Verso la forma normale di Birkhoff.

Stiamo considerando

$$H(\underline{x}, \underline{y}) = \sum_{j=1}^{n} \frac{\omega_j}{2}\left(x_j^2 + y_j^2\right) + \sum_{\ell=3}^{+\infty} f_\ell(\underline{x}, \underline{y})$$

dove $f_\ell(\underline{x}, \underline{y}) \in \mathcal{P}_\ell$, cioè $f_\ell(\underline{x}, \underline{y}) = \sum_{|\underline{j}|+|\underline{m}|=\ell} c_{\underline{j},\underline{m}}\, \underline{x}^{\underline{j}} \cdot \underline{y}^{\underline{m}}$

dove $(\underline{x}, \underline{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ (oppure $\mathbb{C}^n \times \mathbb{C}^n$), $c_{\underline{j},\underline{m}} \in \mathbb{R}$ (o in $\mathbb{C}$) e

$$\underline{x}^{\underline{j}} = x_1^{j_1} \cdot \ldots \cdot x_n^{j_n} \quad , \quad \underline{y}^{\underline{m}} = y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} .$$

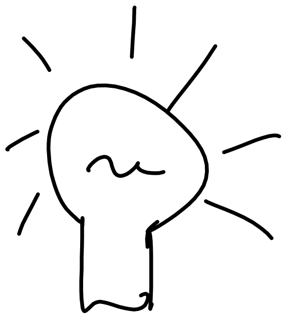
La parte quadratica è t.c.

$$\sum_{j=1}^{n} \frac{\omega_j}{2}\left(x_j^2 + y_j^2\right) = \underline{\omega} \cdot \underline{I}$$

dove $I_1, \ldots, I_n$ sono le azioni dell'osc. armonico

$$\begin{cases} x_j = \sqrt{2I_j}\cos\varphi_j \\ y_j = \sqrt{2I_j}\sin\varphi_j \end{cases}$$

è una trasf. can.



Se $\exists$ una trasf. can. t.c. $(\underline{x}, \underline{y}) = \mathcal{C}(\underline{p}, \underline{q})$ e $H(\mathcal{C}(\underline{p}, \underline{q})) = K(\underline{p})$, allora il sistema sarebbe banalmente integrabile.

coord. di azione e angoli

Proposizione: Siano $f \in P_r$, $g \in P_s$, $\alpha \in \mathbb{R}$ (o $\mathbb{C}$)

allora: $\alpha f \in P_r$, $f' \in P_{r-1}$, $f \cdot g \in P_{r+s}$,

$\{f,g\} \in P_{r+s-2}$ $\left(\text{perché } \{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}\right)$

Proposizione: $T_\chi : P \to P$ è lineare (ovvia!),

preserva il prodotto (cioè $\forall f,g \in P$, $T_\chi(f \cdot g) = T_\chi f \cdot T_\chi g$)

e preserva le par. di Poisson (cioè $\forall f,g \in P$, $T_\chi\{f,g\} = \{T_\chi f, T_\chi g\}$)

Prop.: La serie di Lie è una trasformata di Lie.

Dim.- sia $X = \{0, \dots, 0, X_2, 0, \dots, 0, \dots\}$ dove $X_2 \in P_{2+2}$

allora $T_X f = \sum_{s=0}^{+\infty} \sum_{j=1}^{s} \frac{j}{s} L_{X_j} f_{s-j} =: \sum_{s=0}^{+\infty} f_s$ dove $f_0 = f$ -

$$\Rightarrow f_s = \frac{2}{s} L_{X_2} f_{s-2} \Rightarrow f_{s-2} = \frac{2}{s-2} L_{X_2} f_{s-22} \dots$$

Oss.: $f_m = 0 \ \forall m = 1, \dots, 2-1$, poiché $f_m = \cancel{\sum_{j=1}^{m} \frac{j}{m} L_{X_j} f_{m-j}} = 0$ ($m < 2$)

$\Rightarrow f_s \neq 0$ solo se $s = n2$ con $n \in \mathbb{N}$ -

$$\Rightarrow f_{n2} = \frac{\cancel{2}}{n\cancel{2}} L_{X_2} f_{(n-1)2} = \frac{1}{n(n-1)} L^2_{X_2} f_{(n-2)2} = \dots = \frac{1}{n \cdot (n-1) \cdot \dots \cdot 1} L^n_{X_2} \overset{f}{\overset{\|}{f_0}}$$

$$T_x f = \sum_{n=0}^{+\infty} f_{n2} = \sum_{n=0}^{+\infty} \frac{1}{n!} L^n_{X_2} f =: \exp L_{X_2} f$$

poiché $\Phi^t_{X_2} f = \sum_{n=0}^{+\infty} \frac{t^n}{n!} L^n_{X_2} f$ C.V.D.

Corollario: $\exp L_X \cdot$ è <u>lineare e preserva prodotto</u> e $\{\cdot,\cdot\}$.

Teorema [di scambio]: $\mathcal{C} = \exp L_X \cdot$ induce una trasf. canonica e vale la seguente identità:

$$H \circ \mathcal{C}(\underline{X},\underline{Y}) := H(\underline{x},\underline{y})\Big|_{(\underline{x},\underline{y}) = \exp L_X(\underline{X},\underline{Y})} = \exp L_X H \Big|_{(x,y)=(\underline{X},\underline{Y})}$$

Dim. è banale conseguenza delle proposizioni precedenti.

$$T_{(0,\dots,0,X,0,\dots)}$$

ad. es. $x_1 = \exp L_\chi X_1$, $x_1^2 = (\exp L_\chi X_1)^2$

$$\Rightarrow x_1^2 = \exp L_\chi (X_1^2)$$

$$(T_\chi f)\cdot(T_\chi g) = T_\chi(f\cdot g)$$

Analogamente se abbiamo

$$\alpha f(\underline{x},\underline{y})\Big|_{(\underline{x},\underline{y})=\exp L_\chi(\underline{X},\underline{Y})} +$$

$$+\beta g(\underline{x},\underline{y})\Big|_{(\underline{x},\underline{y})=\exp L_\chi(\underline{X},\underline{Y})}$$

$$\exp L_\chi\big(\alpha f(x+y)+\beta g(x,y)\big)\Big|_{(x,y)=(\underline{X},\underline{Y})} =$$

Applico alla Ham. (def. come serie formale) la prop: che $\exp L_\chi$ commuta con somme e prodotti —

C.V.D.

—— ∘ ——

Es.: Mostriamo che il metodo di integrazione leap-frog ($\Phi_H^t = \Phi_{A+B}^t = \Phi_A^{t/2} \circ \Phi_B^t \circ \Phi_A^{t/2} + O(t^3)$) è di ordine 2

$$\Phi_{A+B}^{\tau} \underline{z} = \exp(\tau L_{A+B})\underline{z} = \underline{z} + \tau L_{A+B}\underline{z} + \frac{\tau^2}{2} L_{A+B}^2 \underline{z} + O(\tau^3)$$

$$= \underline{z} + (\tau L_A \underline{z} + \tau L_B \underline{z}) + \frac{\tau^2}{2}\left(L_A^2 \underline{z} + L_A L_B \underline{z} + L_B L_A \underline{z} + L_B^2 \underline{z}\right) + O(\tau^3)$$

$$\Phi_A^{\tau/2}\left(\Phi_B^{\tau}\left(\Phi_A^{\tau/2}\underline{z}\right)\right) = \Phi_A^{\tau/2}\left(\Phi_B^{\tau}\left(\underline{z}+\frac{\tau}{2}L_A\underline{z}+\frac{\tau^2}{8}L_A^2\underline{z}+O(\tau^3)\right)\right)$$

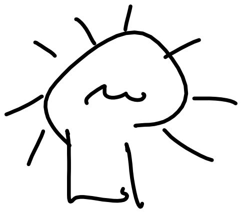
$$= \Phi_A^{\tau/2}\left(\underline{z}+\frac{\tau}{2}L_A\underline{z}+\frac{\tau^2}{8}L_A^2\underline{z}+\tau L_B\underline{z}+\frac{\tau^2}{2}L_BL_A\underline{z}+\frac{\tau^2}{2}L_B^2\underline{z}+O(\tau^3)\right)=$$

$$= \underline{z}+\frac{\tau}{2}L_A\underline{z}+\frac{\tau^2}{8}L_A^2\underline{z}+\tau L_B\underline{z}+\frac{\tau^2}{2}L_BL_A\underline{z}+\frac{\tau}{2}L_A\underline{z}+\frac{\tau^2}{4}L_A^2\underline{z}$$

$$+\frac{\tau^2}{2}L_AL_B\underline{z}+\frac{\tau^2}{2}L_BL_A\underline{z}+\frac{\tau^2}{8}L_A^2\underline{z}+\frac{\tau^2}{2}L_B^2\underline{z}+O(\tau^3)=$$

$$= \underline{z}+\tau\left(L_A\underline{z}+L_B\underline{z}\right)+\tau^2\left(\frac{1}{2}L_A^2\underline{z}+\frac{1}{2}L_AL_B\underline{z}+\frac{1}{2}L_BL_A\underline{z}+\frac{1}{2}L_B^2\underline{z}\right)+O(\tau^3) = \Phi_{A+B}^{\tau}\underline{z}+O(\tau^3)$$

Forma normale di Birkhoff



Proviamo a rimuovere la dipendenza dagli angoli

nelle variabili complesse significa avere la dipendenza solo da $z_j (i\bar{z}_j)$

1. con le serie di Lie
2. rimuovendo i termini perturbativi a gradi via via crescenti.

Introduciamo $\forall j$ $z_j = \frac{1}{\sqrt{2}}(x_j + i y_j)$ dove (x, y) sono coord. can.

momenti

coord

$$\Longrightarrow \begin{cases} x_j = \dfrac{1}{\sqrt{2}}\left(z_j - i\left(i\bar{z}_j\right)\right) \\ y_j = \dfrac{-i}{\sqrt{2}}\left(z_j + i\left(i\bar{z}_j\right)\right) \end{cases}$$

Sono canoniche le coord.

$$\left(\underline{z},\ i\underline{\bar{z}}\right)$$

mom. coord.

Infatti,

$$\{x_i, x_j\}_{(\underline{z}, i\underline{\bar{z}})} = 0 = \{y_i, y_j\}_{(\underline{z}, i\underline{\bar{z}})} = \{x_i, y_j\}_{(\underline{z}, i\underline{\bar{z}})} \quad \forall\, i \neq j$$

$$\{x_j, x_j\}_{(\underline{z}, i\underline{\bar{z}})} = \{y_j, y_j\}_{(\underline{z}, i\underline{\bar{z}})} = 0$$

$$\boxed{\frac{\partial z_j}{\partial z_j} \cdot \frac{\partial i\bar{z}_j}{\partial i\bar{z}_j} = 1}$$

$$\forall j \quad \{x_j, y_j\}_{(\underline{z}, i\underline{\bar{z}})} = \frac{-i}{2}\left(i\{z_j, i\bar{z}_j\}_{(\underline{z}, i\underline{\bar{z}})} - i\{i\bar{z}_j, z_j\}_{(\underline{z}, i\underline{\bar{z}})} \right) = -\frac{i}{2}(i + i) = \frac{-i \cdot \cancel{2} i}{\cancel{2}} = 1.$$

Oss.: $\begin{cases} z_j = \dfrac{\sqrt{2 I_j}}{\sqrt{2}}\left(\cos\varphi_j + i\sin\varphi_j\right) = \sqrt{I_j}\, e^{i\varphi_j} \\ i\left(\overline{z}_j\right) = i\sqrt{I_j}\, e^{-i\varphi_j} \end{cases}$ A: è canonica!

$$\Rightarrow I_j = \frac{x_j^2 + y_j^2}{2} = \sqrt{I_j}\, e^{i\varphi_j} \cdot \sqrt{I_j}\, e^{-i\varphi_j} = z_j \overline{z}_j$$

Procedimento di costruzione della forma normale

indice del "passo di normalizzazione"

$$H^{(0)}(\underline{z}, i\overline{\underline{z}}) = \sum_{j=1}^{n} \omega_j (z_j \overline{z}_j) + \sum_{\ell=1}^{+\infty} f_\ell^{(0)}(\underline{z}, i\overline{\underline{z}})$$

$$= \mathcal{Z}_0 + \sum_{s=1}^{+\infty} f_s^{(0)}$$

di grado $s+2$ in $(\underline{z}, i\underline{\bar{z}})$

Consideriamo la Ham. dopo $r-1$ passi di normalizzazione:

$$H^{(r-1)}(\underline{z}, i\underline{\bar{z}}) = \sum_{s=0}^{r-1} \mathcal{Z}_s + \sum_{s=r}^{+\infty} f_s^{(r-1)}$$

$\mathcal{Z}_s(z_1 i\bar{z}_1, \ldots, z_n i\bar{z}_n)$

sono pol. di grado $s+2$ in $(\underline{z}, i\underline{\bar{z}})$

Descriviamo l'r-esimo passo.

Lemma: siano $f \in \mathcal{P}_r$, $g \in \mathcal{P}_s \Rightarrow L_g f = \{f, g\} \in \mathcal{P}_{r+s-2}$

Introduciamo $H^{(r)} = \exp L_{\chi_r} H^{(r-1)}$,

dove $\chi_r \in \mathcal{P}_{r+2}$ in $(\underline{z}, i\underline{\bar{z}})$ t.c.

eq. omologica $\longrightarrow$ $\boxed{L_{\chi_r} Z_0 + f_r^{(r-1)} = Z_r}$ dove $Z_r(z_1, i\bar{z}_1, \dots, z_n, i\bar{z}_n)$

Lemma: sia $\underline{\omega}$ t.c. $\boxed{(\underline{\ell} - \underline{\bar{\ell}}) \cdot \underline{\omega} \neq 0}$ $\forall\ |\underline{\ell}| + |\underline{\bar{\ell}}| = r+2$ e $\underline{\ell} \neq \underline{\bar{\ell}}$

$|\underline{\ell}| + |\underline{\bar{\ell}}| = \ell_1 + \dots + \ell_n + \bar{\ell}_1 + \dots + \bar{\ell}_n$

allora
$$\chi_r = \sum_{\substack{|\underline{\ell}| + |\underline{\bar{\ell}}| = r+2 \\ \underline{\ell} \neq \underline{\bar{\ell}}}} \frac{c_{\underline{\ell}, \underline{\bar{\ell}}}\, \underline{z}^{\underline{\ell}} (i\underline{\bar{z}})^{\underline{\bar{\ell}}}}{i(\underline{\ell} - \underline{\bar{\ell}}) \cdot \underline{\omega}}$$

$\underline{\ell} \in \mathbb{N}^n$ $(\ell_1, \dots, \ell_n)$

$\underline{\bar{\ell}} \in \mathbb{N}^n$ $(\bar{\ell}_1, \dots, \bar{\ell}_n)$

dove $f_2^{(2-1)}(\underline{z}, i\underline{\bar{z}}) = \sum_{|\underline{\rho}|+|\underline{\bar{\rho}}|=2+2} c_{\underline{\rho},\underline{\bar{\rho}}}\, \underline{z}^{\underline{\rho}} (i\underline{\bar{z}})^{\underline{\bar{\rho}}}$

e $\mathcal{Z}_2(\underline{z}, i\underline{\bar{z}}) = \sum_{2|\underline{\rho}|=2+2} c_{\underline{\rho},\underline{\rho}}\, \underline{z}^{\underline{\rho}} (i\underline{\bar{z}})^{\underline{\rho}}$ $\leftarrow = 0$ se 2 è dispari

Dim.: Poniamo $\chi_2 = \sum_{|\underline{\rho}|+|\underline{\bar{\rho}}|=2+2} \Theta_{\underline{\rho},\underline{\bar{\rho}}}\, \underline{z}^{\underline{\rho}} (i\underline{\bar{z}})^{\underline{\bar{\rho}}}$ $\leftarrow$ incogniti

$$\Rightarrow L_{\chi_2} \mathcal{Z}_0 = \{\mathcal{Z}_0, \chi_2\}(\underline{z}, i\underline{\bar{z}}) = \sum_{|\underline{\rho}|+|\underline{\bar{\rho}}|=2+2} \left(\sum_{j=1}^{n} \cancel{\frac{\partial\, \omega_j \cancel{z_j} (-i)\, \cancel{i\bar{z}_j}}{\partial (i\bar{z}_j)}} \cdot \Theta_{\underline{\rho}\underline{\bar{\rho}}}\, \frac{\rho_j \underline{z}^{\underline{\rho}} (i\underline{\bar{z}})^{\underline{\bar{\rho}}}}{\cancel{z_j}} - \omega_j (-i)\, \cancel{i\bar{z}_j} \cdot \Theta_{\underline{\rho}\underline{\rho}}\, \bar{\rho}_j \frac{\underline{z}^{\underline{\rho}} (i\underline{\bar{z}})^{\underline{\bar{\rho}}}}{\cancel{i\bar{z}_j}} \right)$$

$$= -i \sum_{|\underline{\ell}|+|\bar{\underline{\ell}}|=r+2} \boxed{(\underline{\ell}-\bar{\underline{\ell}})\cdot\underline{\omega}}\ \Theta^{(H)}_{\underline{\ell},\bar{\underline{\ell}}}\ \underline{z}^{\underline{\ell}}\ i\underline{z}^{\bar{\underline{\ell}}} = 0 \text{ se } \underline{\ell}=\bar{\underline{\ell}}$$

imponiamo che sia $= -f_2^{(r-1)} + Z_2$

$\Rightarrow$ in corrisp. a ogni monomio $\underline{z}^{\underline{\ell}}\ i\underline{z}^{\bar{\underline{\ell}}}$

con $\underline{\ell} \neq \bar{\underline{\ell}}$ $\Rightarrow$ $-i(\underline{\ell}-\bar{\underline{\ell}})\cdot\underline{\omega}\ \Theta^{(H)}_{\underline{\ell},\bar{\underline{\ell}}} + c_{\underline{\ell},\bar{\underline{\ell}}} = 0$

$$\Rightarrow \Theta^{(H)}_{\underline{\ell},\bar{\underline{\ell}}} = \frac{c_{\underline{\ell},\bar{\underline{\ell}}}}{i(\underline{\ell}-\bar{\underline{\ell}})\cdot\underline{\omega}} \quad \forall\, \underline{\ell} \neq \bar{\underline{\ell}}$$