid locally in the case where $f(\mathcal{L})$ has singular image by I.4.11, i.e. $f(\mathcal{L})$ is regularly embedded in $\mathcal{X}$, whence homotopic to a tubular neighbourhood in the same. Everything else is just a corollary of the previous tabulations combined with II.3.6-7. $\square$

II.4. One end

We get the case of one end out of the way first, and in the process establish some notation with,

$$N^{\vee}_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}(n), \ m \geq n \in \mathbb{Z}$$

the co-normal bundle of our invariant curve L in our formal scheme X. On $\mathbb{P}^1$ we take homogeneous coordinates $[S, T]$, $s = S/T$ the standard coordinate at $o = [0, 1]$ etc., and over the cone $\mathbb{V}(N^{\vee}_{L/X})$, take x, y local generators of $\mathcal{O}(m)$, $\mathcal{O}(n)$ respectively at 0 with ξ, η the same at infinity, so that $x = t^m\xi$, $y = t^n\eta$. The condition of one end means that on the cone we may suppose that the foliation is given by $s^2\frac{\partial}{\partial s}$. As such, to start we do some linear algebra, i.e. write,

$$\partial \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and see how the matrix transforms, from which we deduce that we can write,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - ms & 0 \\ c(s) & d - ns \end{bmatrix}, \, ad = 0\,, \, \deg c \leq m - n$$

Equally, however, not both a and d can be zero. Now what's important is to deduce that the $x = 0$ plane is the weak branch, so suppose this is false, and form the blow up $\rho : \tilde{X} \to X$ in the reduced structure of $\mathrm{sing}(\mathcal{F})$ which to first order would be $y = s = 0$, with $\tilde{L}$ the proper transform of L. Consequently $N^{\vee}_{\tilde{L}/\tilde{X}}$ contains as a sub-bundle $\mathcal{O}(m)$, and the H-N filtration/Grothendieck decomposition is,

$$N^{\vee}_{\tilde{L}/\tilde{X}} = \mathcal{O}(m) \oplus \mathcal{O}(n+1)$$

even if $m = n$ since $H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$. The condition $\partial x = -msx \bmod I_L^2$ is completely stable under blowing up & proper transform, but after $(m-n)+1$ blow ups, our matrix becomes,

$$\begin{bmatrix} \lambda - (m+1)s & O \\ 0 & -ms \end{bmatrix}, \; \lambda \neq 0$$

and to 1st order, inside the weak branch, $\partial = -ms\frac{\partial}{\partial x} + s^2\frac{\partial}{\partial s}$, so $m = 0$, which, incidentally, even proves $n = 0$ if we were in the desired case, from which we deduce,

II.4.1 Fact *Either* mod I_L^2 *the germ of the weak branch coincides with* $x = 0$*, and* $n = 0$*, or* $N_{L/X} = \mathcal{O} \oplus \mathcal{O}(b)$, $b \in \mathbb{N} \cup \{0\}$ *and* L *moves in a covering family.*

The latter case is invariably to be considered good and will be passed over without comment. As such we're reduced to the situation $N^{\vee}_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}$, $m \in \mathbb{N}$ with the $\mathcal{O}(m)$ function $x = 0$ coinciding with the weak branch. In this situation, and wholly generally, an easy induction gives,

II.4.2 Fact *There is a unique bundle* $\mathcal{O}_X(1)$ *on* X *lifting* $\mathcal{O}_{\mathbb{P}^1}(1)$ *and generating* Pic *of the same. Furthermore, for any thickening* $X_p = \mathit{Spec}\, \mathcal{O}_X/I_L^p$, $p \in \mathbb{N}$*, sections of* $\mathcal{O}_{X_p}(q)$, $q \geq -1$ *can be lifted to* X.

What in general, however, is rather false is that sections of $\mathcal{O}_{X_p}(q)$, $q \leq -2$ can be lifted. Indeed even from X_2 to X_3. This can be interpreted at the level of blowing up. Specifically, the canonical map $\mathcal{O}(m) \to N^{\vee}_{L/X}$ determines a unique curve $\tilde{L}$ in the exceptional divisor E of the blow up $\tilde{X}$ in L, and we have an exact sequence,

$$0 \longrightarrow \mathcal{O}_{\tilde{L}}(-E) = \mathcal{O}_{\tilde{L}} \longrightarrow N^{\vee}_{\tilde{L}/\tilde{X}} \longrightarrow N^{\vee}_{\tilde{L}/E} = \mathcal{O}_{\tilde{L}}(-m) \longrightarrow 0$$

Of course $\tilde{L} \xrightarrow{\sim} L \xrightarrow{\sim} \mathbb{P}^1$ is the unique negative section of a Hizerbruch surface, and being able to lift to 2nd order, amounts to asking that the H-N filtration of $N^\vee_{L/X}$ is $\mathcal{O}(m) \oplus \mathcal{O}$. Should this happen, it makes sense to ask the same question on $\tilde{X}$ blown up in $\tilde{L}$, so in the first instance we might say that the H-N filtration is *stable after 1 blow up*, in the second *after 2*, and so forth. Whence we formulate,

II.4.3 Claim *Suppose quite generally (i.e. nothing to do with $\mathcal{F}$), $N^\vee_{L/X} = \mathcal{O} \oplus \mathcal{O}(m),\ m \in \mathbb{N}$ is stable after p blow ups, then the unique map $\mathcal{O}(m) \to N^\vee_{L/X}$ can be lifted to X_{p+2}.*

proof For $\pi : \tilde{X} \to X$ as above, $N^\vee_{\tilde{L}/\tilde{X}} = \mathcal{O} \oplus \mathcal{O}(m)$ is stable after $p-1$ blow ups, so $\mathcal{O}(m) \to N^\vee_{\tilde{L}/\tilde{X}}$ lifts to $\tilde{X}_{p+1}$. We denote the nilpotent sub-scheme that it cuts out by $\tilde{V}_{p+1}$, with E_k the kth-thickening of E by itself, and $\tilde{V}_k$ the kth neighbourhood of $\tilde{L}$ in $\tilde{V}_{p+1}$ for $k \leq p+1$. Now consider the following diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_{\tilde{L},E}(-(p+1)E) & \to & \mathcal{O}_E(-(p+1)E) & \to & \mathcal{O}_{\tilde{L}}(-(p+1)E) = \mathcal{O}_{\tilde{L}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_{\tilde{V}_{p+1},E_{p+1}}(-E) & \to & \mathcal{O}_{E_{p+1}}(-E) & \to & \mathcal{O}_{\tilde{V}_{p+1}}(-E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{\tilde{V}_p,E_p}(-E) & \to & \mathcal{O}_{E_p}(-E) & \to & \mathcal{O}_{\tilde{V}_p}(-E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Notice that $\pi_*\mathcal{O}_{E_{p+1}}(-E) = I_{L,X}/I^{p+2}_{L,X}$, and we assert that $\pi_*\mathcal{I}_{\tilde{V}_{p+1},E_{p+1}}(-E)$ defines the required ideal. To see this one checks by induction on $0 \leq k \leq p+1$, that $R^i\pi_*\mathcal{I}_{\tilde{L},E}(-kE) = 0,\ i > 0$, so that the whole diagram is still exact after π_*. Consequently the said ideal cuts out a unique regularly embedded subscheme, $\pi_* V_{p+1}$, say, of X_{p+2} in which L has trivial normal bundle. $\square$.

Needless to say the claim is well adapted to our needs, Indeed back at our $\mathcal{F}$ invariant curve L in X with one end, on blowing up in it the curve $\tilde{L}$ is necessarily invariant. Better still on $\tilde{X}$ the singularities consist of the proper transform of sing$\tilde{\mathcal{F}}$ and an isolated point. A local calculation shows that $\tilde{L}$ passes through the former and not the latter. In addition the function s is still non-zero on $\tilde{L}$, so by I.4.11, $\tilde{L}$ factors through the induced weak branch, which is in fact the proper transform $\tilde{W}$ of W. Now our linear algebra prior to II.4.1 applies to deduce that the H-N filtration of $N^\vee_{L/X}$ is stable after 1 blow up, and by induction arbitrarily so. Consequently,

II.4.4 Fact *There is a smooth formal subscheme W_L of X whose completion at the end coincides with the weak branch, and $N_{L/W_L} \xrightarrow{\sim} \mathcal{O}_L$. In particular when X arises from the global context II.3.8, W_L is algebraic, with the induced foliation being in conics.*

This passes over some subtleties in the deformation theory of algebraic stacks, but an adequate solution to this is explained in [II] III.2.

II.5. Two movable ends

We continue to address the case where the image L of $\mathbb{P}^1$ in II.3.8 is smooth. Our next goal concerns not only two ends, but a not wholly negative normal bundle, i.e. $N^\vee_{L/X}$ has H-N filtration $\mathcal{O}(m) \oplus \mathcal{O}(n),\ m \in \mathbb{N},\ n \leq 0$. Now, independently of the extra hypothesis we retake the notation of II.4, only now a generator of the foliation on the cone is $\partial = s\frac{\partial}{\partial s}$, and for $m \geq n$ the foliation is described by a matrix,

$$\partial \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},\ a, d \in \mathbb{C},\ \deg c \leq m - n$$

It may happen that c cannot be taken to be zero. This occurs if $d - a \in \{0, \ldots, m - n\}$, in which case it's a constant times s^{d-a}. In any of our two end situations $ad = 0$, and there is at most one non-zero eigenvalue λ (which for convenience is allowed to be zero), with a well defined semi-simple matrix,

$$A_S(0) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$$

at zero, transforming as,

$$A_S(\infty) = \begin{bmatrix} -m - \lambda & 0 \\ 0 & -n \end{bmatrix} \text{ or } \begin{bmatrix} -m & 0 \\ 0 & -n - \lambda \end{bmatrix}$$

for the standard orientation of loops at zero and infinity. Equally, however, the matrix at infinity must be singular, and always taking $m \in \mathbb{N}$ we have the following,
II.5.1 Possibilities *Either, without loss of generality* (I) $n = 0$ *or indeed* (II) $n \neq 0$ *albeit* $\lambda = -m$ *and in any case,*

$$A_S(0) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$$

Notice, however, that in the second hypothesis the matrix at ∞ has the eigenvalue in the bottom right corner. In the particular case of $n < 0$ we eliminate this by blowing up. Proceeding as II.4.3, with $\rho : \tilde{X} \to X$ the blow up in the component of $\mathrm{sing}(\mathcal{F})$ through ∞ with $\tilde{L}$ the proper transform, we have an exact sequence,

$$0 \longrightarrow \mathcal{O}(m) \longrightarrow N^\vee_{\tilde{L}/\tilde{X}} \longrightarrow \mathcal{O}(n+1) \longrightarrow 0$$

which necessarily splits, so the H-N filtration is stable under this operation - which, incidentally, it may not be if we try the same stunt at 0 or punctually. Plainly we need to make n blow ups to get what we want, and with the final one the singularity changes from that of I.4.7 to log-flat weak branching. Regardless the situation is still convenient but now, $N^\vee_{L/X} = \mathcal{O}(m) \oplus \mathcal{O},\ m > 0$, and whether here, or in the previous, we can think about employing II.4.3 with $\pi : \tilde{X} \to X$ the blow up in L, and $\tilde{L}$ the canonical section of the resulting Hizerbruch surface. Under this operation $\tilde{L}$ meets the induced singular locus in the proper transform of $\mathrm{sing}(\mathcal{F})$ at an end of type I.4.6, and in another log-flat end, with weak branching if that's what we had to start with. As such we get two ends of the same type with $A_S(0) \mapsto \tilde{A}_S(0)$, say, $A_S(\infty) \mapsto \tilde{A}_S(\infty)$, without change. The H-N filtration may not, however, be stable. Should this occur then $\lambda = -a$, $a \in \{1, \dots, m-1\}$, as such,

II.5.2 Possibilities *The applicability of II.4.3, or otherwise, is described by,*

(I) $-\lambda \notin \{1, \dots, m-1\}$, *and the situation is stable after infinite blowing up, so by op. cit., L moves in the $x = 0$ direction.*

(II) $-\lambda \in \{1, \dots, m-1\}$, *but it still succeeds in being stable under blowing up, so, again L moves in the $x = 0$ direction.*

(III) $-\lambda \in \{1, \dots, m-1\}$, *and the situation is only stable under $p \in \mathbb{N} \cup \{0\}$ blow ups.*

Notice that the smooth formal divisor D in which L moves, has $D.L = -m < 0$, so it must be invariant, since otherwise the tangency locus between $\mathcal{F}$ and D gives a section of $H^0(D, \mathcal{O}_D(K_{\mathcal{F}} + D))$. Using I.3.3 & I.4.11 we can describe its form. The situation is as follows,

II.5.3 Fact *Let D be the invariant divisor constructed above in cases (I) or (II), or, better, $\tilde{D}$ if we started with $N^\vee_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}(n),\ n < 0$, and effected $-n$ blow ups in* $\mathrm{sing}(\mathcal{F})$ *through the end at ∞ to work on $\rho : \tilde{X} \to X$ with D its image in X. Furthermore let $\mathcal{D}, \tilde{\mathcal{D}}, \tilde{\mathcal{X}}$ etc. be the necessarily global objects occuring when things arise from the a priori global situation* II.3.8. *Then in all cases the induced foliation on D is a pencil of conics, and either,*

(a) $N^\vee_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}$, *and the completion of D in the end at zero is the unique plane not in $\mathcal{E}$ containing the given component of* $\mathrm{sing}(\mathcal{F})$*, and idem at infinity if $\lambda \neq -m$.*

(b) $N^\vee_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}(n),\ n < 0$, *and, again, the completion of D in the end at zero is the unique plane not in $\mathcal{E}$ containing the given component of* $\mathrm{sing}(\mathcal{F})$*, while at ∞ we have a singularity of type* I.4.7, *but the completion of D at ∞ is locally the divisor $z = 0$ of* I.4.7 *with the equations of the conics those of op. cit.*

proof At zero, let $X = 0$ be the local equation of a Jordan coordinate distinct from s, and $x = 0$ the local equation for D. By construction x is invariant with non-zero eigenvalue equal to that of X. Furthermore in the $\xi = 0$ plane the semi-simple foliation is smooth, given by a fibration Y =constant, which we may also take to be Jordan. Whence we can write, $x = x_0(z) + Xh$, and apply ∂_S to conclude,

$$(\lambda h + \partial_S h)X = x\nu,\ \nu(0) \neq O$$

so $X|x$, and deduce that they're the same. A quick check of possible Jordan forms reveals, however, that the nilpotent part at zero lies in the ideal of D, so the end at zero has a removable singularity for the induced foliation in D. Whence D, or slightly more precisely, $\mathcal{D}$, is a pencil of conics. This completely proves (a), and much of (b). As to what remains: on $\tilde{X}$ the induced foliation has a log-flat singularity with weak branching. Consequently if the singularity at ∞ is removable in D then either we have an invariant rational curve for every point of a component of $\mathrm{sing}(\mathcal{F})$ that contracts to the end $f(\infty)$, and this is what is asserted, or we don't. If, though, we don't the induced foliation in D would have a saddle node, which is nonsense for a conic pencil, or the curves stop intersecting the weak branching locus at a tame beast on $\tilde{X}$. In the latter situation, the fact that L is smooth implies, I.3.7 & I.4.7, that D is smooth on X, which would contradict (a) for nearby curves. Furthermore the induced singularity in $\tilde{D}$ at infinity must indeed be removable, since otherwise, we'd again get a conic pencil with a saddle. □

II.6. Rigid Ends

From our previous discussion we see that to study rational curves L which do not move we may assume that we have $N^\vee_{L/X} = \mathcal{O}(m) \oplus \mathcal{O}(n),\ m, n \in \mathbb{N}$ with two ends at which the eigenvalues are $-m$ and $-n$ respectively. One should, however, not be lulled into a false sense of security since,

II.6.1 Remark/Warning *We are only studying curves not in $\mathcal{E}$ up to finitely many exceptions, and some of these exceptions may be rigid yet have both positive and negative terms in $N^\vee_{L/X}$. Furthermore we'll eventually have to be more precise about* II.5.2(III), *since this only reduces to the current scenario after an operation depending on L.*

Being that as it may, in our current situation we can by [A3] effect a formal contraction $\rho : X \to X_0$, with $(X_0, \mathcal{F})$ Gorenstein. By the theorem of formal functions $\Gamma(\mathcal{O}_{X_0})$ is complete in the filtration $F^p\Gamma(\mathcal{O}_{X_0})$, where,

$$\frac{F^p\Gamma(\mathcal{O}_{X_0})}{F^{p+1}\Gamma(\mathcal{O}_{X_0})} = H^0(L, S^p N^\vee_{L/X})$$

By definition $(X_0, \mathcal{F})$ still has canonical singularities, and for ∂ a generator of the foliation on X_0, equivalently all of X, we can make a Jordan decomposition $\partial = \partial_S + \partial_N$ of the action on $\Gamma(\mathcal{O}_{X_0})$, or, if one prefers, globally on X. In particular, with the notations of II.4, there are functions F, G on X with the following properties,

(a) $\partial_S F = -mF,\ F$ lifts $x = 0$ over a standard affine patch around 0, and mod I_L^2 is $t^m\xi$ at infinity.

(b) $\partial_S G = nG,\ G$ lifts $\eta = 0$ over a standard affine neighbourhood of infinity, and mod I_L^2 is $s^n y$ at zero.

with this in mind, the assertion is,

II.6.2 Claim *Assuming, as we may, the normalisation $\partial s = s,\ t = 1/s$, then $t^m|F$ at infinity, while $s^n|G$ at zero.*

proof If in a minor abuse of notation t, ξ, η are Jordan coordinates at ∞, the eigenfunctions of $-m$ are $t^{m+k\eta}\eta^k$, and similarly around zero. □

Consequently the (formal) Zariski closure of $F = 0$ on a neighbourhood of zero, respectively of $G = 0$ on a neighbourhood of ∞, are smooth invariant divisors $D_0,\ D_\infty$ defined by global sections of $\mathcal{O}_X(-m)$, and $\mathcal{O}_X(-n)$ respectively. Furthermore, by the previous argument, I.3.2, around 0, D_0 is necessarily the unique invariant plane at 0 not in $\mathcal{E}$ containing $\mathrm{sing}(\mathcal{F})$, and similarly at infinity. In particular we can perform a formal flop.

The operation of formal flopping is described as follows. Firstly, let's aim for the standard notation, so let X_- be our original X, X what was the contraction X_0, and ρ_- the map. Now form the formal stack, $X_-(\sqrt[m]{D_0},\ \sqrt[n]{D_\infty})$, and blow up in the pull-back of L. The moduli of this $\tilde{X}$ is a formal scheme, in fact a weighted blow up of X, but the given stack $\tilde{\mathcal{X}}$ is rather nicer, since we can blow down in the other direction to a smooth formal stack, which we simplify by putting a minimal smooth structure to obtain a formal stack $\hat{\mathcal{X}}_+$, étale over its moduli X_+ in codimension 1. As such if $D_{0,-},\ D_{\infty,-}$ (i.e. our old $D_0,\ D_\infty$) go to $D_{0,+},\ D_{\infty,+}$ they no longer meet the flopped curve L_+, or at the stack level $\mathcal{L}_+$, which itself is an invariant generalised projective stack $\mathcal{P}^1(m,n)$, so potentially not generically scheme like if $(m,n) \neq 1$. Notice also that $X_-, X_+ \to X$ are at least formally relatively projective, i.e. they're Prof's, cf. [II] IV.2.3, and whence we summarise the situation as follows,

II.6.3 Fact *Suppose the image of $f : \mathcal{L} \to \mathcal{X}$ is smooth, with $N^\vee_{\mathcal{L}/\mathcal{X}} > 0$, and on the scheme like formal neighbourhood $X_- \to \mathcal{X},\ \lambda \neq -1$, then in fact $\mathcal{L} = L$ is everywhere scheme like, and there is a formal flop,*

$$\begin{tikzcd} & (\tilde{X},\tilde{\mathcal{F}}) \arrow[dl] \arrow[dr] & \\ (X_-,\mathcal{F}_-) \arrow[dr, "-D_{0,-};-D_{\infty,-}\text{ rel. ample }\rho_-"'] & & (X_+,\mathcal{F}_+) \arrow[dl, "\rho_+\ D_{0,+};D_{\infty,+}\text{ rel. ample}"] \\ & (X,\mathcal{F}) & \end{tikzcd}$$

Up to some obvious technical lacunae, i.e. relative projectivity of the ρ's, and algebraicity of the D's; this equally describes the global flop. It is furthermore an anti-flop for $\mathcal{E}_-$, with $\mathcal{E}_+$ a weighted blow up of $\mathcal{E}_-$ containing the flopped curve as exceptional divisor. Nevertheless $\mathcal{E}_+$ has simple normal crossings, and even the convenience of the singularities is preserved.

proof/elaboration Since we're working under rather restrictive hypothesis, the only possibility for f to have non-scheme like image is the generalised weighted projective stack $\mathcal{P}^1(2,2)$, or, what amounts to the same thing, $[\mathbb{P}^1/\mathbb{Z}/2]$ for the standard $x \leftrightarrow -x$ action. In this case our scheme like formal neighbourhood $X \to \mathcal{X}$ comes equipped with a $\mathbb{Z}/2$ action, and we can arrange for our global generator ∂ of $\mathcal{F}$ over X to satisfy $\partial^\sigma = -\partial$ for σ the generator of $\mathbb{Z}/2$. In particular, $\partial = -\partial_S^\sigma - \partial_N^\sigma$ is still a Jordan decomposition, so $\partial_S^\sigma = -\partial_S$, and eigenfunctions go to eigenfunctions. Whence this can only happen if $\lambda = -1$. This establishes all that's been said, but also more. Indeed, we must by uniqueness, I.3.2, have $D^{\sigma}_{0,-} = D_{\infty,-}$, and idem for the germs $\mathcal{E}_{0,-}$; $\mathcal{E}_{\infty,-}$ of $\mathcal{E}$ through 0 and ∞. Also the fixed points of σ for our conventional choice of coordinates are 1, -1, which are still distinct from the singularities of the flipped curve. Plainly at the level of the completion $\hat{\mathcal{X}}$ of $\mathcal{X}$ in $f(\mathcal{L})$ there is only one divisor $\mathcal{D}_-$, say, and the flopped stack $\mathcal{L}_+$ is again a $\mathcal{P}^1(2,2)$ with a single point in $\mathrm{sing}(\mathcal{F})$, which is scheme like. Better still the intermediary flap $\tilde{\mathcal{X}}$ is just the blow up in $\mathcal{L}_-$, and we summarise by the obvious diagram,

II.6.4 Fact *In the missing case where $f(\mathcal{L})$ is non-scheme like, we have a flop in formal stacks described by,*

$$\begin{tikzcd}[column sep=small] & (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \arrow[dl] \arrow[dr] & \\ (\hat{\mathcal{X}}_-, \mathcal{F}_-) \arrow[dr, "-\mathcal{D}_- \text{ relatively ample } \rho_-"'] & & (\hat{\mathcal{X}}_+, \mathcal{F}_+) \arrow[dl, "\rho_+ \ \mathcal{D}_+ \text{ relatively ample}"] \\ & (\hat{\mathcal{X}}, \mathcal{F}) & \end{tikzcd}$$

Again it is a $\mathcal{E}_-$ anti-flop, the singularities are still convenient, and everything globalises, modulo the aforesaid technical lacunae understood at the level of the moduli.

II.7. Singular curves

Our considerations so far have ignored the possibility that the map $f : \mathcal{L} \to \mathcal{X}$ is ramified, or has intersecting branches in its image. As such, suppose to begin with, that the image of L in the scheme like neighbourhood X has a node, with by definition, both branches smooth. The only way this can occur, as ever up to finitely many exceptions with no factoring through $\mathcal{E}$, is at singularities of type I.4.7, with the branches described by I.4.3. In particular the eigenvalue is in $\mathbb{N}$, and by blowing up we reduce to the case that it's equal to 1, then by a further blow up contradict the simple connectedness of $\mathbb{P}^1$.

Consequently let's turn to cusps but under the initial hypothesis that the two ends 0 and ∞ are disjoint. The unique possibility here that we need to deal with occurs when at least one of the ends has Jordan form,

$$\partial = ps\frac{\partial}{\partial s} + qx(1 + a(y))\frac{\partial}{\partial x} \ , \ a(0) \neq 0$$

and $p, q \in \mathbb{N} \setminus \{1\}$. If we resolve the singularity around our cusp which we necessarily suppose of the form, $(s^p, cs^q, 0)$, $c \in \mathbb{C}^{\times}$, then the induced end at 0 has a nil matrix for it's semi-simple part in the normal direction. Whence the cusp must move, so that in the limit $c \to 0$, we find an invariant rational curve given around $f(0)$ by $x = y = 0$. Now the ends are disjoint, so we may aswell suppose that at ∞, both our original curve, and its smooth deformation limit are smooth. However, by our previous considerations the deformation limit must move in the divisor $x = 0$, so in fact the curve even moves in the whole space. Notice though,

II.7.1 Remark *Families of cusps with $K_{\mathcal{F}}.\mathcal{L} = 0$ in the $y = 0$ plane can occur, but only when there are non-scheme like points in* $\mathrm{sing}(\mathcal{F})$.

This leaves the possibility of $f(0) = f(\infty)$, with at least one branched cusp. So, say, one branch smooth, one branch cusped. On resolving the cusp we either get a node, which we know doesn't happen, or we find that we again have a nil semi-simple matrix in the normal direction at the resolved cusp, so we obtain an inexistent node as a deformation limit. Both branches cusped reduces to this case by way of a deformation limit, and so,

II.7.2 Fact *Apart from finitely many exceptions (all in the weak branching locus by the way) if there is an invariant singular 1-dimensional substack $\mathcal{L}_0$ not wholly in* $\mathrm{sing}(\mathcal{F})$ *with $K_{\mathcal{F}}.\mathcal{L}_0 = 0$, then $\mathcal{L}_0$ (or some multiple of it) moves in a family covering $\mathcal{X}$.*

II.8. Constant eigenvalue

So far we've managed to conclude that an invariant rational type stack will, up to finitely many exceptions, move unless the formal scheme like neighbourhood X of L ($\xrightarrow{\sim} \mathbb{P}^1$) has eigenvalues $-a$ and $-b$ at the ends, with $N^{\vee}_{L/X}$ either $\mathcal{O}(a) \oplus \mathcal{O}(b)$ or $\mathcal{O}(a+b) \oplus \mathcal{O}$. In the latter case we have a further integer of interest p, which is the number of times we need to blow up in L, and the extremal horizontal curve in the exceptional divisor, before equating this situation with the former. Slightly more conveniently this latter situation can be described in terms of a weighted blow up $X(L) \to X$, which is equivalently the blow down of the first $p-1$ exceptional divisors in the direction from whence they came. If we put a minimal smooth stack structure on $X(L)$, say, $X(L)^p$, then around the extremal curve L_p in the exceptional divisor P (albeit that it may not be globally extremal) everything is scheme like, with the monodromy concentrating on another horizontal curve, denoted L_p^∞, around which it is identically $\mathbb{Z}/p$. Manifestly the construction of $X(L)^p$ globalises with no loss of projectivity of the moduli. This construction allows us to focus on the $\mathcal{O}(a) \oplus \mathcal{O}(b)$ case, which we'll call *rigid*, while the latter case we'll denote *frigid*.

Regardless, as we've said, in either case we have a Jordan decomposition, $\partial_S^L + \partial_N^L$ of a local generator ∂ of the foliation on X. In the rigid case this is particularly easy to describe since we have embedding coordinate $x_0, \dots, x_a; y_0 \dots y_b$ on the contraction of L, equivalently generators of $\Gamma = \Gamma(\mathcal{O}_X)$, and,

II.8.1 Formula $\partial_S^L = (i-a)x_i \frac{\partial}{\partial x_i} + (b-j)y_j \frac{\partial}{\partial y_j}$, $\partial_N^L = y_b x_i \alpha(x_a, y_b) \frac{\partial}{\partial x_i} + x_a y_j \beta(x_a, y_b) \frac{\partial}{\partial y_j}$

for a priori arbitrary functions of two variables α and β. Equally the Jordan decomposition around L must restrict to a Jordan decomposition at the ends. Furthermore when one of the ends, say ∞, has constant eigenvalue we equally have a Jordan decomposition of the foliation around the singular component $Z_\infty \ni \infty$, or, more correctly, a neighbourhood of ∞ after completion in Z_∞ such that around both L and Z_∞ we can simultaenously find a generator of the foliation. Indeed the issue is as follows: Jordan decomposition is a notion of fields not of foliations, so while it's true that the semi-simple direction is largely independent of the choice of generator, the nilpotent direction may not be. As such fix an affine neighbouhood U_∞ of Z_∞ with trivial Picard group, and consider the formal scheme Ξ, obtained by completion, whose trace is $U_\infty \cap Z_\infty$, and finitely many rigid or frigid curves L_i with an end in $U_\infty \cap Z_\infty$. Now contract all the L_i, so that we get a formal scheme $\pi : \Xi \to \Xi^\#$. Plainly, if we can trivialise $K_\mathcal{F}$ around $\Xi^\#$ then it trivialises over Ξ. By construction, however, the trace of $\Xi^\#$ is affine, so the question is of first order, and since the singularity of the trace is a plane node, we have an isomorphism,

$$\mathcal{O}^\times_{\Xi_0} \xrightarrow{\sim} \mathcal{O}^\times_{U_\infty \cap Z_\infty} \coprod_i \mathcal{O}^\times_{L_i}$$

Furthermore, to give a line bundle on the trace of $\Xi^\#$ is to give a line bundle on Ξ_0 which is trivial on each L_i, so, indeed, we can find a generator ∂ of the foliation, which is equally a generator around both $U_\infty \cap Z_\infty$ and the L_i on some formal neighbourhood, so that we get a uniform Jordan decomposition $\partial_S + \partial_N$ on the said neighbourhood, which in turn gives rise to,

II.8.2 Case (a) $\partial_N \neq 0$

In this case one observes by the explicit formula II.8.1 that if the curve L is rigid then it is defined as the intersection of a pair of divisors - necessarily the D_0, D_∞ post II.6.2 if the eigenvalue at 0 is unramified- around which ∂_N is non-saturated. Since there are only finitely many such divisors, the number of possibilities for a rigid L is finite. Similarly, if it's frigid the same statement is true on $X(L)$, and the only extra possibility for a divisor along which ∂_N is non-saturated is the exceptional divisor P. On the other hand, this is transverse to D_0 at 0, and transverse to D_∞ at ∞, so for unramified eigenvalue this is irrelevant, and the number of frigid curves is constant too.

II.8.3 Case (b) $\partial_N = 0$

In this case the situation is uniformly semi-simple at both ends, so if we have infinitely many curves, we may sub-divide so that they're all either frigid or rigid, and we obtain a formal substack $\mathcal{Y}$ of the completion of $\mathcal{X}$ in the component Z_0 of $\mathrm{sing}(\mathcal{F})$ through the end at zero, and as large a collection of curves $\mathcal{L}_i$ as we may choose such the the slopes of the H-N filtration of $N_{\mathcal{L}_i/\mathcal{X}}$ are constant in i. Now we simply approximate $\mathcal{Y}$ around Z_0 by way of global sections of some ample bundle H, supposing, as we may, that $\mathcal{Y}$ is non-algebraic. Consequently for $n \in \mathbb{N}$ we can find divisors $D \in |nH|$ whose restriction to $\mathcal{Y}$ vanishes on Z_0 to order n^2, so if $D|_{\mathcal{L}_i} \neq 0$ then, $\deg_H(\mathcal{L}_i)$ is at least of order n. In particular D vanishes on all of the $\mathcal{L}_i$ which enjoy $\deg_H(\mathcal{L}_i) \leq O(n)$. Whence on $\mathcal{Y}$ we get a Cartier divisor $D|_\mathcal{Y}$ with cycle at least,

$$n^2 Z_0 \ + \sum_{\deg_H(\mathcal{L}_i) \leq O(n)} m_i \mathcal{L}_i$$

On the other hand the slopes of $N_{\mathcal{L}_i/\mathcal{X}}$ are constant, so we must have,

$$m_i \geq n^2 (1 - \frac{\deg_H \mathcal{L}_i}{n}), \quad \deg_H \mathcal{L}_i \leq O(n)$$

and whence by Bézout's theorem, or better, its various refinements, [F] 12.3, an inequality of $\mathbb{Q}$ cycles,

$$H^2 \gg \sum_{\deg_H(\mathcal{L}_i) \leq O(n)} (1 - \frac{\deg_H \mathcal{L}_i}{n})\mathcal{L}_i$$

so the degree of the $\mathcal{L}_i$'s is uniformly bounded, whence the absurdity that $\mathcal{Y}$ is algebraic. Thus, to summarise, we have,

II.8.4 Fact *Suppose the eigenvalue of a rigid or frigid curve is constant at one of its ends, i.e. if, say, the end is zero with Z_0 the component of $\mathrm{sing}(\mathcal{F})$ through the same, the map $Z_0 \to |K_{\mathcal{F}}^{\otimes 2}|$ given by symmetric functions is constant, then there are only finitely many rigid or frigid curves with an end in Z_0.*

II.9 The Primitive Beast

The situation that remains to be understood occurs when we are presented with two (possibly identical) components Z_0 and Z_∞ of $\mathrm{sing}(\mathcal{F})$, together with an infinite (infact Zariski dense) set of frigid or rigid curves $\mathcal{L}_i$ with ends in the same. Furthermore, we may suppose that in terms of the two unique formal divisors $S_0,$ containing Z_0 or $T_\infty,$ D_∞ containing Z_∞ the $\mathcal{L}_i$ factor through D_0 and D_∞ at the ends. The primitive situation that we wish to achieve is that every invariant curve meeting all but finitely many of our $\mathcal{L}_i$'s is itself rigid or frigid. As we've currently set up the problem this is impossible since the divisors $S_0,$ S_∞ coincide with components of our background divisor $\mathcal{E}$ through $\mathrm{sing}(\mathcal{F})$. Manifestly components of $\mathcal{E}$, and other things, will have to be eliminated. As a consequence we'll have to allow a more general set up, which to begin with we describe over the Gorenstein covering stack $(\mathcal{X}', \mathcal{F}) \to (X, \mathcal{F})$ of the moduli. In particular it may occur that around Z_0 and/or Z_∞ the situation is non-gorenstein (actually the monodromy here will be $\mathbb{Z}/2$, and will only occur at one of the ends) then over and above this we will have some cyclic monodromies $\mathbb{Z}/n_0$, $\mathbb{Z}/n_\infty$ corresponding to the extra monodromy at the generic points of Z_0 and Z_∞ respectively. The non-gorenstein case is a bit more fastidious, so let's ignore it for the moment. As a result if d is the gcd of n_0 and n_∞ a rigid, or frigid, curve in this scenario will (up to finitely many exceptions) admit an étale covering by the generalised weighted projective stack $\boldsymbol{\mathcal{P}}^1(\frac{n_0}{d}, \frac{n_\infty}{d})$. Furthermore, since we'll allow no more extra non-scheme structure than this, any smooth invariant curve not factoring through $\mathrm{sing}(\mathcal{F})$ is either étale covered by $\mathbb{P}^1$, a weighted projective stack of the form $\boldsymbol{\mathcal{P}}^1(n)$, or, again, a $\boldsymbol{\mathcal{P}}^1(\frac{n_0}{d}, \frac{n_\infty}{d})$. In any case, since all these objects are simply connected with co-homology and Picard groups as expected, cf. [II] I.9, the whole of the previous discussion goes through up to an appropriate change in the definition of $\mathcal{O}(1)$. As such the only really important thing to notice is the change in the behaviour of the eigenvalue which we normalise by a choice of standard loops at the ends, i.e. $\frac{\partial_0 s}{s} = 1.$ and $\frac{\partial_\infty t}{t} = 1$ for s, t local equations for $S_0,$ S_∞ respectively with $p = n_0/d,$ $q = n_\infty/d$ a rigid or frigid curve has eigenvalues $-a/q,$ $-b/p;$ $a, b \in \mathbb{N}$ at 0 and ∞, while its normal bundle is either $\mathcal{O}(\frac{a}{pq}) \oplus \mathcal{O}(\frac{b}{pq})$ or $\mathcal{O}(\frac{a+b}{pq}) \oplus \mathcal{O}$ for $\mathcal{O}(\frac{1}{pq})$ the generator of $\mathrm{Pic}\boldsymbol{\mathcal{P}}^1(p, q)$, and, of course, this is to be understood on an étale neighbourhood if

$d \neq 1$.

Now let's consider how this more general set up relates to our initial situation, and convenient singularities. In particular if we're not a priori supposing I.3.1, then we ask whether the invariant curves intersecting $K_{\mathcal{F}}$ in zero, and factoring through S_0 are finite or infinite in number. In the former case, we don't care, and do nothing, in the latter case, either they're all frigid or rigid, and, again, we do nothing, or they move, in which case the Zariski closure, say $\bar{S_0}$, is a divisor, and the induced foliation is a pencil of conics. Actually, rather better, the induced log-object $(\bar{S_0}, Z_0 \cup W_0, \mathcal{F})$, where Z_0, W_0 are the components of $\mathrm{sing}(\mathcal{F})$ not invariant by the induced foliation is generically an isotrivial family of $\mathbb{G}_m$'s over Z_0 and/or W_0. Now since the eigenvalue is non-constant along Z_0 it's equally so along W_0, and around W_0 we have the canonical formal invariant planes, i.e. S_0 and an other S_1, containing it. Consequently we may ask the same question for S_1 as we did for S_0, and continue if we find lots of invariant curves leading to a well defined $\bar{S_1}$. Notice, in particular, on continuing by induction, that inside any $\bar{S_i}$ we never find curves of which a neighbourhood is non-gorenstein. Indeed an open subset of any $\bar{S_i}$ containing a dense set of its invariant curves can be contracted to an open neighbourhood U_i of some component W_i of $\mathrm{sing}(\mathcal{F})$. On this component we can put a smooth stack structure such that the foliation has canonical singularities, but is non-gorenstein over its moduli, i.e. there is a non-trivial character χ of the monodromy group G of the generic point of W_i such that $\partial^\sigma = \chi(\sigma)\partial$ for an appropriate local generator ∂ of the foliation. The co-dimension of the singular locus is, however, two so $\chi = 1$ or -1, and the eigenvalue is identically -1. As such associated to S_1 there is a genuinely different component W_1 of $\mathrm{sing}(\mathcal{F})$, etc., and in any case, the creation of the W_i's plainly terminates. if the process terminates in Z_∞ then there is nothing more to do. Otherwise we ask the same questions creating a bunch of invariant divisors $T_{\infty-i}$ and components V_i of $\mathrm{sing}(\mathcal{F})$ with non-constant eigenvalue. Plainly there is an open set containing the frigid and/or rigid initial curves, and all but finitely many of the new ones, which can be contracted to Z_0 and/or Z_∞, and lead to our goal of a primitive neighbourhood of the beast, i.e. one where any $K_{\mathcal{F}}$ nil invariant curves meeting the rigid or frigid ones have the same property. Nevertheless we want to do this in a reasonably global way that avoids causing too much trouble. Furthermore, we're happy to do it even at the cost of losing projectivity of the moduli, not to mention a lot of arbitrariness away from the frigid or rigid curves. Consequently, we may even start with some blowing up so that the induced log-object $(\bar{S_0}, Z_0 \cup W_0, \mathcal{F})$ is smooth with canonical singularities, so, in particular, there is a projection σ_0 to Z_0. As a result, we can write,

$$K_{\mathcal{F}}|_{\bar{S_0}} = Z'' + C$$

where Z'' is the part of $\mathrm{sing}(\mathcal{F}) \cap \bar{\mathrm{S}}_0$ invariant by the induced foliation, and C is contracted to a not necessarily unique canonical model. Indeed, the non-uniqueness has its advantages since if Γ is any irreducible invariant curve not factoring through Z'' we can insist that Γ does not intersect the support of C, so it cannot intersect Z'' either, i.e. Z'' is a bunch of fibres of σ_0. Now we want to flop curves in the support of C, but the problem arises that for the curve C_i which should be contracted in

$\bar{S_0}$, to flop it, we'd prefer $\bar{S_0}.C_i < 0$. On the other hand let's just guarantee this by blowing up enough an appropriate curve in the same fibre of σ_0. Now we can flop: C_i is contractible, whence there is a well defined semi-simple foliation on a formal neighbourhood of it in $\mathcal{X}$, and since $\bar{S_0}.C_i < 0$ the divisor associated to it is not nil for the semi-simple foliation so we can find a formal divisor F such that C_i has a split neighbourhood (cf. [II] IV.4) defined by $\bar{S_0}$ and F. Once we flop C_i we preserve the smoothness of our new $\bar{S_0}$ by explicit computation of what this kind of flop looks like. Around Z'' the situation is a bit different. To see why a singular curve flops, observe that by I.4.11/12, S_0 must infact be a weak branch about the curve. It will suffice to work on an étale neighbourhood, so say the curve is a generalised weighted projective line $\mathcal{P}^1(a,b)$ with a, b relatively prime, and $\mathcal{O}(\frac{1}{ab})$ the generator of its Picard group. There is a distinguished case (which is actually easier) of log-type, but the principle is the same, so we'll ignore it and concentrate on the 'general' case where the curve meets exactly one other curve in the fibre of S_0. As such we can arrange that around 0, i.e. $\mathbb{Z}/a$ monodromy, we have coordinates s, y_0 in S_0, and at infinity coordinates t, y_∞ such that the generator ∂ (necessarily existent by II.4.2) has the form,

$$\partial|_{S_0} = y_0^u s^v \left(s\frac{\partial}{\partial s} - \frac{n}{b} y_0 \frac{\partial}{\partial y_0}\right)$$

Now we require to compute mod (x_0, y_0^{u+1}), where x_0 is the local equation for the weak branch, so we can suppose $x_\infty = z^m x_0$, $y_\infty = z^n x_0$ are equations at infinity mod $(x_\infty, y_\infty^{u+1})$ with z the transition function for $\mathcal{O}(\frac{1}{ab})$, and $m, n \in \mathbb{N}$. Consequently for ∂ to exist we must have,

$$\frac{\partial x_0}{x_0} = 1 + \sum_{i=1}^{u-1} f_i(s) y^i + f(s) y^u$$

where apart from the irrelevant $\deg f_i \le n_i/b$, and by the way $vb = nu + a$, the leading term in f which is precisely of degree v is $-\frac{m}{b}s^v$, and infact to this precision the Jordan form at 0 can be taken to be,

$$x_0\left(1 - \frac{m}{b}s^v y_0^u\right)\frac{\partial}{\partial x_0} + y_0^u s^v \left(s\frac{\partial}{\partial s} - \frac{n}{b} y_0 \frac{\partial}{\partial y_0}\right)$$

From which it's easy to see that the flop exists. Indeed profiting from the possibility of taking roots, we can suppose that $n = 1$, blow up in the curve, repeat the same analysis, and conclude by induction on m, which has now gone down to $m - 1$, in the spirit of II.4.3. Whence, not only is there a flop, but it's a flap, arising from extracting a mth root of x_0 and a nth root of y_0. We therefore eventually reduce to the situation where σ_0 expresses $\bar{S_0}$ as a $\mathbb{P}^1$-bundle over Z_0, which we then contract. The only effect of this is to make the new Z_0 non-scheme like, but otherwise wholly smooth, and we just continue, with the only change being that we contract at the next stage a bundle of weighted projective stacks.

We have thus modified our original situation as follows: there is a stack $\rho : \tilde{X} \to \mathcal{X}$ obtained by a sequence of blow ups in invariant centres, whence necessarily with projective moduli, together with a proper map $\pi : \tilde{X} \to \mathcal{X}_{pr}$, where the latter may not have projective moduli. All these maps are isomorphisms in a neighbourhood containing the generic point of all but finitely many of our original rigid and/or frigid curves determined by the ends Z_0, Z_∞, with $\mathcal{X}_{pr}$ having the wholly desirable property that any curves in the planes S_0^{pr}, T_∞^{pr} intersecting $K_{\mathcal{F}}$ in zero, must themselves be frigid or rigid, and of course, these are the only possibilities for curves meeting our original curves. Notice, of course, that when the ends Z_0 and Z_∞ are the same component in $\mathcal{X}_{pr}$ then apart from the non-gorenstein case with $\mathbb{Z}/2$ monodromy, the curves now have nodes.

Next, let's just write down the Jordan decomposition of a generator ∂ at 0 in the rigid case, where we take standard coordinates on our weighted projective stack analogous to II.4, with $x = 0,\ y = 0$ for the H-N directions, then,

$$\partial_S^0 = \frac{\partial}{\partial s} - \frac{a}{q}\frac{\partial}{\partial x}$$
$$\partial_N^0 = xy^{\tilde{p}}\alpha(s^{\tilde{a}}x^{\tilde{q}}, y^{\tilde{p}})\frac{\partial}{\partial x} + s^{\tilde{a}}x^{\tilde{q}}y\beta(s^{\tilde{a}}x^{\tilde{q}}, y^{\tilde{p}}\frac{\partial}{\partial y}$$

where $\tilde{a} = a/(a,q)$, $\tilde{q} = q/(a,q)$ etc., and the local monodromy is $\mathbb{Z}/p$ acting by way of $(s,x,y) \mapsto (\theta s, \theta x, y)$, $\theta^p = 1$. In particular one sees that either the eigenvalue at zero or infinity ramifies at its end unless, $\tilde{p} = \tilde{q} = 1$, and since this can only happen finitely many times, we may suppose that it doesn't occur, so that,

$$\frac{\partial_N}{xy} = \alpha(s^{\tilde{a}}x, y)\frac{\partial}{\partial x} + s^{\tilde{a}}\beta(s^{\tilde{a}}x, y)\frac{\partial}{\partial y}$$

Consequently if the eigenvalue doesn't ramify in zero, ∂_N/xy is a smooth foliation with tangent direction transverse to our curve, but tangent to any other invariant curve through this end, and similarly at infinity. The frigid situation is even more desirable. Indeed if $d \in \mathbb{N}$ is the number of times that we have to blow up to make it rigid, with $\tilde{a}$ as above, then modulo $(d+2)$th powers of the ideal of the curve, the eigenvalue looks like,

$$-\tilde{a} + (s^{\tilde{a}}x - y^{d+1})f$$

for f a function and s, x, y appropriate coordinates at 0. Whence, this ramifies, and there can only be finitely many such curves.

Apart, therefore, from a smooth rigid curve in $\mathcal{X}_{pr}$ with no other curves at its ends, we must eliminate,

II.9.1 Outstanding Cases *That may involve non-ramified eigenvalues*

(i) *The ends at* 0 *and* ∞ *meet, and the curve forms a node in* $\mathcal{X}_{pr}$.

(ii) *There is another smooth rigid curve joining the distinct ends at* 0 *and* ∞.

(iii) *There is only one end, i.e. we're in the* $\mathbb{Z}/2$ *monodromy situation, and there is another curve through the end with the same property.*

In all these cases, a neighbourhood, whether of the node, or the two curves simultaenously, is elliptic, so we cannot a priori guarantee the existence of a single generator of the foliation on the whole neighbourhood, and whence of a unique well defined nilpotent direction. Nevertheless, the situation is not arbitrary, and in the respective cases the following occur,

(i) Organise the reduction to $\mathcal{X}_{pr}$ so that at the penultimate stage there is no extra monodromy at zero, and only $\mathbb{Z}/q$ at infinity. Then there are $a, b \in \mathbb{N}$ equal to the eigenvalues at the end, and another positive integer d, satisfying, $q^2 = a(d - 1/b)$

(ii) Both the eigenvalue and its inverse at either end are in $\mathbb{Z}$.

(iii) Exactly as per (ii).

Since we're supposing non-constant eigenvalue at either end, we therefore conclude,
II.9.2 *Let $(\mathcal{X}, \mathcal{F})$ be a given foliated gorenstein normal 3-fold with $K_{\mathcal{F}}$ nef., then any map $f : \mathcal{L} \to \mathcal{X}$ from a smooth 1 dimensional stack bi-rational at its generic point (more generally a bundle of classifying stacks over the same) and invariant, with $0 = K_{\mathcal{F}}\cdot_f \mathcal{L} > -\chi(\mathcal{L})$ either moves in a family covering $\mathcal{X}$ or has bounded degree or on $\mathcal{X}_{pr}$ defines a smooth rigid curve which meets no other invariant curve whose intersection with $K_{\mathcal{F}}$ is zero.*
proof/elaboration We've discussed everything, except maybe when $\mathcal{L}$ moves in a family which covers $\mathcal{X}$. Depending on how big the normal bundle of the generic member of the family is, i.e. either ample and the foliation is given by a vector field, ample$\oplus$trivial and we have a family of vector fields, or trivial, so we get some sort of Ricatti thing, we conclude. $\Box$

III. RESIDUAL MEASURES

III.1 Stoke's Theorem

We'll require Stoke's theorem in a little more generality than is standard, albeit that the following remarks undoubtedly constitute the sort of thing that is well known to experts. A more than sufficient degree of generality amounts to currents with measure coefficients on a smooth proper analytic stack $\mathcal{X}/\mathbb{C}$. Evidently a minor variant of [KM] implies the existence of a moduli map $\pi : \mathcal{X} \to X$, with $X/\mathbb{C}$ a proper complex analytic space. Equally we have well defined sheaves of smooth forms $\mathcal{A}^{*,*}$ and normal currents $\mathcal{N}_{*,*}$, i.e. $L_1^{*,*} + d(L_1^{*,*})$, where the lower numeration indicates indexing by dimension, albeit that at times a co-dimension indexing $\mathcal{N}^{*,*}$ may at times be convenient/appear. In any case the local nature of the Poincaré lemma and standard considerations of hypercohomology immediately imply the usual sort of smoothing in co-homology, i.e.

$$H^q(\mathcal{X}_{an}, \mathbb{C}) = \frac{ZA^q(\mathcal{X})}{dA^{q-1}(\mathcal{X})} = \frac{ZN^q(\mathcal{X})}{dN^{q-1}(\mathcal{X})}$$

where A^*, N^* are global forms and normal currents respectively, while the Z at the front indicates those of the same which are closed. Consequently given a globally closed normal dimension p current $d\mu$ we can find a smooth $2\dim_{\mathbb{C}}\mathcal{X} - p$ form τ and a L_1 function γ such that,

$$d\mu = \tau + d\gamma$$

As a result if $d\mu$ is a positive (p,p) current, then in fact $d\gamma$ has measure coefficients, i.e.

$$|d\gamma(f)| << \|f\|_\infty$$

where f is a smooth test form, and, as ever $<<$ is less than up to an inessential constant. Now observe that for U_i, $i \in I$ a properly nested sequence of open sets indexed by some Lebesgue measurable I with the property that the union of the boundaries has positive Lebesgue measure, the measure of the boundary of U_i with respect to $d\mu \wedge \omega^p$, ω smooth positive, is zero with probability one. Consequently we may without difficulty,

(a) 'Cover' $\mathcal{X}$ by disjoint analytic neighbourhoods Δ_α such that $\mathcal{X} \setminus \bigcup \Delta_\alpha$ has not only zero Lebesgue measure, but zero $d\mu \wedge \omega^p$ measure.

(b) Apply the said partition in order to write,

$$d\mu = \sum_\alpha \mathbb{I}_{\Delta_\alpha} d\mu = \tau + \sum_\alpha d(\mathbb{I}_{\Delta_\alpha}\gamma)$$

(c) Take the Δ_α to be polydiscs, regularises the $\mathbb{I}_{\Delta_\alpha}\gamma$ in the usual way, so as to obtain a sequence of regularisation $\gamma_n \to \gamma$ such that for U_i, $i \in I$ as before,

$$d\mu(\mathbb{I}_{U_i} f) = \lim_n \int_{U_i} f d\mu_n$$

with probability one independently of the test function f, where needless to say, $d\mu_n = \tau + d\gamma_n$.

As such we arrive to our immediate goal,

III.1.1 Fact *Let $d\mu$ be a closed positive (p,p) current, f a smooth (or for that matter sufficiently regular) function, and $\mathcal{U}$ an open substack whose boundary has zero $d\mu \wedge \omega^p$ measure, then $d\mu|_{\partial\mathcal{U}}$ is well defined, and Stoke's holds, i.e.*

$$\int_{\mathcal{U}} df\, d\mu = \int_{\partial\mathcal{U}} f\, d\mu$$

All applications will of course resolve around an essentially random choice of $\mathcal{U}$ amongst a Lebesgue measurable set of possibilities. Such a choice will, unsurprisingly, be irrelevant and whence will be omitted from the notations and/or discussion without comment.

III.2 Remarks on Kắhlerianity

All applications of the current chapter will revolve around dimension $(1,1)$ currents with measure coefficients, and, in this situation, there is never any need to know whether the stack $\mathcal{X}$ is Kắhler or not. Whence the Kắhler free nature of the discussion in III.1. Nevertheless it seems worth a brief detour to give some simple conditions under which this holds. The essentially necessary and sufficient condition is that the moduli X should itself be Kắhler, but, rather than get into a long winded discussion about what this should mean let's just say that X embeds into a Kắhler manifold with ω the restriction of the Kắhler form.

With this hypothesis we'll prove by induction on the dimension of substacks that $\mathcal{X}$ is Kắhler. Whence call a reduced substack $\mathcal{Y}$ in $\mathcal{X}$ Kắhler if there is a non-negative *appropriate* smooth $(1,1)$ form on $\mathcal{X}$ strictly positive on a neighbourhood of $\mathcal{Y}$. The technical definition of appropriate is as follows: for every substack $\mathcal{Z}$ of $\mathcal{X}$ there is a smooth function (infact many) $\|1\!\!1_{\mathcal{Z}}\|$ which after blowing up in $\mathcal{Z}$ (or if one prefers resolve $\mathcal{I}_{\mathcal{Z}}$ by blow ups in smooth centres) looks like, i.e. up to a smooth unit, $|f|$ for $f = 0$ a local equation of the total exceptional divisor. A $(1,1)$ form is appropriate if it looks like,

$$N\omega + \sum_i N_i dd^c \|1\!\!1_{\mathcal{Z}_i}\|^2$$

for N, $N_i > 0$, and $\mathcal{Z}_i$ some finite bunch of substacks. Plainly our induction is on the dimension of $\mathcal{Y}$, and the existence of appropriate smooth $(1,1)$ forms around the same.

Beginning the induction in dimension zero with $x_1, \ldots, x_n$ coordinates at a point, we take a bump function ρ on a small neighbourhood of its image in X, and note that if G is the stabiliser, then,

$$\|1\!\!1_{\mathcal{Z}}\|^2 = \sum_{\sigma \in G} \rho \|x\|^{2\sigma}$$

is a perfectly good example of what we mean with $\|x\|^2 = \sum |x_i|^2$. As such,

$$N\omega + dd^c\|\mathbb{I}_{\mathcal{Z}}\|^2$$

for N sufficiently large gets us off and running since ρ itself is defined on X so $dd^c\rho$, $d\rho d^c\rho$ are dominated by multiples of ω.

Now suppose a not necessarily pure dimensional substack $\mathcal{Y}$ is given, then at its generic points it's étale over its own moduli with $\mathcal{Z} \subset \mathcal{Y}$ a proper substack of every component the locus where this is false. In particular there is a non-negative appropriate $(1,1)$ form $\omega_{\mathcal{Z}}$ strictly positive on a neighbourhood of $\mathcal{Z}$, and for M, N sufficently large,

$$M\omega + N\omega_{\mathcal{Z}} + dd^c\|\mathbb{I}_{\mathcal{Y}}\|^2$$

is strictly positive on a neighbourhood of $\mathcal{Y}$, while being globally non-negative. As such we conclude,

III.2.1 Fact *Suppose that the moduli X of a smooth separated stack $\mathcal{X}/\mathbb{C}$ embeds in a Kähler manifold (e.g. X is projective) then $\mathcal{X}$ is Kähler.*

It therefore goes without saying that all the results of harmonic theory go through verbatim for proper Kähler stacks. It therefore seems appropriate to briefly remark on how algebraic stacks put the essentials of vanishing in rather clear light, particularly the role of $\mathbb{Q}$ divisors, i.e.

III.2.2 Divertimento *Let $X/\mathbb{C}$ be a projective variety with at worst quotient singularities, and $\pi_0 : \mathcal{X}_0 \to X$ the minimal smooth stack structure on X of [V] 2.5 with L a line bundle on X such that as $\mathbb{Q}$ divisors,*

$$\pi_0^* L = H + \sum a_i D_i$$

for $\bigcup D_i$ simple normal crossing on $\mathcal{X}_0$, H big and nef., and $0 \le a_i < 0$, then,

$$H^q(X, L^\vee) = 0 \ , \ \text{for, } q < \dim X$$

proof By the exactness of $(\pi_0)_*$, cf. [AV], it suffices to prove $H^q(\mathcal{X}_0, \pi_0^* L^\vee) = 0$. We first do the case of H ample. Plainly for any smooth stack $\pi : \mathcal{X} \to X$ with X as moduli, either of the standard proofs of Kodaira vanishing work. The analytic one gives that the ℓ_p-norm for $p < 2$ of any harmonic (n,q), $q \ge 1$ form with values in H is zero whenever H is an ample bundle by III.2.1, and the algebraic one by the exactness of π_*, which under the same hypothesis easily implies that H is 'co-homologically ample', [II] I.3.5. As such if we put $a_i = p_i/q_i$, $(p_i, q_i) = 1$, and form a stack $\pi : \mathcal{X} \to \mathcal{X}_0$ by taking successively q_ith roots of the D_i in some order we can suppose that,

$$H^q(\mathcal{X}, K_{\mathcal{X}} \otimes \pi^* H) = 0 \, , \, q > 0$$

which immediately gives the case where $a_i = 1 - 1/q_i$, $\forall i$ by the ubiquitous exactness of π_*. The next easiest case occurs when no integer between p_i and $q_i - 1$ has a common factor with q_i. When this happens, one looks at the long exact sequence associated to the short exact sequence,

$$0 \to K_{\mathcal{X}}(H - \frac{(n_1+1)}{q_1} D_1) \to K_{\mathcal{X}}(H - \frac{n_1}{q_1} D_1) \to K_{\frac{D_1}{q_1}}(H - \frac{(n_1+1)}{q_1} D_1) \to 0$$

where we suppose that D_1 is the last divisor of which we extract a root $p_1 \neq q_1 - 1$, and $0 \leq n_1 \leq (q_1 - 1) - p_1$, $n_1 \in \mathbb{N}$. In particular we write $\pi_1 : \mathcal{X} \to \mathcal{X}_1$, for the map between $\mathcal{X}$ and the stack $\mathcal{X}_1$ where we extracted the q_ith roots for $i \geq 2$. If we view $\frac{1}{q_1} D_1$ as a $B_{\mathbb{Z}/q_1\mathbb{Z}}$ bundle over its image D_1 in $\mathcal{X}_1$ then,

$$H^q(\frac{1}{q_1} D_1, K_{\frac{1}{q_1} D_1}(H - \frac{(n_1+1)}{q_1} D_1)) = H^q(D_1, K_{D_1}(\pi_1)_*(H - \frac{(n_1+1)}{q_1} D_1))$$

Furthermore the class of H in the relative Picard group of $\mathcal{X}$ over $\mathcal{X}_1$ is, by the definition of H as a bundle on $\mathcal{X}$, $-\frac{a_1}{q_1} D_1$, whence we're reduced to computing,

$$H^q(\mathbb{Z}/q_1, \mathbb{C}(\chi))$$

where χ is the $\mathbb{Z}/q_1$ character, $1/q_1 \mapsto \zeta^{p_1+n_1+1}$, for ζ a primitive q_1th root of unity. By hypothesis, however, this character is fully faithful, so the said groups are zero for all q.

In general, everything reduces to this case by a double induction, i.e. at the D_1 level one may ask for the smallest $m_1 > p_1$ such that m_1 and q_1 have a common factor, and one assumes the result for $\frac{m_1}{q_1} D_1$, while the other induction is simply on the number of D_i's.

To get from H ample to H big and nef., one simply takes an honest ample A, produces an $\epsilon \in \mathbb{Q}_+$ such that $H - \epsilon A = B$ is effective, and for very small δ writes,

$$H = (1-\delta)(H + \frac{\epsilon\delta}{1-\delta} A) + \delta B$$

Certainly B may be very far from normal crossing, so it's necessary to use a resolution of singularities, but since we can take δ as small as we please, there's no problem. More generally the L_2-vanishing theorem reduces to this one since algebraically this is the right technical condition to permit the use of a resolution of singularities. $\Box$

There is, of course, really only one vanishing theorem & even one proof if one understood better the relation between complex conjugation & conjugation under Frobenius. As to how one then goes from the basic theorem to its various refinements isn't really that important.

III.3 Segre Classes

Our goal is to associate to a closed (p,p) current $d\mu$ with measure coefficients and a closed substack $\mathcal{Y}$ of a proper stack $\mathcal{X}/\mathbb{C}$ a positive residual $(p-1,p-1)$ current $s_{\mathcal{Y},d\mu}$ with measure coefficients supported on the projective cone $P(C_{\mathcal{Y}/\mathcal{X}})$ generalising the standard segre class when $d\mu$ is a current of integration. In the course of the discussion, it will emerge that there is a well defined notion of currents with measure coefficients on any proper stack, at least provided that it embeds in a (not necessarily proper) smooth stack. As such for convenience we'll start with $\mathcal{X}$ smooth, and indeed Kăhler if $p > 1$, but in reality only the properness of $\mathcal{Y}$ is really important. Before getting underway let's establish,

II.3.1 Notation/Definition *For $\mathcal{Y} \hookrightarrow \mathcal{X}$ as above let $\|\mathbb{I}_{\mathcal{Y}}\|$ be a function on X as described in* III.2, *or indeed of the form $|f|$ up to a bounded unit for f a local equation of the total exceptional divisor obtained by blowing up in $\mathcal{Y}$. Marginally confusingly we'll also employ the notation $\mathbb{I}_*$ for the characteristic function of Lebesgue measurabe sets.*

The key point is to verify,

III.3.2 Claim *Let ω be Kăhler in a neighbourhood of $\mathcal{Y}$, then for any non-negative integers i, j with $i = j = p$,*

$$\lim_{\epsilon \to 0} \int_{\|\mathbb{I}_{\mathcal{Y}}\| \leq \epsilon} (\frac{dd^c \|\mathbb{I}_{\mathcal{Y}}\|^2}{\epsilon^2})^i \omega^j d\mu$$

is a well defined, non-infinite, number.

proof The argument is, of course, nothing other than a regurgitation of the standard argument for the existence of Le-long numbers, i.e. the case where $\mathcal{Y}$ is a point, which depends not on the fact that $\mathcal{Y}$ is a point, but that it has no boundary. Evidently the case $i = 0$ is trivial, and $\omega^j d\mu \geq 0$ so we're reduced, for a possibly different measure, to the case $i = p$. As such if we put,

$$\sigma^{\epsilon}_{\mathcal{Y},d\mu} := \int_{\|\mathbb{I}_{\mathcal{Y}}\| \leq \epsilon} (\frac{dd^c \|\mathbb{I}_{\mathcal{Y}}\|^2}{\epsilon^2})^p d\mu$$

then for $\epsilon > \delta$, Stokes implies,

$$\sigma^{\epsilon}_{\mathcal{Y},d\mu} - \sigma^{\delta}_{\mathcal{Y},d\mu} = \int_{\delta < \|\mathbb{I}_{\mathcal{Y}}\| \leq \epsilon} (dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2)^p d\mu$$

However $dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2$ is as good as positive on $\mathcal{X} \setminus \mathcal{Y}$, i.e. bounded below by the restriction of a smooth form, so indeed $s_{\mathcal{Y},d\mu}(1) := \lim_{\epsilon \to 0} \sigma^{\epsilon}_{\mathcal{Y},d\mu}$ exists, and in fact,

$$s_{\mathcal{Y},d\mu}(1) = \lim_{\epsilon \to 0} \int_{\|\mathbb{I}_{\mathcal{Y}}\| = \epsilon} d^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2 (dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2)^{p-1} d\mu$$

$$\sigma^{\epsilon}_{\mathcal{Y},d\mu} = s_{\mathcal{Y},d\mu}(1) + \int_{0 < \|\mathbb{I}_{\mathcal{Y}}\| \leq \epsilon} (dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2)^p d\mu$$

We can also check, as the notation suggests, that there's no dependence on the choice of $\|\mathbb{I}_{\mathcal{Y}}\|$, so say $\|\tilde{\mathbb{I}_{\mathcal{Y}}}\|$ is another such, with, without loss of generality $\|\tilde{\mathbb{I}_{\mathcal{Y}}}\| \geq \|\mathbb{I}_{\mathcal{Y}}\|$, then, in the obvious notation,

$$\sigma^{\epsilon}_{\mathcal{Y},d\mu} - \sigma_{\tilde{\mathcal{Y}},d\mu}{}^{\epsilon} = \int_{\|\tilde{\mathbb{I}_{\mathcal{Y}}}\| \leq \epsilon} (dd^c u)^p d\mu + O(\int_{c\epsilon < \|\tilde{\mathbb{I}_{\mathcal{Y}}}\| \leq \epsilon} (dd^c \log \|\mathbb{I}_{\mathcal{Y}}\|^2)^p d\mu)$$

for some $c > 0$, and u smooth, or even just bounded. As such we conclude to the said lack of dependence by way of the essential positivity of $dd^c \log \| \mathbb{1}_{\mathcal{Y}} \|^2$, and the Skoda El-Mir theorem, albeit that we'll review the latter theorem from our current standpoint momentarily. $\square$

Several important technical points emerge from the proof, of which the first is,

$$\lim_{\epsilon \to 0} \int_{\| \mathbb{1}_{\mathcal{Y}} \| \le \epsilon} dd^c \| \mathbb{1}_{\mathcal{Y}} \|^2 \omega^{p-1} d\mu = 0$$

This rather benign looking observation implies a result that would otherwise be false without the properness of $\mathcal{Y}$, i.e.

III.3.3 Fact *Suppose $\mathcal{Y}$ has a smooth neighbourhood (Kåhler if $p > 1$) then the notion of current on $\mathcal{Y}$ is well defined (independently of the choice of neighbourhood) and if $d\mu = \mathbb{1}_{\mathcal{Y}} d\mu$, then $d\mu = i_* d\nu$ for $d\nu$ positive (p,p) on $\mathcal{Y}$, and i the embedding on $\mathcal{Y}$ in $\mathcal{X}$.*

proof We first look at the locus $\mathcal{Y}_{reg}$ where $\mathcal{Y}$ is smooth, and profiting from,

$$\mathbb{1}_{\mathcal{Y}} d\mu = \mathbb{1}_{\mathcal{Y}_{reg}} d\mu + \mathbb{1}_{\mathcal{Y} \setminus \mathcal{Y}_{reg}} d\mu$$

aim to do $\mathbb{1}_{\mathcal{Y}_{reg}} d\mu$ first. On $\mathcal{Y}_{reg}$ there is an unambiguous notion of current, and what must be verified (to which I'm indebted to M. Paun for bringing to my attention) is that,

$$\int d\| \mathbb{1}_{\mathcal{Y}} \| d^c \| \mathbb{1}_{\mathcal{Y}} \| \tau \mathbb{1}_{\mathcal{Y}_{reg}} d\mu = 0$$

for τ smooth on the neighbouhood. This is, however, clear, since up to an unimportant error, for ϵ small,

$$d\| \mathbb{1}_{\mathcal{Y}} \| d^c \| \mathbb{1}_{\mathcal{Y}} \| \mathbb{1}_{\mathcal{Y}} d\mu << dd^c \| \mathbb{1}_{\mathcal{Y}} \|^2 \mathbb{1}_{\mathcal{Y}} d\mu$$

so there's nothing to do. As such on $\mathcal{Y}_{reg}$, $\mathbb{1}_{\mathcal{Y}_{reg}} d\mu$ really is an honest (p,p) current on $\mathcal{Y}_{reg}$, which has a unique extension by zero over $\mathcal{Y} \setminus \mathcal{Y}_{reg}$ independent of the embedding, and we conclude by a Noetherian induction. $\square$

Of course, as we'll discuss later, $s_{\mathcal{Y},d\mu}(1)$ only depends on $\mathbb{1}_{\mathcal{X} \setminus \mathcal{Y}} d\mu$, but equally the above discussion works in somewhat greater generality, i.e.

III.3.4 Fact *For any $n \in \mathbb{N}$, $\frac{d\| \mathbb{1}_{\mathcal{Y}} \| d^c \| \mathbb{1}_{\mathcal{Y}} \|}{\| \mathbb{1}_{\mathcal{Y}} \|^{2(1-1/n)}}$ is integrable with respect to $\mathbb{1}_{\mathcal{X} \setminus \mathcal{Y}} d\mu$ and indeed,*

$$\lim_{\epsilon \to 0} \int_{\| \mathbb{1}_{\mathcal{Y}} \| \le \epsilon} \frac{d\| \mathbb{1}_{\mathcal{Y}} \| d^c \| \mathbb{1}_{\mathcal{Y}} \|}{\| \mathbb{1}_{\mathcal{Y}} \|^{2(1-1/n)}} \tau \mathbb{1}_{\mathcal{X} \setminus \mathcal{Y}} d\mu = 0$$

for any bounded τ. Idem but for $\frac{d\| \mathbb{1}_{\mathcal{Y}} \| d^c \| \mathbb{1}_{\mathcal{Y}} \|}{\| \mathbb{1}_{\mathcal{Y}} \|^2 \log^2 \| \mathbb{1}_{\mathcal{Y}} \|}$.

proof Let's do the latter. For ϵ small it's sufficient to look at,

$$- \int_{\| \mathbb{1}_{\mathcal{Y}} \| \le \epsilon} dd^c \log |\log \| \mathbb{1}_{\mathcal{Y}} \|| \omega^{p-1} d\mu$$

as ever ω Kãhler if $p > 1$. This is, however, equal to,

$$\frac{1}{|\log\epsilon|}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon}\frac{d^c\|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|}\omega^{p-1}d\mu-\lim_{\delta\to 0}\frac{1}{|\log\delta|}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\delta}\frac{d^c\|\mathbb{I}_{\mathcal{Y}}\|}{\|\mathbb{I}_{\mathcal{Y}}\|}\omega^{p-1}d\mu$$

with the interior residue plainly going to zero. □

Which, in itself, reproves the Skoda El-Mir theorem in this particular case (or more generally for almost psh functions whose polar set has no boundary), i.e.

III.3.5 Fact *Let α be a $2p-1$ form whose coefficients are L_2 with respect to $\omega^p d\mu$ then,*

$$\lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon}\alpha d\mu = 0$$

proof The measure $\frac{d\epsilon}{\epsilon|\log\epsilon|}$ on $\mathbb{R}_+$ is unbounded on neighbourhoods of zero, so integrate the integral under the limit with respect to it, apply III.3.4 and the Cauchy-Schwarz inequality. □

This not unuseful miscellany dealt with, let's re-interpret the key claim III.3.2 by way of,

III.3.6 Fact *Let $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow up of $\mathcal{X}$ in $\mathcal{Y}$ then $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}}d\mu$ has a well defined proper transform $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}}d\tilde{\mu}$, i.e. the extension of $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}}d\mu$ by zero across the exceptional divisor, which itself is closed (p,p).*

For $\mathcal{Y} \hookrightarrow \mathcal{X}$ smooth, this is exactly the content of the previous claim, i.e. what has to be verified is that $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}}d\tilde{\mu}$ has finite mass. Now what one might quibble about is exactly what the definition is if $\mathcal{Y}$ is arbitrary. We can, however, resolve $\mathcal{I}_{\mathcal{Y}}$ by a sequence of blow ups in smooth centres to get some $\rho_1 : \mathcal{X}_1 \to \mathcal{X}$, and certainly there is therefore a well defined $d\mu^1$ extending $\mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}}d\mu$ by zero. Furthermore if $\rho_2 : \mathcal{X}_2 \to \mathcal{X}$ is another such, with $d\mu^2$ the obvious thing, there is a modification $\mathcal{X}^{\#}$, as ever by blow ups in smooth centres, with projections $\rho_i : \mathcal{X}^{\#} \to \mathcal{X}$ dominating both, together with a $d\mu^{\#}$ such that, $d\mu^i = (\rho_i)_* d\mu^{\#}$. In this sense the discussion is wholly unambiguous, but it's also unambiguous no matter how, even locally, we choose to embed $\tilde{\mathcal{X}}$ in a smooth. The key point is that,

$$\lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon} d^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2(dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2)^{i-1}\omega^j d\mu\ ,\ i+j=p$$

exists, whence anything of the form,

$$\lim_{\epsilon\to 0}\int_{\|\mathbb{I}_{\mathcal{Y}}\|=\epsilon} fd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2(dd^c\log\|\mathbb{I}_{\mathcal{Y}}\|^2)^{i-1}\omega^j d\mu$$

exists for f bounded. Whence, no matter how one chooses to define it $(\rho_i)_* d\mu^i = \mathbb{I}_{\mathcal{X}\backslash\mathcal{Y}}d\tilde{\mu}$, and the final re-interpretation of III.3.2 presents itself,

III.3.7 Fact/Definition *There is a well defined closed positive $(p-1,p-1)$ current $s_{\mathcal{Y},d\mu}$, or $d\sigma_{\mathcal{Y},d\mu}$ in infinitesimal form, on $P(C_{\mathcal{Y}/\mathcal{X}})$ with total mass $s_{\mathcal{Y},d\mu}(1)$. The said measure will be called the segre class and/or residual measure, while, sometimes we'll write $\delta_{\mathcal{Y}} \wedge d\mu$, especially if $\mathcal{Y}$ is Cartier in $\mathcal{X}$.*

Indeed we've already seen in III.3.3 that there's no problem, even locally, talking about measures on singular things provided that we have some knowledge of the behaviour in the normal direction in a smooth embedding, which is indeed the case here.

III.4 Intermission/Explanation

It seems reasonable to pause and to ask what's going on, since as everyone knows it's not possible to define a wedge of δ-functions. The key, however, to observing that what's gone before isn't nonsense is, as we've said, to notice that there's no dependence on the part of $d\mu$ supported on $\mathcal{Y}$. Indeed, manifestly, $\delta_{\mathcal{Y}} \wedge \mathbb{I}_{\mathcal{Y}} d\mu = 0$, so things don't descend to co-homology, although, on the plus side, there exists the possibility of defining a cap product of currents modulo rational equivalence (i.e. limits of δ-functions of rational functions on sub-stacks). Specifically in the case of $(1,1)$ measures $d\mu$, and divisors $\mathcal{D}$ we'd be looking at,

$$\delta_{\mathcal{D}} \cap d\mu = (i_{\mathcal{D}})_*(\delta_{\mathcal{D}'} \cap \mathbb{I}_{\mathcal{D}} d\mu) + \delta_{\mathcal{D}} \wedge d\mu$$

for $\mathcal{D}' \hookrightarrow \mathcal{D}$ a not necessarily effective Cartier divisor, rationally equivalent to $\mathcal{O}_{\mathcal{D}}(\mathcal{D})$. Granted this only works for capping divisors with currents, but by way of the main regularisation theorem of [D1] this can actually be extended to arbitrary positive co-dimension $(1,1)$ currents, at least in the wholly scheme like situation. In any case, this is certainly deserving of attention, albeit that the need for a suitable regularisation theorem in all dimensions means that we'll postpone it for another time, and content ourselves by noting the agreement in dimension $(1,1)$ with standard intersection theory, i.e.

II.4.1 Divertimento *Say everything scheme like to avoid notational complication, with D an irreducible Cartier divisor not containing an irreducible curve C with $D \cap C$ their intersection as a positive zero cycle in the sense of algebraic intersection theory, then we have an identity of measures,*

$$\delta_D \wedge \delta_C = \delta_{D \cap C}$$

proof From what we've already seen it's clear that $\delta_D \wedge \delta_C$ is a positive measure supported on the set theoretic intersection of D and C, so all that's at stake is making sure that the local multiplicities are correct. To compute these, observe, quite generally, that if $Z \subset X$ is a closed subscheme of pure co-dimension $e+1$ defined locally by an ideal $(z_1, \ldots, z_d)$ with $\|z\|^2 = \sum |z_i|^2$ then, locally,

$$\delta_Z = dd^c(\log \|z\|^2 (dd^c \log \|z\|^2)^e)$$

To see this, just blow up in Z to get the exceptional divisor $\pi : P(C_{Z/X}) \to Z$ again of pure dimension, with $\|z\|^2$ yielding a Fubini-Study type metricisation $\bar{L}$ of the tautological bundle on the projective cone, according to which, for f a compact test form on X,

$$\int_X \log \|z\|^2 (dd^c \log \|z\|^2)^e dd^c f = \int_{P(C_{Z/X})} \pi^* i_Z^* f c_1(\bar{L})^e$$

while for a generic point z of Z,

$$\int_{P(C_{Z/X})\otimes\mathbb{C}(z)} c_1(\bar{L})^e = 1$$

Applying these considerations to the computation of $\delta_D \wedge \delta_C$ on a compactly supported test function f with $g = 0$ a local equation for D we find,

$$\delta_D \wedge \delta_C(f) = \lim_{\epsilon\to 0} \int_{P(C_{C/X})\cap(|g|=\epsilon}) f c_1(\bar{L})^{dimX-2} d^c \log |g|^2$$

so that the mass of $\delta_D \wedge \delta_C$ around a point of $D \cap C$ is infact the tautological degree of $P(C_{D\cap C/D})$ at the same, as required. □

Of course the said discussion is much more general than the assertion, so let's observe that modulo notational complication we've even deduced,

III.4.2 Sub-divertimento *Let $\mathcal{C}$ be a pure 1-dimensional closed substack of $\mathcal{X}/\mathbb{C}$ with $\mathcal{Y} \hookrightarrow \mathcal{X}$ a closed substack not containing any generic point of $\mathcal{C}$, then as measures,*

$$\delta_{\mathcal{Y}} \wedge \delta_{\mathcal{C}} = \sum_{c\in\mathcal{Y}\cap\mathcal{C}} s(c)\delta_c$$

where $s(c)$ is the standard segre class of $\mathcal{Y}$ at geometric points c of the intersection computed, for example, as per [II] I.8.6

III.5 Deformation to the Normal Cone

We retake verbatim the notations of III.3, so that $\mathcal{Y} \hookrightarrow \mathcal{X}$ is a closed proper stack and $d\mu$ a positive closed (p, p) current. As one might imagine the deformation of $d\mu$ to the cone $C_{\mathcal{Y}/\mathcal{X}} \to \mathcal{Y}$ is slightly more informative than the segre class. The deformation construction is, of course, the standard one of [F] §5, i.e. blow up $\mathcal{X} \times \mathbb{P}^1$ in $\mathcal{Y} \times 0$ with say $\mathcal{W}$ the total space, and $\mathcal{E}$ the total exceptional divisor $\xrightarrow{\sim} P(C_{\mathcal{Y}/\mathcal{X}} \oplus \mathbb{1})$. In particular if ρ is the projection of $\mathcal{X} \times \mathbb{P}^1$ to $\mathcal{X}$, then $\rho^* d\mu$ is positive $(p + 1, p + 1)$ and we have a well defined positive closed (p, p) current $d\sigma_{\mathcal{E},d\mu}$ as per III.3.7. The deformation of $d\mu$ to the normal cone is then,

III.5.1 Definition *Let things be as above, and identify $C_{\mathcal{Y}/\mathcal{X}} \hookrightarrow P(C_{\mathcal{Y}/\mathcal{X}} \oplus \mathbb{1})$ with the complement of the hyperplane at infinity, then the deformation, or better specialisation, of $d\mu$ to the normal cone is the positive closed (p, p) current,*

$$d\mu^! := \mathbb{1}_{C_{\mathcal{Y}/\mathcal{X}}} d\sigma_{\mathcal{E},d\mu}$$

Superficially this looks like we need to suppose $\mathcal{X}$ Kãhler even if $p = 1$, but, in reality, we don't, i.e.

III.5.2 Fact *This is well defined for $p = 1$ even if $\mathcal{X}$ isn't Kãhler.*

proof For $t = 0$ a local equation for 0 on $\mathbb{P}^1$ we need to consider the integral,

$$\int_{\|\mathbb{1}_{\mathcal{Y}}\|^2+|t|^2\le\epsilon} \frac{dd^c(\|\mathbb{1}_{\mathcal{Y}}\|^2 + |t|^2)}{\epsilon^2} \Psi\rho^* d\mu$$

for Ψ on $\mathcal{X} \times \mathbb{P}^1$ strictly positive in a neighbourhood of $\mathcal{Y} \times 0$. Plainly we can take Ψ to be of the form $\rho^*\psi + dd^c|t|^2$ for $\psi > 0$ on $\mathcal{X}$. Consequently,

$$\Psi\rho^* d\mu = \rho^*(\psi d\mu) + dd^c|t|^2 d\mu$$

is still closed positive $(1,1)$, which is what we needed to define $d\sigma_{\mathcal{E},d\mu}$. $\square$

In addition the computation of the segre class commutes with specialisation to the normal cone, i.e.

III.5.3 Fact *let* $[0] \hookrightarrow C_{\mathcal{Y}/\mathcal{X}}$ *be the zero section, then,*

$$s_{[0],d\mu^!} = s_{\mathcal{Y},d\mu}$$

proof We'll only do the $(1,1)$ case, since apart from being the only one that we require, it's notationally less fastidious. As such for f a test function on a neighbourhood of $\mathcal{Y}$, by definition,

$$\begin{aligned} s_{[0],d\mu^!}(f) &= \lim_{\epsilon\to 0} \int_{\|\mathbb{1}_{[0]}\|=\epsilon} f d^c \log \|\mathbb{1}_{[0]}\|^2 d\mu^! \\ &= \lim_{\epsilon\to 0} \lim_{\delta\to 0} \int_{|t|=\delta} d^c \log |t|^2 \int_{\|\mathbb{1}_{[0]}\|=\epsilon} f d^c \log \|\mathbb{1}_{[0]}\|^2 d\mu \end{aligned}$$

where we profit from from the fact that $t = 0$ is a local equation for $\mathcal{E}$ around $[0]$. Now the function $\|\mathbb{1}_{[0]}\|$ is exactly $\|\mathbb{1}_{\mathcal{Y}}\|(\|\mathbb{1}_{\mathcal{Y}}\| + |t|)^{-1}$ so identifying the fibre over $t \neq 0$ of the deformation with $\mathcal{X}$ the integral can be re-written as,

$$\lim_{\epsilon\to 0} \lim_{\delta\to 0} \int_{\|\mathbb{1}_{\mathcal{Y}}\|=\frac{\epsilon\delta}{(1-\delta)}} f(1-\epsilon) d^c \log \|\mathbb{1}_{\mathcal{Y}}\|^2 d\mu$$

whence the assertion. $\square$.

The specialised class $d\mu^!$ has other agreeable properties which reflect the purely algebraic fact that for $\mathcal{V} \hookrightarrow \mathcal{X}$ closed, $C_{\mathcal{Y}\cap\mathcal{V}/\mathcal{V}}$ is pure dimensional. For instance if $\nu : C_{\mathcal{Y}/\mathcal{X}} \to \mathcal{Y}$ is the projection,

III.5.4 Fact *Say* $d\mu$ *is dimension* $(1,1)$ *for the sake of a clean statement, then* $(\mathbb{1}_{\mathcal{Y}} d\mu)^!$ *is just the inclusion by zero, and* $\nu_*(\mathbb{1}_{\mathcal{X}\setminus\mathcal{Y}} d\mu)^! = 0$.

proof The first part just amounts to pulling back $\rho^* \mathbb{1}_{\mathcal{Y}} d\mu$ along $\mathcal{Y} \times 0$ in $\mathcal{Y} \times \mathbb{P}^1$, so this is clear. For the second one takes ω to be a $(1,1)$ form on $\mathcal{X}$ and writes,

$$\int_{\|\mathbb{1}_{[0]}\|\leq\epsilon} \nu^*(\omega|_{\mathcal{Y}}) d\mu^! = \lim_{\delta\to 0} \int_{|t|=\delta} d^c \log |t|^2 \int_{\|\mathbb{1}_{\mathcal{Y}}\|\leq\frac{\epsilon\delta}{(1-\delta)}} \omega d\mu = 0$$

from which we conclude. $\square$

III.6 Transverse Invariant Measures

It's opportune to remark to what extent the previous discussion improves in the presence of a foliation, i.e. $(\mathcal{X}, \mathcal{F})$ is a smooth analytic stack $/\mathbb{C}$ foliated by curves and $d\mu_{\mathcal{X}/\mathcal{F}}$ is an invariant measure. For example,

III.6.1 Fact *let* $\mathcal{Y} \hookrightarrow \mathcal{X}$ *be a proper invariant sub-stack, then* $s_{\mathcal{Y},d\mu}$ *is supported on* $\mathrm{sing}(\mathcal{F}) \cap \mathcal{Y}$.

This is clear from the global existence of $s_{\mathcal{Y},d\mu}$ combined with the existence of local foliation coordinates. The said coordinates also imply that $s_{\mathcal{Y},d\mu}$ can be defined without $\mathcal{Y}$ being proper, e.g.

III.6.2 Fact *Let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a locally closed and (to fix ideas) irreducible divisor with generic point transverse to $\mathcal{F}$ with $\partial\mathcal{Y}$ contained in the smooth locus, then $s_{\mathcal{Y},d\mu}$ is well defined.*

proof The hypothesis imply that around $\partial\mathcal{Y}$ we're looking at a polydisc Δ with $x = 0$ a local equation for $\mathcal{Y}$, and the other coordinates z_i defining the foliation by way of a fibration. We need only consider over Δ an integral of the form,

$$\lim_{\epsilon\to 0}(\sigma_\epsilon := \int_{|x|=\epsilon} d^c \log |x|^2 d\mu_{\mathcal{X}/\mathcal{F}})$$

so that for $\epsilon > \delta$, Stokes gives that $\sigma_\epsilon - \sigma_\delta$ is an integral of $d^c \log |x|^2$ over $|x| \leq \epsilon$, but max$|z_i|$ =constant. The latter real hypersurface is invariant so $d\mu_{\mathcal{X}/\mathcal{F}}$ restricted to it is zero. $\Box$

The same sort of idea works in a lot of other situations, which in full generality, we'll postpone till IV.4. For the moment let us content ourselves with the most basic extension, with everything local about an analytic neighbourhood (harmlessly denoted $\mathcal{X}$) of a proper connected component Z of $\mathrm{sing}(\mathcal{F})$, *i.e.*

III.6.3 Fact *Let $\mathcal{W}$ be an invariant substack of a neighbourhood $\mathcal{X}$ as above, then $s_{\mathcal{W},d\mu_{\mathcal{X}/\mathcal{F}}}$ is well defined and supported on* $\mathrm{sing}(\mathcal{F})$.

proof For $\|\mathbb{1}_{\mathcal{Z}}\|$, $\|\mathbb{1}_{\mathcal{W}}\|$ as per usual, we may as well say that our neighbourhood is of the form, $\|\mathbb{1}_{\mathcal{Z}}\| \leq 1$. Consequently, as ever consider,

$$\sigma^\epsilon_{\mathcal{W},d\mu} := \int_{\|\mathbb{1}_{\mathcal{W}}\|=\epsilon,\, \|\mathbb{1}_{\mathcal{Z}}\|\leq 1} d^c \log \|\mathbb{1}_{\mathcal{W}}\|^2 d\mu_{\mathcal{X}/\mathcal{F}}$$

Whence, for a suitably small (but independent of ϵ) normalisation of $\|\mathbb{1}_{\mathcal{Z}}\|$, we can write for $\delta < \epsilon$,

$$\sigma^\epsilon_{\mathcal{W},d\mu} - \sigma^\delta_{\mathcal{W},d\mu} = \int_{\delta\leq\|\mathbb{1}_{\mathcal{W}}\|\leq\epsilon,\|\mathbb{1}_{\mathcal{Z}}\|\leq 1} dd^c \log \|\mathbb{1}_{\mathcal{W}}\|^2 d\mu_{\mathcal{X}/\mathcal{F}}$$
$$+ \int_{\delta\leq\|\mathbb{1}_{\mathcal{W}}\|\leq\epsilon,\, \|\mathbb{1}_{\mathcal{Z}}\|=1} w d\mu_{\mathcal{X}/\mathcal{F}}$$

where the 1-form w is bounded in a neighbourhood of $\|\mathbb{1}_{\mathcal{Z}}\| = 1$. As such our set up is precisely as in the proof of III.3.2, so the limit $\sigma_{\mathcal{W},d\mu}$ as $\epsilon \to 0$ of $\sigma^\epsilon_{\mathcal{W},d\mu}$ exists. A priori it defines a measure supported on $\mathcal{W}$, but as per III.6.1 this is easily seen to be zero off Z. $\Box$

IV. FOLIATED RESIDUES

IV.1 Filtrations on co-homology

Sticking to the convention that a foliation is a saturated sub-sheaf of $\mathcal{T}_{\mathcal{X}}$ we necessarily have a short exact sequence of the form,

$$0 \longrightarrow \Omega^1_{\mathcal{X}/\mathcal{F}} \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow K_{\mathcal{F}}\mathcal{I}_{\mathcal{Z}} \longrightarrow 0$$

where $\mathcal{Z}$ is the singular sub-stack of $\mathcal{F}$, and the slightly abusive notation $\Omega^1_{\mathcal{X}/\mathcal{F}}$ is used to denote the necessarily reflexive subsheaf of $\Omega^1_{\mathcal{X}}$ which forms the kernel of the penultimate map on the right. Manifestly, however, we have similar short exact sequences for every $1 \leq m \leq n = \dim X$, which take the form,

$$0 \longrightarrow \Omega^m_{\mathcal{X}/\mathcal{F}} \longrightarrow \Omega^m_{\mathcal{X}} \longrightarrow Q_m \longrightarrow 0$$

where generically $\Omega^m_{\mathcal{X}/\mathcal{F}}$ is $\bigwedge^m \Omega^1_{\mathcal{X}/\mathcal{F}}$, but if needs be, we saturate it to guarantee that Q_m is torsion free. In particular associated to the standard DeRham complex,

$$\Omega^\bullet_{\mathcal{X}} : 0 \longrightarrow \mathcal{O}_{\mathcal{X}} \underset{d}{\longrightarrow} \Omega^1_{\mathcal{X}} \underset{d}{\longrightarrow} \dots \underset{d}{\longrightarrow} \Omega^n_{\mathcal{X}} \underset{d}{\longrightarrow} 0$$

we have a filtration by sub-complexes defined by,

IV.1.1 Definition

$$F^p\Omega^m_{\mathcal{X}} = \begin{cases} 0\,, & \text{if } p > m \\ \Omega^p_{\mathcal{X}/\mathcal{F}}\,, & \text{if } p = m \\ \Omega^m_{\mathcal{X}/\mathcal{F}} & \text{if } p < m \end{cases}$$

where, of course, $0 \leq p \leq n$, and the one term that we haven't defined $\Omega^0_{\mathcal{X}/\mathcal{F}}$ is nothing other than $\mathcal{O}_{\mathcal{X}}$. Consequently in the usual way we get spectral sequences,

$$A_1^{p,q} := \mathbb{H}^{p+q}(\mathcal{X}, \mathrm{gr}^p\Omega^\bullet_{\mathcal{X}}) \Longrightarrow \mathbb{H}^{p+q}(\mathcal{X}, \Omega^\bullet_{\mathcal{X}}) = H^{p+q}(\mathcal{X}, \mathbb{C})$$

$$A_{1,c}^{p,q} := \mathbb{H}_c^{p+q}(\mathcal{X}, \mathrm{gr}^p\Omega^\bullet_{\mathcal{X}}) \Longrightarrow \mathbb{H}_c^{p+q}(\mathcal{X}, \Omega^\bullet_{\mathcal{X}}) = H_c^{p+q}(\mathcal{X}, \mathbb{C})$$

Fortunately our interest is not in the spectral sequences themselves, but in their E_1 terms, so that it's worth noting,

IV.1.2 Notation $\mathrm{gr}^p\Omega^\bullet_{\mathcal{X}}[p] = \{\Omega^p_{\mathcal{X}/\mathcal{F}} \underset{\nabla}{\longrightarrow} Q_{p+1}\}$

where the torsion free sheaf Q_{p+1} is simply the reflexive hull of $\Omega^p_{\mathcal{X}/\mathcal{F}} \otimes K_{\mathcal{F}}$, and whence the notation ∇ for the differential, whereby it should be thought of as a connection along the leaves.

Implicit in this description is the C^∞ preparation theorem, or, more correctly, the corollary that the ring $\mathcal{A}_{\mathcal{X}}$ of smooth functions is flat over $\mathcal{O}_{\mathcal{X}}$. In particular, therefore, if we consider the canonical $\bar\partial$ resolution of $\mathcal{O}_{\mathcal{X}}$, i.e.

$$\mathcal{A}^{0,*}_{\mathcal{X}} : \mathcal{A}^{0,0}_{\mathcal{X}} \underset{\bar\partial}{\longrightarrow} \mathcal{A}^{0,1}_{\mathcal{X}} \underset{\bar\partial}{\longrightarrow} \dots\dots \underset{\bar\partial}{\longrightarrow} \mathcal{A}^{0,n}_{\mathcal{X}}$$

then the graded complex associated to the total complexes $F^p\Omega^\bullet_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{A}^{0,*}_{\mathcal{X}}$ is simply the total complex of $\mathrm{gr}^p\Omega^\bullet_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{A}^{0,*}_{\mathcal{X}}$, which is acyclic with respect to taking global sections, and, of course, similarly for compact support. If, however, we consider resolving $\mathcal{O}_{\mathcal{X}}$ by distributions, i.e.

$$\mathcal{D}^{0,*}_{\mathcal{X}} : \mathcal{D}^{0,0}_{\mathcal{X}} \underset{\bar\partial}{\longrightarrow} \mathcal{D}^{0,1}_{\mathcal{X}} \underset{\bar\partial}{\longrightarrow} \dots\dots \underset{\bar\partial}{\longrightarrow} \mathcal{D}^{0,n}_{\mathcal{X}}$$

then this is no longer flat, and we get a rather different answer. Indeed with the usual notation that $D^{\bullet;*}_{\mathcal{X}}$ is global distributions, then by virtue of the injectivity of $\mathcal{D}$ the graded complex associated to the total complex becomes,

$$\mathrm{gr}^p(D^{\bullet;*}_{\mathcal{X}})[p]_q = \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(Q_{n-p}, \mathcal{D}^{n,q}_{\mathcal{X}}) \oplus \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\Omega^{n-p-1}_{\mathcal{X}/\mathcal{F}}, \mathcal{D}^{n,q-1}_{\mathcal{X}})$$

and the differential is $\nabla^\vee \oplus \bar{\partial}$. To identify the co-homology of this complex, observe that it is simply homotopy classes of maps between the complexes,

$$\mathrm{gr}^{n-p-1}\Omega^\bullet_{\mathcal{X}}[n-p] \;\text{ and }\; \mathcal{D}^{n,*}_{\mathcal{X}}[q]$$

and since $\mathcal{D}^{n,*}_{\mathcal{X}}$ is an injective resolution of $K_{\mathcal{X}}$ this is just $\mathbb{E}\mathrm{xt}^q_{\mathcal{O}_{\mathcal{X}}}(\mathrm{gr}^{n-p-1}\Omega^\bullet_{\mathcal{X}}[n-p], K_{\mathcal{X}})$. Consequently, we obtain rather different spectral sequences,

$$D^{p,q}_1 = \mathbb{E}\mathrm{xt}^q_{\mathcal{O}_{\mathcal{X}}}(\mathrm{gr}^{n-p-1}\Omega^\bullet_{\mathcal{X}}[n-p], K_{\mathcal{X}}) \Longrightarrow H^{p+q}(\mathcal{X}, \mathbb{C})$$

$$D^{p,q}_{1,c} = \mathbb{E}\mathrm{xt}^q_{c,\mathcal{O}_{\mathcal{X}}}(\mathrm{gr}^{n-p-1}\Omega^\bullet_{\mathcal{X}}[n-p], K_{\mathcal{X}}) \Longrightarrow H^{p+q}_c(\mathcal{X}, \mathbb{C})$$

Again our interest is in the initial term rather than the spectral sequences themselves, and we may note that the usual pairings between ext's and co-homologies imply the same at the hyper level, so that we have,

IV.1.3 Pairings

$$\begin{aligned} <\quad >:\; & A^{p,q}_{1,c} \times D^{n-1-p,n+1-q}_1 \longrightarrow \mathbb{C} \\ & A^{p,q}_1 \times D^{n-1-p,n+1-q}_{1,c} \longrightarrow \mathbb{C} \end{aligned}$$

Better still forms always map to distributions, and in the particular case that $p = n-1$ or 0, it's rather easy to determine the relation. Indeed, we have,

$$A^{n-1,q}_1 = \mathbb{H}^q(\mathcal{X}, \mathrm{gr}^{n-1}\Omega^\bullet_{\mathcal{X}}[n-1])$$

and both the terms in $\mathrm{gr}^{n-1}\Omega^\bullet_{\mathcal{X}}$ are bundles, so this is just,

$$\mathbb{E}\mathrm{xt}^q(\mathcal{O} \to K_{\mathcal{F}}[1], K_{\mathcal{X}})$$

from which we obtain a long exact sequence,

$$\to \mathrm{Ext}^q(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}}) \to A^{n-1,q}_1 \longrightarrow D^{n-1,q}_1 \to \mathrm{Ext}^{q+1}(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}}) \to$$

where we can, and will, think of the $\mathrm{Ext}^q(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}})$'s as controlling the obstruction to co-homological smoothing of invariant distributions. Unfortunately, however, the smoothing involves a boundary which may well behave in a less than desirable way if $\mathcal{Z}$ is non-compact. As such the particularly important case of $q = n-1$ deserves special mention by way of,

IV.1.4 Definition *Suppose $\mathcal{Z}$ is compact, then we have a diagram* **RES** *with exact rows,*

$$\begin{array}{ccccc} A^{n-1,n-1}_{1,c} & \longrightarrow & D^{n-1,n-1}_{1,c} & \longrightarrow & \mathrm{Ext}^n_c(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}}) \\ \downarrow & & \downarrow & & \| \\ A^{n-1,n-1}_1 & \longrightarrow & D^{n-1,n-1}_1 & \xrightarrow{Res} & \mathrm{Ext}^n(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}}) \end{array}$$

If one wants to describe the map Res explicitly, then there are a couple of possibilities, particularly if we focus on classes in $D^{n-1,n-1}_1$ defined by transverse invariant measures $d\mu_{\mathcal{X}/\mathcal{F}}$. The first one is to observe that this defines a class in $\mathrm{Hom}(\mathcal{I}_{\mathcal{Z}}, \Omega^{n-1}_{\mathcal{X}/\mathcal{F}} \otimes \mathcal{D}^{0,n-1})$, so a fortiori in $\mathrm{Ext}^{n-1}(\mathcal{I}_{\mathcal{Z}}, \Omega^{n-1}_{\mathcal{X}/\mathcal{F}})$, and certainly therefore in $\mathrm{Ext}^n(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}})$. Alternatively, as the name Res suggests, we can use duality, and starting from a section w of $H^0(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}})$ lift to a smooth $(1,0)$ form ω over $\mathcal{X} \setminus \mathcal{Z}$ so that,

$$Res(d\mu_{\mathcal{X}/\mathcal{F}})(\omega) = -\lim_{\epsilon \to 0} \int_{\|\mathbb{I}_{\mathcal{Z}}\| = \epsilon} \omega d\mu_{\mathcal{X}/\mathcal{F}}$$

Regrettably the above analysis fails to address the basic issue of whether $Res(d\mu)$ can be a priori defined in $\mathrm{Ext}^n_c(K_{\mathcal{F}} \otimes \mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}})$ without supposing that $\mathcal{Z}$ is compact. There are also various possible refinements that we can introduce by way of the log-complex $\Omega^{\bullet}_{\mathcal{X}}(\log \mathcal{D})$ for $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ a foliated log-stack, but this still fails to grasp the above nettle, so we'll simply consider such things as we need them.

IV.2 Invariant Bundles

We may as well immediately specialise the previous discussion to what, after blowing up, becomes the critical case, viz: our stack $\mathcal{X}$ is a neighbourhood of a simple normal crossing divisor $\mathcal{D}$ containing the support of $\mathcal{Z}$. Unsurprisingly the discussion is most elegant when the triple $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ has log-canonical singularities, but for the moment this isn't immediately relevant. In any case we can basically replace at this juncture $H_c(\mathcal{X})$ by local co-homology groups $H_{\mathcal{D}}(\mathcal{X})$. If we understand the latter meromorphically, then strictly speaking this isn't true, but since we're primarily interested in pairings that depend only on things of the form $\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n_{\mathcal{D}}$, $n \in \mathbb{N}$, this makes no essential difference. Regardless we wish to consider what it should mean to have a $\mathcal{F}$ invariant bundle on $\mathcal{X} \setminus \mathcal{D}$. Certainly such a bundle, indeed let's say line bundle to fix ideas, should be equipped with a connection along the leaves, but this is only to be defined off $\mathcal{D}$ so we should admit poles, and whence,

IV.2.1 Definition *A line bundle L on $\mathcal{X}$ is said to be invariant on $\mathcal{X} \setminus \mathcal{D}$ with meromorphic poles, if there is a connection,*

$$\nabla : L \longrightarrow L \otimes K_{\mathcal{F}}(a_j D_j)$$

with $a_j \in \mathbb{N} \cup \{0\}$, D_j the components of $\mathcal{D}$, and, of course, the rule: $\nabla(fs) = f\nabla(s) + \partial_{\mathcal{F}}(f)s$ must be satisfied for $\partial_{\mathcal{F}}$ the composition of holomorphic d with restriction to $K_{\mathcal{F}}$, f a function, and s a section.

Before progressing notice that bundles with meromorphic connections arise from bundles with holomorphic connections in a couple of natural ways, viz:

(a) If we start from a foliation $(\mathcal{X}_0, \mathcal{F}_0)$ with arbitrary singularities, and suppose $\rho : (\mathcal{X}, \mathcal{F}) \to (\mathcal{X}_0, \mathcal{F}_0)$ is a resolution with canonical singularities, then, in general, a bundle with holomorphic connection along $\mathcal{F}_0$ becomes after pull-back a bundle with meromorphic connection along $\mathcal{F}$.

(b) This time we start with a space $(\mathcal{X}, \mathcal{F})$, even with canonical singularities, and let W be an invariant subvariety, then if $\mathrm{sing}(\mathcal{F}) \cap \mathrm{W}$ is a divisor, a $\mathcal{F}$ invariant bundle on $\mathcal{X}$ restricts, in general, to a bundle with meromorphic connection for the induced foliation on W.

Apart from these being our motivating examples, the immediate upshot of our definition is that we have a well defined class,

$$c_1(L,\nabla) \in H^1_{\mathcal{D}}(K_{\mathcal{F}}\mathcal{I}_{\mathcal{Z}})$$

Indeed for s_α some local generator of L on a scheme like open U_α the ∇s_α patch to a section of $K_{\mathcal{F}}(a_j D_j)\otimes\mathcal{O}_{\mathcal{Z}}$, which maps to a class in $H^1(X, K_{\mathcal{F}}(a_j D_j)\mathcal{I}_{\mathcal{Z}})$, and the local co-homology is just the direct limit of these over all divisors supported on $\mathcal{D}$. in particular for $d\mu_{\mathcal{X}/\mathcal{F}}$ an invariant measure, IV.1.3 guarantees that we have a pairing,

$$< c_1(L,\nabla), d\mu_{\mathcal{X}/\mathcal{F}} >\in \mathbb{C}$$

and what we require is to try and factor this through Res. Manifestly if $\mathcal{D}$ is empty, then by what we've said there is actually a class $\nabla \log s \in H^0(K_{\mathcal{F}}\otimes\mathcal{O}_{\mathcal{Z}})$ affording chern, and by the $p=0$ version of IV.1.4 an exact sequence,

$$H^0(K_{\mathcal{F}}\otimes\mathcal{O}_{\mathcal{Z}}) \longrightarrow A^{0,2}_{1,c} \longrightarrow D^{0,2}_{1,c}$$

together with a map $H^1_{\mathcal{D}}(K_{\mathcal{F}}\mathcal{I}_{\mathcal{Z}}) \to A^{0,2}_{1,c}$ arising from the degeneration at E_2 (for trivial dimension reasons) of the troncation bête spectral sequence all commuting with the formation of $c_1(L,\nabla)$, and so,

$$< c_1(L,\nabla), d\mu_{\mathcal{X}/\mathcal{F}} >= Res(d\mu_{\mathcal{X}/\mathcal{F}})(\nabla \log s)$$

The general situation is, however, rather more delicate, and the first thing to do is consider the situation around a component D_0 which isn't invariant. Here we should suppose that $(\mathcal{X},\mathcal{D},\mathcal{F})$ has log-canonical singularities so that D_0 is smooth and everywhere transverse to $\mathcal{F}$. Consequently we require to adapt the co-homological smoothing of $d\mu_{\mathcal{X}/\mathcal{F}}$ to the pole order a_0 of the connection around D_0. As such we have to filter $\Omega^\bullet_{\mathcal{X}/\mathcal{F}}$ according to,

$$\tilde{F}^{n-1}\Omega^{n-1}_{\mathcal{X}} = \Omega_{\mathcal{X}/\mathcal{F}}(-D')$$

where, a little more generally, D' is some divisor of high order of vanishing along the non-invariant components of our supposed log-canonical triple $(\mathcal{X},\mathcal{D},\mathcal{F})$. Unsurprisingly, therefore, we get a slightly different group $\tilde{A}^{n-1,q}$ at the E_1 level of the spectral sequence abutting to algebraic DeRham co-homology, which may be written in terms of hyperext by way of,

$$\tilde{A}^{n-1,q} = \mathbb{E}\mathrm{xt}^q(\mathcal{O}_{\mathcal{X}} \to K_{\mathcal{F}}(D')[1], K_{\mathcal{X}})$$

and from there an exact sequence,

$$\tilde{A}^{n-1,n-1} \longrightarrow D^{n-1,n-1} \underset{\widetilde{Res}}{\longrightarrow} \mathrm{Ext}^n(K_{\mathcal{F}}\otimes\mathcal{O}_{\mathcal{Z}}, K_{\mathcal{X}}) \oplus \mathrm{Ext}^n(K_{\mathcal{F}}\otimes\mathcal{O}_{D'}(D'), K_{\mathcal{X}})$$

where the direct sum decomposition on the right follows from the smoothness of $\mathcal{F}$ around D'. In order to reduce $\widetilde{Res}$ to Res we appeal to,

IV.2.2 Fact *Suppose things are as above, but in addition the measure $d\mu_{\mathcal{X}/\mathcal{F}}$ has zero segre class around the support of D' then $\widetilde{Res}$ factors through Res.*

proof Just as post IV.1.4 we can use duality to reduce the calculation of the projection of $\widetilde{Res}$ to $\mathrm{Ext}^n(K_{\mathcal{F}}\otimes\mathcal{O}_{D'}(D'), K_{\mathcal{X}})$ to that of an integral of the form,

$$\lim_{\epsilon \to 0} \int_{\| \mathbb{1}_{|D'|} \| = \epsilon} \tau d\mu_{\mathcal{X}/\mathcal{F}}$$

where $|D'|$ is the support of D', and if $x = 0$ is a local equation for the necessarily smooth (if not connected) divisor D' then τ has the form $\rho \frac{dx}{x^n}$, for ρ smooth, and $n \in \mathbb{N} \cup \{0\}$. Manifestly if we can calculate this for ρ compactly supported then a fortiori we're done, while the cases $n \leq 1$ follow by III.3.7. On the other hand everything is smooth, so we may even suppose that x, z_i are a local coordinate system with the foliation given by $\frac{\partial}{\partial x}$, so this follows immediately by integration by parts. $\Box$

Clearly with the hypothesis of IV.2.2 there's going to be no problem in factoring the calculation of $< c_1(L, \nabla), d\mu_{\mathcal{X}/\mathcal{F}} >$ through Res as a result of meromorphic poles around the non-invariant part, since no matter how high their order, we can smooth to a still higher order. In order to deal with the invariant terms we proceed rather differently, and independently of any foliation hypothesis, appeal to,

IV.2.3 Fact [Ko] *Let ϕ be a smooth $(n-1, n-1)$ form on a n dimensional polydisc with $x_1 \ldots x_k$ an equation of a simple normal crossing divisor, and τ a $(1,0)$ form such that for $p_i \in \mathbb{N} \cup \{0\}$, $1 \leq i \leq k$, $x_1^{p_1} \ldots x_k^{p_k} \tau$ is smooth, then for each $1 \leq i \leq k$ the limit,*

$$\lim_{\epsilon \to 0} \int_{\substack{|x_i| = \epsilon \, |x_j| \geq \epsilon \\ j \neq i}} \tau \wedge \phi$$

is well defined, and independent of the choice of coordinates.

Notice, critically, that the support of τ need not be compact, and in fact something slightly stronger is even true. Namely for each i there is a smooth function τ_i, independent of the coordinate defining the divisor $x_i = 0$ such that,

$$\lim_{\epsilon \to 0} \int_{\substack{|x_i| = \epsilon \, |x_j| \geq \epsilon \\ j \neq i}} \tau \wedge \phi = \lim_{\epsilon \to 0} \int_{\substack{|x_i| = \epsilon \, |x_j| \geq \epsilon \\ j \neq i}} \tau_i \bigwedge_{1 \leq j \neq i \leq k} \frac{dx_j d\bar{x}_j}{x_j^{p_j}} \bigwedge_{a > k} dx_a d\bar{x}_a$$

In particular if for a specific choice of coordinate system τ_i is zero almost everywhere, then all of these residues are zero. This has a manifest application to computing the pairing of any class α in $A_1^{n-1,n-1}$ with $c_1(L, \nabla)$ since as post IV.1.4 the pairing may be written in the form,

$$- \lim_{\epsilon \to 0} \int_{\| \mathbb{1}_{|\mathcal{D}|} \| = \epsilon} \tau \wedge \alpha$$

of course we should a priori insist that α vanishes to sufficiently high order on non-invariant components, and since τ satisfies precisely the hypothesis of IV.2.3, the calculation around the invariant components when performed in coordinates adapted to the foliation over the smooth locus of the same is zero by the above discussion, and the obvious type reason that a is $(n-1, n-1) \oplus (n, n-2)$, whence it's zero everywhere. Consequently, we deduce,

IV.2.4 Fact *Let $d\mu_{\mathcal{X}/\mathcal{F}}$ be a transverse invariant measure, and (L, ∇) a line bundle with leafwise meromorphic connection, then if we have log-canonical foliation singularities around the polar locus and $d\mu_{\mathcal{X}/\mathcal{F}}$ has zero segre class on any non-invariant poles, the pairing, $< c_1(L, \nabla), d\mu_{\mathcal{X}/\mathcal{F}} >$ depends only on $Res(d\mu_{\mathcal{X}/\mathcal{F}})$.*

Notice that this is and isn't the algebraic De-Rham theorem in the relative context $\pi : \mathcal{X} \to [\mathcal{X}/\mathcal{F}]$. It plainly isn't stricta dictum by virtue of the fact that we supposed zero segre class along non-invariant divisors. This is, however, trivially rectified since such divisors must be everywhere transverse to $\mathcal{F}$ when the singularities are log-canonical, so one can smooth the residual measure on the divisor, propagated along the foliation, and arrive to pure dependence on $Res(d\mu_{\mathcal{X}/\mathcal{F}})$ by IV.2.3. If one doesn't do this, as Kontsevich pointed out to me, the answer could be infinite in terms of integrals such as those post IV.1.4. On the other hand this is a slightly irrelevant nicety since the structure here is no more or less complicated than a punctured one dimensional disc. The real issue is that except for very special cases π, or more correctly $[\mathcal{X}/\mathcal{F}]$ has many definitions as a stack in analytic spaces around the singularities. Consequently the words 'invariant bundle' are ambiguous, depending on whether one asks for connections with values in $K_{\mathcal{F}}\mathcal{I}_{\mathcal{Z}}$ or just $K_{\mathcal{F}}$. Both possibilities have enough injectives, and both give a sensible notion of π_*, manifestations of which are the different spectral sequences of IV.1. The more important part, therefore, of IV.2.4, which functorially speaking cannot be considered a relative algebraic De-Rham theorem, is that this is the unique possible ambiguity.

IV.3 Isolated Residues

The purpose of this chapter is wholly exemplary. It's objects are two fold: to indicate the difficulties involved in calculating the residue symbol, and to avoid people bombarding me with e-mails claiming that they cannot understand the proof for surfaces in [M1]. Whence let $(\mathcal{X}, \mathcal{F})$ be a foliated smooth stack with not just canonical singularities but satisfying the embedded resolution property I.6.7, with I.3.1 in force. As such an isolated singularity where the residue symbol cannot be immediately related to the segre class $s_{\mathcal{E},d\mu}$ along the exceptional divisor has semi-simple part of rank $\dim\mathcal{X} - 1$ with, say, eigenfunctions $x_1, \ldots, x_{n-1}$. Consequently if y is a local equation for $\mathcal{E}$, which, by the way, is necessarily smooth at such a singularity, with ∂ a local generator for the foliation, then $\partial y = y^{p+1}u(y) \bmod(x_1, \ldots, x_{n-1})$, $p \in \mathbb{N}$ with u a unit. Plainly the extra dependence of ∂y on the x_i's is pretty irrelevant, e.g. we can just blow up till we get log-flat or $y^{p+1}|\partial y$. Whence we can normalise things so that,

$$\partial = y^{p+1}\frac{\partial}{\partial y} + \lambda_i x_i \frac{\partial}{\partial x_i} + \text{stuff}$$

where $\lambda_i \in \mathbb{C}^\times$, and stuff will prove to be on the irrelevant side. As ever this formula may not be analytically convergent, but a sufficiently good approximation to it modulo the maximal ideal of the point will be more than adequate. Indeed ignoring this subtlety for the moment consider what might be termed the *basic* *trick*, i.e. using $s_{\mathcal{E},d\mu}=0$ gives,

$$\int_{\substack{|y|=\epsilon\\ |x_i|\le\epsilon}} d^c\log|y|^2 d\mu_{\mathcal{X}/\mathcal{F}} = -\sum_i \int_{F_i} d^c\log|y|^2 d\mu_{\mathcal{X}/\mathcal{F}}$$
$$= -\sum_i \frac{1}{2\pi}\int_{F_i} \Im(y^p\frac{dx_i}{\partial x_i}) d\mu_{\mathcal{X}/\mathcal{F}}$$

where F_i is the face $|x_i| = \epsilon$, $|y| \le \epsilon$, $|x_j| \le \epsilon$, $i \ne j$, and what we require is to ensure that on F_i, $|\partial x_i|$ is bounded below by ϵ. The only obstruction to this is the first order terms in the Jordan decomposition, which can easily be killed by sequential weighted blowing up determined by Jordan blocks. Furthermore the conclusion only requires a very good approximation modulo the maximal ideal, and so we obtain,

$$0 \le \int_{|y|=\epsilon\,,\,|x_i|\le\epsilon} d^c\log|y|^2 d\mu_{\mathcal{X}/\mathcal{F}} \le \epsilon^p o(\epsilon)$$

where the $o(\epsilon)$ is an unknown function, necessarily going to zero with ϵ by virtue of the hypothesis $s_{\mathcal{E},d\mu}=0$. Consequently we deduce,

IV.3.1 Fact *Suppose $(\mathcal{X},\mathcal{F})$ is a foliated stack with the embedded resolution property and $z\in\mathrm{sing}(\mathcal{F})$ an isolated singularity then for $\lambda_z(d\mu_{\mathcal{X}/\mathcal{F}})$ just the standard Lelong number of a transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ we have,*

$$\lambda_z(d\mu_{\mathcal{X}/\mathcal{F}}) = 0 \Longrightarrow Res_z(d\mu_{\mathcal{X}/\mathcal{F}}) = 0$$

Notice that this argument does not easily generalise to non-isolated singularities, i.e.

IV.3.2 Difficulties The non-isolated problems are at least,

(a) Unlike standard residue theory one cannot just reduce from non-isolated to isolated by a generic cutting argument, since this would amount to finding invariant functions at each point of the singularity. Even ignoring convergence issues, this cannot always be done. Indeed in each dimension there are new 'non-cuttable' singularities proper to the dimension in the form of higher saddles, e.g. $x\frac{\partial}{\partial x} + y^p(z\frac{\partial}{\partial z} + y^{q+1}\frac{\partial}{\partial y})$, $p,q\in\mathbb{N}$ which don't even have this property formally, while untame beasts cannot posses it analytically. Whence,

(b) The basic trick is in general obstructed by the curvature of $\mathcal{E}$ around $\mathcal{Z}$.

(c) For surfaces, the unique non-trivial case of IV.3.1 occurs at saddle nodes. Here a better theorem is available, [Br1], *i.e.* there are no invariant measures around nodes except for the invariant curves through the singularities. The proof, by way of holonomy (and a considerable body of knowledge about nodes on surfaces, [H&], [MR]) reveals the basic trick as an upside down, and 1st order, version of the holonomy computation, albeit that such 'equivalence' no longer holds in dimension 3.

IV.4 The intervention of LCR

The fundamental difficulty in the discussion so far, excepting IV.3, is that it fails to make any distinction between Res as defined in IV.1.4 being computed along an invariant measure as opposed to an invariant distribution. In an ideal world this would have been resolved by III.6.3. Indeed, appealing to embedded LCR, I.6.7, and it's variants, cf. post VI.4.2, it is formally true that about every irreducible component Z of $\mathrm{sing}(\mathcal{F})$ there is a well defined weak branching formal sub-stack $\mathfrak{W}_Z$ of the completion in Z. In the situation where these actually converged in an analytic neighbourhood of Z (and to fix ideas we momentarily pretend that an irreducible component is also a connected one) to an analytic stack $\mathcal{W}$ we could appeal to III.6.3 to conclude that there was a well defined residual measure $s_{\mathcal{W},d\mu_{\mathcal{X}/\mathcal{F}}}$ for any transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$. Better still, apart from being supported on Z, this measure has the property that for any class $\alpha \in A^{0,2}_{1,c}$ there is a bounded function f such that,

$$Res(\mathbb{1}_{\mathcal{X}\setminus\mathcal{W}} d\mu_{\mathcal{X}/\mathcal{F}})(\alpha) = s_{\mathcal{W},d\mu_{\mathcal{X}/\mathcal{F}}}(f)$$

where the left hand side is the residue pairing IV.1.4. Plainly, at least as far as the part of the measure outside $\mathcal{W}$ is concerned, this substantially improves the situation and allows us to conclude (a postiori from the properness of Z) that the residue symbol is locally well defined. As such, we'll suppose from here to the end of this chapter that all foliated spaces $(\mathcal{X}, \mathcal{F})$ which appear have canonical singularities, and we will liberally apply the convention I.3.1.

Unfortunately, the hypothesis of III.6.3 are unrealistic even formally, as the example, I.5, of the 3-D beast shows. By way of embedded LCR this can be described rather well. For a given connected component Z of $\mathrm{sing}(\mathcal{F})$, there is a weak branching stack $\mathfrak{W}^1$ of minimal co-dimension which is equally the weak branching stack of a certain union Z^1 of irreducible components (strictly speaking there may actually be more than one such branch, and more than one such co-dimension, but the different such branches are sufficiently disjoint that for notational simplicity, we'll suppose that there's just the one). Just as per the 3-D beast, there is simply no reason to suppose that $\mathfrak{W}^1$ exists in a reasonable way except as a sub-stack of the completion in Z^1. By way of embedded LCR, we can suppose that the induced foliation in $\mathfrak{W}^1$ is in LCR form, with components of $\mathrm{sing}(\mathcal{F})$ contained in $\mathfrak{W}^1$ not only forming a simple normal crossing divisor, but with any non-invariant components for the induced foliation constituting a log-canonical boundary. In the same way there is a $\mathfrak{W}^2$ of the next smallest co-dimension (strictly speaking, we have exactly the same notational caveat as before), defined a priori around a certain number of irreducible components, Z'. It is, however, possible to formally extend $\mathfrak{W}^2$. Indeed where the components around which it is defined meet $\mathfrak{W}^1$, $\mathfrak{W}^2$ is, in their completion, naturally the weak branch of minimal dimension in $\mathfrak{W}^1$, consequently it extends around a sub-stack of the singular locus of $\mathfrak{W}^1$, which has exactly the same property as before, *i.e.* outwith the union of this sub-stack with Z', there is no reason to believe that $\mathfrak{W}^2$ can be defined. Consequently, we denote the sub-stack of $\mathrm{sing}(\mathcal{F})$ where we can reasonably expect $\mathfrak{W}^2$ to be defined by Z^2. Continuing in this way, we obtain what may be thought of as a decreasing filtration $\mathfrak{W}^p$, albeit that we can only talk about inclusions after completing in sub-schemes where everything is defined. Notice, that this description allows the possibility that a weak branch is an irreducible component of $\mathrm{sing}(\mathcal{F})$ where the generic linearisation of a generator of the foliation has full rank, what's important though is,

IV.4.1 Summary *Associated to a connected component of* $\mathrm{sing}(\mathcal{F})$ *in the presence of embedded LCR there is a natural tree structure on the weak branches* $\mathfrak{W}^p$ *which in turn come with a sub-scheme* Z^p *outwith the completion in which, it is unrealistic to expect, even formally, that* $\mathfrak{W}^p$ *can be defined.*

Consequently (and there are plenty of problems besides this) the minimal improvement that we must make on III.6.3 is to define a residual measure under the hypothesis that $\mathfrak{W}^p$ is analytically convergent on a neighbourhood of Z^p, and even here we have to remember that Z^p contains components of the singular locus of the induced foliation in $\mathfrak{W}^{p-1}$, and where these components are not all of the said singular locus, we still can't expect the thing to converge. Let us therefore note this by way of a sub-scheme $V^p \subset Z^p$, at which we find an intersection of Z^p with components of the singular locus of some $\mathfrak{W}^q$ (including the ambient space $\mathfrak{W}^0$) at which even further formal extension of $\mathfrak{W}^p$ is unrealistic.

Let us, therefore, return to the proof of III.6.3, and suppose that this were the only obstruction to analytic convergence. Irrespectively, the above tree structure on the $\mathfrak{W}^p$ lead to a decreasing filtration $F^p A^{0,2}_{1,c}$, and to classes w^p defined by the $\mathfrak{W}^p$, at least provided we enlarge our topology to allow the images of standard opens on local blow ups. With this in mid, if we had analytic convergence of the $\mathfrak{W}^p$ on a full neighbourhood of the Z^p, so supposing no problem at the aforesaid V^p's, the term in III.6.3 which is no longer under control is an integral of the form,

$$\int_{\|\mathbb{I}_{Z^p}\|=\delta} \rho w^p d\mu_{\mathcal{X}/\mathcal{F}}$$

for ρ with compact support around V_p, and δ fixed, *i.e.* in op. cit. this is the boundary integral involving the bounded form w, and δ does not go to zero. For the moment we're not able to undertake this calculation, but it is opportune to note,

IV.4.2 Remark *The strategy of* III.6.3 *doesn't quite work around the* V^p*, since although* Z^p *is proper, we'll have to 'cut' across any other singular components through* V^p *in order to define a residual measure around* $\mathfrak{W}^p$*.*

As such, what we wish to do is to put LCR in 'measure theory form', *i.e.* LCR gives us formal weak branches $\mathfrak{W}^p$, which we require to thicken up a little to residual measures. Now, as bad luck would have it, IV.4.2, or better careful attention around the V^p's, is a wholly realistic description of the formal obstructions that will be encountered, but is still insufficient for analytic convergence. The phenomenon is well known, already for surfaces, where convergence need never be better than 'sectorial'. In the above set up this arises as follows: having blown everything up as necessary, the singular locus of a $\mathfrak{W}^p$ is a simple normal crossing (infact log-canonical for the induced foliation) divisor in $\mathfrak{W}^{p+1}$, and even arises from restriction of a divisor on the ambient space/$\mathfrak{W}^0$ with the same smooth components, so, locally, say, coordinate functions $y_1, \dots, y_n$, which define sectors after a choice of argument functions $\theta_1, \dots, \theta_n \in [0, 2\pi)$, *i.e.* open sets of the form $\theta_j \in I_j$, for $I_j \subset [0, 2\pi)$, alternatively closed sectors allowing the inclusion of the origin may be employed. Consequently, the best way to think about this is in terms of convergence at germs around boundary points on a real blow up $\mathcal{X}_{\mathbb{R}} \to \mathcal{X}$ in what are locally the analytic hypersurface $y_i = 0$, which in turn arises from resolving ideals around $\mathrm{sing}(\mathcal{F})$. The type of convergence that we require is summarised by,

IV.4.3 Plausibility (LCR$^+$) *With notations and definitions as above, so in particular supposing* LCR, *then at boundary points on $\mathcal{X}_{\mathbb{R}}$ not lying over V^p there is a a sufficiently small neighbourhood S on which there are invariant analytic stacks W^p of the corresponding sector, C^∞ up to the boundary, having formal completion $\mathfrak{W}^p$. In particular, after analytic blowing up, there ought to be a not necessarily reduced Cartier divisor $\mathcal{E}$ containing a given connected component of the induced singular locus, in which we can make a real blow up to ensure convergence of the $\mathfrak{W}^p$'s in the above sense.*

proof/evidence The statement is true in dimension 2, and it's easy enough to conclude from what will be said in combination with I.3.11 that it holds in dimension 3. As such, the obvious stratagem is to proceed by an induction on the dimension in the spirit of I.6.7. In this respect it's a bit easier to think of a purely local decreasing filtration by weak branches $\mathfrak{W}^p$ in the completion of a closed point x. In particular, rather than suppose that we're away from the above problematic V^p's, we'll suppose that each $\mathfrak{W}^p$ is a well defined formal stack on the completion of an analytic neighbourhood of x in the singular locus of the induced foliation in $\mathfrak{W}^{p-1}$, which, needless to say is in LCR form. In dimension 3, and even 4, this changes nothing, but otherwise is a slightly stronger condition than the above hypothesis.

The model example of a formal to analytic convergence statement is the Morse lemma, so that the key thing to show, cf. [SG] VII.8.2 is that the convergence of the $\mathfrak{W}^p$ is an infinitesimally stable condition, *i.e.* supposing that the $\mathfrak{W}^p$ do (sectorially) converge then any 1st order analytic deformation has the same property. After which, one proceeds in the usual way, viewing the singularity in question as a deformation of a known convergent singularity (essentially it's formal normal form), and deduces convergence by integrating the vector field preserving the convergence condition implied by infinitesimal stability. That this latter step can be carried out over a closed, rather than simply open, interval follows from the fact that considered as formal power series in the above $y_1, \dots, y_n$ the functions defining the $\mathfrak{W}^p$'s enjoy coefficients which are never particularly big, *e.g.* the coefficient of $y_1^{a_1} \dots y_n^{a_n}$ is an analytic function bounded by a power of $(a_1)! \dots (a_n)!$.

Unfortunately, despite its name, infinitesimal stability is still an analytic condition. Indeed, if to fix ideas, we suppose that our weak branch $\mathfrak{W}^1$ of minimal co-dimension is a divisor, then we require to solve (sectorially) an equation of the form,

$$X - D(X) = b$$

where b is a given in one less variable, and D is a vector field, again, in one less variable. By induction, the form of D, can be improved, sectorially, to, $g\partial$, where the foliation induced by the field ∂ forms a LCR triple with the divisor defined by $g = 0$. There is, however, even here, a subtlety, since a weak branch need not be divisorial, which leads to issues involving resonances amongst eigenvalues, albeit that from a practical point of view IV.3 means we can eschew this on 3-folds.

Needless to say, such an equation is without formal obstructions, but it's already the case that sectorial convergence is the best that one can do in dimension 1, *e.g.*,

$$X(y) - u(y)y^p X'(y) = b(y)$$

$u(y)$ a unit, and $p \in \mathbb{N}$, need not admit an analytic solution X without restricting $\arg(y)$. In general, therefore, we'd like to integrate a vector field with embedded LCR singularities on neighbourhoods of infinity on the universal cover of the complement of $y_1 \dots y_n = 0$, then reduce to the 1-dimensional case on specialising to fibres. $\Box$

This discussion can also be usefully extended to the structure of weak branching stacks $\mathfrak{W}^p$ around the loci V^p where they may not even be formally defined. Indeed if we look to the 3-D beast, we can study it on a formal scheme of the form $\mathrm{Spf}A$, for A complete in $\mathrm{sing}(\mathcal{F})$. As such, with notations as per I.5.1, there are distinguished opens $\mathrm{Spf}A_{\{z\}}$, and $\mathrm{Spf}A_{\{y\}}$, on either of which we have local equations x_z, respectively x_y for the weak branch, yet these cannot be patched to a local equation on $\mathrm{Spf}A$. The reason for this is that an untame beast is formally obstructed at the level of Pic. The reasoning of IV.4.3 leads to a further clarification in this respect, *i.e.* if we make a small deformation of a tame, and analytically convergent beast, then seek a local equation for it on the $y \neq 0$ patch, we see that this can only be done on introducing a choice of $\arg(y)$, albeit that after such a choice the deformation remains convergent. This example is representative, and leads to a general paradigm on the convergence of higher beasts, *i.e.* the $\mathfrak{W}^p$'s around the V^p's. Specifically, we have (at least after local blowing up) a local Cartier divisor Y (*e.g.* this could be associated to weak branches of higher co-dimension, rather than just an exceptional divisor arising from blowing up) a local equation for which is an eigenfunction on some component of the induced singular locus in $\mathfrak{W}^{p-1}$ through V^p other than those in Z^p, and whose eigenvalue(s) λ go to zero at V^p. This determines an open $\lambda \neq 0$, with λ algebraic by the way, where the beast may be very wild, and even formal equations for it may well have essential singularities around $\lambda = 0$. The situation off Y, however, isn't that bad. Certainly, we cannot find local equations for the $\mathfrak{W}^p$ which are analytic everywhere, but we can expect to find such after a determination of the argument of local equations for V^p, and, better still, the example of the 3-D beast suggests that on different sectors these equations will agree up to a power (depending on the neighbourhood) of $\|\mathbb{I}_Y\|$. Consequently let us make,

IV.4.3 bis Plausibility and the Beast *Let everything be as above, and introduce the components Y^p of the singular locus of the induced foliation in $\mathfrak{W}^{p-1}$ through V^p other than Z^p then after a choice of argument for local equations defining Y^p and/or V^p, we can even find a local sectorial covering S_α off Y^p such that on S_α there is an analytic stack $\mathcal{W}^p$ whose completion in $Z^p \cap S_\alpha$ is $\mathfrak{W}^p$, with further patching of the defining functions $\|\mathbb{I}_{W^p, S_\alpha}\|$ up to a positive power of $\|\mathbb{I}_{Y^p}\|$ determined by the width of the neighbourhood around Z^p under consideration.*

proof/further evidence In the formal completion at V^p, beasts have a deceptively simple form, and again the important thing to consider is the infinitesimal stability of this form. This is easy enough in dimension 3, but, the general difficulties are exactly as per IV.4.3. Furthermore, while the convergence off V^p discussed above should hold after real blowing up in a single, not necessarily reduced, Cartier divisor (corresponding to sectors determined by Y^p) that for beasts may necessitate further real blowing, or equivalently, a choice of argument for the degenerating eigenfunction(s) λ. The differentials of these, are, however of small modulus for any transverse invariant measure, so from a computational point of view this isn't terribly important. □

Having thus failed to prove either of these statements, let us take comfort in the evidence provided by dimensions 2, and 3, and investigate whether we can still define residual measures around weak branches even if the convergence were no better than sectorial. For the moment, and bearing in mind that the precise fashion for addressing the formal obstructions implicit in IV.2 is postponed till later, let us also keep the complication down by supposing that we have sectorial convergence of all our weak branches $\mathfrak{W}^p$ on a connected component Z of $\mathrm{sing}(\mathcal{F})$, so that we can focus on just one of them. As such, let, $\coprod S_\alpha \to \mathcal{X}_{\mathbb{R}}$ be an open cover of the real blow up such that locally our given formal weak branch $\mathfrak{W}$ may be identified with the completion of an honest analytic stack W_α which is C^∞ up to the boundary. It is, however, far from necessarily true that the W_α patch. Indeed, to fix ideas, suppose that W_α were a divisor, then on S_α we get a local equation x_α. Certainly, therefore, we can put $x_\alpha = g_{\alpha\beta} x_\beta$ for some transition functions $g_{\alpha\beta}$, but these functions may very well fail to be bounded on $\mathcal{X}_{\mathbb{R}}$, not just on the boundary, but even on the interior. Nevertheless, what is true on overlaps is that,

$$\frac{dx_\alpha}{x_\alpha} - \frac{dx_\beta}{x_b} = f_{\alpha\beta}\partial^\vee + \omega_{\alpha\beta}$$

where $\partial^\vee$ is dual to a local generator for the foliation, $\omega_{\alpha\beta}$ is the image of a bounded 1-form in $K_{\mathcal{F}}$, and $f_{\alpha\beta}$ is asymptotically flat, *i.e.* for any $k \in \mathbb{N}$, there is a sufficiently small neighbourhood of the boundary such that $|f_{\alpha\beta}| \ll \|\mathbb{1}_{\mathcal{E}}\|^k$. More generally, for any weak branch (necessarily formally smooth given embedded LCR, and a lack of formal obstructions) defined by local equations $x_{\alpha 1}, \ldots, x_{\alpha n}$, we get the same phenomenon, *i.e.* if $\log \|\mathbb{1}_{W_\alpha}\|$ is a local potential of the usual form such as $\log(|x_{\alpha 1}|^2 + \ldots + |x_{\alpha n}|^2)$, or $\log\max\{|x_{i\alpha}|\}$, or whatever, then,

$$d^c \log \|\mathbb{1}_{W_\alpha}\| - d^c \log \|\mathbb{1}_{W_\beta}\| = f_{\alpha\beta}\partial^\vee + \omega_{\alpha\beta} \in K_{\mathcal{F}}$$

for $f_{\alpha\beta}$, $\omega_{\alpha\beta}$, $\partial^\vee$ as above. If, therefore, we take a sufficiently random 'cover' (*i.e.* exactly as per the discussion of Stoke's theorem in III.1) S_α of $\mathcal{X}_{\mathbb{R}}$ to avoid creating any more problems than we already have, then we can try to piece together the measures defined on each S_α by way of,

$$\sigma^{\epsilon_\alpha}_{W,d\mu} := \int_{\|\mathbb{1}_{W_\alpha}\| = \epsilon_\alpha} \frac{d^c \|\mathbb{1}_{W_\alpha}\|}{\|\mathbb{1}_{W_\alpha}\|} d\mu_{\mathcal{X}/\mathcal{F}}$$

We now proceed exactly as in III.6.3, and note that in S_α, for $\delta_\alpha < \epsilon_\alpha$,

$$\sigma^{\epsilon_\alpha}_{W,d\mu} - \sigma^{\delta_\alpha}_{W,d\mu} = \int_{\delta_\alpha \le \|\mathbb{1}_{W_\alpha}\| \le \epsilon_\alpha} dd^c \log \|\mathbb{1}_{W_\alpha}\|^2 d\mu_{\mathcal{X}/\mathcal{F}} - \int_{\partial S_\alpha, \delta_\alpha \le \|\mathbb{1}_{W_\alpha}\| \le \epsilon_\alpha} d^c \log \|\mathbb{1}_{W_\alpha}\| d\mu_{\mathcal{X}/\mathcal{F}}$$

We divide the ultimate integral, according to what might be termed an exterior boundary: $\partial\mathcal{X} \cap S_\alpha$, which we can certainly handle as per the proof of op. cit., with the genuinely new term being that which occurs on the rest of the boundary, and in particular that which has the form $\underline{\theta}$ =constant, for $\underline{\theta}$ a choice of arguments

for a vector of holomorphic functions. Our partition of $\mathcal{X}_{\mathbb{R}}$ is such, however, that this interior boundary also occurs, with opposite sign, as the interior boundary of some unique ∂V_β or part thereof, or the interior boundary of $\mathcal{X}_{\mathbb{R}}$ itself. Given that every weak branch contains an appropriate component of $\mathrm{sing}(\mathcal{F})$, everything can be supposed to vanish on the latter, while the former cancel on summing over α, and letting the ϵ_α go to zero thanks to the aforesaid asymptotic flatness. Consequently,
IV.4.4 Fact *Suppose in the presence of* LCR$^+$ *a weak branch* $\mathfrak{W}$ *is well defined as a formal substack of a connected component of* $\mathrm{sing}(\mathcal{F})$*, then for any transverse invariant measure* $d\mu_{\mathcal{X}/\mathcal{F}}$ *there is a well defined (although bear in mind* III.4) *residual measure* $s_{W,d\mu}$*. If, moreover,* $\mathfrak{W}$ *is the totality of all weak branches containing the component, then provided* $1\!\!\mathrm{I}_{W_\alpha} d\mu_{\mathcal{X}/\mathcal{F}} = 0$ *for* W_α *analytic substacks of finitely many sectors* S_α*, then the said measure computes any residue, i.e. for* $a \in A^{0,2}_{1,c}$ *there is a bounded function* f *such that,*

$$Res(d\mu_{\mathcal{X}/\mathcal{F}})(a) = s_{W,d\mu_{\mathcal{X}/\mathcal{F}}}(f)$$

This rather useful localisation of the calculation of residues not withstanding,, convergence by sector, even everywhere, lacks an important corollary of actual convergence, *viz:* when we have actual convergence, there is a function of the form $\log \|1\!\!\mathrm{I}_{\mathbf{W}}\|$ defined on a neighbourhood of a connected component of $\mathrm{sing}(\mathcal{F})$, such that for ρ with compact support in the said neighbourhood,

$$dd^c(\log \|1\!\!\mathrm{I}_{\mathbf{W}}\| d\mu_{\mathcal{X}/\mathcal{F}})(\rho) = s_{W,d\mu_{\mathcal{X}/\mathcal{F}}}(\rho) - \int \rho c_1(\mathfrak{W}) d\mu_{\mathcal{X}/\mathcal{F}}$$

For $c_1(\mathfrak{W})$ a chern type class associated to $\mathfrak{W}$. Plainly, such a potential could easily exist even when the situation is obstructed, *e.g.* it does after all suffice that things should converge $d\mu$ a,e, When, of course such a potential exists, we're effectively in the situation of III.6.3, and everything is much easier. Whence, let us make,
Definition IV.4.5 *Suppose the residual measure of a weak branching stack* $\mathfrak{W}$ *is well defined around a connected component of* $\mathrm{sing}(\mathcal{F})$*, then we say that* $\log \|1\!\!\mathrm{I}_{\mathbf{W}}\|$ *is a potential for* $s_{W,d\mu}$ *if for* ρ *with compact support, the above relation holds with* $c_1(\mathfrak{W})$ *absolutely integrable against* $d\mu$*.*

The same usage, will also be employed if for $\mathfrak{W}^p$ the tree of weak branches associated to a connected component of $\mathrm{sing}(\mathcal{F})$ in the presence of embedded LCR, $\log \|1\!\!\mathrm{I}_{W^p}\|$ is well defined in a neighbourhood of Z^p, albeit that it is implicitly supposed that this leads to a well defined residual measure around the V^p.

IV.5 Harmonic theory on foliated varieties

We continue to let $(\mathcal{X}, \mathcal{F})$ be a foliated smooth (Kä̃hler) stack, but we fix a (supposed existent) transverse invariant measure $d\mu$. We could, of course, avoid this and proceed à la Connes, [Co], to do things independent of a particular realisation. Nevertheless we have specific measures with specific properties in mind so we'll proceed somewhat more simplistically.

As ever one builds up from $C^\infty(\mathcal{X})$. Nevertheless distributions and so forth should all be understood relative to $d\mu$. In particular we're only interested in the initial terms derived from the filtration of IV.1 on smooth forms, i.e.

$$\mathrm{gr}^{i,\bar{j}} = \frac{F^{i,\bar{j}} A^*_{\mathcal{X}}}{F^{i,\bar{j}-1} A^*_{\mathcal{X}} + F^{i-1,\bar{j}} A^*_{\mathcal{X}}} \quad i,j \in \{0,1\}$$

where $A^*_{\mathcal{X}}$ is global smooth forms on $\mathcal{X}$. Consequently when we talk about a distribution $\mathrm{dr}^{i,\bar{j}}$ this will mean an element of $\mathrm{gr}^{1-i,1-\bar{j}}$ calculated along $d\mu$, i.e.

IV.5.1 Definition *A distribution of type $(i,\bar{j})$ of regularity class (p,q) is an element of the dual of $gr^{1-i,1-\bar{j}}$ in the sobolev (p,q) norm along $d\mu$, i.e. the qth derivatives are $\omega d\mu$ ℓ_p, ω Kăhler. In general all statements about sobolev spaces, ℓ_1 spaces, etc. are to be understood in this sense.*

The restriction of the standard differential operators gives, therefore, a Hodge diamond,

$$\begin{array}{ccccc} & & \mathrm{gr}^{0,\bar{0}} & & \\ & \partial \swarrow & & \searrow \bar{\partial} & \\ \mathrm{gr}^{1,0} & & & & \mathrm{gr}^{0,\bar{1}} \\ & \bar{\partial} \searrow & & \swarrow \partial & \\ & & \mathrm{gr}^{1,\bar{1}} & & \end{array}$$

so that if we think of $\mathrm{gr}^{i,\bar{j}}$ as embedded in smooth sections of $K^i_{\mathcal{F}} \otimes \bar{K_{\mathcal{F}}}^j$, with $\partial_{\mathcal{F}}$ a local generator of the foliation these can be written as $\partial_{\mathcal{F}} \otimes \partial^\vee_{\mathcal{F}}$ etc. Now the $*$ operator from $\mathrm{gr}^{1,0}$ to $\mathrm{gr}^{0,\bar{1}}$ is plainly just complex conjugation. Whence if H is the class of a Kăhler metric, and we desire the standard formula,

$$\int \omega \wedge *\tau d\mu = \int <\omega,\tau> H d\mu$$

we see that the right choice of metric on $K_{\mathcal{F}}$ is a norm of the form smooth$\times 1/\|\mathrm{I\!I}_{\mathcal{Z}}\|$ where $\mathcal{Z}$ is the singular substack with nilpotent structure, and $\|\mathrm{I\!I}_{\mathcal{Z}}\|$ is as per III.3.1. As a particular consequence, in terms of the local generator $\partial_{\mathcal{F}}$ the Laplacian on $\mathrm{gr}^{0,0}$ has the form,

$$\Delta f = \frac{1}{\|\mathrm{I\!I}_{\mathcal{Z}}\|^2} \partial_{\mathcal{F}} \bar{\partial}_{\mathcal{F}}(f)$$

and not as one might suspect $\partial_{\mathcal{F}}\bar{\partial}_{\mathcal{F}}(f)$. More generally, on identifying an element of $\mathrm{gr}^{i,\bar{j}}$ with its image in $K^i_{\mathcal{F}} \otimes \bar{K_{\mathcal{F}}}^j$ the other Laplacians look like,

$$\Delta^{1,0} f = \partial_{\mathcal{F}}(\frac{1}{\|\mathrm{I\!I}_{\mathcal{Z}}\|^2} \bar{\partial}_{\mathcal{F}}(f)) \ , \ \ \Delta^{1,1} f = \partial_{\mathcal{F}} \bar{\partial}_{\mathcal{F}}(\frac{1}{\|\mathrm{I\!I}_{\mathcal{Z}}\|^2} f)$$

where we've been deliberately vague about which Laplacian we're using, since, infact, the Hodge-Riemann bilinear relations continue to hold, so they're all the same.

Certain things are, therefore, pure formalism. For example for $s \in \mathbb{N}$ we have sobolev spaces,

$$\mathrm{gr}^{i,\bar{j}}_{2,2s} := \{f : \|\Delta^p f\|^2 < \infty \, , \, 0 \le p \le s\}$$

and their duals $\mathrm{gr}^{i,\bar{j}}_{2,-s}$ with an inverse limit $\mathrm{gr}^{i,\bar{j}}_{2,\infty}$ and a direct limit $\mathrm{gr}^{i,\bar{j}}_{2,-\infty}$. Thus, essentially by definition, if $\Delta f = g$ for $g \in \mathrm{gr}^{*}_{2,s}$, then $f \in \mathrm{gr}^{*}_{2,s+2}$. What, however, is far from true is that the inclusions,

$$\mathrm{gr}^{*}_{2,s} \longrightarrow \mathrm{gr}^{*}_{2,t}$$

for $s > t$ are compact. Indeed, if the foliation is a fibration, then such a map is no more or less compact than the identity on a countable Hilbert space, which is evident nonsense. On the other hand, for a fibration the transverse invariant measures which are reduced to a point in the sense of [II] A.3.9 are simply the pull-backs of δ-functions on the base, and for such extremal measures the Sobolev inclusions are just the standard ones, whence compact. Unfortunately, for arbitrary foliations the extremal measures are not as well behaved. They are , however, much better behaved than other measures, so let's clarify things by way of,

IV.5.2 Definition *We will say that a transverse invariant measure $d\mu$ is reduced to a point if (bearing in mind* [II] A.3.19) *some (whence any) faithful $\underline{E}QF$ slicing of the foliation groupoid, op. cit.* A.1.6, *gives rise to an invariant measure which is reduced to a point in the sense of op. cit.* A.3.9.

Equivalently for $\mathcal{X}$ not necessarily compact, but $d\mu$ Radon, this amounts to the sub-ring of $\mathcal{L}_\infty(\omega d\mu)$, ω positive $(1,1)$, of invariant functions being simply the constants, whence if $\mathcal{X}$ is compact, the $\mathcal{L}_\infty(\omega d\mu)$ harmonic functions being reduced to the constants. More generally, there is a very weak 'regularity' statement,

IV.5.3 Fact *Suppose $\mathcal{X}$ is compact, ω is Kắhler and $d\mu$ is reduced to a point, then for any $1 \leq p \leq \infty$, the $\mathcal{L}_p(\omega d\mu)$ harmonic functions are the constants*

proof It plainly suffices to do the case $p = 1$, given the $p = \infty$ case. On the other hand for $\mathcal{X}$ compact, and f real harmonic ℓ_1,

$$\max\{f, N\},\ \min\{f, -N\},\ N \in \mathbb{R}_+$$

are sub, respectively super, harmonic, whence harmonic ℓ_1 by way of integration by parts. Better still, we deduce that we can write f as a difference $f_+ - f_-$, with not only f_+, f_- non-negative harmonics, but, infact, limits of bounded harmonic functions, so, indeed, both f_+ and f_- must be constants. □

Needless to say, this allows us to easily identify the Harmonic projector,

$$H_\omega : \mathcal{L}_p(\omega d\mu) \to \mathbb{C} : f \mapsto \int_{\mathcal{X}} f \omega d\mu$$

and, whence, by a wholly standard exercise in adjunction,

IV.5.4 Fact *Suppose $\mathcal{X}$ is compact, and $d\mu$ is reduced to a point, then for any $1 \leq p < \infty$, there is a pseudo-Hodge decomposition,*

$$\mathcal{L}_p(\omega d\mu) = \mathbb{C} \oplus \overline{\Delta(\mathcal{L}_{p,2}(\omega d\mu)}$$

where the further subscript $p, 2$ is as per IV.5.1.

Such considerations for $p = 2$, and indeed any k, are, without even supposing that $d\mu$ is reduced to a point verging on the blindingly obvious. To imagine, however, that what's at stake is simply some ℓ_p version of projecting in Hilbert spaces would be to miss the point, *i.e.*

IV.5.5 Remark What's actually at stake, and what underlies the weak regularity statement IV.5.3 is that for $\mathcal{X}$ compact and/or $p \geq 1$ we can conclude information about the derivative (relative to $d\mu$) of a Borel measurable function starting from information about the Laplacian. As such, it is not clear, and it may very well be false that IV.5.4 can be extended to $p = \infty$. Specifically, IV.5.3 may very well be false if $\mathcal{L}_1(\omega d\mu)$ is replaced by $\mathcal{L}_\infty(\omega d\mu)^\vee$. Of course, IV.5.4 is only a priori a weak Hodge decomposition, but morally the arguments of the next chapter suggest that for $p < \infty$ it can be extended to a genuine Hodge decomposition, i.e., the Laplacian has closed range whenever $d\mu$ is reduced to a point, and that more generally this could be achieved for any function space whose dual satisfies IV.5.3. Strong Hodge-decomposition for $p = \infty$ is, however, in general false even for $d\mu$ reduced to a point, e.g. take a general Kronecker foliation on an abelian surface in it's unique invariant measure, and blow up in a point.

IV.6 Logs and Points

The root of the difficulty presented by IV.5.5 is that it's simply unclear, and as we've said quite possibly false, that harmonic measures $\gamma \in \mathcal{L}_\infty(\omega d\mu)^\vee$ are closed. Indeed, to be precise, these conditions are defined by an adjunction, and so are more correctly written as $\Delta^*\gamma = 0$, respectively, $d^*\gamma = 0$. The latter is, however, rather strong, since supposing that we had a priori blown up in some singularities, the foliation is given by an actual smooth groupoid, $\tilde{\mathcal{F}}_{\mathrm{hol}} \rightrightarrows \tilde{\mathcal{X}}$, [II] A.1.9, if not smooth equivalence relation, so that $d^*\gamma = 0$ implies a factorisation of the form,

$$\gamma(f) = \gamma_0(\int_{\mathcal{F}} f\omega), \ \ f \in \mathcal{L}_\infty(\omega d\mu)$$

where γ_0 is an invariant measure, and the relative integral is as per op. cit. A.3.17. As such,

IV.6.1 Fact *Let $(\mathcal{X}, \mathcal{F})$ be a not necessarily compact analytic space foliated by curves, with $d\mu$ an invariant Radon-measure reduced to a point, and ω a positive $(1,1)$ form absolutely integrable against $d\mu$, then any functional $\gamma \in \mathcal{L}_\infty(\omega d\mu)^\vee$ satisfying $d^*\gamma = 0$ is a constant.*

proof/precisions In effect, if $\mathcal{X}$ were compact, and ω Kähler, then the condition $d^*\gamma = 0$ can be re-written on identifying γ with $\lim_{n\to\infty} f_n$ for $f_n \in \mathcal{L}_1(\omega d\mu)$ as, $df_n \to 0$, *i.e.*

$$\int_{\mathcal{X}} f_n d\alpha d\mu \to 0 \ ,$$

for any bounded 1 form α with compact support, and this continues to have perfect sense irrespective of compactness and/or Kählerianity. On the other hand, the smoothness of the groupoid, $(s,t) : \tilde{\mathcal{F}}_{\mathrm{hol}} \rightrightarrows \tilde{\mathcal{X}}$ gives after slicing by a transversal H, $\lim_n s^* f_n|_H = \lim_n t^* f_n|_H$. As such γ arises from a functional γ_0 on $\Gamma(H, \mathcal{L}_\infty(\mu))$ which factors through ϵ_* as defined in [II] A.3.6. By hypothesis, however, there are no such functionals except μ itself, and multiples there of. $\Box$

This leads to a quite general argument whereby we can pass from regularity of the derivative of a function to regularity for the function itself. To begin with, let's do,

IV.6.2 Claim *Let $(\mathcal{X}, \mathcal{F})$ be, to simplify the discussion, a proper foliated stack with ω Kähler, with $d\mu$ a transverse invariant measure reduced to a point, then*

for $\mathcal{E} \hookrightarrow \mathcal{X}$ a divisor such that $\mathbb{I}_{\mathcal{E}} d\mu = 0$ and $d\mu$ has zero segre class about $\mathcal{E}$, $\log \|\mathbb{I}_{\mathcal{E}}\|$ *is ℓ_1 with respect to $\omega d\mu$.*
proof Without loss of generality $\|\mathbb{I}_{\mathcal{E}}\| \leq 1$, and we consider the regularisations,

$$\ell_\epsilon := -\log \max\{\|\mathbb{I}_{\mathcal{E}}\|, \epsilon\} \geq 0\,.$$

Should the assertion be false, then,

$$\|\ell_\epsilon\| := \int_{\mathcal{X}} \ell_\epsilon \omega d\mu \to \infty\,, \ \text{ as } \ \epsilon \to 0\,.$$

On the other hand, after subsequencing, we have a measure $\gamma \in \mathcal{L}_\infty(\omega d\mu)^\vee$ defined by,

$$\gamma := \lim_{\epsilon \to 0} \frac{\ell_\epsilon}{\|\ell_\epsilon\|}$$

which in turn may be integrated against $c_1(\mathcal{E})$, so that by Stokes,

$$0 \leq \lim_{\epsilon \to 0} \frac{1}{\|\ell_\epsilon\|} \int_{\|\mathbb{I}_{\mathcal{E}}\| \geq \epsilon} \frac{d\|\mathbb{I}_{\mathcal{E}}\| d^c \|\mathbb{I}_{\mathcal{E}}\|}{\|\mathbb{I}_{\mathcal{E}}\|^2} d\mu < \infty$$

An immediate application of Cauchy-Schwartz, however, gives for any test form α,

$$| < d\gamma, \alpha > | \leq \lim_{\epsilon \to 0} \frac{1}{\|\ell_\epsilon\|} \left(\int_{\|\mathbb{I}_{\mathcal{E}}\| \geq \epsilon} \frac{d\|\mathbb{I}_{\mathcal{E}}\| d^c \|\mathbb{I}_{\mathcal{E}}\|}{\|\mathbb{I}_{\mathcal{E}}\|^2} d\mu \right)^{\frac{1}{2}} \left(\int_{\mathcal{X}} \alpha \wedge \bar{\alpha} d\mu \right)^{\frac{1}{2}}$$

So, indeed, γ is closed. Whence, by IV.6.1, $\frac{\ell_\epsilon}{\|\ell_\epsilon\|}$ converges, weakly, to a constant in $\mathcal{L}_\infty(\omega d\mu)^\vee$, infact 1 since the relative integral of ω is bounded on transversals, which is absurd for $\mathbb{I}_{\mathcal{E}} d\mu = 0$. $\square$

We can push this a good bit further, *i.e.*
IV.6.3 Better Claim *Let everything be as in* IV.6.2 *then there is a $\delta > 0$ such that* $\|\mathbb{I}_{\mathcal{E}}\|^{-\delta}$ *is absolutely integrable with respect to $\omega d\mu$.*
proof Fix $t > 0$, and for $\epsilon > 0$ proceed as before by way of the regularised max,

$$L_{t,\epsilon} := (\max\{\|\mathbb{I}_{\mathcal{E}}\|, \epsilon\})^{-t}$$

with $\Lambda_{t,\epsilon}$ it's ℓ_1 norm. Irrespectively of whether this goes to ∞ as $\epsilon \to 0$, although obviously we may suppose that it does for every $t > 0$, we can subsequence in ϵ to obtain elements $\gamma_t \in \mathcal{L}_\infty(\omega d\mu)^\vee$ defined by,

$$\gamma_t := \lim_{\epsilon \to 0} \frac{L_{t,\epsilon}}{\Lambda_{t,\epsilon}}$$

Equally, for a test form α, we can estimate the derivative by way of,

$$| < d\gamma_t, \alpha > | \leq \lim_{\epsilon \to 0} \frac{t}{\Lambda_{t,\epsilon}} \left(\int_{\|\mathbb{I}_{\mathcal{E}}\| \geq \epsilon} \frac{d\|\mathbb{I}_{\mathcal{E}}\| d^c \|\mathbb{I}_{\mathcal{E}}\|}{\|\mathbb{I}_{\mathcal{E}}\|^{2+t}} d\mu \right)^{\frac{1}{2}} \left(\int_{\mathcal{X}} \frac{\alpha \wedge \bar{\alpha}}{\|\mathbb{I}_{\mathcal{E}}\|^t} d\mu \right)^{\frac{1}{2}}$$

Once again, Stokes gives that the first of these integrals is bounded, up to an irrelevant constant, by $\frac{\Lambda_{t,\epsilon}}{t}$, so,

$$\|d\gamma_t\| \ll t^{\frac{1}{2}}$$

By construction, though, the γ_t are not only bounded in $\mathcal{L}_\infty(\omega d\mu)^\vee$, but satisfy $\gamma_t(1) = 1$, so that letting $t \to 0$, and subsequencing leads to the same contradiction as before. □

In the course of the discussion we have also established,

IV.6.4 Corollary *Again let things be as in* IV.6.3 *then for* $\delta > 0$ *as per* IV.6.4 $\frac{d\|\mathbb{I}_{\mathcal{E}}\|d^c\|\mathbb{I}_{\mathcal{E}}\|}{\|\mathbb{I}_{\mathcal{E}}\|^{2+\delta}}$ *is absolutely integrable with respect to* $d\mu$.

The evident candidate for divisors to which we might like to apply this discussion are those which occur in the situation where $(\mathcal{X}, \mathcal{F})$ is a blow up in the singular locus of a foliation, and $\mathcal{E}$ is the total exceptional divisor. Indeed, given LCR this is actually rather more useful than it may seem, *e.g.*

IV.6.5 Fact (supposing LCR) *Let everything be as in* IV.6.3 *with* $\mathcal{E}$ *a divisor containing* $\operatorname{sing}(\mathcal{F})$*, and for ease of exposition suppose further that the segre class of* $d\mu$ *around every proper co-dimension 2 subvariety is zero, then for any non-invariant proper divisor* A*,* $\log\|\mathbb{I}_A\|$ *is* ℓ_1 *with respect to* $\omega d\mu$.

proof Denote by $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ a LCR resolution of singularities, then $\rho^* A = \tilde{A} + \mathcal{E}$, for $\tilde{A}$ everywhere transverse to the induced foliation, and a most likely different, albeit in no essential way, exceptional divisor $\mathcal{E}$. With the given hypothesis, by IV.6.3, $\log\|\mathbb{I}_{\mathcal{E}}\|$ is ℓ_1 with respect to $\tilde{\omega} d\mu$ for $\tilde{\omega}$ Kăhler on the blow up. Whence, it suffices to prove that the same is true of $\log\|\mathbb{I}_{\tilde{A}}\|$. The foliation though is smooth around $\tilde{A}$, so by III.3.4 $\log\|\mathbb{I}_{\tilde{A}}\|$ has ℓ_1-derivatives. Plainly, $d\mu$, or more correctly it's extension by zero across the exceptional divisor, cannot be supported in the wholly transverse divisor $\tilde{A}$, so we quickly proceed to a contradiction: $\log\|\mathbb{I}_{\tilde{A}}\|$ not ℓ_1 implies there are ℓ_1 functions f_n bounded in $\mathcal{L}_\infty(\omega d\mu)^\vee$ tending to a measure supported on $\tilde{A}$, with $f_n(1) = 1$, and $df_n \to 0$, so $f_n \to 1$, which is rubbish. □

In turn this proposition can be used to address the integrability of the potential functions, $\log\|\mathbb{I}_{\mathbf{W}}\|$, associated to weak branching subschemes $\mathfrak{W}$, IV.4.5, *i.e.*

IV.6.6 Fact *Let everything be as in* IV.6.3/5 *with* $\log\|\mathbb{I}_{\mathbf{W}}\|$ *a weak branching potential in the sense of* IV.4.5, *then for* $\mathcal{X}$ *enjoying projective moduli,* $\log\|\mathbb{I}_{\mathbf{W}}\|$ *is* ℓ_1 *with respect to* $\omega\mathbb{I}_{\mathcal{X}\setminus\mathbf{W}}d\mu$*, or better for* $d\mu$ *without support in* $\mathfrak{W}$.

proof A priori we wish to consider for $\epsilon > 0$ integrals of the form,

$$\int_U |\log\max\{\|\mathbb{I}_{\mathbf{W}}\|, \epsilon\}| c_1(\bar{A}) d\mu$$

where U is an open neighbourhood of a component of $\operatorname{sing}(\mathcal{F})$ on which the potential is defined, and $\bar{A}$ is an ample divisor metricised in such a way that the chern class is the Kăhler form. Depending on A, but certainly independent of ϵ, we can again take a resolution of singularities, $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$, inducing $\rho : \tilde{U} \to U$, such that $\rho^* A = \tilde{A} + E$, where the proper transform $\tilde{A}$ of A is smooth and everywhere transverse to the induced foliation. Better still, $\log\|\mathbb{I}_{\mathbf{W}}\| = \log\|\mathbb{I}_{\tilde{W}}\| + \log\|\mathbb{I}_F\|$, for F some subscheme supported on the exceptional divisor. Therefore,

(a) We need only prove that $\log\|\mathbb{I}_{\tilde{W}}\|$ is ℓ_1 on $\tilde{U}$.

(b) For next to any metricisation of $\tilde{A}$, *e.g.* smooth,

$$\lim_{\epsilon\to 0}\int_{\tilde{U}} |\log\max\{\|\mathbb{I}_{\tilde{W}}\|, \epsilon\}| c_1(\tilde{A}) d\mu$$

exists. Indeed (a) is clear, as is the essential of (b) since $\tilde{W} \cap \tilde{A} = \emptyset$, but there is a slight subtlety on the boundary, which we'll subordinate to a slightly different discussion. Specifically, in order to avoid worrying about what $c_1(\tilde{A})$ looks like in a neighbourhood of $\partial\tilde{U}$ close to $\tilde{W}$, let us adopt the expedient of extending the potential function $\log\|\mathbb{I}_{\tilde{W}}\|$ to a neighbourhood of $\rho^* A$. Since, the said function is wholly bounded close to $\tilde{A}$ this costs nothing, as does supposing that our potential takes negative values. Furthermore, we'll take A to contain $\mathrm{sing}(\mathcal{F})$, so we can suppose that we're working on a neighbourhood of the form $\rho^*\|\mathbb{I}_A\| < t$, for some $t > 0$, and we proceed via integration by parts, *i.e.*

$$\begin{aligned}
&\frac{1}{2}\int_{\|\mathbb{I}_A\|<t} |\log\max\{\|\mathbb{I}_{\tilde{W}}\|,\epsilon\}| c_1(\bar{A})d\mu \\
&= \int_{\|\mathbb{I}_A\|=t} \log\max\{\|\mathbb{I}_{\tilde{W}}\|,\epsilon\}\frac{d^c\|\mathbb{I}_A\|}{\|\mathbb{I}_A\|}d\mu - \int_{\|\mathbb{I}_{\tilde{W}}\|\geq\epsilon} \frac{d\|\mathbb{I}_A\|}{\|\mathbb{I}_A\|}\frac{d^c\|\mathbb{I}_{\tilde{W}}\|}{\|\mathbb{I}_{\tilde{W}}\|}d\mu \\
&- \lim_{\delta\to 0}\int_{\|\mathbb{I}_A\|=\delta} \log\max\{\|\mathbb{I}_{\tilde{W}}\|,\epsilon\}\frac{d^c\|\mathbb{I}_A\|}{\|\mathbb{I}_A\|}d\mu \\
&= \int_{\|\mathbb{I}_A\|=t} \log\max\{\|\mathbb{I}_{\tilde{W}}\|,\epsilon\}\frac{d^c\|\mathbb{I}_A\|}{\|\mathbb{I}_A\|}d\mu - \int_{\|\mathbb{I}_A\|=t} \log\|\mathbb{I}_A\|\frac{d^c\|\mathbb{I}_{\tilde{W}}\|}{\|\mathbb{I}_{\tilde{W}}\|}d\mu \\
&+ \int_{\|\mathbb{I}_{\tilde{W}}\|=\epsilon} \log\|\mathbb{I}_A\|\frac{d^c\|\mathbb{I}_{\tilde{W}}\|}{\|\mathbb{I}_{\tilde{W}}\|}d\mu - \int_{\|\mathbb{I}_{\tilde{W}}\|<\epsilon} \log\|\mathbb{I}_A\| c_1(\tilde{W})d\mu + O(1) \\
&\leq O(1)
\end{aligned}$$

where the first intervention of $O(1)$ occurs as $\delta \to 0$ by way of the disjointness of $\tilde{A}$ and $\tilde{W}$. The second has two aspects: firstly by way of the chern type class $c_1(\tilde{W})$ as found in the proof of III.6.3. This class is certainly smooth if W is a divisor, and in general, as per III.3.2, it's negative is bounded below by a smooth, so it's overall contribution is bounded above IV.6.5. Finally, there's an ultimate $O(1)$ coming from the boundedness of $d^c\log\|\mathbb{I}_{\tilde{W}}\|$ on the boundary as in III.6.3, while by accident rather than design the other integrals, though a priori of large modulus, are, nevertheless, bounded above. □

Putting all of this together, we can proceed to the computation of the segre class of weak branches, *i.e.*

IV.6.7 Fact *Let all of the hypothesis of* IV.6.3-6 *hold, then the segre class* $s_{W,d\mu}$ *around the weak branch in question is zero.*

proof We take $\mathcal{E}$ to be a divisor arising from a priori blowing up in $\mathrm{sing}(\mathcal{F})$, so that $\log\|\mathbb{I}_{\mathcal{E}}\|$ is ℓ_1, and we organise some further blowing up in such a way that our weak branching potential $\log\|\mathbb{I}_{\mathbf{W}}\|$ is defined on a neighbourhood $\|\mathbb{I}_{\mathcal{E}}\| < t$ of $\mathcal{E}$. The previous integration by parts therefore gives for any $\epsilon > 0$,

$$\begin{aligned}
\int_{\|\mathbb{I}_{\mathbf{W}}\|=\epsilon} |\log\|\mathbb{I}_{\mathcal{E}}\||\frac{d^c\|\mathbb{I}_{\mathbf{W}}\|}{\|\mathbb{I}_{\mathbf{W}}\|}d\mu &= \int_{\|\mathbb{I}_{\mathcal{E}}\|=t} \log\max\{\|\mathbb{I}_{\mathbf{W}}\|,\epsilon\}\frac{d^c\|\mathbb{I}_{\mathcal{E}}\|}{\|\mathbb{I}_{\mathcal{E}}\|}d\mu \\
+ \int_{\|\mathbb{I}_{\mathcal{E}}\|<t} \log\max\{\|\mathbb{I}_{\mathbf{W}}\|,\epsilon\}c_1(\mathcal{E})d\mu &- \int_{\|\mathbb{I}_{\mathbf{W}}\|\geq\epsilon, \|\mathbb{I}_{\mathcal{E}}\|<t} \log\|\mathbb{I}_{\mathcal{E}}\| c_1(W)d\mu
\end{aligned}$$

On the other hand $\log\|\mathbb{I}_{\mathbf{W}}\|$ is ℓ_1, so all the integrals on the right exist, and are bounded as $\epsilon \to 0$, while, $s_{W,d\mu}$ is a measure wholly supported on $\mathcal{E}\cap W$, whence $\log\|\mathbb{I}_{\mathcal{E}}\|$ could only be integrable against it if it were zero. □

IV.7 Computing Residues

Finally, we can bring together the previous considerations in order to compute the symbol Res. To this end we should be clear under what hypothesis the results of IV.6 can be guaranteed. In the first place IV.6.2/3 work not only as stated, but would equally hold on a neighbourhood of the divisor $\mathcal{E}$ in question provided that $d\mu_{\mathcal{X}/\mathcal{F}}$ were reduced to a point on the said neighbourhood. Evidently, we have in mind the situation where LCR holds, and we've already blown up in $\mathrm{sing}(\mathcal{F})$, with $\mathcal{E}$ the exceptional divisor. Consequently, there is a certain sense in which we can work 'locally', albeit semi-local would be more accurate, since we will have to address the issues in IV.2. The condition on nullity of segre classes for proper sub-varieties of co-dimension 2 in IV.6.5 is, however, overkill, at least in so much as it is required in IV.6.6. Indeed for a generic choice of ample divisor, which is all IV.6.6 needs, this could be reduced to the nullity of the segre class in $\mathrm{sing}(\mathcal{F})$, but even without proving this, and leaving things as they stand, we'll still get IV.6.6 (and whence IV.6.7) provided the segre classes vanish on an 'appropriate sub-variety of co-dimension 2', where the definition of appropriate arises from the centres employed in the LCR resolution of the triple $(\mathcal{X}, A, \mathcal{F})$, for some ample A (understood on the ambient space if we're working on some neighbourhood and/or local sub-stack) containing $\mathrm{sing}(\mathcal{F})$. Consequently, all of our problems really centre around the well definedness of the weak branching potentials, *i.e.*

IV.7.1 Fact *Suppose* LCR *and that every weak branching scheme* $\mathfrak{W}$ *as found in* I.6.7, cf. *also* VI.4, *converges on a neighbourhood of* $\mathrm{sing}(\mathcal{F})$*, and that the transverse invariant measure* $d\mu_{\mathcal{X}/\mathcal{F}}$ *has no segre classes nor support in the singularities of the log-triple* $(\mathcal{X}, A, \mathcal{F})$ *for a generic ample divisor containing* $\mathrm{sing}(\mathcal{F})$*, then in the sense of* IV.1.4, $Res(d\mu_{\mathcal{X}/\mathcal{F}}) = 0$.

proof We'll proceed by induction on the dimension for $(U, \mathcal{F})$ an analytic space containing $\mathrm{sing}(\mathcal{F})$, or for that matter $\mathcal{E}$ a divisor as per I.3.1, but critically with not only $\mathrm{sing}(\mathcal{F})$ proper, but U embedded in an algebraic stack with projective moduli. Irrespectively, if $\mathfrak{W}$ is the totality of weak branches, then as noted at the start of IV.4: $Res(\mathbb{1}_{U\setminus\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}) = 0$, provided that the residual measures associated to each irreducible component of $\mathfrak{W}$ vanish.

Now for any component W of $\mathfrak{W}$, and $\epsilon > 0$, we have the formula,

$$s_{W,d\mu}(1) = \int_{\|\mathbb{1}_{\mathbf{W}}\|=\epsilon} \frac{d^c\|\mathbb{1}_{\mathbf{W}}\|}{\|\mathbb{1}_{\mathbf{W}}\|} d\mu + \int_{\|\mathbb{1}_{\mathbf{W}}\|\leq\epsilon} c_1(W) d\mu$$

for $c_1(W)$ the chern type class associated to W. This is smooth for W a divisor, and in general absolutely integrable with respect to $d\mu_{\mathcal{X}/\mathcal{F}}$. As such if, for ω Kähler, we identify $d\mu_{\mathcal{X}/\mathcal{F}}$ with a Radon measure, $d\mu(x)$, on the space $X_0 \otimes \mathbb{C}(d\mu_{\mathcal{X}/\mathcal{F}})$ of extremal invariant measures for $\mathcal{L}_\infty(\mathbb{1}_{U\setminus W} d\mu_{\mathcal{X}/\mathcal{F}})$, [II] A.3.14, then,

$$s_{W,\mathbb{1}_{U\setminus W} d\mu_{\mathcal{X}/\mathcal{F}}} = \int_{X_0\otimes\mathbb{C}(d\mu)} s_{W,x} d\mu(x)$$

where x denotes a measure reduced to a point on U, and $s_{W,x}$ is the associated residual measure. Whence, it suffices to check that $s_{W,x}$ is zero for every measure which is reduced to a point, which, for hypothesis as stated, follows from IV.6.7.

This leaves us with the part of $d\mu_{\mathcal{X}/\mathcal{F}}$ supported in $\mathfrak{W}$, which follows by induction on dimension, or more correctly in the presence of similar convergence hypothesis on sub-weak branches, given LCR. and IV.2.4. $\square$

Plainly, the principle problem with this proposition lies in the hypothesised convergence of the weak branches, which we'll begin to address with,

IV.7.2 Fact *Suppose* LCR *and that the transverse invariant measure* $d\mu_{\mathcal{X}/\mathcal{F}}$ *has no segre classes nor support in the singularities of the log-triple* $(\mathcal{X}, A, \mathcal{F})$ *for a generic and sufficiently ample divisor containing* $\mathrm{sing}(\mathcal{F})$*, and that after resolving the singularities of such a triple by analytic blowing up, we can make a real blow up,* $\mathcal{X}_{\mathbb{R}}$*, in a Cartier divisor* $\mathcal{E}$ *supported on the exceptional locus, such that every weak branch* $\mathfrak{W}^p$ *about a connected component of* $\mathrm{sing}(\mathcal{F})$ *is locally on* $\mathcal{X}_{\mathbb{R}}$ *the completion of an analytic sub-stack* W^p*, then* $Res(d\mu_{\mathcal{X}/\mathcal{F}}) = 0$.

proof Proceeding as in the proof of IV.7.1, we take U to be a sufficiently small neighbourhood of a connected component Z of $\mathrm{sing}(\mathcal{F})$, and in light of IV.2.4, we may, as above, suppose that $d\mu$ is reduced to a point on the neighbourhood U without support in weak branches. Once again, we've not unduly complicated the situation by supposing that each of our weak branches is sectorially convergent around an entire connected component of $\mathrm{sing}(\mathcal{F})$. Apart from allowing us to simply concentrate on showing the nullity of the residual measure around a single branch $\mathfrak{W}$, it also allows us to focus on the real problem that we can no longer realise the residual measure by way of a potential as found in IV.4.5. We do, however, have convergence by sector, so let's say that more generally we have a cover, or better partition since we'll be proceeding via Stokes as in III.1, S_α in some topology (inevitably finer than the standard one, since we'd like to consider the images of standard opens on analytic or real blow ups) together with local potentials $\log \|\mathbb{1}_{W_\alpha}\|$ for $s_{W,d\mu}$ restricted to S_α. Consequently, for a sufficiently random choice of S_α, we may harmlessly try our luck with,

$$\phi := \sum_\alpha \mathbb{1}_{V_\alpha} \log \|\mathbb{1}_{W_\alpha}\|$$

Plainly this is just asking for trouble since, we've introduced a massive additional discontinuity on the boundaries of our sectors S_α. Surprisingly enough there's a trick to avoid this. Specifically, we want to show that $s_{W,d\mu}$ is zero, and at the end of the day the essence of how this is done in IV.6.7 was to exhibit a function which was integrable against $s_{W,d\mu}$, while identically $+\infty$ on it's support. To, this end if we succeed in performing the double integration by parts of IV.6.6 for $|\phi|$ against $f^*\gamma d\mu$ for a rational map f well defined around $\mathrm{sing}(\mathcal{F})$, and γ the standard Fubini form on $\mathbb{P}^1$ then $-\log \|f\|^2$, for $\|f\|$ the Fubini-Study norm of the zero section, is such a function whenever the fibre over 0 contains $\mathrm{sing}(\mathcal{F})$.

Now, given a foliation with canonical singularities, the vanishing of $s_{W,d\mu}$ is independent of whatever blowing up we require to perform a LCR resolution of a triple $(\mathcal{X}, A, \mathcal{F})$, for A a divisor whose generic points are transverse to $\mathcal{F}$, cf. post VI.4.2. Consequently, blowing up as required, leads to plenty of rational maps f, well defined around $\mathrm{sing}(\mathcal{F})$ whose fibre over 0 contains $\mathrm{sing}(\mathcal{F})$. Choose one, and of course make sure that the support of the divisor $f^{-1}(0)$ satisfies all the prescriptions, apart from ampleness, of the total transform of the divisor A appearing in IV.6.6. Furthermore, by hypothesis, we can even choose f, so that our sectors are defined by a choice of it's argument. As such, if we get the double integration by parts of IV.6.6 underway then we have a real spot of luck, *i.e.* for $\epsilon_\alpha > 0$,

$$\begin{aligned}\int_{S_\alpha} |\log\max\{\|\mathrm{1\!I}_{\tilde W_\alpha}\|,\epsilon_\alpha\}|f^*\gamma d\mu = &\int_{\|f\|=\mathrm{cst}\cap \mathrm{S}_\alpha} \log\max\{\|\mathrm{1\!I}_{\tilde W_\alpha}\|,\epsilon_\alpha\}\frac{d^c\|f\|}{\|f\|}d\mu \\ &+0(1) - \int_{S_\alpha, \|\mathrm{1\!I}_{\tilde W_\alpha}\|>\epsilon_\alpha} \frac{d\|f\|}{\|f\|}\frac{d^c\|\mathrm{1\!I}_{\tilde W_\alpha}\|}{\|\mathrm{1\!I}_{\tilde W_\alpha}\|}d\mu \\ &\pm \int_{\arg(f)=\mathrm{cst}} \log\max\{\|\mathrm{1\!I}_{\tilde W_\alpha}\|,\epsilon_\alpha\}\frac{d^c\|f\|}{\|f\|}d\mu\end{aligned}$$

where the signs, as in IV.6.6, on the first of the two integrals on the right are of paramount importance, but that on the last irrelevant, since, infact, $d^c\|f\|$ is identically zero on any real hypersurface where its argument is constant. As before the $0(1)$ comes from the segre class around the fibre of f over 0, albeit, and rather critically, this is zero except at components where $\mathcal{F}$ is smooth and transverse. Having got this far, there are no further problems in applying Stoke's beyond those that we have already encountered pre IV.4.4 in deducing the well definedness of the residual measure around $\mathfrak{W}$ from purely sectorial convergence, so we deduce, for $t > -\infty$, that the following is bounded by a constant depending on f,

$$\begin{aligned}&\int_U (|\max\{\phi,t\}|f^*\gamma - |\log\|f\||c_1(\mathfrak{W}))d\mu \\ +&\int_{\|f\|=\mathrm{cst}} |\max\{\phi,t\}|\frac{d^c\|f\|}{\|f\|}d\mu + \int_{\phi=t}|\log\|f\||d^c\phi d\mu\end{aligned}$$

where the chern type class is just the sum over α of the chern type classes on S_α extended by zero. We know though that $\log\|f\|$ is ℓ_1 by IV.6.5, so everything that we don't know in this expression is non-negative, and we've achieved our goal. □

This is a manifest improvement on IV.7.1, but it still sweeps the rather substantial difference between irreducible and connected components of $\mathrm{sing}(\mathcal{F})$ under the carpet, which we finally address by way of,

IV.7.3 Fact *Suppose* LCR$^+$ *and that the transverse invariant measure* $d\mu_{\mathcal{X}/\mathcal{F}}$ *has no segre classes nor support in the singularities of the log-triple* $(\mathcal{X}, A, \mathcal{F})$ *for a generic and sufficiently ample divisor containing* $\mathrm{sing}(\mathcal{F})$*, then in the sense of* IV.1.4, $Res(d\mu_{\mathcal{X}/\mathcal{F}}) = 0$.

proof Needless to say we have the same initial reductions as IV.7.1, *i.e.* we can suppose that $d\mu$ is reduced to a point on a neighbourhood of a connected component of $\mathrm{sing}(\mathcal{F})$ without support in weak branches, although one should note that we appeal to the well definedness of the residue symbol, IV.1.4, rather than the well definedness of the residual measures at weak branches, which for the moment we don't actually know.

Now, with notations as per IV.4.2, we can, for a given weak branch $\mathfrak{W}^p$, use LCR$^+$ to form a potential function, $\log\|\mathrm{1\!I}_{W^p}\|$, around a neighbourhood U^p of Z^p. This is done with the exact same procedure as in IV.7.2, *i.e.* extend local potentials on a sectorial partition S_α by zero. Once again, we suppose sufficient a priori blowing up to permit us to conclude the existence of a meromorphic function

f, well defined around Z^p, whose fibre over 0 contains Z^p. In principle, we may have problems around V^p, so we do the basic trick on $\|1\!\!1_{V^p}\| > 0$. There are no extra residues because of our a priori information IV.6.2-5, and so we obtain for $t > -\infty$, and a constant $c > 0$ to be chosen that,

$$\int_{\|f\|<c} (|\max\{\log\|1\!\!1_{W^p}\|, t\}| f^*\gamma - |\log\|f\|| c_1(\mathfrak{W}^p)) d\mu$$
$$- \int_{\|f\|=c} \max\{\log\|1\!\!1_{W^p}\|, t\} \frac{d^c\|f\|}{\|f\|} d\mu - \int_{\|1\!\!1_{W^p}\|=e^t} \log\|f\| d^c \log\|1\!\!1_{W^p}\| d\mu$$

is bounded by,

$$-\int_{\|f\|=c, \|1\!\!1_{W^p}\|>e^t} \log\|f\| \frac{d^c\|1\!\!1_{W^p}\|}{\|1\!\!1_{W^p}\|} d\mu + 0(1)$$

Up to now, we've profited from the boundedness of the differential in this integral off $\mathrm{sing}(\mathcal{F})$ to conclude that this term is bounded. We can, though, simply exploit our freedom to choose c, equivalently normalise $\|f\|$, to conclude that it's zero. Whence, amongst what is bounded, all unknowns are still non-negative, so, in particular, for all $t > -\infty$,

$$\int_{\|1\!\!1_{W^p}\|=e^t} |\log\|f\|| d^c \log\|1\!\!1_{W^p}\| d\mu$$

This not only proves that the residual measure around $\mathfrak{W}^p$ is well defined, but that it is zero. This measure, however, continues to compute any residue, since exactly as per IV.4.4 the contribution around the boundaries of different sectors cancel, even at V^p. $\Box$.

V. HYPERBOLICITY (EMPHASIS ON 3-D)

V.1 Parabolic Measures

Consider the following statements for maps to a foliated $\mathbb{Q}$-gorenstein projective triple $(X, B, \mathcal{F})$ with canonical singularities and conventions on the weights of the necessarily non-invariant boundary B as per [II] I.6.1.,

V.1.1 Finiteness *There exists a proper closed subvariety Z of X such that the subspace of* $\mathrm{Hom}(\Delta, X)$ *consisting of invariant maps not factoring through B, with $f^{-1}(B_i)$ a divisor of weight divisible by that of B_i, for every component B_i of B, and not arbitrarily close to Z in the compact open sense is compact.*

Now suppose this statement is false, then by [III] II.3,

V.1.2 Summary *We can find a transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ with the following properties,*

(a) $(K_{\mathcal{F}} + B).d\mu_{X/\mathcal{F}} \leq 0$

(b) $D.d\mu_{X/\mathcal{F}} \geq 0,\ \forall$ *effective Cartier divisors D.*

(c) *For any proper generically finite map $\rho : \mathcal{X} \to X$ in the 2-category of algebraic stacks there is a closed positive transverse invariant measure $d\nu_{\mathcal{X}}$ satisfying (b) on $\mathcal{X}$ and $\rho_* d\nu_X = d\mu_{X/\mathcal{F}}$. Furthermore if we have a foliated triple such that $(\mathcal{X}, \mathcal{B}, \mathcal{F})$ is unramified in the leaf direction, then we have (a) on $\mathcal{X}$ too.*

In particular for the stack $\rho : (\mathcal{X}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ with its divisorial ramification over B of [II] I.7.4, we get a transverse invariant measure satisfying $K_{\tilde{\mathcal{F}}}.d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$, and whence equality if $K_{\tilde{\mathcal{F}}}$ is nef. Consequently we introduce,

V.1.3 Definition *Call an invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ on a minimal model $(\mathcal{X}, \mathcal{F})$ of a foliated gorenstein stack with projective moduli parabolic if $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} = 0$, and we have conditions (b) & (c) of* V.1.2.

A useful initial remark is to clarify the condition (c) for bi-rational modification, so let's say $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ a blow up in an irreducible centre $\mathcal{Y}$ of co-dimension p with $\mathcal{E}$ the exceptional divisor. As such we define the multiplicity,

$$m_{\mathcal{Y}}(d\mu_{\mathcal{X}/\mathcal{F}}) = \sup\{\mathcal{E}.d\nu_{\mathcal{X}} \ : \ \rho_* d\nu_{\mathcal{X}} = d\mu_{\mathcal{X}/\mathcal{F}},\ d\nu_{\mathcal{X}} \text{ satisfies (b)}\}$$

and we assert,

V.1.4 Claim *If* $\dim \mathcal{Y} <$ *numerical Kodaira dimension of $K_{\mathcal{F}}$, then $m_{\mathcal{Y}}(d\mu_{\mathcal{X}/\mathcal{F}}) =$* 0.

proof The proposition is absolutely nothing to do with most of the hypothesis, and is infact a simple counting argument. Indeed let $\epsilon \in \mathbb{Q}_{>0}$, and $d \in \mathbb{N}$ be sufficiently large and divisible, then for ν the numerical Kodaira dimension of a nef K on the n-dimensional moduli X, with H ample,

$$h^0(X, K^d \otimes H^{d\epsilon}) \backsimeq \frac{d^n \epsilon^{n-\nu} k^\nu H^{n-\nu}}{\nu!}(1 + O(\epsilon))$$

On the other hand to vanish to order m on a sub-scheme Y of co-dimension p is determined inductively by exact sequences,

$$0 \to H^0(X, K^d \otimes H^{d\epsilon}\mathcal{I}_Y^{i+1}) \to H^0(X, K^d \otimes H^{d\epsilon}\mathcal{I}_Y^i) \to H^0(K^d \otimes H^{d\epsilon} \otimes \frac{\mathcal{I}_Y^i}{\mathcal{I}_Y^{i+1}})$$

for $0 \leq i < m$, from which we deduce,

$$h^0(X, K^d \otimes H^{d\epsilon}\mathcal{I}_Y^m) \gtrsim h^0(X, K^d \otimes H^{d\epsilon}) - \text{const}\frac{m^p d^{n-p}}{p!(n-p)!}$$

where the implied constant depends on Y. Whence if we're aiming for multiplicity $d\delta$, then we need,

$$\delta^p << \epsilon^{n-\nu}$$

and by hypothesis, $n - \nu \leq p - 1$, so if we take, as we may δ of order $\epsilon^{n-\nu/p}$ then δ is at least of order $\epsilon^{1-1/p}$, so we find,

$$\rho^*(K + \epsilon H) \gg \epsilon^{1-1/p}\mathcal{E}$$

on $\mathcal{X}$, from which,

$$\epsilon^{1/p}H > d\mu_{\mathcal{X}/\mathcal{F}} \gg m_{\mathcal{Y}}(d\mu_{\mathcal{X}/\mathcal{F}}) \geq 0$$

for every $\epsilon > 0$. $\square$.

As such, and particularly given that our primary interest centres on Kodaira dimension $n-1$, it's invariably the case that we may suppose that the multiplicity is zero. In such a scenario, it's often the case that V.1.2 (c) is a bit empty. Specifically suppose $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ is a blow up in a lci substack $\mathcal{Y}$, so that the exceptional divisor $\rho : \mathcal{E} \to \mathcal{Y}$ is relatively smooth. Consequently for any remotely sensible metricisation of $\mathcal{O}_{\mathcal{E}}(1) := \mathcal{O}_{\mathcal{E}}(-\mathcal{E})$ the class $\mathcal{O}_{\mathcal{E}}(1)^{p-2}.\rho^* s_{\mathcal{Y},d\mu}$ is a positive measure, and for $1\!\!1_{\mathcal{Y}} d\mu_{\mathcal{X}/\mathcal{F}} = 0$ we get a perfectly sensible 'proper transform',

$$1\!\!1_{\mathcal{X}\backslash\mathcal{E}} d\mu_{\mathcal{X}/\mathcal{F}} + (i_{\mathcal{E}})_* \mathcal{O}_{\mathcal{E}}(1)^{p-2}.\rho^* s_{\mathcal{Y},d\mu}$$

which incidentally is just $\rho^* d\mu_{\mathcal{X}/\mathcal{F}}$ as an 'analytic cycle' if one were to use the theory of residual measure to adapt [F] 6.7 to measures. Nevertheless when $1\!\!1_{\mathcal{Y}} d\mu_{\mathcal{X}/\mathcal{F}}$ isn't 0 the existence of a proper transform is a highly non-trivial hypothesis, whose relation to residual measure merits clarification. To fix ideas and/or consider a case that captures all the essentials , suppose that $d\mu_{\mathcal{X}/\mathcal{F}}$ arises from a countable Zariski dense set of curves $\mathcal{C}_i \hookrightarrow \mathcal{X}$ of unbounded degree, i.e. after subsequencing,

$$d\mu_{\mathcal{X}/\mathcal{F}} = \lim_i \frac{1}{H.\mathcal{C}_i}\mathcal{C}_i$$

where $\mathcal{C}_i$ is equally used to denote its current of integration, or if one prefers over C_i, where the latter is smooth, scheme like, and dominates $\mathcal{C}_i$ by way of f_i. Plainly therefore we have an a priori notion of proper transform for, say, $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ the blow up in $\mathcal{Y}$ as before, given by,

$$d\mu_{\tilde{\mathcal{X}}/\tilde{\mathcal{F}}} = \lim_i \frac{1}{H.\mathcal{C}_i}\tilde{\mathcal{C}}_i$$

where $\tilde{\mathcal{C}}_i$ is the proper transform of the still Zariski dense set of $\mathcal{C}_i$'s not factoring through $\mathcal{Y}$, with extra subsequencing as necessary. In order to relate this to the theory of residual measure consider for $\mathcal{Z} \hookrightarrow \mathcal{X}$ a closed substack with $\|\mathbb{I}_{\mathcal{Y}}\|$, $\|\mathbb{I}_{\mathcal{Z}}\|$ as per III.3.1, and $\delta > 0$, the various integrals over a given $\mathcal{C}_i$,

$$\begin{aligned}
&\int_{1/\log|\log\|\mathbb{I}_{\mathcal{Z}}\||<\|\mathbb{I}_{\mathcal{Y}}\|<\delta} dd^c \log\|\mathbb{I}_{\mathcal{Y}}\|^2 = \\
&\int_{\delta=\|\mathbb{I}_{\mathcal{Y}}\|>1/\log|\log\|\mathbb{I}_{\mathcal{Z}}\||} (d^c \log\|\mathbb{I}_{\mathcal{Y}}\|^2 + d^c \log\log^2|\log\|\mathbb{I}_{\mathcal{Z}}\||) \\
- &\int_{\delta\geq\|\mathbb{I}_{\mathcal{Y}}\|>1/\log|\log\|\mathbb{I}_{\mathcal{Z}}\||} dd^c \log\log^2|\log\|\mathbb{I}_{\mathcal{Z}}\|| - s_{\mathcal{Y},\mathcal{C}_i}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0})
\end{aligned}$$

Now divide out by the degree of the $\mathcal{C}_i$ take the limit over i, then let δ go to zero to get,

$$\begin{aligned}
-\mathcal{E}.\mathbb{I}_{\mathcal{E}\cap\rho^{-1}(\mathcal{Z})}d\mu_{\mathcal{X}/\mathcal{F}} &= \lim_{\delta\to 0} s^{\delta}_{\mathcal{Y},d\mu}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|<\exp(-\exp(\frac{1}{\delta}))}) - m_{\mathcal{Y},d\mu_{\mathcal{X}/\mathcal{F}}}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0}) \\
&\leq s_{\mathcal{Y},d\mu}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0}) - m_{\mathcal{Y},d\mu_{\mathcal{X}/\mathcal{F}}}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0})
\end{aligned}$$

where s^{δ} is the segre type class, but calculated at a fixed radius $\|\mathbb{I}_{\mathcal{Y}}\| = \delta$, while m, evidently for multiplicity, is the multiplicity type measure arising from the normalised limits of the $s_{\mathcal{Y},\mathcal{C}_i}$, and the terms involving $\log\log|\log\|\mathbb{I}_{\mathcal{Z}}\||$ all disappear because $dd^c \log|\log\|\mathbb{I}_{\mathcal{Z}}\||$ is absolutely integrable with respect to $d\mu$ irrespective of any choices. Obviously an equality would have been preferable, but we'll invariably be able to get by by via the generally valid formula,

$$\mathcal{O}_{\mathcal{E}}(1).\mathbb{I}_{\mathcal{E}}d\mu_{\tilde{\mathcal{X}}/\tilde{\mathcal{F}}} = s_{\mathcal{Y},d\mu}(1) - m_{\mathcal{Y},d\mu}(1)$$

Consequently, one should note that the possibility exists to refine our definition of parabolic measure to include,

V.1.5 (V.1.2 bis.) Summary/Definition *A parabolic measure could, for all practical purposes, be asked to satisfy the following further naturality condition for blow ups $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ in lci centres $\mathcal{Y}$ with exceptional divisor $\mathcal{E}$,*

$$\text{(d)} \quad -\mathcal{E}.\mathbb{I}_{\mathcal{E}\cap\rho^{-1}(\mathcal{Z})}d\mu_{\tilde{\mathcal{X}}/\tilde{\mathcal{F}}} \leq s_{\mathcal{Y},d\mu}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0}) - m_{\mathcal{Y},d\mu}(\mathbb{I}_{\|\mathbb{I}_{\mathcal{Z}}\|=0})$$

for $\mathcal{Z} \hookrightarrow \mathcal{X}$ any closed substack, and $\|\mathbb{I}_{\mathcal{Z}}\|$ as per III.3.1.

V.2 Parabolic Decomposition

Suppose we're presented with a parabolic measure $d\mu_{\mathcal{X}/\mathcal{F}}$ on $(\mathcal{X}, \mathcal{F})$ with, say, $\mathcal{X}$ smooth of dimension 3 to fix ideas. Notice that by definition the singularities of $\mathcal{F}$ are canonical so any invariant divisor $\mathcal{D}$ that moves in a pencil defines not just a rational map, but an honest 1st integral, $\lambda : \mathcal{X} \to \mathcal{C}$, so if $\mathcal{D}_t$ is generic in the pencil $\mathcal{O}_{\mathcal{D}_t}(\mathcal{D}_t)$ is at worst torsion. Consider, therefore, the condition,

$$\mathrm{I\!I}_{\mathcal{D}} d\mu_{\mathcal{X}/\mathcal{F}} \neq 0\,, \text{ and } \mathcal{D}.\mathrm{I\!I}_{\mathcal{D}} d\mu_{\mathcal{X}/\mathcal{F}} < 0$$

Now certainly $d\mu_{\mathcal{X}/\mathcal{F}}$ may have support on curves, which are necessarily invariant, and countably many, so write,

$$d\mu_{\mathcal{X}/\mathcal{F}} = d\nu + \sum_i \lambda_i L_i\,,\ \mathrm{I\!I}_{\mathcal{C}} d\nu = 0 \text{ for all curves } \mathcal{C}$$

where L_i are the classes of integration of the said curves. Consequently, if we still have $\mathcal{D}.d\nu < 0$ then $\mathcal{D}$ is the only effective Cartier divisor that meets $\mathrm{I\!I}_{\mathcal{D}} d\nu$ negatively, and indeed, $\mathcal{C}.\mathrm{I\!I}_{\mathcal{D}} d\nu \geq 0$ for all effective Cartier divisors $\mathcal{C}$ on $\mathcal{D}$. Now let's proceed to an understanding of $\mathrm{I\!I}_{\mathcal{D}} d\nu$ in the usual way, i.e. as in II.1 consider the diagram,

$$\begin{array}{ccc}
(\mathcal{Y}_0, \mathcal{G}_0) & \xleftarrow{\ \pi\ } & (\mathcal{Y}, \mathcal{G}) & & \\
 & & \Big\downarrow \rho & & \\
 & & (\tilde{\mathcal{D}}, \tilde{\mathcal{F}}) & & \\
 & & \Big\downarrow & & \\
 & & (\mathcal{D}, \mathcal{F}) & \longrightarrow & (\mathcal{X}, \mathcal{F})
\end{array}$$

where $\tilde{\mathcal{D}} \to \mathcal{D}$ is the normalisation, $\rho : \mathcal{Y} \to \tilde{\mathcal{D}}$ a relatively canonical model contracting curves E_i, and $\pi : \mathcal{Y} \to \mathcal{Y}_0$ the actual canonical model contracting curves C_j. As such we get a formula of the form,

$$\rho^* K_{\mathcal{F}} = \pi^* K_{\mathcal{G}_0} + Z + \sum_i a_i E_i + \sum_j b_j C_j$$

$a_i \in \mathbb{Q}_{>0}$, $b_j \in \mathbb{Q}_{\geq 0}$, Z supported on $\mathrm{sing}(\mathcal{F})$ with parts Z', Z'' invariant and non-invariant by $\tilde{\mathcal{F}}$. In particular it's perhaps better to think in terms of relatively canonical & canonical models for $(\tilde{\mathcal{D}}, Z'', \tilde{\mathcal{F}})$. Regardless if $(\mathcal{Y}_0, \mathcal{G}_0)$ or more generally $(\mathcal{Y}_0, Z'', \mathcal{G}_0)$ is fibred in rational curves or weighted projective stacks then $\mathrm{I\!I}_{\mathcal{D}} d\nu$ has a fairly obvious shape, and we make no further comment. At the other end of the spectrum if we have general type then $\mathrm{I\!I}_{\mathcal{D}} d\nu$ doesn't exist by the index theorem, while rather more delicately the classification theorem in dimension 2, [M2], tells us that that it cannot exist for numerical Kodaira dimension 1 either without being supported on curves, or arising in the obvious way from an elliptic fibration. If the Kodaira dimension in zero then the cases where $\mathcal{G}_0$ is a smooth foliation is distinguished. Here there are invariant measures unique in homology, say $d\nu_0$, with the property that $d\nu_0^2 = 0$, so that $\pi^* d\nu_0$ can be identified with its extension by zero across $\bigcup C_j$ when it's not supported on curves, or by the pull-back of a curve when it is. Either way, since $d\nu$ itself isn't supported on curves $\rho^* \mathrm{I\!I}_{\mathcal{D}} d\nu$

may be identified with the extension by zero across $\bigcup E_i$ plus some curves, which in turn is homologous to $\pi^* d\nu_0$ by the index theorem. Furthermore the log-Kodaira dimension must also be zero, so Z'' is empty, and $\rho^*\mathcal{O}_{\mathcal{D}}(\mathcal{D})$ admits a connection along the leaves with meromorphic poles on Z'. On the other hand $Res(\pi^* d\nu_0)$ by construction, so $\mathcal{D}.\mathrm{I\!I}_{\mathcal{D}} d\nu = 0$, contrary to hypothesis. This leaves the possibility that $Z'' = \emptyset$, and $(\mathcal{Y}_0, \mathcal{D}_0)$ is defined by a $\mathbb{G}_m \times \mathbb{G}_m$ or $\mathbb{G}_a \times \mathbb{G}_m$ action, or $Z'' \neq \emptyset$ and we have an isotrivial family in $\mathbb{G}_m$'s, nearly all of which can occur. More precisely consider a $\mathbb{G}_a \times \mathbb{G}_m$ action modeled on $\mathbb{P}^1 \times \mathbb{P}^1$ and suppose $d\nu_0$ is a measure on it not supported on curves, then by [Br1], $Res(d\nu_0) = 0$, so by the index theorem every cycle C supported on invariant curves satisfies $C^2 \leq 0$, which is nonsense, so this case is inexistent. Similarly by op. cit. $Res(d\nu_0)$ would also be zero on $\mathbb{P}^1 \times \mathbb{P}^1$ for a $\mathbb{G}_m \times \mathbb{G}_m$ action given by a field of the form $x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$, $\lambda \notin \mathbb{R}$, the case of $\lambda \in \mathbb{R}$ may, though, a priori occur. It's therefore worth noting that apart from the real eigenvalue, if this example did occur then the index theorem tells us that $\mathrm{sing}(\mathcal{F}) \cap \mathcal{D}$ has dimension zero, while $(\mathcal{Y}, \mathcal{G})$ and $(\tilde{\mathcal{D}}, \tilde{\mathcal{F}})$ coincide, and are modifications of $(\mathcal{Y}_0, \mathcal{G}_0)$ by a sequence of blow ups in invariant centres. In particular $\mathrm{I\!I}_{\mathcal{D}} d\nu$ is not only homologous to a positive sum of invariant curves, but is so in the strong sense that the difference intersects any invariant object in zero, whence, to all intents and purposes we'll be able to assume that they are equal. Consequently, modulo bearing in mind the need for a little care in this case, let us simply write,

$$d\mu_{\mathcal{X}/\mathcal{F}} = d\nu + \sum_i \lambda_i L_i$$

where now $\mathcal{D}.d\nu \geq 0$ for every effective Cartier divisor $\mathcal{D}$, and the L_i are allowed to be not only invariant curves, but invariant measures on any divisor on which the induced foliation is a conic pencil. Whence if $K_{\mathcal{F}}^2 \neq 0$,

V.2.1 Fact *Let things be as above with $K_{\mathcal{F}}^2 \neq 0$, then $d\mu_{\mathcal{X}/\mathcal{F}}$, $d\nu$, and $\sum L_i$ are all parallel when projected into* $\mathrm{NE}_1(\mathcal{X})$.

This is, of course, just an application of III.2.1, but $d\nu$ has further structure, such as,

V.2.2 Further Fact *Let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be an invariant substack of dimension less than the numerical Kodaira dimension of $\mathcal{F}$ then the residual measure $s_{\mathcal{Y},d\nu}$ is zero.*

proof By hypothesis $\mathrm{I\!I}_{\mathcal{Y}} d\nu = 0$, so if $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ is the blow up in $\mathcal{Y}$ with $\mathcal{E}$ the exceptional divisor then,

$$s_{\mathcal{Y},d\nu}(1) = \mathcal{E}.d\tilde{\nu}$$

where $d\tilde{\nu}$ is the extension of $d\nu$ across $\mathcal{E}$ by zero. Arguing as per V.1.4, we're plainly going to be done if we can show that $d\tilde{\nu}$ intersects every Cartier divisor on $\mathcal{X}$ non-negatively. Replacing $\tilde{\mathcal{X}}$ by a resolution via a sequence of blow ups in invariant centres we deduce the existence of an irreducible Cartier divisor $\mathcal{D}$ such that $\mathcal{D}.d\tilde{\nu} < 0$, so $\mathcal{D}.\mathrm{I\!I}_{\mathcal{D}} d\tilde{\nu} < 0$. On the other hand ρ is bi-rational at the generic point of $\mathcal{D}$, and we've already removed everything that might have this property. $\square$.

From which we proceed to,
V.2.3 Main Fact *Either the number of possible L_i's covered by a rational curve is 'finite' - i.e. something appearing in the sum $\sum \lambda_i L_i$ is: one of a finite list, moves in one of finitely many families, or moves in a family covering $\mathcal{X}$.*
proof Most of this has already been done in §III, so that in the notation of III.9.2 we may assume that $\mathcal{X} = \mathcal{X}_{pr}$, and L_i is a rigid curve, étale covered by a generalised weighted projective stack $\mathbb{P}^1(p,q)$, where p, q are the orders of the generic monodromy at either end modulo the gcd of the same. What's critical to know is that $\mathrm{I\!I}_{\mathcal{X}\setminus L_i} d\mu$ has no residual measure around L_i. For $K_{\mathcal{F}}^2 \neq 0$ this follows from V.2.2 & the construction of $\mathcal{X}_{pr}$. For $K_{\mathcal{F}}$ numerically nil, there is never an issue since the eigenvalues at any end are constant. The case $K_{\mathcal{F}} \neq 0$, but $K_{\mathcal{F}}^2 = 0$ is rather subtle. One has to first prove by specialisation ad nauseam to the normal cone of L_i that the only way that this could happen is if $d\nu$ were infinitely tangent to the formal schemes D_0 and D_∞ of II.6 at the $\mathcal{X}_{pr}$ level, then go back up to $\mathcal{X}$ and proceed as per II.8.3 to extract a contradiction. We don't, however, need this for our main 3-D application, so we omit further discussion. Now we can just construct the flap $\tilde{\mathcal{X}}_{pr}$ of L_i, i.e. the weighted blow up mapping down to the flop, let $\mathcal{E}$ be the exceptional divisor and conclude,

$$\mathcal{E}.\mathrm{I\!I}_{\mathcal{E}} d\tilde{\mu} \geq 0$$

which is nonsense, since $\mathcal{O}_{\mathcal{E}}(-\mathcal{E})$ is ample on $\mathcal{E}$. □

V.3 The Diffuse Part

Plainly at this juncture our knowledge of the 'atomic' part $\sum L_i$ that may appear in the parabolic decomposition of a parabolic measure $d\mu_{\mathcal{X}/\mathcal{F}}$ is rather good. Indeed,
V.3.1 Summary *We have the following alternatives,*

(i) *Some rational L_i moves in a family covering $\mathcal{X}$, so, in particular, if $K_{\mathcal{F}}^2 \neq 0$, there is a foliated surface $(\mathcal{S}, \mathcal{G})$ and a rational map $(\mathcal{X}, \mathcal{F}) \dashrightarrow (\mathcal{S}, \mathcal{G})$ compatible with the foliations, whence, 'much' of $d\mu_{\mathcal{X}/\mathcal{F}}$ descends to $\mathcal{S}$, and in any case $K_{\mathcal{X}}.K_{\mathcal{F}}^2 < 0$.*

(ii) *Some L_i is actually elliptic, and wholly contained in the smooth locus, so its normal bundle is flat. Whence, again if $K_{\mathcal{F}}^2 \neq 0$ then such a curve is parallel to $K_{\mathcal{F}}^2$ and $K_{\mathcal{X}}.K_{\mathcal{F}}^2 = 0$.*

(iii) *The number of L_i's is 'finite' in the sense of V.2.3, although one should bear in mind that there may be curves wholly contained in the $\mathrm{sing}(\mathcal{F})$, so it's important to remember that the definition of invariant measure supposes the vanishing of differentials in the co-normal bundle along $d\mu$, so this can only happen for curves that are singular solutions in the sense of* I.4.10 *i.e. contained in the weak branch and invariant by the induced foliation.*

Otherwise we can further enhance our understanding of $d\mu_{\mathcal{X}/\mathcal{F}}$ under some additional hypothesis, i.e.

V.3.2 Fact *If $K_{\mathcal{F}}^2 \neq 0$, then either the 'diffuse' part $d\nu$ doesn't exist or $K_{\mathcal{X}}.K_{\mathcal{F}}^2 = 0$.*

proof Suppose $K_{\mathcal{X}}.K_{\mathcal{F}}^2 \neq 0$, then $d\nu$ is parallel to $K_{\mathcal{F}}^2$, so $K_{\mathcal{X}}.d\nu \neq 0$. The co-normal bundle is, however, invariant so by IV.7.3 $s_{\mathcal{Y},d\nu} \neq 0$ for $\mathcal{Y} \hookrightarrow \mathcal{X}$ a substack of co-dimension at least 2, contradicting V.2.2. □.

In particular,

V.3.3 corollary *If $K_{\mathcal{X}}.K_{\mathcal{F}}^2 > 0$ then $d\mu_{\mathcal{X}/\mathcal{F}}$ is a (possibly countable) sum of 'finitely many' curves in the sense of* V.2.3 *which are either rational invariant not factoring through* $\mathrm{sing}(\mathcal{F})$ *or contained in the weak branch, and invariant by the induced foliation on the same.*

V.4 The Weak Branch

For this section we'll suppose $K_{\mathcal{X}}.K_{\mathcal{F}}^2 \neq 0$, and that we have the alternative V.3.1 (iii). As such there could be a number of curves L_i whose completion around $\mathrm{sing}(\mathcal{F})$ factors through the weak branching scheme. Necessarily we can suppose that this happens at a single irreducible, even smooth, 2-dimensional component $\mathfrak{W}$ of the weak branch which is non-algebraic, so that by I.4.11 the number of L_i involved is finite stricta dictum, and, to re-iterate, may contain some components of $\mathrm{sing}(\mathcal{F})$. In any case denote, slightly abusively, by $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ the sum $\sum_{i\in I'} \lambda_i L_i$, where I' is the set of L_i in question. Our basic assertion is,

V.4.1 Claim *$\mathfrak{W}$ extends as a formal scheme with trace $\bigcup L_i$, $i \in I'$, and for any $i \in I'$, $L_i.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$.*

proof For $L_i \subset \mathrm{sing}(\mathcal{F})$ we've already done this. Otherwise as per I.2.2, it's harmless to blow up in components of $\mathrm{sing}(\mathcal{F})$ sufficiently to guarantee that the L_i in question is contractible, so that we can find $\mathfrak{W}$ around L_i by way of Jordan decomposition on the contraction.

For the next part we proceed as follows: extract a dth root of $\mathfrak{W}$ around the completion in the trace so extended to get a formal stack $\hat{\mathcal{X}}_d$ which is pseudo reflecting around $\mathfrak{W}$, then blow up in the pre-image of the given L_i. Having done this, kill all the pseudo-reflections and sew the thing back into the ambient $\mathcal{X}$ to get a weighted blow up $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ with exceptional divisor $\mathcal{E}$. By construction $\tilde{\mathcal{X}}$ has no more monodromy around the proper transform $\tilde{\mathfrak{W}}$ than $\mathcal{X}$ has around $\mathfrak{W}$, and has extra non-scheme like structure with monodromy $\mathbb{Z}/d$ concentrated on some curve L_i^∞ wholly disjoint from $\tilde{\mathfrak{W}}$. Furthermore any part of the $d\tilde{\mu}$ guaranteed by V.1.2 which is neither on $\mathcal{E}$ or factors through $\tilde{\mathfrak{W}}$ meets $\mathcal{E}$ in L_i^∞ by I.4.11, whence,

$$L_i.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} + \mathcal{E}.\mathbb{I}_{\mathcal{E}\backslash\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq -\frac{1}{d} s_{L_i,\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}}(1)$$

Within $\mathcal{E}$, we've a priori taken $\mathcal{O}_{\mathcal{E}}(-\mathcal{E})$ to be ample for curves $L_i \subsetneq \mathrm{sing}(\mathcal{F})$, while otherwise the foliation in $\mathcal{E}$ is of conic type with integral projection to L_i, so, $-\mathcal{E}.\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, and we let $d \to \infty$. □

The critical use of this proposition is as follows: call the induced foliation in $\mathfrak{W}$ $\mathcal{G}$, then $K_{\mathcal{G}} = K_{\mathcal{F}}|_{\mathfrak{W}} - Z$, where Z is some bunch of curves in $\mathrm{sing}(\mathcal{F})$ that may perfectly well contain some of the L_i's . Nevertheless thanks to V.4.1, we know $K_{\mathcal{G}}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$, so we can examine the structure of $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ by doing minimal model theory on $\mathfrak{W}$. Even though $\mathfrak{W}$ is formal, there's no problem constructing a contraction $\pi : \mathfrak{W} \to \mathfrak{W}_0$ which contracts finitely many $\mathcal{G}$-invariant components of the trace such that $K_{\mathcal{G}_0}.C \geq 0$ for every curve C in the trace of $\mathfrak{W}_0$ unless, of course, $\mathfrak{W}$ were a pencil of conics in the induced foliation, which would contradict its non-algebraicity. As a result we can apply the 'index theorem' (more precisely elementary considerations about quadratic forms) to conclude that $K_{\mathcal{G}_0} \neq 0$ unless every component of $\mathrm{sing}(\mathcal{F}) \cap \mathfrak{W}$ is $\mathcal{G}$ invariant, and deduce in any case $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$, $K_{\mathcal{F}}|_{\mathfrak{W}}$ are parallel. Furthermore the inequality $K_{\mathcal{G}}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$ becomes an identity, and every $L_i \subset \mathrm{sing}(\mathcal{F})$ must be rational if there is more than one component, with any intersections being simple normal crossing and at most one for any point of sing$\mathcal{G}$. As such we have two possibilities: $\pi_* \mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ is supported on at least one rational curve disjoint from all the others, which would then have to move, contradicting the non-algebraicity of $\mathfrak{W}$, or $\pi_* \mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ is elliptic -equivalently the curves $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ form one of the Dynkin diagrams of [M1] IV.4.3. In addition, $D.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$ for every Cartier divisor D. If one wishes, it's possible to proceed as per the discussion of II.9 to flop in $\mathcal{X}$ rather than just contract in $\mathfrak{W}$, albeit that we may loose projectivity of the moduli if we proceed as per op. cit., and whence replace $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ by a single irreducible smooth elliptic or irreducible nodal substack. In principle one then wishes to deduce that the curve moves. This is not, indeed cannot be (just consider an arbitrary representation of $\mathbb{Z}^2$ in Diff$(\mathbb{C})$), a purely formal question, and, unfortunately, neither of the proofs in the case of $\mathfrak{W}$ an algebraic surface generalises to the current situation. Indeed [M1] IV.5.1 is highly global, and, although [Br3] is a little better it needs meromorphic forms with poles along the curve in the co-normal bundle of $\mathcal{G}$. Nevertheless we're happy enough to conclude,

V.4.2 Fact *If $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \neq 0$, and $\mathfrak{W}$ is non-algebraic then infact $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ defines a divisor of elliptic fibre type within $\mathfrak{W}$, and $\mathcal{D}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$ for every effective Cartier divisor on $\mathcal{X}$. In particular $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ is parallel to $K^2_{\mathcal{F}}$.*

VI ALGEBRAIC SURFACES

VI.1 Jets

Although jets are a well documented subject, we'll need some notation, so we may aswell establish this by doing some definitions into the bargain. Our ultimate interest is in stacks over $\mathbb{C}$, and since the semi-stability arguments to be employed aren't quite valid arbitrarily we may aswell suppose for simplicity that this our base. We begin with,

VI.1.1 Definition [GG] *For* $n \in \mathbb{N} \cup \{0\}$*, let* $\Delta_n = \operatorname{Spec} \mathbb{C}[z]/(z^{n+1})$*, and for* $\mathcal{X}/\mathbb{C}$ *an algebraic stack define a fibred category* $V_{\mathcal{X},n}$ *over* $\mathbb{C}$*-affine schemes T by way of,*

$$V_{\mathcal{X},n}|_T = \operatorname{Hom}(\Delta_n|_T, \mathcal{X}|_T)$$

Plainly $V_{\mathcal{X},n}$ is representable as a locally closed substack of $\operatorname{Hilb}(\Delta_n \times_{\mathbb{C}} \mathcal{X})$, or at least it would be if the latter were known to exist. What's at stake, however, in this particular case is rather easy, and one just checks it by hand - indeed, $V_{\mathcal{X},n}$ is a substack of the cone $\mathfrak{P}_{\mathcal{X}} \to \mathcal{X}$ by way of the forgetful map $\pi_{0,n}$, where $\pi_{i,j}$, or just π if there is no likelyhood of confusion, goes from $V_{\mathcal{X},j}$ to $V_{\mathcal{X},i}$ for $i \leq j$. More importantly $V_{\mathcal{X},n}$ is a principal homogeneous space under $\operatorname{Aut}(\Delta_n)$, and essentially what we'd like to do is construct the quotient. Unfortunately the quotient includes the special case of $\mathbb{G}_m$ acting on $T_{\mathcal{X}}$, and as is well known, one only gets a sensible answer after excluding the zero section. Ultimately, however, this is the sum total of complication even in the general setting, so let us first note the compactified answer by way of,

VI.1.2 Further Definition ([A&] or [D2]) *Suppose in addition that* $\mathcal{X}/\mathbb{C}$ *is smooth, and define inductively a tower* P_n *of* $\mathbb{P}^{dim \mathcal{X}-1}$ *bundles together with embeddings* $P_n \hookrightarrow \mathbb{P}(\Omega_{P_{n-1}})$*,* $n \in \mathbb{N}$ *according to,*

(1) $P_0 = \mathcal{X}$

(2) $P_1 = \mathbb{P}(\Omega_{\mathcal{X}})$

(3) $P_{n+1} = \mathbb{P}(J_{\mathcal{X},n+1})$*, where* $J_{\mathcal{X},n+1}$ *is the quotient of* Ω_{P_n} *defined by the restriction to* P_n *of the universal sub-bundle on* $\mathbb{P}(\Omega_{P_{n-1}})$*.*

In particular we have projections, also denoted $\pi_{i,j}$, with the same possibility of omission, from P_j to P_i, $i \leq j$, tautological bundles L_n, along with short exact sequences,

$$0 \longrightarrow L_n \longrightarrow J_{\mathcal{X},n+1} \longrightarrow \Omega_{P_n/P_{n-1}} \longrightarrow 0\ ,\ n \in \mathbb{N}$$

giving rise to distinguished divisors $\infty_{n+1} \hookrightarrow P_{n+1}$, linearly equivalent to $L_{n+1} - L_n$. Notation established, up to dropping $\mathcal{X}$ which will be fixed, we assert,

VI.1.3 Claim *Let* V_n^* *be* V_n *complemented by the pull-back of the zero section from* V_1 *then* $\mathit{Aut}(\Delta_n) \times V_n^* \rightrightarrows \mathit{Aut}(\Delta_n)$ *is a representable groupoid and* $[V_n^*/\mathit{Aut}(\Delta_n)]$ *exists as an algebraic stack and is isomorphic to* $P_n \backslash \bigcup_{n \geq p \geq 2} \pi_{pn}^{-1}(\infty_p)$*.*

proof Consider first the local question, i.e. $\mathcal{X}$ an affine, then $\mathrm{Aut}(\Delta_n)$ acts on V_n^* without stabilisers, so a fine quotient exists by [KM]. Better still the lack of stabilisers automatically means everything globalises so $[V_n^*/\mathrm{Aut}(\Delta_n)]$ exists as an algebraic stack, and the projection from V_n^* is flat, so the quotient is even smooth if V_n is, i.e. if $\mathcal{X}$ is smooth. Furthermore the inductive definition of the P_n gives a map by way of nth derivation $V_n^* \to P_n$, and one easily checks that this induces an isomorphism on closed points,

$$[V_n^*/\mathrm{Aut}(\Delta_n)](\mathbb{C}) \xrightarrow{\sim} P_n \setminus \bigcup_{n \geq p \geq 2} \pi_{pn}^{-1}(\infty_p)(\mathbb{C})$$

and since everything is smooth, we conclude. $\square$

We wish to make this rather more explicit, so to this end observe that $\mathrm{Aut}(\Delta_n)$ maps naturally to $\mathrm{Aut}(\Delta_{n-1})$ with kernel $\mathbb{G}_a$ for $n \geq 2$, and of course $\mathrm{Aut}(\Delta_1)$ is isomorphic to $\mathbb{G}_m$. There's also an embedding of $\mathbb{G}_m$ in $\mathrm{Aut}(\Delta_n)$ by way of $z \mapsto \lambda z$, but this isn't a normal subgroup, so we have to be a little cautious as to how we use it. Nevertheless, we'll frequently identify $\mathbb{G}_m$ with its image in $\mathrm{Aut}(\Delta_n)$, and use the embedding of $\mathcal{O}_{V_n}$ that it provides, i.e.

$$\mathcal{O}_{V_n}(m) := \{f \in \mathcal{O}_{V_n} \ : \ f^\lambda = \lambda^m f, \, \lambda \in \mathbb{G}_m\}$$

In any case, we can use this to construct the quotient inductively, so that we get diagrams of the form,

$$\begin{array}{ccccc} V_n^*/\mathbb{G}_a & \longrightarrow & V_n^*/\mathrm{Aut}(\Delta_n) & \longrightarrow & P_n \setminus \bigcup_{n \geq p \geq 2} \pi_{pn}^{-1}(\infty_p) \\ \downarrow & & \downarrow & & \downarrow \\ V_{n-1}^* & \longrightarrow & V_{n-1}^*/\mathrm{Aut}(\Delta_{n-1}) & \longrightarrow & P_{n-1} \setminus \bigcup_{n-1 \geq p \geq 2} \pi_{pn}^{-1}(\infty_p) \end{array}$$

where the last square is fibred by virtue of flat descent. Now let's simplify the notation a little by putting ∞_p equal to $\pi_{ip}^{-1}(\infty_p)$, irrespective of $i > p$, A_n the kernel of the natural map from $\mathrm{Aut}(\Delta_n)$ to $\mathrm{Aut}(\Delta_1)$, and note the isomorphism,

$$\mathcal{O}_{V_n^*}^{A_n} \xrightarrow{\sim} \mathcal{O}_{V_1^*} \otimes_{\mathcal{O}_{P_1}} \varinjlim_{\mathbf{t}} \mathcal{O}_{P_1}(t_2\infty_2 + \ldots + t_n\infty_n)$$

where, just to clear up notation , $\mathcal{O}_{P_1}(t_2\infty_2 + \ldots + t_n\infty_n)$ is just the push forward under π_{n1} of $\mathcal{O}_{P_n}(t_2\infty_2 + \ldots + t_n\infty_n)$. At this point $\mathbb{G}_m$ acts on the whole thing, so we get a map,

$$\mathcal{O}_{V_n^*}^{A_n} \cap \mathcal{O}_{V_n}(t_1) \longrightarrow \pi_* L_1^{t_1}(\varinjlim_{\mathbf{t}} t_2\infty_2 + \ldots + t_n\infty_n)$$

Now unfortunately A_n doesn't act on the individual $\mathcal{O}_{V_n}(t)$'s, but it acts on the suns of such, so that finally we get a map,

$$(\bigoplus_{t_1 \le m} H^0(\mathcal{X}, \pi_* \mathcal{O}_{V_n}(t_1)))^{A_n} \to \bigoplus_{t_1 \le m} \varinjlim_{(t_2,\dots,t_n)} H^0(\mathcal{X}, \pi_* L_1^{t_1}(t_2\infty_2 + \dots t_n\infty_n))$$

where the directed limit structure comes from tensoring with various canonical sections $\mathbb{I}_{\infty_p}$, so certainly all the implied maps on the right are injective. Better still we don't need the full range of $(t_2, \dots, t_n)$ going all the way to infinity in order to describe where an element on the left may end up. Indeed if $x_1, \dots, x_l$ are local coordinates on $\mathcal{X}$, then, in an obvious notation we get coordinates $d^k x_j$, $O \le k \le n$ on V_n where $d^k x_j$ of a jet is its kth, or more correctly $1/k!$ times it, derivative of its jth projection. Whence elements of $\pi_* \mathcal{O}_{V_n}(t_1)$ are generated as an $\mathcal{O}_{\mathcal{X}}$ module by elements of the form,

$$(d^1\mathbf{x})^{P_1} \dots (d^n\mathbf{x})^{P_n}$$

where P_i is some multi-indice, so $(d^1\mathbf{x}) = \prod_{j=1}^{l} (d^1 x_j)^{p_{1j}}$, etc., with $|P_i| = \sum_j p_{ij}$, and necessarily $|P_1| + 2|P_2| + \dots + n|P_n| = t_1$. In order to compute the order of pole of such a thing around any ∞_p, we can proceed using appropriate liftings to $\mathrm{Aut}(\Delta_n)$ of the various kernels $\mathrm{Aut}(\Delta_p) \to \mathrm{Aut}(\Delta_{p-1})$, $2 \le p \le n$, say, $\xi_p(z) = z + \xi z^p$, and observe that each ξ_p acts on the basis $d^k x_j$ not just in an upper semi-triangular way, but fixing the $d^k x_j$ for $k < p$. Suppose now, and without loss of generality, that we're working on the affine patch $d^1 x_1 \ne 0$, then we require to express the $d^k x_j$, $2 \le k \le n$, $2 \le i \le l$ in terms of the $d^1 x_j / d^1 x_1$, the eigenfunctions corresponding to the divisors at infinity, and $d^k x_1$'s. We know, however, that the quotient exists and is smooth, so in the maximal ideal of a suitable point, say, $d^k x_j =$ for $k, j \ne 1$, the eigenfunctions cannot be in the square of the maximal ideal, since there's obviously at least one which isn't, and since the action is already linearisable on an affine by way of the ξ_p's we must be able to express the $d^k x_j$'s $2 \le k \le n$, $2 \le i \le l$ as a linear $\mathcal{O}_{V_1^*}$ sum of the $d^k x_1$'s and the eigenfunctions. Consequently the pole of an element of $\mathcal{O}_{V_n}(t_1)$ which happened to be invariant under A_n around ∞_i would be at worst $p_i + p_{i+1} + \dots + p_n$, for some integers p_i satisfying $p_1 + 2p_2 + \dots + np_n = t_1$, so in particular, no bigger than t_1. As such after tensoring with appropriate $\mathbb{I}_{\infty_p}$'s we finally obtain a map,

$$(\bigoplus_{t_1 \le m} H^0(\mathcal{X}, \pi_* \mathcal{O}_{V_n}(t_1)))^{A_n} \longrightarrow \bigoplus_{t_1 \le m} H^0(P_n, L_n^{\otimes t_1})$$

The advantage of this over the inductive definition of the L_i is that the left hand side is rather computable. Indeed the Euler characteristic of $\bigoplus_{t_1 \le m} \pi_* \mathcal{O}_{V_n}(t_1)$ is for m sufficiently large and divisible, asymptotic to,

$$\frac{m^{dim \mathcal{X}(n+1)}}{\{(n+1) dim \mathcal{X}\}! (n!)^{dim \mathcal{X}}} {}^n C_{dim \mathcal{X}} (K_{\mathcal{X}}^{top} + O(\frac{1}{n}))$$

with appropriate corrections as per [T] when things aren't sufficiently divisible. Better still if $\mathcal{X}$ has small dimension, i.e. 2, we can compute the higher cohomology rather easily thanks to the semi-stability of $\Omega_{\mathcal{X}}$ with respect $K_{\mathcal{X}}$. Indeed this is effectively what's done in [GG], and it works in this generality thanks to [TY] or [Me]. Whence,

VI.1.4 Fact (cf. [GG]) *Let $\mathcal{S}$ be a minimal 2-dimensional algebraic stack, then for m sufficiently large, about 9 will do,*

$$\bigoplus_{t_1=0}^{\infty} H^0(\mathcal{S}, \pi_* \mathcal{O}_{V_n}(t_1))$$

is an algebra of transcendence degree $2(n+1)$.

Better still on taking A_n invariants we can profit from its filtration by $\mathbb{G}_a$'s to conclude that we loose precisely n dimensions. Infact we even see that the tautological bundles L_n are big provided $c_1^2 > \frac{2}{n+1} c_2$, which isn't bad given the approximations that we've employed, albeit that with a little fooling around for small n one sees that definition VI.1.2 allows direct computation of the Euler characteristic of sums of the L_i's, which indeed hovers around $2/n+1$. Notice also that a similar, and rather easier, argument shows with the hypothesis of VI.1.4 that $L_n|_{P_n \otimes \mathcal{K}}$ is big for any n with $\mathcal{K}$ an appropriately general element of a high multiple of the linear system $K_{\mathcal{S}}$, and so we have,

VI.1.5 Fact/Summary *Let $\mathcal{S}$ be a smooth 2 dimensional algebraic stack with projective moduli, which is minimal and of general type, then its nth tautological jet bundle L_n is big for $c_1^2(\mathcal{S}) > \frac{2}{n+1} c_2(\mathcal{S})$. In addition for $\mathcal{K} \hookrightarrow \mathcal{S}$ a generic closed substack defined by a sufficiently large multiple of $K_{\mathcal{S}}$, $L_n|_{P_n \otimes \mathcal{K}}$ is big for any n.*

VI.2 Value Distribution

We wish to study the value distribution of holomorphic lines, or more generally ramified covers of the same around divisors in surfaces. It is, therefore, notationally and conceptually convenient to pretend that the punctured disc is a stack. More precisely rather than talking about log-stacks $(\mathcal{S}, \mathcal{D})$ we'll make,

VI.2.1 Convention/Definition *Unless there is a possibility for confusion we'll often use the word stack to include log-stacks $(\mathcal{S}, \mathcal{D})$, i.e. if $\mathcal{D}$ is a simple normal crossing divisor, we'll think of this as a stack $\tilde{\mathcal{S}}$ with infinite stabiliser around $\mathcal{D}$ in just the same way that the classical theory of 1-dimensional orbifolds thinks of a punctured disc. Technically speaking $\tilde{\mathcal{S}}$ doesn't exists even in the analytic topology, but the only thing that we'll have to be careful about is that for $f : \mathcal{C} \to \mathcal{S}$ a map between stacks in this broad sense, the pre-images of points with infinite stabiliser should have the same property, while the generic stabiliser should never be infinite.*

This is, as we've said, a notational/conceptual issue which allows us to re-phrase value distribution theory in terms of maps between stacks. In this respect let's note that for $\mathcal{C}$ a 'smooth complete curve' in the sense of VI.2.1, and L_n the tautological bundle on any jet space $\pi : P_n \to \mathcal{X}$ over a stack of any dimension, with $f^{(n)} : \mathcal{C} \to P_n$ the derivative of a map $f : \mathcal{C} \to \mathcal{X}$ we have,

$$L_{n \cdot f^{(n)}} \mathcal{C} \leq -\chi_{\mathcal{C}}^{top}$$

where $\chi_{\mathcal{C}}^{top}$ is the topological Euler characteristic of $\mathcal{C}$. The log-case of VI.1.5 is known for $\mathcal{S}$ a minimal log-stack of general type, so for H ample on the moduli,

$$H._f\mathcal{C} \leq \frac{1}{\kappa_{\mathcal{S}}}\chi_{\mathcal{C}}^{top}$$

for some $\kappa_{\mathcal{S}} < 0$, unless $f^{(n)}$ factors through the base locus of some linear system on the form $L_n^N \otimes \pi_n^* H^\vee$, for n, and N sufficiently large, i.e f satisfies a nth order ODE on $\mathcal{S}$.

We can argue similarly not just for a complete curve $\mathcal{C}$ and algebraic maps f, but for ramified covers $p : \mathcal{C} \to \mathcal{S}$. In this context the Euler characteristic is a function of $r \in \mathbb{R}_+$, which for $\mathcal{C}(r)$ the pre-image of the disc of radius r is given by,

$$-\chi_{\mathcal{C}}(r) = -\deg(p)\log r + \sum_{0<|p(c)|<r} \log|\frac{r}{p(c)}|\mathrm{ord}_c(\mathrm{Ram}_p)$$

where, for simplicity, we suppose p étale over 0, and conventionally $\mathrm{ord}_c(\mathrm{Ram}_p) = 1$ if the stabiliser is infinite. With this in mind, and $f^{(n)}$ the nth derivative we have,

$$\fint_{\mathcal{C}(r)} f^{(n)*}c_1(L_n) \leq_{exc} -\chi_{\mathcal{C}}(r) + \text{(small error)}$$

where the integral on the left is the standard Nevanlinna sort of integral, and the subscript exc. means r outside a set of finite Lebesgue measure, cf. [III] I.2.13 & I.4.4. Whence, again if $f^{(n)}$ doesn't factor through the base locus we get,

$$\fint_{\mathcal{C}(r)} f^*c_1(H) \leq_{exc} \frac{1}{\kappa_{\mathcal{S}}}\chi_{\mathcal{C}}(r) + \text{(small error)}$$

There is also a similar statement for maps of discs to $\mathcal{S}$, but unfortunately it's only valid for discs that are not arbitrarily close in the compact open sense to the base locus. In particular, and rather unfortunately, one doesn't have the situation that one finds above whereby discs failing to satisfy an analogue to the latter inequality would have to factor through at least one irreducible component of the base, nor indeed do big discs even need to accumulate on an irreducible component, but only a connected one. This suggests that we make,

VI.2.2 Definition *Say that there is an isoperemtric inequality in Nevanlinna's sense for $\mathcal{S}$ if there is a proper closed substack $\mathcal{Z}$, and a constant $\kappa_{\mathcal{S}} < 0$ such that for every map $f : \mathcal{C}/\mathbb{C} \to \mathcal{S}$ from a ramified covering of the line not factoring through $\mathcal{Z}$ we have,*

$$\fint_{\mathcal{C}(r)} f^*c_1(H) \leq_{exc} \frac{1}{\kappa_{\mathcal{S}}}\chi_{\mathcal{C}}(r)$$

Notice that the given formulation implies the boundedness in moduli of maps from honest complete curves. Either way if the inequality is false, there is some sequence of maps f_m such that as $m \to \infty$,

$$\chi_{f_m} := \inf_{\substack{E\subset\mathbb{R}_+\\ \lambda(E)<\infty}} \sup_{r\notin E} \chi_{\mathcal{C}_m}(r)\{\fint_{\mathcal{C}_m(r)} f_m^* c_1(H)\}^{-1}$$

where λ is ordinary Lebesgue measure, without loss of generality, increases and satisfies $\lim_m \chi_{f_m} \geq 0$. Now quite generally given a sequence of maps g_m from appropriate $\mathcal{C}_m$'s to a stack $\mathcal{X}$ we can for $N \in \mathbb{N}$ define the Zariski closure $\mathcal{V}_N$ of g_m, $m \geq N$, to get a decreasing chain,

$$\ldots \supseteq \mathcal{V}_N \supseteq \mathcal{V}_{N+1} \supseteq \ldots$$

of reduced stacks, which must eventually stabilise at some $\mathcal{V}_\infty$, which we call the Zariski closure of the sequence. In the particular case that the isoperimetric inequality in Nevanlinna's sense is supposed to fail on a surface $\mathcal{S}$, we can find a divisor $\mathcal{D}$ in some jet space P_n containing the Zariski closure $f_m^{(n)}$. Either $\mathcal{D}$ is the closure of this sequence, or it is not, if it's not we take a component $\mathcal{D}^1$ of the closure which contains an infinite subsequence. Notice that $\mathcal{D}^1$ itself dominates some P_i for $i \leq n-2$, or maps to an algebraic curve in $\mathcal{S}$. Either way we can replace $\mathcal{D}^1$ by the divisor in the corresponding P_{i+1}, respectively an algebraic curve. There is some minor possibility that the $f_m^{(i+1)}$'s aren't Zariski dense in this replacement, but we'll just continue until we eventually do get some ODE of order $i+1$ (including the case $i=-1$) such that an infinite subsequence of the f_m^{i+1}'s factor through, and are Zariski dense, in the ODE considered as a divisor in P_{i+1}. Plainly by the same subterfuge be it for $\mathcal{D}$, or something smaller, we can even insist that every subsequence of the appropriate derivatives of the f_m's is also Zariski dense. As such, cleaning up the notations by supposing that the $f_m^{(n)}$'s and all subsequences thereof are dense in the nth order ODE $\mathcal{D}$, we first discuss the order 0 case. One of two things can happen: either the 0th order ODE is a curve of genus $g \geq 2$, more generally has normalisation étale covered by a complete 1-dimensional hyperbolic orbifold, and we immediately get a contradiction by the so called second main theorem of Nevanlinna theory, or it's rational or elliptic. In the latter case all such curves have to be included in $\mathcal{Z}$ anyway, so modulo the set of all such being finite we have a contradiction again. Whence supposing the said finiteness, our ODE is of order at least 1, and we have a foliation by curves of the form $(\mathcal{D}, L_n|_{\mathcal{D}})$. Supposing LCR, I.6.1, we can clean this up to a model $\pi : (\mathcal{X}, \mathcal{F}) \to (\mathcal{D}, L_n|_{\mathcal{D}})$ with log-canonical singularities, and no radial singularities, then pass to a minimal model $\rho : (\mathcal{X}, \mathcal{F}) \dashrightarrow (\mathcal{X}_0, \mathcal{F}_0)$. The semi-stability of $\Omega_{\mathcal{S}}$ with respect to $K_{\mathcal{S}}$ continues to imply that $K_{\mathcal{F}}$ is big on a generic cut by some divisor in a high multiple of the $K_{\mathcal{S}}$ linear system, so $K_{\mathcal{F}_0}$ has Kodaira dimension at least $n = \dim \mathcal{X} - 1$. Furthermore $K_{\mathcal{F}}$ and $K_{\mathcal{X}}$ contain in the cone they span a big divisor, so the same is true for $K_{\mathcal{F}_0}$ and $K_{\mathcal{X}_0}$. Indeed we can suppose that $\mathcal{F}_0$ is not of general type, and harmlessly replace $\mathcal{X}$ by a modification that actually maps to $\mathcal{X}_0$, so we must have $K_{\mathcal{X}_0}.K_{\mathcal{F}_0}^{n-1} > 0$, which by II.2.1 is enough to guarantee that for some $\epsilon > 0$, $K_{\mathcal{F}_0} + \epsilon K_{\mathcal{X}_0}$ is big. Now in carrying out these various modifications in the 2-category of stacks we may have to introduce some extra monodromy, whence the various maps $f_m^{(n)}$ may only lift, be it on $\mathcal{X}$ or $\mathcal{X}_0$ on some ramified cover $\tilde{\mathcal{C}}_m$ of

$\mathcal{C}_m$. At which point we may get some slightly different Euler characteristics $\chi_{\tilde{f}_m}$ in the obvious notation. The difference between the limiting values of χ_{f_m} and $\chi_{\tilde{f}_m}$ is completely dealt with by the various computations of how invariant maps meet singularities in [III] V.4., and by op. cit. it's still the case that $\lim_m \chi_{\tilde{f}_m} \geq 0$, where, without loss of generality, we sub-sequence so that these increase. Now define harmonic measures on $\mathcal{X}_0$ by the formulae,

$$T_m(r) : A^{(1,1)}(\mathcal{X}_0) \longrightarrow \mathbb{C} : \omega \longmapsto (\fint_{\mathcal{C}_m(r)} f^* c_1(H))^{-1} \fint_{\mathcal{C}_m(r)} \tilde{f}^* \omega$$

after possible subsequencing to dump any $\tilde{f}_m$'s lying in the loci which is modified during the minimal model algorithm, we can then subsequence in m and r to obtain a closed transverse invariant measure $d\mu_{\mathcal{X}_0/\mathcal{F}_0}$ satisfying,

$$d\mu_{\mathcal{X}_0/\mathcal{F}_0} = \lim T_m(r) \ , \ K_{\mathcal{F}_0}.d\mu_{\mathcal{X}_0/\mathcal{F}_0} = 0$$

where the latter inequality follows from [III] II.3.1. The measure $d\mu_{\mathcal{X}_0/\mathcal{F}_0}$ is therefore parabolic satisfying all of the hypothesis of V.1.2, including the bis., V.1.5. Similarly there is a measure $d\mu_{\mathcal{X}/\mathcal{F}}$ on $\mathcal{X}$, or indeed any invariant modification of it, satisfying $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$, so $K_{\mathcal{F}}.d\mu_{\mathcal{X}/\mathcal{F}} = 0$ since $d\mu_{\mathcal{X}/\mathcal{F}}$ intersects every Cartier divisor non-negatively. Furthermore any part of $d\mu_{\mathcal{X}/\mathcal{F}}$ in the loci modified by the minimal model algorithm (cf. the next section for a precise definition of 'any part') must push forward by the projection π to $\mathcal{S}$ to zero or one of finitely many rational curves. Consequently it is notationally unambiguous to assert,

VI.2.3 Claim *Suppose LCR* I.6.1 *then $\pi_* d\mu_{\mathcal{X}_0/\mathcal{F}_0}$ is at worst a countable sum of rational or elliptic curves, i.e. $\pi_* d\mu_{\mathcal{X}_0/\mathcal{F}_0} = \sum a_i \mathcal{L}_i$, for $\mathcal{L}_i$ the current of integration over a closed substack of $\mathcal{S}$ whose normalisation has non-negative topological Euler characteristic.*

The proof of the claim will occupy the next several sections. Curiously,

VI.2.4 Remark *If similarly we suppose that solutions of the initial ODE, considered as invariant discs did not converge in the compact open topology provided they weren't arbitrarily close to some closed substack of $\mathcal{D}$, i.e. almost solutions of another ODE, then again by [III] II.3.1 we'd be able to produce an invariant measure for $\mathcal{F}_0$ with the same properties, and whence the same claim holds. Nevertheless it doesn't seem to follow from this rather strong hyperbolicity of solutions to the ODE that we get VI.2.2 due to the possible degeneracies in the associated Kobayashi type metric.*

VI.3 Inductive Decomposition

Quite generally when presented with a measure $d\mu$, say of type $(1,1)$ to fix ideas on a stack $\mathcal{X}$ with projective moduli $\pi : \mathcal{X} \to X$ we can decompose into smaller pieces. More precisely if we filter the chow scheme of X (bear in mind the existence of the Hilbert scheme of $\mathcal{X}$ isn't actually a theorem, whence the subtrefuge) by dimension d and degree δ, and starting in dimension 1 ask,

VI.3.1 Question *For fixed δ, $\epsilon > 0$ and H ample, how many curves $\mathcal{C}$ of degree δ satisfy $H.\mathbb{I}_{\mathcal{C}} d\mu \geq \epsilon$?*

Plainly the answer is necessarily finite, since if $\mathcal{C}_i, i \in I$ is the collection of such curves then since $1\!\!1_{\mathcal{C}_i \cap \mathcal{C}_j} d\mu = 0$, we must have $H.d\mu \geq \# I \epsilon$, and so independently of the degree we find countably many curves $\mathcal{C}_i$ with $1\!\!1_{\mathcal{C}_i} d\mu \neq 0$, and we can replace $d\mu$ by $d\mu^1$ defined by,

$$d\mu^1 := d\mu - \sum_i 1\!\!1_{\mathcal{C}_i} d\mu$$

Now by construction $d\mu^1$ has no support on curves and we can ask,

VI.3.2 Question bis. *For fixed δ, $\epsilon > 0$ how many surfaces $\mathcal{S}$ of degree δ satisfy $H.1\!\!1_{\mathcal{S}} d\mu^1 \geq \epsilon$?*

At this juncture if we have two distinct surfaces $\mathcal{S}_i, \mathcal{S}_j$ then $\mathcal{S}_i \cap \mathcal{S}_j$ has dimension at most 1 so $1\!\!1_{\mathcal{S}_i \cap \mathcal{S}_j} d\mu^1 = 0$, so that again we find, independently of the degree countably many such surfaces $\mathcal{S}_i$, and we write,

$$d\mu^2 := d\mu^1 - \sum_i 1\!\!1_{\mathcal{S}_i} d\mu^1$$

and, of course, $d\mu^2$ cannot have support on either curves or surfaces. This process can certainly be continued inductively to obtain,

VI.3.3 Fact *In each dimension d there are countably irreducible substacks $\mathcal{Y}_i^d$ satisfying $1\!\!1_{\mathcal{Y}_i^d} d\mu^{d-1} \neq 0$ where $d\mu^e$ is defined inductively by,*

$$d\mu^e := d\mu^{e-1} - \sum_i 1\!\!1_{\mathcal{Y}_i^e} d\mu^{e-1}$$

In particular by III.3.3, *each $1\!\!1_{\mathcal{Y}_i^e} d\mu^{e-1}$ is a closed positive current on $\mathcal{Y}_i^e$, and not just a current on $\mathcal{X}$ with support.*

It's evidently natural enough to refer to the terms $1\!\!1_{\mathcal{Y}_i^e} d\mu^{e-1}$ appearing in the implied decomposition of a measure as its *components*. Notice in particular, by construction, a component which we'll denote $1\!\!1^0_{\mathcal{Y}_i^e} d\mu$ satisfies $1\!\!1_{\mathcal{V}}.1\!\!1^0_{\mathcal{Y}_i^e} d\mu = 0$ for any substack $\mathcal{V}$ not containing $\mathcal{Y}_i^e$.

Now if we apply the discussion in the presence of a foliation $\mathcal{F}$ together with transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ then it goes without saying that all the components are supported on invariant subvarieties. If moreover, the foliation arises from an ODE $\mathcal{D}$ of order n on an algebraic surface $\mathcal{S}$ then we have more structure still. Specifically for P_n the nth jet bundle we have, on supposing LCR, a situation of the form,

$$\begin{array}{ccc} & (\mathcal{X}, \mathcal{F}) & \\ \sigma \swarrow & \downarrow \rho & \\ & & \\ P_n \hookleftarrow & X_n & (= \mathcal{D}) \end{array}$$

where $\rho : (\mathcal{X}, \mathcal{F}) \to X_n$ is a resolution of the induced foliation by curves on X_n. Consequently if $d\nu$ is a component of $d\mu_{\mathcal{X}/\mathcal{F}}$ supported on a substack $\mathcal{Y}$, then $\sigma(\mathcal{Y})$ defines a subvariety of P_n. One can then ask for the necessarily unique P_i

(including, by the way, P_0, i.e. our original surface) such that the image of $\sigma(\mathcal{Y})$ is a divisor dominating P_{i-1} or, alternatively $\mathcal{Y}$ simply pushes forward to a point. This ultimate degenerate case is plainly agreeable, since it says that our component on $\mathcal{X}$ has nothing to do with $\mathcal{S}$, while the other cases allow us to think of $\mathcal{Y}$ itself as an ODE, $\mathcal{D}'$, say, of smaller order m. Plainly we can use LCR to form diagrams of the form,

$$\begin{array}{ccc} (\mathcal{X},\mathcal{F}) & \xleftarrow[\iota]{} & (\mathcal{Y},\mathcal{F}|_{\mathcal{Y}}) \\ \downarrow & & \downarrow \\ P_n & \longrightarrow & P_m \leftarrow (\mathcal{Y}_0,\mathcal{G}) \end{array}$$

where ι is an embedding, $(\mathcal{Y}_0,\mathcal{G})$ is a resolution of the foliation singularities of the mth order ODE defined by $\mathcal{Y}$, and $\mathcal{X}$ a possibly bigger modification of our original $\mathcal{X}$ in order to guarantee that $(\mathcal{Y},\mathcal{F}|_{\mathcal{Y}})$ maps to $\mathcal{Y}_0$. In any case the important thing is that the components of a transverse invariant measure associated to such an ODE are themselves naturally what might be termed dense, or better *diffuse* components of another ODE, i.e. of the form $d\mu^t op$, in the notation of VI.3.3. Plainly, however, this discussion is completely useless unless we can control the diffuse component. To this end we have,

VI.3.4 Fact *Suppose LCR & let $\sigma : (\mathcal{X},\mathcal{F}) \to P_n$ be a resolution of singularities of an ODE $\mathcal{D}$ of order $n \geq 1$ on an algebraic surface with $d\nu$ the diffuse component of a transverse invariant measure with $K_{\mathcal{F}}.d\nu \leq 0$, then infact $d\nu$ doesn't exist.*

proof By [II] IV.7.4 there is a birational map $\rho : (\mathcal{X},\mathcal{F}) \dashrightarrow (\mathcal{X}_0,\mathcal{F}_0)$ by way of contractions and flips to a minimal model, such that $K_{\mathcal{F}} = \rho^* K_{\mathcal{F}_0} + \mathcal{E}_0$ for $\mathcal{E}_0 \subset \mathcal{X}$ a divisor supported on invariant substacks, and on which the induced foliation is even by rational curves. Since $d\nu$ is diffuse, $\mathcal{E}_0.d\nu \geq 0$, so infact $0 = K_{\mathcal{F}_0}.\rho_* d\nu = \mathcal{E}_0.d\nu$. On the other hand, as remarked prior to VI.2.3 the cone generated by $K_{\mathcal{F}}$ & $K_{\mathcal{X}}$, or better $K_{\mathcal{F}_0}$ & $K_{\mathcal{X}_0}$, contains a big divisor, so $K_{\mathcal{X}}.d\nu = (K_{\mathcal{X}} - K_{\mathcal{F}}).d\nu > 0$. Whence by IV.7.3 there is a sub-stack $\mathcal{Z} \hookrightarrow \mathcal{X}$ of co-dimension at least 2 such that the residual measure $s_{\mathcal{Z},d\nu} \neq 0$. Now form the blow up $\tilde{\rho} : (\tilde{\mathcal{X}},\tilde{\mathcal{F}}) \to (\mathcal{X},\mathcal{F})$, or if one prefers resolve $\mathcal{I}_{\mathcal{Z}}$ by a sequence of blow ups in smooth invariant centres, and let $\mathcal{E}$ be the total exceptional divisor with $d\tilde{\nu}$ the extension of $d\nu$ across $\mathcal{E}$ by zero, so that $s_{\mathcal{Z},d\nu}(1) = \mathcal{E}.d\tilde{\nu}$. On the other hand $d\tilde{\nu}$ still intersects every Cartier divisor $\mathcal{D}$ on $\tilde{\mathcal{X}}$ non-negatively. Indeed if $\mathcal{D}.d\tilde{\nu} < 0$, then $\mathbb{I}_{\mathcal{D}} d\tilde{\nu} \neq 0$. At the same time $d\tilde{\nu}$ has by construction no support on $\mathcal{E}$, so $\tilde{\rho}$ must be bi-rational at the generic point of $\mathcal{D}$, whence $\mathbb{I}_{\rho(\mathcal{D})} d\nu \neq 0$, contrary to it's construction. Better still by VI.1.5, $K_{\mathcal{F}_0}$ has Kodaira dimension at least $\dim\mathcal{X} - 1$, so arguing as per V.2.2 we deduce an absurdity. □

Which extends to,

VI.3.5 Refined Fact *Suppose LCR & let $\sigma : (\mathcal{X},\mathcal{F}) \to P_n$ be a resolution of an ODE $\mathcal{D}$ of order $n \geq 1$ with $d\mu_{\mathcal{X}/\mathcal{F}}$ a parabolic invariant measure, then $d\mu_{\mathcal{X}/\mathcal{F}}$ has no components supported in the smooth locus of $\mathcal{F}$ other than those defined by ODEs of order at most 0 (so say conventionally points have order -1), and those of order 0 are rational or elliptic curves on $\mathcal{S}$.*

proof Let $d\nu$ be a component supported on an invariant substack $\mathcal{Y}$ whose generic point does not lie in $\mathrm{sing}(\mathcal{F})$. Furthermore let $\rho : (\mathcal{X}, \mathcal{F}) \dashrightarrow (\mathcal{X}_0, \mathcal{F}_0)$ be a minimal model as before with $K_{\mathcal{F}} = \rho^* K_{\mathcal{F}_0} + \mathcal{E}_0$. Since $d\mu_{\mathcal{X}/\mathcal{F}}$ is parabolic $K_{\mathcal{F}_0}.\rho_* d\mu_{\mathcal{X}/\mathcal{F}} = 0$. Notice, moreover, that at every stage of the minimal model algorithm one either contracts, or flips, an invariant subvariety on which the induced foliation is by rational curves. In particular unless $\mathcal{Y}$ pushes forward to a rational curve or a point on our surface $\mathcal{S}$, it's generic point is unaffected by the algorithm, and we get a component $d\nu_0$ of $\rho_* d\mu_{\mathcal{X}/\mathcal{F}}$ which necessarily satisfies $K_{\mathcal{F}_0}.d\nu_0 = 0$.

Better still if $(\mathcal{Y}_0, \mathcal{F}_0|_{\mathcal{Y}_0})$ is the induced foliation, and $\sigma^\# : (\mathcal{V}, \mathcal{G}) \to P_m$ a resolution of singularities of the induced ODE, supposed of non-negative order, we have a rational map (which may be highly non-birational) of foliated varieties $(\mathcal{Y}_0, \mathcal{F}_0|_{\mathcal{Y}_0}) \dashrightarrow (\mathcal{V}, \mathcal{G})$, so that if $\rho^\# : (\mathcal{V}, \mathcal{G}) \dashrightarrow (\mathcal{V}_0, \mathcal{G}_0)$ is a minimal model, we can appeal to [II] IV.11.3 to form a diagram,

$$\begin{array}{rccc} & (\tilde{\mathcal{X}}_0, \mathcal{F}_0) \hookleftarrow & (\tilde{\mathcal{Y}}_0, \mathcal{F}_0|_{\tilde{\mathcal{Y}}_0}) \underset{\tilde{\sigma}^\#}{\rightarrow} & (\mathcal{V}_0, \mathcal{G}_0) \\ \text{invariant modification} = \tilde{\rho} \downarrow & & \downarrow & \uparrow \\ & (\mathcal{X}_0, \mathcal{F}_0) \hookleftarrow & (\mathcal{Y}_0, \mathcal{F}_0|_{\mathcal{Y}_0}) \dashrightarrow & (\mathcal{V}, \mathcal{G}) \end{array}$$

where $\tilde{\sigma}^\#$ is an honest map. Now for $\tilde{\mathcal{G}}_0$ the induced foliation on $\tilde{\mathcal{Y}}_0$ which we may suppose smooth with canonical singularities, we have,

$$K_{\tilde{\mathcal{G}}_0} = K_{\mathcal{F}_0}|_{\tilde{\mathcal{Y}}_0} - \mathcal{D}$$

for $\mathcal{D}$ some effective Cartier divisor supported on $\tilde{\mathcal{Y}}_0$. Furthermore we have a map from $\tilde{\sigma}^\# K_{\mathcal{G}_0}$ to $K_{\tilde{\mathcal{G}}_0}$ away from $\mathrm{sing}(\mathcal{G}_0)$ so at worst,

$$K_{\tilde{\mathcal{G}}_0} = \tilde{\sigma}^{\#*} K_{\mathcal{G}_0} + \mathcal{D}_+ - \mathcal{D}_-$$

where the Cartier divisor $\mathcal{D}_-$ maps down to $\mathrm{sing}(\mathcal{G}_0)$. Now take an irreducible component D_- of $\mathcal{D}_-$ and inductively blow up in the reduced structure of its image until such times that we get a map of germs,

$$\mathrm{Spec}\ \mathcal{O}_{\tilde{\mathcal{Y}}_0, D_-} \underset{\tilde{\sigma}^\#}{\longrightarrow} \mathrm{Spec}\ \mathcal{O}_{\tilde{\mathcal{V}}_0, E_-}$$

for E_- a Cartier divisor on an appropriate modification of $\mathcal{V}_0$. In terms of the induced canonical bundle $K_{\tilde{\mathcal{G}}_0^\#}$ on $\tilde{\mathcal{V}}_0$ we must have an honest map $\tilde{\sigma}^{\#*} K_{\tilde{\mathcal{G}}_0^\#}$ to $K_{\tilde{\mathcal{G}}_0}$ around D_-, and since the singularities of $(\mathcal{V}_0, \mathcal{G}_0)$ are canonical we conclude that D_- doesn't exist. Consequently,

$$K_{\mathcal{F}_0}|_{\tilde{\mathcal{Y}}_0} = K_{\tilde{\mathcal{G}}_0} + \mathcal{D} = \tilde{\sigma}^{\#*} K_{\mathcal{G}_0} + \mathcal{D} + \mathcal{D}_+$$

Now $d\nu_0$ is diffuse in $\mathcal{Y}_0$, so infact $K_{\mathcal{G}_0}.(\tilde{\sigma}^\#)_* d\nu_0 = 0$, and we can appeal to VI.3.4, or slightly more correctly its proof, to deduce that $d\nu_0$ cannot exist for an ODE of order at least 1, while if the order is zero, the minimal model is just the normalisation of the corresponding 1-dimensional substack of $\mathcal{S}$, so it must be rational or elliptic. $\square$

VI.4 Singular Components

To complement our previous investigation, we have to consider the possibility that a transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ on a foliated stack $(\mathcal{X}, \mathcal{F})$ develops components inside the singular locus itself, so, say a closed irreducible substack $\mathcal{Z} \hookrightarrow \mathrm{sing}(\mathcal{F})$. Now the definition here is that all smooth 1-forms vanishing in $K_{\mathcal{F}}$ vanish on $\mathcal{Z}$, which is not particularly easy to work with. As such the following is rather useful, i.e.

VI.4.1 Lemma (local on $\mathcal{X}$) *Let $\mathcal{V}$ be a closed invariant irreducible substack containing $\mathcal{Z}$ such that for ∂ a local generator of the foliation $\partial(\mathcal{I}_{\mathcal{V}})$ generates $\mathcal{I}_{\mathcal{V}}$, then $d\mu$ is invariant by the induced foliation on $\mathcal{V}$.*

proof Let ω be a 1-form mapping to zero on the induced canonical of $\mathcal{V}$ with N the kernel of $\Omega^1_{\mathcal{X}} \to K_{\mathcal{F}}$. A priori, since $\mathcal{V}$ is irreducible we have $\partial(\omega) \in \mathcal{I}_{\mathcal{V}}$. However by hypothesis $\partial(\mathcal{I}_{\mathcal{V}})$ generates $\mathcal{I}_{\mathcal{V}}$ so if $f_1, \dots, f_t$ generate $\mathcal{I}_{\mathcal{V}}$ we can solve,

$$\partial(g_i df_i) = g_i \partial(f_i) = \partial(\omega)$$

for some functions g_i. Whence $\omega - g_i df_i \in N$, and since the component is the honest push-forward of a measure on $\mathcal{Z}$ by III.3.3, we are done. □

We can build on this to discern what happens to the component under blowing up. Specifically we have a weak branching scheme $\mathfrak{W}_{\mathcal{Z}}$, a priori at the generic component, but infact with well defined formal Zariski closure by EmbLCR, albeit that this is initially irrelevant. Now suppose $\mathcal{V}$ is a smooth substack of $\mathrm{sing}(\mathcal{F})$ containing $\mathcal{Z}$, and let $\rho : \tilde{\mathcal{X}} \to \mathcal{X}$ be the blow up in $\mathcal{V}$, with $\mathcal{E}$ the exceptional divisor, then we assert,

VI.4.2 Claim *Any singular component $d\nu$ of the proper transform of $d\mu$ dominating our given component, i.e. $\rho_* d\nu = \mathbb{1}^0_{\mathcal{Z}} d\mu$, factors via the proper transform of $\mathfrak{W}_{\mathcal{Z}}$.*

proof The discussion is generic on $\mathcal{Z}$, so take $z_1, \dots, z_m$ functions such that dz_1,..., dz_m generate $\Omega_{\mathcal{Z}}$ locally. By hypothesis for ∂ a local generator, $\partial(z_i) \in \mathcal{I}_{\mathcal{V}}$, so after blowing up $\partial(z_i) \in \mathcal{I}_{\mathcal{E}}$. Whence for $\partial_{\mathcal{E}}$ the generator of the induced foliation $\partial_{\mathcal{E}}(z_i) = 0$. As such $dd^c|z_i|^2 d\nu = 0$ by VI.4.1 whenever the induced foliation on $\tilde{\mathcal{X}}$ is log-flat around $\mathcal{E}$ at the generic point of the support. This is, however, necessarily the case at all points outside the proper transform of $\mathfrak{W}_{\mathcal{Z}}$. □

Now while this doesn't quite allow us to assert that every singular component on the modification factors through the proper transform of the weak branch, it does allow us to assert that if we have a singular component on $\mathcal{X}$ then there is a singular component on $\tilde{\mathcal{X}}$, with similar sorts of properties, factoring through the proper transform of $\mathfrak{W}_{\mathcal{Z}}$. The consequence of this is that we may safely employ EmbLCR or more correctly its proof, cf. I.6.7. Specifically (starting from an empty background divisor to avoid further confusion) we arrive to the situation where $\mathfrak{W}_{\mathcal{Z}}$ is smooth with log-canonical singularities. We may however be unlucky and find that $\mathcal{Z}$ (strictly speaking a different $\mathcal{Z}$, but still the support of a singular component) is still singular in $\mathfrak{W}_{\mathcal{Z}}$. In which case we put $\mathfrak{W}_{\mathcal{Z}} = \mathfrak{W}^0_{\mathcal{Z}}$, with $\mathfrak{W}^1_{\mathcal{Z}}$ the weak branch at the generic point of $\mathcal{Z}$ in $\mathfrak{W}^0_{\mathcal{Z}}$. We can, without difficulty, modify the ambient space so that $\mathfrak{W}^1_{\mathcal{Z}}$ is also smooth with induced log-canonical singularities, and, of course, continue until we find ourselves in the situation where our component is generically smooth for the induced foliation in $\mathfrak{W}^p_{\mathcal{Z}}$. This latter statement has perfect sense, since components are always inside the trace of any $\mathfrak{W}^i_{\mathcal{Z}}$, and the trace of this is just some bunch of components of the singular locus of $\mathfrak{W}^{i-1}_{\mathcal{Z}}$.

The unfortunate thing in this discussion is that $\mathfrak{W}^p_{\mathcal{Z}}$, or indeed any of the $\mathfrak{W}^i_{\mathcal{Z}}$ for $i > 0$ may not actually come from the weak branches in the ambient space. Necessarily there is a maximal i for which this is true, so our initial goal is simply to understand '$\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$' for $\mathfrak{W}$ a weak branch in $\mathcal{X}$, which although not immediately good enough to understand our given singular component is plainly very much closer to it than what we know on $\mathcal{X}$. Before proceeding let's pull,

VI.4.3 Rabbit out of the Hat *A numerically relevant component for $\mathfrak{W}$ of $d\mu_{\mathcal{X}/\mathcal{F}}$ (i.e. one that has $\mathcal{Y}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \neq 0$, for $\mathcal{Y}$ a component of the trace, and, irrespective of how the latter is defined) is supported on a substack $\mathcal{V}$ of $\mathcal{X}$ lying in the pseudo trace of $\mathfrak{W}$, i.e. the completion of $\mathcal{V}$ in the trace of $\mathfrak{W}$ is non-empty, and factors through $\mathfrak{W}$.*

As such $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ will just really be a short hand for encoding the components of $d\mu_{\mathcal{X}/\mathcal{F}}$ supported on substacks in the pseudo trace of $\mathfrak{W}$, or, if one prefers, infinitely tangent to it. To understand $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$, however, one needs some knowledge of how the eigenvalues degenerate on $\mathfrak{W}$. Fortunately this comes out in the wash from the proof of EmbLCR, or more correctly the above variant. Indeed there are a bunch of smooth weak branches, albeit not necessarily very weak, in the ambient space $\mathcal{X}$ which we can suppose with log-canonical singularities for their induced foliations, which we take to be nodes of a directed graph $\mathbb{W}$. We put a directed arrow $\mathfrak{W}' \to \mathfrak{W}''$, if the latter has bigger dimension than the former, and the completion of $\mathfrak{W}'$ in the trace of $\mathfrak{W}''$ is non-empty, and locally a weak branch in $\mathfrak{W}''$ completed in the said intersection. The directed graph $\mathbb{W}$ is filtered by co-dimension by directed subgraphs $F^p\mathbb{W}$ in the obvious way, and an important thing to note is that the formal stack $\mathfrak{W}i$ which is the union over all nodes of pure co-dimension i is actually smooth. Plainly given VI.4.3, we can talk about $\mathbb{I}_{F^p\mathbb{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ etc. in the obvious way.

VI.5 Numerical Relevance

We take up where we left off, so that we have not just our foliated stack $(\mathcal{X}, \mathcal{F})$ - for the moment still with projective moduli- but our directed graph $\mathbb{W}$, and various filtrations $F^\bullet\mathbb{W}$. As such for $\mathfrak{W}$ a node we wish to investigate the numerical properties of $\mathbb{I}_{\mathbf{W}} d\nu$ for $d\nu$ a component of our transverse invariant measure $d\mu_{\mathcal{X}/\mathcal{F}}$ supported on a substack $\mathcal{V}$. Our initial aim is to prove VI.4.3, so to begin with we'll forget our previous discussion and just pretend we can define $\mathbb{I}_{\mathbf{W}} d\nu$ as a germ of a closed measure around the trace of $\mathfrak{W}$ by way of asymptotic expansion, cf. IV.4. A postiori, however, a wholly algebraic definition will emerge. In any case plainly $\mathbb{I}_{\mathbf{W}} d\nu$ should only be considered numerically relevant if $\mathcal{Y}.\mathbb{I}_{\mathbf{W}} d\nu \neq 0$, for $\mathcal{Y}$ a Cartier divisor on $\mathfrak{W}$ corresponding to some irreducible component of the trace. We therefore have the following possibilities,

VI.5.1 Case I *$\mathcal{V}$ is contained in the pseudo trace of $\mathfrak{W}$.*

Obviously this is purely algebraic, wholly satisfactory, and we move on. In particular since $d\nu$ is a component $\mathfrak{W} \cap \mathcal{V}$ must be a Zariski dense formal substack of $\mathcal{V}$. This is still true no matter how much we blow up, so we may aswell, albeit that it's only for convenience, suppose that $\mathcal{V}$ is smooth, and we distinguish,

VI.5.1 bis Case II *Otherwise*

Now what we'll do next is actually a very general fact with next to no hypothesis beyond projectivity of the moduli, and the strict difference of dimension between $\mathfrak{W}\cap\mathcal{V}$ and $\mathcal{V}$. Specifically, we may aswell suppose that the trace is $\mathcal{Y}$, so that $\mathcal{Y}\cap\mathcal{V}$ is either a Cartier divisor in a given component of $\mathfrak{W}\cap\mathcal{V}$ or all of the component. In the latter case, the component is far from Zariski dense in $\mathcal{V}$, so we may aswell say that $\mathbb{I}_{\mathbf{W}}d\nu$ is a sum $\sum \mathbb{I}_{\mathbf{W}_i}d\nu$ over the components which are not in $\mathcal{Y}\cap\mathcal{V}$, then say $\mathcal{Y}.\mathbb{I}_{\mathbf{W}_i}d\nu \neq 0$, and replace $\mathcal{Y}\cap\mathcal{V}$ by a, possibly non-reduced, but regularly embedded algebraic stack $\tilde{\mathcal{Y}} \hookrightarrow \mathfrak{W}_i$. As such we get ideals I_n defining $\mathfrak{W}_i$ in the completion of $\mathcal{V}$ in $\tilde{\mathcal{Y}}$ together with a standard thickening sequence,

$$0 \longrightarrow I_{n+1} \longrightarrow I_n \longrightarrow N^{-n}_{\tilde{\mathcal{Y}}/\mathfrak{W}_i} \longrightarrow 0$$

To make sections, however, of $H^0(\mathcal{V}, H^m I_n)$, for H ample on the moduli, is cheap, e.g. it's no problem to take n of order m^2, yet if $\mathcal{E}_n$ is the exceptional divisor on the blow up $\mathcal{V}_n \to \mathcal{V}$ in I_n we must have,

$$\frac{m}{n}H.d\nu \geq \frac{1}{n}\mathcal{E}_n.d\nu_n \geq \mathcal{Y}.\mathbb{I}_{\mathbf{W}}d\nu > 0$$

where $d\nu_n$ is the extension by zero of $d\nu$ on $\mathcal{V}_n$, and we profit from the fact that $d\nu$ is a component. This, though, is nonsense so we must be in Case I. Better still the essential of the contradiction is,

VI.5.2 Fact/Definition *In the above notation, the following are equivalent,*

(I) *$d\nu$ is numerically relevant, i,e, the limit of the above intersection numbers $1/n\mathcal{E}_n.d\nu$ which we take to be the definition of $\mathcal{Y}.\mathbb{I}_{\mathbf{W}}d\nu$ is non-zero.*

(II) *The support $\mathcal{V}$ of $d\nu$ is in the pseudo trace of $\mathfrak{W}$.*

So as promised we have VI.6.3, and we'd ideally like to deduce a nefness statement for $\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ from the same for $d\mu_{\mathcal{X}/\mathcal{F}}$, and, perhaps rather surprisingly, this is indeed possible, i.e.

VI.5.3 Fact *Suppose, as it does, $d\mu_{\mathcal{X}/\mathcal{F}}$ satisfies V.1.2, then for any component $\mathcal{Y}$ of its trace, $\mathcal{Y}.\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$.*

proof This is a reasonably intricate extension of V.4.1, therefore, we'll proceed in stages. Firstly we'll do the case when $\mathfrak{W}$ considered as a node of $\mathbb{W}$ has no directed arrows coming out, i.e. it's at the bottom of our Christmas tree equivalently there is no degeneration in the eigenvalues, before proceeding to the general case. Furthermore, even in the general case, we'll firstly give a slightly heuristic argument in order to convey the idea, which we'll subsequently make rigorous.

So let's get underway with the mildly heuristic argument, according to which we'll hypothesise a well defined residual measure in a neighbourhood associated to $\mathfrak{W}$ in a neighbourhood of the component $\mathcal{Y}\cap\mathfrak{W}$ of $\mathrm{sing}(\mathcal{F})$, so, for example $\mathfrak{W}$ algebraic, but the condition is radically weaker than this, cf. IV.4. Now take $p\in\mathbb{N}$, and let I_p be the ideal in $\mathcal{X}$ of the pth thickening of $\mathcal{Y}\cap\mathfrak{W}$ by $\mathfrak{W}$, i.e. if x_i define $\mathfrak{W}$ in the completion of $\mathcal{X}$ in the trace of $\mathfrak{W}$, and $y=0$ is the local equation for $\mathcal{Y}\cap\mathfrak{W}$ (which, without loss of generality can be taken as coming from a global Cartier divisor) we're just looking at the ideal (x_i, y^p). This has a rather easy resolution by a sequence of blow ups $\mathcal{X}_{i+1} \to \mathcal{X}_i$ in smooth centres. Indeed if we blow up in $\mathcal{Y} \cap \mathfrak{W}$, with $\mathcal{X}_0 = \mathcal{X}$, and the former $\mathcal{X}_1$, then the pull-back of I_p is resolved everywhere, except at the proper transform of $\mathfrak{W}$, $\mathfrak{W}_1$, say, itself isomorphic to $\mathfrak{W}$, where it becomes 'I_{p-1}' for the induced weak branch. Plainly $\mathcal{X}_2 \to \mathcal{X}_1$ is just the blow up in the appropriate component of the singular locus, and so forth. Denote the various maps $\mathcal{X}_j \to \mathcal{X}_i$ by ρ_{ij}, $j \geq i$, and $\mathcal{E}_i$ the exceptional divisor for $\mathcal{X}_i \to \mathcal{X}_{i-1}$, $p \geq i \geq 1$, then the resolution of I_p is,

$$\mathcal{E}^p := \sum_{i=1}^{p} \rho_{ip}^* \mathcal{E}_i$$

so that in particular on $\mathfrak{W}_p(\xrightarrow{\sim} \mathfrak{W})$, $\mathcal{O}_{\mathfrak{W}_p}(\mathcal{E}^p) \xrightarrow{\sim} \mathcal{O}_{\mathfrak{W}}(p\mathcal{Y})$. Furthermore by hypothesis we have well defined measures $d\mu^p_{\mathcal{X}/\mathcal{F}}$ with the property that $\mathcal{E}^p.d\mu^p_{\mathcal{X}/\mathcal{F}} \geq 0$. This global fact can be refined by breaking down $d\mu^p_{\mathcal{X}/\mathcal{F}}$ into pieces. Specifically the working hypothesis on the existence of a residual measure associated to $\mathfrak{W}$ implies, as per III.3.3, that there are well defined germs of closed measures $\mathbb{1}_{\mathcal{X}\backslash \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$, $\mathbb{1}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ in a neighbourhood of $\mathcal{Y} \cap \mathfrak{W}$, and this is good enough to divide up most of the calculation. Identifying $\mathfrak{W}$ with $\mathfrak{W}^p$, and leaving the latter alone, we certainly have,

$$\frac{1}{p} c_1(\mathcal{E}^p).\mathbb{1}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} = \mathcal{Y}.\mathbb{1}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$$

while notationally taking $\mathcal{Y}$ to be the same as some global Cartier divisor with $\|\mathbb{1}_{\mathbf{W}}\|, \|\mathbb{1}_{\mathcal{Y}}\|$ as in III.3.1 (remember we're being heuristic as regards $\mathfrak{W}$) we can consider the extension of $\mathbb{1}_{\mathcal{X}\backslash \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ on $\mathcal{X}_p$, and notationally identify it with the same to deduce,

$$\begin{aligned} \frac{1}{p} c_1(\mathcal{E}^p).\mathbb{1}_{\mathcal{X}\backslash \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} &= \frac{1}{p} s_{\mathcal{Y}\cap \mathbf{W}, \mathbb{1}_{\mathcal{Y}\backslash \mathbf{W}} d\mu}(1) \\ &+ \frac{1}{p} \lim_{\epsilon \to 0} \int_{\substack{\|\mathbb{1}_{\mathbf{W}}\| = \epsilon \\ \|\mathbb{1}_{\mathcal{Y}}\| \leq \epsilon^{\frac{1}{p}}}} d^c \log \|\mathbb{1}_{\mathbf{W}}\|^2 \mathbb{1}_{\mathcal{X}\backslash \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \\ &+ \lim_{\delta \to 0} \int_{\substack{\|\mathbb{1}_{\mathcal{Y}}\| = \delta \\ \|\mathbb{1}_{\mathbf{W}}\| \leq \delta^{\frac{1}{p}}}} d^c \log \|\mathbb{1}_{\mathcal{Y}}\|^2 \mathbb{1}_{\mathcal{X}\backslash \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \end{aligned}$$

The first of these terms is unimportant, while the quasi-heuristic hypothesis in force for the moment says that the first of these integrals is bounded independently of p, so dividing by p certainly makes it go to zero. The latter we calculate by Stokes. Indeed fix $\eta > 0$, then,

$$\begin{aligned} \int_{\substack{\|\mathbb{1}_{\mathcal{Y}}\| = \delta \\ \|\mathbb{1}_{\mathbf{W}}\| \leq \delta^{1/p}}} \mathbb{1}_{\mathcal{X}\backslash \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} &= -\int_{\substack{\|\mathbb{1}_{\mathcal{Y}}\| = \eta \\ \|\mathbb{1}_{\mathbf{W}}\| \leq \delta^{1/p}}} d^c \log \|\mathbb{1}_{\mathcal{Y}}\|^2 \mathbb{1}_{\mathcal{X}\backslash \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \\ &- \int_{\substack{\delta^{1/p} \leq \|\mathbb{1}_{\mathcal{Y}}\| \leq \eta \\ \|\mathbb{1}_{\mathbf{W}}\| = \delta^{1/p}}} d^c \log \|\mathbb{1}_{\mathcal{Y}}\|^2 \mathbb{1}_{\mathcal{X}\backslash \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} + o(\eta) \end{aligned}$$

where, as ever, $o(\eta)$ goes to zero with η. Plainly for η fixed the first of these integrals goes to zero with δ, while we're at a singularity with weak branching so the second, given the hypothesis is $O(\eta)$, albeit with a constant depending on p. Nevertheless for p fixed, we may take η sufficiently small to conclude,

$$\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} = O(\frac{1}{p})$$

This leaves the contribution coming from $\mathbb{I}_{\mathcal{E}^p}d\mu_{\mathcal{X}/\mathcal{F}}$ to take care of. A quick and efficient way to do this is to blow down, in order, the proper transforms of the exceptional divisors $\mathcal{E}_1,\ldots,\mathcal{E}_{p-1}$ in the same direction, then add the minimum amount of extra monodromy required to get a smooth stack $\mathcal{X}^p$. As ever this is a weighted blow up resolving I_p, and we can identify the total exceptional divisor $\mathcal{E}^p_\#$ with $\frac{1}{p}\mathcal{E}^p$ by way of the map between the moduli of $\mathcal{X}_p$ and $\mathcal{X}^p$, with the extra $\mathbb{Z}/p$ monodromy concentrating on a 'hyperplane at infinity', $\mathcal{Y}_\infty$ inside $\mathcal{E}^p_\#$. Outside $\mathcal{E}^p$ nothing changes, so infat it suffices to look at $c_1(\mathcal{E}^p_\#).\mathbb{I}_{\mathcal{E}^p_\#\backslash\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}}$. Now the projection from $\mathcal{E}^p_\#$ to $\mathcal{Y}$ is a first integral of the induced foliation, and since VI.4.1 applies to $\mathcal{E}^p_\#$ off the proper transform of $\mathfrak{W}$, $\mathbb{I}_{\mathcal{E}^p_\#\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ intersects every $(1,1)$ form coming from $\mathcal{Y}$ in zero. Necessarily though $c_1(-\mathcal{E}^p_\#)+\rho^*\omega$ is a positive smooth form for some sufficiently positive ω on $\mathcal{Y}$, so,

$$\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{E}^p\backslash\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}} = c_1(\mathcal{E}^p_\#).\mathbb{I}_{\mathcal{E}^p_\#\backslash\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}} \le 0$$

From which we deduce,

$$0 \le \frac{1}{p}c_1(\mathcal{E}^p).d\mu^p_{\mathcal{X}/\mathcal{F}} \le c_1(\mathcal{Y}).\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}} + O(\frac{1}{p})$$

and we conclude by letting p go to infinity.

Let's consider, therefore, what we need to make this rigorous by way of the definition of $\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ in terms of components of the pseudo trace, or more accurately for $\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ as the non-pseudo trace components. In this context there is nothing non-rigorous in regard to the computation of $\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathbf{W}}d\mu_{\mathcal{X}/\mathcal{F}}$ or $\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{E}^p\backslash\mathbf{W}}d\mu^p_{\mathcal{X}/\mathcal{F}}$, while for $d\nu$ a component,

$$\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p \le c_1(\mathcal{E}^1).d\nu^1 = s_{\mathcal{Y}\cap\mathfrak{W},d\nu}(1)$$

where $d\nu^i$ is extension by zero. Consequently to calculate the limit as p goes to infinity of $\frac{1}{p}c_1(\mathcal{E}^p).\mathbb{I}_{\mathcal{X}\backslash\mathbf{W}}\ d\mu_{\mathcal{X}/\mathcal{F}}$ we can divide by components, and apply the dominated convergence theorem, i.e. it suffices to show,

$$\lim_{p\to\infty}\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p = 0$$

for $d\nu$ a component supported on an irreducible substack $\mathcal{V}$ not contained in the pseudo trace. As such take $n\in\mathbb{N}$, and deform everything to the normal cone of

$\mathcal{E}^n_\#$ in $\mathcal{X}^n$, III.5, so that the deformation of $\mathfrak{W}$ itself may now be identified with an honest substack $C_{\mathcal{Y}/\mathbf{W}}$ of $C_{\mathcal{E}^n_\#/\mathcal{X}^n}$. In addition there is a specialised measure $d\nu^!$ supported on the pure dimensional cone $C_{\mathcal{E}^n_\# \cap \mathcal{V}/\mathcal{V}}$, without support on the zero section $[0_n]$. Better still, cf. op. cit., for $p \geq n$, the calculation of $\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p$ commutes with specialisation to the cone. A convenient way to re-write the calculation is to blow up in $[0_n] \cap C_{\mathcal{Y}/\mathbf{W}}$, with say C_1 the exceptional divisor, then in the proper transform $C^1_{\mathcal{Y}/\mathbf{W}}(\xrightarrow{\sim} C_{\mathcal{Y}/\mathbf{W}}) \cap C_1$, etc., $(p-n)$ times. This gives us a modification $\tilde{C}^p_{\mathcal{E}^n_\#/\mathcal{X}^n} \to C_{\mathcal{E}^n_\#/\mathcal{X}^n}$, with,

$$\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p = \frac{n[0_n]}{p}.d\nu^! + \frac{1}{p-n}\sum_{i=1}^{p-n} C_i.(d\nu^!)^i$$

where $(d\nu^!)^i$ is the extension by zero of $d\nu^!$ to $\tilde{C}^i_{\mathcal{E}^n_\#/\mathcal{X}^n}$. This reformulation has the advantage that although $C_{\mathcal{Y}/\mathbf{W}}$ is a component of the singular locus of the specialised foliation, we can, for n sufficiently large, find a generator $\partial^!$ of the same together with a local equation $y=0$ for $[0_n]$ about $C_{\mathcal{Y}/\mathbf{W}} \cap [0_n]$ such that $\partial^!(y) = 0$. Whence if we write,

$$d\nu^! = \mathbb{I}_{C_{\mathcal{Y}/\mathbf{W}}} d\nu^! + \mathbb{I}_{C_{\mathcal{E}^n_\#/\mathcal{X}^n} \setminus C_{\mathcal{Y}/\mathbf{W}}} d\nu^!$$

then what were previously heuristic considerations apply rigorously, to conclude,

$$\frac{1}{p-n}\sum_{i=1}^{p-n} C_i.(\mathbb{I}_{C_{\mathcal{E}^n_\#/\mathcal{X}^n} \setminus C_{\mathcal{Y}/\mathbf{W}}} d\nu^!)^i = O(\frac{1}{p-n})$$

Furthermore, the intersections $[O_n].\mathbb{I}_{C_{\mathcal{Y}/\mathbf{W}}} d\nu^!$ are no bigger than the $\mathcal{E}_n.d\nu^n$ appearing in VI.5.2(I), so for an a priori choice of n sufficiently large the contribution of $\mathbb{I}_{C_{\mathcal{Y}/\mathbf{W}}} d\nu^!$ in the formula for $\frac{1}{p}c_1(\mathcal{E}^p).d\nu^p$ is $o(n)$, so that we deduce,

$$\lim_{p\to\infty} \frac{1}{p}c_1(\mathcal{E}^p).d\nu^p \leq o(n)$$

for every n, and whence it's actually zero as required.

More generally we wish to consider the situation of an arbitrary node in the directed graph $\mathbb{W}$, or equivalently its apex, i.e. the node which is not the sink of any directed arrow. Again for $\mathcal{Y} \hookrightarrow \mathfrak{W}$ a component of its trace, and in particular a Cartier divisor, we make appropriate modifications of $\mathcal{X}$. The weighted blow up formulation $\mathcal{X}^p \to \mathcal{X}$ with $\mathcal{E}^p_\#$ the exceptional divisor is convenient. The calculation of $c_1(\mathcal{E}^p_\#).\mathbb{I}_{\mathcal{X}^p \setminus \mathcal{E}^p_\#} d\mu^p_{\mathcal{X}/\mathcal{F}}$ just proceeds as before with no changes, and what requires a little more care is to know that $c_1(\mathcal{E}^p_\#).\mathbb{I}_{\mathcal{E}^p_\# \setminus \mathbf{W}} d\mu^p_{\mathcal{X}/\mathcal{F}} \leq 0$. Certainly it's still the case that the projection of $\mathcal{E}^p_\#$ to $\mathcal{Y}$ is a first integral for the induced foliation, so it's more than adequate to know that the induced foliation of $\mathcal{E}^p_\#$, or appropriate substacks, leaves any component of $\mathbb{I}_{\mathcal{E}^p_\# \setminus \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ invariant. To this end observe that the filtration $F_i\mathbb{W}$ by increasing dimension gives a decreasing filtration $F^i\mathcal{E}^p_{\#}$ of $\mathcal{E}^p_{\#}$. Indeed the proper transform of the nodes of a given dimension, say dim$\mathfrak{W} + i$ intersected with $\mathcal{E}^p_{\#}$ precisely define $F^i\mathcal{E}^p_{\#}$. On $\mathcal{E}^p_{\#} \setminus F^1\mathcal{E}^p_{\#}$ VI.4.1 continues to apply, so there's nothing to do. More generally on $F^p\mathcal{E}^p_{\#} \setminus F^{p+1}\mathcal{E}^p_{\#}$, we can again apply VI.4.1 to conclude that any component with generic support in this open is invariant by the induced foliation on the proper transform $\tilde{\mathfrak{W}}_i$ of some weak branch $\mathfrak{W}_i$. Around here, however, $F^i\mathcal{E}^p_{\#}$ is the exceptional divisor on $\tilde{\mathfrak{W}}_i$, and since we're off the degeneracy locus $F^{i+1}\mathcal{E}^p_{\#}$, VI.4.1 applies again, so that once more the component is invariant by the induced foliation in $F^i\mathcal{E}^p_{\#}$, which continues to have the induced projection as first integral, from which we conclude. □

Notice that the proof works in slightly greater generality. Specifically, suppose $\mathcal{V} \hookrightarrow \mathcal{X}$ is a smooth invariant substack around which we may extend $\mathfrak{W}$, i.e. $\mathcal{V}$ is a piori in the pseudo trace, but, for whatever reason $\mathfrak{W}$ has a well defined extension around $\mathcal{V}$. Suppose further that $\mathcal{V}$ is Cartier in $\mathfrak{W}$, e.g. blow up in it, which we might aswell say is the case. Furthermore, albeit this is unnecessary, to keep ourselves strictly within the previous discussion let's say $\mathfrak{W}$ is smooth around $\mathcal{V}$, then as before we have a well defined smooth weighted blow up $\rho : \mathcal{X}^p \to \mathcal{X}$ with exceptional divisor $\mathcal{E}^p_{\#}$, and,

VI.5.4 Fact *Let things be as above, suppose that* $c_1(\mathcal{E}^p_{\#}).\mathbb{I}_{\mathcal{E}^p_{\#}\setminus \mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$ *for* p *sufficiently large, and all the singularities in the induced foliation on* $\mathfrak{W}$ *extended around* $\mathcal{V}$ *are in the components of* $\mathrm{sing}(\mathcal{F})$ *contained in the original* $\mathfrak{W}$ *then* $\mathcal{V}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$.

proof The calculation of the $\mathcal{E}^p_{\#}$ intersection on components outside the pseudo trace of $\mathfrak{W}$ considered as a formal substack extended around $\mathcal{V}$ goes as before, since $\mathcal{V}$ is invariant so the intersection concentrates at the singularities, and is given by a necessarily vanishing residue. One should note, however, that the pseudo trace, to be more precise that of $\tilde{\mathfrak{W}}$ defined to be the extension as a formal stack around $\mathcal{V}$, may not be the same as that of our original $\mathfrak{W}$. Indeed viewed from $\mathcal{V}$ our considerations of numerical relevance VI.5.1/2 says that a component supported on $\mathcal{A}$ should be considered such if its completion in its intersection with $\mathcal{V}$ is non-empty and factors through $\tilde{\mathfrak{W}}$. Plainly though this intersection may not contain any points of the trace of $\mathfrak{W}$, so in the obvious variant of our notation we have to organise a sub-calculation,

$$\mathcal{V}.\mathbb{I}_{\tilde{\mathbf{W}}} d\mu_{\mathcal{X}/\mathcal{F}} = \mathcal{V}.\mathbb{I}_{\mathbf{W}}\mathbb{I}_{\tilde{\mathbf{W}}} d\mu_{\mathcal{X}/\mathcal{F}} + \mathcal{V}.\mathbb{I}_{\tilde{\mathbf{W}}\setminus\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$$

In particular if the component of $d\nu$ associated to $\mathcal{A}$ is in the $\mathbb{I}_{\tilde{\mathbf{W}}\setminus\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ part, then the intersection of $\mathcal{A}$ with the trace of $\mathfrak{W}$ is empty. On the other hand $\mathcal{O}_{\mathcal{A}}(\mathcal{V})$ is an invariant Cartier divisor for which the construction of a leafwise holomorphic connection depends only on the embedding of $\mathcal{V}$ in $\tilde{\mathfrak{W}}$. By hypothesis $\mathcal{A}$ doesn't meet the singularities of $\tilde{\mathfrak{W}}$, so, infact, $\mathcal{V}.d\nu = 0$. The remaining part of the calculation is, of course, $\mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\setminus\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq$), which we're supposing is true. □

VI.6 Weak Flips

For $\mathcal{X}$ a stack the words *rational curve* will be employed rather than the more long winded 1-dimensional irreducible closed substack whose normalisation has positive Euler characteristic. Now consider the situation of the previous section with $\mathfrak{W}$ a node of $\mathbb{W}$, and $\mathcal{G}$ the induced foliation. We know that $K_{\mathcal{G}}$ is related to $K_{\mathcal{F}}$ by a formula of the form $K_{\mathcal{G}} = K_{\mathcal{F}}(-\mathcal{E})$, for $\mathcal{E}$ a Cartier divisor in $\mathfrak{W}$ supported on the trace. Better still, we know that $\mathcal{E}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, so,

$$K_{\mathcal{G}}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq K_{\mathcal{F}}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$$

If, therefore, to fix ideas, our singular component of interest $d\nu$ were generically smooth for $\mathcal{G}$, then to get $K_{\mathcal{G}}.d\nu = 0$ we basically need to know that $K_{\mathcal{G}}$ is nef. Before elaborating let's consider how this applies to the situation where $(\mathcal{X}, \mathcal{F})$ arises from an ODE on algebraic surface. In this situation associated to the substack $\mathcal{V}$ on which $d\nu$ is supported there is a corresponding ODE, which expressed as a foliation by curves has a minimal model of the form $(\mathcal{V}_0, \mathcal{H}_0)$. As soon as we get $K_{\mathcal{G}}.d\nu = 0$ for any sort of induced foliation which is generically smooth, then we can argue exactly as in VI.3.5 to construct a modification $\widetilde{\mathcal{V}}$ of $\mathcal{V}$, mapping by π to $\mathcal{V}_0$ with $K_{\mathcal{H}_0}.\pi_* d\nu = 0$, and so conclude that the push forward of $d\nu$ to the original surface is rational or elliptic. Of course it may happen that $(\mathfrak{W}, \mathcal{G})$ is not of 'nef. type', but in this case it's 'minimal model' will be étale covered by a bundle of generalised weighted projective stacks in the radial foliation by lines, so apart from giving us the desired conclusion in the surface case, it also eliminates the possibility of any 'sub-singular components' since anything which is algebraically integrable does not exhibit weak branching.

Plainly, therefore, nef. is good. Unfortunately we'll need a bunch of definitions and remarks to explain what it should mean. As one can imagine there isn't that much to do beyond this, since if $\mathfrak{W}$ were algebraic we'd already be done by [II] IV.7.4, and what's involved is really just a certain amount of unraveling to check that it works for formal stacks. Even here formal stack must be understood strictly since we'll need to carry with us the pseudo-trace. Whence,

VI.6.1 Definition *An irreducible component $\mathcal{V}$ of a formal stack $\mathfrak{W}$ will be said to be in the pseudo trace of $\mathfrak{W}$ if,*

(a) *$\mathcal{V}$ is a stack, with not necessarily projective moduli.*

(b) *It intersects at least one component which isn't a stack.*

(c) *The completion of $\mathcal{V}$ in this intersection factors through the non-stack like components.*

A connected formal stack will be called pseudo irreducible if exactly one component is an irreducible formal stack which is not itself a stack, and every other component lies in the pseudo trace.

Basically we're planning on proving the minimal model theorem for foliated pseudo irreducible formal stacks, which sounds worse than it actually is, i.e. what we're doing is keeping track of of tangency conditions. As such we'll view $\mathcal{F}$ as a not necessarily saturated foliation on $\mathfrak{W}$, together with a distinguished closed substack of the trace around which $\mathfrak{W}$ is genuinely formal, each component of this distinguished subset is a Cartier divisor in $\mathfrak{W}$, and the induced foliation at this juncture has log-canonical singularities. This leads to,

VI.6.2 Abusive Definition (i.e. not particularly functorial with respect to the ideas) *By a foliated (pseudo irreducible) foliated stack* $(\mathfrak{W}, \mathcal{F})$ *we mean,*

(a) *A pseudo irreducible formal stack* $\mathfrak{W}$ *together with a* $\mathbb{Q}$*-gorenstein, not necessarily saturated, foliation by curves* $\mathcal{F}$ *all understood functorially with respect to the ideas.*

(b) *A distinguished open formal substack* $\mathfrak{V}$*, which itself is an irreducible proper formal stack which is not a stack, every component of the trace of which is Cartier in* $\mathfrak{V}$ *(whence in* $\mathfrak{W}$*), and at which* $\mathcal{F}$ *is never saturated.*

The induced saturated foliation on $\mathfrak{V}$ *will be written* $\mathcal{G}$*, and the trace of* $\mathfrak{V}$ *will be refered to as the strict trace, with trace employed for components where things are strcitly formal. Furthermore,*

(c) *We say that* $(\mathfrak{W}, \mathcal{F})$ *has log-canonical singularities iff the singularities of* $(\mathfrak{V}, \mathcal{G})$ *are log-canonical in the functorial sense.*

(d) *We'll write* $K_{\mathcal{F},\mathbf{W}}$ *for the class* $K_{\mathcal{F}}(-\mathcal{Y})$*, where* $\mathcal{Y}$ *is the necessarily effective Cartier divisor on* $\mathfrak{W}$ *defined by* $i_*\mathcal{O}_{\mathfrak{V}}/(K_{\mathcal{F}} - K_{\mathcal{G}})$*, with* i *the inclusion of* $\mathfrak{V}$ *in* $\mathfrak{W}$*.*

The model for this monologue is our original node $\mathfrak{W}$ of $\mathbb{W}$ together with all the components in its pseudo trace, except that $\mathfrak{W}$ has become $\mathfrak{V}$ and the union $\mathfrak{W}$. It may happen though that the trace can be extended. From a notational/definition point of view this risks a certain confusion, as such one should note that when this occurs it is not assumed to change the distinguished component $\mathfrak{V}$ in the model example. More importantly the key instance where this occurs is when the induced foliation in a pseudo trace component $\mathcal{V}$ is wholly by rational curves on which $K_{\mathcal{F}}$ is nil. To see why this is possible one appeals, as ever, to [EGA] III, III.3.4.2 in the spirit of I.2.2 to reduce to cases that one knows, e.g. a smooth rational curve in $\mathcal{X}$ with a strictly convex neighbourhood, where one can then appeal to the Jordan decomposition of a global generator on some étale cover of the neighbourhood. More generally the same argument works generically for bundles of such, so that one can do a Noetherian induction as per the proof of I.6.7 to conclude that $\mathfrak{V}$ can be extended around all of $\mathcal{V}$. As such $\mathfrak{W}$ unlike $\mathfrak{V}$ may be a complete mess, indeed it's not even Noetherian, but of course it's a direct limit of things of 'formal finite type', which is all that matters, and which if one wants, one can add as VI.6.2(e).

Now it stands to reason that one cannot just prove something by a definition in this kind of generality, irrespective of its length. Whence we'll need that $(\mathfrak{W}, \mathcal{F})$ is *dominated by a projective* and indeed *terminally* so. More precisely there should be a stack $(\tilde{\mathcal{X}}, \mathcal{F})$ with projective moduli, canonical singularities, and $K_{\mathcal{F}}$ nef, together with a foliated pseudo irreducible formal stack $(\tilde{\mathfrak{W}}, \mathcal{F}) \subset (\tilde{\mathcal{X}}, \mathcal{F})$ precisely as per the above model example (indeed $\tilde{\mathcal{X}}$ will be an invariant modification of our original $\mathcal{X}$) together with an honest map $\rho : \tilde{\mathfrak{W}} \to \mathfrak{W}$ which is an isomorphism at every generic point of $\mathfrak{W}$ such that we have,

VI.6.3 Condition/Definition *The locus where the moduli of* $\tilde{\mathfrak{w}}$ *is not isomorphic to* $\mathfrak{W}$ *is* $K_{\mathcal{F},\mathbf{W}}$ *terminal, i.e. if we find ourselves at a point where* ρ *isn't an isomorphism then it will be a point of* $\mathfrak{V}$*, so* $K_{\mathcal{F},\mathbf{W}}$ *should be understood as* $K_{\mathcal{G}}$

terminality functorially with respect to the ideas. Infact the map ρ at the stack level will be a sequence of blow ups in smooth centres in the strict trace, and wholly transverse to $\mathcal{G}$.

It will, of course, be the case that $K_{\mathcal{F}}$ really descends to $\mathfrak{W}$, so there's no ambiguity in the notation, and we make,

VI.6.4 Definition *We say that $K_{\mathcal{F},\mathbf{W}}$ is nef., if for every curve $\mathcal{C}$ in the pseudo trace with $K_{\mathcal{F}}.\mathcal{C} = 0$, $K_{\mathcal{F},\mathbf{W}}.\mathcal{C} \geq 0$.*

The critical thing is, therefore,

VI.6.5 Fact *Let everything be as above (in particular $(\mathfrak{W}, \mathcal{F})$ is terminally dominated by a projective, and its singularities are log-canonical) then if $K_{\mathcal{F},\mathbf{W}}$ is not nef there is an invariant rational stack $\mathcal{L}$ (generically in the smooth locus of $\mathcal{G}$) in $\mathfrak{W}$ satisfying,*

$$K_{\mathcal{F}}.\mathcal{L} = 0 \ , \ K_{\mathcal{F},\mathbf{W}}.\mathcal{L} \in [-2, 0)$$

so in particular $\mathcal{L}$ has non-empty intersection with the strict trace.

proof If $K_{\mathcal{F},\mathbf{W}}$ is not nef., there is some component of the trace or pseudo trace $\mathcal{V}$ such that on the proper transform $\tilde{\mathcal{V}}$ of $\mathcal{V}$, $\rho^* K_{\mathcal{F},\mathbf{W}}$ is not non-negative on the half space $\mathrm{NE}(\tilde{\mathcal{V}})_{K_{\mathcal{F}}=0}$. In particular there is an extremal ray R in $\mathrm{NE}(\tilde{\mathcal{V}})_{K_{\mathcal{F}}=0}$ satisfying $\rho^* K_{\mathcal{F},\mathbf{W}}.R < 0$. Quite generally R is a limit of sums of irreducible algebraic 1-cycles, $Z_i = \sum_j a_{ij} Z_{ij}$, $i \to \infty$. The Néron-Severi group is, however, finite dimensional of say dimension t, and for fixed i, $\sum_j \mathbb{R}_+ Z_{ij}$ is a polyhedral cone, so we need at most t Z_{ij}'s in our formula. After subsequencing we may suppose that for each $1 \leq j \leq t$, $a_{ij} Z_{ij}$ converges to some R_j as $j \to \infty$, so $R = R_1 + \ldots + R_t$. By hypothesis $K_{\mathcal{F}}$ is nef., so infact all the R_i must be parallel since R is extremal. Whence we can suppose that R is a limit of irreducible 1-cycles $a_i Z_i$, so in particular, $K_{\mathcal{F},\mathbf{W}}.\rho_* Z_i$is eventually strictly negative.

Now denote by $f_i : \mathcal{C}_i \to \mathfrak{W}$ the normalisation of the irreducible 1-dimensional substack of $\mathfrak{W}$ supported on $\rho(Z_i)$. Certainly $\mathcal{C}_i$ must meet the strict trace, so we can talk about whether the completion $\hat{\mathcal{C}}_i$ in this intersection (which is quite possibly all of $\mathcal{C}_i$) meets the locus where $\mathcal{G}$ on $\mathfrak{V}$ is smooth or not. If, however, it didn't it would be wholly singular, so by the log-canonicity of the singularities we'd get $K_{\mathcal{F},\mathbf{W}}.\mathcal{C}_i = K_{\mathcal{G}}.\mathcal{C}_i \geq 0$. Consequently $\mathcal{C}_i$ doesn't factor through the singular locus, and we can form the pull-back of the infinitesimal birational groupoid $\mathfrak{G}_s^{\#} \times_{f_i} \mathcal{C}_i$ of [II] II.1.3. Where one observes apart from the triviality of using $\#$ so as to preserve $\tilde{}$ for proper transforms, that the bi-rational groupoid construction is well defined independently of any saturation hypothesis, whence certainly for $\mathcal{F}$, while the correction for the singularities of $\mathcal{G}$ is a local question, so infact is well defined even if $\mathcal{V}$ were only in the pseudo trace. The formal stack $\mathfrak{G}_s^{\#} \times_{f_i} \mathcal{C}_i$ contains $\mathcal{C}_i$ as a section with positive (infact $-K_{\mathcal{F},\mathbf{W}}$) normal bundle, so the Chow lemma, e.g. [II] II.2.2, to the effect that its field of meromorphic functions has transcendence degree at most 2 continues to hold. What, however, is less obvious is that the said degree actually is 2. This is where VI.6.3 enters, since it guarantees (profiting from, say, the convergent frobenius theorem to avoid some technical issues of definition) that ρ is bi-rational at the generic point of $\mathfrak{G}_s^{\#} \times_{f_i} \mathcal{C}_i$. Indeed this is clear from the description of ρ in terms of blow ups, but it also follows from the terminality condition itself since any modification in a centre tangent to the foliation, yet not invariant by it leads to a non-canonical singularity. As such there is a well defined proper transform $\widetilde{\mathfrak{G}_s^\# \times_{f_i} \mathcal{C}_i}$ mapping finitely to $\mathcal{X} \times \mathcal{C}_i$, so indeed the space of rational functions really does have dimension 2. Consequently we find an honest 2 dimensional normal stack $s : \mathcal{S}_i \to \mathcal{C}_i$ together with a section $t(\mathcal{C}_i)$, say, such that the completion of $\mathcal{S}_i$ in $t(\mathcal{C}_i)$ is $\mathfrak{G}_s^\# \times_{f_i} \mathcal{C}_i$ with it's natural section.

On the normal surface $\mathcal{S}_i$ we have an honest Mumford type intersection theory, NS^M or NE^M for the cone, and as per [II] II.3.3 for every $c_i \in \mathcal{C}_i(\mathbb{C})$ we have an invariant (possibly wholly singular) rational curve $\mathcal{L}_{c_i}$ (i.e. a curve in the fibre of s through c_i) satisfying the basic estimate,

$$N.\mathcal{L}_{c_i} \leq 2\frac{N._{f_i}\mathcal{C}_i}{-K_{\mathcal{F},\mathbf{W}}\cdot_{f_i}\mathcal{C}_i}$$

for every nef. (in the sense of Mumford) divisor N in NS^M. In particular taking N to be $K_{\mathcal{F}}$, we see that $K_{\mathcal{F}}$ is eventually $\leq$ epsilon for i sufficiently large, and since it takes discrete non-negative values $K_{\mathcal{F}}$ is actually zero on every fibre of $s : \mathcal{S}_i \to \mathcal{C}_i$. What remains, therefore, is to produce some $\mathcal{L}_{c_i}$ on which $K_{\mathcal{F},\mathbf{W}}$ is negative. The key point, irrespective of any projectivity assumptions, is that the effective Mumford cone doesn't contain lines, which allows us to assert,

VI.6.6 Claim *There is a decomposition,*

$$\mathrm{NE}^M(\mathcal{S}_i) = \mathrm{NE}^M(\mathcal{S}_i)_{K_{\mathcal{F},\mathbf{W}}\geq 0} + \sum_n \mathbb{R}_+[\mathcal{L}_n]$$

where the $\mathcal{L}_n$ are $K_{\mathcal{F},\mathbf{W}}$ invariant (i.e. fibres of s) rational substacks of $\mathcal{S}_i$ satisfying $K_{\mathcal{F}}.\mathcal{L}_n = 0$, $-2 \leq K_{\mathcal{F},\mathbf{W}}.\mathcal{L}_n < 0$.

proof Following [K1] III.1.2, we prove the closure of the right hand side is the left hand side, which is enough since $\mathcal{S}_i$ is a surface. Whence if things are false there is a nef. (Mumford sense) Cartier divisor H on $\mathcal{S}_i$ strictly positive on the right hand side, and vanishing on some extremal ray D of $\mathrm{NE}^M(\mathcal{S}_i)$. As such, arguing as before, we produce a sequence $a_m D_m$ of irreducible 1-cycles on $\mathcal{S}_i$ converging to D, but with the additional property that not only is $-K_{\mathcal{F},\mathbf{W}}.D_m > 0$ but $(-K_{\mathcal{F},\mathbf{W}}.D_m)^{-1}(H.D_m) \to 0$ as $m \to \infty$. Continuing in the same vein, one therefore finds for each point $d \in \mathcal{D}_m(\mathbb{C})$, $\mathcal{D}_m$ a stack with support D_m, a rational invariant curve $\mathcal{L}_{md}$ on $\mathcal{S}_i$ satisfying,

$$N.\mathcal{L}_{md} \leq 2\frac{N.D_m}{-K_{\mathcal{F},\mathbf{W}}.D_m}$$

for every nef. divisor N on $\mathcal{S}_i$. As such if we take N to be H we deduce that $H.\mathcal{L}_{md}$ goes to zero as $m \to \infty$ for every d. We already know, however, that $K_{\mathcal{F}}.\mathcal{L}_{md} = 0$ for all m and d, while quite generally any invariant curve satisfies $K_{\mathcal{F},\mathbf{W}}.\mathcal{L}_{md} \geq -2$, so that for m sufficiently large we contradict the strict positivity of H on the cone on the right hand side. □

Back at VI.6.5 therefore, we quickly conclude that indeed there must be an invariant rational curve $\mathcal{L}$ satisfying $0 > K_{\mathcal{F},\mathbf{W}}.\mathcal{L} \geq -2$. It must, of course, meet the strict trace non-emptily, so if $\hat{\mathcal{L}}$ is its completion in the same, then $\mathcal{G}$ must be smooth at the 'generic point' of $\hat{\mathcal{L}}$, or technically better $\hat{\mathcal{L}}$ doesn't factor through sing$(\mathcal{G})$, since $\mathcal{G}$ has log-canonical singularities. □

We will refer to the curves guaranteed by VI.6.5 as $K_{\mathcal{F},\mathbf{W}}$ *negative curves*. Notice that if $\mathcal{L}$ is such, and the distinguished open formal substack $\mathfrak{V}$ is smooth then it's extension around $\mathcal{L}$ (whose existence has already been noted) is smooth. Indeed if $\mathcal{L}$ factors through the strict trace there is nothing to do, otherwise sing$(\mathfrak{W}) \cap \mathcal{L}$ is a closed invariant substack of $\mathcal{L}$ so it certainly contains the unique point of sing$(\mathcal{G}) \cap \hat{\mathcal{L}}$ (use the adjunction formula of II.3 together with the weakness of $\tilde{\mathfrak{W}}$, II..3.5) where, as ever, $\hat{\mathcal{L}}$ is the completion in the intersection. By definition, though, $\mathfrak{W}$ is smooth at sing$(\mathcal{G}) \cap \hat{\mathcal{L}}$, so this is nonsense. Consequently the analysis of [II] §III goes through verbatim, or strictly speaking a slightly more involved variant to deal with generalised weighted projective stacks, to conclude,

VI.6.7 Fact *The deformations of $K_{\mathcal{F},\mathbf{W}}$ negative curves in $\mathfrak{W}$ are unobstructed. Better still the local germ of deformation is (up to étale covering) that of an invariant curve for a bundle of radial foliations in generalised weighted projective stacks over a locally smooth component of sing$(\mathcal{G})$, and the formal neighbourhood structure in $\mathfrak{W}$ splits,* cf. [II] IV.4.

At which point,

VI.6.8 Difficulties *We face the following problems,*

(a) *We need to guarantee that the local germ of deformation converges.*

(b) *Even after we do (a) we have to avoid a foliated variant of a Hironaka type example [H] - which certainly exist by the way.*

Both problems are resolved by VI.6.3. In case (a), as in the proof of VI.6.5, it guarantees that if ρ is an isomorphism at the generic point of the germ of deformation - infact even on the original curve- then the said germ has a well defined proper transform. By the definition of $\tilde{\mathfrak{W}}$ all we have to do is check that the dimension of the deformation space is correct, which itself reduces to a surface question, i.e. does a rational curve in the broad sense of nodal and every sub-curve of negative arithmetic genus & trivial normal bundle move, which indeed it does, so this is okay. As to (b), suppose we're presented with a $K_{\mathcal{F},\mathbf{W}}$ negative curve $\mathcal{L}^0$. If it doesn't move, we don't care. It may, however, move and break up into $\mathcal{L}_1^0, \ldots, \mathcal{L}_1^n$. Certainly all the $\mathcal{L}_1^i$ are invariant, $K_{\mathcal{F}}.\mathcal{L}_i^0$ for each i, so we can find one of them $\mathcal{L}^1$, say, which is a $K_{\mathcal{F},\mathbf{W}}$ negative curve. There is, though, no guarantee that a Hironaka example won't form, i.e. this will continue ad nauseam. Again, VI.6.3 comes to the rescue, since all of the individual specialisations $\mathcal{L}^0 \rightsquigarrow \mathcal{L}^1 \rightsquigarrow \mathcal{L}^2 \rightsquigarrow$, etc. can be lifted to $\tilde{\mathcal{X}}$ without any loss (indeed there may be a gain) in the number of extra components introduced when a given $\mathcal{L}^i$ breaks. Consequently,

VI.6.9 Definition/Summary *Given a $K_{\mathcal{F},\mathbf{W}}$ negative irreducible curve $\mathcal{L}^0$ there exists another (possibly the same) $K_{\mathcal{F},\mathbf{W}}$ negative irreducible curve $\mathcal{L}$ such that every germ of deformation of $\mathcal{L}$ that may exist by VI.6.7 not only converges, but every curve obtained by the deformation is also $K_{\mathcal{F},\mathbf{W}}$ negative and irreducible. Such a curve without breaking will be termed pseudo extremal.*

Now pseudo extremality, as opposed to extremality, is all that one needs for the analysis of [II] III.3-6 to go through, i.e.

VI.6.10 Fact *The deformations of a pseudo extremal $K_{\mathcal{F},\mathbf{W}}$ negative curve $\mathcal{L}$ in $\mathfrak{W}$ sweep out a bundle of radially foliated generalised weighted projective stacks $\mathcal{P}$, with the base of the bundle $\mathcal{Z}$ a smooth substack of the strict trace. Indeed $\mathcal{Z}$ is a connected component of* $\mathrm{sing}(\mathcal{G})$, *and the normal bundle of $\mathcal{P}$ in $\mathfrak{W}$ is, up to étale covering, locally on $\mathcal{Z}$ a sum of relatively negative line bundles whenever $\mathcal{P} \neq \mathfrak{W}$.*

The formal form of the flip theorem, [II] IV.2 & IV.5, therefore goes through without change. Indeed the only quibble that one might have is at the pseudo trace, but since the flip is actually a flap this is guaranteed by Theorem 3.1 of [A3], and the unicity of contraction, which extends op. cit. to stacks. Whence,

VI.6.11 Fact *Suppose the $\mathcal{P}$ of* VI.6.10 *is not all of $\mathfrak{W}$ then putting $\mathfrak{W}_- = \mathfrak{W}$ there is a flip/flap diagram of pseudo irreducible formal stacks,*

$$\begin{array}{ccccc} & & \mathfrak{W}^{\#} & & \\ \text{weighted blow up} \swarrow & & & & \searrow \text{weighted blow down} \\ & \mathfrak{W}_- & - - - - \rightarrow & \mathfrak{W}_+ & \\ K_{\mathcal{F},\mathbf{W}_-} \text{ negative} \searrow & & & & \swarrow K_{\mathcal{F},\mathbf{W}_+} \text{ positive} \\ & & \mathfrak{W}_0 & & \end{array}$$

In addition having started from the situation that the distinguished irreducible open $(\mathfrak{V}_-, \mathcal{G}_-)$ is smooth with log-canonical singularities, the same is true of $(\mathfrak{V}_+, \mathcal{G}_+)$, while the centre of $\mathfrak{W}^{\#} \to \mathfrak{W}_+$ is contained in the strict trace of $\mathfrak{W}_+$ and everywhere transverse to $\mathcal{G}_+$.

On a slightly technical point: one should note that 'weighted blow up' on the left includes the possibility of extracting a root of a Cartier divisor if $\mathcal{P}$ is such or the identity if this is unnecessary. This allows us to assert that the distinguished open $(\mathfrak{V}^{\#}, \mathcal{G}_-)$ again is smooth with log-canonical singularities. Plainly if $\mathfrak{W}_-$ is terminally dominated by a projective in the sense of VI.6.3 & prequel then after sufficient modification of the ambient space (including extraction of invariant roots if necessary) there is, thanks to the transversality of the $\mathfrak{W}^{\#} \to \mathfrak{W}_+$ centre, no problem to conclude the same for $\mathfrak{W}_+$ since everything is actually $\mathcal{F}$ invariant, while $K_{\mathcal{F}}$ itself obviously descends to $\mathfrak{W}_+$ by a standard theorem of formal functions type argument. Taken in combination with VI.6.5, together with the termination of flipping thanks to the destruction of a singular component of $\mathcal{G}$ at each stage, we obtain,

VI.6.12 *Let $(\mathfrak{W}, \mathcal{F})$ be a weak foliated pseudo irreducible formal stack inside a stack $(\mathcal{X}, \mathcal{F})$ with projective moduli and $K_{\mathcal{F}}$ nef. such that the distinguished open $(\mathfrak{V}, \mathcal{G})$ is smooth with log-canonical singularities, then there are diagrams,*

(a) *The minimal model algorithm,*

$$(\mathfrak{W}, \mathcal{F}) = (\mathfrak{W}_0, \mathcal{F}) \dashrightarrow (\mathfrak{W}_1, \mathcal{F}) \dashrightarrow \ldots \dashrightarrow (\mathfrak{W}_n, \mathcal{F})$$

The arrows here are the $K_{\mathcal{F},\mathbf{W}}$ minimal model algorithm as described above, so either $K_{\mathcal{F},\mathbf{W}}$ is nef. in the sense of VI.6.4 or $(\mathfrak{W}_n, \mathcal{F})$ is a radially foliated bundle of generalised weighted projective stacks, with the base of the bundle the unique *smooth component of* $\mathrm{sing}(\mathcal{G})$. *In any case all the* $(\mathfrak{V}_i, \mathcal{G})$*'s are smooth with log-canonical singularities.*

(b) *Domination by a projective for all* i,

$$(\mathcal{X}_i, \mathcal{F}) \underset{j_i}{\hookleftarrow} (\tilde{\mathfrak{W}}_i, \mathcal{F}) \underset{\rho_i}{\twoheadrightarrow} (\mathfrak{W}_i, \mathcal{F})$$

Indeed each of the $(\tilde{\mathfrak{W}}_i, \mathcal{F})$*'s are as in the exemplary discussion of a foliated pseudo irreducible stack post* VI.6.2, *with the* j_i*'s embeddings in stacks with projective moduli, while the* ρ_i*'s are as per* VI.6.3.

(c) *Composition by invariant modification,*

$$\begin{array}{ccc} (\mathcal{X}_i, \mathcal{F}_i) & \xleftarrow[j_i]{} & (\tilde{\mathfrak{W}}_i, \mathcal{F}) \\ \Big\downarrow & & \Big\downarrow \\ (\mathcal{X}_{i-1}, \mathcal{F}_{i-1}) & \xleftarrow[j_{i-1}]{} & (\tilde{\mathfrak{W}}_{i-1}, \mathcal{F}) \end{array}$$

All of the vertical maps $i \in \mathbb{N}$ *are modifications in* $K_{\mathcal{F}}$ *nil invariant centres. Infact they are compositions of weighted blow ups in the broad sense post* VI.6.11.

It's worth closing by way of,

VI.6.13 Remark *As one can imagine from the emphasis on* $K_{\mathcal{F}}$ *centres, it would certainly have been preferable to preserve the immersion* $(\mathfrak{W}, \mathcal{F}) \hookrightarrow (\mathcal{X}, \mathcal{F})$*, i.e. flop. Already in dimension 3 if one wants to preserve projectivity of the moduli then it's necessary to create nodes. Of course, one can think to finesse the situation in the spirit of* II.9 *combined with* VI.6.3 *but this is in general impossible when the weak branch has co-dimension at least 2. The technical difficulty in working with nodes is rather small, but would still require several more pages of justification than those required to define* $\mathfrak{W}$*. Irrespectively, the key would remain* VI.6.5.

VI.7 Inductive Remarks

To begin with, for $\mathfrak{V}$ a node of the weak branching graph $\mathbb{W}$ as defined in VI.4 let's see how VI.6.12 applies to the study of $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$, where $\mathbb{I}_{\mathbf{W}}$ is now understood functorially with respect to the ideals for $\mathfrak{W}$ the associated pseudo irreducible formal stack with $\mathfrak{V}$ the distinguished open, and this even agrees with the previous rabbit out of a hat notation VI.4.3. Anyway, as post VI.4.2 we can modify $\mathcal{X}$ as we please, and so assume that $\mathcal{X}$ is the $\mathcal{X}_n$ of VI.6.12. Plainly if $(\mathfrak{W}, \mathcal{F})$ is a modification of a radially foliated weighted projective stack then we're rather happy, and the discussion finishes, otherwise there is a map $\rho : (\mathfrak{W}, \mathcal{F}) \to (\mathfrak{W}_+, \mathcal{F})$ such that $\rho^* K_{\mathcal{F}, \mathbf{W}_+}$ is non-negative on all $K_{\mathcal{F}}$ nil curves in the pseudo trace of $\mathfrak{W}$. Now let $d\nu$ be a component of $\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}}$ supported on a closed substack $\mathcal{V}$ factoring through $\mathfrak{W}$. Everything has projective moduli, and we just project $d\nu$ into $\mathrm{NS}_1(\mathcal{V})$, i.e. view it as an operator on bundles, whence this projection $[d\nu]$ is in $\mathrm{NE}_1(\mathcal{V})_{K_{\mathcal{F}}=0}$, so by construction $\rho^* K_{\mathcal{F}, \mathbf{W}_+}.d\nu \geq 0$. We further assert,

VI.7.1 Claim *For every* $d\nu$ *as above,* $\rho^* K_{\mathcal{F}, \mathbf{W}_+}.d\nu = K_{\mathcal{F}, \mathbf{W}_+}.\rho_* d\nu = 0$.

proof By VI.5.3, we know that $K_{\mathcal{F}, \mathbf{W}}.\mathbb{I}_{\mathbf{W}} d\nu \leq 0$, while $K_{\mathcal{F}, \mathbf{W}}$ is related to $K_{\mathcal{F}, \mathbf{W}_+}$ by a formula of the form,

$$K_{\mathcal{F}, \mathbf{W}} = \rho^* K_{\mathcal{F}, \mathbf{W}_+} + \mathcal{E}$$

where without loss of generality $\mathcal{E}$ is an effective Cartier divisor in the ambient space, and $\mathcal{E} \cap \mathfrak{W}$ is an effective Cartier divisor supported on $K_{\mathcal{F}}$ nil invariant rational curves. It may, however, happen that $\mathcal{E} \cap \mathfrak{W}$ isn't contained in the strict trace, so VI.5.3 doesn't apply, and to conclude what is obviously sufficient, i.e. $\mathcal{E}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, we may have to try to use VI.5.4. Whence let $\mathcal{Y}$ be a component of the trace in $\mathcal{E} \cap \mathfrak{W}$ which isn't strict, with $\mathcal{E}_{\mathcal{Y}}$ the component of $\mathcal{E}$ affording $\mathcal{Y}$ which by blowing up we may not only assume is irreducible, as we may indeed suppose $\mathcal{Y}$ smooth, but that,

$$N_{\mathcal{Y}/\mathcal{X}} \xrightarrow{\sim} N_{\mathcal{Y}/\mathbf{W}} \oplus N_{\mathcal{Y}/\mathcal{E}_{\mathcal{Y}}}$$

Now to prove that $\mathcal{Y}.\mathbb{I}_{\mathbf{W}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, it will suffice, thanks to VI.5.3., to do it on any modification which only involves blowing up in things inside the strict trace. Necessarily $\mathcal{Y}$ is covered by $\mathcal{F}$ invariant rational curves $\mathcal{C}_t$ all meeting $\operatorname{sing}(\mathcal{G})$ without factoring through it, so by blowing up enough we can assume that $N^{\vee}_{\mathcal{Y}/\mathcal{X}}|_{\mathcal{C}_t}$ is ample for generic $\mathcal{C}_t$, where, without loss of generality, generic means not factoring through the strict trace. Furthermore if I_p is the sheaf of ideals used to define the weighted blow ups $\mathcal{X}^p \to \mathcal{X}$ employed in VI.5.3, adapted to $\mathcal{Y}$ as in VI.5.4, then we have an exact sequence,

$$0 \longrightarrow N^{-p}_{\mathcal{Y}/\mathbf{W}} \longrightarrow I_p/I_p^2 \longrightarrow N^{\vee}_{\mathcal{Y}/\mathcal{E}_{\mathcal{Y}}} \longrightarrow 0$$

and an extension of ample by ample is ample, so without loss of generality $I_p/I_p^2|_{\mathcal{C}_t}$ is ample for $\mathcal{C}_t$ not factoring through the strict trace, which we'll denote $\mathcal{Z}$. Consequently if $\pi : \mathcal{E}^p_{\#} \to \mathcal{Y}$ is the projection of the exceptional divisor of $\mathcal{X}^p$ then we organise our computation of $\mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}} d\mu^p_{\mathcal{X}/\mathcal{F}}$ according to,

$$\mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}} d\mu^p_{\mathcal{X}/\mathcal{F}} = \mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}\backslash\pi^{-1}(\mathcal{Z})} d\mu^p_{\mathcal{X}/\mathcal{F}} + \mathcal{E}^p_{\#}.\mathbb{I}_{\pi^{-1}(\mathcal{Z})}\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}} d\mu^p_{\mathcal{X}/\mathcal{F}}$$

The part off $\mathcal{Z}$ pushed forward to $\mathcal{Y}$ comes from a measure $d\lambda$ on $\mathcal{Y}/\mathcal{F}$, i.e. the moduli (even coarsely speaking) of our rational curves, so that,

$$\mathcal{E}^p_{\#}.\mathbb{I}_{\mathcal{E}^p_{\#}\backslash\mathbf{W}\backslash\pi^{-1}(\mathcal{Z})} d\mu^p_{\mathcal{X}/\mathcal{F}} = \int_{\mathcal{Y}/\mathcal{F}} d\lambda(t) \int_{\pi^{-1}(\mathcal{C}_t)} \mathcal{E}^p_{\#}.d\mu_t$$

for $d\mu_t$ some invariant measure on $\pi^{-1}(\mathcal{C}_t)$. However, $\mathcal{O}_{\pi^{-1}(\mathcal{C}_t)}(-\mathcal{E}^p_{\#})$ is ample, by construction, so this is at most zero. The part over $\mathcal{Z}$ is as before, i.e. here the induced foliation on $\mathcal{E}^p_{\#}$ has π as a first integral, hence we may argue exactly as in the proof of VI.5.3 by way of vigorous application of VI.4.1 to deduce that the conditions of VI.5.4 hold. □

This completes the discussion if our original singular component of interest meets the locus where the induced foliation on a single weak branching stack in $\mathcal{X}$ is smooth. Unfortunately, as we've noted post VI.4.2 the general situation is that we need to go down a chain $\mathfrak{V}^1 \supset \ldots \supset \mathfrak{V}^p$ with $\mathfrak{V}^{i+1}$ a weak branch in $\mathfrak{V}^i$ before concluding to generic smoothness. This is more notationally fastidious than anything else. For example given that $\mathfrak{V}^1$ is defined as the smallest weak branching stack in $\mathcal{X}$ containing our component the notation $\mathfrak{V}^1 \supset \mathfrak{V}^2$ is unambiguous, but $\mathfrak{V}^1 \supset \mathfrak{V}^2 \supset \mathfrak{V}^3$ may not be, depending on whether $\mathfrak{V}^3$ extends to a weak branch in $\mathfrak{V}^1$ or not. Consequently we do some relabeling/throwing out so that $\mathfrak{V}^2$ is actually the smallest weak branch in $\mathfrak{V}^1$ containing our component, idem for $\mathfrak{V}^3$ in

$\mathfrak{Y}^2$ etc., and write this as $\mathfrak{Y}^1 > \ldots > \mathfrak{Y}^p$. As such $\mathfrak{Y}^{i+1}$ in $\mathfrak{Y}^i$ has the additional property that if it contains a component of the singular locus of the induced foliation $\mathcal{G}^{i-1}$, say, on $\mathfrak{Y}^{i-1}$ then it actually equals it. Necessarily we can suppose that each $(\mathfrak{Y}^i, \mathcal{G}^i)$ is smooth with log-canonical singularities, indeed even EmbLCR, and we have associated pseudo irreducible formal stacks $(\mathfrak{W}^i, \mathcal{F})$ defined in the obvious way as post VI.6.2, i.e. throw in the pseudo trace components of $\mathfrak{Y}^i$, and extend $\mathfrak{Y}^i$ were possible. A moments reflection shows that $(\mathfrak{W}^i, \mathcal{F})$ is actually contained in $(\mathfrak{W}^{i-1}, \mathcal{F})$, and plainly what we'll have to do is prove that $K_{\mathcal{F},\mathbf{W}^p}$ is nef. in some suitable sense. Certainly $K_{\mathcal{F},\mathbf{W}^p}$ is wholly well defined as a bundle on $\mathfrak{W}^p$, but just as the definition of nefness for $K_{\mathcal{F},\mathbf{W}^1}$ was only for $K_{\mathcal{F}}$ nil curves we'll have to do something similar here - this results from the relations governing $K_{\mathcal{F},\mathbf{W}^{i+1}}$ in terms of $K_{\mathcal{F},\mathbf{W}^i}$. In this respect observe, apart from the ubiquitous we're done if $\mathfrak{W}^i$ is foliated in conics (which we'll systematically ignore from now on to simplify the discussion), that the minimal model algorithm has nothing to do with weak branching, i.e. no flipping variety ever meets a weak branch, so infact $K_{\mathcal{F},\mathbf{W}^1}$ is $K_{\mathcal{F},\mathbf{W}^1_+}$ around $\mathfrak{W}^2$, in an obvious extension of the notations pre VI.7.1. This suggests (for notational reasons if nothing else) that we proceed back to front, since VI.6.3 is already a given for the image $\mathfrak{W}^2_1$ of $\mathfrak{W}^2$ in $\mathfrak{W}^1_+$. Now define *nefness* for $\mathfrak{W}^2$ to mean that $K_{\mathcal{F},\mathbf{W}^2}$ is non-negative on curves which are $K_{\mathcal{F},\mathbf{W}^1_+}$, and $K_{\mathcal{F}}$ nil, or better: just define a $K_{\mathcal{F},\mathbf{W}^2}$ *negative curve* to be an invariant rational curve in $\mathfrak{W}^2$ on which $K_{\mathcal{F}}$, $K_{\mathcal{F},\mathbf{W}^1_+}$ are nil, and $K_{\mathcal{F},\mathbf{W}^2}$ is negative. Rather helpfully all of VI.6.7-11 is valid, independently of VI.6.5 so without having verified that it satisfies the nefness criterion, we can nevertheless define $\mathfrak{W}^2_+$ by the condition that we've exhaustively flipped all pseudo extremal $K_{\mathcal{F},\mathbf{W}^2}$ negative curves in $\mathfrak{W}^2_1$, and whence there are no more such. In addition we can use all the other arrows in VI.6.12 so that $\mathfrak{W}^1_+$ gets modified just as $\mathcal{X}$ in op. cit. by weighted blow up in $K_{\mathcal{F},\mathbf{W}^1_+}$ invariant centres, and needless to say $\mathcal{X}$ gets modified too. Subsequently we define $K_{\mathcal{F},\mathbf{W}^3}$ negative curves in the obvious way, flip all of these, and continue till we eventually get,

VI.7.2 Diagram

$$\begin{array}{ccccccccc}
\mathfrak{W}^p & \subset & \ldots\ldots & \subset & \mathfrak{W}^2 & \subset & \mathfrak{W}^1 & \subset & \mathcal{X} \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\mathfrak{W}^p_1 & \subset & \ldots\ldots & \subset & \mathfrak{W}^2_1 & \subset & \mathfrak{W}^1_+ & & \\
\downarrow & & & & \downarrow & & & & \\
\mathfrak{W}^p_2 & \subset & \ldots\ldots & \subset & \mathfrak{W}^2_+ & & & & \\
\downarrow & & & & & & & & \\
\vdots & & & & & & & & \\
\downarrow & & & & & & & & \\
\mathfrak{W}^p_+ & & & & & & & &
\end{array}$$

where all the arrows $\rho^!_{*\bullet} : \mathfrak{W}^!_* \to \mathfrak{W}^!_\bullet$ are a sequence of weighted blow ups in centres contained in the strict trace, and wholly transverse to the induced foliation around the distinguished open $\mathfrak{Y}^!_\bullet$, and any extensions thereof. Now for each $1 \leq i \leq p$, $j < i$, and each component $\mathcal{V}$ of the trace or pseudo trace of $\mathfrak{W}^i$, let $\mathrm{NE}(\mathcal{V})_j$ be the curves in $\mathrm{NE}(\mathcal{V})$ on which $(\rho_+)^* K_{\mathcal{F},\mathbf{W}^k_+}$ is nil for all $k \leq j$, including $K_{\mathcal{F}}$ nil, $k = 0$. Having thus defined everything, it remains to check that it works.

Obviously it would be very disappointing if the following proved insufficient,

VI.7.3 Fact *$(\rho^p_+)^* K_{\mathcal{F},\mathbf{W}^p_+}$ is non-negative on* $\mathrm{NE}(\mathcal{V})_{p-1}$ *for all $\mathcal{V}$ in the trace or pseudo trace of $\mathfrak{W}^p$.*

proof VI.6.5 has been constructed to generalise easily to these circumstances. Since, however, we're determined to sail close to the border where this sort of thing fails, we give some extra details. We go by induction on q, $1 \le q \le p$, with $q = 1$ being precisely VI.6.5. To do the case q (≥ 2), supposing $q - 1$ we use VI.7.2, but with q instead of p.

On starts therefore, for an appropriate $\mathcal{V}$, with an extremal ray R in $\mathrm{NE}(\mathcal{V})_{q-1}$, supposed to have $(\rho^q_+)^* K_{\mathcal{F},\mathbf{W}^q_+}.R < 0$, while the inductive hypothesis allows us to suppose R is a limit of irreducible 1-cycles $a_i Z_i$ such that,

$$\frac{(\rho^r_+)^* K_{\mathcal{F},\mathbf{W}^r_+}.Z_i}{-(\rho^q_+)^* K_{\mathcal{F},\mathbf{W}^q_+}.Z_i} \to 0\ ,\ i \to \infty$$

for each $r < q$, including $K_{\mathcal{F}}$ for $r = 0$. The log-canonical nature of the singularities of $(\mathfrak{Y}^q_+, \mathcal{G}^q)$ together with the structure of the left most column of VI.7.2, i.e. $K_{\mathcal{F},\mathbf{W}^q_+}$ terminality, allows us to play the bi-rational groupoid trick. Whence for $f_i : \mathcal{C}_i \to \mathfrak{W}^q_+$ the normalisation of the substack supported on $(\rho^q_+)(Z_i)$ we produce a normal surface $s : \mathcal{S}_i \to \mathcal{C}_i$ together with a section whose normal bundle is $\mathcal{O}_{\mathcal{C}_i}(-K_{\mathcal{F},\mathbf{W}^q_+})$, so that for every $c_i \in \mathcal{C}_i(\mathbb{C})$ there is an invariant rational curve $\mathcal{L}_{c_i}$ satisfying the basic estimate,

$$N.\mathcal{L}_{c_i} \le 2 \frac{N._{f_i}\mathcal{C}_i}{-K_{\mathcal{F},\mathbf{W}^q_+}\cdot_{f_i}\mathcal{C}_i} \quad (\to 0,\ \text{as } i \to \infty)$$

for every nef N (as ever Mumford sense) divisor on $\mathcal{S}_i$. In particular, as per VI.6.5, $K_{\mathcal{F}}$ is nil on every $\mathcal{L}_{c_i}$, and more generally on every fibre of s.

To get from here to a contradiction we require to prove that all other $K_{\mathcal{F},\mathbf{W}^r_+}$'s, for $r < q$ are nef., stricta dictum, on $\mathcal{S}_i$. To do this one does an induction within an induction à la VI.6.6. Indeed with VI.6.6 as is, and the nullity of $K_{\mathcal{F}}$ on the fibres of s, one sees that $K_{\mathcal{F},\mathbf{W}^1_+}$ is already nef. on $\mathcal{S}_i$, so for $i \gg 0$ the basic estimate applies to deduce that $K_{\mathcal{F},\mathbf{W}^1_+}$ is nil on all the fibres of s too. Now reprove VI.6.6, but for $K_{\mathcal{F},\mathbf{W}^2_+}$ with the polyhedral part generated by $K_{\mathcal{F},\mathbf{W}^2_+}$ negative curves (bearing in mind this includes $K_{\mathcal{F}}$, $K_{\mathcal{F},\mathbf{W}^1_+}$ nullity). Since there are no such curves, $K_{\mathcal{F},\mathbf{W}^2_+}$ is nef., and so forth till we get up to $K_{\mathcal{F},\mathbf{W}^q_+}$ nef, so as to contradict $K_{\mathcal{F},\mathbf{W}^q_+}.\mathcal{C}_i < 0$. $\square$

By the usual expedient of projecting into Néron-Severi, it goes, without saying, that we have,

VI.7.4 Corollary *For every component $d\nu$ of a transverse parabolic measure $d\mu$ supported on a substack factoring through the trace or pseudo trace of some $\mathfrak{W}^q$, $(\rho^q_+)^* K_{\mathcal{F},\mathbf{W}^q_+}.d\nu \ge 0$, provided $(\rho^r_+)^* K_{\mathcal{F},\mathbf{W}^r_+}.d\nu \ge 0$, for each $0 \le r < q$.*

The next thing to observe is that the proof of VI.5.3 doesn't depend on the fact that $\mathcal{X}$ is a stack, but that $d\mu_{\mathcal{X}/\mathcal{F}}$ satisfies V.1.2 (b) & (c) - (a) & (d) are irrelevant- provided that Cartier divisor is understood in the more expedient sense of directly relevant to the proof of VI.5.3 (4), so, infact one can even tone down V.1.2 (c) to (notations as per op. cit.) ρ an invariant (including root extraction) modification of pseudo irreducible formal stacks. Consequently since VI.5.3 is as valid on any modification which in current notation is $\mathcal{G}^1$ invariant seen from $\mathfrak{Y}^1$ as the $\mathcal{X}$ we first thought of, we can induct through the $\mathfrak{W}^i$ to conclude,

VI.7.5 Fact *For any $1 \leq q \leq p$, and any component of the strict trace of $\mathfrak{W}^q$, $\mathcal{Y}.\mathbb{I}_{\mathbf{W}^q} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, where, as ever $\mathbb{I}_{\mathbf{W}^q} d\mu_{\mathcal{X}/\mathcal{F}}$ is the sum of components of the trace or pseudo trace of $\mathfrak{W}^q$, and, of course, these are the only numerically relevant components as per* VI.5.1/2.

Now we have to carefully use VI.7.2 to induct up to p, by way of the formulae,

$$(\rho_+^q)^* K_{\mathcal{F},\mathbf{W}_+^q} + (\rho_{q-1}^q)^*(\mathcal{E}_{q-1} + \mathcal{E}') = (\rho_{q-1}^q)^* K_{\mathcal{F},\mathbf{W}_+^{q-1}}$$

where the effective Cartier divisors $\mathcal{E}_{q-1}, \mathcal{E}'$ are the differences between $K_{\mathcal{F},\mathbf{W}_{q-1}^q}$ and $K_{\mathcal{F},\mathbf{W}_+^q}$, respectively $K_{\mathcal{F},\mathbf{W}_+^{q-1}}$. As such $\mathcal{E}'$ is in the strict trace of $\mathfrak{W}_{q-1}^q$, and ρ_{q-1}^q is even an isomorphism here, so VI.7.5 is well adapted to dealing with this, as it is for any components of $\mathcal{E}_{q-1}$ in the strict trace. Where we have to be cautious is at a component $\mathcal{E}$, say, which lies outside the strict trace of $\mathfrak{W}_{q-1}^q$. In this respect observe that the role of the projectivity of $\mathcal{X}$ is limited. The place (in this context) where we need this is the implication that numerically relevant things must be of pseudo trace type. This, however, we can do a priori in $\mathcal{X}$ with a possibly different notion of pseudo trace which pushes forward under $\rho_{*\bullet}^!$ to what we want. Consequently the difficulty rests in knowing that if we start associating weighted blow ups, $\sigma_+^{n,q-1} : \mathfrak{W}_+^{n,q-1} \to \mathfrak{W}_+^{q-1}$ to $\mathcal{E}$ inside $\mathfrak{W}_+^{q-1}$ with associated exceptional divisors $\mathcal{E}_\#^n$ and modify the whole of VI.7.2 appropriately then $\mathcal{E}_\#^n.\mathbb{I}_{\mathbf{W}_+^{n,q-1}} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, since otherwise VI.7.1 goes through without change. To this end, therefore, consider a generic curve $\mathcal{C}$ in $\mathcal{E}$, evidently rational & invariant, which doesn't lie in the strict trace of $\mathfrak{W}_{q-1}^q$. Viewed in $\mathfrak{W}_+^q$ it's either in the strict trace or it isn't. If it is then we're happy since all the $\rho_{*\bullet}^!$'s pull back strict trace to strict trace, so we've got what we need by VI.7.5, which is valid for any modifications which seen from $\mathfrak{Y}^q$ are $\mathcal{G}^q$ invariant. Otherwise we look at the proper transform $\tilde{\mathcal{C}}$ of $\mathcal{C}$ in $\mathfrak{W}_{q-2}^{q-1}$. If this is in the strict trace of $\mathfrak{W}_+^{q-2}$ then we stop. If it isn't then we appeal to the nullity of $K_{\mathcal{F},\mathbf{W}_+^{q-2}}$ along $\mathcal{C}$ and II.3. Indeed the index set J of op. cit. can be viewed as the number of curves contracted by ρ_{q-1+}^{q-1} meeting $\mathcal{C}$, which has to become an inded set I viewed from $\mathfrak{W}_+^{q-2}$, so infact J is empty, i.e. ρ_{q-1+}^{q-1} is an isomorphism around $\mathcal{C}$. Thus,

VI.7.6 Possibilities *Continuing we get exactly one of,*

(a) *We find some $\mathfrak{W}_+^r$, $1 \leq r \leq q$ containing the proper transform $\mathcal{E}^r$ of $\mathcal{E}$ in its strict trace. Better still for each $r < s \leq q$ an invariant rational curve $\mathcal{C}$ in the proper transform $\mathcal{E}_+^s$ of $\mathcal{E}$ in $\mathfrak{W}_+^s$ is either wholly contained in the strict trace, or meets it, infact in a unique point, and all the singularities of the induced foliation on $\mathfrak{W}_+^s$ around $\mathcal{E}_+^s$ are contained in the strict trace.*

(b) *$\mathcal{E}$ even meets the smooth locus of $\mathcal{X}$, with everything as in (a), except that $r = 0$.*

Now to conclude we argue as follows. To begin with, albeit that it's convenient rather than important, everything in VI.7.2 is smooth around $\mathcal{E}$ and its proper transforms for the habitual reason, i.e. singularity is an invariant condition, yet everything is smooth around the distinguished opens $(\mathfrak{Y}^!, \mathcal{G}^!)$. Next we a priori blow up a lot in sing$(\mathcal{G}^q)$ to guarantee that around an invariant rational curve $\mathcal{C}$ in $\mathcal{E}$, not in the strict trace, the ampleness of various bundles such as , $N^{\vee}_{\mathcal{C}/\mathbf{W}^{\bullet}_{+}}|_{\mathcal{C}}$, $N_{\mathbf{W}^{\bullet}_{+}/\mathbf{W}^{!}_{+}}|_{\mathcal{C}}$, etc. The entire discussion is $\mathcal{G}^q$ invariant, so this changes nothing. Our starting point therefore (with the obvious notation) is $\mathcal{E}^{r+1}_r$ inside $\mathfrak{W}^r_+$ mapping to $\mathcal{E}^r_+$, so that necessarily (II.3.5) the points where this isn't an isomorphism are in the strict trace of $\mathfrak{W}^{r+1}_r$. Next, in a notation that's rapidly getting out of hand, form for each n weighted blow ups $\mathfrak{W}^{r,n}_+ \to \mathfrak{W}^r_+$ associated to the nth power of the pull back of the ideal of $\mathcal{E}^{r+1}_+$ viewed in $\mathfrak{W}^r_+$, with $\mathcal{E}^{r,n}_{\#}$ the total exceptional divisor, and $\mathcal{Z}$ the locus where $\mathcal{E}^{r+1}_r$ isn't isomorphic to $\mathcal{E}^{r+1}_+$. At which point we have everything we need: $\mathcal{E}^{r,n}_{\#}.\mathbb{1}_{\mathbf{W}^{r,n}_+} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$, by VI.7.6 (a) or (b), $\mathcal{E}^{r,n}_{\#}.\mathbb{1}_{\mathcal{E}^{r,n}_{\#} \setminus \mathbf{W}^{r,n}_r} d\mu_{\mathcal{X}/\mathcal{F}} \leq 0$ as in VI.7.1, i.e. off $\mathcal{Z}$ we have our ampleness argument, while over $\mathcal{Z}$ the induced foliation on $\mathcal{E}^{r,n}_{\#}|_{\mathcal{Z}} \to \mathcal{Z}$ has $\mathcal{Z}$ as a first integral, and the numerically relevant components all lie in $\mathbb{1}_{\mathbf{W}^{r+1}_r} d\mu_{\mathcal{X}/\mathcal{F}}$ by VI.5.4, or more correctly its proof. More generally the same argument works for any scheme structure, or better sequence of scheme structures, supported on $\mathcal{E}^{r+1}_+$ which when pulled back to $\mathfrak{W}^{r+1}_r$ differ from that on $\mathfrak{W}^{r+1}_+$ only around the strict trace of $\mathfrak{W}^{r+1}_r$. All of which is equally valid for any of steps,

$$\begin{array}{ccc} \mathfrak{W}^{s+1}_s & \longrightarrow & \mathfrak{W}^s_+ \\ \downarrow & & \\ \mathfrak{W}^{s+1}_+ & & \end{array}$$

for any $q-1 \geq s \geq r$, with the exception of the positivity of the ambient weighted blow ups on $\mathfrak{W}^s_+$, which we obviously do by induction down from r. Whence,

VI.7.7 Fact *For every* $1 \leq q \leq p$, $(\rho^q_{q-1})^* \mathcal{E}_{q-1}.\mathbb{1}_{\mathbf{W}^q} d\mu_{\mathcal{X}/\mathcal{F}} \geq 0$.

Thus by VI.7.4 we obtain,

VI.7.8 Final Fact *For any singular component* $d\nu$ *of a parabolic invariant measure in the strict trace of* $\mathfrak{W}^p$, $(\rho^p_+)^* K_{\mathcal{F},\mathbf{W}^p_+}.d\nu = 0$.

In the particular case that our foliated stack $(\mathcal{X}, \mathcal{F})$ comes from an ODE on a 2-dimensional algebraic stack $\mathcal{S}$ with projective moduli by VI.4.2, and as noted pre VI.6.1, this provides the missing piece of the jigsaw required to extend VI.3.5 to singular components, so that indeed in the presence of LCR (I.6.1), VI.2.3 holds as asserted. To conclude from here to the statements of the introduction is an immediate application of [III].

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