KAM Theory and Applications in Celestial Mechanics – First Lecture: Chaotic Motions in Hamiltonian Systems

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Integrability *à la* Liouville

Theorem (Liouville):

Let $H = H(\mathbf{p}, \mathbf{q})$ be the Hamiltonian of a system with n degrees of freedom. **If** there are n constant of motions $J_1 , \ldots , J_n$ such that

- they are in **involution**, i.e. $\{J_i , J_j\} = 0 \; \forall \; i , j = 1 , \ldots , n$;
- they are **independent**, i.e.

$$\mathrm{rank} \left(\frac{\partial(J_1 , \ldots , J_n)}{\partial(p_1 , \ldots , p_n , q_1 , \ldots , q_n)} \right) \neq 0 \; ;$$

then there is a canonical transformation $(\mathbf{p}, \mathbf{q}) = \Psi(\mathbf{J}, \boldsymbol{\alpha})$, such that in the new coordinates $H = H(\mathbf{J})$.

Let us recall that the Poisson bracket $\{\cdot, \cdot\}$ with respect to the canonical coordinates $(\mathbf{p}, \mathbf{q})$ is defined so that $\{f , g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right)$.

- The proof of Liouville's theorem is **constructive** and Hamilton's equations are **integrable** in the new coordinates, i.e. $J_i(t) = J_i(0)$ and $\alpha_i(t) = \alpha_i(0) + \frac{\partial H}{\partial J_i} t \, , \; \forall \; i = 1 , \ldots , n$.

Integrability *à la* Arnold–Jost

Theorem (Arnold–Jost):

Let $H = H(\mathbf{p}, \mathbf{q})$ be a Hamiltonian satisfying the hypotheses of Liouville's theorem. Moreover, **if**

- there are n constant values $c_1\,, \ldots\,, c_n$ such that each equation $J_1(\mathbf{p}, \mathbf{q}) = c_1\,, \ldots\,, J_n(\mathbf{p}, \mathbf{q}) = c_n$ implicitly defines a regular and **compact** manifold in the phase–space,

then there is a local canonical transformation $(\mathbf{p}, \mathbf{q}) = \Psi(\mathbf{I}, \boldsymbol{\vartheta})$ defined on **action–angle coordinates** (i.e. Ψ is defined on $\mathcal{G} \times \mathbb{T}^n$, with $\mathcal{G}$ open subset of $\mathbb{R}^n$), such that in the new coordinates $H = H(\mathbf{I})$.

Remark:

The proof of Arnold–Jost theorem is **constructive** and Hamilton's equations are **integrable** in the new coordinates, i.e. $I_i(t) = I_i(0)$ and $\vartheta_i(t) = \vartheta_i(0) + \omega_i t$, with **angular velocities** $\omega_i = \frac{\partial H}{\partial I_i}$, $\forall\ i = 1\,, \ldots\,, n$. Thus, some regions of the phase space are filled by **invariant tori** and the motion over them is related to the **frequencies** $\omega_i/(2\pi)\ \forall\ i = 1\,, \ldots\,, n$.

Integrability *à la* Arnold–Jost (an example with 1 d.o.f.)

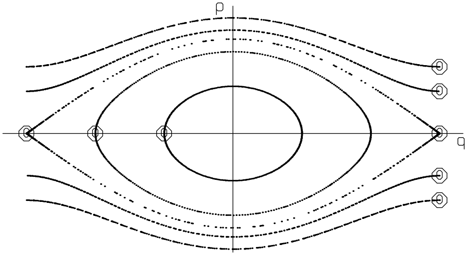

Figure: Eight orbits of the Poincaré map for the simple pendulum equation, i.e. $\ddot{x} = -\sin x$. All the initial conditions are marked with the symbol 0 (in a "◯"). For each orbit, 400 points are plotted at regular interval of time equal to $2\pi/\sqrt{2}$ in the phase space with $(q, p) = (x, \dot{x})$ coordinates, being $q \in [-\pi, \pi]$.

- the orbits lie on the energy level $H(p, q) = \frac{p^2}{2} - \cos q = E$.
- Two sepatrices (meeting in the hyperbolic point $(p, q) = (0, \pm\pi)$) are between the "**librational tori**" (with $E \in (-1, 1)$) and the "**rotational tori**" ($E > 1$).

A first example of chaotic motion in a Poincaré map

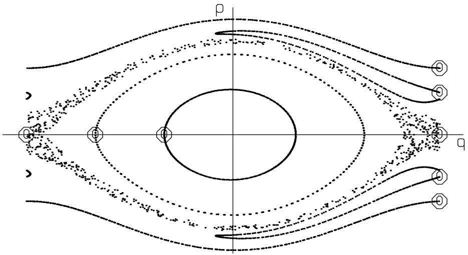

Figure: Eight orbits of the Poincaré map for the forced pendulum equation $\ddot{x} = -\sin x - \varepsilon\cos(\Omega t)$ with $\varepsilon = 0.05$ and $\Omega = \sqrt{2}$. All the initial conditions (marked with the symbol 0 (in a "◯") are the same as in the previous figure. For each orbit, the points are plotted at regular interval of time equal to $2\pi/\Omega$.

Remark:

After having added a small perturbation depending on time, orbits starting close to an hyperbolic point **do not lie on a regular 1D–curve**.

A first example of chaotic motion in a Poincaré map

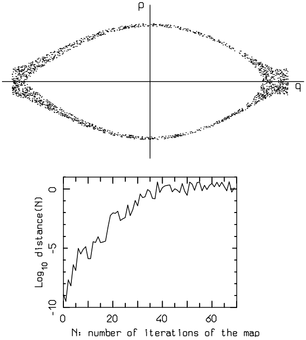

Figure: Same Poincaré map as before. Behavior of the distance between two chaotic orbits as a function of the number of iterations of the map.

- **Definition**: a region of the phase space is said to be **chaotic**, when it is very sensitive to the initial conditions; i.e. $d(t) \simeq e^{\lambda t} d(0)$, where d is the distance between two motions and $\lambda > 0$.

Basics for symplectic maps

- **Definition**: a differentiable change of coordinates $(\mathbf{P},\mathbf{Q})=\mathcal{C}(\mathbf{p},\mathbf{q})$ is said to be **symplectic** if it preserves the Poisson brackets, i.e. $\{P_i,P_j\}=\{Q_i,Q_j\}=0$ and $\{P_i,Q_j\}=-\delta_{i,j}\,,\ \forall\ i\,,\,j=1\,,\,\ldots\,,\,n\,,$ being $\delta_{i,i}=1$ and $\delta_{i,j}=0$ when $i\neq j\,.$
- **Theorem**: let Φ_H^t be the flow induced by a Hamiltonian H after the time t , i.e. $\Phi_H^t(\mathbf{p}(0),\mathbf{q}(0))=(\mathbf{p}(t),\mathbf{q}(t))$ with $\dot{p}_j=-\frac{\partial H}{\partial q_j}$ and $\dot{q}_j=\frac{\partial H}{\partial p_j}\,.$ Therefore, Φ_H^t **is symplectic** $\forall\ t\,.$
- **Corollary**: the Poincaré map Φ_H^T **is symplectic** (with a fixed time-step T) also when Hamiltonian H explicitly depends on time.
- **Theorem**: a 2D map is symplectic **if and only if** is **area-preserving**.
- **Definition**: a point $(\bar{p},\bar{q})$ is said to be an **hyperbolic point** for a 2D symplectic map $\mathcal{M}$, when
  - $(\bar{p},\bar{q})$ is a fixed point for the map, i.e. $\mathcal{M}(\bar{p},\bar{q})=(\bar{p},\bar{q})\,;$
  - the map $\mathcal{M}$ can be linearly approximated near the fixed point by the operator $\mathrm{d}\mathcal{M}_{(\bar{p},\bar{q})}$, that admits two eigenspaces $\mathcal{E}^+$ and $\mathcal{E}^-$, such that

$$\mathrm{d}\mathcal{M}_{(\bar{p},\bar{q})}\,\xi=\lambda\xi\ \ \forall\ \xi\in\mathcal{E}^+\ , \qquad \mathrm{d}\mathcal{M}_{(\bar{p},\bar{q})}\,\xi=\frac{\xi}{\lambda}\ \ \forall\ \xi\in\mathcal{E}^-\ ,$$

  where the eigenvalue $\lambda\in(0,1)\,.$

Stable and unstable manifolds

- **Definition**: the sets $W^{+}_{(\bar{p},\bar{q})}$ and $W^{-}_{(\bar{p},\bar{q})}$, such that

$$W^{\pm}_{(\bar{p},\bar{q})} = \left\{ (p,q) \;:\; \lim_{k\to\infty} \mathcal{M}^{\pm k}(p,q) = (\bar{p},\bar{q}) \right\} ,$$

are said to be the stable and unstable manifold, respectively, for the hyperbolic point $(\bar{p},\bar{q})$ of the map $\mathcal{M}$.

Theorem:

The stable and unstable manifolds $W^{\pm}_{(\bar{p},\bar{q})}$ are tangent to the eigenspaces $\mathcal{E}^{\pm}$, respectively. Moreover, *locally*, $W^{\pm}_{(\bar{p},\bar{q})}$ are regular graphs of $\mathcal{E}^{\pm}$, resp.

Corollary:

The stable manifold $W^{+}_{(\bar{p},\bar{q})}$ cannot intersect itself; the same holds true for the unstable manifold $W^{-}_{(\bar{p},\bar{q})}$.

Stable and unstable manifolds: the integrable case ($\varepsilon = 0$)

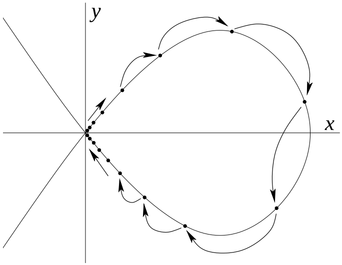

Figure: Stable and unstable manifolds for the Poincaré map $\Phi_H^{2\pi}$ related to the Hamiltonian $H(x, y, t) = (y^2 - x^2)/2 + (1 + \varepsilon \cos t)x^3/3$ when $\varepsilon = 0$.

Both the stable manifold and the unstable one lie on the separatrix given by the implicit equation $H = (y^2 - x^2)/2 + x^3/3 = 0$.

- **Definition**: an orbit (point) is said to be **homoclinic** if it is included in the intersection of the stable manifold with the unstable one.

Splitting of the stable and unstable manifolds (case $\varepsilon \neq 0$)

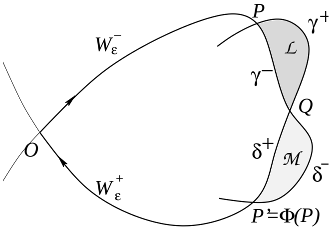

Figure: Schematic representation of the intersections between the stable and unstable manifolds for the Poincaré map $\Phi = \Phi_H^{2\pi}$ related to the Hamiltonian $H(x, y, t) = (y^2 - x^2)/2 + (1 + \varepsilon \cos t)x^3/3$ in the perturbed case, i.e. with $\varepsilon \neq 0$. The consecutive lobes $\mathcal{L}$ and $\mathcal{M}$ have the same area.

- **Remark**: if $\varepsilon \neq 0$ the stable manifold W_ε^+ do not superpose to the unstable one W_ε^- ; they cross each other in the homoclinic points.
- **Proposition**: two consecutive lobes (including the regions between the stable and the unstable manifolds) have the same area.

Splitting of the stable and unstable manifolds (case $\varepsilon \neq 0$)

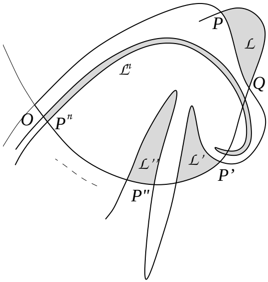

Figure: Schematic representation of the intersections between the stable and unstable manifolds for the same Poincaré map Φ of the previous slide. The lobes $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$, ..., $\mathcal{L}^n$ include the same area.

- **Remark**: the hyperbolic point is an accumulation point for the homoclinic orbits. Since all the lobes have the same area and their "bases" are shorter and shorter, they are **stretched in the direction of the stable (unstable) manifold**, which cannot be crossed.

Stable and unstable manifolds for the standard map

- **Definition**: the change of coordinates $\mathcal{M}_\varepsilon : \mathbb{R}\times\mathbb{T} \mapsto \mathbb{R}\times\mathbb{T}$ is called **standard map**, when $(p', q') = \mathcal{M}_\varepsilon(p, q)$ with

$$p' = p + \varepsilon \sin q \ , \qquad q' = q + p' \quad \mathrm{mod}\ 2\pi \ .$$

- **Remark**: one can easily check that the (standard) map $\mathcal{M}_\varepsilon$ is **symplectic** and, then, it is **area-preserving**.
- **Remark**: for all $\varepsilon \neq 0$ the origin is an hyperbolic point for $\mathcal{M}_\varepsilon$.
- **Remark**: $\mathcal{M}_\varepsilon$ is 2π–periodic also with respect to the action p .
- **Stable [unstable] manifold W_ε^+ $\left[W_\varepsilon^-\right]$; drawing "N iterations":**
  - for the initial iteration with $n = 1$, draw a short segment connecting the hyperbolic point to another point belonging to the eigenspace tangent to W_ε^+ $\left[W_\varepsilon^-\right]$;
  - for each segment of the $n-1$ iteration, if its length is $> L$ (being L suitably fixed) split it in a grid of sub-segments shorter than L; for each (sub)segment, consider both the vertexes, compute the pair of transformed points along the map $\mathcal{M}_\varepsilon^{-1}$ $\left[\mathcal{M}_\varepsilon\right]$ and connect them;
  - repeat the previous operation while $n \leq N$.

Stable and unstable manifolds for the standard map

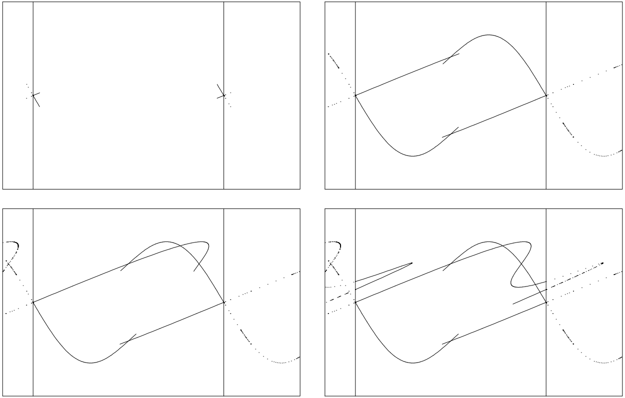

Figure: Stable and unstable manifolds for the standard map $\mathcal{M}_\varepsilon$ with $\varepsilon = 2.36$. Each box represents the phase space $[0, 2\pi] \times [-\pi, \pi]$; the shaded parts are reported to make clearer the 2π–periodicity in the angle. Top–left, top–right, bottom–left and bottom–right boxes contain the drawing of the manifolds with 1, 2, 3 and 4 iterations, respectively.

Stable and unstable manifolds for the standard map

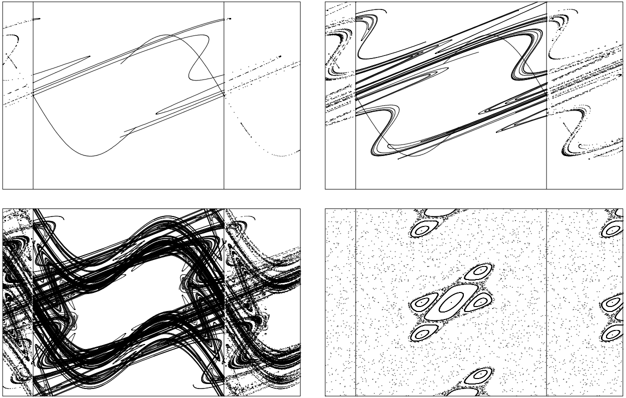

Figure: Stable and unstable manifolds (for the previous standard map $\mathcal{M}_\varepsilon$) are compared to some orbits. Top–left, top–right and bottom–left boxes contain the drawing of the manifolds with 5 , 8 and "as many as possible" iterations, resp. In the bottom–right box, some orbits of $\mathcal{M}_\varepsilon$ are plotted to highlight that **the manifolds fills the chaotic region**.

Poincaré sections

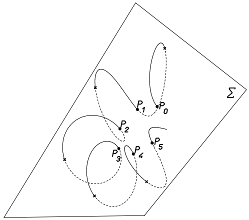

Figure: Schematic representation of a Poincaré section. The flow define a map Π such that P_0 is related to P_1, because P_1 is the outgoing intersection (between the orbit and the surface Σ) next to P_0 (ingoing intersections are discarded). For the same reason, $\Pi(P_1) = P_2, \ldots, \Pi(P_4) = P_5$.

- **Remark**: consider a n degrees of freedom Hamiltonian H and a $(2n-1)$D surface Σ, such that for all points $P \in \Sigma$ the flow Φ_H^t is transversal to Σ in P. It is possible to define a Poincaré section, i.e. a map $\Pi : \Sigma \mapsto \Sigma$ as in figure above.

The Hénon–Heiles model

The Hénon–Heiles model is described by the following Hamiltonian:

$$H(\mathbf{p},\mathbf{q}) = \frac{\omega_1}{2}\left(p_1^2+q_1^2\right)+\frac{\omega_2}{2}\left(p_2^2+q_2^2\right)+q_1^2 q_2-\frac{1}{3}q_2^3 \ .$$

Remarks

- For small values of the canonical coordinates (q_1, q_2), the system is well approximated by a pair of harmonic oscillators, i.e. $H(\mathbf{p},\mathbf{q}) \simeq \omega_1\left(p_1^2+q_1^2\right)/2+\omega_2\left(p_2^2+q_2^2\right)/2$, with $\omega_1 > 0$ e $\omega_2 > 0$.
- The section surface Σ is defined so that $q_1 = 0$. When the energy level $H = E > 0$, Σ is obviously transversal to the Hamiltonian flow.
- The Poincaré sections are usually represented for a fixed energy level $H = E > 0$ and on the plane (p_2, q_2), where each point locates an initial condition (and, so, an orbit), because $q_1 = 0$ and p_1 is given by the equation $H(p_1, p_2, 0, q_2) = E$.
- When the energy level $E < E_e$, where the escape energy value $E_e = \min\left\{\omega_1^3/24+\omega_1^2\omega_2/8\,,\ \omega_2^3/6\right\}$, then the points (p_2, q_2) of the Poincaré sections are bounded so that $\omega_2(p_2^2+q_2^2)/2 - q_2^3/3 \le E$.

Poincaré sections for the Hénon–Heiles model

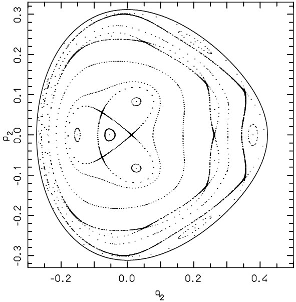

Figure: Poincaré sections for the Hénon–Heiles model in a so called "non-resonant" case ($\omega_1 = 1$ and $\omega_2 = (\sqrt{5} - 1)/2$). The energy level is fixed so that $E = 0.030$. In this case the escape energy value is $E_e = 0.03934466$. The most external curve is the "border" orbit, i.e. $\omega_2(p_2^2 + q_2^2)/2 - q_2^3/3 = E_e$.

- **Remark**: small chaotic regions are visible close to hyperbolic points, but most of the orbits lie on regular 1D–curves.

Poincaré sections for the Hénon–Heiles model

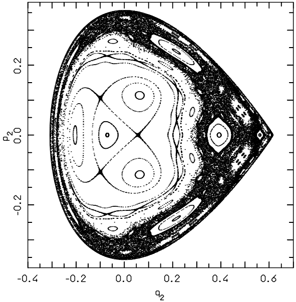

Figure: Poincaré sections for the Hénon–Heiles model with the same values of the angular velocities ω_1 and ω_2 as in the previous slide. The energy level is fixed so that $E = 0.039344$, that is very close to the escape energy value.

- **Remark**: by increasing the energy (and so the perturbation) the chaotic regions gets larger, but (according to the Hénon words) **islands of ordered motion still persist, although they are in a chaotic sea**.

Chaos is everywhere! Does this mean everything is chaotic?

- Poincaré claimed that the **general problem of dynamics** is given by a Hamiltonian system of the type $H(\mathbf{p},\mathbf{q}) = h(\mathbf{p}) + \varepsilon f(\mathbf{p},\mathbf{q})$ where ε is a **small parameter** and $(\mathbf{p},\mathbf{q})$ are **action–angle coordinates** (that are defined on $\mathcal{G} \times \mathbb{T}^n$, with $\mathcal{G}$ open subset of $\mathbb{R}^n$).
  Poincaré proved that a Hamiltonian of the type $H(\mathbf{p},\mathbf{q}) = h(\mathbf{p}) + \varepsilon f(\mathbf{p},\mathbf{q})$ is **generically non-integrable**.
  His proof is based on the fact that **resonances are everywhere dense** when $\left(\frac{\partial^2 h}{\partial p_i \partial p_j}\right)_{i,j}$ is non-degenerate.
- Each resonance shows hyperbolic points, homoclinic orbits, stable/unstable manifolds, chaotic regions.
- Why ordered regions can still be detected in the Hénon–Heiles model? **They cannot be integrable** (according to Poincaré theorem of non-existence of the first integrals for generic Hamiltonians).

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$\Longrightarrow$ **KAM THEORY!**