Modello di Henon-Heiles

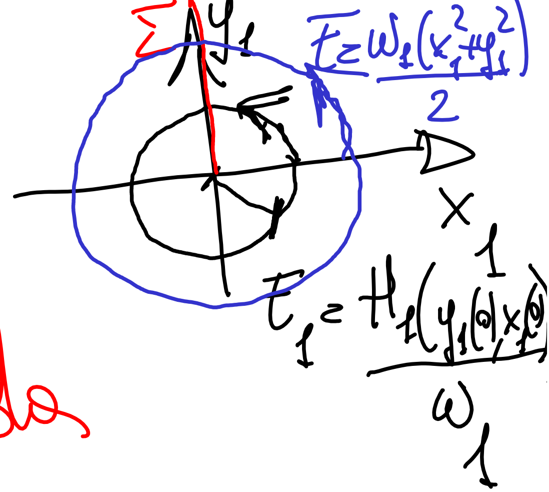
$$H(\underline{y}, \underline{x}) = \frac{1}{2}\omega_1(x_1^2+y_1^2) + \frac{1}{2}\omega_2(x_2^2+y_2^2) + \cancel{x_1^2 x_2} - \frac{x_2^3}{3}$$

termine di accoppiamento tra (y_1, x_1) e (y_2, x_2)

momentaneamente leviamo

Senza accoppiamento: $H(\underline{y}, \underline{x}) = H_1(y_1, x_1) + H_2(y_2, x_2)$

dove $H_1(y_1, x_1) = \frac{\omega_1}{2}(x_1^2 + y_1^2)$



La sezione di Poincaré Σ è dotata da

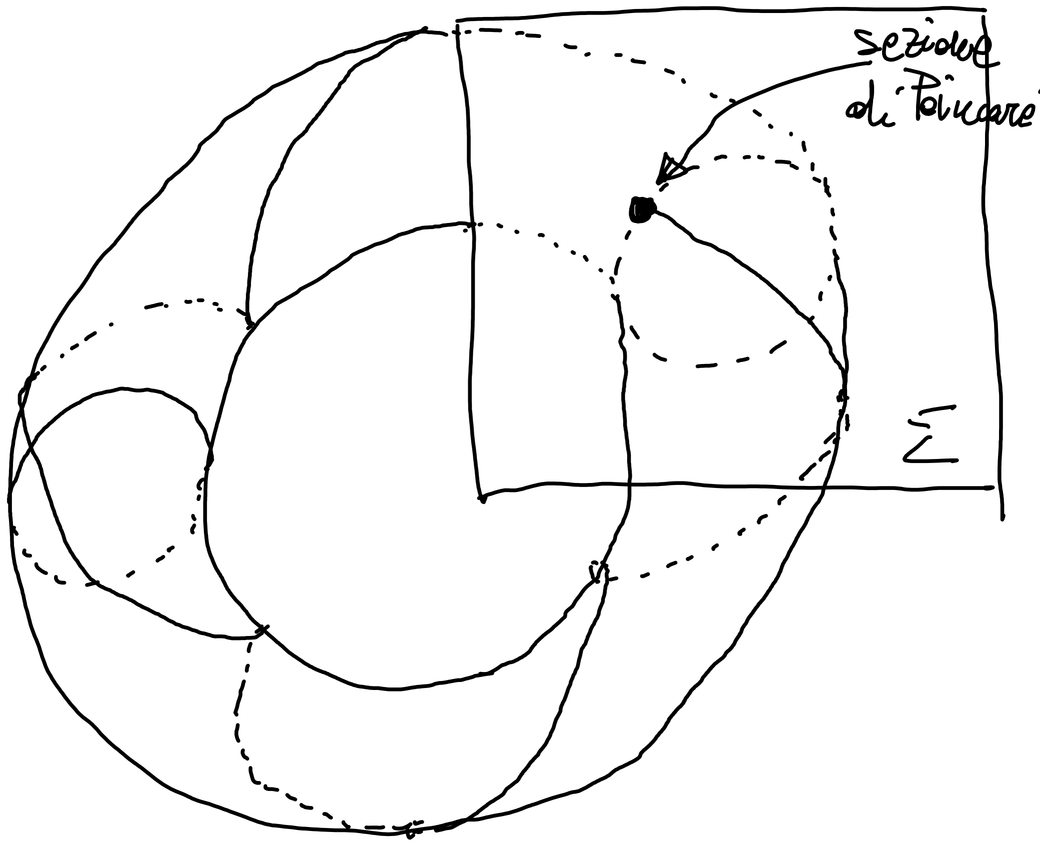
$$x_1 = 0 \;, \quad \dot{x}_1 = \frac{\partial H_1}{\partial y_1} = \omega_1 y_1 > 0$$

inoltre $H_2(y_2, x_2) = \frac{\omega_2}{2}(x_2^2 + y_2^2) - \frac{x_2^3}{3}$

Siccome $\{H_1, H_2\} = 0$, $\{H_1, H\} = \{H_2, H\} = 0$

$\Rightarrow$ 2 cost. del moto in involuzione e indip. $\Rightarrow$ sist. INTEGR.

Siccome $H_2(y_2, x_2)$ è cost. del moto allora $H_2(y_2, x_2) = E_2 = H_2(y_2(0), x_2(0))$



Costruzione della forma normale di Birkhoff con Mathematica

$$Z_0 = \omega_1 z_1 \bar{z}_1 + \omega_2 z_2 \bar{z}_2 \qquad \left(i\underline{\bar{z}}, \underline{z} \right)$$

momenti

var. canoniche

coord.

Rappr. -

Mathematica	scrittura nostra
z1	z_1
z2	z_2
izb1	$i\bar{z}_1$
izb2	$i\bar{z}_2$
$1 =$ w1	ω_1
$\frac{\sqrt{5}-1}{2} =$ w2	ω_2
znf0	$-i\omega_1 z_1 i\bar{z}_1 - i\omega_2 z_2 i\bar{z}_2$
fs[[1]]	$f_1 \in P_3$ pol. om. di grado 3
fs[[2]]	$f_2 \in P_4$ " " " " 4
⋮	⋮

$$z_1 = \frac{1}{\sqrt{2}}(x_1 + iy_1)$$

$$i\bar{z}_1 = \frac{i}{\sqrt{2}}(x_1 - iy_1)$$

$$\frac{1}{\sqrt{2}}\left(z_1 - i(i\bar{z}_1)\right) = x_1$$

iz1b^l1b*iz2b^l2b Mathematica

z1^l1*z2^l2* ⇐ l1, l2, l1b, l2b

omosolv[f]

passonormi[fs, r]

scrittura nostra

esponenti $l_1, l_2, \bar{l}_1, \bar{l}_2 \Rightarrow z_1^{l_1} z_2^{l_2} (i\bar{z}_1)^{\bar{l}_1} (i\bar{z}_2)^{\bar{l}_2}$

risolve l'eq. omologica $\left\{\sum_{j=1}^{2} \omega_j z_j \bar{z}_j, \chi\right\} + f = Z$

dip. solo da $z_1\bar{z}_1, z_2\bar{z}_2$

ci fa passare da $H^{(r-1)}$ a $H^{(r)} = \exp L_{\chi_2} H^{(r-1)}$

$$H^{(r)} = \exp L_{\chi_2} Z_0 + \exp L_{\chi_2} Z_1 + \dots + \exp L_{\chi_2} Z_{r-1}$$

$$+ \exp L_{\chi_2} f_r^{(r-1)} + \exp L_{\chi_2} f_{r+1}^{(r-1)} + \dots$$

Zuf0

fs[[1]]

fs[[r-1]]

fs[[r]]

fs[[r+1]]

Con il troncamento all'indice R1-1

Rappresentiamo tutta la Ham. con
Zn$\mathcal{f}$O , $\mathcal{f}$s

Hamiltoniana iniziale e anche ai passi successivi (ma cambiano le $\mathcal{f}$s) $=$ Zn$\mathcal{f}$O $+$ $\mathcal{f}$s[[1]] $+$ $\mathcal{f}$s[[2]] $+ \dots$

Supponiamo di essere al passo $r=2$

$L_{\chi_2} Z_0 \in \mathcal{P}_4$ va a sommarsi a $f_2^{(1)}$, $\frac{1}{2} L^2_{\chi_2} Z_0 \in \mathcal{P}_6$ termini da calcolare $[R-1/2]$

$$\frac{1}{j!} L^j_{\chi_2} f^{(r-1)}_s \quad \text{va a sommarsi a} \quad f^{(r)}_{s+jr}$$

$$\in \mathcal{P}_{s+jr+2} \qquad\qquad \in \mathcal{P}_{s+jr+2}$$

$\Longrightarrow$ j_{fin} deve ess. il max. j t.c. $s+jr \leq R-1$

$$j_{fin} = \left\lfloor \frac{R-1-s}{r} \right\rfloor$$

per $j=0$, lieterm $= Z_0$

per $j=1$, summand $= \{Z_0, \chi_2\}$, lieterm $= \{Z_0, \chi_2\}$, $f^{(2)}_2 \leftarrow L_{\chi_2} Z_0$

" $j=2$, summand $= \{L_{\chi_2} Z_0, \chi_2\}$, lieterm $= \frac{1}{2} L^2_{\chi_2} Z_0$, $f^{(2)}_{22} \leftarrow \frac{1}{2} L^2_{\chi_2} Z_0$

" $j=3$, " $= \{\frac{1}{2} L^2_{\chi_2} Z_0, \chi_2\}$, " $= \frac{1}{3} \cdot \frac{1}{2} L^3_{\chi_2} Z_0$, $f^{(2)}_{32} \leftarrow \frac{1}{3!} L^3_{\chi_2} Z_0$

Schema di integrazione "semi-analitico" delle eq. di Ham. per il modello di H.-H.

$$\left(\underline{y}(0), \underline{x}(0)\right) \xrightarrow{\mathcal{A}^{-1}\circ\left(\mathcal{C}^{(r)}\right)^{-1}\circ\mathcal{D}^{-1}} \left(\underline{I}(0), \underline{\theta}(0)\right)$$

$$\Phi^{t}_{H^{(r)}}(\underline{I},\underline{\theta}) \simeq \Phi^{t}_{\mathcal{Z}^{(r)}(\underline{I})}$$

$$\left(\underline{y}(t), \underline{x}(t)\right) \xleftarrow{\mathcal{D}\circ\mathcal{C}^{(r)}\circ\mathcal{A}} \left(\underline{I}(t)=\underline{I}(0), \underline{\vartheta}(t)=\underline{\vartheta}(0)+\underline{\omega}t\right)$$

dove $\omega_j = \dfrac{\partial \mathcal{Z}^{(r)}}{\partial I_j}\ \forall\, j=1,\ldots,4$

Faremo questo schema in rosso per tempi $t=j\Delta$ con $j=0,\ldots,N=T/\Delta$

dove $(i\bar{\underline{z}}, \underline{z}) = \mathcal{A}(\underline{I}, \underline{\vartheta})$ con

$$z_j = \sqrt{I_j}\, e^{-i\vartheta_j} \quad \forall j \quad \text{infatti} \quad \{z_j, i\bar{z}_j\}_{(\underline{I},\underline{\vartheta})} = -i\cancel{\sqrt{I_j}}\,\cancel{e^{-i\vartheta_j}} \cdot \frac{i\,\cancel{e^{i\vartheta_j}}}{2\cancel{\sqrt{I_j}}} - \frac{\cancel{e^{-i\vartheta_j}}}{2\cancel{\sqrt{I_j}}} \cdot i^2 \cancel{e^{i\vartheta_j}}\, \cancel{\sqrt{I_j}}$$

$$= -\frac{i^2}{\cancel{2}} \cdot \cancel{2} = 1 \text{ Ok!}$$

$$(i\bar{\underline{z}}^{(0)}, \underline{z}^{(0)}) = \mathcal{C}^{(r)}(i\bar{\underline{z}}^{(r)}, \underline{z}^{(r)}) \quad \text{dove}$$

Il teor. di scambio

$$\mathcal{C}^{(r)}(i\bar{\underline{z}}, \underline{z}) = \exp L_{\chi_r}\left(\exp L_{\chi_{r-1}}\left(\dots \exp L_{\chi_1}(i\bar{\underline{z}}, \underline{z})\right)\right)\dots$$

$$H^{(r)} = \exp L_{\chi_r}\left(\exp L_{\chi_{r-1}}\left(\dots \exp L_{\chi_1}(H^{(0)})\right)\right)\dots$$

Infine, abbiamo che $(\underline{y}, \underline{x}) = D(i\overline{\underline{z}}, \underline{z})$ dove

$$\forall j = 1, \dots, n \quad \begin{aligned} z_j &= \frac{1}{\sqrt{2}}(x_j + i y_j) \\ i\bar{z}_j &= \frac{i}{\sqrt{2}}(x_j - i y_j) \end{aligned} \implies \begin{cases} y_j = \dfrac{-i}{\sqrt{2}}\left(z_j + i(i\bar{z}_j)\right) \\ x_j = \dfrac{1}{\sqrt{2}}\left(z_j - i(i\bar{z}_j)\right) \end{cases} \forall j = 1, \dots, n$$

è la def. di D

$$\implies x_j(t) = \frac{1}{\sqrt{2}} \exp L_{\chi_2}\left(\dots \left(\exp L_{\chi_1} z_j^{(2)}\right)\right) - \frac{i}{\sqrt{2}} \exp L_{\chi_2}\left(\dots \left(\exp L_{\chi_1} i\bar{z}_j^{(2)}\right)\right)$$

$$= \exp L_{\chi_2}\left(\dots \left(\exp L_{\chi_1}\left(\frac{z_j^{(2)}(t)}{\sqrt{2}} - \frac{i}{\sqrt{2}}\left(i\bar{z}_j^{(2)}(t)\right) \dots \right)\right)\right)$$

$$= \exp L_{\chi_2}\left(\dots \left(\exp L_{\chi_1}\left(\frac{\sqrt{I_j(0)}}{\sqrt{2}} e^{-i(\vartheta_j(0) + \omega_j t)} - \frac{i}{\sqrt{2}}\left(i\sqrt{I_j(0)}\, e^{i(\vartheta_j(0) + \omega_j t)}\right) \dots \right)\right)\right)$$

Formule analoghe si avranno per il calcolo di $y_i(t)$

Controlleremo la (quasi) conservazione dell'energia $E = H(\underline{y}(t), \underline{x}(t))$, dove

$$H(\underline{y},\underline{x}) = \frac{\omega_1}{2}\left(x_1^2+y_1^2\right) + \frac{\omega_2}{2}\left(x_2^2+y_2^2\right) + x_1^2 x_2 - \frac{x_2^3}{3}$$

Nota: siccome $i\bar{z}_j \cdot z_j = i\sqrt{I_j}\, e^{+i\vartheta_j}\sqrt{I_j}\, e^{-i\vartheta_j} = i I_j$.

Nota: siccome $z_j = \sqrt{I_j}\, e^{-i\vartheta_j}$ lungo la legge del moto $\underline{I}(t) = \underline{I}(0) = \underline{I}^*$

$$\underline{\vartheta}(t) = \underline{\vartheta}(0) + \underline{\omega}\cdot t = 0 + \underline{\omega}^*\cdot t \Longrightarrow z_j(t) = \sqrt{I_j^*}\, e^{-i\omega_j^* \cdot t}$$

Calcolo approssimato dell'energia

$$H = \omega_1 I_1 + \omega_2 I_2 + O\left(\|I\|^{3/2}\right) \simeq \omega_1 I_1 + \omega_2 I_2$$

nel caso $I_1^* = I_2^* = 0.016$ e $\omega_1 = 1$, $\omega_2 = \frac{\sqrt{5}-1}{2} \simeq 0.618$

$$\Rightarrow E \simeq \left(1 + \frac{\sqrt{5}-1}{2}\right) I_1^*$$

Calcolo approssimato di $\max_t |E(t) - E(0)|$

Per $R1 = 5 \Rightarrow H^{(R1-1)} = \mathcal{Z}^{(4)} + R^{(4)}$

$$= Z_0 + Z_1 + Z_2 + Z_3 + Z_4 + R^{(4)}$$

$$Z_3 + Z_4 = O\left(\|z\|^3\right) = O\left(\|I\|^3\right), \quad R^{(4)} = O\left(\|z\|^7\right) = O\left(\|I\|^{7/2}\right)$$

$$\Rightarrow \max_t |E(t) - E(0)| = O\left(\|I\|^{7/2}\right)$$