
Teorema (di Arnold-Jost): Siano $\Phi_1(p,q), \dots, \Phi_n(p,q)$ un sistema completo di integrali primi in involuzione per la Ham. $H = H(p,q)$. Sia $M_{\underline{c}}$ una componente n-dim. connessa e compatta dell'insieme $\{(p,q) : \underline{\Phi}(p,q) = \underline{c}\}$, allora

$\exists$ un intorno U di M_c (nello spazio delle fasi) t.c.

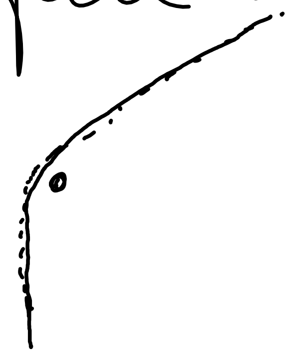
$\exists$ una trasformazione canonica $\underline{\Psi} : G \times \mathbb{T}^n \to U$ dove $G \subseteq \mathbb{R}^n$ aperto

t.c. $(\underline{p}, \underline{q}) = \underline{\Psi}(\underline{I}, \underline{\varphi})$ (cioè sono n angoli!) e $H(\underline{\Psi}(\underline{I}, \underline{\varphi})) = H(\underline{I})$

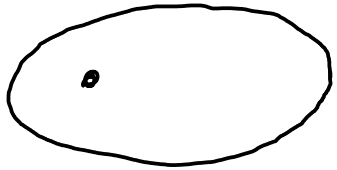
$$\Rightarrow I_j(t) = I_j(0) \quad \text{e} \quad \varphi_j(t) = \omega_j(t - t_0) \quad \text{dove } \omega_j = \dot{\varphi}_j = \frac{\partial H}{\partial I_j}(\underline{I}(0))$$

Oss. che ad esempio nel problema di Keplero NON siamo su una superficie compatta quando

$$\overline{E} = \underbrace{\frac{1}{2} m \underline{\dot{x}} \cdot \underline{\dot{x}} - \frac{k}{\|\underline{x}\|}}_{} > 0$$



Il moto si svolge su una superficie compatta quando $E<0$



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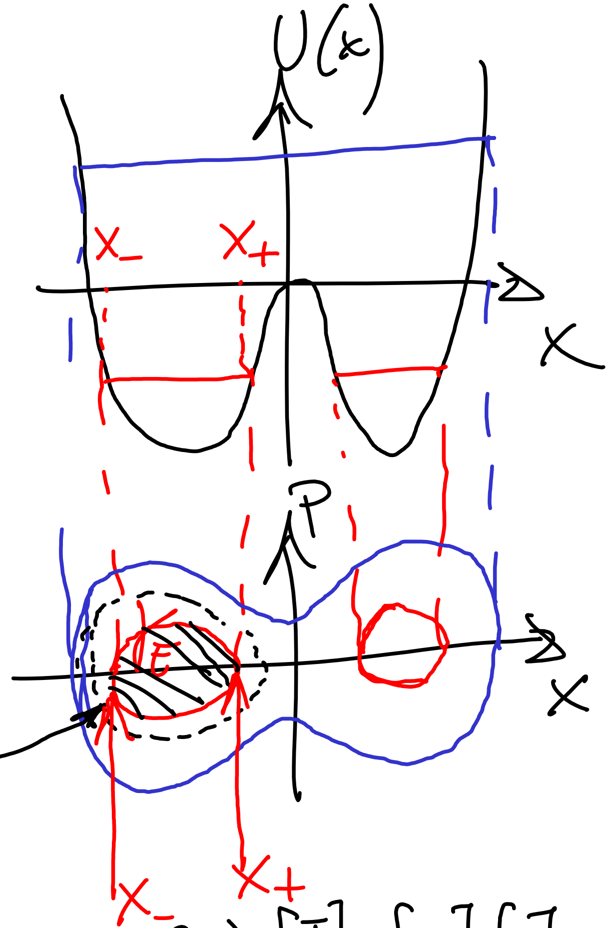
Esempio: $E = \frac{1}{2} m \dot{x}^2 + U(x)$

in fig. $U(x) = -\frac{a}{2} x^2 + \frac{b}{4} x^4$ con $a, b > 0$



Cerchiamo di introdurre coord. di angolo (e azione)

Introduciamo $I = \frac{1}{2\pi} \oint_{\gamma_E} p(E, x)\, dx$ è un'azione, cioè $[I] = [p][x]$

$\Rightarrow [I] = [E] \cdot [t]$, poiché $\dot{p} = -\frac{\partial H}{\partial q} \Rightarrow \frac{[p] \cdot [q]}{[t]} = [E]$

$\Rightarrow [p] \cdot [q] = [E] \cdot [t] = [\text{azione}!]$

$\Rightarrow$ se $[q]$ è un angolo $\Rightarrow [p] = [\text{azione}]$ -

$\Rightarrow S = \int p(I, x)\, dx$, l'angolo associato

sarà $\varphi = \frac{\partial S}{\partial I}$. Si osservi che

$$\oint_{e(\gamma_E)} p\, dq = \oint_{\gamma_E} I\, d\varphi = I \int_0^{2\pi} d\varphi = 2\pi I$$

Siccome $I = \frac{1}{2\pi} \oint p\, dx = \frac{1}{2\pi} 2 \int_{x_-}^{x_+} dx \sqrt{2m(E - U(x))}$ dove

$p = \frac{\partial}{\partial \dot{x}} \left(\frac{1}{2} m \dot{x}^2 - U(x) \right) = m\dot{x} \Rightarrow E = \frac{p^2}{2m} + U(x) \rightarrow p = \pm \sqrt{2m(E - U(x))}$

e $x_\pm$ t.c. $U(x_\pm) = E$.

Inoltre, abbiamo che $S(I, x) = \int dx\, p(I, x) = \int dx \sqrt{2m(E(I) - U(x))}$

$\varphi = \frac{\partial S}{\partial I}$, $p = \frac{\partial S}{\partial x} = \sqrt{2m(E(I) - U(x))}$

$\Rightarrow$ il periodo $T = \frac{2\pi}{\omega}$ dove $\omega = \dot{\varphi} = \frac{\partial H}{\partial I}$

$\Rightarrow T = 2\pi / \left(\frac{dH}{dI} \right)$. Siccome $H(E) = E \Rightarrow \frac{dH}{dI} \cdot \frac{dI}{dE} = 1 \rightarrow \frac{dI}{dE} = \left(\frac{dH}{dI} \right)^{-1}$

$$\Rightarrow \boxed{T = 2\pi \cdot \frac{d\bar{I}}{d\bar{E}}}$$

Infatti, $T = \frac{\cancel{2\pi} \cdot 2 \cdot d}{\cancel{2\pi} \; d\bar{E}} \int_{x_-}^{x_+} \sqrt{2m\left(E - U(x)\right)}\, dx$

$$= 2 \int_{x_-}^{x_+} \frac{\cancel{2}m\, dx}{\cancel{2}\sqrt{2m\left(E - U(x)\right)}} = 2 \int_{x_-}^{x_+} \frac{dx}{\sqrt{\frac{2}{m}\left(E - U(x)\right)}},$$

che è la formula per il calcolo del periodo per moti (periodici) in problemi meccanici 1D.

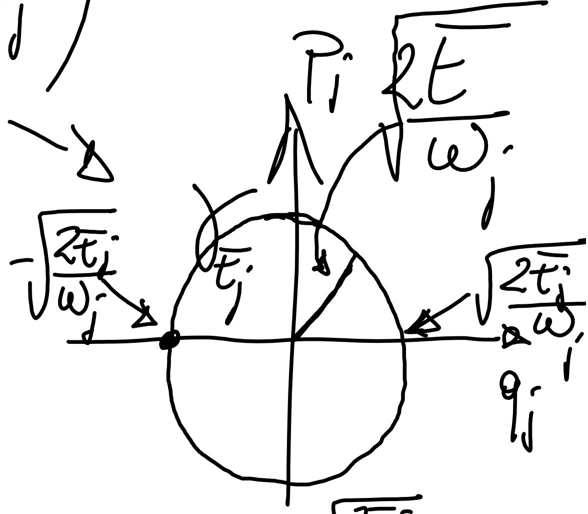
[Inciso: $\frac{1}{2} m\dot{x}^2 + U(x) = E \Rightarrow \dot{x} = \pm\sqrt{\frac{2}{m}\left(E - U(x)\right)} \Rightarrow dt = \frac{\pm dx}{\sqrt{\frac{2}{m}\left(E - U(x)\right)}}$

$\Rightarrow T = 2\int_{x_-}^{x_+} \frac{dx}{\sqrt{\frac{2}{m}\left(E - U(x)\right)}}$, cioè]

Altro esempio: $H = \sum_{j=1}^{n} \frac{\omega_j}{2}\left(p_j^2+q_j^2\right)$ [Sistema di n oscillatori armonici]

$\Rightarrow H = \sum_{j=1}^{n} E_j$ dove $E_j = \frac{\omega_j}{2}\left(p_j^2+q_j^2\right)$

Poniamo $I_j = \frac{1}{2\pi}\oint_{\gamma_{E_j}} p_j \, dq_j =$



$$= \frac{1}{\cancel{2}\pi}\cancel{2}\int_{-\sqrt{\frac{2E_j}{\omega_j}}}^{\sqrt{\frac{2E_j}{\omega_j}}} \sqrt{\frac{2E_j}{\omega_j} - q_j^2} = \frac{1}{\pi}\sqrt{\frac{2E_j}{\omega_j}}\int_{-\sqrt{\frac{2E_j}{\omega_j}}}^{\sqrt{\frac{2E_j}{\omega_j}}} \sqrt{1-\frac{\omega_j q_j^2}{2E_j}}\, dq_j$$

$$\Longrightarrow I_j = \frac{1}{\pi}\frac{2E_j}{\omega_j}\int_{-\sqrt{\frac{2E_j}{\omega_j}}}^{\sqrt{\frac{2E_j}{\omega_j}}} \sqrt{1-\frac{\omega_j^2 q_j^2}{2E_j}}\,\sqrt{\frac{\omega_j}{2E_j}}\,dq_j = \quad \text{sostituisco} \quad \sqrt{\frac{\omega_j}{2E_j}}\,q_j = \sin t$$

$$\Longrightarrow \sqrt{\frac{\omega_j}{2E_j}}\,dq_j = \cos t\,dt$$

$$I_j = \frac{1}{\pi}\frac{2E_j}{\omega_j}\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2 t}\,\cos t\,dt = \frac{1}{\pi}\frac{2E_j}{\omega_j}\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt$$

$$= \frac{1}{\cancel{\pi}}\frac{\cancel{2}E_j}{\omega_j}\,\frac{1}{\cancel{2}}\cdot\cancel{\pi} = \frac{E_j}{\omega_j} \Longrightarrow E_j = \omega_j I_j$$

$$S = \sum_{j=1}^{n}\int p(I_j, q_j)\,dq_j = \sum_{j=1}^{n}\int\sqrt{\frac{2\cancel{\omega_j} I_j}{\cancel{\omega_j}} - q_j^2}\,dq_j \Longrightarrow \varphi_j = \frac{\partial S}{\partial I_j} = \int\frac{\cancel{2}\,dq_j}{\cancel{2}\sqrt{2I_j - q_j^2}}$$

$$\varphi_j = \frac{1}{\sqrt{2I_j}} \int \frac{dq_j}{\sqrt{1-\frac{q_j^2}{2I_j}}} = \arcsin\left(\frac{q_j}{\sqrt{2I_j}}\right)$$

$$\Rightarrow \begin{cases} q_j = \sqrt{2I_j}\,\sin\varphi_j & , \text{ siccome } p_j = \sqrt{2I_j - q_j^2}\,) \\ p_j = \sqrt{2I_j}\,\cos\varphi_j & \forall\, j = 1, \dots, n \end{cases}$$

$$\Rightarrow H = \sum_{j=1}^{n} \frac{\omega_j}{2}\left(p_j^2 + q_j^2\right) = \sum_{j=1}^{n} \frac{\omega_j}{\cancel{2}}\, \cancel{2} I_j = \sum_{j=1}^{n} \omega_j I_j$$

Def.: flusso Hamiltoniano $\phi_H^t(p,q)$ la mappa t.c. $\phi_H^t(p(0),q(0)) = (p(t),q(t))$

dove $\tau \mapsto (p(\tau), q(\tau))$ è sol. delle eq. di Ham. per H

Proposizione: Il flusso Hamiltoniano induce una trasf. canonica

Dim.: Vogliamo dim. che ϕ_H^t preserva le par. di Poisson fondamentali, cioè $\frac{d}{dt}\{\phi_H^t q_i, \phi_H^t p_j\}_{p,q} = \frac{d}{dt}\{\phi_H^t q_i, \phi_H^t q_j\}$

$\frac{d}{dt}\{\phi_H^t q_i, \phi_H^t p_j\} = 0$. Osserviamo che

$$\lim_{h \to 0} \frac{\{\phi_H^{t+h} q_i, \phi_H^{t+h} p_j\} - \{\phi_H^t q_i, \phi_H^t p_j\}}{h}$$

, siccome $\dot{f} = \{f, H\}$ la der. temporale dipende solo dalla posizione $\phi^t(p,q)$

$\Rightarrow$ mi basta provare che $\frac{d}{dt}\left\{\Phi_H^t q_i, \Phi_H^t p_j\right\}\Big|_{t=0} = 0$

$\forall (p, q)$

$$\Rightarrow \frac{d}{dt}\left\{\Phi_H^t q_i, \Phi_H^t p_j\right\}_{t=0} = \lim_{t\to 0} \frac{\left\{\Phi_H^t q_i, \Phi_H^t p_j\right\} - \left\{q_i, p_j\right\}}{t}$$

$$= \lim_{t\to 0} \frac{\left\{q_i + \Phi_H^t q_i, p_j + \Phi_H^t p_j\right\} - \left\{q_i, p_j\right\}}{t} = \lim_{t\to 0} \frac{\cancel{\{q_i, p_j\}} + \cancel{t}\{\dot q_i, p_j\} + \cancel{t}\{q_i, \dot p_j\} - \cancel{\{q_i, p_j\}}}{\cancel{t}}$$

$$\Rightarrow = \left\{\dot q_i, p_j\right\} + \left\{q_i, \dot p_j\right\} = \left\{\frac{\partial H}{\partial p_i}, p_j\right\} + \left\{q_i, -\frac{\partial H}{\partial q_j}\right\} = \frac{\partial^2 \cancel{H}}{\partial q_j \partial p_i} - \frac{\partial^2 \cancel{H}}{\partial p_i \partial q_j} = 0$$

Analogamente (ma un po' più veloce), $\frac{d}{dt}\{q_i, q_j\} = \{\{q_i, q_j\}, H\} =$

$$= -\{\{q_j, H\}, q_i\} - \{\{H, q_i\}, q_j\} = -\left\{\frac{\partial H}{\partial p_j}, q_i\right\} + \left\{\frac{\partial H}{\partial p_i}, q_j\right\} =$$

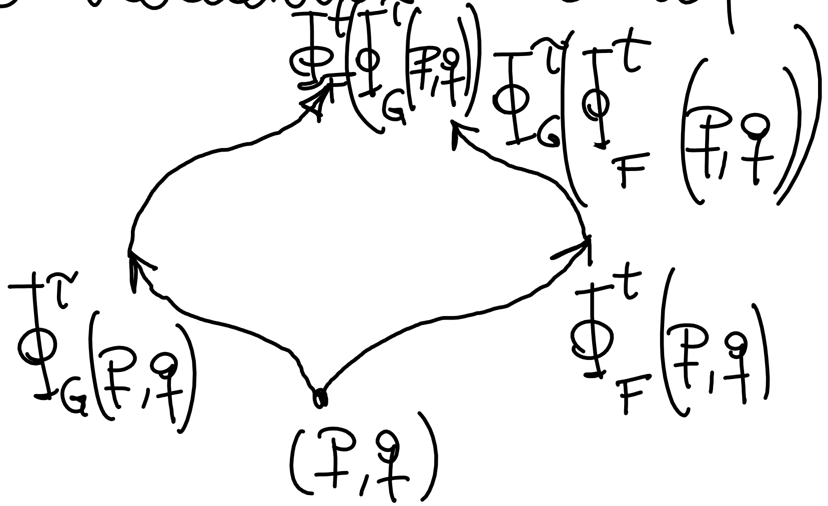
$$= \frac{\partial^2 H}{\partial p_i \partial p_j} - \frac{\partial^2 H}{\partial p_j \partial p_i} = 0$$

(dove abbiamo usato Jacobi). C.V.D.

Siano $F(\underline{p}, \underline{q})$, $G(\underline{p}, \underline{q})$ 2 variabili dinamiche

Poniamo $\underline{z} = (\underline{p}, \underline{q})$ e introduciamo

$L_F \cdot = \{\cdot, F\}$, cioè $\dot{p}_j = \{p_j, H\} = L_H p_j$, $\dot{q}_j = L_H q_j$



Consideriamo $\Phi^t_F \Phi^\tau_G \underline{z} - \Phi^\tau_G \Phi^t_F \underline{z} = \cancel{\underline{z}} + \tau \cancel{L_G \underline{z}} + \frac{\tau^2}{2} \cancel{L^2_G \underline{z}} + t \cancel{L_F \underline{z}} + \frac{t^2}{2} \cancel{L^2_F \underline{z}} +$

$$+ t\tau L_F L_G \underline{z} - \left(\cancel{\underline{z}} + t \cancel{L_F \underline{z}} + \frac{t^2}{2} \cancel{L^2_F \underline{z}} + \tau \cancel{L_G \underline{z}} + \frac{\tau^2}{2} \cancel{L^2_G \underline{z}} + t\tau L_G L_F \underline{z} \right)$$

$$+ o\left(\|(t,\tau)\|^2 \right)$$

$$= t \cdot \tau \left(L_F L_G \underline{z} - L_G L_F \underline{z} \right) + o\left(\|(t,\tau)\|^2 \right)$$

Oss.: $\underline{\text{se } \{F, G\} = 0}$ allora

$$\Phi^t_F \Phi^\tau_G \underline{z} - \Phi^\tau_G \Phi^t_F \underline{z} = o\left(\|(t,\tau)\|^2 \right),$$ poiché

$$L_F L_G \underline{z} - L_G L_F \underline{z} = \{\{\underline{z}, G\}, F\} - \{\{\underline{z}, F\}, G\} = \{\{\underline{z}, G\}, F\} + \{\{F, \underline{z}\}, G\}$$

(per Jacobi) $= -\{\{G, F\}, z\} = -\{0, z\} = 0$.

Proposizione: Siano F, G t.c. $\{F, G\} = 0$

allora (1) $F(\Phi_G^\tau p, \Phi_G^\tau q) = F(p, q)$;

(2) $\forall t, \tau$ si ha che $\Phi_F^t \Phi_G^\tau (p, q) = \Phi_G^\tau \Phi_F^t (p, q)$

Dim. per la (1) $\dot F = \frac{d}{d\tau} F(\Phi_G^\tau (p, q)) = \{F, G\} = 0$.

per la 2 si faccia la der. direzionale per $f(\alpha) = (\Phi_F^{\alpha t} \phi_G^{\alpha\tau} - \phi_G^{\alpha\tau} \Phi_F^{\alpha t})(p, q)$ con $\alpha \in [0, 1]$

$\Rightarrow f'(\alpha) = (\underline{0}, 0)$ e $f(0) = (\underline{0}, \underline{0}) \Rightarrow \forall \alpha \quad f(\alpha) = (\underline{0}, \underline{0})$

C.V.D.