Verso la forma normale di Birkhoff.

Stiamo considerando

$$H(\underline{x},\underline{y}) = \sum_{j=1}^{n} \frac{\omega_j}{2}\left(x_j^2+y_j^2\right) + \sum_{l=3}^{+\infty} f_l(\underline{x},\underline{y})$$

dove $f_l(\underline{x},\underline{y}) \in \mathcal{P}_l$, cioè $f_l(\underline{x},\underline{y}) = \sum_{|\underline{j}|+|\underline{m}|=l} c_{\underline{j},\underline{m}}\, \underline{x}^{\underline{j}} \cdot \underline{y}^{\underline{m}}$

dove $(\underline{x},\underline{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ (oppure $\mathbb{C}^n \times \mathbb{C}^n$), $c_{\underline{j},\underline{m}} \in \mathbb{R}$ (o in $\mathbb{C}$) e

$$\underline{x}^{\underline{j}} = x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}, \qquad \underline{y}^{\underline{m}} = y_1^{m_1} \cdot \ldots \cdot y_n^{m_n}.$$

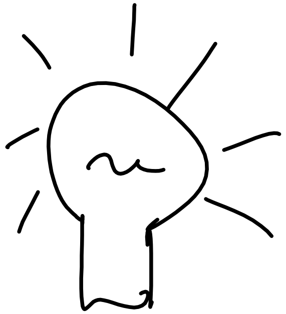
La parte quadratica è t.c.

$$\sum_{j=1}^{n} \frac{\omega_j}{2}\left(x_j^2 + y_j^2\right) = \underline{\omega} \cdot \underline{I}$$

dove $I_1, \ldots, I_n$ sono le azioni dell'osc. armonico

$$\begin{cases} x_j = \sqrt{2 I_j} \cos \varphi_j \\ y_j = \sqrt{2 I_j} \sin \varphi_j \end{cases}$$

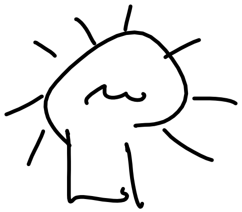
è una trasf. can.



Se $\exists$ una trasf. can. t.c. $(\underline{x}, \underline{y}) = \mathcal{C}(\underline{p}, \underline{q})$ e $H(\mathcal{C}(\underline{p}, \underline{q})) = K(\underline{p})$, allora il sistema sarebbe banalmente integrabile.

coord. di azione e angoli

Forma normale di Birkhoff



Proviamo a rimuovere la dipendenza dagli angoli

1. con le serie di Lie
- rimuovendo i termini perturbativi a gradi via via crescenti.

nelle variabili complesse significa avere la dipendenza solo da $z_j(i\bar{z}_j)$

Introduciamo $\forall j$ $z_j = \frac{1}{\sqrt{2}}(x_j + i y_j)$ dove $(\underline{x}, \underline{y})$ sono coord. can.

momenti

coord.

$$\Longrightarrow \begin{cases} x_j = \dfrac{1}{\sqrt{2}}\left(z_j - i\left(i\bar{z}_j\right)\right) \\[2mm] y_j = \dfrac{-i}{\sqrt{2}}\left(z_j + i\left(i\bar{z}_j\right)\right) \end{cases}$$

Sono canoniche le coord.

$$\Big(\underbrace{\underline{z}}_{\text{mom.}},\ \underbrace{i\underline{\bar{z}}}_{\text{coord.}}\Big)$$

Infatti,

$$\{x_i, x_j\}_{(\underline{z}, i\underline{\bar{z}})} = 0 = \{y_i, y_j\}_{(\underline{z}, i\underline{\bar{z}})} = \{x_i, y_j\}_{(\underline{z}, i\underline{\bar{z}})} \quad \forall\, i \neq j$$

$$\{x_j, x_j\}_{(\underline{z}, i\underline{\bar{z}})} = \{y_j, y_j\}_{(\underline{z}, i\underline{\bar{z}})} = 0$$

$$\frac{\partial z_j}{\partial z_j} \cdot \frac{\partial i\bar{z}_j}{\partial i\bar{z}_j} = 1$$

$$\forall j \quad \{x_j, y_j\}_{(\underline{z}, i\underline{\bar{z}})} = \frac{-i}{2}\left(i\{z_j, i\bar{z}_j\}_{(\underline{z}, i\underline{\bar{z}})} - i\{i\bar{z}_j, z_j\}_{(\underline{z}, i\underline{\bar{z}})} \right) = -\frac{i}{2}(i+i) = \frac{-i \cdot \cancel{2} i}{\cancel{2}} = 1.$$

Oss.:
$$\left\{\begin{aligned} z_j &= \frac{\sqrt{2 I_j}}{\sqrt{2}}\left(\cos\varphi_j + i\sin\varphi_j\right) = \sqrt{I_j}\, e^{i\varphi_j} \\ i\,\bar{z}_j &= i\sqrt{I_j}\, e^{-i\varphi_j} \end{aligned}\right.$$

A: è canonica!

$$\Rightarrow I_j = \frac{x_j^2 + y_j^2}{2} = \sqrt{I_j}\, e^{i\varphi_j} \cdot \sqrt{I_j}\, e^{-i\varphi_j} = z_j \bar{z}_j$$

Procedimento di costruzione della forma normale

$$(\square) \quad H^{(0)}(\underline{z}, i\underline{\bar{z}}) = \sum_{j=1}^{n} \omega_j (z_j \bar{z}_j) + \sum_{\ell=1}^{+\infty} f_\ell^{(0)}(\underline{z}, i\underline{\bar{z}})$$

indice del "passo di normalizzazione"

$$= \mathcal{Z}_0 + \sum_{s=1}^{+\infty} f_s^{(0)}$$

di grado $s+2$ in $(\underline{z}, i\underline{\bar{z}})$

Consideriamo la Ham. dopo $r-1$ passi di normalizzazione:

$$H^{(r-1)}(\underline{z}, i\underline{\bar{z}}) = \sum_{s=0}^{r-1} \mathcal{Z}_s + \sum_{s=r}^{+\infty} f_s^{(r-1)}$$

$\mathcal{Z}_s(z_1 i\bar{z}_1, \dots, z_n i\bar{z}_n)$

sono pol. di grado $s+2$ in $(\underline{z}, i\underline{\bar{z}})$

Descriviamo l'r-esimo passo.

Lemma: Siano $f \in \mathcal{P}_r$, $g \in \mathcal{P}_s \Rightarrow L_g f = \{f, g\} \in \mathcal{P}_{r+s-2}$

Introduciamo $H^{(r)} = \exp L_{\chi_r} H^{(r-1)}$,

dove $\chi_r \in \mathcal{P}_{r+2}$ in $(\underline{z}, i\underline{\bar{z}})$ t.c.

eq. omologica $\longrightarrow$ $\boxed{L_{\chi_r} Z_0 + f_r^{(r-1)} = Z_r}$ dove $Z_r(z_1 i\bar{z}_1, \dots, z_n i\bar{z}_n)$

Lemma: Sia $\underline{\omega}$ t.c. $\boxed{(\underline{\ell} - \underline{\bar{\ell}}) \cdot \underline{\omega} \neq 0}$ $\forall\ |\underline{\ell}| + |\underline{\bar{\ell}}| = r+2$ e $\underline{\ell} \neq \underline{\bar{\ell}}$

$|\underline{\ell}| + |\underline{\bar{\ell}}| = \ell_1 + \dots + \ell_n + \bar{\ell}_1 + \dots + \bar{\ell}_n$

allora $$\chi_r = \sum_{\substack{|\underline{\ell}| + |\underline{\bar{\ell}}| = r+2 \\ \underline{\ell} \neq \underline{\bar{\ell}}}} \frac{c_{\underline{\ell}, \underline{\bar{\ell}}}\, \underline{z}^{\underline{\ell}} (i\underline{\bar{z}})^{\underline{\bar{\ell}}}}{i(\underline{\ell} - \underline{\bar{\ell}}) \cdot \underline{\omega}}$$

$\underline{\ell} \in \mathbb{N}^n$ $(\ell_1, \dots, \ell_n)$

$\underline{\bar{\ell}} \in \mathbb{N}^n$ $(\bar{\ell}_1, \dots, \bar{\ell}_n)$

dove $f_r^{(r-1)}(\underline{z}, i\underline{\bar{z}}) = \sum_{|\underline{\ell}|+|\underline{\bar{\ell}}|=r+2} c_{\underline{\ell},\underline{\bar{\ell}}}\, \underline{z}^{\underline{\ell}} (i\underline{\bar{z}})^{\underline{\bar{\ell}}}$

e $\mathcal{Z}_r(\underline{z}, i\underline{\bar{z}}) = \sum_{2|\underline{\ell}|=r+2} c_{\underline{\ell},\underline{\ell}}\, \underline{z}^{\underline{\ell}} (i\underline{\bar{z}})^{\underline{\ell}}$ $\leftarrow = 0$ se r è dispari

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Proposizione: $H^{(r)} = \exp L_{\chi_r} H^{(r-1)}$ (dove $H^{(0)}$ è dato da $(\square)$) è della forma

$H^{(r)} = \mathcal{Z}^{(r)} + R^{(r)}$ dove la parte di forma normale

$\mathcal{Z}^{(r)} = \sum_{s=0}^{r} Z_s(z_1 \cdot \bar{z}_1, \ldots, z_n \cdot \bar{z}_n)$ e il resto $R^{(r)} = \sum_{s=r+1}^{+\infty} f_s^{(r)}(\underline{z}, i\underline{\bar{z}})$

con $Z_s \in \mathcal{P}_{s+2}$, $f_s^{(r)} \in \mathcal{P}_{s+2}$, questo è vero se

$Z_0 = \sum_{j=1}^{n} \omega_j z_j \bar{z}_j$ con $\underline{\omega}$ non risonante $\left(\underline{k} \cdot \underline{\omega} \neq 0 \ \forall \underline{k} \in \mathbb{Z}^n \setminus \{\underline{0}\} \right)$

Dim.: $\exp L_{\chi_r} H^{(r-1)}$ con χ_r che soddisfa l'eq. omologica $L_{\chi_r} Z_0 + f_r^{(r-1)} = Z_r \Rightarrow \chi_r \in \mathcal{P}_{r+2}$

$$\Rightarrow H^{(r)} = \exp L_{\chi_r} H^{(r-1)} = \sum_{s=0}^{r-1} \sum_{j=0}^{+\infty} \frac{1}{j!} L^j_{\chi_r} Z_s + \sum_{s=r}^{+\infty} \sum_{j=0}^{+\infty} \frac{1}{j!} L^j_{\chi_r} f_s^{(r-1)}$$

Introduciamo $f_s^{(r)} = f_s^{(r-1)} \ \forall s \geq r$ e poi li ridefiniremo (tante volte) con il simbolo $\hookleftarrow$ che significa che $a \hookleftarrow b$ è ridefivisto come $a = a + b$

Per induzione $f_s^{(r)} = f_s^{(r-1)} \in \mathcal{P}_{s+2}$, dopo aver ricordato $L_{\chi_2} f_s^{(r-1)} \in \mathcal{P}_{r+s+2}$ e $L_{\chi_2} Z_s \in \mathcal{P}_{r+s+2}$

$$\Rightarrow \frac{1}{j!} L^j_{\chi_2} f_s^{(r-1)} \in \mathcal{P}_{s+jr+2} \quad \text{e} \quad \frac{1}{j!} L^j_{\chi_2} Z_s \in \mathcal{P}_{s+jr+2} .$$

Ne segue $\forall j \geq 1$ $\quad f^{(r)}_{s+jr} \hookleftarrow \frac{1}{j!} L^j_{\chi_2} Z_s \quad$ e $\quad f^{(r)}_{s+jr} \hookleftarrow \frac{1}{j!} L^j_{\chi_2} f_s^{(r-1)}$

In modo più classico diamo le definizioni con una sola formula dei nuovi termini

$$f_s^{(r)} = \frac{1}{\left\lfloor \frac{s}{2} \right\rfloor !} L_{\chi_2}^{\lfloor s/2 \rfloor} Z_m + \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor - 1} \frac{1}{j!} L_{\chi_2}^{j} f_{s-j2}^{(r-1)} \quad \forall s \geq 2 \, .$$

dove $s = \left\lfloor \frac{s}{2} \right\rfloor \cdot 2 + m$

$$\Rightarrow R^{(r)} := \sum_{s=r+1}^{+\infty} f_s^{(r)} \quad \text{definiti come sopra}$$

e $f_2^{(r)}$ è il nuovo termine di forma normale. Infatti, $f_2^{(r)} = \frac{1}{1!} L_{\chi_2} Z_0 + f_2^{(r-1)} = Z_2$ per l'eq. omologica, dove $Z_2(z_1\bar{z}_1, \ldots, z_n\bar{z}_n)$. C.V.D.

Ci aspettiamo che $\mathcal{R}^{(r)} = \sum_{s=r+1}^{+\infty} f_s^{(r)} = O\left(\|\underline{z}\|^{r+3}\right)$,

"piccolo" rispetto a $Z^{(r)} = O\left(\|\underline{z}\|^2\right)$ per $\underline{z} \to \underline{0}$.

Schema di integrazione "semi-analitica" della forma normale di Birkhoff.

Siano $\left(\underline{z}^{(0)}, i\underline{\bar{z}}^{(0)}\right)$ prima di cominciare la costr. alla Birkhoff.

conosciuto

$$\left(\underline{z}^{(r-1)}, i\underline{\bar{z}}^{(r-1)}\right) := \exp L_{\chi_r}\left(\underline{z}^{(r)}, i\underline{\bar{z}}^{(r)}\right) := \psi^{(r)}\left(\underline{z}^{(r)}, i\underline{\bar{z}}^{(r)}\right)$$

$$\Rightarrow \left(\underline{z}^{(0)}, i\underline{\bar{z}}^{(0)}\right) = \mathcal{C}^{(r)}\left(\underline{z}^{(r)}, i\underline{\bar{z}}^{(r)}\right) := \psi^{(1)} \circ \ldots \circ \psi^{(r)}\left(\underline{z}^{(r)}, i\underline{\bar{z}}^{(r)}\right) \quad \forall r \geq 1$$

$$\Rightarrow \mathcal{D}\left(\underline{z}^{(0)}, i\,\underline{\bar z}^{(0)}\right) = \mathcal{C}^{(r-1)}\left(\Psi^{(r)}\left(\underline{z}^{(r)}, i\,\underline{\bar z}^{(r)}\right)\right) =$$

$$= \mathcal{C}^{(r-1)}\left(\exp L_{\chi_r}\left(\underline{z}^{(r)}, i\,\underline{\bar z}^{(r)}\right)\right)$$

$$= \exp L_{\chi_r}\left(\mathcal{C}^{(r-1)}\left(\underline{z}^{(r)}, i\,\underline{\bar z}^{(r)}\right)\right) = \ldots$$

$$= \exp L_{\chi_r} \cdots \exp L_{\chi_1}\left(\underline{z}^{(r)}, i\,\underline{\bar z}^{(r)}\right)$$

dove $\omega_j = \dfrac{\partial H}{\partial I_j}\left(\underline{I}(0)\right)$

$$\Phi^t_H \simeq \begin{tikzcd}[column sep=huge, row sep=large]
\left(\underline{x}(0), \underline{y}(0)\right) \arrow[r, "\mathcal{A}^{-1}\circ\left(\mathcal{C}^{(r)}\right)^{-1}\circ\mathcal{D}^{-1}"] \arrow[d, squiggly] & \left(\underline{I}(0), \underline{\varphi}(0)\right) \arrow[d, "\Phi^t_{H^{(r)}} \simeq \Phi^t_{Z^{(r)}(\underline{I})}"] \\
\left(\underline{x}(t), \underline{y}(t)\right) & \left(\underline{I}(t)=\underline{I}(0), \underline{\varphi}(t)=\underline{\varphi}(0)+\underline{\omega}t\right) \arrow[l, "\mathcal{D}\circ\mathcal{C}^{(r)}\circ\mathcal{A}"]
\end{tikzcd}$$

Verso uno studio analitico delle proprietà della forma normale di Birkhoff

Sia $g = \sum_{|\underline{\ell}|+|\underline{\bar{\ell}}|=s+2} c_{\underline{\ell},\underline{\bar{\ell}}}\, \underline{z}^{\underline{\ell}} (i\underline{\bar{z}})^{\underline{\bar{\ell}}}$, allora

$$\|g\| = \sum_{|\underline{\ell}|+|\underline{\bar{\ell}}|=s+2} |c_{\underline{\ell},\underline{\bar{\ell}}}| \Rightarrow \sup_{\underline{z}\in B_\rho(\underline{0})} |g(\underline{z}, i\underline{\bar{z}})| \le \|g\|\, \rho^{s+2}$$

Lemma: Siano $f \in \mathcal{P}_{r+2}$, $g \in \mathcal{P}_{s+2}$

$$\|L_g f\| = \|\{f, g\}\| \le (r+2)\cdot(s+2)\, \|f\|\cdot\|g\|$$

Dim.: Sia l'esp. di g come prima e quella di f t.c.

$$f = \sum_{|\underline{\ell}'|+|\underline{\bar\ell}'| = r+2} c_{\underline{\ell}',\underline{\bar\ell}'}\, \underline{z}^{\underline{\ell}'}\, \underline{i\bar z}^{\underline{\bar\ell}'}$$

$$\Rightarrow \{f, g\} = \sum_{j=1}^{n} \sum_{|\underline{\ell}|+|\underline{\bar\ell}| = s+2} \sum_{|\underline{\ell}'|+|\underline{\bar\ell}'| = r+2} c_{\underline{\ell},\underline{\bar\ell}} \cdot c'_{\underline{\ell}',\underline{\bar\ell}'}\, \underline{z}^{\underline{\ell}+\underline{\ell}'}\, \underline{i\bar z}^{\underline{\bar\ell}+\underline{\bar\ell}'} \cdot \left(\frac{\bar\ell_j \ell'_j}{i\bar z_j z_j} - \frac{\ell_j \bar\ell'_j}{z_j i\bar z_j} \right)$$

dove

$$\sum_{j=1}^{n} \left| \bar\ell_j \ell'_j - \ell_j \bar\ell'_j \right| \le \sum_{j=1}^{n} |\bar\ell_j||\ell'_j| + |\ell_j||\bar\ell'_j| \le \max\{|\ell_j|, |\bar\ell_j|\} \cdot \sum_{j=1}^{n} |\ell'_j| + |\bar\ell'_j|$$

$$\le (r+2) \cdot \sum_{j=1}^{n} |\ell_j| + |\bar\ell_j| \le (r+2) \cdot (s+2)$$

$$\|\{f,g\}\| \le \sum_{|\underline{\ell}|+|\underline{\bar{\ell}}|=s+2} \ \sum_{|\underline{\ell}'|+|\underline{\bar{\ell}}'|=r+2} (r+2)\cdot(s+2)\,|c_{\underline{\ell},\underline{\bar{\ell}}}|\,|c'_{\underline{\ell}',\underline{\bar{\ell}}'}|$$

$$\le (r+2)\cdot(s+2)\cdot\sum_{|\underline{\ell}|+|\underline{\bar{\ell}}|=s+2}|c_{\underline{\ell},\underline{\bar{\ell}}}|\cdot\|f\|$$

$$\le (r+2)\cdot(s+2)\cdot\|f\|\cdot\|g\| \qquad \text{C.V.D.}$$

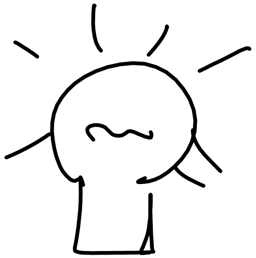
Lemma: Sia $\underline{\omega}$ diofanteo, cioè t.c. $|\underline{k}\cdot\underline{\omega}| \ge \dfrac{\gamma}{|\underline{k}|^{\tau}}$ $\forall\, \underline{k}\in\mathbb{Z}^n\setminus\{0\}$ con $\gamma>0$, $\tau\ge n-1$, allora

$$\|\chi_2\| \le \frac{\|f_2^{(r-1)}\|}{\gamma}(r+2)^{\tau}, \quad \text{dove } L_{\chi_2} Z_0 + f_2^{(r-1)} = Z_2$$

Dim. sia $$f_r^{(r-1)} = \sum_{|\underline{\ell}|+|\bar{\underline{\ell}}|=r+2} c_{\underline{\ell},\bar{\underline{\ell}}}\, \underline{z}^{\underline{\ell}} \left(i\bar{\underline{z}}\right)^{\bar{\underline{\ell}}}$$

$$\Rightarrow \|\chi_r\| = \sum_{\substack{|\underline{\ell}|+|\bar{\underline{\ell}}|=r+2 \\ \underline{\ell}\neq\bar{\underline{\ell}}}} \frac{|c_{\underline{\ell},\bar{\underline{\ell}}}|}{\left|(\underline{\ell}-\bar{\underline{\ell}})\cdot\underline{\omega}\right|} \le \frac{(r+2)^{\tau}}{\gamma}\left\|f_r^{(r-1)}\right\|$$

C.V.D.



Si vede che le serie generate dalla forma normale di Birkhoff divergono per $r \to +\infty$ su ogni $B_\varrho(\underline{0})$ con $\varrho > 0$.

$$\|X_r\| \simeq r^{\tau} \|f_r^{(\tau-1)}\| \Rightarrow f_{r+1}^{(\tau)} \approx L_{X_r} Z_1$$

$$\Rightarrow \|f_{r+1}^{(\tau)}\| \lesssim \tau \cdot r \|X_r\| \lesssim r^{\tau+1} \|f_r^{(\tau-1)}\|$$

$$\ldots \lesssim \left(r \cdot (r-1) \cdot (r-2) \cdot \ldots \cdot 1 \right)^{\tau+1} \|f_1^{(0)}\|$$

$$= O\left((r!)^{\tau+1} \|f_1^{(0)}\| \right)$$

costanti

$$\sup_{\underline{z} \in B_\rho(0)} |R^{(r)}| \gtrsim (r!)^{\tau+1} \cdot C \cdot \rho^{r+3} \xrightarrow[r \to +\infty]{} +\infty$$

$$\forall \rho > 0 \ -$$