Useful technicalities about the Kepler problem: passing from position and velocities to orbital elements and viceversa.

Let us recall some very well known things, this will help us to fix the notation. When we refer to the Kepler problem, we mean that we study the motion of a point mass P attracted toward a center of attraction (put in the origin of our frame) with a force that is inversely proportional to the square of the distance (from the origin). Thus, the vector $\underline{x}$ representing the coordinates of $\vec{OP}$ in our frame will satisfy the following Newton equation:

$$\boxed{\ddot{\underline{x}} = -\frac{\Gamma}{\|\underline{x}\|^2} \cdot \frac{\underline{x}}{\|\underline{x}\|}},$$

where Γ is nothing but the product of the gravitational constant times the mass of the center of attraction. It is well known that the motion of the point P does not depend on its mass in the Kepler problem (because the attraction exerted by P on the body located in the origin is neglected!); therefore, it is convenient to assume that the mass of P is equal to 1, so to ~~[illegible]~~ avoid to consider one more parameter.

We are going to consider the solutions of the Newton equation just for bounded orbits, i.e., in the case the total energy is negative:

$$\boxed{\mathcal{E} = \frac{1}{2}\,\underline{\dot{x}}\cdot\underline{\dot{x}} - \frac{\Gamma}{\|\underline{x}\|} < 0\,.}$$

In such a case, it is well known that the orbits are ellipses which are travelled in such a way to satisfy both the second and the third Kepler laws.

Let us introduce the linear momentum vector

$$\boxed{\underline{p} = \underline{\dot{x}}}$$

(let us recall that the mass of P is assumed to be equal to 1) and the distance

(1) $$\boxed{\rho = \|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}\,},$$

then the condition about the energy can be reformulated as follows:

$$(2) \quad \boxed{\mathcal{E} = \frac{1}{2}\underline{p}\cdot\underline{p} - \frac{\Gamma}{\rho} < 0}.$$

There is a transformation

$$(\underline{x},\underline{p}) \longmapsto (a, e, i, \Omega, \omega, M)$$

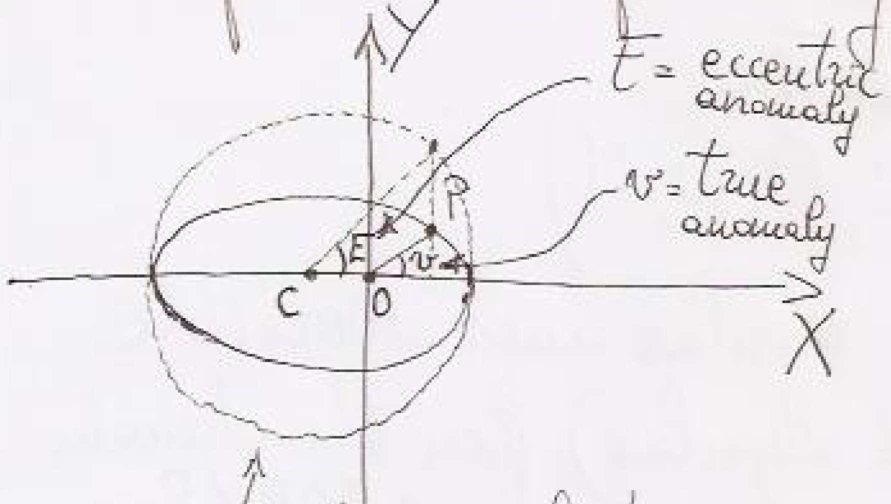
that is generically regular and invertible (we will see where it is singular) on the domain

$$\left\{(\underline{x},\underline{p}) \in \mathbb{R}^3\setminus\{\underline{0}\}\times\mathbb{R}^3 : \frac{1}{2}\underline{p}\cdot\underline{p} - \frac{\Gamma}{\|\underline{x}\|} < 0 \ \wedge\ \underline{x}\wedge\underline{p}\neq\underline{0}\right\},$$

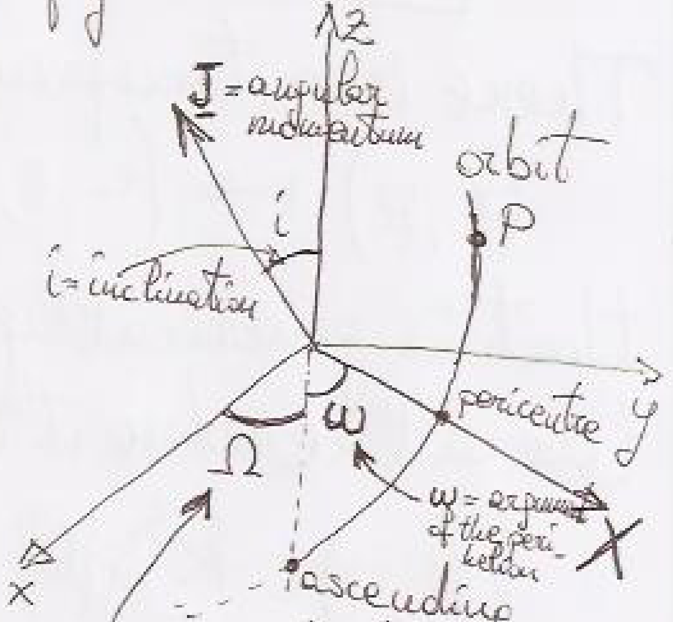
where
- a → semi-major axis
- e → eccentricity
- i → inclination
- Ω → longitude of the node
- ω → argument ~~anomaly~~ of the ~~node~~ perihelion
- M → mean anomaly

are known as the orbital elements. Actually, all of them but M locate the Keplerian ellipse while M allows to determine the instantaneous position of P on that ellipse.

In order to understand the physical meaning of each orbital element is convenient to refer to the following figures



See also the equivalent Figure 2.7 of the Murray & Dermott's book. ("Solar System Dynamics")



See also the equivalent Figures 2 and 2.13 of the Laskar's notes and of the "Solar System Dynamics" book.

Refer also to the figure at the end of the present document.

The figure on the left represents the usual orbital ellipse in the invariant plane OXY that is normal to the angular momentum

(3) $$\boxed{\underline{J} = \underline{x} \wedge \underline{p}}$$

Of course, $\underline{J}$ must be $\neq \underline{0}$.

In that plane, the X axis is directed as the perihelion.

Moreover, the axes X and Y are ~~direct~~ such that after having denoted with $\underline{i}$ and $\underline{j}$ the corresponding unit vectors, respectively, and $\underline{k} = \frac{\underline{J}}{\|\underline{J}\|}$, the ortonormal basis $\underline{i}, \underline{j}, \underline{k}$ satisfies also the relation $\boxed{\text{Orb. el. 5}}$

$$\underline{i} \wedge \underline{j} = \underline{k} \, .$$

The semimajor axis a and the eccentricity e are nothing but the semi-major axis and the eccentricity of the ellipse described by the orbit of P in the plane OXY. They are easy to compute, by using the following relations:

$$\mathcal{E} = -\frac{\Gamma}{2a}$$ (energy as a function of the semi-major axis)

$$L = \sqrt{\Gamma a}$$ (definition of the first action of the Delaunay coordinates)

$$G = L\sqrt{1-e^2}$$ (definition of the second action, that is $G = \|\underline{J}\|$, of the Delaunay coordinates).

The latter one is equivalent to

$$\|\underline{J}\| = \sqrt{\Gamma a\left(1-e^2\right)} \, .$$

The formulas above can be rearranged so to calculate the first two orbital elements, in the following way:

(4) $$\boxed{a = -\frac{\Gamma}{2\mathcal{E}}}\ ,$$

(5) $$\boxed{e = \sqrt{1 - \left(\frac{\underline{J}\cdot\underline{J}}{\Gamma\cdot a}\right)}}\ .$$

The inclination can be easily computed by using the relation between the second action and the third one of the Delaunay variables, that is:

$$\Theta = G\cos i$$

Since the third action $\Theta = J_3$, that is the projection of $\underline{J}$ on the third axis, then

(6) $$\boxed{\cos i = \frac{J_3}{\|\underline{J}\|}}\ .$$

Of course, the inclination could be easily given by $i = \mathrm{acos}(J_3/\|\underline{J}\|)$. However since we plan to use the transformation to orbital elements together with the transformation backwards to $(\underline{x}, \underline{p})$, in the context of a numerical integrator it is much faster to store both the cosine and the sine of the inclination. Thus, we are going to provide an equation which determines directly $\sin i$, that is

(7) $$\boxed{\sin i = \frac{\sqrt{J_1^2 + J_2^2}}{\|\underline{J}\|}} \, .$$

Let us remark that, from the equation above it follows that $0 \leq i \leq \pi$; this is not a fictitious limitation, because of the definition of the inclination (as the angle between the vertical axis and the angular momentum, see the figure at page 4).

Let us now recall that there ~~axis~~ is a singularity intrinsic to the problem, i.e.,

$$\underline{x} = \underline{0}$$

where the potential is not defined.
In order to avoid that the orbit can include the singularity we requested that the angular momentum is different from zero, i.e.

$$\underline{J} = \underline{x} \wedge \underline{p} \neq \underline{0} \, .$$

Moreover, we restricted ourselves to consider just the keplerian ellipses, by requiring that the energy is negative, i.e.

$$\mathcal{E} < 0 \, .$$

Indeed, the orbital elements can be defined also for parabolic orbits or hyperbolic ones, but we are not interested on them.

~~Fu/bilder~~ There are also other singularities in the transformation leading to the orbital elements; actually the new singularities we are going to describe are not intrinsic to the problem and they could be removed with a different choice of the new elements (i.e., in a similar way to what is done by introducing Poincaré canonical coordinates instead of the Delaunay ones).

A first singularity occurs when $i = 0$; in fact, looking at the figure in page 4, it is obvious then Ω is not defined when the inclination is equal to zero (or π).

A second singularity arises when $e = 0$; of course, when the eccentricity is equal to zero, the pericentre is not defined and then the angle ω looses sense.

For both those (fictitious) singularities, we will arbitrarily define the values of the corresponding angles (that are Ω and ω); actually, this will allow the transformation to be invertible in those cases and an integration method based on those change of coordinates will be well defined as well.

In order to define Ω is convenient to first locate the unit vector $\underline{n}$ related to the direction of the ascending node. This is easy to be done, after having remarked that $\underline{n}$ is orthogonal to both the z-axis and the angular momentum $\underline{J}$. Thus,

(8) $\boxed{\underline{n} = \frac{\underline{k} \wedge \underline{J}}{\|\underline{k} \wedge \underline{J}\|}}$ (that is well defined, when $\underline{k} \wedge \underline{J} \neq \underline{0}$, i.e. when $\sin i \neq 0$),

where, of course, $\underline{k} = (0, 0, 1)^T$. By definition,

$$\underline{n} \cdot \underline{k} = 0 ,$$

therefore, the coordinates of $\underline{n}$ are such that $\underline{n} = (n_1, n_2, 0)^T$.

Therefore, we set

(9) $\boxed{\cos\Omega = n_1}$, (10) $\boxed{\sin\Omega = n_2}$, when $\sin i \neq 0$.

Otherwise, if $\sin i = 0$, then we define

(8a) $\boxed{\underline{n} = (1, 0, 0)}$, and, accordingly,

(9a) $\boxed{\cos\Omega = 1}$, (10a) $\boxed{\sin\Omega = 0}$.

In order to determine the argument of the pericentre ω, it is convenient to introduce the so called Laplace-Runge-Lenz vector:

$$\boxed{\underline{A} = \underline{p} \wedge \underline{J} - \frac{\Gamma \underline{x}}{\rho}} \tag{11}$$

It is easy to prove that $\underline{A}$ is a constant of motion; in fact,

$$\dot{\underline{A}} = \dot{\underline{p}} \wedge \underline{J} + \cancel{\underline{p} \wedge \dot{\underline{J}}} - \frac{\Gamma \dot{\underline{x}}}{\rho} + \frac{\Gamma \underline{x}}{\rho^3} \frac{\cancel{2}(x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3)}{\cancel{2}\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$= \left(-\frac{\Gamma}{\rho^2}\frac{\underline{x}}{\rho}\right) \wedge \underline{J} - \frac{\Gamma \dot{\underline{x}}}{\rho} + \frac{\Gamma \underline{x} \cdot \dot{\underline{x}}}{\rho^2}\frac{\underline{x}}{\rho} = -\frac{\Gamma \underline{x}}{\rho^3} \wedge (\underline{x} \wedge \underline{p})$$

$$- \frac{\Gamma \dot{\underline{x}}}{\rho} + \frac{\Gamma \underline{x} \cdot \underline{p}}{\rho^3}\underline{x} = -\cancel{\frac{\Gamma \underline{x} \cdot \underline{p}}{\rho^3}\underline{x}} + \frac{\Gamma \underline{x} \cdot \underline{x}}{\rho^3}\underline{p} - \frac{\Gamma \dot{\underline{x}}}{\rho} + \cancel{\frac{\Gamma \underline{x} \cdot \underline{p}\, \underline{x}}{\rho^3}}$$

$$= \cancel{\frac{\Gamma \rho^2}{\rho^3}\underline{p}} - \cancel{\frac{\Gamma \underline{p}}{\rho}} = 0,$$

where we used the obvious equations $\dot{\underline{x}} = \underline{p}$, $\dot{\underline{p}} = -\frac{\Gamma}{\rho^2}\frac{\underline{x}}{\rho}$, $\rho^2 = \underline{x} \cdot \underline{x}$ and the rule of the vector triple product

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} \quad \forall \underline{a}, \underline{b}, \underline{c} \in \mathbb{R}^3.$$

In order to calculate the Laplace-Runge-Lenz vector, it is convenient to evaluate it at the pericentre (since it is a constant of motion, one evaluation will fix it for ever). By referring to the figures ~~of~~ at page 4 one immediately realizes that, when P is on the X-axis with positive abscissa, then

$$\frac{\underline{x}}{\rho} = \underline{e}_X, \quad \underline{p} \wedge \underline{J} = \frac{\underline{J} \cdot \underline{J}}{\rho}\underline{e}_X,$$

being $\underline{e}_X$ the unit vector related to the pericentre direction, therefore, we ~~know~~ obtain

$$\underline{A} = \left(\frac{\|\underline{J}\|^2}{\rho} - \Gamma\right)\underline{e}_X = \left(\frac{\Gamma \cancel{a}\,\cancel{(1-e^2)}^{1+e}}{\cancel{a}\cdot\cancel{(1-e)}} - \Gamma\right)\underline{e}_X = \Gamma e\,\underline{e}_X\ ,$$

where we used the equations $\|\underline{J}\| = \sqrt{\Gamma a(1-e^2)}$ and, at the pericentre, $\rho = a(1-e)$. Thus, we can conclude that

$$\boxed{\underline{A} = \Gamma e\ \underline{e}_X}\ .$$

This means that the Laplace-Runge-Lenz vector ~~is directed~~ points in the direction of the pericentre and its size is proportional to the eccentricity e. Therefore, if the eccentricity is different from zero, then the pericentre axis is determined by the following equation:

$$(12)\quad \boxed{\underline{e}_X = \frac{\underline{A}}{\Gamma e}}\ ;$$

This allows us to provide explicit expressions for both the cosine and the sine of the argument of the pericentre:

$$(13)\quad \boxed{\cos\omega = \underline{n}\cdot\underline{e}_X}\ ,\quad (14)\quad \boxed{\sin\omega = \mathrm{sign}(\underline{e}_X\cdot\underline{k})\cdot\|\underline{n}\wedge\underline{e}_X\|}\ ,$$

where we used the well known relation $\|\underline{n}\wedge\underline{e}_X\| = \|\underline{n}\|\,\|\underline{e}_X\|\,|\sin\omega|$ and, from the figure at page 4, the easy remark that $\omega\in(0,\pi)$ when the pericentre is above the horizontal plane Oxy.

On the other hand, if the eccentricity is equal to zero, then we set

(12a) $\boxed{\underline{e}_x = \underline{n}}$ $\Rightarrow$ (13a) $\boxed{\cos\omega = 1}$, (14a) $\boxed{\sin\omega = 0}$.

Thus, we ended the definition of the orbital elements describing the orbital ellipse.

Now, we are going to determine the instantaneous position onto that ellipse. Actually, the position of P is located by the "true anomaly" angle v (refer to the figure at page 4), which is related to the eccentric anomaly E, that in turn, is obtained by the mean anomaly M.

The mean anomaly is a "virtual angle" (and, so, is not appearing in the figure) which satisfies the property of having constant angular velocity:

$$\dot{M} = \frac{2\pi}{T}$$, where the period T is fixed by the third Kepler law: $T = 2\pi\sqrt{\frac{a^3}{\Gamma}}$

Thus, it will be easy to follow the evolution with respect to time of the unique orbital element that is not constant, by using the equation

(15) $$\boxed{\dot{M} = \sqrt{\frac{\Gamma}{a^3}}}\ .$$

The eccentric anomaly can be easily obtained by using the classical formula (~~see~~ equivalent to ~~f~~ equations (5.17) and (2.42) of the Laskar's notes and the "Solar System Dynamics" book):

$$\boxed{\rho = a\left(1 - e\cos E\right)}\ .$$

Thus, <u>if the eccentricity is different from zero</u>, we can compute

(16) $$\boxed{\cos E = \frac{1 - \rho/a}{e}}\ .$$

In order to ~~evaluate~~ determine if $E \in [0, \pi]$, it is convenient to refer to the scalar product $\underline{x} \cdot \underline{p}$, so to finally write (alternatively, $\mathrm{sign}\left((\underline{e}_X \wedge \underline{x}) \cdot \underline{J}\right)$ could be used)

(17) $$\boxed{E = \mathrm{sign}(\underline{x} \cdot \underline{p})\ \mathrm{acos}\left(\frac{1 - \rho/a}{e}\right)}\ .$$

The previous relation is easier to handle with respect to another, which is more commonly known:

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\,\tan\frac{v}{2}.$$

Giving the eccentric anomaly as a function of the true one that could be computed after some ~~rotation~~ scalar product of ~~coordinates~~ (we omit these calculations), that ~~th~~ are similar to the following ones in the case $e=0$).

Finally, the mean anomaly can be determined by using the famous Kepler equation:

$$(18)\quad \boxed{M = E - e\sin E}\,.$$

If the eccentricity is equal to zero, then one immediately realizes that

$$\boxed{v = E = M}$$

(it is enough to look at the equations in this page).

~~In~~ The true anomaly can be ~~understood~~ calculated as follows:

$$(16a)\quad \boxed{\cos v = \frac{\underline{e}_X \cdot \underline{x}}{\|\underline{x}\|} = \frac{\underline{e}_X \cdot \underline{x}}{\rho}}$$

By the way, let us recall that the line of the pericentre coincides with that of the ascending node, when $e=0$ (i.e., $\underline{e}_X = \underline{n}$ if $e=0$). Moreover, we remark that $v \in (0,\pi)$ if $\underline{e}_X \wedge \underline{x}$ is pointing in the same direction with respect to $\underline{J}$. Therefore, we can conclude

$$(18a)\quad \boxed{M = \mathrm{sign}\big((\underline{e}_X \wedge \underline{x}) \cdot \underline{J}\big)\ \mathrm{acos}\left(\frac{\underline{e}_X \cdot \underline{x}}{\rho}\right)}.$$

This concludes the calculation of the orbital elements.

It is convenient to make a more synthetic description of the whole procedure, so to summarize what a C-function must do. Let us call orbel_xp2el such a function. It is important to fix a priori a threshold ε on some test (let us recall that in a numerical code $e \neq 0$ even if $\underline{J} \cdot \underline{J} = \Gamma a$, because of the round-off errors). Such a value of ε should be fixed so to be sligthly larger than the error-machine value (e.g., for double-type numbers, $\varepsilon = 10^{-15}$ could be a good choice). Moreover, just pure numbers must be compared with ε (i.e. quantities that have physically adimensional units of measure).

Description of the function "orbel_xp2el"

- Input variables: $\underline{x}$, $\underline{p}$, Γ, ε
- Output variables: a, e, $\cos i$, $\sin i$, $\cos\Omega$, $\sin\Omega$, $\cos\omega$, $\sin\omega$, M
  (use memory adresses!!)
- Local variables: ρ, $\mathcal{E}$, $\underline{J}$, $\|\underline{J}\|$, $\underline{n}$, $\underline{A}$, $\underline{e}_x$, $\cos E$, E, $\sin E$ and many other temporary variables useful to speed up the calculations.

- Working procedure
  - Calculation of $P, \mathcal{E}, \underline{J}$ according to formulas (1), (2), (3), resp.
  - If $\boxed{\mathcal{E} \geq 0}$ then
    - print an error message
    - stop the execution of the whole program
  - If $\boxed{\underline{J} = \underline{0}}$ then
    - print another error message
    - stop the execution of the whole program
  - Calculation of $a, e, \cos i, \sin i$ according to formulas (4), (5), (6) and (7), resp.
  - If $\boxed{|\sin i| > \varepsilon}$ then
    - compute $\underline{n}, \cos\Omega, \sin\Omega$ according to (8), (9) and (10), resp.
  - else (case $|\sin i| \leq \varepsilon$)
    - compute $\underline{n}, \cos\Omega, \sin\Omega$ according to the alternative formulas (8a), (9a), (10a), resp.

- If $\boxed{e > \varepsilon}$ then
  - compute $\underline{A}$, $\underline{e}_x$, $\cos\omega$, $\sin\omega$ according to formulas (11), (12), (13), (14), resp.
  - compute $\cos E$, E, and finally, M using formulas (16), (17) and (18), resp. ($\cos E$ must be reset to -1 or 1 if it exceeds the interval $[-1, 1]$ because of roundoff errors)
- else (i.e. in the case $e \leq \varepsilon$)
  - compute $\underline{e}_x$, $\cos\omega$, $\sin\omega$ according to formulas (12a), (13a), (14a), resp.
  - compute $\cos E$ (that is equal to $\cos v$) and, finally, M using formulas (16a) and (18a), resp. ($\cos E$ must be reset to -1 or 1 if it exceeds $[-1, 1]$ because of roundoff errors).
- Return to the calling function.

• Let us some more remarks.
It should be clear (from the previous pages) that the whole procedure could be rearranged so to produce the Delaunay variables (as output), which are $L, G, \Theta, M, \omega, \Omega$ but we are not

interested in it.

Moreover, we could even produce the Poincaré canonical coordinates, that are

$$\Lambda = L,\ \xi = \sqrt{2L}\sqrt{1-\sqrt{1-e^2}}\cos(-\omega-\Omega),$$

$$p = \sqrt{2L}\sqrt{\sqrt{1-e^2}(1-\cos i)}\cos(\Omega),\ \lambda = M+\omega+\Omega,$$

$$\eta = \sqrt{2L}\sqrt{1-\sqrt{1-e^2}}\sin(-\omega-\Omega),\ q = \sqrt{2L}\sqrt{(1-e^2)(1-\cos i)}\sin(-\Omega).$$

The Poincaré coordinates are not singular ~~with respect to~~ in the cases when $e=0$, or $i=0$.

Nevertheless, the impact (should not be) of such singularities when the code runs the transformation to orbital elements and the change of coordinates backwards to positions and velocities (we mean, from a numerical point of view)

. One more technical remark.

It is useful to include in the ~~function~~ code some functions doing the (a) scalar product between two vectors, (b) the vectorial (= cross) product, (c) the norm, (d) the product between matrices, (e) the product between a matrix and a vector, and so on.

let us repeat the main idea about the strategy of the algorithm performing the numerical integration of the Kepler problem, that can be described by the following scheme:

$$\begin{array}{ccc}
\left(\underline{x}(t_0), \underline{p}(t_0)\right) & \xmapsto{\text{passing to orbital elements}} & \left(a, e, i, \Omega, \omega, M(t_0)\right) \\
 & & \Big\downarrow \text{flow of the motion on the Keplerian ellipse} \\
\left(\underline{x}(t_1), \underline{p}(t_1)\right) & \xleftarrow{\text{transformation backwards from orb. elements}} & \left(a, e, i, \Omega, \omega, M(t_1)\right)
\end{array}$$

During the flow corresponding to the motion on the Keplerian ellipse (characterized by the values of a, e, i, Ω, ω), ~~changes~~ just the mean motion angle can change its value, but this is done in a very easy way to compute. In fact, after having recalled equation (15), one immediately realizes that

$$(19) \quad \boxed{M(t_1) = \sqrt{\frac{\Gamma}{a^3}}\,(t_1 - t_0) + M(t_0)}\,.$$

Now, we are going to describe the last step of our scheme, i.e., the transformation backwards from orbital elements to position and momenta (or, equivalently, velocities). | Orb. el. 21

First, of all the Kepler equation (18) must now be solved with respect to the unknown E (i.e., the eccentric anomaly). By rewriting that equation in the form

$$(20)\quad \boxed{\begin{array}{l} f(E) - M = 0 \, , \\ f(E) = E - e\sin E \end{array}}$$

where $f(E) = E - e\sin E$, one immediately realizes that the solution $\exists !$ because of the implicit function theorem, that can be applied because

$$f'(E) = 1 - e\cos E \geq 1 - e > 0$$

(recall that $0 \leq e < 1$, because we are not considering parabolas or hyperbolas).

The solution can be easily found by applying numerical methods. Let us give some hints about that.
First let us recall that the Newton method is extremely fast, but it could not converge to the exact solution if it is started ~~too far~~ from an initial approximation

that is to far with respect to the solution.
On the other hand, the bisection method certainly converge to the solution but in a much slower way with respect to the Newton method.
Actually, they can be implemented in such a way ~~that~~ to iterate initially the bisection procedure, until we are sure that we produced an approximation good enough to safely start the Newton method.
This can be done, according to the following algorithm.

- Initially define
$$E_- = M - e \,, \qquad E_+ = M + e$$
(because it is easy to remark that the solution $E = \bar{E}$ of the Kepler equation (20) satisfies the relation $\bar{E} \in [E_-, E_+]$
- do the following cycle of operations
  - (re) define the mid-point $E_M = (E_+ + E_-)/2$
  - if $(E_+ - E_-)/2 < \varepsilon$ (where ε means, again, a tolerance error) then stops the algorithm, because E_M is a good enough numerical solution

- redefine the values of E_- and E_+ according to the bisection procedure

- repeat while the condition

$$\boxed{\frac{e\cdot(E_+-E_-)}{2(1-e)} \geqslant 1}$$

is satisfied

- Redefine the mid-point $E_M = (E_+ + E_-)/2$
- Apply repeatedly the Newton method by redefining E_M by adding the quantity

$$\boxed{\Delta E_M = -\frac{f(E_M)}{f'(E_M)}}$$

while the condition

$$\boxed{\Delta E_M \geqslant \varepsilon}$$

is satisfied

- When the execution of the code ends the previous cycle of operations, then the last calculated value of E_M is a good enough numerical approximation of the true solution $\bar{E}$.

For more details, about the numeri- | Orb. el. 24
cal methods to solve the Kepler equation, one can refer to section 1.2 of the "Appunti di Meccanica Celeste" by A. Giorgilli.
After having determined the eccentric anomaly, one can easily compute the coordinates in the orbital plane:

$$(21)\ \boxed{X = a(\cos E - e)}\ ,\ (22)\ \boxed{Y = a\sqrt{1-e^2}\,\sin E}\ .$$

(See formula (2.41) of the "Solar System Dynamics" book)
We need to use the values of the distance ρ and the angular velocity β, so to be able to evaluate the velocity in the orbital plane:

$$(23)\ \boxed{\dot{X} = -\frac{\beta a^2}{\rho}\sin E}\ ,\ (24)\ \boxed{\dot{Y} = \frac{\beta a^2\sqrt{1-e^2}}{\rho}\cos E}\ ,$$

(see the equivalent formula (2.68) of the "Solar System Dynamics" book).
where

$$(24)\ \boxed{\rho = a(1-e\cos E)}\ ,\ (25)\ \boxed{\beta = \sqrt{\frac{\Gamma}{a^3}}}\ .$$

Now, it is "just" matter to express the ~~coordinat~~ positions $\underline{x}$ and momenta $\underline{p}$ in the reference frame $Oxyz$ as a function of the positions (X,Y) and velocities $(\dot{X},\dot{Y})$ ~~in~~ the ~~ref~~ orbital plane. In order to do that, let us first remark that we have to apply to the coordinates (of the orbital plane) the following transformations:

(I) a rotation R_ω of an angle ω around the direction $\underline{J}$ (which is meant as the vertical axis):

$$(26) \quad \boxed{R_\omega = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

(II) a rotation R_i of an angle i around the direction $\underline{n}$ locating the ascending node (which is meant as a temporary x-axis); then

$$(27) \quad \boxed{R_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix}};$$

(III) a rotation R_Ω of an angle Ω around the vertical direction of the $Oxyz$ frame, that is

$$(28) \quad R_\Omega = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the composition of rotations gives another rotation, then we can introduce the "total" orthogonal matrix

$$\boxed{\mathcal{R} = R_\Omega \cdot R_i \cdot R_\omega} \qquad (29)$$

(In order to check it, one can refer to sect. 1.3.3 of Giorgilli's book or to sect. 2.8 of "Solar System Dynamics book).

More explicitly, by making some calculations, one obtains

$$\mathcal{R} = \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \mathcal{R}_{13} \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & \mathcal{R}_{23} \\ \sin i\sin\omega & \sin i\cos\omega & \mathcal{R}_{33} \end{pmatrix} \qquad (30)$$

where $\mathcal{R}_{13} = \sin\Omega\sin i$, $\mathcal{R}_{23} = -\cos\Omega\sin i$, $\mathcal{R}_{33} = \cos i$ are useless (see the equivalent formula at the end of section 1.3.3 of Giorgilli's book).

In fact, we obtain the wanted positions and moments by making the following final calculations:

$$(31)\quad \boxed{\underline{x} = \mathcal{R}\cdot\begin{pmatrix} X \\ Y \\ 0 \end{pmatrix}}, \qquad (32)\quad \boxed{\underline{p} = \mathcal{R}\begin{pmatrix} \dot{X} \\ \dot{Y} \\ 0 \end{pmatrix}}$$

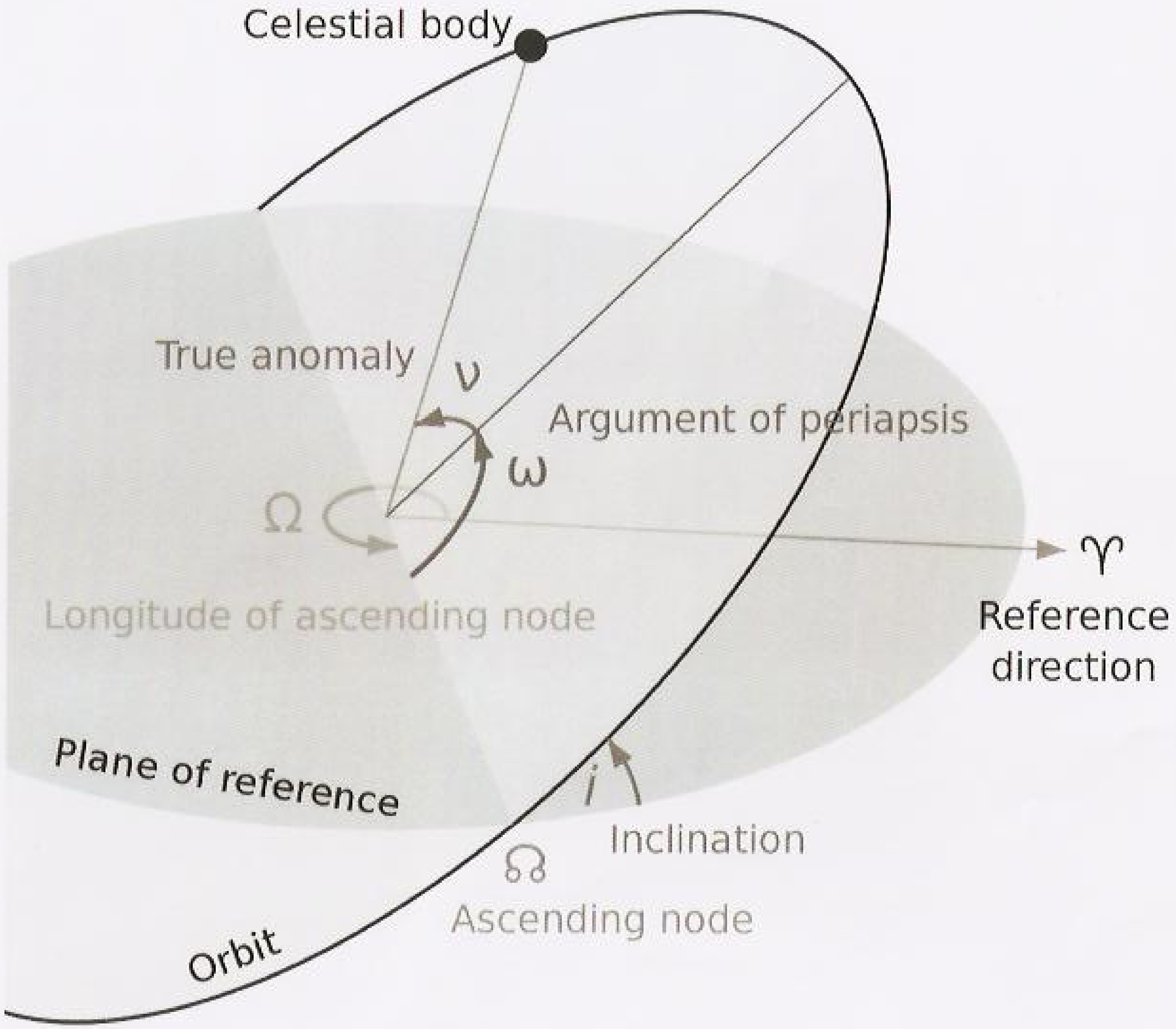
Also in this case, it is convenient to summarize the computational procedure so to describe what a C-function must do. Let us call orbel_el2xp such a function.

Description of the function "orbel_el2xp"

- Input variables: a, e, $\cos i$, $\sin i$, $\cos\Omega$, $\sin\Omega$, $\cos\omega$, $\sin\omega$, M, Γ
- Output variables: $\underline{x}$, p
- Local variables: E, $\cos E$, $\sin E$, X, Y, ρ, β, $\mathring{X}$, $\mathring{Y}$, and the matrices R_Ω, R_i, R_ω, R or just the first 2 columns of R.
- Working procedure
  - Determine the value of E by solving the Kepler equation (20); it is convenient to do that by calling another function that executes the algorithm at pages 22 and 23 (thus, by linking the bisection method to the Newton one)

- Calculation of $X, Y, \dot{X}, \dot{Y}$
  by using formulas (21), (22), (23), (24), resp.; and (25), (26)
- Calculation of R, that can be done either
  - by using formulas (26), (27), (28), (29)

  or, alternatively,
  - by using formulas (30)
- Finally, compute the wanted vectors $\underline{x}$ and ~~y~~ p, by using the formulas (31) and (32), respectively
- Return to the calling function -


Celestial body
True anomaly
ν
Argument of periapsis
ω
Ω
♈
Reference
direction
Longitude of ascending node
Plane of reference
i
Inclination
☊
Ascending node
Orbit