Andrejewski lectures
Göttingen, November 2004

Operator Algebras and Index Theorems in Quantum Field Theory

Roberto Longo
University of Rome "Tor Vergata"

Part I: General motivations and an index theorem

Ingredients for QFT analysis. Two paths from the finite-dimensional classical calculus to QFT:

$$\begin{tikzcd} \boxed{\text{Classical, finite dim.}} \arrow[r] \arrow[d] & \boxed{\text{Variational calculus}} \arrow[d] \\ \boxed{\text{Quantum, finite dim.}} \arrow[r] & \boxed{\text{Quantum Field Th.}} \end{tikzcd}$$

Ordinary manifolds $\to$ variational calculus did not require a new calculus; e.g. derivative makes sense replacing points by functions.

Classical $\to$ quantum mechanics does require a new structure (noncommutativity) and a new calculus.

Standard quantization:

$$\text{functions} \to \text{selfadjoint operators}$$
$$\text{Poisson brackets} \to \text{commutators}$$
$$x_h \to P_h$$
$$-i\frac{\partial}{\partial x_h} \to Q_h$$

position and momentum with Heisenberg commutation relations $[P_h, Q_k] = i\delta_{hk} I$.

Connes quantized, <u>finite-dimensional</u> calculus:

CLASSICAL	QUANTUM
Variable	Operator
Differential	$[F, \cdot]$
Integral	$⨍$ (Dixmier trace)
Infinitesimal	Compact operator
...	...

Concerning Quantum Field Theory, we shall consider the underlying structure illustrated in the following table:

Non-trivial map

$$\boxed{\textit{points}} \longrightarrow \boxed{\textit{fields}}\ ,$$

(second quantization functor), cf. the multiplicative structure of the index.

Superselection sectors as QFT analogs of elliptic operator. Atiyah–Singer index thm:

$$\boxed{\text{analytic index}(D) = \text{geom.–topol. index}(D)}$$

D elliptic operator.

Analytic index = Fredholm index, integer!

Geometric index: invariant under deformations.

Major consequence: integrality of the geometical index.

Operator Algebras: proper noncommutative setting for measure theory, topology and geometry.

QFT proper noncommuative setting with <u>infinitely many degrees of freedom</u>.

$$\boxed{\begin{array}{c} \textit{Fredholm linear operators} \rightarrow \mathrm{End}(M) \\ \textit{Elliptic operators} \rightarrow \textit{Localized endomorphisms} \\ \textit{Fredholm index} \rightarrow \textit{Jones index/DHR dimension} \\ \textit{Geometric index} \rightarrow \text{???} \end{array}}$$

look geometric counterparts of index.

NC manifold: net

$$\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \ .$$

Endomorphism ρ localized $\mathcal{O}_0$ is local

$$\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O}), \quad \mathcal{O} \supset \mathcal{O}_0,$$

cf. locality characterization of differential operators.

Roberts cohomology: geometrical description of the superselection structure of $\mathcal{A}$, non-abelian cohomology ring $H^1_R(\mathcal{A})$. How to get dimension?

Consider only localized unitary cocycles associated with translations $H^1_\tau(\mathcal{A})$ which describes the covariant superselection sectors. Denoting by $\mathfrak{S}_{KMS}$ the set of extremal KMS states for the time evolution, at inverse temperature β, satisfying Haag duality, we have a pairing

$$\mathfrak{S}_{KMS} \times H^1_\tau(\mathcal{A}) \ni \varphi \times [u] \to \langle \varphi, [u] \rangle = \int u(i\beta) \mathrm{d}\varphi \in \mathbb{R}$$

where $\int u \mathrm{d}\varphi \equiv \varphi(u)$.

QFT index theorem.

Let $\mathcal{V}$ be a $d+1$ dimensional globally hyperbolic spacetime manifold with a bifurcate Killing horizon and $\mathcal{R}$ a “wedge” of $\mathcal{V}$ (Kay-Wald setting). Typical examples: Schwartzschild $\subset$ Schwartzschild-Kruskal, Rindler $\subset$ Minkowski.

$\kappa = \kappa(\mathcal{R})$ be the *surface gravity*

$\mathcal{A}(\mathcal{O})$ local von Neumann algebras with usual properties. Restriction to the horizon is a Möb covariant (Wiesbrock, Summers, Verch, Guido, Roberts, R.L.) and expected uniquely conformal (Weiner, Carpi)

φ (KMS) Hartle-Hawking state (often unique KMS for Killing evolution). Hawking temperature $\beta^{-1} = \kappa(\mathcal{R})/2\pi$

ρ and σ endomorphisms localized on the horizon with KMS states φ_ρ, φ_σ

$\mathrm{d}F \equiv \frac{1}{2}(F(\varphi_\rho|\varphi_\sigma) + F(\varphi_{\bar{\rho}}|\varphi_{\bar{\sigma}}))$ incremental free energy (can be defined!), then

$$\boxed{\log d(\rho) - \log d(\sigma) = \frac{2\pi}{\kappa(\mathcal{R})}\mathrm{d}F}$$

analytical index = geometric index

Part II: Topological sectors

Diff($\mathbf{S^1}$) and its covers. Diff(S^1) = smooth oriented diffeomorphisms of S^1. The Lie algebra of Diff(S^1) is Vect(S^1) The Virasoro algebra is the (complexification of) the unique, non-trivial one-dim. central extension of Vect(S^1). Generators:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}$$

and $[L_n, c] = 0$.

Diff$^{(n)}(S^1) = n-$cover of Diff(S^1). Set for a fixed $n > 0$

$$L^{(n)}_{\pm m} \equiv \frac{1}{n}L_{\pm mn} \ ,$$

$$L^{(n)}_0 \equiv \frac{1}{n}L_0 + \frac{c}{24}\frac{(n^2-1)}{n} \ .$$

The map

$$\begin{cases} L_m \mapsto L^{(n)}_m \\ c \mapsto nc \ , \end{cases}$$

gives an embedding of the Virasoro algebra into itself. There corresponds an embedding of $\mathrm{Diff}^{(n)}(S^1)$: There is a unique continuous isomorphism $M^{(n)}$ of $\mathrm{Diff}^{(n)}(S^1)$ into $\mathrm{Diff}(S^1)$ such that for all $g \in \mathrm{Diff}^{(n)}(S^1)$ the following diagram commutes

$$\begin{array}{ccc} S^1 & \xrightarrow{M_g^{(n)}} & S^1 \\ {\scriptstyle z^n}\downarrow & & \downarrow{\scriptstyle z^n} \\ S^1 & \xrightarrow{\underline{g}} & S^1 \end{array}$$

i.e. $M_g^{(n)}(z)^n = \underline{g}(z^n)$ for all $z \in S^1$. $\underline{g} \in \mathrm{Diff}(S^1)$ corresponds to g.

Conformal nets on S^1. Let $\mathcal{A}$ be a conformal net on S^1.

U (projective) rep. of $\mathrm{Diff}(S^1)$. Given $n \in \mathbb{N}$, projective unitary representation $U^{(n)}$ of $\mathrm{Diff}^{(n)}(S^1)$

$$U^{(n)} \equiv U \cdot M^{(n)} \ .$$

central charge $c \rightarrow$ central charge nc.

Topological sectors and an index theorem

$\zeta \in S^1$ and

$$h : S^1 \setminus \{\zeta\} \simeq \mathbb{R} \to S^1$$

smooth injective map which is smooth also at $\pm\infty$, i.e. $\exists \lim_{z\to\zeta^\pm} \frac{\mathrm{d}^n h}{\mathrm{d}z^n}$ $\forall n$.

$I \in \mathcal{I}$, $\zeta \notin I$, set

$$\Phi_{h,I}^{(\zeta)} \equiv \mathrm{Ad}U(k) \ ,$$

$k \in \mathrm{Diff}(S^1)$ and $k(z) = h(z)$ for all $z \in I$

$\Phi_{h,I}^{(\zeta)}$ $\underline{\text{does not depend on}}$ $k \in \mathrm{Diff}(S^1)$

$$\Phi_h^{(\zeta)} : I \mapsto \Phi_{h,I}^{(\zeta)}$$

is $\underline{\text{well defined (soliton)}}$ rep. of $\mathcal{A}_0 \equiv \mathcal{A} \restriction \mathbb{R}$.

Now f smooth, locally injective map

$$f : S^1 \to S^1, \quad \mathrm{deg} f = n \geq 1.$$

$\zeta \in S^1 \to n$ right inverses h_i, $i = 0, 1, \dots n-1$, for f;

$$h_i : S^1 \smallsetminus \{\zeta\} \to S^1, \quad f(h_i(z)) = z,$$

(smooth at $\pm\infty$). Order h_i e.g. counterclockwise

Example. $f(z) = z^n$, $h_i(z) = \sqrt[n]{z}$ (n choices)

We get a soliton $\pi_f^{(\zeta)}$ of $\mathcal{A}_0 \otimes \mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_0$

$$\boxed{\pi_{f,I}^{(\zeta)} \equiv \chi_I \cdot (\Phi_{h_0,I}^{(\zeta)} \otimes \Phi_{h_1,I}^{(\zeta)} \otimes \cdots \otimes \Phi_{h_{n-1},I}^{(\zeta)})}$$

$\chi_I : \mathcal{A}(I_0) \otimes \cdots \otimes \mathcal{A}(I_{n-1}) \to \mathcal{A}(I_0) \vee \cdots \vee \mathcal{A}(I_{n-1})$ (split property), $I_k \equiv h_k(I)$.

Reordering the h_i's $\to$

$$\pi_{f,p} = \pi_f \cdot \beta_p$$

- $\deg f_0 = \deg f \implies \pi_{f_0} \simeq \pi_{f,p}$ (some $p \in \mathbb{S}_n$)

• $\pi_{f,p}$ depends only on deg$f$ and p up to unitary equivalence.

• Index$(\pi_f) = \mu_{\mathcal{A}}^{n-1}$.

• The conjugate of π_f is given by

$$\bar{\pi}_f = \pi_{\bar{f},p}$$

where $\bar{f}(z) \equiv \overline{f(\bar{z})}$ and $p : m \mapsto -m$ in $\mathbb{Z}_n$.

Set

$$\boxed{\tau_f \equiv \pi_f^{(\zeta)} \restriction (\mathcal{A} \otimes \mathcal{A} \cdots \otimes \mathcal{A})^{\mathbb{Z}_n}}$$

is <u>DHR representation</u> independent of ζ up to unitary equivalence.

Thm. (Xu, L.) (a): τ_f depends only on $n =$ deg$f$ up to unitary equivalence.

(b): τ_f is diffeomorphism covariant; projective unitary representation $U^{(n)} = U \cdot M^{(n)}$

(c): $$\boxed{\mathrm{Index}(\tau_f) = n^2\mu_{\mathcal{A}}^{n-1}}$$

(d): τ_f direct sum of n irreducibles $\tau_f^{(0)}, \tau_f^{(1)}, \dots, \tau_f^{(n-1)}$

(e):

$$\boxed{\begin{gathered} \mathrm{spin}(\tau_f^{(i)}) = \frac{i}{n} + \frac{n^2-1}{24n}c \ , \\ \mathrm{Index}(\tau_f^{(i)}) = \mu_{\mathcal{A}}^{n-1} \ , \end{gathered}}$$

Some consequences

Rationality implies modularity

Split & $\mu_{\mathcal{A}} < \infty$ imply strong additivity

Sectors with infinite statistics

Dichotomy rational/uncountably many sectors

Fusion for cyclic/permutation orbifold (Kac, Xu, L.)

Example Let $\mathcal{A}$ be the local conformal net on S^1 associated with the $U(1)$-current algebra. In the real line picture $\mathcal{A}$ is given by

$$\mathcal{A}(I) \equiv \{W(f) : f \in C^\infty_\mathbb{R}(\mathbb{R}),\ \text{supp} f \subset I\}''$$

associated with the vacuum state ω

$$\omega(W(f)) \equiv e^{-||f||^2}, \quad ||f||^2 \equiv \int_0^\infty |\tilde{f}(p)|^2 p \mathrm{d}p$$

$\{\alpha_q, q \in \mathbb{R}\}$ Buchholz-Mack-Totodorovsectors

$$\alpha_q(W(f)) \equiv e^{2i \int Ff} W(f), \quad F \in C^\infty, \quad \int F = q\ .$$

Now consider $\mathcal{A}\otimes\mathcal{A}$. irreducible sectors of $\mathcal{A}\otimes\mathcal{A}$ are $\alpha_q \otimes \alpha_{q'}$, index $= 1$.

Yet, the index 2 subnet $(\mathcal{A}\otimes\mathcal{A})^{\text{flip}}$ has an irreducible sector with infinite index, as $\mathcal{A}$ is not completely rational. (compare with Fredenhagen sectors).

<u>Compare</u>: in higher dimension mass gap $\Longrightarrow$ finite index (Buchholz-Fredenagen).

Part III: Noncommutative spectral invariants

Bekenstein formula. The entropy S of a black hole is proportional to the area A of its horizon

$$\boxed{S = A/4}$$

Note: S is proportional to the *area*, not to the volume as a naive microscopic interpretation of entropy would suggest (logarithmic counting of possible states).

This dimensional reduction has led to the *holographic principle* by t'Hooft, Susskind, ...

The horizon is not a physical boundary, but a submanifold where coordinates pick critical values $\to$ *conformal symmetries*

The proportionality factor 1/4 is fixed by Hawking temperature (*quantum* effect).

Discretization of the horizon (Bekenstein): horizon is made of cells or area ℓ^2 and k degrees of freedom ($\ell =$ Planck length):

$$A = n\ell^2,$$
$$\text{Degrees of freedom} = k^n,$$
$$S = Cn \log k = C\frac{A}{\ell^2} \log k,$$
$$\mathrm{d}S = C \log k$$

Conclusion.

Black hole entropy

$\downarrow$

Two-dimensional conformal quantum field theory with a "fuzzy" point of view

Legenda: Fuzzy = *noncommutative geometrical*

Weyl's theorem and ellipticity. M compact oriented Riemann manifold, Δ Laplace operator on $L^2(M)$. The eigenvalues of M can be thought as "resonant frequencies" of M and capture most of the geometry of M (M. Kac).

Weyl theorem: heat kernel expansion as $t \to 0^+$

$$\boxed{\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots)}$$

or, by Tauberian theorems, asymptotic density distribution of eigenvalues of Δ as $\lambda \to +\infty$,

$$N(\lambda) \sim \frac{\mathrm{vol}(M)}{(4\pi)^{n/2}\Gamma((n/2)+1)}\lambda^{n/2}$$

$N(\lambda)$ eigenvalues $\leq \lambda$, Γ Euler Gamma-function.

The *spectral invariants* n and $a_0, a_1, \ldots$ encode geometric information and in particular

$$a_0 = \mathrm{vol}(M), \qquad a_1 = \frac{1}{6}\int_M \kappa(m) d\mathrm{vol}(m),$$

κ scalar curvature. $n = 2$: a_1 is proportional to the Euler characteristic $= \frac{1}{2\pi}\int_M \kappa(m) d\mathrm{vol}(m)$ by Gauss-Bonnet theorem.

Infinite dim. quantum systems; log-ellipticity.
h positive selfadjoint operator on Hilbert space $\mathcal{H}$ and H the Fermi second quantization of h on the exponential of $\mathcal{H}$. Then as $t \to 0^+$:

$$\frac{\log \mathrm{Tr}(e^{-tH})}{\mathrm{Tr}(e^{-th})} = \frac{\mathrm{Tr} \log(1 + e^{-th})}{\mathrm{Tr}(e^{-th})} = O(t)$$

A positive linear operator H on a Hilbert space is *log-elliptic* (or QFT elliptic) if there exists $n > 0$ and $a_i \in \mathbb{R}$, $a_0 \neq 0$, such that

$$\boxed{\log \mathrm{Tr}(e^{-tH}) \sim \frac{1}{t^{n/2}}(a_0 + a_1 t + \cdots) \quad \text{as } t \to 0^+}$$

$n \equiv$ *dimension* of H, $a_i \equiv i^{\mathrm{th}}$ *spectral invariant* of H. Thus

finitely many degrees of freedom $\to$ ellipticity
infinitely many degrees of freedom $\to$ log-ellipticity

and log-ellipticity captures the spectral invariants of the existing one-particle Hamiltonian.

Modularity and log-ellipticity of $\mathcal{A}$. A conformal net $\mathcal{A}$ is two-dimensional *log-elliptic* if its conformal Hamiltonian L_0 is log-elliptic with dimension 2, i.e.

$$\log \mathrm{Tr}(e^{-2\pi t L_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots) \quad \text{as } t \to 0^+$$

log-ellipticity is essentially the *nuclearity condition* of Buchholz and Wichmann (and we fix the dimension).

With ρ rep. of $\mathcal{A}$, set $L_{0,\rho}$ conf. Hamiltonian of ρ,

$$\chi_\rho(\tau) = \mathrm{Tr}\left(e^{2\pi i \tau (L_{0,\rho} - c/24)}\right) \quad \mathrm{Im}\,\tau > 0.$$

specialized character, c the central charge.

$\mathcal{A}$ is *modular* if $\mu_{\mathcal{A}} < \infty$ and

$$\chi_\rho(-1/\tau) = \sum_\nu S_{\rho,\nu}\chi_\nu(\tau),$$
$$\chi_\rho(\tau + 1) = \sum_\nu T_{\rho,\nu}\chi_\nu(\tau).$$

with S, T the (algebraically defined) Verlinde-Rehren matrices generating a representation of $SL(2,\mathbb{Z})$. One has:

• Modularity $\Longrightarrow$ complete rationality

• Modularity holds in all computed rational case, e.g. $SU(N)_k$-models

• $\mathcal{A}$ modular, $\mathcal{B} \supset \mathcal{A}$ irreducible extension $\Longrightarrow$ $\mathcal{B}$ modular.

• All conformal nets with central charge $c < 1$ are modular.

Modular nets as NC manifolds (∞ degrees of freedom)

2-dim. cpt manifold M	conformal net $\mathcal{A}$
supp$(f) \subset I$	$x \in \mathcal{A}(I)$
Laplacian Δ	conf. Hamiltonian L_0
Δ elliptic	L_0 log-elliptic
area vol(M)	NC area $a_0(2\pi L_0)$
Euler charact. $\chi(M)$	NC Euler char. $12a_1$

Thm. (Kawahigashi, L.) $\mathcal{A}$ is modular. The following asymptotic formula holds as $t \to 0^+$:

$$\log \mathrm{Tr}(e^{-2\pi t L_0}) \sim \frac{\pi c}{12}\frac{1}{t} - \frac{1}{2}\log \mu_{\mathcal{A}} - \frac{\pi c}{12}t$$

Thus $\mathcal{A}$ is two-dimensional log-elliptic with non-commutative area $a_0 = 2\pi c/24$

In any representation ρ, as $t \to 0^+$:

$$\boxed{\log \mathrm{Tr}(e^{-2\pi t L_{0,\rho}}) \sim \frac{\pi c}{12}\frac{1}{t} + \frac{1}{2}\log \frac{d(\rho)^2}{\mu_{\mathcal{A}}} - \frac{\pi c}{12}t}$$

Note: spectral density $L_0 \to$ normalized index

$$\log d(\rho) - \frac{1}{2}\log \mu_{\mathcal{A}} = \lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t} t \log \mathrm{Tr}(e^{-tL_{0,\rho}}) \ .$$

Conjecture 1:

$$\boxed{\mathcal{A} \text{ c. rational} \Leftrightarrow \lim_{t\to 0^+} \frac{\mathrm{d}}{\mathrm{d}t} t \log \mathrm{Tr}(e^{-tL_0}) > -\infty}$$

Conjecture 2:

$$\boxed{\mathcal{A} \text{ modular} \Leftrightarrow \mathcal{A} \text{ completely rational}}$$

By Kohlbecker's Tauberian theorem as $\lambda \to \infty$

$$\log N(\lambda) \sim 2\pi\sqrt{\frac{c}{6}\lambda}$$

where $N(\lambda)$ is the number of eigenvalues (with multiplicity) of $L_{0,\rho}$ that are $\leq \lambda$. (Partial version of Cardy's formula on the 2-dim. Minkowski space).

Entropy. From the physics viewpoint it is natural to define $S_{\mathcal{A}}$, the *entropy of* $\mathcal{A}$, as the leading coefficient of the expansion of $\log \mathrm{Tr}(e^{-2\pi t L_0})$, thus

$$\begin{aligned} &a_0 = S_{\mathcal{A}} \ , \\ &a_1, a_2, \cdots = \text{higher order corrections to } S_{\mathcal{A}} \ . \end{aligned}$$

By definition, the entropy is proportional to the noncommutative area: it is just a matter of reading the same formula from different point of views. Meaning of spectral invariants:

Inv.	*Value*	*Geometry*	*Physics*
a_0	$\pi c/12$	NC area	Entropy
a_1	$-\frac{1}{2}\log\mu_{\mathcal{A}}$	NC Euler charact.	1^{st} order entr.
a_2	$-\pi c/12$	2^{nd} spectral invariant	2^{nd} order entr.

$a_2 = -a_0$, consequence of modular symmetry.

Incremental free energy. $\mathcal{A}$ modular implies a strong Kac-Wakimoto formula

$$\log \mathrm{Tr}(e^{-2\pi t L_{0,\rho}}) - \log \mathrm{Tr}(e^{-2\pi t L_{0,\sigma}}) \\ = \log d(\rho) - \log d(\sigma) + o(t)$$

cf. QFT index theorem: here true difference of free energy

$$\mathrm{d}F = \frac{2\pi}{\kappa}\Big(\log d(\rho) - \log d(\sigma)\Big) = \frac{\pi}{6\kappa}\Big(\chi_\sigma - \chi_\rho\Big)$$

Relation to black hole entropy. I. Microscopic derivation of black hole entropy and its relation to conformal symmetries and central charge is discussed Strominger, Vafa and others. We illustrate our discussion by the work of Carlip. Yet we use here only the value of the central charge and not Cardy's formula nor the boundary term of the energy.

For a black hole in the above class considered by Carlip we have

$$S_{\mathcal{A}} = A/4$$

where A is the area of the black hole horizon. Thus

$$\begin{tikzcd}
\text{Entropy} \arrow[r, "\text{physics}"] & a_0 \arrow[d, "\text{modular}"', "\text{nets}"] & \arrow[l, "\text{geometry}"'] 4\pi \cdot \text{NC area} \\
& 2\pi c/12 \arrow[d, "\text{black hole}"', "\text{models}"] & \\
& A/4 &
\end{tikzcd}$$

The mean free energy (topological increment of the second spectral invariant). $\mathcal{A}$ conformal net in any representation. We divide S^1 into n equally spaced cells, namely we consider the n-interval $E_n \equiv \sqrt[n]{S^+}$, S^+ upper semicircle. Each interval I_k contains minimal information (as the cells of Planck length).

Two canonical evolution associated with E_n corresponding to the rotations on the full S^1:

First: rescaled rotations $R(\frac{1}{n}\vartheta)$, rescaled conformal Hamiltonian $\hat{L}_0^{(n)} \equiv \frac{1}{n}L_0$

Second: associated with $U^{(n)}$ (rep. of $\mathrm{Diff}^{(n)}(S^1)$), Hamiltonian $L_0^{(n)} = \frac{1}{n}L_0 + \frac{c}{24}\frac{(n^2-1)}{n}$, takes care of "boundary effects". The geometrical complexity should be encoded in the difference between the two terms.

Associated free energy: difference of the free energy given the corresponding partition functions at infinite temperature:

$F_n \equiv t^{-1}\log\mathrm{Tr}(e^{-t2\pi L_0^{(n)}}) - t^{-1}\log\mathrm{Tr}(e^{-t2\pi \hat{L}_0^{(n)}})$

thus

$$F_n = \frac{c}{24}\frac{(n^2-1)}{n}2\pi$$

hence: model independent formula for the mean free energy associated to the "discretization of S^1".

$$\boxed{F_{\mathrm{mean}} = 2\pi\frac{c}{24}}$$

Note:

$$a_2(2\pi L_0^{(n)}) - a_2(2\pi \hat{L}_0^{(n)}) = F_n$$

NC geometrical meaning of F_{mean}.

Two-dimensional conformal QFT, both chiral components contribute to the topological entropy and physical topological entropy duplicates:

$$\boxed{F_{\text{mean}} = 2\pi \frac{c}{12}}$$

Relation to black hole entropy. II As above

$$\boxed{F_{\text{mean}} = A/4}$$

model independent, no modularity assumption.

The modular group of a n-interval von Neumann algebra. Model independent, in arbitrary representation, of Schroer and Wiesbrock formula for $U(1)$-current algebra.

$E \equiv \sqrt[n]{I}$ symmetric n-interval of S^1, $E = \{z \in S^1 : z^n \in I\}$. $I_0, I_1, \cdots I_{n-1}$ n connected components of E, $I_k = R(2\pi k/n)I_0$. $\mathcal{A}$ split conformal net on S^1, in a irreducible representation. Split isomorphism:

$$\chi_E : \mathcal{A}(I_0)\vee\cdots\vee\mathcal{A}(I_{n-1}) \to \mathcal{A}(I_0)\otimes\cdots\otimes\mathcal{A}(I_{n-1}) \,.$$

Rotation invariant product state φ on $\mathcal{A}(E)$:

$$\varphi \equiv (\varphi_0 \otimes \varphi_1 \otimes \cdots \otimes \varphi_{n-1}) \cdot \chi_E \ ,$$

φ_k normal faithful state on $\mathcal{A}(I_k)$ and $\varphi_k = \varphi_0 \cdot \mathrm{Ad}U(R(2k\pi/n))$.

$\Phi_k : \mathcal{A}(I_k) \to \mathcal{A}(I)$ isomorphism associated with z^n, namely

$$\Phi_k(x) \equiv U(h_k)xU(h_k)^*, \quad x \in \mathcal{A}(I_k)$$

where $h_k \in \mathrm{Diff}(S^1)$ s.t. $h_k(z) = z^n$, $z \in I_k$.

$\varphi_k \equiv \omega_I \cdot \Phi_k$, where ω vacuum state (or KMS state), φ_E the associated rotation invariant product state on $\mathcal{A}(E)$. Then the modular group σ^{φ_E} is given by

$$\sigma_t^{\varphi_E} = \mathrm{Ad}U^{(n)}(\Lambda_I(-2\pi t))\restriction_{\mathcal{A}(E)}$$

Λ_I lift to Möb$^{(n)}$ of "dilation" of I.

indeed, with $V(t) \equiv U^{(n)}(\Lambda_I(-2\pi t))$,

$$\boxed{\mathrm{Ad}V(t)\restriction_{\mathcal{A}(E)} = \sigma_t^{\varphi_E}, \quad \mathrm{Ad}V(-t)\restriction_{\mathcal{A}(E')} = \sigma_t^{\varphi_{E'}}}$$

Index and entropy. Abstract mathematical results concerning Jones index in the Kosaki framework and Connes-Haagerup noncommutative measure theory.

N_1, N_2 commuting factors on a Hilbert space $\mathcal{H}$, $N_1 \vee N_2 = B(\mathcal{H})$, i.e. $M_1 \equiv N_2' \supset N_1$, $M_2 \equiv N_1' \supset N_2$ irreducible subfactors

$\varphi_i = (\cdot\, \xi_i, \xi_i)$ state on N_i, ξ_i cycl. separ. for N_i

$V(t) = e^{-itK}$ a one-parameter unitary group on $\mathcal{H}$ s.t.

$$\mathrm{Ad}V(t)\restriction_{N_1} = \sigma_t^{\varphi_1}, \quad \mathrm{Ad}V(-t)\restriction_{N_2} = \sigma_t^{\varphi_2},$$

where σ^{φ_i} is the modular group of (N_i, φ_i).

Thm. $[M_1 : N_1] = (e^K\xi_1, \xi_1)(e^{-K}\xi_2, \xi_2)$

If $\exists$ unitary U s.t. $UN_1U^* = N_2$, $\varphi_2 = \varphi_1 \cdot \mathrm{Ad}U$ and $UV(t)U^* = V(-t)$, then

$$(e^K\xi_1, \xi_1) = (e^{-K}\xi_2, \xi_2) = [M_1 : N_1]^{\frac{1}{2}} ,$$

thus

$$\boxed{K = -\log\frac{\mathrm{d}\varphi_1 \cdot \varepsilon_1}{\mathrm{d}\varphi_2} + \frac{1}{2}\log[M_1 : N_1]}$$

where $\varepsilon_1 : M_1 \to N_1$ expectation (finite-index case). Thus

$$(K\xi_2, \xi_2) = -(\log\frac{\mathrm{d}\varphi_1 \cdot \varepsilon_1}{\mathrm{d}\varphi_2}\xi_2, \xi_2) + \frac{1}{2}\log[M_1 : N_1]$$

$=$ Araki entropy $+$ Pimsner-Popa entropy.

Entropy and spectral invariants with the proper Hamiltonian. We replace the conformal Hamiltonian L_0 with the "local" Hamiltonian

$$K_1 \equiv i(L_1 - L_{-1}) \ ,$$

the generator of the one-parameter dilatation unitary group associated with the upper semicircle S^+.

Dynamics	rotations	dilations
Hamiltonian	L_0	K_1
State	ω	$\omega{\restriction}\mathcal{A}(I)$
Pos. energy	$L_0 \geq 0$	KMS condition

Dilations satisfy the equilibrium condition at Hawking temperature and are natural to be considered.

We will now consider the "n-cell" dynamics

dilations $\longrightarrow$ n-dilations in analogy with the passage rotation $\to$ n-rotation with the action of $\mathrm{Diff}^{(n)}(S^1)$ and compute noncommutative spectral invariants in complete generality.

$\mathcal{A}$ split local conformal net on S^1, $E \equiv E_n = \sqrt[n]{I}$ and K_n the infinitesimal generator of $V^{(n)}(t) = U^{(n)}(\Lambda_I(-2\pi t))$

$$K_n \equiv i(L_1^{(n)} - L_{-1}^{(n)}) = \tfrac{i}{n}(L_n - L_{-n}),$$

$E_n' = \sqrt[n]{I'}$. $\varphi_{E_n} = (\cdot\xi_n, \xi_n)$ canonical rotation-invariant product state on $\mathcal{A}(E_n)$. We have:

$$\boxed{(e^{-2\pi K_n}\xi_n, \xi_n) = d(\rho)\mu_{\mathcal{A}}^{\frac{n-1}{2}}}$$

thus

$$\log(e^{-\frac{2\pi i}{n}(L_n - L_{-n})}\xi_n, \xi_n) \\ = \tfrac{n-1}{2}\log(\sum_i d(\rho_i)^2) + \log d(\rho)$$

Let $\hat{\varphi}_{E_n} = \varphi_{E_n} \cdot \varepsilon_{E_n}$ the state on $\hat{\mathcal{A}}(E_n)$ extended by the expectation $\varepsilon_{E_n} : \hat{\mathcal{A}}(E_n) \to \mathcal{A}(E_n)$.

Then

$$\boxed{K_n = -\frac{1}{2\pi}\Big(\log\Big(\frac{\mathrm{d}\widehat{\varphi}_{E_n}}{\mathrm{d}\varphi_{E'_n}}\Big) + \tfrac{n-1}{2}\log\mu_{\mathcal{A}} + \log d(\rho)\Big)}$$

The quantity

$$Z_n(t) \equiv (e^{-tK_n}\xi_n, \xi_n)$$

is the geometric partition function associated to the symmetric n-interval partition of S^1, thus

$$\begin{aligned} F_{n,\mu} \equiv -t^{-1}\log Z_n(t)|_{t=2\pi} \\ = -\tfrac{n-1}{4\pi}\log\mu_{\mathcal{A}} - \tfrac{1}{2\pi}\log d(\rho) \end{aligned}$$

is the associated n-μ-*free energy*. Dividing by the numbers of cells (intervals) we get *mean* μ-*free energy*.

$$\boxed{F_{\text{mean},\mu} = -\tfrac{1}{4\pi}\log\mu_{\mathcal{A}}}$$

The 0^{th} *and* 1^{st} *spectral invariants* are then defined by

$$a_{0,\mu} \equiv \lim_{n\to\infty} \frac{t \log Z_n(t)}{n}|_{t=2\pi}$$
$$a_{1,\mu} \equiv \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\frac{t \log Z_n(t)}{n}|_{t=2\pi}$$

Note that $-\frac{\mathrm{d}}{\mathrm{d}t}\log Z_n(t)$ is the n-μ-*energy* $H_{n,\mu}$ associated with $Z_n(t)$. Due to the thermodynamical relation

$$\text{Free energy} = T \cdot \text{Entropy} - \text{Energy}$$

where T is the temperature, we thus define the mean $n - \mu$-*entropy* by $S_{n,\mu} = t(F_{n,\mu} + H_{n,\mu})$. We have:

$$S_{n,\mu} = S(\widehat{\varphi}_{E_n}|\varphi_{E'_n})$$

Araki relative entropy.

We have the "local" spectral invariants

$$\begin{cases} a_{0,\mu} = \frac{1}{2}\log \mu_{\mathcal{A}} \ , \\ a_{1,\mu} = -S_{\text{mean},\mu} = \log \mu_{\mathcal{A}} - \lim_{n\to\infty}\frac{1}{n}S(\widehat{\varphi}_{E_n}|\varphi_{E'_n}) \end{cases}$$

Final comment. It would be interesting to relate our setting with Connes' Noncommutative Geometry. A link should be possible in a supersymmetric context, where cyclic cohomology appears. In this respect model analysis with our point of view, in particular in the supersymmetric frame, may be of interest. Note also that Connes' spectral action concerns the Hamiltonian spectral density behavior.