Bloch Hyperbolicity

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Abstract

Recent study of the hyperbolicity of algebraic varieties has centered on purely qualitative aspects such as non-existence of maps from the complex line. Occasionally such properties thanks to lemmas of R. Brody and M. Green give quantitative information, but in general cannot be related to A. Bloch's dictum, "Nihil est in infinito quod non prius fuerit in finito". We remedy this for quasi-projective surfaces (X, D) of general type and $s_2(X, D) > 0$. Our techniques also yield essentially optimal bounds for the degeneration of the Kobayashi metric, and the heights of algebraic points over fields of functions in characteristic zero.

0 Introduction

The origins of the modern study of the hyperbolicity of higher algebraic varieties appears to be A. Bloch's substantial generalisation of Montel's theorem that maps of the disc to $\mathbb{P}^1\backslash\{0, 1, \infty\}$ are normal to the case of $\mathbb{P}^n\backslash\{\text{hyperplanes}\}$. Specifically Bloch, [B], or perhaps more correctly H. Cartan in his thesis [C], given the suggestion of obscurity in the former, showed that a sequence of maps to $\mathbb{P}^n\backslash\{n+2\,\text{hyperplanes}\}$ which does not have a convergent subsequence must be eventually arbitrarily close, in the compact open sense, to the diagonal hyperplanes, i.e. identify $\mathbb{P}^n$ with $X_0 + \cdots + X_{n+1} = 0$ in $\mathbb{P}^{n+1}$, and the $n + 2$ planes with the coordinate hyperplanes in the latter then for $I \subset \{0, \ldots, n + 1\}$, $2 \leq |I| \leq n$, $\sum\limits_{i \in I} X_i = 0$ is a diagonal hyperplane.

Evidently any extension of such a result should pertain to at most projective varieties X with simple normal crossing boundary D (actually log-canonical singularities would be sufficient) such that (X, D) has logarithmic general type. Equally, even under this hypothesis, a moments thought shows that Bloch's result will not necessarily hold. Indeed if say $D = \varphi$, and X contains a rational curve then one can trivially make sequences of discs which converge to discs with bubbles. More generally for $D \neq \varphi$, this will happen if either $X \backslash D$ or some $D_i \backslash \bigcup_{j \neq i} D_j$ contains $\mathbb{A}^1$'s where D_i are the components of D, while divergent sequences arbitrarily close to D can be formed if $D_i \backslash \bigcup_{j \neq i} D_j$, contains holomorphic lines, e.g. elliptic curves or $\mathbb{G}_m$'s. Necessarily these obstructions don't occur in Bloch's theorem, but in general they are unavoidable. Certainly in dimension 2, at the minor cost of allowing quotient singularities we can work on a model such that there is no a priori reason for being able to construct larger and larger discs close to the boundary D. However in general it's a delicate question, which to answer even conjecturally would require a number of conjectures of which the most probable and tractable is the minimum model programme itself. Whence in a general formulation of Bloch's approach to hyperbolicity let us allow the possibility of arbitrary proximity to the boundary D. We must also allow the possibility of discs with bubbles (which incidentally is possible iff there are $\mathbb{A}^1$'s in $X \backslash D$, supposing $D_i \backslash \bigcup_{j \neq i} D_j$ not containing the same) and so using Gromov's compactness theorem, [Gr], introduce a space $\overline{\mathrm{Hom}}(\Delta, X)$ of discs with bubbles except at the origin, which compactifies the subspace of $\mathrm{Hom}(\Delta, X)$ of maps from the disc of "bounded" Nevanlinna degree, §2.1. Consequently the optimal, modulo the aforesaid questions on the boundary D, extension of Bloch's theorem would ask: if a sequence in $\overline{\mathrm{Hom}}(\Delta, X \backslash D)$ doesn't enjoy a convergent subsequence in $\overline{\mathrm{Hom}}(\Delta, X)$ is it arbitrarily close of $\overline{Z} \cup D$ where $Z \subset X \backslash D$ is a proper closed subvariety. Risking confusion with many other possible definitions of the same let us call this property *hyperbolic modulo Z*.

Observe that this definition immediately implies Kobayashi's, and indeed somewhat more, i.e., Kobayashi considers the invariant metric on $T_X(-\log D)$, a priori defined as a pseudo-metric off D, which assigns to a tangent vector the reciprocal of the largest disc through it, 2.3.2. Hyperbolic modulo Z in the Bloch sense immediately implies that off Z this function isn't just a metric, Kobayashi's definition, but that the said metric is continuous. Manifestly it also implies such properties as every holomorphic line $f : \mathbb{C} \to X \backslash D$ factors through Z. In the particular case that both $X \backslash D$ and $D_i \backslash \bigcup_{j \neq i} D_j$ contain no such holomorphic lines then by the reparametrisation lemma of Brody, together with a lemma of M. Green these definitions are equivalent, but in general they need not be, although one can probably get from the Kobayashi definition to the Bloch definition with some extra information such as quantification of the degeneration of the metric around Z and D. The other quasi converse is that with the exception of the Bloch theorem itself there seem to be no known examples of hyperbolicity in this sense beyond those deducible from the lack of lines together with reparametrisation and Green's lemma, or a minor $\mathbb{Q}$-factorial variation thereof. An ample illustration of this is the relative difficulty involved in the proofs that maps $f : \mathbb{C} \to \mathbb{P}^n \backslash \{\text{hyperplanes}\}$ factor through the diagonals, which even in the quantitative version of a defect relation in value distribution theory is at worst straightforward, in comparison to Bloch's proof which is anything but. Consequently as an initial example we consider what is often considered the key case of the Bloch theorem, $n = 2$, which in any case is that which is most pertinent to our ultimate interest of algebraic surfaces.

To quickly summarise our approach to the Bloch theorem, 3.2.1, we must introduce the bundle of 2-jets, P_2, which is a $\mathbb{P}^1$-bundle over the projective logarithmic tangent space P_1. These spaces have associated tautological bundles L_i, which satisfy a tautological inequality 1.2.8 which for $f \in \mathrm{Hom}(\Delta, X)$ is morally of the form,

$$L_{i \cdot f}\, \Delta(r) \leq \log{(1-r^2)} + \mathrm{rad}_{f,D}\,(r)$$

where the degree is measured in the sense of Nevanlinna, and $\Delta(r)$, $r \in (0,1)$ is the disc of radius r while $\mathrm{rad}_{f,D}\,(r)$ counts the points of intersection, as ever in Nevanlinna's sense, of f with D without multiplicity. Our starting point/Bloch's finishing point is that, $L_2 \otimes L_1^2(-H)$, where incidentally H is of course the tautological bundle on $\mathbb{P}^2$ but more pertinently the log-canonical bundle of $\mathbb{P}^2 \backslash \{4 \text{ planes}\}$, has a global section, which happens to correspond to the Wronskian, but this is of very limited importance, and this gives at once the Nevanlinna defect relation, unless $f : \mathbb{C} \to \mathbb{P}^2$, factors through a line, which is trivially easy to analyse. In the Bloch situation however, these observations only tell us that if a sequence f_n doesn't admit a convergent subsequence then the second derivatives f_n'' are eventually arbitrarily close to the divisor cut out by the Wronskian. The tautological inequality together with the observation that one may associate a closed positive current T to such a sequence should it fail to have accumulation points (although in this special case as a particular consequence of the geodesics defined by the Wronskian being algebraic, harmonic rather than closed would suffice) gives $L_i \cdot T \leq 0$, while a sequence of blowing up and down, actually a flip of the foliation on P_1 defined by the Wronskian with the obvious definition of that notion, allows us to get the inequality $H \cdot T \leq 0$ thanks to the invariance of T by the same and its smoothness on the birational model $\mathbb{P}(\Omega_{\mathbb{P}^2})$. This latter conclusion being modulo the first derivatives f_n' not being arbitrarily close to the foliations defined by any pair of lines, which we'll easily deal with by way of our main lemma.

Now for arbitrary surfaces (X, D) of log-general type, one can certainly look at the tautological bundles L_i on some logarithmic jet spaces P_i, and hope to find $a_i \in \mathbb{N}$, such that $\bigotimes_{i=n}^{n} L_i^{a_i}(-(K_X+D))$ has a global section for some $n \gg 0$. However just as the example of the Bloch theorem illustrates this will be of little value unless it can be eliminated systematically that the lower derivatives $f^{(k)}$, $k < n$, are not arbitrarily close to divisors in the lower jet spaces P_{k-1}, the divisor in question, unlike the exception of the 1$^{\text{st}}$ stage of the Bloch theorem, not being either necessarily irreducible or purely horizontal over the previous jet space. The first case of interest is, therefore, nothing other than a divisor F in $\mathbb{P}(\Omega_X(\log D))$. Again in contrast to the Bloch theorem the horizontal components of F need not have degree 1, i.e. they correspond to arbitrary 1^{st} order ODE's rather than just linear ones/foliations. We can of course reduce to the latter case, but only at the cost of replacing maps from the disc by ramified covers of the same, the fact that such a thing is covered by the disc being irrelevant since the key issue is how it is exhausted. Consequently our natural objects of study are very much inverse to the uniformisation theorem, i.e. maps and indeed sequences of maps,

$$\begin{array}{ccc} Y_k & \xrightarrow[f_k]{} & X \\ \downarrow{\scriptstyle p_k} & & \\ B & & \end{array}$$

where $p_k : Y_k \to B$ is a ramified cover of $B = \mathbb{P}^1$, $\mathbb{C}$ or the disc Δ. Needless to say the disc theory is the most interesting, and so we tend to concentrate on it. However even the case of $B = \mathbb{P}^1$, and $\dim X = 2$, will yield new information by way of our Main Lemma, namely,

Main Lemma (4.1) *Let X be a projective surface, and D a simple normal crossing divisor with, $f_k : Y_k/B \to X$ a sequence of maps, such that the derived current T' associated to the derivatives $f'_k : Y_k \to \mathbb{P}(\Omega_X(\log D))$ is supported in a divisor all of whose horizontal components have degree 1. Then after a sufficiently large birational modification $\rho : (\widetilde{X}, \widetilde{D}) \to (X, D)$*[1]*, with $\widetilde{T}'$ the derived current associated to the proper transforms of the f_k and L the tautological bundle on $\mathbb{P}(\Omega_{\widetilde{X}}(\log \widetilde{D}))$ we have,*

$$L.\, \widetilde{T}' \geq (K_{\widetilde{X}} + \widetilde{D}).\, \widetilde{T}$$

provided $\widetilde{T}'$ is closed, and its pushforward $\widetilde{T}$ satisfies, $D_1 \cdot \widetilde{T} \geq 0$ for all D_1 in the support of $\widetilde{D}$.

The proof goes very much as per our sketch of the Bloch theorem with lots of blowing up to handle the foliation singularities although surprisingly the main complication arises from non-vertical components of the support of the derived current which lead to even more blowing up. All the machinery for handling the horizontal components being already present in [M1] and [Br], where one should note that unlike related results about holomorphic lines due to the necessary presence here of ramification the precise lower bound is absolutely essential for applications, and would not be possible by the coarse estimates on the deviation of flatness along the leaves of normal bundles to foliations in the former, but only thanks to

[1] Sufficiently large means determined by X, D, the said divisor etc., but not the maps f_k, while $\widetilde{D}$ is permitted to be very much bigger than $\rho^{-1}(D)$, although always with simple normal crossings.

the best possible estimate of the latter. Contrary to first appearances the closure of the appropriate current is always guaranteed in any application of interest, 1.3, and so we obtain,

COROLLARY 1.(3.1) *Let (X,D) be an algebraic surface of logarithmic general type with $s_2(X,D) > 0$ (i.e. $c^2_{1,\log} > c_{2,\log}$) then for some proper Zariski closed subset Z of $X\backslash D$, (X,D) is hyperbolic modulo Z.*

What's more even when the assumption of positivity of intersection with the components of D isn't satisfied, the methodology still gives, something, i.e.

COROLLARY 2.(4.2) *Let (X,D) be a minimal model of a logarithmic general type surface with $s_2(X,D) > 0$ and Z the closed subset of corollary 1, with $\rho : (\widetilde{X},\widetilde{D}) \to (X,D)$ a minimal resolution such that $\widetilde{D}$ (the total reduced pull-back of D) and $\widetilde{Z}$, the proper transform of Z, have simple normal crossings. Further let E_i be the divisors contracted by ρ and $a_i \in \mathbb{Q}_{\geq 0}$ their logarithmic discrepancies then for all $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that for any $f \in \mathrm{Hom}(\Delta, X\backslash D)$,*

$$\prod_i \|1\!\!1_{E_i}\|^{a_i}(0)\, \|d\widetilde{f}\|_{\widetilde{\log}}(0) \leq \frac{C(\varepsilon)}{\|1\!\!1_{\widetilde{Z}}\|^{1+\varepsilon}\, \|1\!\!1_{\widetilde{D}}\|^{\varepsilon}\,(0)}$$

where $\widetilde{f} : \Delta \to \widetilde{X}$ is the proper transform, and $\| \ \|_{\widetilde{\log}}$ is a norm on $T_{\widetilde{X}}(-\log \widetilde{D})$.

This theorem on the degeneration of the Kobayashi metric can almost certainly be improved by replacing $x^{-\varepsilon}$ with $|\log x|$. However that would involve unravelling the definitions and an admission that blowing up cannot wholly take the place of analysis, or perhaps only the more acceptable admission that algebraic resolution of singularities is not quite the resolution of singularities which M. Gromov envisages as underpinning estimates in analysis.

Regardless the main lemma can be thought of as a rather surprising extension of the principle that one dimensional hyperbolic objects have curvature -1. Surprising in so much as the object in question is a sequence of approximate solutions of an ODE, which even in the substantially more trivial case of $B = \mathbb{P}^1$ yields,

COROLLARY 3.(4.3) *Let (X_k, D_k) be a quasi-projective curve over a function field k in characteristic zero and $\varepsilon > 0$ then there is a constant $C(\varepsilon)$ such that for all algebraic points $f \in X(\overline{k})\backslash D$,*

$$h_{\omega_X + D}(f) \leq (1+\varepsilon)\,(\mathrm{discr}(f) + \mathrm{rad}_{f,D}) + C(\varepsilon)\,(k(f):k)$$

where the radical is the intersection of f with D counted without multiplicity calculated in any suitable model.

The case when the pair (X_k, D_k) is isotrivial is of course just the logarithmic Riemann-Hurwitz formula with, as Vojta observes, [V1], the special case of $\mathbb{P}^1\backslash\{0,1,\infty\}$ being Mason's "a,b,c" theorem for polynomials. The constant $1+\varepsilon$ is of course best possible but the $C(\varepsilon)$ appears ineffective beyond hope, which is rather contrary to Vojta's own $2+\varepsilon$, [V2], which apart from being valid in characteristic p (more algebraically currents can be replaced by the Leray spectral sequence in De Rham cohomology in our proof, but that doesn't help much, although one should note that the extension [K] of [V2] to positive characteristic is wrong, but easily fixable on noting that the various foliation singularities which cannot be resolved never contribute more than the degree over the base) is essentially effective, i.e. uniform in the moduli.

The debt of this article to Brunella and Gromov is enormous, but pales to nothing before the labours of Cécile.

1 Nevanlinna Theory

1.1 Preliminaries

The entire content of what we require is summarised by way of the operator,

$$\deg : A^{1,1}(\overline{\Delta}) \to \mathbb{R} : \eta \mapsto \int_0^1 \frac{dt}{t} \int_{\Delta_t} \eta \tag{1.1.1}$$

where the space of forms in question are smooth on the unit disc Δ, and continuous on its boundary, with Δ_t the disc of radius t. For what it's worth the only necessary regularity assumption is that the right hand side makes sense, so bounded on $\overline{\Delta}$ would be another adequate choice of a suitable space of forms. The rather curious point here is that from an intersection theory point of view it's the degree function rather than simply the integral over the disc which is the correct analogue of the degree, equivalently pushforward to a point, equaivalently integral for compact Riemann surfaces. As such, following Fulton, it would be philosophically preferable to write $\int_{\overline{\Delta}} \eta$ for $\deg(\eta)$, however, this is likely to be notationally very confusing so for ease of exposition we'll avoid this. The important things to note are that for Δ_r the disc of any radius we can identify this with Δ via multiplication m_r by r and so define,

$$\deg(r) : A^{1,1}(\overline{\Delta}_r) \to \mathbb{R} : \eta \mapsto \deg(m_r^* \eta) \tag{1.1.2}$$

while the ultimate justification for the definition of the degree lies in Jensen's formula, i.e. for $\overline{L}$ a line bundle on $\overline{\Delta}$ with metric and s a meromorphic section of L,

$$\deg\left(c_1(\overline{L})\right) = -\int_{\partial\overline{\Delta}} \log \frac{\|s\|}{\|s\|\,(0)} \frac{d\theta}{2\pi} + \sum_{0\,<\,|z|\,<\,1} \mathrm{ord}_z(s) \log\left|\frac{1}{z}\right| \tag{1.1.3}$$

where for convenience, i.e. to avoid a more complicated formula, we suppose 0 is neither a zero nor pole of s. Consequently in a typical application of f a map from Δ to a projective variety X and $\overline{D}$ a metricised effective divisor,

$$\overline{D} \cdot_f \Delta(r) := \deg(r)\,(f^* c_1(\overline{D}))\,, \quad r < 1 \tag{1.1.4}$$

is basically positive provided f doesn't factor through D. There is however a very important caveat in that more precisely,

$$\overline{D} \cdot_f \Delta(r) \geq \log \| f^* \,1\!\!1_D \| \,(0) + 0_D(1) \tag{1.1.5}$$

where the implied constant is certainly independent of f, but for f_n a sequence of maps with $f_n(0)$ progressively closer to D the assertion $\overline{D} \cdot_{f_n} \Delta(r) \geq 0_{\overline{D}}(1)$ is not in general possible. It is precisely this apparently minor difficulty that distinguishes the Nevanlinna theory of discs from that of the special case of maps from the complex line, which to all intents and purposes behave as sequences of maps from discs but with fixed origin. In any case for any metricised divisor $\overline{D}$, effective or not, we'll extend the definition of $\overline{D} \cdot_f \Delta(r)$ in the obvious way, and there remains to note the natural extension of all of this to ramified coverings $p : Y \to \Delta$, i.e. for $r < 1$,

$$\deg_Y(r) : A^{1,1}(Y(r)) \to \mathbb{R} : \eta \mapsto \deg(r)\,(p_* \eta) \tag{1.1.6}$$

where of course $Y(r)$ is the pre-image of the disc of radius r.

1.2 Comparison of conformal structures

The natural conformal structure on the disc is of course the Poincaré metric, ψ, which is most conveniently written as $dd^c \log(1-|z|^2)^{-2}$. From our point of view the issue is its degree which in turn we consider as the discriminant of the disc of radius r, i.e.,

$$\mathrm{discr}(r) := \deg(r)\,(\psi) = \log(1-r^2)^{-1}\,. \tag{1.2.1}$$

More generally for $p : Y \to \Delta$ a covering, unramified over the origin for convenience, the discriminant should take account of the p-ramification, i.e.

$$\mathrm{discr}^{\#}_{Y/\Delta}(r) := \sum_{0\,<\,|p(y)|\,<\,1} \mathrm{ord}_y(\mathrm{Ram}_p) \log \frac{1}{|p(y)|} \tag{1.2.2}$$

which again is natural enough on bearing in mind that it's the weighted order function $\log \frac{1}{|p(y)|}\, \mathrm{ord}_y(\;)$ rather than simply the order function which is the natural valuation associated to a point $y \in Y$. Not surprisingly we put these both together to define,

$$\mathrm{discr}_{Y/\Delta}(r) := \mathrm{discr}(r) + \mathrm{discr}^{\#}_{Y/D}(r) \tag{1.2.3}$$

which ought to be thought of as the size of $p^*\psi$ modified by the ramification in an appropriate way to give a conformal structure on Y.

Now given a map $f : Y \to X$ with X say projective to fix ideas, although this is wholly unnecessary, we have a conformal, or perhaps better pseudo conformal because of the ramification, structure induced on Y by way of,

$$T_Y \to f^* T_X$$

which we wish to compare with the natural one on Y, i.e. $T_Y \to p^* T_\Delta$. Indeed we should work in a little greater generality by permitting a simple normal crossing boundary D on X, with respect to which we consider the comparison map,

$$F : Y \to P(T_X(-\log D) \oplus 1\!\!1) : y \mapsto \left[\frac{x_1'}{x_1}, \ldots, \frac{x_k'}{x_k}, x_{k+1}, \ldots, x_n, p'\right]$$

where $y \mapsto (x_i(y))_{1 \leq i \leq n}$ is a representation of f in local coordinates with $x_1 \ldots x_k = 0$ the divisor D. The introduction of the divisor D, necessitates a final definition, that of the radical or intersection of f with D counted without multiplicity,

$$\mathrm{rad}_{f,D}(r) := \sum_{0 < |p(y)| < r} \min\{1, \mathrm{ord}_y(f^* D)\} \log \frac{1}{|p(y)|} \tag{1.2.4}$$

where for convenience we suppose that $p^{-1}(0) \cap f^{-1}(0) = \varphi$, a condition for which we'll subsequently use the slightly abusive short hand $f(0) \notin D$. An appropriate way to metricise things is to take any metric on $T_X(-\log D)$, which we'll denote $\| \;\|_{\log}$, and a metric on $\mathcal{O}_X(D)$ of sup norm at most $1/e$ – otherwise various things in the error term have to get divided out by the said sup. At this point we have a metricised tautological bundle $\overline{M}$ on $P(T_X(-\log D))$ represented by the metricised section at ∞ and the comparison of conformal structures simply consists in estimating $\overline{M} \cdot_F Y(r)$, i.e.

Fact 1.2.5. ([M2] V.3.1) Notations as above, for some constant $C > 0$ depending on (X, D) the metrics etc., but wholly independent of f,

$$\begin{aligned} \overline{M} \cdot_F Y(r) \quad &+ \quad \frac{1}{2} \log \left(1 + \frac{\|df\|^2_{\log}(0)}{|dp|^2(0)}\right) \\ &\leq \quad \frac{1}{2} \log \frac{(r\, S_f'(r))'}{r} + \mathrm{rad}_{f,D}(r) + \mathrm{discr}^{\#}_{Y/\Delta}(r) \\ &+ \quad \log^+ \overline{H} \cdot_f Y(r) + \log^+ |\log \|f^* 1\!\!1_D\|(0)| + C \end{aligned}$$

where $\overline{H}$ is a positive metricised ample bundle, the short hand $\cdot\,(0)$ means sum in the fibre over 0, and S_f is an increasing C^2 function bounded by,

$$C\, (\overline{H} \cdot_f Y(r) + \log \log^2 f^*\, 1\!\!1_D(0) + C)\,.$$

This may look a little complicated but morally it asserts that the left hand side is bounded by the discriminant as the following lemma shows,

Subfact 1.2.6. Let S' be a positive increasing differentiable function on $(0,1)$ and ψ any Kitchine function, e.g. $\log^{1+\varepsilon}$, then,

$$E := \{r \in (0,1) : S'(r) \geq S(r)\, \psi(S(r))\, (1 - r^2)^{-1}\}$$

has finite hyperbolic measure.

Proof. Pick $r_0 \in (0,1)$ with $S(r_0) > 0$ provided we're not in the trivial case of $S \equiv 0$, then,

$$\mu(E) \le \mu(0,r_0) + \int_{r_0}^{1} \frac{S'}{S\,\psi(S)}\,dr \le \mu(0,r_0) + \int_{S(r_0)}^{\infty} \frac{dx}{x\,\psi(x)} < \infty\,.$$

The ultimate inequality being the definition of Kitchine function. $\square$

The unfortunate thing is that we have to exclude a set of finite measure, and there doesn't appear to be any obvious to do this independently of f. Nevertheless this kind of estimate is extremely robust and works very well for sequences of functions. A typical example will arise from a sequence of functions and a sequence of varieties, so say a sequence of increasing differentiable functions S_{nk} then,

VARIATION 1.2.7. Notations as above for $r \in (0,1)$ outside a set of finite hyperbolic measure,

$$S'_{nk}(r) \le \{S_{nk}(r) + 2e\} \log^{n^2+k^2+1}\{S_{nk}(r) + 2e\}\,(1-r^2)^{-1}\,.$$

Proof. Indeed writing $F_{nk}(r) = S_{nk}(r) + 2e$, we have $\log F_{nk}(0) > 1$, and the set in question has measure bounded by,

$$\sum_{n,k} \int_{F_{nk}(0)}^{\infty} \frac{dx}{x \log^{n^2+k^2+1} x} = \sum_{n,k} \frac{1}{(n^2+k^2)(\log F_{nk}(0))^{n^2+k^2}} < \infty\,.$$

The use of hyperbolic measure $(1-r^2)^{-1}dr$ is of course rather natural since its induced from the Poincaré metric, however lots of similar estimates are possible for any "interesting" measure, i.e. one for which $(0,1)$ has infinite measure. Indeed bearing in mind that the error term is an estimate for $\log\left(1 + \frac{\|df\|^2_{\log}}{|dp|^2}\right)$ integrated over $\partial Y(r)$ amusing isoperimetric type inequalities can be derived using the induced length function. However our main application is to apply 1.2.5 via the projection onto the divisor at ∞. So blowing up in the zero section, denoting by $\overline{L}$ the metricised tautological bundle on $P(T_X(-\log D))$ and $f' : Y \to P(T_X(-\log D))$ the "logarithmic derivative", we have the,

1.2.8. TAUTOLOGICAL INEQUALITY. Notations as above,

$$\overline{L}\cdot_{f'} \Delta(r) + \log \frac{\|df\|_{\log}(0)}{|dp|} + N_{f'}(r) \le \text{same error as 1.2.5}$$

where $N_{f'} = \sum_{\substack{0 < |p(y)| < r \\ f(y) \notin D}} \mathrm{ord}_y(\mathrm{Ram}_f) \log \frac{1}{|p(y)|}$ with as ever the convenient assumption that f is unramified over 0.

In fact the tautological inequality is extremely close to an equality, although it's more so with 1.2.5 where the error is always bounded below by zero. In the case for compact curves C and maps $f : C \to X$ the analogous assertion is simply $\deg(f^*L) + \deg(\widetilde{\mathrm{Ram}_f}) = 2g(C) - 2 + \mathrm{rad}_{f,D}$, where $\widetilde{\mathrm{Ram}_f}$ is the ramification away from D. Regardless what's important is that even if L has no sections or X isn't negatively curved (i.e. $\overline{L}\cdot_{f'} \Delta(r) \geq \varepsilon\, \overline{H}\cdot_f \Delta(r)$) or whatever there is still functorial information to be had.

1.3 Closed vs. Harmonic

Our main application is going to involve a sequence of triples,

$$\begin{array}{ccc} Y_n & \xrightarrow[f_n]{} & X \\ \downarrow{\scriptstyle p_n} & & \\ \Delta & & \end{array}$$

which might conveniently be abbreviated $f_n : Y_n/\Delta \to X$, where for $\overline{H}$ positive metricised ample on X, and $r \in (0,1)$ fixed, $\overline{H}\cdot_{f_n} Y_n(r) \to \infty$. A fortunate property, therefore, of the intersection product is that it evidently extends to $(1,1)$ forms on X and so we have natural positive currents,

$$T_{f_n}(r) : A^{1,1}(X) \to \mathbb{R} : \eta \mapsto \deg(r)\,(f_n^*\eta)$$

normalising by the degree, $(\overline{H}\cdot_{f_n} Y_n(r))^{-1}\, T_{f_n}(r)$ is then a bounded sequence of such, and whence after subsequencing enjoys a weak limit $T(r)$. An immediate consequence of Jensen's formula/definition of the degree function is that $T(r)$ is harmonic. On the other hand we would like to apply,

1.3.1. Brunella's Residue Estimate. ([Br]) Let $(X, \mathcal{F})$ be a foliated algebraic surface with canonical foliation singularities and D the algebraic leaves through the singularities then for T a closed positive current invariant by $\mathcal{F}$,

$$N(D).\, T \leq 0$$

where N is the co-normal bundle to $\mathcal{F}$.

In Brunella's theorem the hypothesis of closure is critical, since it uses the Baum-Bott philosophy, [BB], that the normal bundle to a foliation is flat along its leaves away from the singularities. This translates into special representations for the chern class of the same in Dolbeault cohomology, but not in a way that respects Hodge decomposition, i.e. there's no $\partial\overline{\partial}$ lemma. Now while there may exist conjectures in dynamics which suggest that 1.3.1 might hold under the weaker condition of harmonicity, rather than believing in fairies it's more appropriate to establish,

Fact 1.3.2. ([M2] V.2.5) Notations as above, and suppose further that our triples $f_n : Y_n/\Delta \to X$ arise from lifting maps $g_n : \Delta \to W$, i.e. we have for each n commutative diagrams,

$$\begin{array}{ccc} Y_n & \xrightarrow[f_n]{} & X \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p} \\ \Delta & \xrightarrow[g_n]{} & W \end{array}$$

with $g_n(0)$ bounded away from the ramification, for n sufficiently large, of the supposed generically finite map p, then for $R \in [r, 1)$ outside a set of finite hyperbolic measure $T(R)$ is closed.

In op.cit. the proof only discusses the case of p_n the identity. The extension however is trivial provided one can get a suitable estimate on the derivative in a neighbourhood of the origin, op.cit. V.1.7, which the hypothesis on the exact way that the p_n arise easily ensures. In general one must use a different way to normalise T involving $\mathrm{discr}^{\#}_{Y/\Delta}(r)$, but since this is of purely philosophical interest there seems little point in getting involved with it. The important thing to bear in mind is that while we don't enjoy the sinicure of studying a single map $f : Y \to X$ from a ramified cover of the line, nor can we hope to reparametrise to this situation à la Brody without losing data, cohomologically in the limit our situation isn't that disimilar. A particularly neat consequence of which is that we can divide up T according to its support thanks to the following theorem of Skoda-El Mir,

THEOREM 1.3.3. (cf. [D]) *If T is a closed positive current on X, and $Z \subset X$ a closed subvariety, then $\mathbb{1}_Z T$ is again closed.*

The final thing to bear in mind is that T enjoys reasonable functoriality properties. Specifically consider a commutative diagram with the vertical map on the right dominant,

$$\begin{array}{ccc} \widetilde{Y}_n & \xrightarrow[\widetilde{f}_n]{} & Z \\ \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho} \\ Y_n & \xrightarrow[f_n]{} & X \\ \downarrow{\scriptstyle p_n} & & \\ \Delta & & \end{array}$$

Evidently we consider $\widetilde{p}_n : \widetilde{Y}_n \to \Delta$ as the composition $\rho_n^* p_n$, and, for any $r \in (0, 1)$, $\rho_* T_{\widetilde{f}_n}(r) = (\widetilde{Y}_n : Y)\, T_{f_n}(r)$, whence, FACT 1.3.4. Let $\widetilde{T}, T$ be the associated currents then up to suitable normalisation, $\rho_* \widetilde{T} = T$.

Here we're deliberately vague. Obviously $(\widetilde{Y}_n : Y)$ constant in n is a without loss of generality hypothesis. What's more delicate is what ample bundle we use. The ample degree on Z will always dominate that on X, and when the converse holds the proposition is up to an unimportant choice of constant unambigous. Typical situations where this will be true are ρ generically finite and $f_n(0)$ bounded away from a suitable closed subvariety or Z the projective/log projective tangent space, and f_n some subsequence of divergent discs. In any case the worst that could happen is that normalising by the Z degree, $\rho_* \widetilde{T} = 0$, which with one exception typically implies trivially what we need to know, so that generally the issue of normalisation will be passed over without comment.

2 Compactness

2.1 Bounded degree

While everything that follows could be done in substantially greater generality, our ultimate interest is the algebraic category, so we may as well just to stick to it, since it affords substantial simplification. The typical situation that will interest us is a sequence of maps $f_n \in \mathrm{Hom}(\Delta, X)$, i.e. from the unit disc to a projective variety X, where $\overline{H}\cdot_{f_n} \Delta(r)$ is uniformly bounded for some $r \in (0,1)$ or even a sequence $r_k \to 1$, with $\overline{H}$ a positively metricised ample bundle. However change of metric only changes the intersection number up to a constant, be the bundle ample or not, so, especially in the presence of infinitely many maps, ignoring the dependence on the metric is wholly harmless. Consequently from the point of view of convergence we might as well say X is a projective space $\mathbb{P}^m$, that the $f_n(0)$ accumulate at, $1 = [1, \ldots, 1]$, and our ample bundle is $\mathcal{O}_{\mathbb{P}^m}(1)$ with the Fubini-Study metric. It is then an easy exercise in manipulating Jensen's formula to show

FACT 2.1.1. ([M2] V.2.4) Notations as above, we may write, $f_n = [f_{on}, \ldots, f_{mn}]$ where, after appropriate subsequencing, for each $0 \le i \le m$, the f_{in} converge in the compact open topology.

Despite appearances, as Brunella pointed out to me, this doesn't guarantee the convergence of f_n in the compact open topology. The classic example is,

$$f_n : \Delta \to \mathbb{P}^1 : z \to \left[z - \zeta, z - \zeta + \frac{1}{n} \right]$$

for any ζ in the disc punctured at the origin. Evidently this problem presents itself quite generally at any point in the common zero locus of the limit $f_0, \ldots, f_n$. On the other hand the maps f_n manifestly "converge", and so the only real question is in what space. The answer is, of course, provided by the Bishop-Gromov compactness theorem which we summarise according to our needs. In the first place we have,

Definition 2.1.2. *A disc with bubbles Δ^b is a connected 1-dimensional complex space with singularities at worst nodes exactly one of whose components is the unit disc Δ, with Δ^b smooth at the origin of the same, and every connected component R_z, $z \in R(\Delta^b) := \Delta \cap \operatorname{sing}(\Delta^b)$, of the complement of Δ is a tree of smooth rational curves.*

Certainly for $f : \Delta^b \to X$, and $\overline{L}$ a metricised line bundle we can easily extend the intersection product by way of,

$$\overline{L} \cdot_f \Delta^b(r) := \overline{L} \cdot_f \Delta(r) + \sum_{\substack{z \in R(\Delta^b) \\ |z| < r}} \log \frac{r}{|z|} \int_{R_z} f^* c_1(\overline{L})$$

so that necessarily we have similar positivity properties for intersections with effective divisors to the intersection product over discs. In addition we can define a graph Γ_f as,

$$\Gamma_f = (\mathrm{id} \times f)\,(\Delta) \bigcup_{z \in R(\Delta^b)} z \times f(R_z) \subset \Delta \times X\,.$$

Moreover the compact open topology on $\operatorname{Hom}(\Delta, X)$ is equivalent to the Hausdorff topology on the graphs of maps, so that quite naturally we can consider $f : \Delta^b \to X$ as a boundary point if its graph is a limit of graphs Γ_{f_n}, i.e.

Definition 2.1.3. *Say that $f_n \in \operatorname{Hom}(\Delta, X)$ converges to $f : \Delta^b \to X$ if for $r \in (0,1) \backslash \{|z| : z \in R(\Delta^b)\}$ the graphs $\Gamma_{f_n} \times_\Delta \overline{\Delta}_r$ converge to $\Gamma_f \times_\Delta \overline{\Delta}_r$ in the Hausdorff metric.*

We can certainly add such boundary points to our Hom sets, to get a new space $\overline{\operatorname{Hom}}(\Delta, X)$ topologised by embedding it in the space of subvarieties of $\Delta \times X$ in the Hausdorff metric over compact subsets of Δ, and with this in mind a convenient formulation of the Bishop-Gromov-Grothendieck compactness theorem is,

Fact 2.1.4. Let $C : (0,1) \to \mathbb{R}_+$ be any function then,

$$\{f \in \overline{\operatorname{Hom}}(\Delta, X) : \overline{H} \cdot_f \Delta(r) \leq C(r)\}$$

is compact. In particular $f_n \in \overline{\operatorname{Hom}}(\Delta, X)$ has a convergent subsequence iff for some subsequence f_k and radii $r_m \to 1$, $\overline{H} \cdot_{f_k} \Delta(r_m)$ is bounded independently of k for each m.

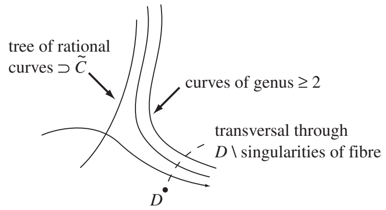
2.2 Green's Lemma

As we've defined it as a subset of the set of subvarieties of $\Delta \times X$ in the Hausdorff topology, $\overline{\operatorname{Hom}}(\Delta, X)$ is a minimal compactification of what may be termed bounded subsets of $\operatorname{Hom}(\Delta, X)$. Now certainly $\overline{\operatorname{Hom}}(\Delta, X)$

$= \mathrm{Hom}(\Delta, X)$ if X doesn't contain rational curves, but conversely we have,

Fact 2.2.1. $\mathrm{Hom}(\Delta, X)$ is a proper subset of $\overline{\mathrm{Hom}}(\Delta, X)$ if the smooth locus of X contains rational curves.

Proof. Taking hyperplane sections, it's clearly sufficient to treat the case of surfaces – actually even curves would do but we'd like to make examples of sequences of discs not wholly contained in the curve. Equally provided we make appropriately generic examples we can always blow them down so we might as well say C is a smooth rational curves in a smooth surface X. Choose a point $\infty \in C$, then we can find an affine neighbourhood U of $C \backslash \infty$, and whence $\mathcal{I}_C \otimes \mathcal{O}_U$ is generated by a single equation, x, say, which defines a pencil $x =$ constant. Passing to a modification $\rho : \widetilde{X} \to X$ we obtain a Lefschetz pencil. Denote by $\widetilde{C}$ the proper transform of C, then for a sufficiently generic choice of pencil, the generic element is a hyperbolic curve, and some component D of the fibre containing $\widetilde{C}$ minus the singularities is hyperbolic and does not blow down to a point under ρ. Around a chain of rational curves in the fibre containing $\widetilde{C}$ as a component which without loss of generality we can suppose meets D our fibration looks like,



(2.2.2)

Consequently if we uniformise the curves of genus ≥ 2 indicated in the figure by the disc with the origin centered on the indicated transversal then we obtain a sequence of discs that converges to a disc with bubbles on $\widetilde{X}$, and by the particular choice of D this blows down to the same on X. $\square$

Now suppose more generally that we're in a quasi-projective situation. The appropriate level of generality is to consider $X \backslash D$, for a pair (X, D) with log-canonical singularities and the components of D, $\mathbb{Q}$-Cartier divisors, although regardless of the singularities if we have a map from $\mathbb{P}^1$ or even just $\mathbb{A}^1$ to the smooth locus of $X \backslash D$ then 2.2.1 goes through verbatim to show that $\mathrm{Hom}(\Delta, S)$ is a strict subset of $\overline{\mathrm{Hom}}(\Delta, X)$, while in the converse direction we may appeal to,

Lemma 2.2.3. ([G]) *Let $f_n : Y \to X \backslash D$ be a sequence of maps converging uniformly on compact subsets of a Riemann surface to $f : Y \to X$, then either, f maps to $X \backslash D$ or D.*

Now necessarily if a sequence $f_n : \Delta \to X$ with degrees uniformly bounded on a sequence of radii $r_k \to 1$ converges to a disc with bubbles, then for some $0 < |\zeta| < 1$, there are maps $\varphi_n : \Delta_{R_n} \to \{z \mid |z - \zeta| < \varepsilon_n\}$ with $R_n \to \infty$ and $\varepsilon_n \to 0$ such that $f_n \circ \varphi_n$ converges uniformly on compact subsets of $\mathbb{C}$ to a rational map $f : \mathbb{A}^1 \to X$ (cf. [S]). Whence if in fact the f_n are maps to $X \backslash D$ then either $X \backslash D$ or D must contain an affine line. Better still if we write $D = \Sigma D_i$ in terms of its components then the line isn't just in D but $D_i \backslash \bigcup_{j \neq i} D_j$ for some i, and should this happen we have,

Fact 2.2.4. Notations as above, if $X \backslash D$ and $D_i \backslash \bigcup_{j \neq i} D_j$ do not contain any affine lines then $\overline{\mathrm{Hom}}(\Delta, X \backslash D) \subset \mathrm{Hom}(\Delta, X)$.

There are plenty of natural situations where the hypothesis that $D_i \backslash \bigcup_{j \neq i} D_j$ contains no lines is guaranteed, e.g. X a toric variety and $X \backslash D$ a torus, while for surfaces we have the wholly general,

Fact 2.2.5. (Log minimal model theorem) Suppose $X \backslash D$ is not fibreed by lines then there is a contraction $\rho : (X, D) \to (X', D')$ within our given category of singularities with $X \backslash D = X' \backslash D'$ such that $D'_i \backslash \bigcup_{j \neq i} D'_j$ does not contain affine lines. Even better if (X, D) has general type then we may even insist that each $D'_i \backslash \bigcup_{j \neq i} D'_j$ is hyperbolic.

The significance of the latter condition will emerge in the next section.

2.3 Hyperbolicity and normality

An unambigous definition of hyperbolicity for a projective variety X is that $\mathrm{Hom}(\Delta, X)$ is compact, which by Brody's reparametrisation lemma is equivalent to the lack of maps $f : \mathbb{C} \to X$. However one often looks to the weaker property that any such non-trivial f should factor through a proper closed subvariety of which the proper infinitesimal analogue is,

Defintion 2.3.1. (a) *For Z a proper subvariety of X we say X is hyperbolic modulo Z if a sequence of maps $f_n : \Delta \to X$ of which a subsequence doesn't converge in $\overline{\mathrm{Hom}}(\Delta, X)$ is, in the compact open sense, contained in arbitrarily small neighbourhoods of Z.*

Observe that by 2.2 the above definition is best possible. Indeed to fix ideas, say X is a surface then since everything which is Stein has a Stein neighbourhood at the very least Z must contain all the rational and elliptic curves, so in fact there must be a finite set of such. If there are only elliptic curves then we could replace $\overline{\mathrm{Hom}}$ by Hom, but this is probably a fairly special situation. In the quasi-projective situation the most generally valid definition is,

Defintion 2.3.1. (b) *Let (X, D) be a projective variety with boundary and Z a proper subvariety of $X \backslash D$ then we say (X, D) is hyperbolic modulo Z if a sequence of maps $f_n : \Delta \to X \backslash D$ of which a subsequence doesn't converge* *in* $\overline{\mathrm{Hom}}(\Delta, X)$ *is, in the compact open sense, contained in arbitrarily small neighbourhoods of* $Z \cup D$.

Again as we've seen this is best possible. On the other hand given that hyperbolicity necessitates general type we could propose a definition excluding the possibility of arbitrary proximity to D, provided D were hyperbolic in the sense of 2.2.5, i.e. at the very least $D_i \backslash \bigcup_{j \neq i} D_j$ didn't contain any holomorphic lines. Equally while such a compactification of $X \backslash D$ exists for surfaces, higher dimension seems anything but obvious, while additionally our methodology involves an interplay between the singularities of Z and D, to such an extent that working on the minimal model would be too restrictive. In any case the hyperbolicity of X – respectively (X, D)-modulo Z leads to a natural metric, at least in the smooth case – respectively log-smooth – on T_X, respectively $T_X(-\log D)$, which we'll give in the more general log case, i.e.

Defintion 2.3.2. (Kobayashi-Royden) *Let* (X, D) *be a smooth projective variety, with simple normal crossing boundary, which is hyperbolic modulo* Z *then for* $x \in X \backslash D \backslash Z$, *and* $\partial \in T_X(-\log D) \otimes \mathbb{C}(x)$,

$$\|\partial\|_{\mathrm{Kob}} = \inf \left\{ \frac{1}{R} \mid f : (\Delta, 0) \to (X \backslash D, x) : f_* \left(\frac{\partial}{\partial z} \right) = R\partial \right\} .$$

Now while the metric is certainly non-degenerate off Z under the hyperbolicity hypothesis, which motivates the same as Kobayashi's definition of hyperbolicity, hyperbolicity in the sense of 2.3.1 implies more. Specifically,

Fact 2.3.3. Let (X, D) be hyperbolic modulo Z, then $\| \; \|_{\mathrm{Kob}}$ is continuous on the total space of $T_X(-\log D)$ off $D \cup Z$.

Proof. Upper semi-continuity is as ever just the property that everything Stein has a Stein neighbourhood. In the other direction, let $x \in X \backslash D \cup Z$, $x_n \to x$, and $(x_n, \partial_n) \to (y, \partial)$ in $T_X(-\log D)$, with $(f_n)_* \left(\frac{\partial}{\partial z}\right) = R_n \partial_n$. Now f_n converges to a disc with bubbles, $f : \Delta^b \to X$, which by definition doesn't bubble at the origin so there is a fixed $\varepsilon > 0$ such that the discs $f_n \mid_{\Delta_\varepsilon}$ converge to $f \mid_{\Delta_\varepsilon}$, so if $f_* \left(\frac{\partial}{\partial z}\right) = R\partial$, then $\varepsilon R_n \to \varepsilon R$, and we're done. □

It's probably worth closing by way of an easy example, viz,

Example 2.3.4. Let $\pi : X \to B$ be a fibration in curves of genus ≥ 2, of a smooth surface over a smooth curve and $D = b_1 + \cdots + b_k$ the singular fibres, with $\pi^{-1}(D)$ simple normal crossing. Necessarily if π isn't isotrivial, $B \backslash D$ is hyperbolic, and of course we suppose this. Now put $|D| = \pi^{-1}(D)_{\mathrm{red}}$, then whether because $(X, |D|)$ has log general type, or quite simply Zariski's lemma we can contract chains of rational curves in $|D|$ to obtain a fibration $\pi : X' \to B$, such that if $|D'| = \bigcup_i C_i$ is the image of D then $C_i \backslash \bigcup_{j \neq i} C_j$ is hyperbolic, with the singularities log-canonical and no worse than $\mathbb{Q}$-factorial. If we'd started from the semi-stable case we could get canonical singularities, but this is fairly irrelevant. Anyway suppose $\mathrm{Hom}(\Delta, X'\backslash|D'|)$ is not relatively compact in $\mathrm{Hom}(\Delta, X')$ then by Brody's reparametrisation lemma we have a map $f : \mathbb{C} \to X'$ which is the uniform limits of maps from big discs, $f_n : \Delta_{R_n} \to X'\backslash|D'|$, $R_n \to \infty$. However appealing to Green's lemma this is nonsense, so $X'\backslash|D'|$ is hyperbolic modulo the empty set, without even the need to worry about maps getting arbitrarily close to the boundary $|D'|$.

3 Differentiation and Jets

3.1 Bogomolov's Lemma

The applicability of the main lemma stems from the following observation of Bogomolov,

LEMMA 3.1.1. ([Bo]) *Let (X, D) be a smooth surface with simple normal crossing boundary of logarithmic general type and $s_2(X, D)$ $(= c_1^2(X, D) - c_2(X, D)) > 0$ then the tautological bundle L on $\pi : \mathbb{P}(\Omega_X(\log D)) \to X$ is big.*

Now let's try and apply this in combination with 1.2.7 and 1.2.8 to the study of $\mathrm{Hom}(\Delta, X)$. Taking $\overline{H}$ to be a positively metricised ample divisor then the lemma implies (indeed it is equivalent) to the existence of some $b \in \mathbb{N}$ and a naturally metricised effective divisor $\overline{F}$ such that $\overline{L}$, metricised as in 1.2, is equal to $\pi^*\overline{H}(\overline{F})$. So that for $f \in \mathrm{Hom}(\Delta, X)$ with $f(0) \notin F$,

$$b\,\overline{L}\cdot_{f'}\Delta(r) \geq \overline{H}\cdot_{f}\Delta(r) + \log \|f'^{*}\,\mathbb{1}_F\|\,(0) + \mathrm{const} \tag{3.1.2}$$

where the constant is just a lower bound for $-\underline{\log \|\mathbb{1}_F\|}$. Now suppose further that $f_n \in \mathrm{Hom}(\Delta, X)$ don't converge in $\overline{\mathrm{Hom}}(\Delta, X)$ then we can subsequence to guarantee $\overline{H}\cdot_{f_n}\Delta(r_0)$, for some fixed r_0, as large a function of n as we like, so that applying 1.2.7 and 1.2.8 we obtain for $r \in (0, 1)$ outside a set of finite hyperbolic measure,

$$\begin{aligned} \log \|df_n\|\,(0) + \overline{L}\cdot_{f_n'}\Delta(r) \quad &\leq \quad (1 + \circ(\overline{H}\cdot_{f_n}\Delta(r))\log(1 - r^2)^{-1} \\ &+ \quad \mathrm{rad}_{f_n,D}(r) + \log|\log f_n^*\,\mathbb{1}_D|\,(0)\,. \end{aligned} \tag{3.1.3}$$

Certainly we can restrict to the case $\mathrm{rad}_{f_n,D} = 0$, e.g. $f_n : \Delta \to X\backslash D$ or D empty, but the real problem concerns what happens at zero. Equally it makes little difference whether we prove that the f_n converge in $\overline{\mathrm{Hom}}$ or that the $\alpha_n^* f_n$ converge, where α_n are a convergent sequence of automorphisms, so provided that the f_n' are not contained in arbitrarily small neighbourhoods of $F \cup \pi^{-1}(D)$, composing with a convergent sequence of automorphisms implies that without loss of generality – although cf. [M2] V.2.7 in order to guarantee that the derivative after composition isn't too small at the origin – that,

$$\overline{H}\cdot_{f_n}\Delta(r) \leq C(r) \tag{3.1.4}$$

and whence a contradiction. The real difficulty arises, therefore, when the sequence f_n becomes arbitrarily close to some divisor, which by the way contains as a strict, and relatively straightforward subcase, the possibility that the f_n are actually in the divisor.

The immediate difficulty is that the divisor F is potentially a complete mess. Concentrating on its horizontal components, these correspond to saturated maps,

$$0 \to M \to \mathrm{Sym}^d\, \Omega_X(\log D)$$

for $d \in \mathbb{N}$ and M a line bundle, which don't factor through a lower symmetric power. Abusing notation slightly, and denoting the component equally by F, with U the universal sub-bundle over the same we have a factorisation,

$$\pi^* M \otimes \mathcal{O}_F \to U \otimes \pi^* \mathrm{Sym}^{d-1}\, \Omega_X(\log D) \otimes \mathcal{O}_F \to \pi^* \mathrm{Sym}^d\, \Omega_X(\log D) \otimes \mathcal{O}_F$$

So passing to a ramified cover $\rho : S \to X$ we have a fibre square,

$$\begin{array}{ccc} \mathbb{P}(\rho^*\Omega_X(\log D)) & \underset{\rho}{\longrightarrow} & \mathbb{P}(\Omega_X(\log D)) \\ \downarrow & & \downarrow \\ S & \underset{\rho}{\longrightarrow} & X \end{array}$$

with all the horizontal components of $\rho^* F$ enjoying degree 1. Further passing to a resolution of singularities not just of S but of $\rho^{-1}(D)$, which again somewhat abusively we'll denote $\rho(S, |D|) \to (X, D)$, we have the natural map,

$$\rho^*\Omega_X(\log D) \to \Omega_S(\log |D|)$$

and a birational map, $\mathbb{P}(\Omega_S(\log |D|) \dashrightarrow \mathbb{P}(\rho^*\Omega_X(\log D))$ which is an isomorphism over the generic point of S. Resolve this latter map by, $\psi : P \to \mathbb{P}(\Omega_S(\log D))$, and consider what's become of the maps f_n. In the first place these do not necessarily lift to S, but we do have a sequence of liftings to ramified coverings,

$$\begin{array}{ccc} Y_n & \underset{\widetilde{f_n}}{\longrightarrow} & S \\ \downarrow p_n & & \downarrow \\ \Delta & \underset{f_n}{\longrightarrow} & X \end{array}$$

with the derivatives $\widetilde{f_n'} : Y_n \to \mathbb{P}(\Omega_S(\log |D|))$ of the proper transforms $\widetilde{f_n}$ lifting $f_n' : \Delta \to \mathbb{P}(\Omega_X(\log D))$ to P. Whence if the f_n are eventually arbitrarily close to some divisor on $\mathbb{P}(\Omega_X(\log D))$ then the $\widetilde{f_n'}$ are the same for its pull-back to P. Consequently since P is isomorphic to $\mathbb{P}(\Omega_S(\log|D|))$ over the generic point the $\widetilde{f}'_n$, defined on Y_n, are arbitrarily close to a divisor each horizontal component of which has degree 1.

This basically completes the reduction of the hyperbolicity of $X\backslash D$ to the main lemma. The only minor thing to note being that if the f_n are not arbitrarily close to some divisor ∂ in X then the same is true for the $\widetilde{f}_n$ at the S level with respect to $\rho^{-1}(\partial)$. Thus by a priori composing with convergent sequences of automorphisms of the disc we can guarantee,

$$E._{\widetilde{f_n}} Y_n(r) \geq \text{const}$$

for any E in the support of $\rho^{-1}(\partial)$, and guarantee the mild positivity condition of the announcement. In any case, given the main lemma, we essentially have for $\varepsilon > 0$,

$$\begin{aligned}(K_S + |D|)._{f_n} Y_n(r) + \log\|df_n\|(0) \quad &\leq \quad \mathrm{rad}_{f_n,D}(r) + \mathrm{discr}^{\#}_{Y_n/\Delta}(r) \\ &+ \quad C(r) + \varepsilon\, H._{f_n}\Delta(r)\end{aligned}$$

and by including the ramification in our excluded set, we can estimate $\mathrm{discr}^{\#}$ in the obvious way by the ramification divisor, so that restricting to one of the natural cases where the radical is zero, we'll obtain,

$$(K_X + D)._{f_n}\Delta(r) \leq C(r) + \varepsilon\, H._{f_n}\Delta(r)$$

from which the hyperbolicity of $X\backslash D$ modulo some effectively computable, if rather unclear, proper Zariski closed subset.

The important things to draw from this discussion are in the first place that we need the full force of the main lemma, i.e. it would be wholly insufficient to simply bound the degree of the maps $f_n : Y_n \to S$ in terms of the ramification, since this would not a priori imply a similar bound for the maps $f_n : \Delta \to X$. Secondly, although in the case of $c_1^2 > c_2$ where the specific problem is that our maps may intersect some horizontal divisor negatively, we might hope to play some numerical game between X and S (all be it, that it's not clear what) to reduce to some easier problem involving less components, this isn't the point. More precisely if we hope to conclude to results on the hyperbolicity of algebraic surfaces, by use of analogues of 3.1.1 for higher jet bundles, then we must be able to exclude systematically, by way of appropriate analogues of the main lemma, the degeneration at the level of lower order jets as the example of the next section illustrates.

3.2 The Bloch Theorem

The Bloch Theorem concerns the hyperbolicity of $(\mathbb{P}^2, D)$, where D is a divisor consisting of 4 lines in general position. The arguements of this section work quite generally for any number of lines, and leads to specific quantification of the value distribution of discs about the same. However the case of at least 5 lines (all be it with worse quantification) follows from 3.1, and things are combinatorially a bit involved so we'll stick to the case of 4.

To begin with denote by P, respectively $P_{\log}$, the bundles $\mu : \mathbb{P}(\Omega_{\mathbb{P}^2}) \to \mathbb{P}^2$, and $\nu : \mathbb{P}(\Omega_{\mathbb{P}^2}(\log D)) \to \mathbb{P}^2$. The bundle P is equally smoothly fibred by λ say, over $(\mathbb{P}^2)^\vee$, which we think of as a smooth connection on $\mathbb{P}^2$, or equivalently a smooth foliation $\mathcal{F}$ on P arising from a splitting of the bundle of 2-jets, J, i.e. the quotient of Ω_P by the universal sub-bundle. Equally if $J_{\log}$ is the quotient of $\Omega_{P_{\log}}(\log \nu^* D)$ by the corresponding universal sub-bundle we have an induced connection corresponding to a singular foliation $\mathcal{F}_{\log}$, with a splitting of the natural exact sequence for $J_{\log}$, i.e.

$$0 \to L_{\log} \to J_{\log} \underset{\overset{\text{split}}{\longleftarrow}}{\longrightarrow} \nu^* K_{\mathbb{P}^2}(D) \otimes L_{\log}^{-2} \to 0$$

where L, and $L_{\log}$ are the respective tautological bundles on P and $P_{\log}$. Now argueing as in 3.1.2 – 3.1.4 this time with respect to the second derivatives $f_n'' : \Delta \to \mathbb{P}(J_{\log})$, for $f_n : \Delta \to \mathbb{P}^2 \backslash D$ a sequence of divergent maps which are not eventually arbitrarily proximate to D, it's immediate that we're done unless the f_n'' are eventually arbitrarily close to the divisor in $\mathbb{P}(J_{\log})$ defined by the connection. In order to reduce to the main lemma we have to relate $P_{\log}$ to P. Observe that over a line C in D, both these bundles posses canonical sections, i.e.

(i) The derivative, dC,

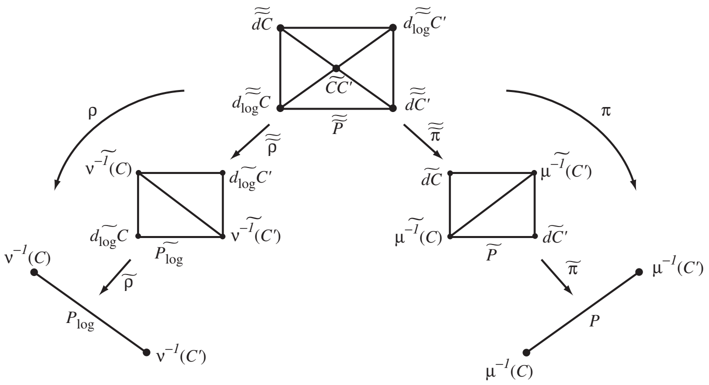
$$0 \to \mathcal{O}_{\mathbb{P}^2}(-C) \to \Omega_{\mathbb{P}^2} \otimes \mathcal{O}_C \to \omega_C \to 0$$

(ii) The residue, $d \log C$,

$$0 \to \mathcal{O}_C(K_X + D) \to \Omega_{\mathbb{P}^2}(\log D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to 0$$

.

The possibilities for resolving the birational map $P_{\log} \dashrightarrow P$ are not unique. The correct one is to first blow up in all the $d \log C$, which by the way are disjoint, then blow in the crossings of the proper transforms of the $\nu^{-1}(C)$, $C \subset |D|$. The final exceptional divisors so obtained are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$'s, so blow down in the other direction, or alternatively they're the blow up in the crossings of the proper transforms of the $\mu^{-1}(C)$ after blowing up in the dC. Note also that the proper transforms of the $\nu^{-1}(C)$, respectively $\mu^{-1}(C)$ blow down in the other direction. Let's summarise all this in terms of some maps, and simplicial complex defined by the intersection of various divisors, which in turn fixes our notation, viz,



Now denote by $K_?$ not just the canonical bundle of any of the above spaces but also of the induced foliations, i.e. the chern class of the corresponding quotient of the cotangent bundle, then:

$$\rho^* K_{\mathcal{F}_{\log}} - \sum_C \widetilde{\widetilde{d\log C}} = K_{\widetilde{\widetilde{\mathcal{F}}}} = \pi^* K_{\mathcal{F}} + \sum_{CC} \widetilde{CC'}$$

where the negative discrepancy on the left reflects the fact that the $\widetilde{d\log C}$ are not invariant, and of course the positive discrepancy on the right that the $\widetilde{CC'}$ are so. Consequently the natural divisor to work with, at least initially, is,

$$\widetilde{\widetilde{D}} := \sum_C \widetilde{\widetilde{dC}} + \sum_{CC} \widetilde{CC'}\,.$$

However we also have other interesting divisors arising from the pencils/foliations defined by any pair of lines. In $(\mathbb{P}^2)^\vee$ these correspond to six lines, which we'll denote by F's, meeting in the 4 points, [C], corresponding to our given curves. This information incorporates itself naturally into our data since $\widetilde{P}$ arises as the fibre product,

$$\begin{array}{ccccc}
\widetilde{P} & \xrightarrow{\widetilde{\lambda}} & (\widetilde{\mathbb{P}^2})^\vee & \longleftarrow & \widetilde{\lambda}(\widetilde{dC}) \\
{\scriptstyle\widetilde{\pi}}\downarrow & \square & \downarrow{\scriptstyle\widetilde{\pi}} & \square & \downarrow \\
P & \underset{\lambda}{\longrightarrow} & (\mathbb{P}^2)^\vee & \hookleftarrow & [C]
\end{array}$$

where the middle vertical arrow is the blow up in the 4, aforesaid points, and we denote by $\widetilde{F}$ the pull-back to $\widetilde{P}$ of the proper transform of the various F's, with $\widetilde{\widetilde{F}}$ its proper transform on $\widetilde{\widetilde{P}}$. At the $\widetilde{\widetilde{P}}$ level all divisors appearing so far are simple normal crossing, with the $\tilde{\tilde{\mathcal{F}}}$ invariant ones being $\widetilde{D}$ together with the $\tilde{\tilde{F}}$'s. Moreover the pull-back to $\tilde{\tilde{P}}$ of the universal subbundle $U_{\log}$ on $P_{\log}$ is still a subbundle of $\Omega_{\tilde{\tilde{P}}}(\log \tilde{\tilde{D}} + \tilde{\tilde{F}})$ (although having a bundle is notationally more important than anything else) so we get an induced jet bundle $\tilde{\tilde{J}}$ with dévissage induced by $\tilde{\tilde{\mathcal{F}}}$,

$$0 \to \rho^* \nu^* K_{\mathbb{P}^2}(D) \otimes \rho^* L_{\log}^{-2} \left(\sum_C \widetilde{\widetilde{d \log C}} + \sum_F \tilde{\tilde{F}} \right) \to \tilde{\tilde{J}} \to K_{\tilde{\tilde{\mathcal{F}}}} \to 0\,. \tag{3.2.2}$$

Now suppose that the derivatives of our maps $f_n' : \Delta \to P_{\log}$ are not eventually arbitrary close to $\underset{F}{\cup}\ \rho(\tilde{\tilde{F}})$ which by the way contain the $d\log C$, then after composition with convergent sequences of automorphisms of the disc the radical of $\tilde{\tilde{F}}$ for the induced maps $\tilde{\tilde{f}}_n : \Delta \to \tilde{\tilde{P}}$ may comfortably be estimated by,

$$\mathrm{rad}_{\tilde{\tilde{f}}_n, \tilde{\tilde{F}}}(r) \leq \tilde{\tilde{F}} \cdot_{\tilde{\tilde{f}}_n} \Delta(r) + \mathrm{const}$$

where the implied constant is independent of n. Consequently applying the tautological inequality for $(\tilde{\tilde{P}}, \tilde{\tilde{D}} + \tilde{\tilde{F}})$ with respect to the induced maps $\tilde{\tilde{f}}_n$ and their derivatives, and argueing as in (3.1.2) – (3.1.4) we're certainly going to bound the degree of the $\tilde{f}_n$, unless the derivatives are eventually arbitrarily proximate to the divisor in $\mathbb{P}(\tilde{\tilde{J}})$ defined by (3.2.2).

At this point let's work in terms of the induced currents in the sense of 1.3.2. Call an appropriate such on $\mathbb{P}(\tilde{\tilde{J}}), T$. We may devide T into two parts, according to the singularities of $\tilde{\tilde{\mathcal{F}}}$ which occur at the intersections $\tilde{\tilde{F}} \cap E_{CC'}$. Denote the said singular curves S, and use the Skoda-El Mir theorem to write,

$$T = \mathbb{I}_S\, T + T_{\mathcal{F}}$$

the current $T_{\mathcal{F}}$ is invariant by $\tilde{\tilde{\mathcal{F}}}$ in the strongest possible sense, i.e. $\tilde{\lambda}_* \tilde{\tilde{\pi}}_* T_{\mathcal{F}} = 0$. Furthermore the full foliation on $\tilde{\tilde{P}}$ is given by,

$$0 \to \tilde{\tilde{\pi}}^* \tilde{\lambda}^* \Omega_{\widetilde{(\mathbb{P}^2)^\vee}}(\log \widetilde{F} + \widetilde{dC}) \to \Omega_{\tilde{\tilde{P}}}(\log \tilde{\tilde{D}} + \tilde{\tilde{F}}) \to K_{\tilde{\tilde{\mathcal{F}}}} \to 0$$

and so,

$$K_{\tilde{\tilde{\mathcal{F}}}} \cdot T_{\mathcal{F}} = (K_{\tilde{\tilde{P}}} + \tilde{\tilde{D}} + \tilde{\tilde{F}})\, T_{\mathcal{F}}$$

while by accident more than design,

$$K_{\tilde{\tilde{\mathcal{F}}}} \cdot \mathbb{I}_S\, T = \sum_{CC'} (\lambda(CC') + E_{CC'} \cdot \mathbb{I}_S\, T) = (K_{\tilde{\tilde{P}}} + \tilde{\tilde{D}} + \tilde{\tilde{F}})\, \mathbb{I}_S\, T$$

where $\lambda(CC')$ is the Lelong number of T along the component of S determined by C and C'. Consequently on combining with the original tautological inequalities for $\mathbb{P}(J_{\log})$ and $\mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))$ we obtain,

$$0 \geq (K_{P_{\log}} + D).\rho_* T + \left(\sum_C d \log C + \widetilde{\widetilde{F}} \right) T \geq (K_{P_{\log}} + D).\rho_* T\,.$$

From which we deduce that the Bloch-Theorem reduces to the main lemma in the specific form,

Fact 3.2.3. Bearing in mind 2.2.4, if a sequence $f_n \in \mathrm{Hom}(\Delta, \mathbb{P}^2 \backslash D)$ which is not eventually arbitrarily close to D doesn't have a subsequence convergent in $\mathrm{Hom}(\Delta, \mathbb{P}^2)$, the derivatives $f'_n : \Delta \to \mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))$ are eventually arbitrarily close to the divisors defined by the pencils of lines defined by any two pairs of lines in D, together with the divisor D itself.

For the sake of illumination let's offer some remarks,

Remarks 3.2.4. (i) The structure of the proof here is very much inverse to Bloch's. The connection corresponds to the Wronskian, the divisors F to various sub-Wronskians. Bloch begins by excluding the possibility that the f_n could be arbitrarily proximate to some appropriate divisors, then he does the some for the sub-Wronskian, and eventually the Wronskian.

(ii) The main lemma in this particular case is an easy exercise, but only because it's straightforward to identify appropriate bundles flat along the various pencils after a modicum of blowing up. However the algebraic nature of the foliations in question, doesn't of itself simplify things that much, except that harmonic rather than closed currents would suffice.

3.3 Foliations

This section is really just a parenthesis, as we'll only summarise in some brevity what we'll need. One should consult [M2] §1 for a full discussion. Despite the fact that all the foliations in 3.2 are just pencils of lines, we deliberately emphasised the former as it's the correct way to understand the relation of 3.2 to the main lemma. All our foliations $\mathcal{F}$, will be foliations by curves. On a smooth variety X this is equivalent to giving a quotient,

$$\Omega_X \to K_{\mathcal{F}} \cdot I_Z \to 0$$

where the line bundle $K_{\mathcal{F}}$ is the canonical class of the foliation, already seen in 3.3, and the singular subscheme Z has codimension at least 2. Evidently there's a one to one correspondence between foliations and horizontal divisors in $\mathbb{P}(\Omega_X)$, à la 3.1, of degree 1. Apart from Brunella's residue theorem which we've already noted in 1.3.1, the pertinent facts regarding singularities are,

Facts 3.3.1. Let $(X, \mathcal{F})$ be a foliation on an algebraic surface then there is a proper birational modification $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ with canonical singularities, in the obvious sense of that notion. Better still if $(X, \mathcal{F})$ has canonical singularities then it is the same for any modification, with the calculation of the discrepancy as follows,

(a) If $\rho : (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ is a blow up in a smooth point and E the exceptional divisor then, $K_{\widetilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}} + E$.

(b) As above but in a singular point, $K_{\widetilde{\mathcal{F}}} = \rho^* K_{\mathcal{F}}$.

In either case the exceptional divisor is invariant under $\widetilde{\mathcal{F}}$.

Apart from the open question of the existence of resolution in higher dimension, there's a lot else in 3.3.1 that doesn't work, as explained in [M1]. However this is of no immediate concern to us.

4 Main Lemma and Corollaries

4.1 Proof of the Main Lemma

To summarise our reductions we are led to consider sequence of triples,

$$\begin{array}{ccc} Y_n & \xrightarrow[f_n]{} & X \\ \downarrow p_n & & \\ \Delta & & \end{array}$$

of maps of finite covers of the disc whose ramification is not worse than the degree such that for some sufficiently large simple normal crossing divisor D on X and foliations $\mathcal{F}^i$, $1 \leq i \leq r$, the derivatives f_n' are arbitrarily close, in the compact open sense to the divisor in $\mathbb{P}(\Omega_X)$ cut out by D and the foliations. Naturally for some ample divisor H on X we suppose the degrees $H\cdot_{f_n} Y_n(s)$ unbounded for some $0 < s < 1$, and that the f_n – or more precisely, bearing in mind 1.1.5, simply the $f_n(0)$ – are not arbitrarily close to some Zariski closed subset V to be specified. With this in mind the main lemma takes the form,

Theorem 4.1.1. (Notations as above) *For any $\varepsilon > 0$ there is a proper birational modification $\rho : (\widetilde{X}, \widetilde{D}) \to (X, D)$*[2] *such that if $f_n' : Y_n \to \mathbb{P}(\Omega_{\widetilde{X}}(\log \widetilde{D}))$ is the logarithmic derivative then for $r \in (0,1)$ outside of a set of finite hyperbolic measure,*

$$(K_{\widetilde{X}} + \widetilde{D})\cdot_{f_n} Y_n(r) \leq L\cdot_{f_n} Y_n(r) + \varepsilon\, H\cdot_{f_n} Y_n(r)$$

where L is the tautological bundle.

[2]Here $\widetilde{D}$ denotes the reduction of the total pull-back of D together with any curves contracted by ρ off D.

Proof. Obviously we blow up until we reach the point where all of our foliations have canonical singularities. Thus we implicitly use the fact, specific to surfaces, that modifications of foliations with canonical singularities still enjoy the same. The same principle even allows us to assume the marginally stronger class of what are classically termed reduced singularities, and whence the technical convenience that any of our foliations has exactly two branches at a given singularity. Regardless we are now in a situation where any exceptional curve on a further blow up must be invariant by all the foliations, so that after further modifications we might as well suppose that we begin from a point where a curve C in the support of D with respect to a given foliation $\mathcal{F}^i$ is either invariant by it or nowhere tangent to it, and in addition that at any crossing of D there passes at least one curve invariant by all the foliations, while at any singularity of any foliation there is at least one curve in the support of D invariant by it.

We now specify the meaning of sufficiently large, namely, that D contains any curve which is invariant and passes through at least one singularity of some $\mathcal{F}^i$, and call this initial modification (X_0, D_0). Now suppose our foliation $\mathcal{F}^i$ is given by a dévissage,

$$0 \to N_0^i \to \Omega_{X_0} \to K_{\mathcal{F}_0^i} \cdot I_{Z_0^i} \to 0$$

where $K_{\mathcal{F}_0^i}$ is to be thought of as the canonical bundle along the foliation, N_0^i the co-normal bundle and Z_0^i the singular subscheme. We can divide our divisor D_0 as $\Delta_0^i + \partial_0^i$ where Δ_0^i is the part invariant by $\mathcal{F}^i$, and consider the corresponding log dévissage,

$$0 \to N_0^i(\Delta_0^i) \to \Omega_{X_0}(\log D_0) \to K_{\mathcal{F}_0^i}(\partial_0^i) \cdot I_{W_0^i} \to 0\,.$$

Given that there is at least one $\mathcal{F}^i$ invariant curve through any crossing of D_0, and D_0 is everywhere transverse to $\mathcal{F}^i$ where it isn't invariant, $W_0^i \subset Z_0^i$. Better still since at every singular point there is at least one $\mathcal{F}^i$ invariant curve C in D_0, if Z_0^i has degree 1 at this point, W_0^i is empty. The structure of W_0^i therefore comes from so called saddle nodes, i.e. for some formal coordinates x, y and $d_i \in \mathbb{N}$, $\mathcal{F}^i$ is given by a derivation of the form,

$$x(1 + \nu_i\, y^{d_i})\,\frac{\partial}{\partial x} + y^{d_i+1}\,\frac{\partial}{\partial y}$$

where $\nu_i \in \mathbb{C}$. The branch $y = 0$ is termed strong, since the foliation accumulates on it, and naturally $x = 0$ the weak branch. Whether or not W_0^i is empty or not is therefore dependent on whether or not, the weak branch is in D_0. Under blowing up the proper transform of the weak branch meets the exceptional divisor in another saddle node whose weak branch is the said proper transform. There is also another singularity on the exceptional divisor but here the degree is 1.

Now certainly amongst the foliations singularities may coincide, but we might as well insist that we've blown up sufficiently that this will be the case iff the branches coincide. In particular if a point is a singularity of two foliations $\mathcal{F}^i$ and $\mathcal{F}^j$ then a given branch is algebraic (whence in the support of D_0) for one iff it's algebraic for the other. Whence define Z_0 to be the set of points in X_0 which are saddle nodes for some foliation $\mathcal{F}^i$ whose weak branch at the point is not algebraic. Define X_1 to be the blow up of X_0 in Z_0, D_1 to be the pull-back of D_0 (by construction this is automatically reduced and contains the exceptional divisor), Z_1 to be the induced saddle nodes with non-algebraic branches, and continue inductively to get a sequence of modifications,

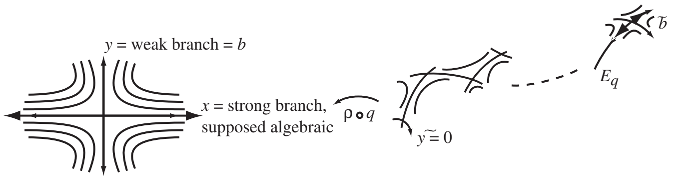
$$(X_n, D_n) \to (X_{n-1}, D_{n-1}) \to \cdots \to (X_1, D_1) \to (X_0, D_0)$$

with $\rho_{pq} : (X_q, D_q) \to (X_p, D_p)$ the projection for $q > p$, and for $z \in Z_0$, E_{zq} the exceptional curve on (X_q, D_q) over z blown down by $\rho_{q,q-1}$. Observe that our foliations $\mathcal{F}^i$ transform according to one of the following possibilities in a neighbourhood of $z \in Z_0$,

(i) $\mathcal{F}^i$ doesn't have a saddle node at z, then all induced singularities on X_q have degree 1. Whence if $\mathcal{F}^i_q$ is the induced foliation and $D_q = \Delta^i_q + \partial^i_q$ the decomposition into invariant and non-invariant parts, $K_{\mathcal{F}^i_q}(\partial^i_q)$ is a quotient of $\Omega_{X_q}(\log D_q)$.

(ii) $\mathcal{F}^i$ has a saddle node at z, with non-algebraic weak branch b, then for a subcheme W^i_q supported at the crossing of $\widetilde{b}$ with the exceptional divisor, say z_q, $K_{\mathcal{F}^i_q}(\partial^i_q)\, I_{W^i_q}$ is a quotient of $\Omega_{X_q}(\log D_q)$.

Pictorially then we have over z a chain of rational curves derived from blowing up a neighbourhood of the origin, i.e.



where intuitively the evident goal of this procedure is to push the saddle node away to infinity.

Now let us turn to our maps $f_n : Y_n \to X$, and their various proper transforms, also denoted f_n, to any X_q. Certainly we have various logarithmic derivatives, $f'_n : Y_n \to \mathbb{P}(\Omega_{X_q}(\log D_q))$ which thanks to the log-tautological inequality, and the general considerations of 1.2 give for all q and $r \in (0,1)$ outside a sort of finite hyperbolic measure positive currents,

$$T'_{nq}(r) : A^{1,1}(\mathbb{P}(\Omega_{X_q}(\log D_q)) \to \mathbb{C} : \eta \mapsto (H \cdot_{f_n} Y(r))^{-1} \int_0^r \frac{dt}{t} \int_{Y(t)} f_n^* \eta$$

converging weakly, after suitable subsequencing, to closed positive currents T'_q, provided of course that we excluded a sufficiently large Zariski subset in order to guarantee the degree on X_0 measured with respect to H, dominates the degree with respect to an ample divisor on X_q. Within

$\mathbb{P}(\Omega_{X_q}(\log D_q))$, which we may as well denote P_q, with π to X_q the projection, we have two kinds of divisors that interest us, one being $\pi^* C$ for C a component of D, the other the closure of the section in codimension 2 defined by the foliation $\mathcal{F}^i$, which we'll denote X_q^i. Necessarily X_q^i is isomorphic to the blow up of X_q in W_q^i, so it's a section away from saddle nodes z, but in general for a node of degree $d_i(z)$ at least 3, it will be a blow up in a non-reduced scheme, so it's singular, and its minimal smooth resolution $\widetilde{X}_q^i$ is around the node isomorphic to $X_{q+d_i(z)}$. By hypothesis T'_q is supported only on these divisors. Before breaking T'_q up according to its support, observe that for any C in D, $(K_X + D).\, C \leq 0$ iff C is rational and meets $D\backslash C$ in at most 2 points. Call such curves parabolic, and observe further that it is a harmless hypothesis to suppose that if C in D is not parabolic and $C' \neq C$ in D meets C then C' is parabolic, i.e. just blow up in the crossings of D, which we might as well have done a priori so that this holds for all D_q, $q \geq 0$. Now we can devide T'_q up into a sum of closed positive currents according to the rule,

$$T'_q = \sum_{C \subset D_q^{n\mathrm{par}}} \pi^* \, \mathbb{1}_C \, T'_q + \sum_{C \subset D_q^{\mathrm{par}}} \pi^* \, \mathbb{1}_{C \backslash D_{np}} \, T'_q + \sum_i \mathbb{1}_{X_q^i \backslash D_q} \, T'_q$$

where $D_q = D_q^{\mathrm{par}} + D_q^{n\mathrm{par}}$ is the decomposition of D_q into its parabolic and non-parabolic components, and $\mathbb{1}_*$ is the characteristic function of the appropriate set.

Given the nature of our preliminaries we take care of $\mathbb{1}_{X_q^i \backslash D_q} \, T'_q$ first. For simplicity let's write this as $\mathbb{1}_{\mathcal{F}_q^i} \, T_q$, by construction $\pi_* \, \mathbb{1}_{\mathcal{F}_q^i} \, T_q$ is a transverse invariant measure for $\mathcal{F}_q^i$, which can have atomic components on at most invariant curves wholly contained in the smooth locus. Locally over the points z, if any, where X_q^i isn't a section we have an exceptional Cartier divisor $E_q^i(z)$ whose pull-back to the minimal smooth resolution $X_{q+d(z)}$ is $\sum_{k=1}^{d_i(z)} \rho^*_{q+d_i,k} \, E_k(z)$. In any case if L_q is the tautological bundle on P_1, then,

$$L_q \otimes \mathcal{O}_{X_q^i} = \mathcal{O}_{X_q^i} \left(K_{\mathcal{F}_q^i} + \partial_q^i - \sum_z E_q^i(z) \right) .$$

Furthermore by a theorem of M. Brunella, [B], for T on X_q a transverse $\mathcal{F}_q^i$ invariant measure, without atomic support $(N_q^i + \Delta_q^i).\, T \leq 0$. However this inequality equally holds trivially for any invariant curve wholly contained in the smooth locus of $\mathcal{F}_q^i$, so it holds for $\pi_* \, \mathbb{1}_{\mathcal{F}_q^i} \, T_q$, which implies,

$$L_q \cdot \mathbb{1}_{\mathcal{F}_q^i} \, T_q \geq \pi^*(K_{X_q} + D_q) \, \mathbb{1}_{\mathcal{F}_q^i} \, T_q - \sum_z E_q^i(z) \cdot \mathbb{1}_{\mathcal{F}_q^i} \, T_q \, .$$

To take care of the error term we employ the following lemma from [M2], namely, FACT. (op.cit. V.3.3) There is a positive rational constant α such that every sufficiently large and divisible multiple of $\alpha\,[\sqrt{q}]^{-1}\,\rho_{q0}^{*}\,H - E_q(z)$ is generated by its global sections off the proper transforms of $E_1(z), \dots, E_{q-1}(z)$.

Bearing in mind that $\pi_* \, \mathbb{1}_{\mathcal{F}_q^i} T_q$ is nef. for any q, we therefore obtain,

$$\sum_z E_q^i(z) \cdot \mathbb{1}_{\mathcal{F}_q^i} T_q \leq (\mathrm{const})\,[\sqrt{q}]^{-1}$$

for some absolute constant depending only on X, so that for some q sufficiently large, $L_q \, \mathbb{1}_{\mathcal{F}_q^i} T_q \geq \pi^*(K_{X_q} + D_q) . \mathbb{1}_{\mathcal{F}_q^i} T_q - \varepsilon$, and we're reduced to obtain a similar inequality for the part of T_q' supported over D.

To this end consider for C in D the canonical section, defined by the short exact sequence,

$$0 \to \mathcal{O}_C(K_X + D) \to \Omega_X(\log D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to 0$$

where the last map is the residue map. For simplicity of notation consider q as fixed, and let P_C be the fibre of P over C with S the said section. Certainly we have,

$$\mathcal{O}_{P_C}(S) = \mathcal{O}_{P_C}(L_q - \pi^*(K_X + D))$$

so an estimate of the form $S \cdot \mathbb{1}_{P_C} T' \geq 0$ will serve us nicely. Better still for C parabolic, $S^2 \geq 0$, so such an estimate trivially holds for any current supported in P_C, in particular for $\mathbb{1}_{P_C \backslash \pi^* D^{n\mathrm{par}}} T$, and we're reduced to study C non-parabolic.

At this point it becomes clear why we wished to simplify the notations. Specifically we consider a sequence of modifications,

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 = P$$

with P_1 the blow up of P_0 in S, so that the proper transform P_C^1 of P_C is isomorphic to P_C whence in this sense we blow up P_1 in S and continue ad infinitum. Here we'll use π_{mn} for the projection between P_n and P_m, and F_n the exceptional divisor blown down by $\pi_{n-1,n}$.

To set the scene on P_1, P_C^1 can be blown down, and the resulting variety is $\mathbb{P}(\Omega_{X_q}(\log D_q'))$ where $D_q = D - C$, with P_C^1 blowing down to the derivative of C. Now since any derived type current with non-trivial push-forward in $\mathbb{P}(\Omega_{X_q}(\log D_q') \otimes \mathcal{O}_C)$ is supported on C this shows that there are no non-exceptional components of the induced current supported on F_q. However this holds much more generally, but let's first set up the notation and clarify the above remarks.

Our sequence $f_k' : Y_k \to P$, has a lifting to any P_n which leads to an induced current T_n (strictly speaking T_{nq}, but q will play no further role). We're interested in the decomposition,

$$T_n = \mathbb{1}_{P_C^n} T_n + \mathbb{1}_{\pi_n^* C \backslash P_C^n} T_n + \mathbb{1}_{P_n \backslash \pi_n^* C} T_n$$

and we've already remarked that $(\pi_{10})_*(\mathbb{1}_{\pi_{01}^* C \backslash P_C^1} T_1) = 0$, however suppose this held more generally, i.e., we have, Claim. $(\pi_{n0})_* (\mathbb{I}_{\pi_n^* C \setminus P_C^n} T_n) = 0$.

Then we may conclude as follows. In the first place this forces $\mathbb{I}_{P_C^n} T_n = \mathbb{I}_{P_C} T_0$, so that $(F_1 + \cdots + F_n) \cdot \mathbb{I}_{P_C^n} T_n = n S \cdot \mathbb{I}_{P_C} T_0$. Furthermore $\mathbb{I}_{\pi_{0n}^* C \setminus P_C^n} T_n$ is homologous to a positive sum of curves contracted by π_{n0}, and $F_1 + \cdots + F_n$ is non-positive on such so,

$$0 \leq (F_1 + \cdots + F_n) \cdot T_n \leq n S \cdot \mathbb{I}_{P_C} T_0 + \sum_{i=1}^{n} F_i \cdot \mathbb{I}_{P_n \setminus \pi_{0n}^* C} T_n$$

where the first inequality holds since $f_n(0)$ is supposed bounded away uniformly from C. Finally note that P_C^n intersects positively any current supported off it, so:

$$\sum_{i=1}^{n} F_i \cdot \mathbb{I}_{P_n \setminus \pi_{0n}^* C} T_n \leq P_C \cdot \mathbb{I}_{P_0 \setminus \pi_{01}^* C} T_0$$

since the latter is fixed independently of n, our claim indeed implies that $S \cdot \mathbb{I}_{P_C} T_0 \geq 0$.

It remains therefore to justify the claim. To this end suppose locally C is given by an equation $x = 0$, and y is the other coordinate. In a neighbourhood of S we have a coordinate $t = \frac{dy}{dx/x}$ (or $\frac{dy/y}{dx/x}$ if we're close to a crossing of D) with respect to which $\pi_{0n} : P_n \to P$ is the minimal smooth resolution of the ideal $I = (t^n, x)$, i.e. $\bigotimes_{i=1}^{n} \pi_{in}^* \mathcal{O}_{P_i}(-F_i) = \pi_{0n}^{-1} I$, so to be an open neighbourhood a distance ε off P_C^n amounts to the condition,

$$\frac{|x|}{|x| + |t|^n} \geq \varepsilon \Leftrightarrow |dy| \text{ (resp. } \left|\frac{dy}{y}\right| \text{ near a crossing)} \leq \left(\frac{1-\varepsilon}{\varepsilon}\right)^{\frac{1}{n}} \frac{|dx|}{|x|^{1-1/n}} .$$

Consequently the claim will follow from an appropriate estimate for the measure, $T\left(\mathbb{I}_{\delta,\varepsilon} \frac{dx\, d\overline{x}}{|x|^{2-2/n}}\right)$ for $\mathbb{I}_{\delta,\varepsilon}$ a small neighbourhood a distance δ around C, but ε off P_C^n. More canonically in terms of the norm $\mathbb{I}_C$, we may observe for δ sufficiently small, determined by n,

$$\frac{1}{n^2} \frac{dx\, d\overline{x}}{|x|^{2-2/n}} \gg\ll 4d\, \mathbb{I}_C^{\frac{1}{n}}\, d^C \mathbb{I}_C^{1/n} + |x|^{\frac{1}{n}} \omega$$

for ω_n a smooth form depending on n. Equally $4d\, \mathbb{I}_C^{1/n}\, d^C \mathbb{I}_C^{1/n}$ is just $dd^C \mathbb{I}_C^{2/n}$ this time up to a factor of $\mathbb{I}_C^{2/n} \times$ smooth, so the following Jensen formula, with $\mathbb{I}_\delta$ denoting points with $\mathbb{I}_C \leq \delta$, will do nicely,

$$\begin{aligned} T_{f_k, Y_k}(r)\left(\mathbb{I}_\delta \frac{dd^c \|\mathbb{I}_C\|^{2/n}}{\delta^{2/n}}\right) &= \int_{\partial Y_k(r)} \mathbb{I}_\delta \left(\frac{\|\mathbb{I}_C\|^{2/n}}{\delta^{2/n}} - 1\right) p_k^* \left(\frac{d\theta}{2\pi}\right) \\ &- \frac{1}{n} \int_{\partial Y_k(r)} \mathbb{I}_\delta \log \frac{\|\mathbb{I}_C\|}{\delta}\, p_k^* \left(\frac{d\theta}{2\pi}\right) \\ &- \mathbb{I}_\delta \left(\frac{\|\mathbb{I}_C\|^{2/n}}{\delta^{2/n}} - 1\right)(0) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{n}\, \mathbb{1}_\delta \log \frac{\|\mathbb{1}_C\|}{\delta}\,(0)\\
&+ \quad \frac{1}{n} \sum_{0<|p_k(y)|<r} \mathrm{ord}_y(f_k^* D) \log \left| \frac{r}{p_k(y)} \right|\\
&- \quad T_{f_k, Y_k}(r)\, (\mathbb{1}_\delta\, C_1(\overline{C}))\,.
\end{aligned}
$$

¿From which we see that the measure $T\left(\mathbb{1}_\delta \frac{dx\, d\overline{x}}{|x|^{2-2/n}}\right)$ goes to zero like $\delta^{2/n}$. Whence if we take $\delta = \varepsilon^2$, we've comfortably established our claim. □

4.2 Degeneration of the Kobayashi metric

Manifestly whenever the condition that a sequence of maps $f_n : \Delta \to X$ or more generally $X \backslash D$ enjoys the property that their derivatives are eventually arbitrarily close to some divisor on $P(T_X)$ but not eventually arbitrarily close to a divisor in D, the main lemma and the tautological inequality apply to bound the K_X-degree. Consequently provided X has general type and the divisor is taken to include the curves contracted to the canonical (not just minimal) model we obtain convergence of a subsequence in $\overline{\mathrm{Hom}}(\Delta, X)$, while evidently in the quasi-projective case we work with the log-canonical model.

The question arises therefore of what becomes of a sequence of maps $f_k : \Delta \to X$ which are arbitrarily close to some divisor ∂ in X. Apart from the need to assume simple normal crossings, the proof of the main lemma provides us with a wholly general estimate. Indeed a moments thought shows that for L the tautological bundle on $\mathbb{P}(\Omega_X(\log D))$ the last step in the proof yields,

FACT 4.2.1. Suppose for $r_0 \in (0,1)$, the degrees $\overline{H}\cdot_{f_k} \Delta_0(r)$ with respect to a positive metricised ample are unbounded in k, then after subsequencing there is a constant $a > 0$ such that for all $n \in \mathbb{N}$ and $r \in (0,1)$ outside a set of finite measure,

$$
L\cdot_{f_k'} \Delta(r) \geq (K_X + \partial)\cdot_{f_k} \Delta(r) - \frac{a}{n}\, H\cdot_{f_k} \Delta(r) + \frac{1}{n} \log \|f_k^*\, \mathbb{1}_\partial\|\,(0)\,.
$$

Now let's turn this into an estimate for the degeneration of the Kobayashi metric on a quasi-projective surface (X, D) which is hyperbolic modulo Z. We necessarily suppose not only that (X, D) has log general type, but that (X, D) is the log-canonical model, or equivalently $K_S + D$ is ample. The unfortunate thing is that the set Z on S could be highly singular and for $D \neq \varphi$, meet the boundary in an extremely messy way. Consequently to apply 4.2.1 we must pass to a resolution $\rho : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth, and the total pull-backs of D and Z have simple normal crossings. Our boundary ∂ then will be the union of $\widetilde{D}$, the reduced total pull-back of D, and $\widetilde{Z}$, any component of the reduced pull-back that isn't in $\widetilde{D}$. An appropriate measure of the singularity of our initial situation is provided by the discrepancy, i.e.

$$K_{\widetilde{X}} + \widetilde{D} = \rho^*(K_X + D) + \sum_i a_i\, E_i$$

where the E_i are the curves contracted by ρ, and $a_i \in \mathbb{Q}_{\geq 0}$. Obviously to keep the situation as natural as possible ρ should be minimal, i.e. first pass to the unique minimal smooth resolution (X', D') of (X, D), then take a minimal resolution for the divisor $D' + Z'$, where the latter is the proper transform of Z. In any case denote by $\| \;\|_{\widetilde{\log}}$ a norm on $T_{\widetilde{X}}(-\log \partial)$ then we have,

Corollary 4.2.2. *Notations as above, then for all $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that for any $f \in \mathrm{Hom}(\Delta, X \backslash D)$,*

$$\prod_i \|1\!\!1_{E_i}\|^{a_i}(0)\ \|d\widetilde{f}\|_{\widetilde{\log}}(0) \leq \frac{C(\varepsilon)}{\|1\!\!1_{\widetilde{Z}}\|^{1+\varepsilon}(0)\, \|1\!\!1_{\widetilde{D}}\|^{\varepsilon}(0)}$$

where $\widetilde{f} : \Delta \to \widetilde{X}$ is the proper transform.

Proof. If the inequality is false, then it's false for a sequence of maps f_k with, $\|d\widetilde{f}_k\|_{\widetilde{\log}}(0) \to \infty$. In the first instance consider the possibility that for some $r \in (0, 1)$, $\overline{H}._{\widetilde{f_k}}\, \Delta(r)$ is bounded, with H ample on $\widetilde{X}$, then after subsequencing we may assume that $f_k \mid_{\Delta(r)}$ converges to a disc with bubbles which does not bubble at the origin. Consequently for some $0 < s < r$ the maps $f_k : \Delta(s) \to X$ converge to some $f : \Delta(s) \to X$. Now given that the f_k actually map to $X \backslash D$, we can apply Green's lemma to conclude that f lies in $X \backslash D$ or D. In the former case clearly $\|df_k\|_{\widetilde{\log}}(0)$ cannot go to infinity, while in the latter case, after possibly shrinking s, we are reduced to a purely local problem, i.e. $X = \Delta \times \Delta$ and $D = 0 \times \Delta$ or $0 \times \Delta \cup \Delta \times 0$, so after projecting even just $X = \Delta$, $D = 0$. However this is basically trivial, since without loss of generality $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, and the f_k correspond to maps, $x_k : \Delta(s) \to \Delta^*$, with $|x_k| \to 0$ on compact sets. On the other hand we can take roots $x_k^{1/n}$ which equally converge to zero, so,

$$\frac{|x_k'|}{|x_k|^{1-1/n}} = n \left| \frac{d}{dz}\, x_k^{\frac{1}{n}} \right|$$

is certainly bounded contrary to assumpion. Thus we're in a position where we can apply 4.2.1 together with 1.2.7 and 1.2.8. Evidently we take the n of 4.2.1 sufficiently large, along with a sufficiently large k to conclude,

$$\begin{aligned} & (K_{\widetilde{X}} + \widetilde{D})._{\widetilde{f_k}}\ \Delta(s) - a\varepsilon\, H._{\widetilde{f_k}}\ \Delta(s) + \log \|d\widetilde{f}_k\|_{\widetilde{\log}}(0) \\ + \quad & \frac{\varepsilon}{2} \log \|f_k^*\ 1\!\!1_{\partial}\|\,(0) \\ \leq \quad & \mathrm{rad}_{\widetilde{f_k}, \widetilde{Z}}(s) + \log(1 - s^2)^{-1}(1 + \circ(H._{\widetilde{f_k}}\ \Delta(s)) \end{aligned}$$

for all $s \in (r, 1)$ outside a set of finite hyperbolic measure, and a perhaps bigger than its previous manifestation in 4.2.1. Now to have a sequence whose degrees with respect to some ample go to infinity is the equivalent for any ample, so we might as well say, $H = \rho^*(K_X + D) \left(- \sum_i \delta_i \, E_i \right)$ for some $\delta_i \in \mathbb{Q}_{>0}$ as small as we like, which by the way, is a trick also used to derive 4.2.1, since there and here we're in the rare situation where the degree might behave very badly on blowing up. Regardless we can estimate the radical in $\widetilde{Z}$ as the degree with respect to the same minus the proximity to zero, and fix a suitable s and δ_i's in order to write,

$$
\begin{aligned}
& (1 - a\varepsilon) \, H \cdot_{f_k} \Delta(s) + \log \prod_i \| \mathbb{1}_{E_i} \|^{a_i} (0) \, \| d\widetilde{f}_k \|_{\widetilde{\log}} (0) \\
\leq \quad & (1 + \varepsilon) \log \| \widetilde{f}_k^* \, \mathbb{1}_{\widetilde{Z}} \| \, (0) + \varepsilon \log \| \widetilde{f}_k^* \, \mathbb{1}_{\widetilde{D}} \| \, (0) + \circ(H \cdot_{f_k} \Delta(s))
\end{aligned}
$$

and we're done. □

4.3 Function fields

The problem of rational points over function fields is quickly translated into geometric terms, i.e. our interest becomes commutative diagrams of the form

$$
\begin{tikzcd}
Y \arrow[r, "f"'] \arrow[dr, "p"'] & X \arrow[d, "\pi"'] \\
& B
\end{tikzcd}
\tag{4.3.1}
$$

where B is a curve, $Y \to B$ a finite cover, and $\pi : X \to B$ not just proper, but also relatively projective. Evidently π is fixed, and is a model of $X \otimes k(B) \to k(B)$, the original variety over our field of functions. Not surprisingly we'll work in the quasi-projective setting, so this means assuming X non-singular and introducing a reduced simple normal crossing divisor D, each of whose components is flat over B. We'll also assume that the singularities of the fibres of π are no worse than nodes.

We'll also work in the quasi-projective setting, so X will be assumed non-singular, and D will be a divisor with simple normal crossings whose components are flat over B, the singularities of the fibres are no worse than nodes, i.e. the reduction of any fibre is simple normal crossing, but the fibre itself need not be reduced, and D together with the singular fibres is simple normal crossing. Necessarily we have a map,

$$\pi^* \, \omega_B \to \Omega_X(\log D)$$

which may not be saturated. Indeed if X_{b_i} are the fibres which are non-reduced, then the saturation is,

$$\pi^* \omega_B \left(\sum_i X_{b_i} - (X_{b_i})_{\mathrm{red}} \right) \to \Omega_X(\log D)\,. \tag{4.3.2}$$

By a quite substantial abuse of notation, define,

DEFINITION 4.3.3. $\Omega_{X/B}(\log D)$ *the quotient of the map* (4.3.2).

This is non-standard notation. If D is empty, the usual definition is the quotient of the non-saturated map.

Evidently $\mathbb{P}(\Omega_{X/B}(\log D)) \hookrightarrow \mathbb{P}(\Omega_X(\log D))$ is the irreducible divisor defined by 4.3.2. Over a singularity of a singular fibre, the reduced fibres of the latter and the former coincide. Now the analogue of the Bishop-Gromov compactness theorem is existence and properness of the components of the Hilbert scheme, which immediately implies that if a sequence of maps $f_k : Y/B \to X/B$ à la 4.3.1, have Zariski dense image and bounded degree with respect to some ample H then the fibration π is isotrivial. So we necessarily restrict ourselves to sequences $f_k : Y/B \to X$ of unbounded degree, but with $(Y_k : B)$ bounded.

Restricting ourselves to characteristic zero, we consider the derivatives $f'_k : Y_k \to \mathbb{P}(\Omega_X(\log D))$, and the associated currents of integration T'_k. Suppose further, as will invariably be the case, that if A is ample on $\mathbb{P}(\Omega_X(\log D))$ then $\liminf H \cdot_{f_k} Y_k / A \cdot_{f'_k} Y_k \neq 0$, then subsequencing as necessary we obtain a closed positive current T on $\mathbb{P}(\Omega_X(\log D))$ as the weak limit of $(H \cdot_{f_k} Y_k)^{-1} T_k$ for which the basic observation is,

LEMMA 4.3.4. *Notations as above, with* $U = \mathbb{P}(\Omega_X(\log D)) \backslash \mathbb{P}(\Omega_{X/B}(\log D))$ *and* ρ *the projection to* X, *then* $\rho_*(\mathbb{1}_u T) = 0$.

Proof. The question is local, and vacuous over the singularities of fibres, so we may as well say b is a coordinate on B, with $\pi^* b = z^n$, for z a coordinate on X. Further let $x_1, \ldots, x_n$ be a coordinate system on X with $x_1 \ldots x_d = 0$ defining D. Consider a polydisc V_ε in U a distance ε away from $\mathbb{P}(\Omega_{X/B}(\log D))$, which amounts to $\|dz\| > \varepsilon$, where,

$$\|dz\|^2 = \frac{|dz|^2}{\sum\limits_{i=1}^{d} \left| \frac{dx_i}{x_i} \right|^2 + \sum\limits_{i=d+1}^{n} |dx_i|^2}\,.$$

Consequently for any $f_k : Y_k/B \to X$, over the open set $(f'_k)^{-1}(V_\varepsilon)$ we have the estimate,

$$\frac{1}{\varepsilon^2} f_k^*(dz\, d\overline{z}) > \sum_{i=1}^{n} f_k^*(dx_i\, d\overline{x}_i)\,.$$

On the other hand, $f_k^*(dz\,d\overline{z}) = p_k^*\left(\frac{db\,d\overline{b}}{n^2|b|^{2(1-1/n)}}\right)$, so for ω some Kähler form on X we have,

$$\int_{(f_k')^{-1}(V_\varepsilon)} \omega \leq \frac{\text{const}}{\varepsilon^2}\,(Y_k : B)$$

where the constant is independent of everything except X and B, and we're done. □

Observe further that for any $f : Y/B \to X/B$ with L the tautological bundle in $\mathbb{P}(\Omega_X(\log D))$ we have the tautological identity,

$$L\cdot_{f'} Y = (2g(Y)-2) + \mathrm{rad}_{f,D} - \sum_{y \notin f^{-1}(D)} \mathrm{ord}_y(\mathrm{Ram}_f)\,.$$

Now if D were empty say, and $\pi : X \to B$ were a family of curves with good reduction then $\mathbb{P}(\Omega_{X/B})$ is just X, and the restriction of L, $\omega_{X/B}$. Consequently 4.3.4 immediately bounds the canonical height in terms of the discriminant of the field of functions $\mathbb{C}(Y)/\mathbb{C}(B)$, or alternatively $2g(Y)-2$. In general we just proceed exactly as in the main lemma, i.e. suppose $\pi : X \to B$ a curve with D satisfying our previous hypothesis on the singularities, which in terms of choosing a model is a non-hypothesis, then we claim,

COROLLARY 4.3.5. *Notations as above, $\forall\, \varepsilon > 0$, there is a constant $C(\varepsilon)$ such that if $f : Y/B \to X/B$ is a map of B-schemes then,*

$$(\omega_{X/B} + D)\cdot_f Y \leq (1+\varepsilon)(2g(Y) - 2 + \mathrm{rad}_{f,D}) + C(\varepsilon)(Y : B)\,.$$

Proof. Fix ε, and suppose the inequality is false. As such we can find a sequence $f_k : Y_k/B \to X/B$ of maps such that,

$$(\omega_{X/B} + D)\cdot_{f_k} Y_k > (1+\varepsilon)(2g(Y_k) - 2 + \mathrm{rad}_{f_k,D}) + k(Y_k : B)\,.$$

Now without loss of generality the f_k are certainly Zariski dense, so we can subsequence in order that we have weak convergence of,

$$\frac{1}{H\cdot_{f_k} Y_k} \int_{f(Y_k)} \to T$$

as closed positive currents. Furthermore if the f_k' are the derivatives, then by the tautological identity their degrees as curves in $\mathbb{P}(\Omega_X(\log D))$ are bounded by their degrees as curves on X, while 4.3.4 works equally well under the hypothesis $(Y_k : B)(H\cdot_{f_k} Y_k)^{-1} \to 0$. Whence we get a current T' satisfying 4.3.4, with $\rho_* T' = T$, where the latter is nef. thanks to the Zariski dense hypothesis. The inequality only gets better under blowing up, so by a substantially simpler variant of the main lemma we have,

$$L.\,T \geq (K_X + D)\cdot T$$

and by definition, $\omega_{X/B} = K_X \otimes \pi^*\omega_B^\vee$, so an application of the tautological identity concludes the proof. □

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